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On Fork-Free t -Perfect GraphsYixin Cao  | Shenghua Wang

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ABSTRACT

In an effort to understand the complexity of the maximum independent set problem, Chvátal introduced t -perfect graphs. While a full characterization of this class remains open, important progress has been made for claw-free graphs [Bruhn and Stein, *Math. Program.* 2012] and P_5 -free graphs [Bruhn and Fuchs, *SIAM J. Discrete Math.* 2017]. We take a further step by characterizing fork-free t -perfect graphs and showing that they are strongly t -perfect and 3-colorable. We also give polynomial-time algorithms for recognizing and coloring fork-free t -perfect graphs.

1 | Introduction

All the graphs discussed in this paper are finite and simple. We use $V(G)$ and $E(G)$ to denote, respectively, the vertex set and the edge set of a graph G . The set of *neighbors* of a vertex u is denoted as $N(u)$, whose cardinality is the *degree* of u . For a proper subset U of $V(G)$, we use $G - U$ to denote the graph obtained from G by deleting all vertices in U and all edges that have an end in U . We use $G - u$ as a shorthand for $G - \{u\}$. We say that F is an *induced subgraph* of G if F can be obtained from G by deleting vertices; otherwise, G is *F -free*. We work primarily with induced subgraphs; unless explicitly stated as t -minors (defined below), all containment and H -freeness are with respect to induced subgraphs. For a set \mathcal{F} of graphs, a graph G is *\mathcal{F} -free* if G is F -free for every $F \in \mathcal{F}$. The *complement* of a graph G has the same vertex set as G and two vertices are adjacent if and only if they are not adjacent in G . A *clique* is a set of pairwise adjacent vertices, and an *independent set* is a set of vertices that are pairwise nonadjacent.

For $\ell \geq 3$, we use C_ℓ , K_ℓ , and P_ℓ to denote the cycle graph, the complete graph, and the path graph, respectively, on ℓ vertices. We use W_ℓ to denote the wheel graph, which is obtained from a C_ℓ by adding a new vertex and making it adjacent to all the vertices on the C_ℓ . Note that a W_3 is a K_4 . A *hole* is an induced cycle with $\ell \geq 4$. An ℓ -cycle, ℓ -hole, or ℓ -wheel is odd if ℓ is odd; note that an odd wheel has an even number of vertices. Indices on an ℓ -hole are understood cyclically.

When Berge [1] introduced perfect graphs in the 1960s, the concept was purely graph-theoretic. One of his conjectures, now known as the weak perfect graph theorem, was resolved with the help of polyhedral combinatorics [2, 3]. Padberg [4] and Chvátal [5] independently characterized the independent set polytope of a perfect graph, i.e., the convex hull of the characteristic vectors of all its independent sets, by showing that this polytope can be described by nonnegativity

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and clique constraints. Chvátal [5] proposed another polytope, defined by nonnegativity, edge, and odd-cycle constraints (formal definition deferred to Section 3). If it coincides with the independent set polytope of a graph G , then G is called *t-perfect*. The strong perfect graph theorem [6], another conjecture from the original paper of Berge on perfect graphs, asserts that a graph G is perfect if and only if G does not contain any odd hole (an induced odd cycle of length at least five) or the complement of an odd hole. Naturally, one seeks a similar characterization for t-perfect graphs. Unfortunately, the progress toward this goal has been slow: even a working conjecture on it is still missing.

The characterization of t-perfect graphs turns out to depend on another graph operation. Let v be a vertex in G . If $N(v)$ is an independent set, then a *t-contraction* on v is the operation of contracting $N(v) \cup \{v\}$ into a single vertex [7]. See Figure 1 for examples. A graph G' is a *t-minor* of G if G' can be obtained from an induced subgraph of G by a sequence of t-contractions. Trivially, every induced subgraph of G is a t-minor of G . The operation t-contraction preserves t-perfection and strong t-perfection [7–9], and hence any t-minor of a (strongly) t-perfect graph is (strongly) t-perfect. Previous work has identified several families of minimally t-imperfect graphs, some of which are reproduced in Figure 2.

Both perfect graphs and t-perfect graphs are inherently related to understanding the complexity of the independent set problem, i.e., finding an independent set of the maximum weight. Indeed, we can solve the maximum independent set problem in polynomial time when the input graph is perfect [10] or t-perfect [11]. There is another line of research on the independent set problem. Algorithms for maximum matching can be used to solve the independent set problem on line graphs. The idea can be generalized to quasi-line graphs and then to claw-free graphs [12, 13]. Since the independent set problem is simple on P_3 -free and P_4 -free graphs, one might ask for which H can the independent set problem be solved in polynomial time when the input graph is H -free. Alekseev [14] showed that all candidates of such H must have a very special structure. Two subsequent additions are the fork [15, 16] and the P_5 [17]. A fork (also called a chair, Figure 3) is obtained by attaching a private neighbor to a degree-one vertex of a claw, or to a degree-two vertex of a P_4 . While the P_5 is a natural generalization of the P_4 , the fork graph is a generalization of both the claw and the P_4 .

Since we know when a claw-free graph [18] or a P_5 -free graph [19] is t-perfect, one naturally wants to ask the same question on fork-free graphs. The main result of the present paper is the list of forbidden t-minors of fork-free t-perfect graphs in Figure 2. Actually, we obtain a stronger result. A graph is *strongly t-perfect* if Chvátal's system on the graph is totally dual integral. Note that every integral solution of Chvátal's system corresponds to an independent set of the

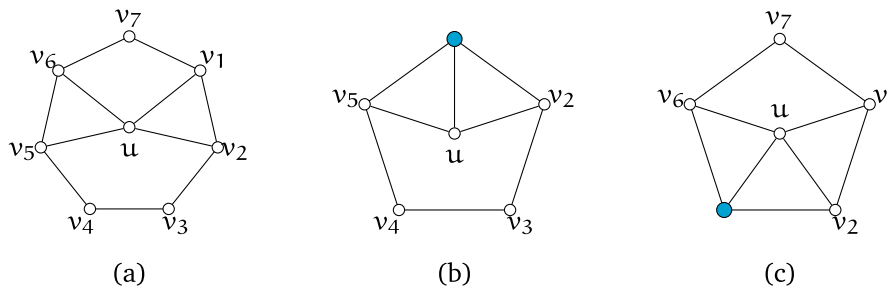


FIGURE 1 | A t-contraction on v_7 or v_4 in (a) leads to the graphs in, respectively, (b) and (c), where the shadow vertices are the newly created. A further t-contraction on a degree-two vertex in (b) or (c) ends with a K_4 . [Color figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com)]

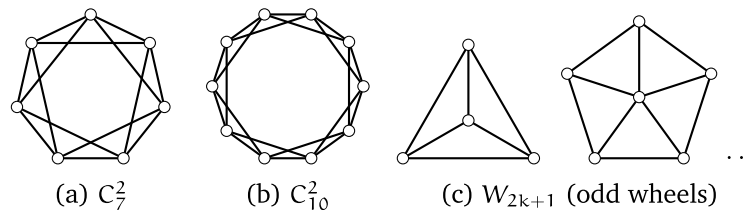


FIGURE 2 | Some minimally t-imperfect graphs.

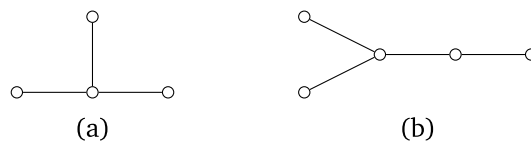


FIGURE 3 | (a) The claw graph and (b) the fork graph.

graph. Thus, it follows from the observation of Edmonds and Giles [20] on total dual integrality that every strongly t -perfect graph is t -perfect. The other direction remains an open problem. In particular, we do not know whether all P_5 -free t -perfect graphs are strongly t -perfect, though it is true for all claw-free t -perfect graphs [9].

Theorem 1.1 (Characterization). *Let G be a fork-free graph. The following are equivalent:*

- i. G is t -perfect.
- ii. G is strongly t -perfect.
- iii. G has no t -minor isomorphic to C_7^2 , C_{10}^2 , or an odd wheel.

Since strong t -perfection implies t -perfection and all the graphs in Figure 2 are known to be t -imperfect [8, 18],¹ we focus on showing that (iii) implies (ii). To demonstrate the strong t -perfection, we consider the dual of Chvátal's system, which is a covering problem. In particular, it looks for a cover of the vertex set by vertices, edges, and odd cycles. The graph is strongly t -perfect if, for any weight function, there is a cover whose cost equals the maximum weight of independent sets [8]. Let G be a fork-free graph that does not contain a C_7^2 , a C_{10}^2 , or any odd wheel as a t -minor. We may assume without loss of generality that G is connected. The graph must be strongly t -perfect if it is also claw-free [18], or if it is perfect (note that W_3 is K_4) [3, 5]. Moreover, if G is not perfect, then the absence of K_4 s and C_7^2 s forces the presence of an odd hole. Hence we may assume G contains both a claw and an odd hole. We show that every odd hole H must be a C_5 , and every other vertex has either exactly two consecutive neighbors or exactly three non-consecutive neighbors on H . Based on the adjacency to vertices on H , we can partition $V(G) \setminus V(H)$ into a few sets. A careful inspection of the edges among them shows that there always exists a cover. Therefore, the graph is strongly t -perfect. Our structural study toward Theorem 1.1 enables us to develop polynomial-time algorithms for recognizing and coloring fork-free t -perfect graphs.

Theorem 1.2 (Recognition). *Given a fork-free graph, we can decide in polynomial time whether it is t -perfect.*

It is conjectured that every t -perfect graph is four-colorable [21, 22]. The conjecture remains wide open; a recent breakthrough of Chudnovsky et al. [23] established that a constant bound suffices for all t -perfect graphs. (The current bound is just under 200,000 colors.) We show that fork-free t -perfect graphs are always 3-colorable.

Theorem 1.3 (Coloring). *If a fork-free graph G is t -perfect, then $\chi(G) \leq 3$, and a 3-coloring of G can be found in polynomial time.*

We would like to ask for characterizations of P_6 -free t -perfect graphs and E -free t -perfect graphs, where the E graph is obtained by attaching a private neighbor to the middle vertex of a P_5 . The claw graph, the fork graph, and the E graph are also known as the $S_{1,1,1}$ graph, the $S_{1,1,2}$ graph, and the $S_{1,2,2}$ graph, respectively. Very recently, Grzesik et al. [24] presented a polynomial-time algorithm for the independent set problem on P_6 -free graphs. Finding a polynomial-time algorithm for the independent set problem on E -free graphs is a major open problem [16, 25]. Characterizing t -perfect graphs inside this class may shed some light on its solution.

2 | Graphs With No Fork, C_7^2 , C_{10}^2 , Odd Wheel as a t -Minor

This section is devoted to a structural study of such connected fork-free graphs that (1) do not contain a C_7^2 , a C_{10}^2 , or any odd wheel as a t -minor, and (2) contain an odd hole and a claw. Let us remark any t -minor of a fork-free graph is fork-free (the proof is omitted because it is not used in the present paper). We will show that every odd hole H is a five cycle, every vertex in $V(G) \setminus V(H)$ has two or three neighbors on H and can be partitioned into five stable sets U_1, U_2, \dots, U_5 according to their neighbors in H . Moreover, we will make some observations on the adjacencies between the sets U_1, U_2, \dots, U_5 . At the end of this section we combine these observations with prior work [6, 18, 26] to prove Theorem 1.3.

When a statement may be of independent interest, we use weaker conditions, e.g., dropping the requirement of containing a claw. The first observation is about the neighborhood of an outside vertex on an odd hole.

Proposition 2.1. *Let H be an odd hole and $u \in V(G) \setminus V(H)$.*

- i. *If u has exactly one neighbor on H , then G contains a fork.*
- ii. *If u has exactly two neighbors on H and they are not consecutive on H , then G contains a fork.*
- iii. *If u has exactly three neighbors on H and they are consecutive on H , then K_4 is a t -minor of G .*
- iv. *If u has exactly four neighbors on H and they induce one or two paths on H , then K_4 is a t -minor of G .*

Proof. For assertions (i) and (ii), we number the vertices on H as v_1, v_2, \dots . Suppose without loss of generality that $uv_3 \in E(G)$. Then u is adjacent to neither v_2 nor v_4 . There is no other neighbor of u on H in (i). In (ii), u cannot be adjacent to both v_1 and v_5 ; we may assume that $uv_1 \notin E(G)$. Then $\{v_3, v_4, u, v_2, v_1\}$ induces a fork.²

For assertions (iii) and (iv), we focus on the subgraph G' induced by $V(H) \cup \{u\}$; see Figure 1. Note that any vertex in $V(H) \setminus N(u)$ has only two neighbors in G' , and they are not adjacent. We do induction on the length of H . In the base case, $|H| = 5$. (Note that in this case, if u has four neighbors on H , then they must be consecutive.) A t -contraction on a vertex in $V(H) \setminus N(u)$ leads to a K_4 . We now consider that $|H| > 5$. We apply a t -contraction on a vertex v in $V(H) \setminus N(u)$, which shortens H into a shorter odd hole, denoted by H' . The length of H' is two shorter than H . If the neighbors of u on H are consecutive, then the two neighbors of v cannot be both adjacent to u (note that $|H| \geq 7$). Thus, u has the same number of neighbors on H' as on H , and they remain consecutive. In the rest, u must have four neighbors on H , and they form two paths. If the two neighbors of v are both adjacent to u , then u has three consecutive neighbors on H' . Otherwise, u still has exactly four neighbors on H' and they form one or two paths on H' . By induction, K_4 is a t -minor of G' , hence of G . \square

The following statement further extends Proposition 2.1(ii). The two ends of any edge of an odd hole can only have one private neighbor, which is not adjacent to any other vertex on the hole.

Proposition 2.2. *Let G be a $\{K_4, \text{fork}\}$ -free graph containing an odd hole H . For any two vertices on H , at most one of their common neighbors is adjacent to only two vertices on H .*

Proof. Suppose for contradiction that there are two distinct vertices x and y such that they have the same pair of neighbors on H . By Proposition 2.1(ii), the neighbors of x on H have to be consecutive. We number the vertices on H as v_1, v_2, \dots such that x is adjacent to v_1 and v_2 . Since G is K_4 -free, $xy \notin E(G)$. Then $\{v_2, x, y, v_3, v_4\}$ induces a fork, a contradiction. \square

The existence of claws have another implication on the neighbors of other vertices on a hole H : there must be a vertex adjacent to three or more vertices on H .

Proposition 2.3. *Let G be a connected fork-free graph containing an odd hole H . The graph G is claw-free if*

- i. G contains neither K_4 nor W_5 ; and
- ii. every vertex $v \in V(G) \setminus V(H)$ has either zero or two neighbors on H .

Proof. Suppose that G satisfies both conditions (i) and (ii). We number the vertices on H as v_1, v_2, \dots . Since G is a fork-free graph, if a vertex $v \in V(G) \setminus V(H)$ has two neighbors on H , then they are consecutive by Proposition 2.1(ii). Thus, for each i , a neighbor of v_i not on H is adjacent to either v_{i-1} or v_{i+1} . By Proposition 2.2, the degree of v_i is at most four, and it cannot be the center of a claw. Suppose for contradiction that G contains a claw. We take a claw T of G whose center has the shortest distance to H , denoted as d , among all claws of G . Note that $d \geq 1$. Let the vertex set of T be $\{c, x, y, z\}$, where c is the center of T .

Case 1, $d = 1$. The vertex c is adjacent to H , and by assumption, it has exactly two neighbors on H . We first note that we can choose T to intersect H . Suppose that none of x, y , and z is on H , and let v be a neighbor of c on H . Since the degree of v is at most four, v has at most one neighbor, say z , in $\{x, y, z\}$. Then $\{c, x, y, v\}$ is another claw. In the rest, without loss of generality, let the two neighbors of c on H be v_1 and v_2 , where $z = v_2$. Since $\{c, x, y, v_2, v_3\}$ cannot induce a fork, at least one of x and y is adjacent to v_3 . Since neither x nor y is adjacent to v_2 , they cannot be both adjacent to v_3 . We may assume that y is adjacent to v_3 , hence to v_4 as well but no other vertex on H . Since $\{c, x, v_2, y, v_4\}$ does not induce a fork, x has to be adjacent to v_4 as well, and its other neighbor on H is v_5 . But then $\{c, y, z, x, v_5\}$ induces a fork, a contradiction.

Case 2, $d = 2$. We may assume that there is a common neighbor p of c and v_1 , and the other neighbor of p on H is v_2 . (Note that $cv_1 \notin E(G)$.)

- Subcase 2.1, p has two or more neighbors in $\{x, y, z\}$, say x and y . By the selection of T , there cannot be any claw in G that has p as the center. Thus, x is adjacent to either v_1 or v_2 , and so is y . Either cxv_1v_2y or cxv_2v_1y is a five-hole, and p is adjacent to all vertices on it. Therefore, G contains a W_5 , a contradiction.
- Subcase 2.2, p has at most one neighbor in $\{x, y, z\}$. (We are in this sub-case when p is one of $\{x, y, z\}$.) Assume without loss of generality that p is adjacent to neither x nor y . Since $\{c, x, y, p, v_2\}$ does not induce a fork, v_2 is

adjacent to at least one of x and y . Since $\{v_2, x, y, p\}$ does not induce a claw, v_2 cannot be adjacent to both x and y . We may assume that $yv_2 \in E(G)$. By Proposition 2.2, the other neighbor of y on H is v_3 . Since $\{c, p, x, y, v_3\}$ does not induce a fork, $xv_3 \in E(G)$. Again, by Proposition 2.2, the other neighbor of x on H has to be v_4 . But then $\{c, p, y, x, v_4\}$ induces a fork, a contradiction.

Case 3, $d \geq 3$. Let $cu_1u_2 \dots$ be a shortest path from c to H . Note that no vertex in $\{x, y, z\}$ is adjacent to u_i with $i > 2$.

- Subcase 3.1, u_1 has at most one neighbor in $\{x, y, z\}$. (We are in this sub-case when u_1 is one of $\{x, y, z\}$.) Assume without loss of generality that u_1 is adjacent to neither x nor y . Since $\{c, x, y, u_1, u_2\}$ does not induce a fork, u_2 is adjacent to at least one of x and y , say y . But then $\{u_2, u_1, u_3, y\}$ induces a claw, and its center u_2 has a shorter distance to H than c , a contradiction to the selection of T .
- Subcase 3.2, u_1 has two or more neighbors in $\{x, y, z\}$, say x and y . Note that $u_1z \notin E(G)$; otherwise $\{u_1, x, y, z\}$ induces a claw, which contradicts the selection of T . If u_2 is adjacent to only x in $\{x, y, z\}$, then $\{c, y, z, x, u_2\}$ induces a fork. If u_2 is adjacent to two vertices in $\{x, y, z\}$, then these two vertices, together with u_2 and u_3 , form a claw that is closer to H than T . If u_2 is adjacent to neither x nor y , then $\{u_1, u_2, x, y\}$ induces a claw that is closer to H than T .

Therefore, there cannot be a claw in G . □

The somewhat conflicting requirements in Propositions 2.1 and 2.3 exclude odd holes longer than five, and force every five-hole to be dominating (i.e., every vertex has neighbors on this hole). We need to recall an observation of Lozin and Milanič [16].

Lemma 2.4 ([16]). *Let G be a $\{W_5, \text{fork}\}$ -free graph. If the subgraph of G induced by $U \subseteq V(G)$ is isomorphic to Figure 4, then every vertex in G has a neighbor in U .*

Proposition 2.5. *Let G be a connected fork-free graph containing an odd hole H . If G contains a claw and does not contain any odd wheel as a t -minor, then $|H| = 5$, and every vertex in G is adjacent to H .*

Proof. We number the vertices of H as v_1, \dots, v_ℓ , where $\ell = 2k + 1$. Since G contains a claw, by Proposition 2.1 and Proposition 2.3, we can find a vertex u that has three or more neighbors on H , and two of them are adjacent. Since G is free of odd wheels, $V(H) \not\subseteq N(u)$. We may assume without loss of generality that u is adjacent to v_3 and v_4 but not v_5 .

We first show $|H| = 5$. If $N(u) \cap V(H) = \{v_2, v_3, v_4\}$, then G contains K_4 as a t -minor by Proposition 2.1(iii). Therefore, there is a neighbor v_i of u with $i \notin \{2, 3, 4\}$. We traverse H from v_4, v_5 till we meet the next neighbor of u ; let it be v_i . Since $\ell \geq 7$ and $i \neq 2$, one of v_3 and v_4 is nonadjacent to all the vertices in $\{v_{i-1}, v_i, v_{i+1}\}$. Since neither $\{v_i, v_{i-1}, v_{i+1}, u, v_4\}$ nor $\{v_i, v_{i-1}, v_{i+1}, u, v_3\}$ induces a fork, $uv_{i+1} \in E(G)$. If u has precisely four neighbors on H , namely, v_3, v_4, v_i , and v_{i+1} , then G contains K_4 as a t -minor by Proposition 2.1(iv). Therefore, u has at least five neighbors on H . As a result, $6 \leq i \leq \ell$. If $i > 6$, then $\{u, v_4, v_j, v_i, v_{i-1}\}$, where v_j is another neighbor of u on H , induces a fork (note that v_j cannot be adjacent to v_4, v_{i-1} , or v_i). In the rest, $i = 6$. If v_5 is the only non-neighbor of u on H , then G contains an odd wheel as a t -minor. Hence, u has at least one neighbor and one non-neighbor in $\{v_8, v_9, \dots, v_\ell, v_1, v_2\}$. We can find a j such that u is adjacent to precisely one of $\{v_j, v_{j+1}\}$. But then $\{u, v_4, v_6, v_j, v_{j+1}\}$ induces a fork (note that there cannot be any edge between v_4, v_6 and v_j, v_{j+1}). Therefore, the length of H has to be five.

By Proposition 2.1(iii, iv), the vertex u has exactly three nonconsecutive neighbors on H . Assume without loss of generality that they are v_1, v_3 , and v_4 . Hence, the subgraph induced by $V(H) \cup \{u\}$ is isomorphic to Figure 4. By Lemma 2.4, every vertex $x \in V(G)$ has a neighbor in $V(H) \cup \{u\}$. If x is only adjacent to u , then $\{v_1, v_2, v_5, u, x\}$ induces a fork. Hence, $N(x) \cap V(H) \neq \emptyset$. This concludes the proof. □

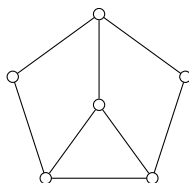


FIGURE 4 | The graph H_2 of Lozin and Milanič [16].

Now consider a connected fork-free graph G that contains an odd hole and a claw, and does not contain any odd wheel as a t -minor. It contains a five-hole H by Proposition 2.5. Let us number the vertices on H as v_1, \dots, v_5 . For $i = 1, \dots, 5$, let U_i be the set of common neighbors of v_{i+2} and v_{i+3} . We show that the five sets U_1, U_2, \dots, U_5 , together with $V(H)$, partition $V(G)$. See Figure 5.

Proposition 2.6. *Let G be a connected fork-free graph containing a five-hole H . If G contains a claw and does not contain any odd wheel as a t -minor, then $\{V(H), U_1, U_2, U_3, U_4, U_5\}$ is a partition of $V(G)$.*

Proof. Let x be an arbitrary vertex in $V(G) \setminus V(H)$. By Proposition 2.5, the vertex x has a neighbor on H . Since G is fork-free and does not contain any odd wheel as a t -minor, x has either exactly two consecutive neighbors on H , or exactly three nonconsecutive neighbors on H (Proposition 2.1). Thus, there is a unique $i \in \{1, \dots, 5\}$ such that $x \in U_i$. On the other hand, no vertex on H is in U_i for all i . □

Since G is K_4 -free, for all $i = 1, 2, \dots, 5$, the set U_i is an independent set. An independent set is *maximal* if it is not a subset of any other independent set. We show that the set $\{v_{i-1}, v_{i+1}\} \cup U_i$ is a maximal independent set of G for every $i = 1, \dots, 5$.

Proposition 2.7. *Let G be a K_4 -free graph containing a five-hole H . If any vertex in $V(G) \setminus V(H)$ has either exactly two consecutive neighbors on H , or exactly three nonconsecutive neighbors on H , then the set $\{v_{i-1}, v_{i+1}\} \cup U_i$ is a maximal independent set of G for every $i = 1, \dots, 5$. Moreover, these are precisely the maximal independent sets that contain two vertices of H .*

Proof. Let S be a maximal independent set of G . Since H is a C_5 , it follows $|S \cap V(H)| \leq 2$. If S contains precisely two vertices from H , they have to be v_{i-1} and v_{i+1} for some i . By the definitions of U_i , we have $S \setminus V(H) \subseteq U_i$. Thus, $S \subseteq \{v_{i-1}, v_{i+1}\} \cup U_i$. Since there is no edge among vertices in U_i , it must hold by equality by the maximality of S . □

If a vertex in U_i has another neighbor on H , then it has to be v_i by Propositions 2.1(iii). We can thus partition U_i into $U_i^+ = U_i \cap N(v_i)$ and $U_i^- = U_i \setminus N(v_i)$. For any vertex $x \in U_i^+$, the set $\{v_i, v_{i-1}, v_{i+1}, x\}$ induces a claw. By Proposition 2.2,

$$|U_i^-| \in \{0, 1\}, i = 1, \dots, 5.$$

By Proposition 2.3, $\bigcup_{i=1}^5 U_i^+$ is not empty. We summarize the adjacency relations among the parts in the following proposition when $U_i^+ \neq \emptyset$. Two disjoint vertex sets are *complete* if all the edges between them are present.

Proposition 2.8. *Let G be a $\{K_4, W_5, \text{fork}\}$ -free graph containing a five-hole H . If any vertex in $V(G) \setminus V(H)$ has either exactly two consecutive neighbors on H , or exactly three nonconsecutive neighbors on H , and U_i^+ is nonempty for some $i = 1, \dots, 5$, then*

- i. U_i is complete to $U_{i-2} \cup U_{i+2}$;
- ii. U_i is complete to $U_{i-1}^- \cup U_{i+1}^-$;
- iii. U_{i+1}^- is complete to U_{i+2}^- ;
- iv. at least one of U_{i+2} and U_{i-2} is empty; and
- v. a vertex in U_i^+ has at most one non-neighbor in U_{i-1}^+ and at most one non-neighbor in U_{i+1}^+ .

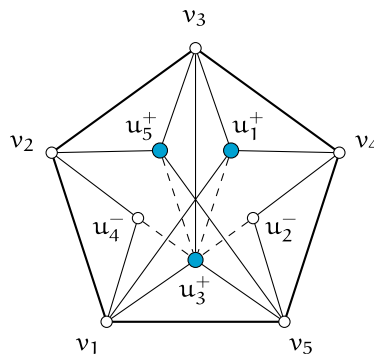


FIGURE 5 | Illustration for the partition of $V(G)$, where $u_i^+ \in U_i^+$ for $i \in \{1, 3, 5\}$ and $u_j^- \in U_j^-$ for $j \in \{2, 4\}$, and the dashed edges are optional. The graph contains forks. [Color figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com)]

Proof. We show the statements for $i = 3$; they hold for other indices by symmetry. Let u_3^+ be a fixed vertex in U_3^+ , which is nonempty by assumption.

- i. Let u_5 be an arbitrary vertex in U_5 . By definition, u_3^+ is adjacent to both v_1 and v_3 but not v_4 , and u_5 is adjacent to v_3 but neither v_1 nor v_4 . The vertices u_3^+ and u_5 must be adjacent as otherwise $\{v_3, u_5, v_4, u_3^+, v_1\}$ induces a fork. Thus, U_3^+ is complete to U_5 , and a similar argument implies that U_3 is complete to U_5^+ . We are done if U_3^- or U_5^- is empty. Hence, assume that $u_5 \in U_5^-$ and let u_3^- be a vertex in U_3^- . Note that $u_3^-u_3^+ \notin E(G)$ (because G is K_4 -free), and we have seen above that $u_3^+u_5 \in E(G)$. By definition, both u_3^- and u_3^+ are adjacent to v_5 and neither is adjacent to v_4 . Thus, $u_3^-u_5 \in E(G)$ as otherwise $\{v_5, v_4, u_3^-, u_3^+, u_5\}$ induces a fork. A symmetric argument applies to U_3 and U_1 .
- ii. Assume that $U_4^- \neq \emptyset$ and u_4 be the vertex in U_4^- . By definition, u_4 is adjacent to v_2 but none of v_3, v_4 , and v_5 . The vertices u_3^+ and u_4 must be adjacent as otherwise $\{v_3, v_4, u_3^+, v_2, u_4\}$ induces a fork. If there is a vertex $u_3^- \in U_3^- \setminus N(u_4)$, then $\{v_5, v_4, u_3^-, u_3^+, u_4\}$ induces a fork. A symmetric argument implies that U_3 is complete to U_2^- .
- iii. We may assume neither U_4^- nor U_5^- is empty. For $j = 4, 5$, let u_j^- be the vertex in U_j^- . By definition, the vertex u_3^+ is adjacent to v_5 but not v_4 , the vertex u_4^- is adjacent to neither v_5 nor v_4 , and the vertex u_5^- is adjacent to neither v_4 nor v_5 . By assertions (i, ii), u_3^+ is adjacent to both u_4^- and u_5^- . Therefore, u_4^- must be adjacent to u_5^- as otherwise $\{u_3^+, u_4^-, u_5^-, v_5, v_4\}$ induces a fork.
- iv. Suppose for contradiction that neither U_1 nor U_5 is empty. For $j = 1, 5$, let u_j be a vertex in U_j . By assertion (i), u_3^+ is adjacent to both u_1 and u_5 . If $u_1u_5 \in E(G)$, then $\{u_3^+, u_5, u_1, v_3\}$ is a clique, a contradiction to that G is K_4 -free. In the rest, $u_1u_5 \notin E(G)$. The vertex v_1 must be adjacent to u_1 as otherwise $\{u_3^+, u_5, v_1, u_1, v_4\}$ induces a fork. By symmetry, $v_5u_5 \in E(G)$. But then u_3^+ has five neighbors on the hole $u_5v_5v_1u_1v_3$, contradicting that G is W_5 -free.
- v. If there are two distinct vertices y and y' in $U_4^+ \setminus N(u_3^+)$, then $\{v_2, y, y', v_3, u_3^+\}$ induces a fork. A symmetric argument applies to U_2^+ . \square

The rest of this section is about coloring. For a positive integer k , we call G k -colorable if we can partition $V(G)$ into k independent sets. The smallest k such that G is k -colorable is the *chromatic number* of G .

Lemma 2.9. *Let G be a fork-free graph. If G does not contain a C_7^2 , a C_{10}^2 , or any odd wheel as a t -minor, then the chromatic number of G is at most three, and an optimal coloring of G can be found in polynomial time.*

Proof. We may assume without loss of generality that G is connected; otherwise, we work on its components one by one. Since G does not contain a K_4 , it cannot contain the complement of any odd hole longer than seven. It does not contain a C_7^2 , which is the complement of C_7 . Therefore, if G does not contain any odd hole, then G is perfect by the strong perfect graph theorem [6], and we can use the algorithm of Fonlupt and Sebö [27] or the algorithm of Chudnovsky et al. [26] to find an optimal coloring. The chromatic number of G is at most three because it is equal to the order of the maximum cliques [6], which is at most three because G is K_4 -free. Otherwise, G contains an odd hole, and thus its chromatic number is at least three. Thus, it suffices to find a three coloring, i.e., a partition of $V(G)$ into three (not necessarily maximal) independent sets. If G is claw-free, then we can use the algorithm of Bruhn and Stein [18] to find an optimal coloring. In the rest, G contains a claw. The algorithm now finds a five-hole H , and partition the vertex set $V(G) \setminus V(H)$ according to their adjacencies with H . We may number the vertices on H such that U_1^+ is nonempty and U_4 is empty. This is possible because of Proposition 2.8(iv) with $i = 1$.

If U_3 is empty, then we partition $V(G)$ into three independent sets $U_5 \cup \{v_4\}$, $U_1 \cup \{v_2, v_5\}$, and $U_2 \cup \{v_1, v_3\}$. If U_5 is empty, then we partition $V(G)$ into three independent sets that are $U_1 \cup \{v_5\}$, $U_2 \cup \{v_1, v_3\}$, and $U_3 \cup \{v_2, v_4\}$. In the rest, neither U_3 nor U_5 is empty. If $U_5^+ \neq \emptyset$, then U_2 is empty because of Proposition 2.8(iv) with $i = 5$. We can partition $V(G)$ into three independent sets $U_1 \cup \{v_2, v_5\}$, $U_3 \cup \{v_3\}$, and $U_5 \cup \{v_1, v_4\}$. The remaining case is when $U_5 = U_5^-$, and we show that this cannot happen. Since neither U_1 nor U_5 is empty, U_3^+ is empty because of Proposition 2.8(iv) with $i = 3$. For $j = 3, 5$, let u_j^- be the only vertex in U_j^- (note that $|U_j^-| \leq 1$ by Proposition 2.2). Let u_1^+ be an arbitrary vertex in U_1^+ ; it is adjacent to both u_3^- , by Proposition 2.8(i), and u_5^- , by Proposition 2.8(ii), both with $i = 1$. If $u_3^-u_5^- \notin E(G)$, then $\{u_1^+, u_3^-, v_4, u_5^-, v_2\}$ induces a fork; otherwise, u_1^+ has three consecutive neighbors on the hole $u_3^-v_5v_4v_3u_5^-$, contradicting Propositions 2.1(iii). The algorithm is thus complete.

All the induced subgraphs we need to check have a constant number of vertices. Both algorithms we call take polynomial time [18, 26]. The rest is clearly doable in polynomial time. Thus, the whole algorithm runs in polynomial time. \square

Theorem 1.3 directly follows from Lemma 2.9 and Theorem 1.1.

3 | Strong t-Perfection

The *independent set polytope* of a graph is defined as the convex hull of the characteristic vectors of all independent sets in it. For a graph G , we define another polytope $P(G)$ as the set of vectors $x \in \mathbb{R}^{V(G)}$ satisfying

$$\begin{aligned} 0 \leq x_v &\leq 1 && \text{for every vertex } v, \\ x_u + x_v &\leq 1 && \text{for every edge } uv, \\ x(V(C)) &\leq \frac{|V(C)|-1}{2} && \text{for every odd cycle } C \text{ in } G, \end{aligned} \tag{1}$$

where $x(S) = \sum_{v \in S} x_v$. Clearly, the characteristic vector of every independent set of G satisfies all the constraints in (1). Thus, the independent set polytope is contained in $P(G)$. While the other direction is not true in general: the vector $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ is in $P(K_4)$ but is not in the independent set polytope of K_4 because it does not satisfy the clique constraint. A graph G is *t-perfect* if $P(G)$ is precisely the independent set polytope of G , and *strongly t-perfect* if the system (1) is totally dual integral. It is well known that every strongly t-perfect graph is t-perfect [8, 20], while the other direction remains an open problem. The K_4 is the smallest graph that is not t-perfect, hence not strongly t-perfect. For a vertex $v \in V(G)$, the polytope $P(G - v)$ is the projection of the intersection of $P(G)$ and the hyperplane $x_v = 0$ on $\mathbb{R}^{V(G-v)}$. Therefore, t-perfection is preserved under vertex deletions. Both t-perfection and strong t-perfection are also invariants with respect to t-contraction [7, 9].

A graph is *perfect* if it does not contain an odd hole or the complement of an odd hole. A graph is perfect if and only if its independent set polytope can be determined by the following linear system [4, 5]:

$$\begin{aligned} x_v &\geq 0 && \text{for every vertex } v, \\ x(K) &\leq 1 && \text{for every clique } K \text{ in } G. \end{aligned} \tag{2}$$

Note that a triangle constraint is both a clique constraint and an odd-cycle constraint. Moreover, the odd-cycle constraints in system (1) can be restricted to induced odd cycles: those on non-induced ones are redundant. In a K_4 -free graph, the clique constraint of system (2) degenerates to vertex, edge, and triangle constraints of system (1). On the other hand, if G is perfect, then it does not contain odd holes, and odd-cycle constraints of system (1) degenerates to triangle constraints. Thus, system (1) and system (2) coincide for a K_4 -free perfect graph. If system (2) is totally dual integral, then G is perfect [5, 20]. Lovász [3] showed that the converse is also true. Therefore, system (2) is totally dual integral if and only if G is perfect; see also [8] for more details.

Proposition 3.1 Folklore. *Every K_4 -free perfect graph is strongly t-perfect.*

Propositions 2.1–2.6 can be summarized as follows. If a connected fork-free graph G contains a claw and an odd hole and does not contain a C_7^2 , a C_{10}^2 , or any odd wheel as a t-minor, then every odd hole H in G has length five, and satisfies the following property.

$$\begin{aligned} \text{Every vertex in } V(G) \setminus V(H) &\text{ has either exactly two consecutive neighbors on } H, \\ &\text{or exactly three nonconsecutive neighbors on } H. \end{aligned} \tag{C5}$$

Interestingly, the other direction also holds true. The main work of this section is to establish the following lemma.

Lemma 3.2. *Let G be a connected fork-free graph that contains a claw and an odd hole. The following statements are equivalent:*

- i. G does not contain a C_7^2 , a C_{10}^2 , or any odd wheel as a t-minor.
- ii. G is $\{K_4, W_5, C_7^2, C_{10}^2\}$ -free, and every odd hole in G has length five and satisfies (C5).
- iii. G is strongly t-perfect.

Before presenting the proof of Lemma 3.2, we use it to prove Theorem 1.1.

Proof of Theorem 1.1. Since strong t -perfection implies t -perfection and C_7^2, C_{10}^2 , and all odd wheels are t -imperfect [8, 18], it suffices to show that if a fork-free graph does not contain a C_7^2 , a C_{10}^2 , or any odd wheel as a t -minor, then it is strongly t -perfect. Suppose that G is such a graph. We show that every component of G is strongly t -perfect, and hence G is strongly t -perfect. Let G' be an arbitrary component of G . Note that G' is fork-free and does not contain a C_7^2 , a C_{10}^2 , or any odd wheel as a t -minor. If G' is claw-free, then it is strongly t -perfect according to Bruhn and Stein ([9], Theorem 2) and ([18], Theorem 3). Note that the complement of C_7 is C_7^2 , and the complement of an odd hole longer than seven contains a K_4 . If G' does not contain an odd hole, then G' is perfect, and hence strongly t -perfect by Proposition 3.1. Now that G' contains a claw and an odd hole, it is strongly t -perfect by Lemma 3.2. \square

The rest of the section is devoted to proving Lemma 3.2. By *duplicating* a vertex v of G we introduce copies of v and make them adjacent to every neighbor of v in G . Note that the copies and v itself form an independent set. Benchetrit [28] proved that the class of strongly t -perfect graphs is closed under vertex duplication.

Lemma 3.3 ([28]). *The graph obtained by duplicating any vertex of a strongly t -perfect graph is strongly t -perfect.*

A graph is *almost bipartite* if it has a vertex that is contained in all odd cycles of the graph. The following can be obtained by applying the much stronger result of Gerards [29].

Lemma 3.4 ([29]). *Almost bipartite graphs are strongly t -perfect.*

The proof of the following can be found in the appendix.

Proposition 3.5. *Let G be a graph with one of the configurations in Figure 6. If G satisfies the condition of Lemma 3.2(ii), then G is strongly t -perfect.*

Throughout the rest of this section, G is a $\{\text{fork}, K_4, W_5, C_7^2, C_{10}^2\}$ -free graph that contains a claw and an odd hole, and every odd hole has length five and satisfies (C5). We fix a five-hole H , and partition the vertices $V(G) \setminus V(H)$ into U_1, \dots, U_5 . For $i = 1, \dots, 5$, the set U_i is further partitioned into U_i^+ and U_i^- . Recall that $|U_i^-| \leq 1$ by Proposition 2.2. By Proposition 2.8, the main uncertain adjacencies are between U_i^+ and U_{i+1}^+ . Thus, the graph has a very simple structure if only one of U_i^+ s or two nonconsecutive of them are nonempty. Indeed, it can be obtained from one of the small graphs (of order at most ten) in Figure 6 by vertex duplications.

Lemma 3.6. *If for any $i = 1, \dots, 5$, one of U_i^+ and U_{i+1}^+ is empty, then G is strongly t -perfect.*

Proof. We may assume without loss of generality that U_2^+ is nonempty, while all of $U_1^+, U_3^+,$ and U_5^+ are empty. Every vertex in U_2^+ is adjacent to $v_2, v_4,$ and v_5 but not v_1 or v_3 by definition.

Suppose first that U_4^+ is also nonempty. Then both U_1 and U_5 are empty by Proposition 2.8(iv). Thus,

$$V(G) \setminus (V(H) \cup U_2^+ \cup U_4^+) = U_2^- \cup U_3^- \cup U_4^-.$$

By Proposition 2.8(i, ii), all the edges between U_2^+ and $U_3^- \cup U_4^-$ are present. Thus, all vertices in U_2^+ have the same neighborhood in $V(G) \setminus U_2^+$. A symmetric argument applies to U_4^+ . Let G be a graph of the pattern in Figure 6(a), where for $i = 2, 3, 4$, the optional vertices u_i^- exists if and only if $U_i^- \neq \emptyset$. By Proposition 3.5, G is strongly t -perfect if it

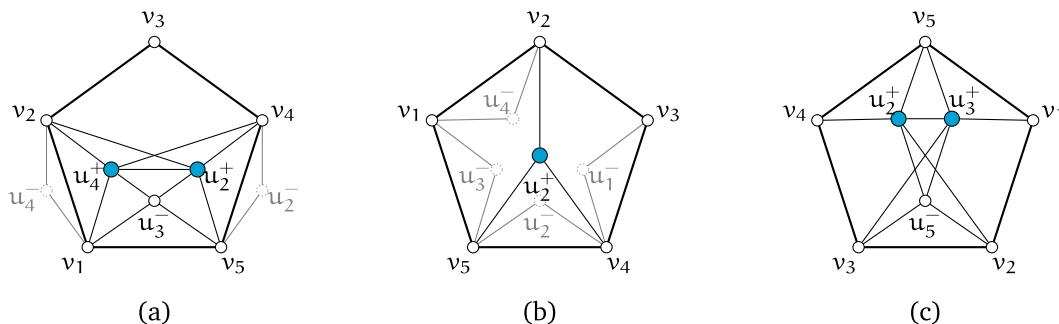


FIGURE 6 | Three configurations for Lemma 3.6 and 3.7. The dotted vertices are optional, and edges incident to them, except to H , are not drawn. [Color figure can be viewed at wileyonlinelibrary.com]

satisfies the condition of Lemma 3.2(ii). We duplicate u_2^+ of G with $|U_2^+|$ vertices, and then duplicate u_4^+ in the resulted graph with $|U_4^+|$ vertices. The final result is G . Therefore, G is strongly t -perfect by Lemma 3.3.

In the rest, U_4^+ is empty. We may assume without loss of generality that $U_5 = \emptyset$. Then

$$V(G) \setminus (V(H) \cup U_2^+) = U_1^- \cup U_2^- \cup U_3^- \cup U_4^-.$$

Every vertex in U_2^+ is adjacent to $U_1^- \cup U_3^- \cup U_4^-$ by Proposition 2.8(i, ii), and nonadjacent to U_2^- by definition. Thus, all vertices in U_2^+ have the same neighborhood in $V(G) \setminus U_2^+$. Let G be a graph of the pattern in Figure 6(b), where for $i = 1, 2, 3, 4$, the optional vertices u_i^- exists if and only if $U_i^- \neq \emptyset$. By Proposition 3.5, G is strongly t -perfect if it satisfies the condition of Lemma 3.2(ii). We duplicate u_2^+ of G with $|U_2^+|$ vertices. The result is G . Therefore, G is strongly t -perfect by Lemma 3.3. \square

Henceforth, we may assume without loss of generality that

$$U_2^+ \neq \emptyset \quad \text{and} \quad U_3^+ \neq \emptyset.$$

By Proposition 2.8(iv) with $i = 2$, at least one of U_4 and U_5 is empty. For the same reason, at least one of U_1 and U_5 is empty. We note that if neither $U_1 \cup U_5$ nor $U_4 \cup U_5$ is empty, then U_2^+ is complete to U_3^+ , and the situation is similar to Lemma 3.6.

Lemma 3.7. *If neither $U_1 \cup U_5$ nor $U_4 \cup U_5$ is empty, then G is strongly t -perfect.*

Proof. We first argue that $U_5 \neq \emptyset$. Suppose otherwise, then neither U_1 nor U_4 is empty. Since both U_3 and U_4 are nonempty, U_1^+ is empty by Proposition 2.8(iv) with $i = 1$. By symmetry, U_4^+ is empty. Therefore $U_1 = U_1^-$ and $U_4 = U_4^-$. Let u_1^- and u_4^- be the only vertex in U_1^- and U_4^- , respectively (note that $|U_j^-| \leq 1$ by Proposition 2.2). By Proposition 2.8(ii) with $i = 3$, the vertex u_3^+ is adjacent to u_4^- . By Proposition 2.8(i) with $i = 3$, the vertex u_3^+ is adjacent to u_1^- . Since $\{u_3^+, u_1^-, v_5, u_4^-, v_2\}$ cannot form a fork, $u_4^- u_1^- \in E(G)$. But then $u_1^- u_4^- v_1 v_5 v_4$ is a five-hole in G and u_3^+ has three consecutive neighbors u_4^-, v_1 , and v_5 on it, contradicting (C5).

Since neither U_2 nor U_3 is empty, U_5^+ is empty by Proposition 2.8(iv) with $i = 5$. Thus, U_5^- is nonempty; let u_5^- be its only vertex (note that $|U_j^-| \leq 1$ by Proposition 2.2). Applying Proposition 2.8(i) twice, with $i = 2, 3$, respectively, we can conclude that u_5^- is adjacent to all the vertices in $U_2 \cup U_3$. We then argue that U_2^+ is complete to U_3^+ . We take an arbitrary vertex u_2^+ from U_2^+ and an arbitrary vertex u_3^+ from U_3^+ . If $u_2^+ u_3^+ \notin E(G)$, then $u_2^+ v_2 v_3 u_3^+ v_5$ is a hole in G on which u_5^- has four neighbors. Thus, U_2^+ is complete to U_3^+ . We next argue that U_2^- is empty. Suppose otherwise and let u_2^- be the only vertex in U_2^- (note that $|U_j^-| \leq 1$ by Proposition 2.2). Note that $u_5^- u_2^- \in E(G)$. Therefore, $u_5^- u_2^- v_5 v_1 v_2$ is a hole in G . But then, u_3^+ has three consecutive neighbors u_2^-, v_5 , and v_1 on the hole, contradicting (C5). Thus, $U_2^- = \emptyset$. A symmetric argument implies U_3^- is empty as well. Since both U_2^+ and U_5^- are nonempty, U_4 is empty by Proposition 2.8(iv) with $i = 2$. A symmetric argument implies U_1 is empty as well. Therefore, $V(G) \setminus V(H) = U_2^+ \cup U_3^+ \cup U_5^-$, and the three parts U_2^+, U_3^+ , and U_5^- are pairwise complete with each other. By Proposition 3.5, the graph G in Figure 6(c) is strongly t -perfect. We duplicate u_2^+ of G with $|U_2^+|$ vertices, and then duplicate u_3^+ in the resulted graph with $|U_3^+|$ vertices. The final result is G . Therefore, G is strongly t -perfect by Lemma 3.3. \square

In the rest, at least one of $U_1 \cup U_5$ and $U_4 \cup U_5$ is empty. We may assume that $U_1 \cup U_5 = \emptyset$; otherwise, we can renumber the vertices on H . We have seen all the maximal independent sets that contains two vertices from H in Proposition 2.7. The following lists other maximal independent sets under this condition.

Proposition 3.8. *If $U_1 \cup U_5$ is empty, then a maximal independent set S of G that contains at most one vertex from H is either*

- i. $U_j^- \cup \{v_j\}$ for some $j = 2, 3, 4$; or
- ii. a pair of nonadjacent vertices $x \in U_3^+$ and $y \in U_2^+ \cup U_4^+$.

Proof. (i) Suppose first that there is one vertex in $S \cap V(H)$. We first exclude v_1 and v_5 . Suppose that $v_1 \in S$. By definition, $U_3 \cup U_4$ is disjoint from S . Thus, $S \subseteq U_2 \cup \{v_1\}$ and cannot be maximal. Likewise, $v_5 \in S$ implies $S \subseteq U_4 \cup \{v_5\}$.

- Case 1, $v_2 \in S$. Then $S \setminus \{v_2\} \subseteq U_2^- \cup U_3$. Since $U_3 \cup \{v_2, v_4\}$ is an independent set, U_2^- cannot be empty, and its only vertex must be in S . It remains to argue that the vertex in U_2^- is adjacent to all the vertices in U_3 . We call Proposition 2.8(ii) with $i = 3$.
- Case 2, $v_3 \in S$. Then $S \setminus \{v_3\} \subseteq U_2 \cup U_3^- \cup U_4$. Since U_2 is complete to U_4 by Proposition 2.8(i) with $i = 2$, one of $S \cap U_2$ and $S \cap U_4$ is empty. Since $U_2 \cup \{v_1, v_3\}$ and $U_4 \cup \{v_3, v_5\}$ are independent sets, by the maximality of S , there is a vertex in $U_3^- \cap S$. By Proposition 2.8(ii) with $i = 2$, the vertex in U_3^- is adjacent to all the vertices in U_2 . Moreover, the vertex in U_3^- is adjacent to all the vertices in U_4 by Proposition 2.8(iii) with $i = 2$ when $U_4^+ = \emptyset$, or by Proposition 2.8(ii) with $i = 4$ otherwise.
- Case 3, $v_4 \in S$. Then $S \setminus \{v_4\} \subseteq U_4^- \cup U_3$, and the argument is similar to that of case 1.

(ii) Now suppose that S is disjoint from $V(H)$. By assumption, $V(G) \setminus V(H) = U_2 \cup U_3 \cup U_4$. We first argue that

$$S \subseteq U_2^+ \cup U_3^+ \cup U_4^+.$$

For $j = 2, 3, 4$, let x_j be the vertex in U_j^- if $U_j \neq U_j^+$. Applying Proposition 2.8(i-iii) with $i = 2$, we can conclude that x_2, x_3 , and x_4 are pairwise adjacent, when they exist. Therefore, at most one of them is in S . On the other hand, if $x_j \in S$ for $j = 2, 3, 4$, then $S \subseteq U_j$ by Proposition 2.8(i, ii). Since this contradicts the maximality of S , we must have $S \subseteq U_2^+ \cup U_3^+ \cup U_4^+$. Since U_2^+ is complete to U_4^+ by Proposition 2.8(i) with $i = 2$, the set S is a subset of either $U_2^+ \cup U_3^+$ or $U_3^+ \cup U_4^+$. By Proposition 2.8(v) (with $i = 3$), each vertex in U_3^+ has at most one non-neighbor in U_2^+ and at most one non-neighbor in U_4^+ . For the same reason, each vertex in U_2^+ or U_4^+ has at most one non-neighbor in U_3^+ . Thus, S is a pair of nonadjacent vertices $x \in U_3^+$ and $y \in U_2^+ \cup U_4^+$. \square

The final step of the proof relies on the duality of linear programming, for which we need to recall some known results. For each weight function $w : V(G) \rightarrow \mathbb{Z}_{\geq 0}$, we can make a linear program out of (1) by adding an objective function $\max \sum_v w(v)x_v$. The dual of this linear program is a covering problem. A w -cover is a family of vertices, edges, and odd cycles in G such that every vertex v in $V(G)$ lies in at least $w(v)$ elements, with repetition allowed. The cost of a w -cover is the sum of the costs of its elements, where the cost of a vertex or an edge is one, and the cost of an odd cycle C is $(|C| - 1)/2$. For a vertex set S , we use $w(S)$ to denote $\sum_{v \in S} w(v)$. We use $\alpha_w(G)$ to denote the maximum value of $w(S)$, with S ranging over all independent sets of G . The following is a consequence of linear programming duality.

Proposition 3.9 ([8]). *A graph G is strongly t -perfect if and only if for every weight function $w : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ there exists a w -cover of cost $\alpha_w(G)$.*

The following observation, implicit from Bruhn and Stein [9], is very helpful in our checking the condition of Proposition 3.9. We provide a proof for the sake of completeness. Note that a vertex set K intersects every maximum-weight independent set of G if and only if $\alpha_w(G - K) < \alpha_w(G)$.

Proposition 3.10 ([9]). *Let G be a graph and $w : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ a weight function. There exists a w -cover of G with cost $\alpha_w(G)$ if*

- *there exists a clique K of at most three vertices such that $\alpha_w(G - K) < \alpha_w(G)$; and*
- *for any weight function $w' : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ with $w'(V(G)) \leq w(V(G))$ and $w - w'$ not identically 0, there exists a w' -cover of cost $\alpha_{w'}(G)$.*

Proof. We may assume without loss of generality that K is inclusion-wise minimal satisfying $\alpha_w(G - K) < \alpha_w(G)$. As a result, $w(v) > 0$ for each $v \in K$: a vertex of zero weight has no impact on $\alpha_w(G)$. We can define another weight function $w' : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ by setting

$$w'(v) = \begin{cases} w(v) - 1 & v \in K, \\ w(v) & \text{otherwise.} \end{cases}$$

Since $w'(V(G)) < w(V(G))$, there exists a w' -cover \mathcal{K} of cost $\alpha_{w'}(G)$ by assumption. Since $|K| \leq 3$, the set $\mathcal{K} \cup \{K\}$ is a w -cover of G and its cost is $\alpha_{w'}(G) + 1 = \alpha_w(G)$. \square

Lemma 3.11. *If $U_1 \cup U_5$ is empty, then G is strongly t -perfect.*

Proof. Suppose for contradiction that G is not strongly t -perfect, and assume without loss of generality that G is a counterexample of the minimum number of vertices. Our first claim is that every proper induced subgraph G' of G is strongly t -perfect. If G' does not contain an odd hole, then it is strongly t -perfect (Proposition 3.1). If G' is claw-free, then G is strongly t -perfect [9, 18]. Thus, G' is strongly t -perfect either because it satisfies one of Lemmas 3.6 and 3.7, or by the selection of G .

By Proposition 3.9, there exists a weight function $w : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ such that G does not have a w -cover of cost $\alpha_w(G)$. Assume w is chosen to minimize $w(V(G))$. The second claim is that the weight is positive. Suppose that $w(v) = 0$ for some vertex $v \in V(G)$. Since every induced subgraph of G is strongly t -perfect, there exists a w -cover \mathcal{K} of $G - v$ with cost $\alpha_w(G - v)$. Since $w(v) = 0$, the cover \mathcal{K} is also a w -cover of G , while $\alpha_w(G - v) = \alpha_w(G)$. But then \mathcal{K} is a w -cover of G with cost $\alpha_w(G)$, a contradiction. As a consequence of the second claim, every maximum-weight independent set is maximal. Recall that all maximal independent sets are listed in Propositions 2.7 and 3.8.

For $j = 2, 3, 4$, let

$$S_j^- = \{v_{j-1}, v_{j+1}\} \cup U_j^-$$

and denote by u_j^- the only vertex contained in U_j^- when it is not empty (note that $|U_j^-| \leq 1$ by Proposition 2.2). For $j = 2, 3$, let u_j^+ be a vertex of the maximum weight from U_j^+ , and

$$S_j^+ = \{v_{j-1}, v_{j+1}, u_j^+\}.$$

We define a set $S_4^+ = \{v_3, v_5, u_4^+\}$ when $u_4^+ \neq \emptyset$, with u_4^+ being a vertex of the maximum weight from U_4^+ . According to Proposition 2.7, all the nine sets S_j^-, S_j^+ , and U_j are independent sets.

From Proposition 3.10 and the selection of the weight function w it can be inferred that $\alpha_w(G - K) = \alpha_w(G)$ for any clique K of at most three vertices. In other words, there exists a maximum-weight independent set S of G disjoint from K . We try to locate a clique of two or three vertices that intersects all maximum-weight independent sets of the graph, thereby producing a contradiction to Proposition 3.10. In the following, we consider potential maximum-weight independent sets. By excluding an independent set we mean that we have evidence that it does not have the maximum weight.

Note that U_4 is not empty; otherwise, $G - v_5$ is bipartite, and G is strongly t -perfect by Lemma 3.4. We take an arbitrary vertex u_4 from U_4 . Note that $u_4 u_2^+ \in E(G)$ by Proposition 2.8(i) with $i = 2$. Let K denote the clique $\{v_2, u_2^+, u_4\}$, and let S be a maximum-weight independent set of G disjoint from K . Note that S has to be $\{v_1, v_4\}$, $\{v_3\} \cup U_3^-$, $\{v_4\} \cup U_4^-$, or one that is disjoint from $V(H)$, i.e., specified in Proposition 3.8(ii).

- Case 1, $S = \{v_1, v_4\}$. (Note that $U_5 = \emptyset$.) Since $\{v_2, v_4, u_3^+\}$ and $\{v_1, v_3, u_2^+\}$ are both independent sets,

$$\begin{aligned} & w(u_2^+) + w(u_3^+) \\ & < w(v_2) + w(v_4) + w(u_3^+) + w(v_1) + w(v_3) + w(u_2^+) - w(v_4) - w(v_1) \\ & = w(\{v_2, v_4, u_3^+\}) + w(\{v_1, v_3, u_2^+\}) - w(S) \\ & \leq \alpha_w(G) + \alpha_w(G) - \alpha_w(G) \\ & = \alpha_w(G). \end{aligned}$$

By the selection of u_2^+ and u_3^+ , a pair of vertices $x \in U_2^+$ and $y \in U_3^+$ cannot have weight $\alpha_w(G)$. In other words, if a maximum-weight independent set is disjoint from $V(H)$, then it comprises a vertex in U_3^+ and a vertex in U_4^+ . On the other hand, from

$$w(v_2) + w(v_3) + w(U_2^- \cup U_3^-) = w(S_2^-) + w(S_3^-) - w(S) \leq \alpha_w(G)$$

we can exclude $\{v_2\} \cup U_2^-$ and $\{v_3\} \cup U_3^-$. Thus, if a maximum-weight independent set contains one vertex from H , then it has to be $\{v_4\} \cup U_4^-$.

- Case 1.1, $\{v_2, v_5\}$ is also a maximum-weight independent set. (Note that $U_1 = \emptyset$.) If $U_4^+ \neq \emptyset$, we use

$$w(u_3^+) + w(u_4^+) < w(S_3^+) + w(S_4^+) - w(\{v_2, v_5\}) \leq \alpha_w(G)$$

to exclude all maximal independent sets disjoint from H . If $U_4^- \neq \emptyset$, we use

$$w(v_4) + w(u_4^-) < w(S_3^-) + w(S_4^-) - w(\{v_2, v_5\}) < \alpha_w(G)$$

to exclude $\{v_4, u_4^-\}$. Since any maximum-weight independent set has to contain two vertices from H , they all intersect the clique $\{v_1, v_5, u_3^+\}$.

- Case 1.2, there exists a maximum-weight independent set $S' = \{x_3, x_4\}$ with $x_3 \in U_3^+$ and $x_4 \in U_4^+$. Note that both $U_3 \cup \{v_2\}$ and $U_4 \cup \{v_3\}$ are not maximal. Therefore, we can use $w(U_3 \cup \{v_2\}) + w(U_4 \cup \{v_3\}) - w(S') < \alpha_w(G)$ to exclude all other pairs $\{x'_3, x'_4\}$ with $x'_3 \in U_3^+$ and $x'_4 \in U_4^+$ (except for S' itself). If U_4^- is empty, then $\{v_1, v_2, x_4\}$ intersects all the possible maximum-weight independent sets. Now that U_4^- is nonempty, we use $w(U_4) + w(S_3^+) - w(S') < \alpha_w(G)$ to exclude $\{v_4, u_4^-\}$. Furthermore, we use $w(S_3^+) + w(S_4^+) - w(S') < \alpha_w(G)$ to exclude $\{v_2, v_5\}$. The clique $\{v_1, v_5, u_3^+\}$ intersects all the remaining maximal independent sets, $S, S', \{v_2, v_4\} \cup U_3, \{v_3, v_5\} \cup U_4$, and $\{v_1, v_3\} \cup U_2$.
- Otherwise (neither of cases 1.1 and 1.2 is true), the clique $\{v_3, v_4\}$ intersects all the possible maximum-weight independent sets.
- Case 2, $S = \{v_3, u_3^-\}$. We use

$$w(u_2^+) + w(u_3^+) < w(U_3) + w(S_2^+) - w(S) < \alpha_w(G)$$

to exclude all pairs $\{x_2, x_3\}$ with $x_2 \in U_2^+$ and $x_3 \in U_3^+$. If U_4^+ is nonempty, we use

$$w(u_3^+) + w(u_4^+) < w(U_3) + w(S_4^+) - w(S) < \alpha_w(G)$$

to exclude all pairs $\{x_3, x_4\}$ with $x_3 \in U_3^+$ and $x_4 \in U_4^+$. From $w(S_3^-) + w(S_4^-) - w(S) < \alpha_w(G)$ we can exclude $\{v_2, v_5\}$ and $\{v_4, u_4^-\}$ (when $U_4^- \neq \emptyset$). If $U_2^- \neq \emptyset$, we use $w(S_2^-) + w(S_3^-) - w(S) < \alpha_w(G)$ to exclude $\{v_2, u_2^-\}$. We are left with $S, \{v_2, v_4\} \cup U_3, \{v_3, v_5\} \cup U_4$, and $\{v_3, v_1\} \cup U_2$. All of them intersect the clique $\{v_3, v_4\}$.

- Case 3, $S = \{v_4, u_4^-\}$. We can use

$$w(v_2) + w(v_5) < w(S_3^-) + w(S_4^-) - w(S) \leq \alpha_w(G)$$

to exclude $\{v_2, v_5\}$. If $U_4^+ \neq \emptyset$, we use

$$w(u_3^+) + w(u_4^+) < w(U_4) + w(S_3^+) - w(S) < \alpha_w(G)$$

to exclude all pairs $\{x_3, x_4\}$ with $x_3 \in U_3^+$ and $x_4 \in U_4^+$.

- Case 3.1, there does not exist a maximum-weight independent set $\{x_2, x_3\}$ with $x_2 \in U_2^+$ and $x_3 \in U_3^+$. If U_2^- is empty, the clique $\{v_3, v_4\}$ intersects all maximum weight independent sets. Now that $U_2^- \neq \emptyset$, we note that $\{u_2^-, u_3^+, u_4^-\}$ intersects all maximum-weight independent sets. To see that it is clique, note that $u_2^- u_4^- \in E(G)$ by Proposition 2.8(i) with $i = 2$, and u_3^+ is adjacent to both u_2^- and u_4^- by Proposition 2.8(ii) with $i = 3$,
- Case 3.2, there exists a maximum-weight independent set $S' = \{x_2, x_3\}$ with $x_2 \in U_2^+$ and $x_3 \in U_3^+$. We can use $w(U_2 \cup \{v_1\}) + w(U_3 \cup \{v_2\}) - w(S') < \alpha_w(G)$ to exclude all other pairs $\{x'_2, x'_3\}$ with $x'_2 \in U_2^+$ and $x'_3 \in U_3^+$

(except for S' itself). If U_2^- is not empty, then we can use $w(U_2) + w(S_3^+) - w(S') < \alpha_w(G)$ to exclude $\{u_2^-, v_2\}$.

Thus, a maximum-weight independent set of G has to be $S, S', \{v_2, v_4\} \cup U_3, \{v_3, v_5\} \cup U_4$, or $\{v_3, v_1\} \cup U_2$.

The clique $\{v_4, v_5, x_2\}$ intersects all maximum-weight independent sets.

- Case 4, $S = \{x_2, x_3\}$, where $x_2 \in U_2^+$ and $x_3 \in U_3^+$. Note that $x_2 \neq u_2^+$ because S is disjoint from K . We can use $w(U_2 \cup \{v_1\}) + w(U_3 \cup \{v_2\}) - w(S) < \alpha_w(G)$ to exclude all other pairs $\{x'_2, x'_3\}$ with $x'_2 \in U_2^+$ and $x'_3 \in U_3^+$ (except for S itself). If U_2^- is nonempty, then we can use $w(U_2) + w(S_3^+) - w(S) < \alpha_w(G)$ to exclude $\{u_2^-, v_2\}$. If no maximum-weight independent set intersects U_4^+ , then the clique $\{x_2, v_4, v_5\}$ intersects all maximum weight independent sets. Suppose that $S' = \{x'_3, x_4\}$ is a maximum-weight independent set with $x'_3 \in x_3$ and $x_4 \in U_4^+$. We can further use $w(U_3 \cup \{v_4\}) + w(U_4 \cup \{v_3\}) - w(S') < \alpha_w(G)$ to exclude all other pairs $\{x''_3, x'_4\}$ with $x''_3 \in U_3^+$ and $x'_4 \in U_4^+$ (except for S' itself). We use $w(S_3^+) + w(S_4^+) - w(S') \leq \alpha_w(G)$ to exclude $\{v_2, v_5\}$. Thus, a maximum-weight independent set of G has to be $S, S', \{v_2, v_4\} \cup U_3, \{v_3, v_5\} \cup U_4$, or $\{v_3, v_1\} \cup U_2$. The set $\{v_4, x_2, x_4\}$ intersects all maximum-weight independent sets. Note that $x_2x_4 \in E(G)$ by Proposition 2.8(i) with $i = 2$.
- Case 5, $S = \{x_3, x_4\}$, where $x_3 \in U_3^+$ and $x_4 \in U_4^+$. The argument is symmetric to Case 4.

This concludes the proof. □

We now prove Lemma 3.2.

Proof of Lemma 3.2. Since C_7^2, C_{10}^2 , and odd wheels are all t-imperfect, (iii) implies (i). By Propositions 2.1–2.6, (i) implies (ii). By Lemmas 3.6, 3.7, and 3.11, (ii) implies (iii). □

4 | Recognition

We now describe an algorithm to decide whether a fork-free graph is (strongly) t-perfect. We may assume without loss of generality that the input graph is connected; otherwise, we work on its components one by one and return “yes” if and only if all components return “yes”. The algorithm is based on Lemma 3.2. The only condition of Lemma 3.2(ii) that cannot be easily checked in polynomial time is that every odd hole has length five. The following proposition bounds the length of the longest odd holes.

Proposition 4.1. *Let G be a $\{K_4, \text{fork}\}$ -free graph containing a five-hole H . If H satisfies (C5), then G contains no odd hole of length exceeding 19.*

Proof. Let H' be a longest odd hole in G . Suppose for contradiction $|H'| \geq 21$. At least $|H'| - 4$ vertices of H' are in $V(G) \setminus V(H)$. Assume without loss of generality that $|U_i \cap V(H')|$ is maximized with $i = 1$. Then $|U_1 \cap V(H')| \geq \lceil \frac{|H'|-4}{5} \rceil \geq 4$. Since U_1 is an independent set, $|U_1 \cap V(H')| \leq \frac{|H'|-1}{2}$. There exists a vertex x in $V(H') \setminus (V(H) \cup U_1)$. By Propositions 2.8(i), (ii), and (v) with $i = 1$, the vertex x has at most one non-neighbor in $U_1 \cap V(H')$. But then x has at least three neighbors in $|H'|$, contradicting that H' is a hole. □

We are now ready to present the recognition algorithm and prove Theorem 1.2.

Proof of Theorem 1.2. The input is a fork-free graph G . We start by checking whether it contains a K_4, W_5, C_7^2 , or C_{10}^2 . Since K_4, W_5, C_7^2 , and C_{10}^2 are not t-perfect, we return “no” if any of them is found. If G does not contain a claw, then we call Bruhn–Schaudt algorithm [30] to decide whether G is t-perfect. We then call the algorithm of Chudnovsky et al. [31] to test whether G contains an odd hole. Since G does not contain a K_4 , it cannot contain the complement of any odd hole longer than seven. It does not contain a C_7^2 , which is the complement of C_7 . Therefore, if G does not contain any odd hole, then G is perfect, and t-perfect (Proposition 3.1), and we return “yes.” In the rest, G contains a claw and an odd hole, and we check the conditions of Lemma 3.2(ii). We enumerate five-holes, and for each of them, test whether it satisfies (C5). If any one does not, then return “no.” Finally, we check whether G contains an odd hole of length between 7 and 19. If any is found, then we can return “no” (by Proposition 2.5). If none is found, every odd hole has length five by Proposition 4.1. Thus, we can return “yes” (by Lemma 3.2). All the induced subgraphs we need to check have a constant number of vertices, and both algorithms we call take polynomial time [30, 31]. Thus, the whole algorithm runs in polynomial time. □

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Endnotes

- ¹The cycle powers C_7^2 and C_{10}^2 are obtained from C_7 and C_{10} , respectively, by adding an edge between any two vertices of distance two.
- ²When we list the vertices of a (potential) fork, we always put the degree-three vertex first, followed by its three neighbors, the last of which has degree two.

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Appendix A

Proof of Proposition 3.5

Suppose, for a contradiction, that G is not strongly t -perfect. By Proposition 3.9, there exists a weight function $w : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ such that G does not have a w -cover of cost $\alpha_w(G)$. Assume w is chosen to minimize $w(V(G))$. Let G' be the subgraph induced by $\{x \in V(G) \mid w(x) > 0\}$; note that G' is not strongly t -perfect by Proposition 3.9. Since G' is K_4 -free, it must contain a C_5 (by Proposition 3.1). Thus, there exists a C_5 all whose vertices receive positive weight under w .

Configuration 6(a). Since t -perfection is closed under taking induced subgraphs, we may assume that all optional vertices in Figure A1a are present. See Figure A1a. The graph G has eight maximal cliques:

$$\{v_1, v_4\}, \{v_2, v_5\}, \{v_2, u_2^-\}, \{v_3, u_3^-\}, \{v_4, u_4^-\}, \{v_1, v_3, u_2^+, u_2^-\}, \{v_3, v_5, u_4^+, u_4^-\}, \{v_2, v_4, u_3^-\}.$$

The only C_5 s are $v_1v_2v_3v_4v_5$ and $u_4^-v_2v_3v_4u_2^-$, which are symmetric. We may assume $w(v_i) > 0$ for all $i \in \{1, \dots, 5\}$.

Suppose first that $\{v_3, u_3^-\}$ is a maximum-weight independent set. Then $w(v_3) \geq w(v_2) + w(v_4) \geq 2$. Consequently, $\{v_1, v_4\}, \{v_2, v_5\}, \{v_2, u_2^-\}$, and $\{v_4, u_4^-\}$ cannot be maximum-weight independent sets of G' . In other words, every maximum-weight independent set of G' contains either v_3 or $\{v_2, v_4\}$. Define $w' : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ by

$$w'(v) = \begin{cases} w(v) - 2 & \text{if } v = v_3, \\ w(v) - 1 & \text{if } v \in \{v_2, v_4\}, \\ w(v) & \text{otherwise.} \end{cases}$$

Thus $\alpha_{w'}(G) = \alpha_w(G) - 2$. We take a w' -cover of cost $\alpha_{w'}(G)$ and add the edges v_2v_3 and v_3v_4 . The resulting family has cost $\alpha_{w'}(G) + 2 = \alpha_w(G)$, hence is a w -cover of cost $\alpha_w(G)$, a contradiction.

In the remaining case, $\{v_3, u_3^-\}$ is not a maximum-weight independent set of G' . If $w(u_4^-) > 0$, then every maximum clique contains a vertex of $K = \{v_1, v_2, u_4^-\}$, violating Proposition 3.10. The argument is symmetric if $w(u_2^-) > 0$. Hence, $w(u_2^-) = w(u_4^-) = 0$. Then every maximum-weight independent set of G' contains exactly two vertices of the cycle $v_1v_2v_3v_4v_5$. Define $w' : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ by

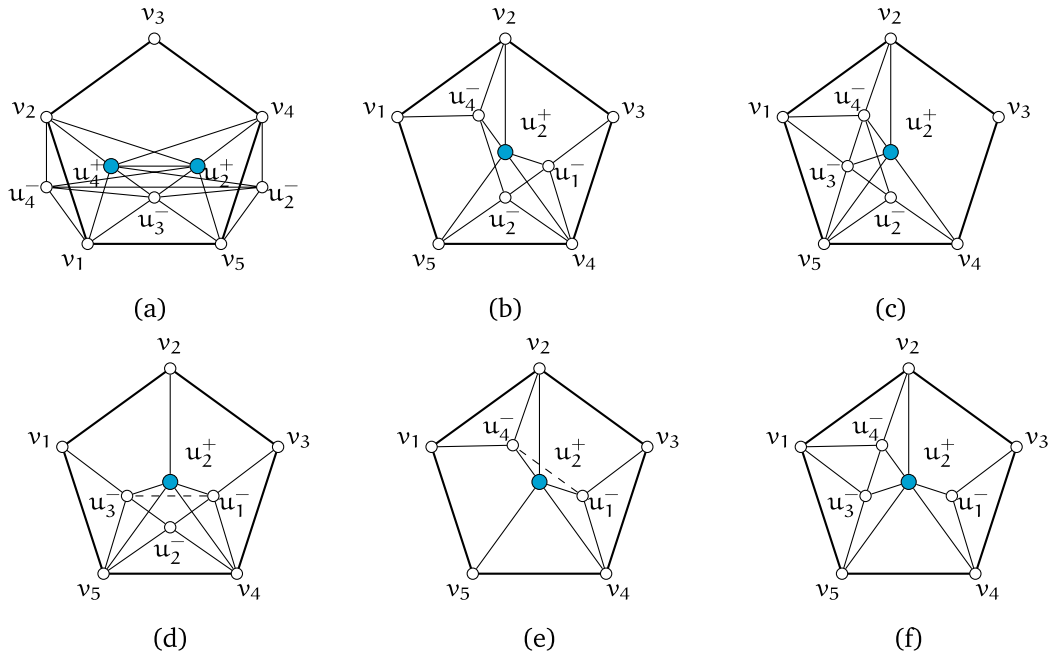


FIGURE A1 | Graphs satisfying configurations in Figure 6. [Color figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com)]

$$w'(v) = \begin{cases} w(v) - 1 & \text{if } v \in \{v_1, \dots, v_5\}, \\ w(v) & \text{otherwise.} \end{cases}$$

Then $\alpha_{w'}(G) = \alpha_w(G) - 2$. Take a w' -cover of cost $\alpha_{w'}(G)$ and add the cycle $v_1v_2v_3v_4v_5$. The cost of the resulting family is $\alpha_{w'}(G) + 2 = \alpha_w(G)$, a contradiction.

Configuration 6(b). Since G satisfies Lemma 3.2(ii), Proposition 2.8 applies. In particular, all the edges between $\{u_2^+, u_2^-\}$ and $\{u_1^-, u_3^-, u_4^-\}$ are present, and the edge $u_3^-u_4^-$ is present whenever the endpoints exist.

First observe that $u_1^-, u_2^-,$ and u_4^- cannot all be present. If they were, then $v_2v_3v_4u_2^-u_4^-$ would be a hole, and u_1^- would have three consecutive neighbors on this hole, contradicting Lemma 3.2(ii) (whether u_1^- is adjacent to u_4^- is irrelevant). See Figure A1b.

If u_1^- is absent, the situation is analogous to Configuration 6(a). See Figure A1c. The two cycles $v_1v_2v_3v_4v_5$ and $u_4^-v_2v_3v_4u_2^-$ are symmetric; for instance, relabel $v'_1 = u_2^-, v'_2 = v_4, v'_3 = v_3, v'_4 = v_2, v'_5 = u_4^-$. If $\{v_3, u_3^-\}$ is a maximum-weight independent set, then every maximum-weight independent set contains either v_3 or $\{v_2, v_4\}$. Otherwise, every maximum-weight independent set contains exactly two vertices of $\{v_1, v_2, v_3, v_4, v_5\}$, or one vertex of $\{v_1, v_2, u_4^-\}$.

Now assume u_1^- is present and $w(u_1^-) > 0$; if not, we reduce to the previous case.

In the second case, u_4^- is absent; see Figure A1d. We may assume $w(v_i) > 0$ for all $i \in \{1, \dots, 5\}$. The hole $v_1v_2v_3v_4v_5$ is the only C_5 when u_3^- is absent or when u_1^- and u_3^- are nonadjacent; otherwise, the two odd holes $v_1v_2v_3v_4v_5$ and $v_1v_5v_4u_1^-u_4^-$ are symmetric. In any case, every maximum-weight independent set contains a vertex of the clique $\{v_3, v_4, u_1^-\}$. By Proposition 3.10, there exists a w -cover of G with cost $\alpha_w(G)$, a contradiction.

In the final case, u_2^- is not present; see Figures A1e and A1f. Note that u_1^- and u_4^- are nonadjacent when u_3^- is present; otherwise, u_3^- would have three consecutive neighbors on the hole $v_1v_5v_4u_1^-u_4^-$, contradicting Lemma 3.2(ii). For the same reason, u_1^- and u_3^- cannot be adjacent. We may assume $w(v_i) > 0$ for all $i \in \{1, \dots, 5\}$. The hole $v_1v_2v_3v_4v_5$ is the only C_5 when u_3^- is present; if u_3^- is absent and $u_1^-u_4^- \in E(G)$, then the two odd holes $v_1v_2v_3v_4v_5$ and $v_1v_5v_4u_1^-u_4^-$ are symmetric.

The hole $v_1v_2v_3v_4v_5$ is the only C_5 when u_3^- is present; if u_3^- is absent and $u_1^-u_4^- \in E(G)$, the only two odd holes $v_1v_2v_3v_4v_5$ and $v_1v_5v_4u_1^-u_4^-$ are symmetric. Again, every maximum-weight independent set contains a vertex of the clique $\{v_3, v_4, u_1^-\}$. By Proposition 3.10, there exists a w -cover of G with cost $\alpha_w(G)$, a contradiction.

Configuration 6(c). Note that this configuration does not have optional vertices. Let $K = \{u_5^-, v_2, v_3\}$. We first show that for every $x \in K$, the subgraph $G - x$ is strongly t-perfect. Since $G - \{u_5^-, v_3\}$ is bipartite, $G - u_5^-$ is almost bipartite and strongly t-perfect by

Lemma 3.4. Since $G - v_2$ is a K_4 -free perfect graph (by the strong perfect graph theorem [6]), it is strongly t -perfect by Proposition 3.1. By symmetry, $G - v_3$ is strongly t -perfect.

Thus, $w(x) > 0$ for every $x \in K$. There are four maximal independent sets in $G - K$, and each can be extended by adding a vertex of K . Thus $\alpha_w(G) \geq \alpha_w(G - K) + t > \alpha_w(G - K)$. By the choice of w , the second condition of Proposition 3.10 is satisfied. Therefore there is a w -cover of G of cost $\alpha_w(G)$, a contradiction.