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DE FINETTI'S PROBLEM WITH FIXED TRANSACTION COSTS AND REGIME SWITCHING*

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Abstract. This paper examines a modified version of de Finetti's optimal dividend problem, incorporating fixed transaction costs and altering the surplus process by introducing two-valued drift and two-valued volatility coefficients. This modification aims to capture the transitions or adjustments in the company's financial status. We identify the optimal dividend strategy, which maximizes the expected total net dividend payments (after accounting for transaction costs) until ruin, as a two-barrier impulsive dividend strategy. Notably, the optimal strategy can be explicitly determined for almost all scenarios involving different drifts and volatility coefficients. Our primary focus is on exploring how changes in drift and volatility coefficients influence the optimal dividend strategy.

Key words. De Finetti's problem, dividend payout, transaction cost, regime switching, two-barrier strategy

MSC codes. 93E20, 91G50, 91G05

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1. Introduction.

De Finetti's optimal dividend problem is a classic stochastic control problem that seeks to determine the optimal timing for paying dividends to maximize the total expected dividends until the point of ruin. Due to discounting, dividends should be paid as soon as possible. However, these decisions must be made carefully to avoid increasing the risk of ruin. Despite its long history since de Finetti's original work [14], research in this area remains active at the intersection of control theory and financial/actuarial mathematics. Recent advances in the theory of stochastic processes and stochastic control have enabled the development of more

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realistic models and their solutions. Various stochastic processes, both Gaussian and non-Gaussian, have been employed as alternatives to the classical Brownian motion and Cramér–Lundberg processes. For a comprehensive review, we refer to [1] and the references therein.

In this paper, we examine a diffusion model described by the stochastic differential equation (SDE) (2.1), where the drift and volatility each take on two different values depending on the state. In the stochastic control literature, SDE (2.1) is associated with what is known as bang-bang control of diffusion. In [8], the optimally controlled process for the bounded velocity follower problem is the solution to (2.1) with $\mu_+ = \mu_-$. In [23], the optimal state equation is given by (2.1). The solution to (2.1) with $\sigma_+ = \sigma_-$ is proposed in [16] as a refracted risk model. Additionally, SDE (2.1) is used in local volatility models in mathematical finance; see, for example, [15].

As a prototype for SDEs with discontinuous coefficients, it is also interesting to investigate various properties of the solution to (2.1). The transition density of solution X to (2.1) with $\sigma_+ = \sigma_-$ is found in [20] and applied to compute the optimal expected costs in the classical control problem treated in [8]. The above transition density is also used in [11] to identify limiting distribution arising from the central limit theorem, under nonlinear expectation, for random variables with the same conditional variance but ambiguous means. Similarly, the transition density of solution X to (2.1) with $\mu_+ = 0 = \mu_-$, called oscillating Brownian motion, is found in [21] and used in [12] for computations concerning the central limit theorem, again under nonlinear expectation, for random variables with mean 0 and varying conditional variance. More recently, an explicit expression for transition density of process X in (2.1) is obtained in [13] using a solution to the exit problem together with a perturbation approach, which can be applied to express the value function for the control problem in [23].

We focus on a Brownian motion–driven surplus process, as defined in (2.1), where the drift and diffusion coefficients vary depending on whether the process is above or below a fixed threshold. In most of the literature, spatially homogeneous processes, such as Brownian motion and Lévy processes, are used, often resulting in the optimality of a simple barrier-type strategy. In contrast, models involving processes with dynamics dependent on their current values are limited and typically yield nonanalytical results. Despite the straightforward dynamics of our surplus process, it has practical applications: it offers a way to model "regime switching," where the regime changes depending on whether the surplus process is above or below the threshold. This formulation is particularly suitable for modeling nonstationary premium rates and volatility that can vary based on the company's financial status. Our regime-switching model differs from the classical regime-switching models in the literature, as it is caused endogenously, while the latter is driven by exogenous factors due to transitions or adjustments in the economic system; see [2], [3], [25], and [27]. Works studying exogenous regime-switching involved optimal dividend problems can be found in [4], [19], and [30]. A work that studies an endogenous regime-switching involved optimal dividend problem appears in [29].

A second extension we consider is the inclusion of fixed transaction costs, which transforms de Finetti's optimal dividend problem (from a regular/singular control problem) into an impulsive control problem. This extension makes the problem more practical but at the same time significantly more challenging. Typically, unlike the barrier strategy used in the absence of fixed costs, the objective becomes demonstrating the optimality of two-barrier strategies, which we call the (z_1, z_2)-strategy (also known as the (s, S)-policy in the inventory control literature). According to such a strategy in the insurance/financial context, a dividend is paid immediately after the surplus reaches above the upper level z_2. In the insurance/financial context, this strategy involves paying a dividend immediately after the surplus exceeds the upper level z_2, reducing the surplus to z_1. Identifying these two barriers and proving that the value function satisfies the associated quasi-variational inequality (QVI) is mathematically challenging. Although impulsive control is popular in inventory control problems for infinite-time horizon scenarios, the inclusion of fixed costs is relatively rare in de Finetti's problem, which is terminated at the time of ruin.

Among these works, several different uncontrolled state processes have been considered: in [10] the surplus is governed by a Brownian motion with drift; in [6] and [24] the income process follows the dynamics of a general diffusion process; in [5] the Cramér–Lundberg risk process is considered; in [17] the surplus process is a jump diffusion; and in [7] the surplus process is a spectrally positive Lévy process.

It is important to note that, to the best of our knowledge, all existing contributions on de Finetti's optimal dividend problem utilize spatially homogeneous processes or diffusion processes with regular drift and volatility coefficients as their uncontrolled reserve processes. The combination of nonregular drift and volatility coefficients with fixed transaction costs makes the problem particularly complex and demanding. In such a setting, the optimal selections of z_1 and z_2 are interdependent and, in this case, are also influenced by the drift/diffusion change-trigger barrier a. There are scenarios where both z_1 and z_2 are either above or below a, as well as cases where z_2 is above a while z_1 is below a. Each scenario requires a different analysis and approach. To tackle this problem, we first solve the exit problem for (2.1) using a martingale approach, which allows us to derive explicit expressions for the expected dividend function under each (z_1, z_2)-strategy. Subsequently, we establish sufficient conditions for optimality. This is followed by a case-by-case analysis, demonstrating that by appropriately selecting the barriers, the candidate value function solves the QVI. As a result, we obtain explicit optimal strategies for all different parameter choices of $\mu_\pm$ and $\sigma_\pm$ in the model, which aids in better understanding and analyzing the connections between the optimal strategies and these parameters.

The contributions of this paper can be summarized from two perspectives: one pertains to the problem formulation and the other to the solution methodology. We subsequently provide a detailed elaboration of these contributions.

From a problem formulation perspective, our main contribution is to solve, for the first time, an optimal dividend problem by simultaneously incorporating two critical features, thereby framing the problem as a challenging impulse control problem: (1) an endogenous state-triggered switching mechanism, and (2) fixed transaction costs. While each feature has been studied in isolation, their combination has not been previously addressed. On the one hand, most existing studies model regime switching exogenously using a Markov chain (e.g., [4], [19], [25], [30]). A recent work by [29] considers endogenous regime switching. However, none of these works incorporate fixed transaction costs. On the other hand, many works include fixed costs (e.g., [5], [7], [10]), but their models are spatially homogeneous and exclude regime switching. Others [6], [24], [17] allow state-dependent coefficients but impose strong regularity assumptions (e.g., Lipschitz continuity, differentiability). In contrast, our model allows for discontinuous drift and volatility coefficients, which change abruptly at a threshold and take distinct values on either side.

From a methodological perspective, existing methods for dividend problems with regime switching (e.g., [19], [29], [30]) rely on regular or singular control and are not applicable to our impulse control setting with fixed transaction costs. Similarly, methods for fixed-cost problems (e.g., [5], [6], [7], [10], [17], [24]) generally assume either spatial homogeneity, constant coefficients, or regular (smooth) state-dependence coefficients, none of which hold in our framework of discontinuous coefficients. To address these challenges, we develop a novel analytical methodology for solving the associated QVI, which includes the following four technical contributions.

First, through extensive technical analysis, we provide a complete and unified characterization of the piecewise convexity/concavity of the function g (Propositions 3.1–3.4), identifying 11 mutually exclusive and collectively exhaustive cases (with 21 subcases) that fully describe the geometry of g and facilitate the explicit construction of candidate optimal strategies.

Second, we introduce a novel construction-based method to characterize the set of candidate optimal impulse dividend strategies $\mathcal{M}_\zeta$ (Theorems 3.1–3.4), yielding closed-form expressions for all relevant quantities and enabling rigorous sensitivity analysis and numerical implementation.

Third, in the proof of Theorem 4.1, we overcome the challenges posed by discontinuous coefficients by constructing new inequalities (see (4.4) and (4.6), etc.) and proposing three calibrated sufficient conditions (of which the union of the first two is also necessary) to verify optimality.

Finally, to establish the monotonicity result in Proposition 3.6, we employ a novel reparameterization technique, transforming the analysis from the "cost" (i.e., β) space to a more tractable "slope of the value function" space. This idea may offer useful insights for other control problems.

The rest of the paper is organized as follows. Section 2 formulates the problem, provides preliminary results, and introduces the two-barrier (z_1, z_2)-impulsive dividend strategy. In section 3, we present a complete and explicit characterization of the optimal strategy among the class of (z_1, z_2)-impulsive dividend strategies. Section 4 is devoted to characterizing the optimal strategy for the targeted dividend control problem. Some lengthy and technical proofs are provided in Appendix A. A version of this paper with all the detailed proofs is available in [28].

2. Problem formulation and preliminary results.

We fix a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ throughout the paper, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by a standard one-dimensional Brownian motion $B = (B_t)_{t\geq 0}$ and satisfies the usual conditions. Fix constants $a \in (0,\infty)$, $\mu_\pm \in \mathbb{R}$, and $\sigma_\pm \in (0,\infty)$.

Consider an SDE with two-valued drift and two-valued diffusion coefficients:

$$\mathrm{d}X_t = (\sigma_+ \mathbf{1}_{\{X_t>a\}} + \sigma_- \mathbf{1}_{\{X_t\leq a\}})\mathrm{d}B_t + (\mu_+ \mathbf{1}_{\{X_t>a\}} + \mu_- \mathbf{1}_{\{X_t\leq a\}})\mathrm{d}t,\ t\geq 0. \tag{2.1}$$

The existence and uniqueness of a strong solution to SDE (2.1) are guaranteed by Theorem 1.3 on page 55 of [22]. We use $X = (X_t)_{t\geq 0}$ to describe the surplus process of a company before paying dividends.

We will investigate an optimal impulsive dividend payout problem. To this end, we first introduce impulsive dividend payout strategies. An impulsive dividend strategy $\pi = (L^\pi_t)_{t\geq 0}$ is an $\mathbb{F}$-adapted nondecreasing, right continuous pure jump process such that $L^\pi_t = \sum_{0\leq s\leq t} \Delta L^\pi_s$ where $\Delta L^\pi_s = L^\pi_s - L^\pi_{s-} \geq 0$ with $L^\pi_{0-} = 0$. Applying an impulsive dividend payout strategy π to the process (2.1), we see the surplus process U^π after paying dividends becomes

$$\mathrm{d}U^\pi_t = (\sigma_+ \mathrm{d}B_t + \mu_+ \mathrm{d}t)\mathbf{1}_{\{U^\pi_t>a\}} + (\sigma_- \mathrm{d}B_t + \mu_- \mathrm{d}t)\mathbf{1}_{\{U^\pi_t\leq a\}} - \mathrm{d}L^\pi_t. \tag{2.2}$$

Define the ruin time of U^π as $T^\pi := \inf\{t\geq 0 \colon U^\pi_t < 0\}$, where $\inf\emptyset = \infty$. We fix a positive constant q to represent the discount rate.

Definition 2.1. *An impulsive dividend payout strategy* $\pi = (L_t^\pi)_{t\geq 0}$ *is called admissible if* $0 \leq \Delta L_t^\pi = L_t^\pi - L_{t-}^\pi \leq U_{t-}^\pi \vee 0$ *for any* $t \geq 0$ *(i.e., the amount of a lump sum of dividend payout is not allowed to make the company bankrupt),*

$$\mathbb{E}_x\left[\sum_{0\leq s\leq T^\pi} e^{-qs}\Delta L_s^\pi \mathbf{1}_{\{\Delta L_s^\pi > 0\}}\right] < \infty,$$

and SDE (2.2) *with any initial value* $U_{0-}^\pi = x$ *admits a unique strong solution. We use* Π *to denote the set of all admissible impulsive dividend payout strategies.*

Let β be a fixed positive constant, which can be interpreted as a fixed transaction cost or penalty parameter. The reward function for an admissible impulsive dividend payout strategy $\pi \in \Pi$ is defined as

$$V_\pi(x) := \mathbb{E}_x\left[\sum_{0\leq s\leq T^\pi} e^{-qs}(\Delta L_s^\pi - \beta)\mathbf{1}_{\{\Delta L_s^\pi > 0\}}\right], \quad x \geq 0. \tag{2.3}$$

Our aim is to determine the optimal value function associated with the impulsive dividend control problem (2.3): $\sup_{\pi\in\Pi} V_\pi(x)$. An impulsive strategy $\pi^* \in \Pi$ is called an optimal impulsive strategy to the problem (2.3) if it satisfies

$$V_{\pi^*}(x) = \sup_{\pi\in\Pi} V_\pi(x) < \infty. \tag{2.4}$$

Clearly, when $x \leq 0$, $T^\pi = 0$ for any admissible strategy π, so $\sup_{\pi\in\Pi} V_\pi(x) = 0$. From now on, we will focus on the case $x > 0$.

By its definition, one can prove that $V_\pi(\cdot)$ is nonnegative and linear growth on $\mathbb{R}^+$. (Please refer to [28] for a detailed proof.) Also, the value function $\sup_{\pi\in\Pi} V_\pi(x)$ is nonincreasing and convex with respect to $\beta \in (0,\infty)$ for any fixed $x > 0$.

2.1. Verification lemma.

We now attempt to characterize the optimal impulsive strategies to the problem (2.4) when $x > 0$.

Lemma 2.1 gives a sufficient condition for an admissible impulsive strategy $\hat{\pi} \in \Pi$ to be the optimal dividend strategy for the control problem (2.4). Indeed, it is shown that any optimal strategy must belong to a subset of Π:

$$\Pi_0 := \left\{\pi \in \Pi;\ \text{for any } t \geq 0, \triangle L_t^\pi \geq \beta \text{ if and only if } \triangle L_t^\pi > 0\right\}. \tag{2.5}$$

Intuitively speaking, one shall not pay a dividend less than β, for otherwise it will give a negative impact on the reward functional.

Let $\mathcal{A}$ be the infinitesimal generator associated with the process X, defined as

$$\mathcal{A}f(x) := (\sigma_+^2\mathbf{1}_{\{x>a\}} + \sigma_-^2\mathbf{1}_{\{x\leq a\}})f''(x)/2 + (\mu_+\mathbf{1}_{\{x>a\}} + \mu_-\mathbf{1}_{\{x\leq a\}})f'(x),$$

for any function f that is continuously differentiable and piecewise C^2 on $\mathbb{R}_+$.

Lemma 2.1 (verification lemma). *Suppose there is a strategy* $\hat{\pi} \in \Pi$ *such that* $V_{\hat{\pi}}$ *is continuously differentiable and piecewise* C^2 *and satisfies* $(\mathcal{A} - q)V_{\hat{\pi}} \leq 0$ *on* $\mathbb{R}_+$ *except for finitely many points,* $V_{\hat{\pi}}(x) \geq 0$ *and* $V_{\hat{\pi}}(x) - V_{\hat{\pi}}(y) \geq x - y - \beta$ *for all* $x > y \geq 0$. *Then* $\hat{\pi}$ *is an optimal impulsive strategy to the problem* (2.3). *Moreover,* $\hat{\pi} \in \Pi_0$.

Proof. For any admissible strategy $\pi = (L_t^\pi)_{t\ge 0} \in \Pi \setminus \Pi_0$, we define a new admissible impulsive strategy $\pi_0 = (L_t^{\pi_0})_{t\ge 0} \in \Pi_0$ where $L_t^{\pi_0} := \sum_{0\le s\le t} \Delta L_s^\pi \mathbf{1}_{\{\Delta L_s^\pi \ge \beta\}}$. By definition, it holds that $\triangle L_t^{\pi_0} \le \triangle L_t^\pi$ for all $t \ge 0$, so $U_t^{\pi_0} \ge U_t^\pi$ for all $t \ge 0$ and consequently, $T^{\pi_0} \ge T^\pi$. Therefore, for any $x \in (0,\infty)$,

$$\begin{aligned}
V_{\pi_0}(x) &= \mathbb{E}_x\left[\sum_{0\le s\le T^{\pi_0}} e^{-qs}(\Delta L_s^\pi - \beta)\mathbf{1}_{\{\Delta L_s^\pi \ge \beta\}}\right]\\
&\ge \mathbb{E}_x\left[\sum_{0\le s\le T^{\pi}} e^{-qs}(\Delta L_s^\pi - \beta)\mathbf{1}_{\{\Delta L_s^\pi \ge \beta\}}\right]\\
&> \mathbb{E}_x\left[\sum_{0\le s\le T^{\pi}} e^{-qs}(\Delta L_s^\pi - \beta)\left(\mathbf{1}_{\{\Delta L_s^\pi \ge \beta\}} + \mathbf{1}_{\{0<\Delta L_s^\pi < \beta\}}\right)\right] = V_\pi(x).
\end{aligned}$$

Hence, it suffices to prove $V_{\hat\pi}(x) \ge V_\pi(x)$ for any $x > 0$ and admissible $\pi \in \Pi_0$.

Fix any $x > 0$ and $\pi \in \Pi_0$. Let $\theta_n := \inf\{t \ge 0 : U_t^\pi > n \text{ or } U_t^\pi < 0\}$. By a version of Ito's formula (see Theorem 4.57 on page 57 of [18] or Theorem 70 in Chapter IV of [26]) we have, for any constant $t > 0$,

$$\begin{aligned}
&e^{-q(t\wedge\theta_n\wedge T^\pi)} V_{\hat\pi}(U^\pi_{t\wedge\theta_n\wedge T^\pi})\\
&= V_{\hat\pi}(x) + \int_0^{t\wedge\theta_n\wedge T^\pi} (\mathcal{A} - q)V_{\hat\pi}(U_s^\pi)\mathrm{d}s + \sum_{0\le s\le t\wedge\theta_n\wedge T^\pi} e^{-qs}\Delta V_{\hat\pi}(U_s^\pi) + M_{t\wedge\theta_n\wedge T^\pi}\\
&\le V_{\hat\pi}(x) - \sum_{0\le s\le t\wedge\theta_n\wedge T^\pi} e^{-qs}(U_{s-}^\pi - U_s^\pi - \beta)\mathbf{1}_{\{\Delta U_s^\pi \ne 0\}} + M_{t\wedge\theta_n\wedge T^\pi}\\
&= V_{\hat\pi}(x) - \sum_{0\le s\le t\wedge\theta_n\wedge T^\pi} e^{-qs}(\Delta L_s^\pi - \beta)\mathbf{1}_{\{\Delta L_s^\pi > 0\}} + M_{t\wedge\theta_n\wedge T^\pi},
\end{aligned}$$

where $(M_t)_{t\ge 0}$ is a continuous local martingale. By nonnegativity, it follows that

$$V_{\hat\pi}(x) \ge \sum_{0\le s\le t\wedge\theta_n\wedge T^\pi} e^{-qs}(\Delta L_s^\pi - \beta)\mathbf{1}_{\{\Delta L_s^\pi > 0\}} - M_{t\wedge\theta_n\wedge T^\pi}.$$

Let τ_n be an increasing localizing stopping time sequence of M with $\lim_{n\to\infty}\tau_n = \infty$. Since $\pi \in \Pi_0$, $(\Delta L_s^\pi - \beta)\mathbf{1}_{\{\Delta L_s^\pi > 0\}} \ge 0$ for any $s \ge 0$. Fatou's lemma gives

$$\begin{aligned}
V_{\hat\pi}(x) &\ge \liminf_{n\to\infty} \mathbb{E}_x\left[\sum_{0\le s\le n\wedge\tau_n\wedge\theta_n\wedge T^\pi} e^{-qs}(\Delta L_s^\pi - \beta)\mathbf{1}_{\{\Delta L_s^\pi > 0\}}\right]\\
&= \mathbb{E}_x\left[\sum_{0\le s\le T^\pi} e^{-qs}(\Delta L_s^\pi - \beta)\mathbf{1}_{\{\Delta L_s^\pi > 0\}}\right] = V_\pi(x),
\end{aligned}$$

which is the desired result. As a byproduct, $\hat\pi \in \Pi_0$. ☐

By this result, our problem now reduces to finding an impulsive strategy $\hat\pi \in \Pi_0$ that fulfills the requirement of Lemma 2.1. We conjecture that the optimal strategy to solve the control problem (2.4) shall be some two-barrier impulsive strategy. To show this, we introduce this kind of strategy in the subsequent section.

2.2. Two-barrier strategies and preliminary results. The two-barrier impulsive dividend strategy corresponding to any pair $0<z_1<z_2$, denoted by $(L_t^{z_1,z_2})_{t\geq 0}$, is the strategy under which a lump sum of dividends is paid out to bring the surplus process down to the level z_1 once the surplus process is greater than or attempts to up-cross the level z_2, and no dividend payout happens if the surplus process is below z_2. For convenience, we also call the strategy a (z_1,z_2)-strategy. Mathematically, the surplus process $(U_t^{z_1,z_2})_{t\geq 0}$ under the (z_1,z_2)-strategy is determined by

$$\begin{cases} L_t^{z_1,z_2} = \sum_{0\leq s\leq t}(U_{s-}^{z_1,z_2} - z_1)\mathbf{1}_{\{U_{s-}^{z_1,z_2}\geq z_2\}}, \quad t\geq 0,\\ \mathrm{d}U_t^{z_1,z_2} = (\mu_+\mathbf{1}_{\{U_t^{z_1,z_2}>a\}} + \mu_-\mathbf{1}_{\{U_t^{z_1,z_2}\leq a\}})\mathrm{d}t - (U_{t-}^{z_1,z_2} - z_1)\\ \qquad \times\mathbf{1}_{\{U_{t-}^{z_1,z_2}\geq z_2\}} + (\sigma_+\mathbf{1}_{\{U_t^{z_1,z_2}>a\}} + \sigma_-\mathbf{1}_{\{U_t^{z_1,z_2}\leq a\}})\mathrm{d}B_t, \quad t\geq 0,\\ U_{0-}^{z_1,z_2} = x. \end{cases}$$

Write the ruin time of U^{z_1,z_2} as $T^{z_1,z_2} := \inf\{t\geq 0 : U_t^{z_1,z_2} < 0\}$, and denote the value function of the two-barrier impulsive strategy $(L_t^{z_1,z_2})_{t\geq 0}$ by

$$V_{z_1}^{z_2}(x) := \mathbb{E}_x\left[\sum_{0\leq s\leq T^{z_1,z_2}} e^{-qs}(\Delta L_s^{z_1,z_2} - \beta)\mathbf{1}_{\{\Delta L_s^{z_1,z_2}>0\}}\right], \quad x\geq 0.$$

We now aim to find an explicit expression for $V_{z_1}^{z_2}(x)$ so that we can apply Lemma 2.1 to derive an optimal strategy to (2.4). To this end, we first define two functions $g^{\pm}\in C^1(\mathbb{R})\cap C^2(\mathbb{R}\backslash\{a\})$ that satisfy the following equation (except at a):

$$(\sigma_+^2\mathbf{1}_{\{x>a\}} + \sigma_-^2\mathbf{1}_{\{x\leq a\}})g''(x)/2 + (\mu_+\mathbf{1}_{\{x>a\}} + \mu_-\mathbf{1}_{\{x\leq a\}})g'(x) = qg(x). \tag{2.6}$$

Define constants

$$\theta_1^{\pm} := (\sqrt{\mu_{\pm}^2 + 2q\sigma_{\pm}^2} + \mu_{\pm})/\sigma_{\pm}^2 > 0, \quad \theta_2^{\pm} := (\sqrt{\mu_{\pm}^2 + 2q\sigma_{\pm}^2} - \mu_{\pm})/\sigma_{\pm}^2 > 0, \tag{2.7}$$

$$c_- = (\theta_1^- - \theta_1^+)/(\theta_2^- + \theta_1^-), \quad 1 - c_- = (\theta_2^- + \theta_1^+)/(\theta_2^- + \theta_1^-) > 0, \tag{2.8}$$

$$c_+ = (\theta_2^+ - \theta_2^-)/(\theta_2^+ + \theta_1^+), \quad 1 - c_+ = (\theta_1^+ + \theta_2^-)/(\theta_2^+ + \theta_1^+) > 0. \tag{2.9}$$

Then, functions $g^{\pm}\in C^1(\mathbb{R})\cap C^2(\mathbb{R}\backslash\{a\})$ defined below satisfy (2.6) (except at a):

$$\begin{cases} g^-(x) = e^{-\theta_1^+(x-a)}\mathbf{1}_{\{x>a\}} + (c_- e^{\theta_2^-(x-a)} + (1-c_-)e^{-\theta_1^-(x-a)})\mathbf{1}_{\{x\leq a\}},\\ g^+(x) = ((1-c_+)e^{\theta_2^+(x-a)} + c_+ e^{-\theta_1^+(x-a)})\mathbf{1}_{\{x>a\}} + e^{\theta_2^-(x-a)}\mathbf{1}_{\{x\leq a\}}. \end{cases} \tag{2.10}$$

It is easy to verify that $g^{\pm}(a) = 1$ and $g^{\pm\prime}(a-) = g^{\pm\prime}(a+)$. Define

$$\begin{aligned} g(x) &:= g^+(x)g^-(0) - g^-(x)g^+(0)\\ &= \left[(1-c_+)g^-(0)e^{\theta_2^+(x-a)} - (g^+(0) - c_+g^-(0))e^{-\theta_1^+(x-a)}\right]1_{\{x>a\}}\\ &\quad + \left[(g^-(0) - c_-g^+(0))e^{\theta_2^-(x-a)} - (1-c_-)g^+(0)e^{-\theta_1^-(x-a)}\right]1_{\{x\leq a\}}. \end{aligned} \tag{2.11}$$

Then, one has $g\in C^1(\mathbb{R})\cap C^2(\mathbb{R}\backslash\{a\})$ and it satisfies (2.6) (except at a).

LEMMA 2.2. *The function g defined by* (2.11) *satisfies $g'>0$ and $g(0)=0$.*

Due to space limits, we cannot give a detailed proof here. Please refer to [28] for a detailed proof.

We next introduce the first hitting time of level $y\in\mathbb{R}$ for the process $(X_t)_{t\geq 0}$ given by (2.1) as

$$T_y := \inf\{t\geq 0 : X_t = y\}.$$

For $y \le x \le z$ with $y \neq z$, applying the generalized Ito's formula (see Theorem 70 of Chapter IV in [26] for more details) we know that the two processes $(e^{-qt}g^{\pm}(X_t))_{t\ge 0}$ are local martingales. Then, it follows from Doob's optional stopping theorem that $g^{\pm}(x) = \mathbb{E}_x[e^{-q(T_y\wedge T_z)}g^{\pm}(X_{T_y\wedge T_z})]$. Solving the above two equations, we have the following result.

Lemma 2.3. *For any $y \le x \le z$ with $y \neq z$, we have*

$$\mathbb{E}_x[e^{-qT_y}\mathbf{1}_{\{T_y<T_z\}}] = (g^+(z)g^-(x) - g^-(z)g^+(x))/(g^-(y)g^+(z) - g^-(z)g^+(y)),$$
$$\mathbb{E}_x[e^{-qT_z}\mathbf{1}_{\{T_z<T_y\}}] = (g^+(y)g^-(x) - g^-(y)g^+(x))/(g^-(z)g^+(y) - g^-(y)g^+(z)).$$

Now we are ready to give the explicit expression for $V_{z_1}^{z_2}$.

Proposition 2.1. *Given $\beta \le z_1 + \beta \le z_2$, we have*

$$V_{z_1}^{z_2}(x) = \begin{cases} \frac{g(z_1)(z_2-z_1-\beta)}{g(z_2)-g(z_1)} + x - z_1 - \beta, & x \ge z_2, \\ \frac{g(x)(z_2-z_1-\beta)}{g(z_2)-g(z_1)}, & 0 \le x < z_2. \end{cases} \tag{2.12}$$

Thanks to Lemma 2.2, $V_{z_1}^{z_2}(\cdot)$ is continuous and strictly increasing on $\mathbb{R}_+$.

Proof. Since both $V_{z_1}^{z_2}(0)$ and $g(0)$ are zero, the claim holds true when $x = 0$. When $x \in (0, z_2)$, one has

$$V_{z_1}^{z_2}(x) = \mathbb{E}_x\left[e^{-qT_{z_2}}\mathbf{1}_{\{T_{z_2}<T_0\}}\right]V_{z_1}^{z_2}(z_2) = (V_{z_1}^{z_2}(z_1) + z_2 - z_1 - \beta)g(x)/g(z_2).$$

Setting $x = z_1$ in the above equation and using the finiteness of $V_{z_1}^{z_2}$, we have $V_{z_1}^{z_2}(z_1) = \frac{g(z_1)}{g(z_2)-g(z_1)}(z_2 - z_1 - \beta)$. Combining the above proves the claim for $x \in (0, z_2)$. When $x \ge z_2$, by the strong Markov property of the process $(U_t^{z_1,z_2})_{t\ge 0}$, we have $V_{z_1}^{z_2}(x) = V_{z_1}^{z_2}(z_1) + x - z_1 - \beta$, and the claim follows by combining the above equations. ∎

For the problem (2.4), we conjecture that its optimal strategy is a (z_1, z_2)-strategy for some (z_1, z_2) satisfying $\beta \le z_1 + \beta \le z_2 < \infty$. To verify our conjecture, we shall first find the optimal strategy among the class of (z_1, z_2)-strategies. By (2.12), one needs to maximize $\frac{z_2-z_1-\beta}{g(z_2)-g(z_1)}$. This motivates us to define

$$\begin{aligned} \mathcal{D}_\zeta &:= \{(x, y) \in [0,\infty)^2 : x + \beta \le y\}, \\ \zeta(z_1, z_2) &:= \frac{z_2 - z_1 - \beta}{g(z_2) - g(z_1)} \ge 0, \quad (z_1, z_2) \in \mathcal{D}_\zeta, \\ \mathcal{M}_\zeta &:= \left\{(z_1, z_2) \in \mathcal{D}_\zeta : \zeta(z_1, z_2) \ge \zeta(x, y) \text{ for all } (x, y) \in \mathcal{D}_\zeta\right\}. \end{aligned} \tag{2.13}$$

Hence, $\mathcal{M}_\zeta$ collects all the global maximizers of $\zeta(z_1, z_2)$ on its domain $\mathcal{D}_\zeta$. We have that $\mathcal{M}_\zeta$ is a nonempty, bounded set, whose proof can be found in [28].

Proposition 2.2. *The set $\mathcal{M}_\zeta$ is not empty. In addition, there exists a finite $z_0 \in (0, \infty)$ such that $\mathcal{M}_\zeta \subseteq \{(x, y) \in [0,\infty)^2 : x + \beta < y < z_0\}$.*

Indeed, we will prove that the set $\mathcal{M}_\zeta$ consists of either one, two, or three elements in Theorems 3.1–3.4.

Remark 2.1. For any $(z_1, z_2) \in \mathcal{M}_\zeta$, we clearly have $\frac{\partial}{\partial z_2}\zeta(z_1, z_2) = 0$, i.e.,

$$g(z_2) - g(z_1) = (z_2 - z_1 - \beta)g'(z_2). \tag{2.14}$$

Substituting this into (2.12) yields another expression:

$$V_{z_1}^{z_2}(x) = \begin{cases} \frac{g(z_2)}{g'(z_2)} + x - z_2, & x \ge z_2, \\ \frac{g(x)}{g'(z_2)}, & 0 \le x < z_2. \end{cases} \tag{2.15}$$

This indicates $V_{z_1}^{z_2}(\cdot) \in C^1(\mathbb{R}_+)$. In addition, if $(z_1, z_2) \in \mathcal{M}_\zeta$ is such that $z_1 > 0$, then $\frac{\partial}{\partial z_1}\zeta(z_1, z_2) = 0$, that is,

$$g(z_2) - g(z_1) = (z_2 - z_1 - \beta) g'(z_1). \tag{2.16}$$

Thanks to $z_2 > z_1$, $g' > 0$ by Lemma 2.2, it follows from (2.14) and (2.16) that $g'(z_1) = g'(z_2)$ for any $(z_1, z_2) \in \mathcal{M}_\zeta$ with $z_1 > 0$. Put

$$\psi(x, y) := \int_x^y \left(1 - \frac{g'(s)}{g'(y)}\right) \mathrm{d}s, \quad x, y \in (0, \infty). \tag{2.17}$$

Then, (2.14) is equivalent to

$$\psi(z_1, z_2) = \beta. \tag{2.18}$$

Therefore, we have $\mathcal{M}_\zeta \subseteq \mathcal{N} := \mathcal{N}^+ \cup \mathcal{N}^-$, where

$$\begin{aligned} \mathcal{N}^+ &:= \{(z_1, z_2) : 0 < z_1 < z_2 < \infty,\ \psi(z_1, z_2) = \beta,\ g'(z_1) = g'(z_2)\}, \\ \mathcal{N}^- &:= \{(0, z_2) : 0 < z_2 < \infty,\ \psi(0, z_2) = \beta\}. \end{aligned}$$

3. Explicit characterization of $\mathcal{M}_\zeta$.

This section is devoted to the characterization of $\mathcal{M}_\zeta$ in four mutually exclusive and collectively exhaustive cases: (1) $\mu_\pm > 0$, (2) $\mu_\pm \le 0$, (3) $\mu_+ \le 0$ and $\mu_- > 0$, and (4) $\mu_+ > 0$ and $\mu_- \le 0$; see Theorems 3.1–3.4 of subsections 3.1–3.4. We also get, in subsection 3.5, several general properties of $\mathcal{M}_\zeta$ that can help to better understand and analyze the connections between the optimal dividend strategy and the model parameters.

To proceed, define five constants x_0, Θ, a_1, a_2, and a_3 as

$$x_0 := \frac{\ln \frac{(g^+(0) - c_+ g^-(0))(\theta_1^+)^2}{(1 - c_+) g^-(0)(\theta_2^+)^2}}{\theta_2^+ + \theta_1^+} + a, \quad \Theta := c_+(\theta_1^+)^2 + (1 - c_+)(\theta_2^+)^2, \tag{3.1}$$

$$a_1 := \frac{2 \ln \frac{\theta_1^-}{\theta_2^-}}{\theta_2^- + \theta_1^-}, \quad a_2 := \frac{\ln \frac{\theta_2^+ + \theta_1^-}{\theta_2^+ - \theta_2^-}}{\theta_1^- + \theta_2^-}, \quad a_3 := \frac{\ln \frac{(1 - c_- c_+)(\theta_1^+)^2 - (1 - c_+) c_- (\theta_2^+)^2}{(1 - c_-)\Theta}}{\theta_1^- + \theta_2^-}, \tag{3.2}$$

whenever they are well-defined (note that $\ln x$ is not well-defined for $x \le 0$). The constants defined in (3.1) and (3.2) are instrumental in establishing the piecewise convexity (concavity) of g (see Propositions 3.1–3.4), a property that is fundamental to the analysis in this paper.

3.1. Explicit characterization of $\mathcal{M}_\zeta$ in the case $\mu_\pm > 0$.

When $\mu_\pm > 0$, we distinguish the following mutually exclusive and collectively exhaustive Cases (i)–(iv).

Case (i) One of the following conditions holds:

- $0 < a_2 \le a \le a_1$ and $c_+ > 0$,
- $0 < a_3 \le a \le a_1 \wedge a_2$ and $c_+ > 0$,
- $0 < a_3 \le a \le a_1$, $c_+ \le 0$, and $\Theta > 0$.

Case (ii) One of the following conditions holds:

- $0 < a \le a_1 \wedge a_2 \wedge a_3$ and $c_+ > 0$,
- $0 < a \le a_1$, $c_+ \le 0$, and $\Theta \le 0$,
- $0 < a \le (a_1 \wedge a_3)$, $c_+ \le 0$, and $\Theta > 0$.

Case (iii) One of the following conditions holds:

- $(a_1 \vee a_2) \le a$ and $c_+ > 0$,
- $(a_1 \vee a_3) \le a < a_2$ and $c_+ > 0$,
- $(a_1 \vee a_3) \le a$, $c_+ \le 0$, and $\Theta > 0$.

Case (iv) One of the following conditions holds:

- $a_1 < a < (a_2 \wedge a_3)$ and $c_+ > 0$,
- $a_1 < a$, $c_+ \leq 0$, and $\Theta \leq 0$,
- $a_1 < a < a_3$, $c_+ \leq 0$, and $\Theta > 0$.

The following proposition gives a complete characterization of the piecewise concavity or convexity of the function g on $\mathbb{R}^+$. Its proof is given in Appendix A.1.

PROPOSITION 3.1. *Suppose that* $\mu_\pm > 0$. *Under Case* (i), $g(x)$ *is concave on* $(0,a)$ *and convex on* (a,∞). *Under Case* (ii), $g(x)$ *is concave on* $(0,x_0)$ *and convex on* (x_0,∞). *Under Case* (iii), $g(x)$ *is concave on* $(0,a_1)$ *and convex on* (a_1,∞). *Under Case* (iv), $g(x)$ *is concave on* $(0,a_1)$, *convex on* (a_1,a), *concave on* (a,x_0), *and convex on* (x_0,∞).

Under Case (i), let $(g')_-^{-1}(x) := \inf\{z \in [0,a]; g'(z) \leq x\}$ for $x \in [g'(a),\infty)$ and $(g')_+^{-1}(x)$ be the inverse function of $[a,\infty) \ni x \mapsto g'(x) \in [g'(a),\infty)$. Define $a_4 := (g')_+^{-1}(g'(0))$. Further, define

$$\phi(x) := \psi((g')_-^{-1}(g'(x)),x) = \int_{(g')_-^{-1}(g'(x))}^{x} \left(1 - \frac{g'(s)}{g'(x)}\right) \mathrm{d}s, \quad x \in [a,\infty). \tag{3.3}$$

Under Case (ii), define the inverse function $(\bar{g}')_-^{-1}$ (resp., $(\bar{g}')_+^{-1}$) the same as $(g')_-^{-1}$ (resp., $(g')_+^{-1}$) but with x_0 in place of a in the above case. We further define $\bar{\phi}$ the same as (3.3) but with $(g')_-^{-1}$ replaced by $(\bar{g}')_-^{-1}$ and x_0 in place of a. Denote by ϕ^{-1} and $\bar{\phi}^{-1}$ the inverse functions of ϕ and $\bar{\phi}$, respectively. The well-definedness of these functions will be confirmed in the proof of the upcoming Theorem 3.1.

Under Case (iii), let $(\tilde{g}')_-^{-1}$ (resp., $(\tilde{g}')_+^{-1}$) be defined the same as $(g')_-^{-1}$ (resp., $(g')_+^{-1}$) but with a_1 in place of a. Define $\tilde{\phi}$ the same as (3.3) but with $(g')^{-1}$ replaced by $(\tilde{g}')^{-1}$ and a_1 in place of a. The inverse functions of $\tilde{\phi}$ is denoted as $\tilde{\phi}^{-1}$.

Under Case (iv), let $(g')_1^{-1}$ be defined the same as $(g')_-^{-1}$ but with a_1 in place of a; let $(g')_2^{-1}$ be the inverse function of $[a_1,a] \ni x \mapsto g'(x) \in [g'(a_1),g'(a)]$; let $(g')_3^{-1}$ be the inverse function of $[a,x_0] \ni x \mapsto g'(x) \in [g'(x_0),g'(a)]$; and let $(g')_4^{-1}$ be the inverse function of $[x_0,\infty) \ni x \mapsto g'(x) \in [g'(x_0),\infty)$. Denote $a_5 := \inf\{x \geq x_0; g'(x) \geq g'(a_1)\}$ and $a_6 := (g')_4^{-1}(g'(a))$. Furthermore, put

$$x_1 := \inf\left\{x \in [a_5,a_6] : \int_{(g')_1^{-1}(g'(x))}^{(g')_3^{-1}(g'(x))} \big(1 - g'(s)/g'(x)\big)\mathrm{d}s \geq 0\right\}, \tag{3.4}$$

$$x_2 := \inf\left\{x \in [a_5,a_6] : \int_{(g')_2^{-1}(g'(x))}^{x} \big(1 - g'(s)/g'(x)\big)\mathrm{d}s \geq 0\right\}, \tag{3.5}$$

$$\omega_1(x) := \int_{(g')_1^{-1}(g'(x))}^{x} \big(1 - g'(s)/g'(x)\big)\mathrm{d}s, \quad x \in [a_1,(g')_2^{-1}(g'(x_2))] \cup [x_2,\infty), \tag{3.6}$$

$$\omega_2(x) := \begin{cases} \displaystyle\int_{(g')_3^{-1}(g'(x))}^{x} (1 - g'(s)/g'(x))\,\mathrm{d}s, & x \in [x_0,x_1), \\ \displaystyle\int_{(g')_1^{-1}(g'(x))}^{x} (1 - g'(s)/g'(x))\,\mathrm{d}s, & x \in [x_1,\infty). \end{cases} \tag{3.7}$$

Theorem 3.1 provides an explicit characterization of $\mathcal{M}_\zeta$, whose proof is presented in Appendix A.2.

THEOREM 3.1. *Suppose that* $\mu_\pm > 0$.

- *Under Case* (i), *we have* $\mathcal{M}_\zeta = \{((g')_-^{-1}(g'(\phi^{-1}(\beta))), \phi^{-1}(\beta))\}$.
- *Under Case* (ii), *we have* $\mathcal{M}_\zeta = \{((\bar{g}')_-^{-1}(g'(\bar{\phi}^{-1}(\beta))), \bar{\phi}^{-1}(\beta))\}$.

- *Under Case* (iii)*, we have* $\mathcal{M}_\zeta = \{((\tilde{g}')_-^{-1}(g'(\tilde{\phi}^{-1}(\beta))), \tilde{\phi}^{-1}(\beta))\}$.
- *Under Case* (iv)*, we have*

$$\mathcal{M}_\zeta = \begin{cases} \{(\tilde{z}_1, \tilde{z}_2)\} & \text{if } \beta \in A_1 \cup A_3, \\ \{(\bar{z}_1, \bar{z}_2)\} & \text{if } \beta \in A_2 \cap \overline{A}_3, \\ \{(\tilde{z}_1, \tilde{z}_2)\} \cup \{(\bar{z}_1, \bar{z}_2)\} & \text{otherwise}, \end{cases}$$

where $(\tilde{z}_1, \tilde{z}_2) := ((g')_1^{-1}(g'(\omega_1^{-1}(\beta))), \omega_1^{-1}(\beta))$, $(\bar{z}_1, \bar{z}_2) := ((g')_3^{-1}(g'(\omega_2^{-1}(\beta))), \omega_2^{-1}(\beta))$, $A_1 := \{\beta > 0 : g'(\omega_1^{-1}(\beta)) < g'(\omega_2^{-1}(\beta))\}$, $A_2 := \{\beta > 0 : g'(\omega_1^{-1}(\beta)) > g'(\omega_2^{-1}(\beta))\}$, $A_3 := (\omega_2(x_1), \infty)$, *and* $\overline{A} := (0,\infty) \setminus A$ *for any set* $A \subseteq (0,\infty)$*. Note that when* $\beta = \omega_1(x_2)$*, one has* $\omega_1^{-1}(\beta) = \{(g')_2^{-1}(g'(x_2)), x_2\}$*, in which case* $\{(\tilde{z}_1, \tilde{z}_2)\}$ *is understood as the two-point set* $\{(\tilde{z}_1, (g')_2^{-1}(g'(x_2))), (\tilde{z}_1, x_2)\}$.

3.2. Explicit characterization of $\mathcal{M}_\zeta$ in the case $\mu_\pm \le 0$. In this case, we intend to offer merely the main results while most of their proofs are omitted because they require no new techniques in comparison to that of subsection 3.1.

When $\mu_\pm \le 0$, one has $\theta_1^- \le \theta_2^-$ and $\theta_1^+ \le \theta_2^+$. The following proposition gives the convexity of the function g on $(0,\infty)$. Its proof is deferred to Appendix A.3.

PROPOSITION 3.2. *Suppose that* $\mu_\pm \le 0$*. The function* g *is convex on* $(0,\infty)$.

Theorem 3.2 explicitly characterizes the set $\mathcal{M}_\zeta$ as a singleton set. Its proof is similar to that of Theorem 3.1 and hence omitted.

THEOREM 3.2. *Suppose that* $\mu_\pm \le 0$*. Then* $\mathcal{M}_\zeta = \{(0, \phi_0^{-1}(\beta))\}$*, where the function* ϕ_0^{-1} *is defined as the inverse function of* ϕ_0 *which is given by*

$$\phi_0(x) := \psi(0, x), \quad x \in [0,\infty). \tag{3.8}$$

3.3. Explicit characterization of $\mathcal{M}_\zeta$ in the case $\mu_+ \le 0$ and $\mu_- > 0$. In this case, one has $\theta_1^+ \le \theta_2^+$ and $\theta_1^- > \theta_2^-$. Proposition 3.3 gives a complete characterization of the piecewise concavity or convexity of the function g on $(0,\infty)$. Its proof is deferred to Appendix A.4.

PROPOSITION 3.3. *Suppose that* $\mu_+ \le 0$ *and* $\mu_- > 0$*. Then the function* g *is concave on* $(0, a_1 \wedge a)$ *and convex on* $(a_1 \wedge a, \infty)$.

Theorem 3.3 explicitly characterizes the set $\mathcal{M}_\zeta$ in the case $\mu_+ \le 0$ and $\mu_- > 0$; its proof is omitted due to its similarity to that of Theorem 3.1.

THEOREM 3.3. *Suppose that* $\mu_+ \le 0$ *and* $\mu_- > 0$.

- *If* $0 < a \le a_1$*, then* $\mathcal{M}_\zeta = \{((g')_-^{-1}(g'(\phi^{-1}(\beta))), \phi^{-1}(\beta))\}$.
- *If* $a > a_1$*, then* $\mathcal{M}_\zeta = \{((\tilde{g}')_-^{-1}(g'(\tilde{\phi}^{-1}(\beta))), \tilde{\phi}^{-1}(\beta))\}$.

3.4. Explicit characterization of $\mathcal{M}_\zeta$ in the case $\mu_+ > 0$ and $\mu_- \le 0$. In this case, one has $\theta_1^+ > \theta_2^+$ and $\theta_1^- \le \theta_2^-$. We distinguish the following mutually exclusive and collectively exhaustive Cases (i) and (ii).

Case (i) One of the following conditions holds:
  - $c_+ > 0$ and $a \ge a_2$,
  - $c_+ > 0$ and $0 < a_3 \le a \le a_2$,
  - $c_+ \le 0$, $a \ge a_3$, and $\Theta > 0$.

Case (ii) One of the following conditions holds:

- $c_+ > 0$ and $0 < a < a_2 \wedge a_3$,
- $c_+ \le 0$ and $\Theta \le 0$,
- $c_+ \le 0$, $0 < a < a_3$, and $\Theta > 0$.

Proposition 3.4 gives a complete characterization of the piecewise concavity or convexity of g on $(0,\infty)$. Its proof is given in Appendix A.5.

Proposition 3.4. *Suppose that $\mu_+ > 0$ and $\mu_- \le 0$. Under Case* (i)*, $g(x)$ is convex on $(0,\infty)$. Under Case* (ii)*, $g(x)$ is convex on $(0,a)$, concave on (a,x_0), and convex on (x_0,∞).*

Under Case (ii), let $[x_0,a_7] \ni x \mapsto (\hat g')_-^{-1}(x) := \inf\{y \in [0,a]; g'(y) \ge g'(x)\}$ and define the inverse function of $g'|_{[a,x_0]}$ as $(\hat g')_+^{-1} : [g'(x_0), g'(a_7)] \to [a,x_0]$ with $a_7 := \sup\{x > x_0 : g'(x) \le g'(a)\}$. Also, put

$$x_3 := \inf\left\{ x \in [x_0,a_7] : \int_0^{(\hat g')_+^{-1}(g'(x))} (1 - g'(s)/g'(x))\,\mathrm{d}s \ge 0 \right\}, \tag{3.9}$$

$$x_4 := \inf\left\{ x \in [x_0,a_7] : \int_{(\hat g')_-^{-1}(g'(x))}^{x} (1 - g'(s)/g'(x))\,\mathrm{d}s \ge 0 \right\}, \tag{3.10}$$

$$\omega_3(x) := \int_0^x (1 - g'(s)/g'(x))\,\mathrm{d}s, \quad x \in [0, (\hat g')_-^{-1}(g'(x_4))] \cup [x_4,\infty), \tag{3.11}$$

$$\omega_4(x) := \begin{cases} \displaystyle\int_{(\hat g')_+^{-1}(g'(x))}^{x} (1 - g'(s)/g'(x))\,\mathrm{d}s, & x \in [x_0,x_3), \\ \displaystyle\int_0^x (1 - g'(s)/g'(x))\,\mathrm{d}s, & x \in [x_3,\infty). \end{cases} \tag{3.12}$$

Theorem 3.4 explicitly characterizes the set $\mathcal{M}_\zeta$ in the case $\mu_+ > 0$ and $\mu_- \le 0$. We omit its proof due to its similarity to that of Theorem 3.1.

Theorem 3.4. *Suppose that $\mu_+ > 0$ and $\mu_- \le 0$.*

- *Under Case* (i)*, we have $\mathcal{M}_\zeta = \{(0, \phi_0^{-1}(\beta))\}$.*
- *Under Case* (ii)*, we have*

$$\mathcal{M}_\zeta = \begin{cases} \{(\hat w_1, \hat w_2)\} & \text{if } \beta \in B_1 \cup B_3, \\ \{(\tilde w_1, \tilde w_2)\} & \text{if } \beta \in B_2 \cap \overline{B}_3, \\ \{(\hat w_1, \hat w_2)\} \cup \{(\tilde w_1, \tilde w_2)\} & \text{otherwise}, \end{cases}$$

where $(\tilde w_1, \tilde w_2) := ((\hat g')_+^{-1}(g'(\omega_4^{-1}(\beta))), \omega_4^{-1}(\beta))$, $(\hat w_1, \hat w_2) := (0, \omega_3^{-1}(\beta))$, $B_1 := \{\beta > 0 : g'(\omega_3^{-1}(\beta)) < g'(\omega_4^{-1}(\beta))\}$, $B_2 := \{\beta > 0 : g'(\omega_3^{-1}(\beta)) > g'(\omega_4^{-1}(\beta))\}$, $B_3 := (\omega_4(x_3), \infty)$, with ω_3^{-1} and ω_4^{-1} being the inverse functions of ω_3 and ω_4, respectively. Note that when $\beta = \omega_3(x_4)$, one has $\omega_3^{-1}(\beta) = \{(\hat g')_-^{-1}(g'(x_4)), x_4\}$, in which case $\{(\hat w_1, \hat w_2)\}$ is understood as $\{(0, (\hat g')_-^{-1}(g'(x_4)), (0, x_4)\}$.

3.5. General properties of $\mathcal{M}_\zeta$.

We showed the characterization of the set $\mathcal{M}_\zeta$ in the previous section. Some properties of the element of $\mathcal{M}_\zeta$ are presented in Propositions 3.5, 3.6, and 3.7. Their proofs are lengthy and nontrivial, so please refer to [28] for details.

Proposition 3.5. *We have* $\lim_{\beta \to 0+} \max_{(z_1,z_2)\in\mathcal{M}_\zeta} (z_2 - z_1) = 0$.

Proposition 3.6. *Both* $\max_{(z_1,z_2)\in\mathcal{M}_\zeta}(z_2 - z_1)$ *and* $\min_{(z_1,z_2)\in\mathcal{M}_\zeta}(z_2 - z_1)$ *are increasing in $\beta \in (0,\infty)$.*

PROPOSITION 3.7. *Let $(z_1, z_2) \in \mathcal{M}_\zeta$. For any fixed $\beta \in (0, a)$, there exists a sufficiently large constant $K > 0$ such that $\beta \le z_1 + \beta < z_2 \le a$ if $\mu_- > K$.*

Remark 3.1. We treat each (z_1, z_2)-strategy (with $(z_1, z_2) \in \mathcal{M}_\zeta$) as a candidate optimal impulsive dividend strategy (whose optimality will be demonstrated in the upcoming Theorem 4.1) to the problem (2.4). As β increases, issuing of a new lump sum of dividends becomes more costly. Hence, it would be sensible to adjust the dividend barriers so that the size of each lump sum of dividends becomes larger. In the extreme case of $\beta = 0$ (paying dividends incurs no costs), it seems reasonable to pay dividends as much and as frequently as possible, implying that $z_2 = z_1$ in the limiting sense. These intuitive perceptions are confirmed in Propositions 3.5–3.6.

Remark 3.2. By Proposition 3.7, if the expected rate of return μ_- (i.e., the drift coefficient when the wealth level is below a) is sufficiently large, then the manager prefers the wealth process to remain below a (rather than above a) to quickly accumulate wealth. Consequently, the upper barrier z_2 of the carefully calibrated optimal impulse dividend strategy $(z_1, z_2) \in \mathcal{M}_\zeta$ is lower than a.

4. Characterization of the optimal impulsive strategy.

The main result of this paper is contained in the following theorem, which, under certain sufficient conditions, characterizes an optimal impulsive strategy to the control problem (2.4).

THEOREM 4.1. *Let (z_1, z_2) be an element of $\mathcal{M}_\zeta$. Then, the (z_1, z_2)-strategy is an optimal impulsive strategy to the control problem* (2.4) *if one of the following conditions holds true:*

(a) $z_2 > a$;
(b) $z_2 \le a$ *and* $\mu_+ - q\,(a - z_2 + g(z_2)/g'(z_2)) \le 0$;
(c) $g''(a+) \ge 0$.

Proof. Let $(z_1, z_2) \in \mathcal{M}_\zeta$. We first prove that

$$V_{z_1}^{z_2}(x) - V_{z_1}^{z_2}(y) \ge x - y - \beta, \quad x \ge y \ge 0. \tag{4.1}$$

By (2.13) and (2.14), we have

$$\frac{x - y - \beta}{g(x) - g(y)} \le \frac{z_2 - z_1 - \beta}{g(z_2) - g(z_1)} = \frac{1}{g'(z_2)}, \quad \beta \le y + \beta \le x < \infty. \tag{4.2}$$

We distinguish the following mutually exclusive and collectively exhaustive cases.

- If $y + \beta > x \ge y \ge 0$, it holds that
$$V_{z_1}^{z_2}(x) - V_{z_1}^{z_2}(y) \ge 0 > x - y - \beta. \tag{4.3}$$
- If $x \ge y \ge z_2$ and $x \ge y + \beta$, it holds that $V_{z_1}^{z_2}(x) - V_{z_1}^{z_2}(y) = x - y > x - y - \beta$.
- If $x \ge z_2 \ge y \ge 0$ and $x \ge y + \beta$, by (4.2) and (4.3), one can get $V_{z_1}^{z_2}(x) - V_{z_1}^{z_2}(y) = x - z_2 + \frac{g(z_2) - g(y)}{g'(z_2)} \ge x - y - \beta$.
- If $z_2 \ge x \ge y + \beta \ge \beta$, by (4.2), one can get $V_{z_1}^{z_2}(x) - V_{z_1}^{z_2}(y) = \frac{g(x) - g(y)}{g'(z_2)} \ge x - y - \beta$.

Combining the above yields (4.1).

We next prove that

$$g(a)/g'(a) \le g(z_2)/g'(z_2) + a - z_2 \quad \text{if} \quad z_2 \le a. \tag{4.4}$$

Actually, by (2.10) and (2.11), it can be verified that

$$1 - \frac{g(x)g''(x)}{(g'(x))^2} = \left[\frac{g(x)}{g'(x)}\right]' = \frac{e^{-(\theta_1^- - \theta_2^-)x}(\theta_1^- + \theta_2^-)^2}{\left(\theta_1^- e^{-\theta_1^- x} + \theta_2^- e^{\theta_2^- x}\right)^2} > 0, \quad x \in [0, a].$$

Suppose $z_2 \le a$. Using Propositions 3.1–3.4 and Theorems 3.1–3.4, one sees that $g''(x) > 0$ for all $x \in [z_2, a)$, so

$$0 < [g(x)/g'(x)]' < 1 \text{ for all } x \in [z_2, a) \quad \text{if} \quad z_2 \le a,$$

which implies (4.4).

By (2.15), $V_{z_1}^{z_2}(x) \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \backslash \{a, z_2\})$. We next verify $(\mathcal{A} - q)V_{z_1}^{z_2}(x) \le 0$ for $x \in (0, \infty)\backslash\{a, z_2\}$. Using (2.11), (2.15), and the fact that $(\mathcal{A} - q)g^{\pm}(x) = 0$ for all $x \ne a$, we have, for any $x \in (0, z_2)\backslash\{a\}$,

$$(\mathcal{A} - q)V_{z_1}^{z_2}(x) = [g'(z_2)]^{-1}\left[g^-(0)(\mathcal{A} - q)g^+(x) - g^+(0)(\mathcal{A} - q)g^-(x)\right] = 0. \tag{4.5}$$

In addition,

$$\begin{aligned}\lim_{x\to z_2+}(\mathcal{A} - q)V_{z_1}^{z_2}(x) &= \lim_{x\to z_2+}(\mu_+\mathbf{1}_{\{x>a\}} + \mu_-\mathbf{1}_{\{x\le a\}}) - qV_{z_1}^{z_2}(z_2)\\ &= \mu_+\mathbf{1}_{\{z_2\ge a\}} + \mu_-\mathbf{1}_{\{z_2<a\}} - qV_{z_1}^{z_2}(z_2),\\ \lim_{x\to z_2-}(\mathcal{A} - q)V_{z_1}^{z_2}(x) &= (\sigma_+^2\mathbf{1}_{\{z_2>a\}} + \sigma_-^2\mathbf{1}_{\{z_2\le a\}})V_{z_1}^{z_2\,\prime\prime}(z_2-)/2\\ &\quad + (\mu_+\mathbf{1}_{\{z_2>a\}} + \mu_-\mathbf{1}_{\{z_2\le a\}}) - qV_{z_1}^{z_2}(z_2).\end{aligned}$$

Combining the above yields

$$\begin{aligned}0 = \lim_{x\to z_2-}(\mathcal{A} - q)V_{z_1}^{z_2}(x) &= (\sigma_+^2\mathbf{1}_{\{z_2>a\}} + \sigma_-^2\mathbf{1}_{\{z_2\le a\}})V_{z_1}^{z_2\,\prime\prime}(z_2-)/2\\ &\quad + (\mu_- - \mu_+)\,\mathbf{1}_{\{z_2=a\}} + \lim_{x\to z_2+}(\mathcal{A} - q)V_{z_1}^{z_2}(x).\end{aligned}$$

Using this and the fact that $V_{z_1}^{z_2\,\prime\prime}(z_2-) \ge 0$ (actually, from the explicit characterizations of $\mathcal{M}_\zeta$ provided in Theorems 3.1–3.4, one knows that $g''(z_2-) \ge 0$, which, by (2.15), is equivalent to $V_{z_1}^{z_2\,\prime\prime}(z_2-) \ge 0$), one has

$$(\mu_- - \mu_+)\,\mathbf{1}_{\{z_2=a\}} + \lim_{x\to z_2+}(\mathcal{A} - q)V_{z_1}^{z_2}(x) \le 0. \tag{4.6}$$

We next prove $(\mathcal{A} - q)V_{z_1}^{z_2}(x) \le 0$ on $(z_2, \infty)\backslash\{a\}$.

- When condition (a) holds true, it follows from (4.6) that

$$\begin{aligned}(\mathcal{A} - q)V_{z_1}^{z_2}(x) &= \mu_+ - qV_{z_1}^{z_2}(x) \le \mu_+ - qV_{z_1}^{z_2}(z_2)\\ &= \lim_{x\to z_2+}(\mathcal{A} - q)V_{z_1}^{z_2}(x) \le -(\mu_- - \mu_+)\,\mathbf{1}_{\{z_2=a\}} = 0, \quad x > z_2.\end{aligned}$$

- When condition (b) holds true, it holds that, for $x \in (z_2, a]$,

$$\begin{aligned}(\mathcal{A} - q)V_{z_1}^{z_2}(x) &= \mu_- - qV_{z_1}^{z_2}(x) \le \mu_- - qV_{z_1}^{z_2}(z_2)\\ &= \left(\mu_- - qV_{z_1}^{z_2}(z_2)\right)\mathbf{1}_{\{z_2<a\}} + \left(\mu_+ - qV_{z_1}^{z_2}(z_2)\right)\mathbf{1}_{\{z_2=a\}} + (\mu_- - \mu_+)\,\mathbf{1}_{\{z_2=a\}}\\ &= \lim_{x\to z_2+}(\mathcal{A} - q)V_{z_1}^{z_2}(x) + (\mu_- - \mu_+)\,\mathbf{1}_{\{z_2=a\}} \le 0,\end{aligned}$$

  and for $x > a$,

$$\begin{aligned}(\mathcal{A} - q)V_{z_1}^{z_2}(x) &= \mu_+ - qV_{z_1}^{z_2}(x) = \mu_+ - q(x - z_2) - qV_{z_1}^{z_2}(z_2)\\ &\le \mu_+ - q(a - z_2) - qg(z_2)/g'(z_2) \le 0.\end{aligned}$$

- When condition (c) holds true, the claim follows if $z_2 > a$ by condition (a); otherwise if $z_2 \le a$, we get from (2.6) that, for $x \in (z_2, a]$,

$$\begin{aligned}(\mathcal{A}-q)V_{z_1}^{z_2}(x) &= \mu_- - qV_{z_1}^{z_2}(x) = \mu_- - q\big((x-z_2)+g(z_2)/g'(z_2)\big)\\ &\le \mu_- - qg(z_2)/g'(z_2) = -\sigma_-^2 g''(z_2-)/2g'(z_2) \le 0,\end{aligned}$$

and for $x > a$, thanks to (2.6) and (4.4),

$$\begin{aligned}(\mathcal{A}-q)V_{z_1}^{z_2}(x) &\le \mu_+ - q\big((a-z_2)+g(z_2)/g'(z_2)\big)\\ &\le \mu_+ - qg(a)/g'(a) = -\sigma_+^2 g''(a+)/2g'(a) \le 0.\end{aligned}$$

The above together with (4.5) implies $(\mathcal{A}-q)V_{z_1}^{z_2}(x) \le 0$ on $(0,\infty)\backslash\{a, z_2\}$ when one of conditions (a), (b), or (c) holds true. Combining with (4.1) and Lemma 2.1, we prove Theorem 4.1. ☐

COROLLARY 4.1. *There is* $(z_1, z_2) \in \mathcal{M}_\zeta$ *such that the* (z_1, z_2)*-strategy is an optimal impulsive strategy to the problem* (2.4), *except for the following two minor cases:*

- $\mu_\pm > 0$, *Case* (iv), $\beta < \omega_1(x_2)$, $g'(\omega_1^{-1}(\beta)) < g'(\omega_2^{-1}(\beta))$;
- $\mu_+ > 0$, $\mu_- < 0$, *Case* (ii), $\beta < \omega_3(x_4)$, $g'(\omega_3^{-1}(\beta)) < g'(\omega_4^{-1}(\beta))$.

In the last two cases, the (z_1, z_2)*-strategy remains optimal if condition* (b) *in Theorem* 4.1 *is satisfied.*

Proof. The proof is a straightforward application of Propositions 3.1–3.4 and Theorems 3.1–3.4 and 4.1; one just needs to check the following facts.

(1) Assume $\mu_\pm > 0$. Then
- in Cases (i) and (ii), $\mathcal{M}_\zeta$ is a singleton set, and we have $z_2 > a$;
- in Case (iii), $\mathcal{M}_\zeta$ is a singleton set, and we have $g''(a+) > 0$,
- in Case (iv), if either one of the conditions
  - $\beta \ge \omega_1(x_2)$,
  - $\beta < \omega_1(x_2)$ and $g'(\omega_1^{-1}(\beta)) \ge g'(\omega_2^{-1}(\beta))$

  holds true, then $\mathcal{M}_\zeta$ is not necessarily a singleton set, but there is at least one $(z_1, z_2) \in \mathcal{M}_\zeta$ with $z_2 > a$.

(2) Assume $\mu_\pm \le 0$. Then $\mathcal{M}_\zeta$ is a singleton set, and we have $g''(a+) > 0$.

(3) Assume $\mu_+ \le 0$ and $\mu_- > 0$. Then $\mathcal{M}_\zeta$ is a singleton set.
- If $0 < a < a_1$, then $z_2 > a$.
- If $a > a_1$, then $g''(a+) > 0$.

(4) Assume $\mu_+ > 0$ and $\mu_- < 0$. Then
- in Case (i), $\mathcal{M}_\zeta$ is a singleton set, and we have $g''(a+) > 0$;
- in Case (ii), if either one of the conditions
  - $\beta \ge \omega_3(x_4)$,
  - $\beta < \omega_3(x_4)$ and $g'(\omega_3^{-1}(\beta)) \ge g'(\omega_4^{-1}(\beta))$

  holds true, then $\mathcal{M}_\zeta$ may not be a singleton set, but there is at least one $(z_1, z_2) \in \mathcal{M}_\zeta$ with $z_2 > a$.

The proof is simple, so we omit the details. ☐

5. Numerical analysis and economic interpretations.

To complement the theoretical results derived in the previous sections, we now present a series of numerical experiments. These analyses serve to visualize the optimal dividend strategy in action and to provide economic intuition for how the firm's decisions are shaped by the underlying model parameters.

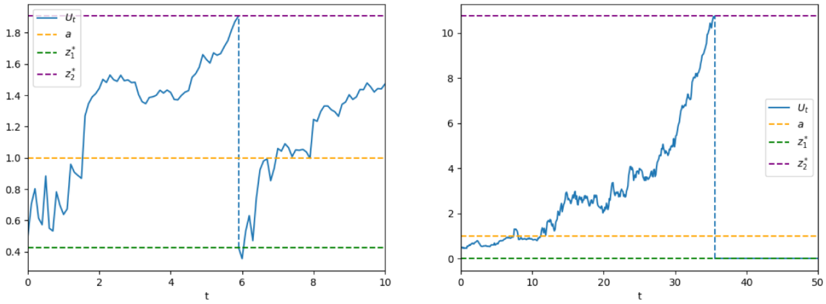

FIG. 1. *A sample path under the optimal strategy.*

5.1. A sample path under the optimal strategy. We begin by simulating a sample path of the optimal controlled surplus process U^{π^*}. The left panel of Figure 1 illustrates this path over a time horizon of $T = 10$. The baseline parameters for this simulation are set as follows: the regime-switching threshold is $a = 1$; the drift and volatility coefficients are $(\mu_-, \sigma_-) = (0.5, 0.5)$ for the lower regime $(U_t \le a)$ and $(\mu_+, \sigma_+) = (0.1, 0.1)$ for the upper regime $(U_t > a)$; and the fixed transaction cost is $\beta = 0.5$. For this parameter set, the explicit characterization of $\mathcal{M}_\zeta$ derived in section 3 identifies the optimal dividend barriers as $z_1^* = 0.4277$ and $z_2^* = 1.9059$. This strategy constitutes the optimal dividend strategy, since condition (a) of Theorem 4.1 holds. This models a company with a high-growth, high-risk "startup" phase (when surplus is below a) that transitions into a low-growth, low-risk "maturity" phase upon expansion (surplus above a). As observed in the left panel of Figure 1, the firm sets a relatively low upper barrier z_2^*. It allows the surplus to enter the more stable (but less profitable) mature phase to safely accumulate funds and pay a dividend of size $\Delta L_t = z_2^* - z_1^*$. The dividend resets the surplus to z_1^*, positioning the firm to releverage its high-growth startup phase. The strategy is thus a dynamic cycle of navigating between risk and stability.

The right panel of Figure 1 represents a completely different economic reality over a time horizon of $T = 50$. Here, the parameters are reversed: $(\mu_+, \sigma_+) = (0.5, 0.5)$ and $(\mu_-, \sigma_-) = (0.1, 0.1)$ with $\beta = 0.5$ and $a = 1$. This models a company that is stable but stagnant when small, and only enters a high-growth, high-risk expansion phase after its surplus exceeds the threshold a. The firm endures a long period of slow growth below a, as the low drift makes it difficult to cross the threshold. Once the threshold is crossed, the firm enters the highly profitable expansion phase. The optimal strategy is to establish a high upper barrier at $z_2^* = 10.4512$, a value consistent with condition (a) of Theorem 4.1, to capitalize on this phase. Upon reaching z_2^*, the firm pays out a very large dividend of $\Delta L_t = z_2^* - z_1^*$. The lower barrier is set at $z_1^* = 0$, the ruin level itself. It signifies that the optimal path is not to restart, but to perform a terminal payout. This aggressive strategy is rational for shareholders aiming to fully extract the firm's value after a successful high-growth period, rather than risk returning to the stagnant, low-growth phase.

5.2. Sensitivity analysis of model parameters. We now investigate the sensitivity of the optimal dividend barriers (z_1^*, z_2^*) with respect to the key model parameters: the transaction cost β, the regime-switching threshold a, the lower regime drift μ_-, and the lower regime volatility σ_-.

The top-left panel of Figure 2 shows the effect of the transaction cost β on the optimal barriers under $(\mu_+, \sigma_+) = (0.1, 0.1)$, $(\mu_-, \sigma_-) = (0.5, 0.5)$, and $a = 1$. The

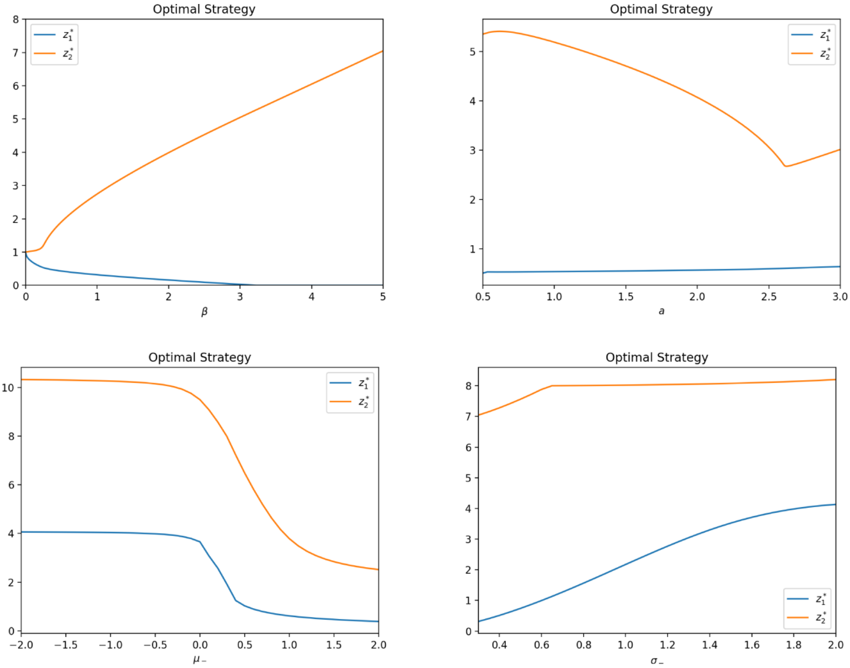

FIG. 2. *Sensitive of parameters β, a, μ_-, and σ_-.*

results are in complete agreement with Propositions 3.5 and 3.6, as well as satisfying condition (a) of Theorem 4.1. As the cost β of paying a dividend increases, the upper barrier z_2^* increases, while the lower barrier z_1^* decreases. Consequently, the dividend size $\Delta L_t = z_2^* - z_1^*$ becomes a strictly increasing function of the cost. The intuition is straightforward: a higher fixed cost per transaction incentivizes the firm to pay dividends less frequently but in larger amounts.

We now examine the sensitivity of the optimal strategy to the regime-switching threshold a under $(\mu_+, \sigma_+) = (0.5, 0.1)$, $(\mu_-, \sigma_-) = (1, 0.5)$, and $\beta = 1$. Each z_2^* value shown in the figure satisfies condition (a) of Theorem 4.1. When a is small, the high-growth zone is narrow, forcing the firm to set a high z_2^* to patiently accumulate surplus in the safe upper regime. As a increases, making the high-growth zone more accessible, the strategy becomes more aggressive by progressively lowering z_2^* to realize profits sooner. The sharp kink at $a = 2.6$ marks a fundamental strategic shift. At and beyond this point, the firm prefers to pay a dividend precisely when it is about to be pushed into the low-growth zone, immediately resetting the process to stay within its preferred, more profitable environment.

The bottom-left panel illustrates the impact of the lower regime's profitability μ_- under $(\mu_+, \sigma_+) = (0.5, 1)$, $\sigma_- = 0.5$, $a = 2$, and $\beta = 1$. Each z_2^* value shown in the figure satisfies condition (a) of Theorem 4.1. When μ_- is negative, the lower regime is a "danger zone." The firm adopts a highly conservative strategy, maintaining high barriers $z_1^* = 4, z_2^* = 10$ to create a large safety buffer and minimize the risk of ruin after a dividend payout. As μ_- increases and surpasses the upper regime's drift μ_+, the lower regime becomes the engine of growth. The firm's strategy flips to become aggressive. It sets both barriers significantly lower to ensure that after a payout, the process returns to the highly profitable lower regime, and it pays dividends more quickly to minimize time spent in the less profitable upper regime.

Finally, the bottom-right panel shows the effect of the lower regime's risk σ_- under $(\mu_+, \sigma_+) = (0.5, 0.5)$, $\mu_- = 1$, $\beta = 1$, and $a = 8$. Both barriers z_1^* and z_2^* are monotonically increasing functions of σ_-. For $\sigma_- > 0.63$, z_2^* satisfies condition (a) of Theorem 4.1; for $\sigma_- < 0.63$, both conditions (b) and (c) hold true. This reflects the firm's response to an increase in operational risk. A higher volatility in the primary operating regime increases the probability of ruin. To mitigate this risk, the firm adopts a more conservative policy by holding more precautionary cash. It raises z_1^* to leave a larger buffer and consequently must also raise z_2^* to ensure that the payout remains large enough to justify the transaction cost. Higher risk, therefore, leads to delayed and larger dividend payments.

Appendix A. Some proofs.

Here we provide the proofs for some results given in the previous sections.

A.1. Proof of Proposition 3.1.

We first discuss the sign of $g''(x)$. When $x \in (0, a)$, using (2.11) we have

$$\begin{aligned} g''(x) &= (g^-(0) - c_- g^+(0))(\theta_2^-)^2 e^{\theta_2^-(x-a)} - (1-c_-)g^+(0)(\theta_1^-)^2 e^{-\theta_1^-(x-a)} \\ &= (1-c_-)e^{(\theta_1^- - \theta_2^-)a}[(\theta_2^-)^2 e^{\theta_2^- x} - (\theta_1^-)^2 e^{-\theta_1^- x}], \end{aligned}$$

which is strictly increasing with its unique zero $a_1 > 0$ given by (3.2). Hence,

(1-1) if $0 < a \le a_1$, one has $g''(x) < 0$ on $(0, a)$;

(1-2) if $a > a_1$, one has $g''(x) < 0$ on $(0, a_1)$ and $g''(x) > 0$ on (a_1, a).

When $x > a$, by (2.11), it holds that

$$g''(x) = (1-c_+)g^-(0)(\theta_2^+)^2 e^{\theta_2^+(x-a)} - (g^+(0) - c_+ g^-(0))(\theta_1^+)^2 e^{-\theta_1^+(x-a)}. \tag{A.1}$$

To proceed, we first discuss the sign of

$$h_1(a) := g^+(0) - c_+ g^-(0) = e^{-\theta_2^- a}(1 - c_+ c_-) - c_+(1-c_-)e^{\theta_1^- a}, \tag{A.2}$$

where we used (2.10) to get the second equality. One can easily verify that $g^-(0) > g^+(0) > 0$. If $c_+ \le 0$, we have $h_1(a) > 0$ for all $a > 0$. If $c_+ > 0$, it follows from $1 - c_+ c_- > 0$ that the function $\mathbb{R}_+ \ni x \mapsto h_1(x)$ is strictly decreasing and has a unique zero $a_2 > 0$ given by (3.2), where we have used the fact that $c_+ > 0$ implies $\theta_2^+ - \theta_2^- > 0$ (see (2.9)). Therefore, if $c_+ > 0$ and $a \in (0, a_2)$, we have $h_1(a) > 0$, and if $c_+ > 0$ and $a \ge a_2$, we have $h_1(a) \le 0$.

(2-1) Suppose $c_+ > 0$, in which case we have the following conclusions.

  (2-1-1) If $c_+ > 0$ and $a \ge a_2$, by (A.1) we have $g''(x) > 0$ for all $x > a$.

  (2-1-2) If $c_+ > 0$ and $0 < a < a_3 \wedge a_2$, then $x_0 > a$, $g''(x) < 0$ on (a, x_0), and $g''(x) > 0$ on (x_0, ∞). Actually, if $c_+ > 0$ and $a \in (0, a_2)$ (hence, $h_1(a) > 0$), the function $g''(x)$ is strictly increasing with its unique zero x_0 given by (3.1). To check whether or not x_0 is greater than a, define

$$\begin{aligned} h_2(a) &:= -g''(a+) = (g^+(0) - c_+ g^-(0))(\theta_1^+)^2 - (1-c_+)g^-(0)(\theta_2^+)^2 \\ &= [(1 - c_- c_+)(\theta_1^+)^2 - (1-c_+)c_-(\theta_2^+)^2]e^{-\theta_2^- a} - (1-c_-)\Theta e^{\theta_1^- a}. \end{aligned} \tag{A.3}$$

It follows from $c_+ > 0$, (2.8), (2.9), and the definition of Θ that

$$-(1-c_-)\Theta = -(1-c_-)\left[c_+(\theta_1^+)^2 + (1-c_+)(\theta_2^+)^2\right] < 0,$$

which together with the fact of $h_2(0)=(1-c_+)[(\theta_1^+)^2-(\theta_2^+)^2]>0$ yields that

$$(1-c_-c_+)(\theta_1^+)^2-(1-c_+)c_-(\theta_2^+)^2>(1-c_-)\Theta>0.$$

Hence, the function $\mathbb{R}_+\ni x\mapsto h_2(x)$ is strictly decreasing and admits a unique zero $a_3>0$ given by (3.2). Hence, if $c_+>0$ and $0<a<a_2\le a_3$, we have $-g''(a+)=h_2(a)>0$ on $a\in(0,a_2)$, which implies $x_0>a$, and hence $g''(x)<0$ on (a,x_0) and $g''(x)>0$ on (x_0,∞). Similarly, if $c_+>0$ and $0<a<a_3<a_2$, one knows that $-g''(a+)=h_2(a)>0$ and $x_0>a$, and hence $g''(x)<0$ on (a,x_0) and $g''(x)>0$ on (x_0,∞).

(2-1-3) If $c_+>0$ and $0<a_3\le a\le a_2$, then $g''(x)>0$ for all $x>a$. Actually, in the case $c_+>0$ and $0<a_3\le a\le a_2$, we get $-g''(a+)=h_2(a)\le 0$, which means $x_0\le a$.

(2-2) Suppose $c_+\le 0$, in which case the function $g''(x)$ is strictly increasing (see (A.1)) with its unique zero x_0 given by (3.1). Let $h_2(a)$ be defined by (A.3).

(2-2-1) If $\Theta\le 0$, $c_+\le 0$, and $a>0$, then $x_0>a$, $g''(x)<0$ on (a,x_0), and $g''(x)>0$ on (x_0,∞). Indeed, if $\Theta\le 0$ and $a>0$, it follows from $h_2(0)>0$ that $-g''(a+)=h_2(a)>0$, which means $x_0>a$.

(2-2-2) If $\Theta>0$, $c_+\le 0$, and $a\in(0,a_3)$, then $x_0>a$, $g''(x)<0$ on (a,x_0), and $g''(x)>0$ on (x_0,∞). Indeed, if $\Theta>0$ and $a\in(0,a_3)$, it follows from $h_2(0)>0$ that the function $\mathbb{R}_+\ni x\mapsto h_2(x)$ is strictly decreasing and $a_3>0$ given by (3.2) is its unique zero. Hence, one knows that $-g''(a+)=h_2(a)>0$, which yields $x_0>a$.

(2-2-3) If $\Theta>0$, $c_+\le 0$, and $a\in[a_3,\infty)$, then $g''(x)\ge 0$ for all $x>a$. Indeed, if $\Theta>0$ and $a\in[a_3,\infty)$, one knows that $-g''(a+)=h_2(a)\le 0$ for $a\in[a_3,\infty)$, which means $x_0\le a$.

Putting all the above together completes the proof of Proposition 3.1.

A.2. Proof of Theorem 3.1. We first consider Case (i) of Proposition 3.1.

To start, we characterize the set $\mathcal{N}$ (actually, if $\mathcal{N}$ is a singleton, then by Proposition 2.2 and $\mathcal{M}_\zeta\subseteq\mathcal{N}$ one has $\mathcal{M}_\zeta=\mathcal{N}$). For any $(z_1,z_2)\in\mathcal{N}$, either $z_1=0$ or $z_1>0$ holds true.

(1) Suppose there is a $(z_1,z_2)\in\mathcal{N}$ with $z_1>0$; then we have (2.18) and $g'(z_1)=g'(z_2)$, which implies $0<z_1\le a\le z_2<\inf\{x\ge a;g'(x)\ge g'(0)\}$. Then $z_1=(g')_-^{-1}(g'(z_1))=(g')_-^{-1}(g'(z_2))$. Hence, (2.18) can be rewritten as

$$\phi(z_2)=\beta, \tag{A.4}$$

where the function $\phi(x)$ is defined by (3.3) with $a\le x\le a_4$ (note that a_4 is guaranteed to be finite since $g'(0)$ is finite and g' is strictly increasing on (a,∞) with $g'(\infty)=\infty$). One can verify that

$$\phi'(x)=\int_{(g')_-^{-1}(g'(x))}^{x}g'(s)\mathrm{d}s\,g''(x)/(g'(x))^2,\quad x\in(a,a_4),$$

which inherits from $g''(x)$ the property of being positive on (a,a_4). That is to say, the function $\phi(x)$ is continuous and strictly increasing on $[a,a_4]$ with $\phi(a)=\psi((g')_-^{-1}(g'(a)),a)=\psi(a,a)=0$. Hence

(1-1) if $\phi(a_4)=\psi\left((g')_-^{-1}\left(g'(a_4)\right),a_4\right)=\psi(0,a_4)>\beta$, by the intermediate value theorem, there exists a unique $z_2=\phi^{-1}(\beta)\in(a,a_4)$ with $z_1=(g')_-^{-1}(g'(z_2))\in(0,a)$ such that (A.4) holds true, and hence the point

(z_1,z_2) with $z_2=\phi^{-1}(\beta)\in(a,a_4)$ and $z_1=(g')_-^{-1}(g'(z_2))$ is the unique solution of (2.18) such that $z_1>0$ and $g'(z_1)=g'(z_2)$; here, ϕ^{-1} denotes the well-defined inverse function of ϕ given by (3.3);

(1-2) if $\phi(a_4)=\psi\left((g')_-^{-1}\left(g'(a_4)\right),a_4\right)=\psi(0,a_4)\le\beta$, there is no solution (z_1,z_2) of (2.18) such that $z_1>0$ and $g'(z_1)=g'(z_2)$.

(2) Suppose there is a $(z_1,z_2)\in\mathcal{N}$ with $z_1=0$; then, (2.14) holds true with $z_1=0$, that is,

$$\beta=\psi(0,z_2)=\begin{cases}\phi_0(z_2):=\int_0^{z_2}\left(1-g'(s)/g'(z_2)\right)\mathrm{d}s, & z_2\in[a,a_4],\\ \phi(z_2), & z_2\in[a_4,\infty).\end{cases}\tag{A.5}$$

It is easy to verify that the function $(0,\infty)\ni x\mapsto\psi(0,x)$ is strictly increasing on (a,∞), $\psi(0,a)<0$, and $\psi(0,\infty)=\infty$. Hence

(2-1) if $\psi(0,a_4)>\beta$, by the intermediate value theorem, there is a unique $z_2\in(a,a_4)$ such that (A.5) holds true, and hence the point $(0,z_2)$ with $z_2\in(a,a_4)$ is the unique solution of (2.18) such that $z_1=0$;

(2-2) if $\psi(0,a_4)\le\beta$, by the intermediate value theorem, there is a unique $z_2\in[a_4,\infty)$ such that (A.5) holds true, and hence the point $(0,z_2)$ with $z_2\in[a_4,\infty)$ is the unique solution of (2.18) such that $z_1=0$.

Summing up the above results, we arrive at the following conclusion.

(a) If $\psi(0,a_4)>\beta$, the set $\mathcal{N}$ is composed of two points, i.e.,

$$\mathcal{N}=\{((g')_-^{-1}(g'(\phi^{-1}(\beta))),\phi^{-1}(\beta)),(0,\phi_0^{-1}(\beta))\}.$$

Here, ϕ_0^{-1} denotes the inverse function of ϕ_0. Due to the fact that $g'(s)>g'(\phi^{-1}(\beta))$ for all $s\in[0,(g')_-^{-1}(g'(\phi^{-1}(\beta))))$, one can verify that

$$\begin{aligned}\beta=\phi(\phi^{-1}(\beta))&=\int_{(g')_-^{-1}(g'(\phi^{-1}(\beta)))}^{\phi^{-1}(\beta)}\left(1-g'(s)/g'(\phi^{-1}(\beta))\right)\mathrm{d}s\\&>\int_0^{\phi^{-1}(\beta)}\left(1-g'(s)/g'(\phi^{-1}(\beta))\right)\mathrm{d}s=\phi_0(\phi^{-1}(\beta)),\end{aligned}$$

which implies

$$\phi_0^{-1}(\beta)>\phi^{-1}(\beta).\tag{A.6}$$

Since both points of $\mathcal{N}$ are solutions to (2.14), by (A.6) and the fact that $g'(x)$ is strictly increasing on (a,a_4), one can get

$$\begin{aligned}\zeta(0,\phi_0^{-1}(\beta))&=1/g'(\phi_0^{-1}(\beta))\\&<1/g'(\phi^{-1}(\beta))=\zeta((g')_-^{-1}(g'(\phi^{-1}(\beta))),\phi^{-1}(\beta)),\end{aligned}$$

which together with the fact that $\emptyset\ne\mathcal{M}_\zeta\subseteq\mathcal{N}$ implies that

$$\mathcal{M}_\zeta=\{((g')_-^{-1}(g'(\phi^{-1}(\beta))),\phi^{-1}(\beta))\}.$$

(b) If $\psi(0,a_4)\le\beta$, the set $\mathcal{N}$ is composed of only one point, i.e.,

$$\mathcal{N}=\{(0,\phi^{-1}(\beta))\}=\{((g')_-^{-1}(\phi^{-1}(\beta)),\phi^{-1}(\beta))\},$$

which combined with the fact that $\emptyset\ne\mathcal{M}_\zeta\subseteq\mathcal{N}$ yields that

$$\mathcal{M}_\zeta=\{((g')_-^{-1}(\phi^{-1}(\beta)),\phi^{-1}(\beta))\}.$$

In Cases (ii) and (iii), one can derive the desired results by adopting a similar argument as the one used for Case (i).

We next discuss Case (iv) of Proposition 3.1. Let x_1 and x_2 be defined by (3.4)–(3.5). To simplify the analysis, we next show the following six claims.

(1) $\{(z_1,z_2)\in\mathcal{N}: z_2\in(a,x_0]\cup[0,a_1]\}\cap\mathcal{M}_\zeta=\emptyset$.
(2) $\{(z_1,z_2)\in\mathcal{N}: z_2\in((g')_2^{-1}(g'(x_2)),a]\}\cap\mathcal{M}_\zeta=\emptyset$.
(3) $\{(z_1,z_2)\in\mathcal{N}: z_2\in[a_1,(g')_2^{-1}(g'(x_2))], z_1\neq(g')_1^{-1}(g'(z_2))\}\cap\mathcal{M}_\zeta=\emptyset$.
(4) $\{(z_1,z_2)\in\mathcal{N}: z_2\in[x_0,x_1), z_1\neq(g')_3^{-1}(g'(z_2))\}\cap\mathcal{M}_\zeta=\emptyset$.
(5) $\{(z_1,z_2)\in\mathcal{N}: z_2=x_1, z_1\notin\{(g')_1^{-1}(g'(z_2)),(g')_3^{-1}(g'(z_2))\}\}\cap\mathcal{M}_\zeta=\emptyset$.
(6) $\{(z_1,z_2)\in\mathcal{N}: z_2\in(x_1,\infty), z_1\neq(g')_1^{-1}(g'(z_2))\}\cap\mathcal{M}_\zeta=\emptyset$.

Obviously, $\{(z_1,z_2)\in\mathcal{N}: z_2\in[0,a_1]\}=\emptyset$. To prove claim (1), assume that $(z_1,z_2)\in\mathcal{N}$ is such that $z_2\in(a,x_0]$. By the definition of $\mathcal{N}$, we have $z_1\in[0,a_1)$. Since

$$g'(s)>g'(z_2)\quad\text{for all}\quad s\in((g')_2^{-1}(g'(z_2)),z_2)\cup[0,(g')_1^{-1}(g'(z_2))),$$

one can verify that

$$\begin{aligned}\beta&=\left(\int_{z_1}^{(g')_1^{-1}(g'(z_2))}+\int_{(g')_1^{-1}(g'(z_2))}^{(g')_2^{-1}(g'(z_2))}+\int_{(g')_2^{-1}(g'(z_2))}^{z_2}\right)(1-g'(s)/g'(z_2))\,\mathrm{d}s\\&<\int_{(g')_1^{-1}(g'(z_2))}^{(g')_2^{-1}(g'(z_2))}(1-g'(s)/g'(z_2))\,\mathrm{d}s=\psi((g')_1^{-1}(g'(z_2)),(g')_2^{-1}(g'(z_2))),\end{aligned}\tag{A.7}$$

which implies that there exists a $(z_1',z_2')\in\mathcal{N}$ such that $0\le z_1'<a_1<z_2'\le a$ and $z_1'=(g')_1^{-1}(g'(z_2'))$. By (A.7), it holds that $z_2'<(g')_2^{-1}(g'(z_2))$. Then, by the fact that $g'(x)$ is strictly increasing on (a_1,a), one can get

$$\zeta(z_1',z_2')=1/g'(z_2')>1/g'(z_2)=\zeta(z_1,z_2),$$

which means that any $(z_1,z_2)\in\mathcal{N}$ such that $z_2\in(a,x_0]$ satisfies $(z_1,z_2)\notin\mathcal{M}_\zeta$. Hence, claim (1) holds true. Claims (2)–(6) can be proved by similar arguments combined with the definition of x_1 and x_2. We hence omit their proofs. By the above claims (1)–(6), we know that $\mathcal{M}_\zeta\subseteq\cup_{i=1}^4\mathcal{R}_i$, where

$$\begin{aligned}\mathcal{R}_1&:=\{(z_1,z_2)\in\mathcal{N}: z_2\in[a_1,(g')_2^{-1}(g'(x_2))], z_1=(g')_1^{-1}(g'(z_2))\},\\\mathcal{R}_2&:=\{(z_1,z_2)\in\mathcal{N}: z_2\in[x_0,x_1), z_1=(g')_3^{-1}(g'(z_2))\},\\\mathcal{R}_3&:=\{(z_1,z_2)\in\mathcal{N}: z_2=x_1, z_1\in\{(g')_1^{-1}(g'(z_2)),(g')_3^{-1}(g'(z_2))\}\},\\\mathcal{R}_4&:=\{(z_1,z_2)\in\mathcal{N}: z_2\in(x_1,\infty), z_1=(g')_1^{-1}(g'(z_2))\}.\end{aligned}$$

The forms of $(\mathcal{R}_i)_{1\le i\le 4}$ motivate us to define ω_1 and ω_2 via (3.6) and (3.7). Note

$$\mathcal{R}_1=\begin{cases}\{(\tilde{z}_1,\tilde{z}_2)\} & \text{if } \beta\le\omega_1(x_2),\\ \emptyset & \text{if else,}\end{cases}$$

and

$$\cup_{i=2}^4\,\mathcal{R}_i=\begin{cases}\{(\bar{z}_1,\bar{z}_2)\} & \text{if } \beta<\omega_2(x_1),\\ \{(\hat{z}_1:=(g')_1^{-1}(g'(\omega_2^{-1}(\beta))),\hat{z}_2:=\omega_2^{-1}(\beta))\} & \text{if } \beta>\omega_2(x_1),\\ \{(\bar{z}_1,\bar{z}_2),(\hat{z}_1,\hat{z}_2)\} & \text{if } \beta=\omega_2(x_1).\end{cases}$$

Suppose $\beta<\omega_2(x_1)$.

- If $g'(\omega_1^{-1}(\beta)) < g'(\omega_2^{-1}(\beta))$, then $\emptyset \neq \mathcal{M}_\zeta \subseteq \{(\tilde{z}_1,\tilde{z}_2),(\bar{z}_1,\bar{z}_2)\}$, and $\zeta(\tilde{z}_1,\tilde{z}_2) = 1/g'(\omega_1^{-1}(\beta)) > 1/g'(\omega_2^{-1}(\beta)) = \zeta(\bar{z}_1,\bar{z}_2)$. Hence, $\mathcal{M}_\zeta = \{(\tilde{z}_1,\tilde{z}_2)\}$.
- If $g'(\omega_1^{-1}(\beta)) > g'(\omega_2^{-1}(\beta))$, then $\emptyset \neq \mathcal{M}_\zeta \subseteq \{(\tilde{z}_1,\tilde{z}_2),(\bar{z}_1,\bar{z}_2)\}$, and

  $$\zeta(\tilde{z}_1,\tilde{z}_2) = 1/g'(\omega_1^{-1}(\beta)) < 1/g'(\omega_2^{-1}(\beta)) = \zeta(\bar{z}_1,\bar{z}_2).$$

  Hence, $\mathcal{M}_\zeta = \{(\bar{z}_1,\bar{z}_2)\}$.
- If $g'(\omega_1^{-1}(\beta)) = g'(\omega_2^{-1}(\beta))$, then $\emptyset \neq \mathcal{M}_\zeta \subseteq \{(\tilde{z}_1,\tilde{z}_2),(\bar{z}_1,\bar{z}_2)\}$, and

  $$\zeta(\tilde{z}_1,\tilde{z}_2) = 1/g'(\omega_1^{-1}(\beta)) = 1/g'(\omega_2^{-1}(\beta)) = \zeta(\bar{z}_1,\bar{z}_2).$$

  Hence, $\mathcal{M}_\zeta = \{(\tilde{z}_1,\tilde{z}_2),(\bar{z}_1,\bar{z}_2)\}$.

Suppose $\beta = \omega_2(x_1)$.

- Note $g'(\omega_1^{-1}(\beta)) > g'(\omega_2^{-1}(\beta))$ cannot hold in this case.
- If $g'(\omega_1^{-1}(\beta)) = g'(\omega_2^{-1}(\beta))$ and $\beta = \omega_2(x_1)$, then $x_2 \le x_1$, $\emptyset \neq \mathcal{M}_\zeta \subseteq \{(\tilde{z}_1,\tilde{z}_2),(\bar{z}_1,\bar{z}_2),(\hat{z}_1,\hat{z}_2)\}$.
  - If $x_2 < x_1$, then $\hat{z}_1 = \tilde{z}_1$ and $\hat{z}_2 = \tilde{z}_2$ (since $\omega_1 \equiv \omega_2$ on $[x_1,\infty)$), and

    $$\zeta(\bar{z}_1,\bar{z}_2) = 1/g'(\omega_2^{-1}(\beta)) = 1/g'(\omega_1^{-1}(\beta)) = \zeta(\tilde{z}_1,\tilde{z}_2).$$

    Hence, $\mathcal{M}_\zeta = \{(\tilde{z}_1,\tilde{z}_2),(\bar{z}_1,\bar{z}_2)\}$.
  - If $x_2 = x_1$, then $\beta = \omega_1(x_2)$ and $(\tilde{z}_1,\tilde{z}_2)$ is understood as a two-point set $\{(\tilde{z}_1,(g')_2^{-1}(g'(x_2))),(\tilde{z}_1,x_2)\}$ with $\tilde{z}_1 = (g')_1^{-1}(g'(\omega_1^{-1}(\beta))) = (g')_1^{-1}(g'(\omega_2^{-1}(\beta))) = \hat{z}_1$. In addition,

    $$\zeta(\bar{z}_1,\bar{z}_2) = 1/g'(x_2) = \zeta(\tilde{z}_1,(g')_2^{-1}(g'(x_2))) = \zeta(\tilde{z}_1,x_2).$$

    Hence, $\mathcal{M}_\zeta = \{(\tilde{z}_1,\tilde{z}_2),(\bar{z}_1,\bar{z}_2)\} = \{(\tilde{z}_1,(g')_2^{-1}(g'(x_2))),(\tilde{z}_1,x_2),(\bar{z}_1,\bar{z}_2)\}$.
- If $g'(\omega_1^{-1}(\beta)) < g'(\omega_2^{-1}(\beta))$, then $x_1 < x_2$, $\emptyset \neq \mathcal{M}_\zeta \subseteq \{(\tilde{z}_1,\tilde{z}_2),(\bar{z}_1,\bar{z}_2),(\hat{z}_1,\hat{z}_2)\}$, and

  $$\zeta(\tilde{z}_1,\tilde{z}_2) = 1/g'(\omega_1^{-1}(\beta)) > 1/g'(\omega_2^{-1}(\beta)) = \zeta(\bar{z}_1,\bar{z}_2) = \zeta(\hat{z}_1,\hat{z}_2).$$

  Hence, $\mathcal{M}_\zeta = \{(\tilde{z}_1,\tilde{z}_2)\}$.

Suppose $\beta > \omega_2(x_1)$.

- If either $x_1 \ge x_2$ or both $x_1 < x_2$ and $\hat{z}_2 \ge x_2$ hold, then $g'(\omega_1^{-1}(\beta)) = g'(\omega_2^{-1}(\beta))$ (since $\omega_1 \equiv \omega_2$ on $[x_1 \vee x_2,\infty)$), and consequently $\hat{z}_1 = \tilde{z}_1$, $\hat{z}_2 = \tilde{z}_2$, and $\cup_{i=1}^4 \mathcal{R}_i = \{(\tilde{z}_1,\tilde{z}_2)\}$. Hence, $\mathcal{M}_\zeta = \{(\tilde{z}_1,\tilde{z}_2)\}$.
- If $x_1 < x_2$ and $\hat{z}_2 < x_2$, then $\omega_1(x_2) = \omega_2(x_2) \ge \omega_2(\hat{z}_2) = \beta > \omega_2(x_1)$, and

  $$\begin{aligned}\beta = \omega_2(\hat{z}_2) &= \left(\int_{\hat{z}_1}^{(g')_2^{-1}(g'(\hat{z}_2))} + \int_{(g')_2^{-1}(g'(\hat{z}_2))}^{\hat{z}_2}\right)\left(1 - \frac{g'(s)}{g'(\hat{z}_2)}\right)\mathrm{d}s \\ &< \int_{\hat{z}_1}^{(g')_2^{-1}(g'(\hat{z}_2))}\left(1 - \frac{g'(s)}{g'(\hat{z}_2)}\right)\mathrm{d}s \quad (\text{since } \hat{z}_2 < x_2) \\ &= \omega_1((g')_2^{-1}(g'(\hat{z}_2))),\end{aligned}$$

  which implies that $\omega_1^{-1}(\beta) < (g')_2^{-1}(g'(\hat{z}_2))$, that is, $g'(\omega_1^{-1}(\beta)) < g'(\hat{z}_2) = g'(\omega_2^{-1}(\beta))$. Therefore $\zeta(\tilde{z}_1,\tilde{z}_2) = 1/g'(\tilde{z}_2) > 1/g'(\hat{z}_2) = \zeta(\hat{z}_1,\hat{z}_2)$. Hence, $\mathcal{M}_\zeta = \{(\tilde{z}_1,\tilde{z}_2)\}$.

A.3. Proof of Proposition 3.2. When $x \in (0, a)$, using (2.11) we have

$$\begin{aligned} g''(x) &= (g^-(0) - c_- g^+(0))(\theta_2^-)^2 e^{\theta_2^-(x-a)} - (1-c_-)g^+(0)(\theta_1^-)^2 e^{-\theta_1^-(x-a)} \\ &= (1-c_-)e^{(\theta_1^- - \theta_2^-)a}\big[(\theta_2^-)^2 e^{\theta_2^- x} - (\theta_1^-)^2 e^{-\theta_1^- x}\big]. \end{aligned} \tag{A.8}$$

Since $\theta_2^- \geq \theta_1^- > 0$, one sees that $g''(x) > 0$ on $(0, a]$. When $x > a$, by (2.11), we have

$$\begin{aligned} g''(x) &= (1-c_+)g^-(0)(\theta_2^+)^2 e^{\theta_2^+(x-a)} - (g^+(0) - c_+ g^-(0))(\theta_1^+)^2 e^{-\theta_1^+(x-a)}, \\ g''(a) &= (1-c_+)g^-(0)(\theta_2^+)^2 - (g^+(0) - c_+ g^-(0))(\theta_1^+)^2 \\ &\geq [(1-c_+)g^-(0) - (g^+(0) - c_+ g^-(0))](\theta_1^+)^2 = [g^-(0) - g^+(0)](\theta_1^+)^2 > 0. \end{aligned} \tag{A.9}$$

These two together imply that $g'' > 0$ on (a, ∞). The proof is complete.

A.4. Proof of Proposition 3.3. When $x \in (0, a)$, we have that (A.8) holds. It is seen that the function

$$\mathbb{R}_+ \ni x \mapsto (1-c_-)e^{(\theta_1^- - \theta_2^-)a}\big[(\theta_2^-)^2 e^{\theta_2^- x} - (\theta_1^-)^2 e^{-\theta_1^- x}\big]$$

is strictly increasing with its unique zero $a_1 > 0$ given by (3.2), where we have used the fact that $\theta_1^- > \theta_2^-$.

(1-1) If $0 < a \leq a_1$, one has $g''(x) < 0$ on $(0, a)$.

(1-2) If $a > a_1$, one has $g''(x) < 0$ on $[0, a_1)$ and $g''(x) > 0$ on $(a_1, a]$.

When $x > a$, we have

$$\begin{aligned} g''(a) &= (1-c_+)g^-(0)(\theta_2^+)^2 - (g^+(0) - c_+ g^-(0))(\theta_1^+)^2 \\ &\geq [(1-c_+)g^-(0) - (g^+(0) - c_+ g^-(0))](\theta_1^+)^2 = [g^-(0) - g^+(0)](\theta_1^+)^2 > 0, \end{aligned}$$

on recalling that $\theta_1^+ \leq \theta_2^+$ and $g^-(0) > g^+(0) > 0$. Using (A.9) and the above we conclude $g''(x) > 0$ on (a, ∞). Putting it all together leads to Proposition 3.3.

A.5. Proof of Proposition 3.4. Using similar arguments as those in the proof of Proposition 3.3, one has the following observations.

(1) We have $g''(x) > 0$ on $(0, a)$.

(2) In what follows, we discuss the sign of $g''(x)$ for $x > a$.

  (2-1) Suppose $c_+ > 0$, in which case we have the following conclusions.

    (2-1-1) If $c_+ > 0$ and $a \geq a_2$, by (A.9) we have $g''(x) > 0$ for all $x > a$.

    (2-1-2) If $c_+ > 0$ and $0 < a < a_3 \wedge a_2$, then $x_0 > a$, $g''(x) < 0$ on (a, x_0), and $g''(x) > 0$ on (x_0, ∞).

    (2-1-3) If $c_+ > 0$ and $0 < a_3 \leq a \leq a_2$, then $g''(x) > 0$ for all $x > a$.

  (2-2) Suppose $c_+ \leq 0$, in which case the function $g''(x)$ is strictly increasing with its unique zero x_0 given by (3.1). We have the following conclusions.

    (2-2-1) If $\Theta \leq 0$, $c_+ \leq 0$, and $a > 0$, then $x_0 > a$, $g''(x) < 0$ on (a, x_0), and $g''(x) > 0$ on (x_0, ∞).

    (2-2-2) If $\Theta > 0$, $c_+ \leq 0$, and $a \in (0, a_3)$, then $x_0 > a$, $g''(x) < 0$ on (a, x_0), and $g''(x) > 0$ on (x_0, ∞).

    (2-2-3) If $\Theta > 0$, $c_+ \leq 0$, and $a \in [a_3, \infty)$, then $g''(x) \geq 0$ for all $x > a$.

Putting all the above together leads to the desired result of Proposition 3.4.

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