H_∞ Fixed-Lag Smoothing and Prediction for Linear Continous-Time Systems[1]

Huanshui Zhang[a], David Zhang[a], Lihua Xie [b 2]

[a] Department of Computing
The Hong Kong Polytechnic University, Hong Kong

[b] BLK S2, School of Electrical and Electronic Engineering
Nanyang Technological University, Singapore 639798
Email: elhxie@ntu.edu.sg

Abstract

This paper addresses the H_∞ fixed-lag smoothing and prediction problems for linear continuous-time systems. We first present a solution to the optimal H_2 estimation problem for linear continuous-time systems with instantaneous and delayed measurements. It is then shown that the H_∞ fixed-lag smoothing and prediction problems can be converted to the latter problem in Krein space. Therefore, the H_2 estimation is extended to give conditions on the existence of a H_∞ fixed-lag smoother and predictor based on innovation analysis and projection in Krein space and a solution for H_∞ smoother or predictor is given in terms of a Riccati differential equation and matrix differential equations.

1 Introduction

The smoothing problem is to estimate a linear combination of system states at time instant t based on measurements up to $t+h$ for any $h>0$. It is usually classified into three categories, namely the fixed-point smoothing, fixed interval smoothing and fixed-lag smoothing. Note that in the discrete-time case where the delay dynamics are finite dimensional, the smoothing problem may be converted into standard filtering through state augmentation [8] or can be solved through a recently developed direct approach [10]. However, for the continuous-time case, the delay dynamics are infinite dimensional, the smoothing problem has been proven to be difficult. The problem of H_∞ fixed-interval smoothing has been solved in [6] for linear continuous-time systems and the optimal fixed interval smoother has been known to be the same as the H_2 optimal smoother [1]. However, for fixed point smoothing and fixed-lag smoothing, the H_2 and H_∞ smoothing will be different. Recently, Mirkin [5] has tackled the infinite horizon H_∞ fixed-lag smoothing problem using a spectral factorization approach. The finite horizon H_∞ fixed lag smoothing and prediction for continuous-time systems remains open.

In this paper, we shall address the finite horizon H_∞ fixed lag smoothing and prediction problems. We first discuss the H_2 estimation for linear continuous-time systems with instantaneous and delayed measurements. Note that the H_2 filtering of systems with state delays has attracted a lot of interest in the past (see, e.g. [2]-[3],[4]) where solutions that are infinite dimensional in nature have been obtained. In the present paper, through re-organized innovation, we derive a solution for the optimal H_2 estimator using innovation analysis and projection in Hilbert space. It is demonstrated that the H_∞ estimation for linear continuous-time systems with instantaneous measurement can be converted to the H_2 estimation for systems with instantaneous measurement and fictitious delayed 'measurement' in Krein space [7]. Therefore, the H_2 estimation is extended to obtain the existence condition of the H_∞ fixed lag smoother. A solution to the fixed lag smoother is given in terms of Riccati differential equation and matrix differential equations. We also present a solution to the prediction problem.

Due to the space limitation, all proofs will be omitted. The detail can be found in [11].

2 Problem Statement

In this section we shall state the H_2 and H_∞ estimation problems to be investigated in the following sections.

2.1 H_2 Optimal Estimation

We shall consider the linear stochastic system described by

$$\dot{\mathbf{x}}(t) = F(t)\mathbf{x}(t)+\Gamma(t)\mathbf{u}(t) \quad (1)$$

$$\mathbf{y}(t) = H(t)\mathbf{x}(t)+\mathbf{v}(t) \quad (2)$$

$$\mathbf{z}_{t-h}(t) = L(t-h)\mathbf{x}(t-h)+\mathbf{v}_z(t) \quad (3)$$

[1]This work is supported by National Nature Science Foundation of China (60174017).

[2]Corresponding author

where $\mathbf{x}(t) \in R^n$ is the state, $\mathbf{u}(t) \in R^r$ is the input noise, $\mathbf{y}(t) \in R^m$ and $\mathbf{z}_{t-h}(t) \in R^p$ are respectively the instantaneous and delayed measurements, $\mathbf{v}(t) \in R^m$ and $\mathbf{v}_z(t) \in R^p$ are the measurement noises. The initial state $\mathbf{x}(0)$ and $\mathbf{u}(t)$, $\mathbf{v}(t)$ and $\mathbf{v}_z(t)$ are uncorrelated white noises with zero means and known covariance matrices $\langle \mathbf{x}(0),\ \mathbf{x}(0)\rangle = P_0$, $\langle \mathbf{u}(t),\ \mathbf{u}(s)\rangle = Q_u(t)\delta(t-s)$, $\langle \mathbf{v}(t),\ \mathbf{v}(s)\rangle = Q_v(t)\delta(t-s)$ and $\langle \mathbf{v}_z(t),\ \mathbf{v}_z(s)\rangle = Q_{v_z}(t)\delta(t-s)$, respectively.

The H_2 optimal estimation problem can be stated as:

H_2 **Estimation Problem**: *Given the measurements up to the time instant $t+h$, $\{\mathbf{y}(s),\ 0 \le s \le t+h;\ \mathbf{z}_s(s+h),\ 0 \le s \le t\}$, find a linear mean square error estimator of $\mathbf{x}(t)$, which is denoted as $\hat{\mathbf{x}}(t \mid t+h,t)$.*

2.2 H_∞ Estimation

We consider the following linear system for the H_∞ estimation problem.

$$\dot{x}(t) = F(t)x(t) + \Gamma(t)u(t) \quad (4)$$
$$y(t) = H(t)x(t) + v(t) \quad (5)$$
$$z(t) = L(t)x(t) \quad (6)$$

where $x(t) \in R^n$, $u(t) \in R^r$, $y(t) \in R^m$, $v(t) \in R^m$ and $z(t) \in R^p$ represent the state, input noise, measurement output, measurement noise and the signal to be estimated, respectively. It is assumed that the input and measurement noises are deterministic signals and are from $L_2[0,T]$ where T is the time-horizon of the estimation problem under investigation.

H_∞ **Estimation Problem:** *Given a scalar $\gamma > 0$, an integer h and the observation $\{y(s),\ s \le t+h\}$, find an estimate $\check{z}(t \mid t+h)$ of $z(t)$, if it exists, such that the following inequality is satisfied:*

$$\sup_{(x_0,u,v)\neq 0} \frac{\int_0^T \left\{[\check{z}(t \mid t+h) - z(t)]^T [\check{z}(t \mid t+h) - z(t)]\right\} dt}{x^T(0)P_0^{-1}x(0) + \int_0^{T+h_0} u^T(t)u(t)dt + \int_0^{T+h} v^T(t)v(t)dt} < \gamma^2 \quad (7)$$

where $h_0 = max\{0,h\}$.

Observe that the above problem includes three cases: (a) H_∞ fixed-lag smoothing problem ($h > 0$); (b) H_∞ filtering problem ($h = 0$) and (c) H_∞ prediction ($h < 0$). The filtering problem has been addressed in [6].

3 H_2 Optimal Estimation

In this section, we shall present a solution to the H_2 estimation for the system (1)-(3) involving instantaneous and delayed measurements using the projection in Hilbert space. The key to our discussion in this section is to organize the instantaneous and delayed measurements in a proper way and introduce an associated innovation sequence.

It is easy to know that the linear space $\{\mathbf{y}(s),\ 0 \le s \le t+h;\ \mathbf{z}_s(s+h),\ 0 \le s \le t\}$, can be reorganized as

$$\mathcal{L}\{\mathbf{y}_f(s), 0 \le s \le t;\ \mathbf{y}(s), t < s \le t+h\} \quad (8)$$

where

$$\mathbf{y}_f(s) \triangleq \begin{bmatrix} \mathbf{y}(s) \\ \mathbf{z}_s(s+h) \end{bmatrix} = \begin{bmatrix} H(s) \\ L(s) \end{bmatrix} \mathbf{x}(s) + \mathbf{v}_f(s) \quad (9)$$
$$\mathbf{y}(s) = H(s)\mathbf{x}(s) + \mathbf{v}(s), \quad (10)$$

with

$$\mathbf{v}_f(s) = \begin{bmatrix} \mathbf{v}(s) \\ \mathbf{v}_z(s+h) \end{bmatrix} \quad (11)$$

being a white noise of zero mean and variance matrix $Q_{v_f}(s) = \begin{bmatrix} Q_v(s) & 0 \\ 0 & Q_{v_z}(s+h) \end{bmatrix}$. It should be noted that $\mathbf{y}_f(s)$ contains measurements of the state $\mathbf{x}(s)$ at time instants s and $s+h$.

Observe that (1) and (9) or (1) and (10) form a standard state space representation.

3.1 Innovation sequence and Riccati equation

First, we introduce the following stochastic sequence:

$$\mathbf{w}(s,t) \triangleq \mathbf{y}(s) - \hat{\mathbf{y}}(s \mid s,t),\ s > t \quad (12)$$
$$\mathbf{w}(t,t) \triangleq \mathbf{y}_f(t) - \hat{\mathbf{y}}_f(t \mid t,t) \quad (13)$$

where $\hat{\mathbf{y}}_f(t \mid t,t)$ is the linear least mean square (LLMS) estimate of $\mathbf{y}_f(t)$ given $\{\mathbf{y}_f(s),\ 0 \le s < t\}$ and $\hat{\mathbf{y}}(s \mid s,t)$ is the LLMS estimate of $\mathbf{y}(s)$ given $\{\mathbf{y}_f(r),\ 0 \le r \le t;\ \ \mathbf{y}(r),\ t < r < s\}$. It is clear that $\mathbf{w}(t,t)$ is the innovation associated with the standard Kalman filtering based on observation $\mathbf{y}_f(t)$. Then we have the following relationships

$$\begin{aligned} \mathbf{w}(s,t) &= \mathbf{y}(s) - H(s)\hat{\mathbf{x}}(s \mid s,t) \\ &= H(s)\mathbf{e}(s,t) + \mathbf{v}(s),\ \ s > t \end{aligned} \quad (14)$$
$$\begin{aligned} \mathbf{w}(t,t) &= \mathbf{y}_f(t) - \begin{bmatrix} H(t) \\ L(t) \end{bmatrix} \hat{\mathbf{x}}(t \mid t,t) \\ &= \begin{bmatrix} H(t) \\ L(t) \end{bmatrix} \mathbf{e}(t,t) + \mathbf{v}_f(t) \end{aligned} \quad (15)$$

where

$$\mathbf{e}(s,t) = \mathbf{x}(s) - \hat{\mathbf{x}}(s \mid s,t),\ s \ge t \quad (16)$$

while $\hat{\mathbf{x}}(s \mid s,t)$ is defined similarly to $\hat{\mathbf{y}}(s \mid s,t)$ and $\hat{\mathbf{y}}_f(t \mid t,t)$. The following lemma shows that $\{\mathbf{w}(\cdot,\cdot)\}$ is indeed an innovation sequence.

Lemma 1

$$\{\mathbf{w}(s,s),\ 0 \le s \le t;\quad \mathbf{w}(s,t),\ t < s \le t+h\} \tag{17}$$

is the innovation which is an uncorrelated white noise and spans the same linear space as

$$\mathcal{L}\{\mathbf{y}_f(s), 0 \le s \le t;\ \mathbf{y}(s),\ t < s \le t+h\} \tag{18}$$

Let $\mathcal{P}^r_{s,t} \triangleq \langle \mathbf{x}(r),\ \mathbf{e}(s,t)\rangle;\ r,s \ge t$ be the cross-covariance matrix of the state $\mathbf{x}(r)$ with the state estimation error $\mathbf{e}(s,t)$, where $\mathbf{e}(s,t)$ is as defined in (14).

Then, from (14) and (15), the innovation covariance matrix

$$Q_w(s,t) \triangleq \langle \mathbf{w}(s,t),\ \mathbf{w}(s,t)\rangle,\ s \ge t$$

can be given as ([1])

$$Q_w(s,t) = \begin{cases} Q_v(s), & s > t \\ Q_{v_f}(t), & s = t \end{cases} \tag{19}$$

Theorem 1 *The cross-covariance matrix $\mathcal{P}^r_{s,t},\ r,s \ge t$ can be calculated as follows:*

1) *For $r = s = t$, $\mathcal{P}^t_{t,t}$ is the solution to the following standard Riccati equation*

$$\begin{aligned} \frac{d\,\mathcal{P}^t_{t,t}}{dt} &= F(t)\mathcal{P}^t_{t,t} + \mathcal{P}^t_{t,t}F^T(t) - \mathcal{K}^t_{t,t}Q_{v_f}(t)\left[\mathcal{K}^t_{t,t}\right]^T + \\ &\quad \Gamma(t)Q_u(t)\Gamma^T(t),\ \ \mathcal{P}^t_{t,t}(0) = P_0 \end{aligned} \tag{20}$$

with

$$\mathcal{K}^t_{t,t} = \mathcal{P}^t_{t,t}\begin{bmatrix} H(t) \\ L(t)\end{bmatrix}^T Q^{-1}_{v_f}(t) \tag{21}$$

2) *For $r = s > t$, $\mathcal{P}^r_{s,t}$ is the solution to the following standard Riccati equation*

$$\begin{aligned} \frac{d\,\mathcal{P}^s_{s,t}}{ds} &= \Phi(s)\mathcal{P}^s_{s,t} + \mathcal{P}^s_{s,t}\Phi^T(s) - \mathcal{K}^s_{s,t}Q_v(s)[\mathcal{K}^s_{s,t}]^T + \\ &\quad \Gamma(s)Q_u(s)\Gamma^T(s) \end{aligned} \tag{22}$$

where the initial condition $\mathcal{P}^t_{t,t}(t)$ is calculated in (20) and

$$\mathcal{K}^s_{s,t} = \mathcal{P}^s_{s,t}H^T(s)Q^{-1}_v(s) \tag{23}$$

3) *For $r > s > t$, $\mathcal{P}^r_{s,t}$ is the unique solution of the linear matrix differential equation*

$$\frac{d\,\mathcal{P}^r_{s,t}}{dr} = \Phi(r)\mathcal{P}^r_{s,t} \tag{24}$$

with the initial value $\mathcal{P}^s_{s,t}$ as calculated in (22).

4) *For $t < r < s$, $\mathcal{P}^r_{s,t}$ is the unique solution of the linear matrix differential equation*

$$\frac{d\,\mathcal{P}^r_{s,t}}{ds} = \mathcal{P}^r_{s,t}A^T(s,t) \tag{25}$$

where the initial condition $\mathcal{P}^r_{r,t}(r)$ is calculated in (22) and

$$A(s,t) = F(s)\left\{I_n - \mathcal{K}^s_{s,t}H(s)\right\} \tag{26}$$

$\mathcal{K}^s_{s,t}$ *is as (23).*

3.2 Optimal estimate $\hat{\mathbf{x}}(t \mid t+h,t)$ with $h \ge 0$

Based on the discussion in the previous subsection, the following result is obtained immediately.

Theorem 2 *Consider the system (1)-(3). The optimal estimator is given by*

$$\begin{aligned} \hat{\mathbf{x}}(t \mid t+h,t) &= \hat{\mathbf{x}}(t \mid t,t) + \int_t^{t+h} \mathcal{P}^t_{s,t}H^T(s)Q_v^{-1} \\ &\quad \times [\mathbf{y}(s) - H(s)\hat{\mathbf{x}}(s \mid s,t)]\,ds \end{aligned} \tag{27}$$

where $\hat{\mathbf{x}}(t \mid t,t)$ is the standard Kalman filtering estimate as given by

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t \mid t,t) &= F(t)\hat{\mathbf{x}}(t \mid t,t) + \mathcal{K}^t_{t,t}\mathbf{w}(t,t) \\ &= \left\{F(t) - \mathcal{K}^t_{t,t}\begin{bmatrix} H(t) \\ L(t)\end{bmatrix}\right\}\hat{\mathbf{x}}(t \mid t,t) + \mathcal{K}^t_{t,t}\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{z}_t(t+h)\end{bmatrix} \end{aligned} \tag{28}$$

with the initial estimate $\hat{\mathbf{x}}(0 \mid 0,0)$, and $\hat{\mathbf{x}}(s \mid s,t),\ s > t$ is given by

$$\begin{aligned} \frac{d\,\hat{\mathbf{x}}(s \mid s,t)}{ds} &= F(s)\hat{\mathbf{x}}(s \mid s,t) + \mathcal{K}^s_{s,t}\mathbf{w}(s,t) \\ &= \left\{F(s) - \mathcal{K}^s_{s,t}H(s)\right\}\hat{\mathbf{x}}(s \mid s,t) + \mathcal{K}^s_{s,t}\mathbf{y}(s), \end{aligned} \tag{29}$$

with the initial condition $\hat{\mathbf{x}}(t \mid t,t)$. In the above, $\mathcal{K}^s_{t,t}$ is as (21) and $\mathcal{K}^s_{s,t}, t < s$ is as (23). The matrix $\mathcal{P}^t_{s,t},\ t \le s$ and $\mathcal{P}^s_{s,t},\ t \le s$ are given in Theorem 1.

4 H_∞ Estimation

In this section, we shall present solutions to the H_∞ fixed-lag smoothing and prediction problems. We first demonstrate that the H_∞ smoothing and prediction can be converted to the H_2 ones in Krein space. The innovation and projection method of the last section is then extended to give a solution to the H_∞ fixed-lag smoothing and prediction.

To start with, we introduce the following cost function in association with the system (4)-(6)

$$\begin{aligned} J_s(t+h) &\triangleq x^T(0)P_0^{-1}x(0) + \int_0^{T+h_0} u^T(s)u(s)ds \\ &+ \int_0^{T+h}[y(s) - H(s)x(s)]^T[y(s) - H(s)x(s)]ds \\ &-\gamma^{-2}\int_0^T [\check{z}(s \mid s+h) - L(s)x(s)]^T \\ &\times[\check{z}(s \mid s+h) - L(s)x(s)]ds \end{aligned} \tag{30}$$

In the case of fixed-lag smoothing ($h > 0$ and thus $h_0 = h$), we define

$$\check{z}(s \mid s+h) = 0,\ x(s) = 0,\ L(s) = 0,\ \ for\ s < 0$$

Then, (30) can be rewritten as

$$\begin{aligned} &J_s(t+h) = x^T(0)P_0^{-1}x(0) + \int^{T+h} u^T(s)u(s)ds \\ &+ \int_0^{T+h} [y_z(s) - H_L(s)\bar{x}(s)]^T \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^{-2}I_p \end{bmatrix} \\ &\times [y_z(s) - H_L(s)\bar{x}(s)]\, ds \end{aligned} \tag{31}$$

where

$$y_z(s) = \begin{bmatrix} y(s) \\ \check{z}(s-h \mid s) \end{bmatrix}, \quad \bar{x}(s) = \begin{bmatrix} x(s) \\ x(s-h) \end{bmatrix} \tag{32}$$

$$H_L(s) = \begin{bmatrix} H(s) & 0 \\ 0 & L(s-h) \end{bmatrix} \tag{33}$$

For the case of prediction ($h < 0$ and thus $h_0 = 0$), we assume

$$y(s) = 0,\ x(s) = 0,\ H(s) = 0,\ \ for\ s < 0$$

In this situation, (30) can be rewritten as

$$\begin{aligned} &J_s(t+h) = x^T(0)P_0^{-1}x(0) + \int_0^T u^T(s)u(s)ds \\ &+ \int_0^T [y_z(s+h) - H_L(s+h)\bar{x}(s+h)]^T \times \\ &\begin{bmatrix} I_m & 0 \\ 0 & -\gamma^{-2}I_p \end{bmatrix} [y_z(s+h) - H_L(s+h)\bar{x}(s+h)]\, ds \end{aligned} \tag{34}$$

where $y_z(s)$, $\bar{x}(s)$, $H_L(s)$ are as in (32) and (33).

4.1 Stochastic system in Krein space

To derive an H_∞ estimator, the following stochastic system associated with (4)-(6) is introduced:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \Phi(t)\mathbf{x}(t) + \Gamma(t)\mathbf{u}(t) && (35) \\ \mathbf{y}(t) &= H(t)\mathbf{x}(t) + \mathbf{v}(t) && (36) \\ \check{\mathbf{z}}(t-h \mid t) &= L(t-h)\mathbf{x}(t-h) + \mathbf{v}_z(t) && (37) \end{aligned}$$

where $\mathbf{u}(t)$, $\mathbf{v}(t)$ and $\mathbf{v}_z(t)$ are uncorrelated white noises, with zero means and $\langle \mathbf{u}(t), \mathbf{u}(s)\rangle = Q_u(t)\delta(t-s)$, $\langle \mathbf{v}(t), \mathbf{v}(s)\rangle = Q_v(t)\delta(t-s)$, $\langle \mathbf{v}_z(t), \mathbf{v}_z(s)\rangle = Q_{v_z}(t)\delta(t-s)$, $\left\langle \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}_z(t) \end{bmatrix}, \begin{bmatrix} \mathbf{v}(s) \\ \mathbf{v}_z(s) \end{bmatrix} \right\rangle = Q_{v_f}(t)\delta(t-s)$, and

$$\begin{aligned} Q_u(t) &= I_r,\ Q_v(t) = I_m,\ Q_{v_z}(t) = -I_p\gamma^2, \\ Q_{v_f}(t) &= \begin{bmatrix} I_m & 0 \\ 0 & -I_p\gamma^2 \end{bmatrix}. \end{aligned} \tag{38}$$

Observe that the introduced stochastic system (35)-(37) is of the same form as that of (1)- (3) for which the H_2 smoothing has been considered. In the following, we'll adopt similar notations as defined in the last section.

Observe that the difference between the stochastic model (35)-(37) and the stochastic model (1)-(3) is that the latter involves a fictitious 'measurement' $\check{\mathbf{z}}$ with error covariance being negative definite. Therefore, in the sequel the innovation and projection associated with the latter system will be discussed in Krein space [7] rather than in the usual Hilbert space. For detail of Krein space, readers are referred to [7].

We now introduce the innovation of the observation $\mathbf{y}_z(t)$ as

$$\mathbf{w}_z(t) = \mathbf{y}_z(t) - \hat{\mathbf{y}}_z(t \mid t) \tag{39}$$

where $\mathbf{y}_z(t)$ is as in (32),

$$\mathbf{y}_z(t) = \begin{bmatrix} \mathbf{y}(t) \\ \check{\mathbf{z}}(t-h \mid t) \end{bmatrix} \tag{40}$$

and $\hat{\mathbf{y}}_z(t \mid t)$, if it exists, is the projection of $\mathbf{y}_z(t)$ on the linear Krein space $\mathcal{L}\{\mathbf{y}_z(s), s \le t\}$. In view of (32), $\mathcal{L}\{\mathbf{y}_z(s), s \le t\}$ is equivalent to

$$\mathcal{L}\{\mathbf{y}_f(s), 0 \le s \le t-h;\ \mathbf{y}(s), t-h < s \le t\} \tag{41}$$

for $h \ge 0$ or

$$\mathcal{L}\{\mathbf{y}_f(s), 0 \le s \le t;\ \check{\mathbf{z}}(s \mid s+h), t < s \le t-h\} \tag{42}$$

for $h < 0$, where

$$\mathbf{y}_f(s) = \begin{bmatrix} \mathbf{y}(s) \\ \check{\mathbf{z}}(s \mid s+h) \end{bmatrix} \tag{43}$$

Thus, $\mathbf{w}_z(t)$ can be rewritten as

$$\mathbf{w}_z(t) = \begin{bmatrix} \mathbf{y}(t) \\ \check{\mathbf{z}}(t-h \mid t) \end{bmatrix} - H_L(t) \begin{bmatrix} \hat{\mathbf{x}}(t \mid t, t-h) \\ \hat{\mathbf{x}}(t-h \mid t, t-h) \end{bmatrix} \tag{44}$$

where

- For $h \ge 0$, $\hat{\mathbf{x}}(t \mid t, t-h)$ and $\hat{\mathbf{x}}(t-h \mid t, t-h)$ are the projections of $\mathbf{x}(t)$ and $\mathbf{x}(t-h)$ on the linear Krein space of (41).
- For $h < 0$, $\hat{\mathbf{x}}(t \mid t, t-h)$ and $\hat{\mathbf{x}}(t-h \mid t, t-h)$ are the projections of $\mathbf{x}(t)$ and $\mathbf{x}(t-h)$ on the linear Krein space of (42).

4.2 Sufficient and necessary condition for the existence of an H_∞ estimator

The following result presents a necessary and sufficient condition for the existence of an H_∞ estimator.

Theorem 3 *Consider the system (4)-(6) and the associated performance criterion (7). Then, for a given scalar $\gamma > 0$, an estimator $\check{z}(t \mid t+h)(0 \leq t \leq T)$ that achieves (7) exists if and only if $\hat{x}(t+h \mid t+h,t)$ exists for $-h \leq t \leq T$ and $\hat{x}(t \mid t+h,t)$ exists for $0 \leq t \leq T$, where for $h \geq 0$, $\hat{x}(t+h \mid t+h,t)$ and $\hat{x}(t \mid t+h,t)$ are respectively the projections of $\mathbf{x}(t+h)$ and $\mathbf{x}(t)$ on the linear Krein space of*

$$\{\mathbf{y}_f(s),\ 0 \leq s \leq t;\ \mathbf{y}(s),\ t \leq s \leq t+h\} \tag{45}$$

and for $h < 0$, $\hat{x}(t+h \mid t+h,t)$ and $\hat{x}(t \mid t+h,t)$ are respectively the projections of $\mathbf{x}(t+h)$ and $\mathbf{x}(t)$ on the linear Krein space of

$$\{\mathbf{y}_f(s),\ 0 \leq s \leq t+h;\ \check{\mathbf{z}}(s \mid s+h),\ t+h \leq s \leq t\} \tag{46}$$

In this situation, an H_∞ estimator $\check{z}(t \mid t+h)$ is given as

$$\check{z}(t \mid t+h) = L(t)\hat{x}(t \mid t+h,t) \tag{47}$$

Remark 1 *From Theorem 3, the computation of an H_∞ estimator $\check{z}(t \mid t+h)$ is converted into the problem of calculating the projection $\hat{x}(t \mid t+h,t)$ in Krein space, which is similar to the H_2 estimation we have studied in the previous section. The difference is that the projection $\hat{x}(t \mid t+h,t)$ in Hilbert space is guaranteed to be solvable under very mild conditions, which may not be the case in Krein space as the covariance $Q_{v_z} < 0$.*

4.3 The calculation of an H_∞ estimator

In order to compute an H_∞ estimator $\check{z}(t \mid t+h)$, we need to check the existence of $\hat{x}(t+h \mid t+h,t)(-h \leq t \leq T)$ and $\hat{x}(t \mid t+h,t)(0 \leq t \leq T)$ and compute $\hat{x}(t \mid t+h,t)$, if it exists.

For the case of filter ($h = 0$), a solution has been given in [6] which is related to the following Riccati differential equation:

$$\frac{d\,\mathcal{P}^t_{t,t}}{dt} = F(t)\mathcal{P}^t_{t,t} + \mathcal{P}^t_{t,t}F^T(t) + \Gamma(t)\Gamma^T(t) - \mathcal{P}^t_{t,t}\left[H^T(t)H(t) - \gamma^{-2}L^T(t)L(t)\right]\mathcal{P}^t_{t,t},\ \ \mathcal{P}^t_{t,t}(0) = P_0 \tag{48}$$

A. Smoother ($h > 0$)

When $h > 0$, the projections $\hat{x}(t \mid t+h,t)$ and $\hat{x}(t+h \mid t+h,t)$ in Krein space are similar to the case of H_2 estimation discussed in the last Section except that the former involves an indefinite covariance matrix. The projections $\hat{x}(t \mid t+h,t)$ and $\hat{x}(t+h \mid t+h,t)$ in Krein space are obtained directly from Theorem 2 as

$$\hat{x}(t \mid t+h,t) = \hat{x}(t \mid t,t) + \int_t^{t+h} \mathcal{P}^t_{s,t}H^T(s)\left[y(s) - H(s)\hat{x}(s \mid s,t)\right]ds \tag{49}$$

and

$$\frac{d\,\hat{x}(s \mid s,t)}{ds} = \left\{F(s) - \mathcal{P}^s_{s,t}H^T(s)H(s)\right\}\hat{x}(s \mid s,t) + \mathcal{P}^s_{s,t}H^T(s)y(s),\ \hat{x}(s \mid s,t)\mid_{s=t} = \hat{x}(t \mid t,t) \tag{50}$$

where $t \leq s \leq t+h$ and $\hat{x}(t \mid t,t)$ is given in (28), with the replacement of $z_t(t+h)$ by $\check{z}(t \mid t+h)$, as

$$\dot{\hat{x}}(t \mid t,t) = \left\{\Phi(t) - \mathcal{P}^t_{t,t}\begin{bmatrix}H(t)\\L(t)\end{bmatrix}^T Q_{v_f}^{-1}\begin{bmatrix}H(t)\\L(t)\end{bmatrix}\right\}\hat{x}(t \mid t,t) + \mathcal{P}^t_{t,t}\begin{bmatrix}H(t)\\L(t)\end{bmatrix}^T Q_{v_f}^{-1}\begin{bmatrix}y(t)\\ \check{z}(t \mid t+h)\end{bmatrix} \tag{51}$$

In the above, Q_{v_f} is as in (38), the matrix $\mathcal{P}^t_{t,t}$ as in (48), $\mathcal{P}^s_{s,t}$ and $\mathcal{P}^t_{s,t}$ ($s \geq t$) are obtained by (22) and (25), respectively, with convariance matrices Q_v and Q_u as given in (38),

$$\frac{d\,\mathcal{P}^s_{s,t}}{ds} = F(s)\mathcal{P}^s_{s,t} + \mathcal{P}^s_{s,t}F^T(s) - \mathcal{P}^s_{s,t}H^T(s)H(s)\mathcal{P}^s_{s,t} + \Gamma(s)\Gamma^T(s) \tag{52}$$

$$\frac{d\,\mathcal{P}^t_{s,t}}{ds} = \mathcal{P}^t_{s,t}A^T(s,t) \tag{53}$$

with the initial condition $\mathcal{P}^t_{t,t}$, and

$$A(s,t) = F(s)\left[I_n - \mathcal{P}^s_{s,t}H^T(s)H(s)\right] \tag{54}$$

Theorem 4 *Consider the system (4)-(6) and the associated performance criterion (7). Given scalar $\gamma > 0$ and $h > 0$. If there exists a bounded solution $\mathcal{P}^t_{t,t}$ to the Riccati equation (48). Then the fixed-lag smoother $\check{z}(t \mid t+h)$ that achieves (7) is solvable, and the smoother $\check{z}(t \mid t+h)$ is given by*

$$\check{z}(t \mid t+h) = L(t)\hat{x}(t \mid t+h,t) \tag{55}$$

where $\hat{x}(t \mid t+h,t)$ is computed by (49)-(51).

B. H_∞ predictor ($h < 0$)

The projections of $\hat{x}(t+h \mid t+h,t)$ and $\hat{x}(t \mid t+h,t)$ with $h < 0$ are different from the case of $h > 0$, but they are dual to each other. So the projections for $h < 0$ can be easily obtained by a similar discussion. In fact, note that for $h < 0$, $\hat{x}(t+h \mid t+h,t)$ and $\hat{x}(t \mid t+h,t)$ are respectively the projections of states $\mathbf{x}(t+h)$ and $\mathbf{x}(t)$ on the linear Krein space of (46). The re-organized innovation asssociated with (46) can be given as

$$\{\mathbf{w}(s,s), 0 \leq s \leq t+h; \mathbf{w}(t+h,s), t+h < s \leq t\} \tag{56}$$

where $\mathbf{w}(s,s)$ has a similar definition as in the case of $h \geq 0$ and $\mathbf{w}(t+h,s)$ ($s > t+h$) is defined as

$$\mathbf{w}(t+h,s) = \check{\mathbf{z}}(s \mid s+h) - L(s)\hat{\mathbf{x}}(s \mid t+h,s) \tag{57}$$

where $\hat{x}(s \mid t+h,s)$ is the projection of the state $\mathbf{x}(s)$ on the linear Krein space of (46). It is then easy to know that

$$\mathbf{w}(t+h,s) = L(s)\mathbf{e}(t+h,s) + \mathbf{v}_z(s+h) \tag{58}$$

where

$$\mathbf{e}(t+h,s) = \mathbf{x}(s) - \hat{\mathbf{x}}(s \mid t+h,s) \tag{59}$$

Similarly, we denote that

$$\mathcal{P}^{r}_{t+h,s} \triangleq \langle \mathbf{x}(r),\ \mathbf{e}(t+h,s)\rangle;\ r,s \geq t+h \tag{60}$$

Then $\mathcal{P}^{r}_{t+h,s}$ can be computed by the Lemma below.

Lemma 2 • *For $r = s = t+h$, $\mathcal{P}^{t+h}_{t+h,t+h}$ is as in (48).*

- *For $r = s > t+h$, $\mathcal{P}^{s}_{t+h,s}$ satisfies*

$$\frac{d\,\mathcal{P}^{s}_{t+h,s}}{ds} = F(s)\mathcal{P}^{s}_{t+h,s} + \mathcal{P}^{s}_{t+h,s}F^{T}(s) + \gamma^{-2}\mathcal{P}^{s}_{t+h,s}H^{T}(s)H(s)\mathcal{P}^{s}_{t+h,s} + \Gamma(s)\Gamma^{T}(s), \tag{61}$$

with the initial condition $\mathcal{P}^{t+h}_{t+h,t+h}$.

- *For $s > r = t+h$, $\mathcal{P}^{t+h}_{t+h,s}$ satisfies*

$$\frac{d\,\mathcal{P}^{t+h}_{t+h,s}}{ds} = \mathcal{P}^{t+h}_{t+h,s}A^{T}(t+h,s) \tag{62}$$

with the initial condition $\mathcal{P}^{t+h}_{t+h,t+h}$ and

$$A(t+h,s) = F(s)[I_n + \gamma^{-2}\mathcal{P}^{s}_{t+h,s}L^{T}(s)L(s)].$$

Lemma 3 *For $h<0$, the projections of $\hat{x}(t+h \mid t+h,t)$ and $\hat{x}(t \mid t+h,t)$ are given by*

$$\hat{x}(t+h \mid t+h,t) = \hat{x}(t+h \mid t+h,t+h) - \gamma^{-2}\int_{t+h}^{t} \mathcal{P}^{t+h}_{t+h,s}L^{T}(s)[\check{z}(s \mid s+h) - L(s)\hat{x}(s \mid t+h,s)]ds \tag{63}$$

and

$$\frac{d\,\hat{x}(s \mid t+h,s)}{ds} = -\gamma^{-2}\mathcal{P}^{s}_{t+h,s}L^{T}(s)\check{z}(s \mid s+h) + \left[F(s) + \gamma^{-2}\mathcal{P}^{s}_{t+h,s}L^{T}(s)L(s)\right]\hat{x}(s \mid t+h,s) \tag{64}$$

where $t+h \leq s \leq t$ and the initial value $\hat{x}(t+h \mid t+h,t+h)$ is as (51) and the matrices $\mathcal{P}^{s}_{t+h,s}$ and $\mathcal{P}^{t+h}_{t+h,s}$ are given by (61)-(62).

Theorem 5 *Consider the system (4)-(6) and the associated performance criterion (7). Given scalar $\gamma > 0$ and $h < 0$, if there exists a bounded solution $\mathcal{P}^{t+h}_{t+h,t+h}$ to the Riccati equation (48) over $0 \leq t \leq N$ and an unique bounded solution $\mathcal{P}^{s}_{t+h,s}$ to (61) over $t+h \leq s \leq t$ and $0 \leq t \leq N$. Then the multi-step prediction $\check{z}(t \mid t+h)$ that achieves (7) is solvable and the predictor $\check{z}(t \mid t+h)$ is*

$$\check{z}(t \mid t+h) = L(t)\hat{x}(t \mid t+h,t) \tag{65}$$

where $\hat{x}(t \mid t+h,t)$ is computed by (64) and (51).

5 Conculsion

This paper has addressed the H_∞ fixed lag smoothing and prediction problems for linear continuous-time systems. It was shown that the H_∞ estimation is in fact an H_2 one for continuous-time systems with delayed 'measurement' in Krein space. Applying innovation analysis and projection theory in Krein space, the H_2 estimation result was extended to give the existence condition and a solution to the H_∞ smoother and predictor.

References

[1] B. D. O. Anderson and J. B. Moore, *Optimal filtering*. Prentice-Hall, Englewood. Cliffs, N.J, 1979.

[2] A. Bagchi, "A martingale approach to state estimation in delay differential systems," *J. Math. Anal. and Applics*, Vol. 56, pp. 195-210, 1976.

[3] N. Briggs and R. Vinter, "Linear filtering for time-delay systems," *IMA J. Math. Contr. and Inf.* , Vol. 6, pp. 167-178, 1989.

[4] H. Kwakernaak, "Optimal filtering in linear systems with time delays," *IEEE Trans. on Automat. Contr.*, Vol. 12, pp. 169-173, 1967.

[5] L. Mirkin, "On the H_∞ fixed-lag smoothing: how to exploit the information preview," *Automatica*, to appear.

[6] K. M. Nagpal and P. P. Khargonekar, "Filtering and smoothing in an H_∞ setting," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 152-166, 1991.

[7] B. Hassibi, A.H. Sayed and T. Kailath, *Indefinite Quadratic Estimation and Control: A Unified Approach to H_2 and H_∞ Theories*, SIAM Studies in Applied Mathematics series, 1998.

[8] Y. Theodor and U. Shaked, "Game theory approach to H_∞ optimal discrete-time fixed-point and fixed-lag smoothing", *IEEE Trans. Automatic Control*, vol. 39, no. 9, pp.1944-1948, 1994.

[9] Y. Theodor, "Robust H_∞ and H_2 estimation and control," PhD Thesis, Tel-Aviv University, 1994.

[10] H. S. Zhang, L. Xie and Y. C. Soh, "A unified approach to linear estimation for discrete-time systems: Part II- H_∞ estimation," *Proc. the 40th IEEE Conf. on Decision and Control, Florida, Dec. 2001.*

[11] H. Zhang, D. David and L. Xie, "H_∞ fixed-lag smoothing and prediction for linear continuous-time systems," Technical Report, Department of Computing, Hong Kong Polytechnic University, 2003.