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COMPARISON THEOREMS FOR MULTIDIMENSIONAL BSDEs WITH JUMPS AND APPLICATIONS TO CONSTRAINED STOCHASTIC LINEAR-QUADRATIC CONTROL*

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Abstract. In this paper, we, for the first time, establish two comparison theorems for multidimensional backward stochastic differential equations with jumps. Our approach is novel and completely different from the existing results for the one-dimensional case. Using these and other delicate tools, we then construct solutions to coupled two-dimensional stochastic Riccati equation with jumps in both standard and singular cases. In the end, these results are applied to solve a cone-constrained stochastic linear-quadratic control problem and a mean-variance portfolio selection problem with jumps. Different from no-jump problems, the optimal (relative) state processes may change their signs, which is of course due to the presence of jumps.

Key words. backward stochastic differential equations with jumps, multidimensional comparison theorem, stochastic Riccati equation with jumps, cone-constrained linear-quadratic control, mean-variance problem

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1. Introduction.

The study of backward stochastic differential equations (BSDEs) can be dated back to Bismut [3], who studied the linear case, as an adjoint equation in the Pontryagin stochastic maximum principle. The general Lipschitz continuous case was later resolved in the seminal paper of Pardoux and Peng [29]. Since then, BSDEs have attracted strong interest of many researchers and found wide applications in partial differential equations, stochastic control, stochastic differential games, and mathematical finance; see, e.g., [6, 8, 9, 10, 12, 17, 31]. In particular, the solvability of quadratic BSDEs in the one-dimensional case was first obtained in Kobylanski [21], and then generalized to the multidimensional case by [11, 16, 26, 39].

BSDEs driven by a Brownian motion and a Poisson random measure, which are called BSDE with jumps (BSDEJ) in this paper, were first tackled by Tang and Li [38], then followed notably by Barles, Buckdahn, and Pardoux [2], Royer [34], and Quenez and Sulem [33] in the Lipschitz case. Quadratic BSDEJs and their applications in utility maximization problems have also been investigated; see, e.g., Antonelli

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and Mancini [1], Kazi-Tani, Possamaï, and Zhou [18], Laeven and Stadje [22], and Morlais [27, 28], among many others. Please refer to Papapantoleon, Possamaï, and Saplaouras [30] for a synopsis of these topics.

BSDEs arising from stochastic linear quadratic (LQ) control problems, called the stochastic Riccati equations (SREs), form an important class of BSDEs. In these BSDEs, the first unknown variable appears on the denominator and the second unknown variable grows quadratically in the generator. These features distinguish them from those well-studied BSDEs with Lipschitz or quadratic growth generators, so that they have to be studied by new methods.

Bismut [4] first found that an optimal control in form of linear state feedback for a stochastic LQ control problem is available, provided that its associated SRE admits a solution in some suitable space. Unfortunately he could not show the existence of such a solution in general. Nowadays much progress has been made in solving SREs. Kohlmann and Tang [20] resolved the existence and uniqueness issues for one-dimensional SREs, then Tang [36, 37] resolved the matrix-valued case using the stochastic maximum principle and the dynamic programming method, respectively. Sun, Xiong, and Yong [35] studied the indefinite case. Inspired by Tang's [37] dynamic programming method, Zhang, Dong, and Meng [40] established the existence and uniqueness of solutions to SREs with jumps (SREJs). Li, Wu, and Yu [23] studied the indefinite case using a so-called relax compensator.

Motivated by the mean-variance (MV) portfolio selection problem with no-shorting constraints, Hu and Zhou [17] studied the cone-constrained stochastic LQ problem and found that the optimal control takes a piecewise (0 is the unique segment point) linear state feedback form. The associated SRE is a two-dimensional, but decoupled, BSDE. Hence it can be treated separately as two one-dimensional BSDEs. The solvability was established with the aid of quadratic BSDE theory and truncation techniques. The decoupling phenomenon lies in the fact that the optimal state process will not change its sign (namely not cross 0), i.e., it will stay positive (resp., negative) if the initial state is positive (resp., negative). Dong [7] generalized the model in [17] to incorporate a jump by the enlargement of filtration framework. The corresponding SRE is a coupled two-dimensional BSDEJ, whose solvability is obtained by solving two recursive systems of BSDEs driven only by Brownian motions. This decomposition approach works only in the filtration enlargement theory; see also Kharroubi, Lim, and Ngoupeyou [19] and Hu, Shi, and Xu [14] for the unconstrained or regime switching case. Czichowsky and Schweizer [5] extended the cone-constrained MV model to a general semimartingale framework, but they cannot solve the two-dimensional SREJ. They claimed that "finding a solution by general BSDE techniques seems a formidable challenge" in [5, Remark 4.8].

This paper is intended as an attempt to cope with the formidable challenge indicated in [5]. Our main contribution is to resolve the solvability of a two-dimensional coupled SREJ in the Wiener–Poisson world via pure BSDE techniques. Although one can consider the more general semimartingale framework, we will focus on the Wiener–Poisson world as SREJ in this case takes more concrete structures for presentation and illustration. We establish the solvability for both standard and singular cases, containing the SREJ emerging in the cone-constrained MV problem as a special example. Since the existing approximation procedures in Kohlmann and Tang [20] and our previous work [13] cannot be applied to the present problem, we provide a new approximation procedure to achieve the goal.

A crucial and novel tool used in the approximation approach is our new comparison theorems for BSDEJs. We establish two comparison theorems which seem to be the first ones for the multidimensional case. The first one requires a locally Lipschitz condition for one generator (see Remark 2.3) and works for bounded state processes, whereas the second one requires the globally Lipschitz condition for both generators and works for square integrable state processes.

Most existing comparison theorems for BSDEJs require the condition $\gamma > -1$ (see Remark 2.2) or even stronger $\gamma > -1+\varepsilon$ in order to utilize the Girsanov theorem; see, e.g., Barles, Buckdahn, and Pardoux [2] and Royer [34]. To the best of our knowledge, Quenez and Sulem's [33] comparison theorem is the only one that relaxes the condition to $\gamma \geq -1$. Without resort to the Girsanov theorem, they used the conditional expectation representation of one-dimensional linear BSDEJs to establish their comparison theorem. Nevertheless, all of these existing comparison theorems for BSDEJs can only deal with the one-dimensional case. In our approximation procedure, however, the SREJ is a fully coupled two-dimensional BSDEJ, therefore comparison theorems for multidimensional BSDEJs are strongly appealing. It is worth pointing out that the conditional expectation representation method used in [33] cannot be applied to multidimensional BSDEJs. In this paper, we propose a completely different approach to establish our comparison theorems for multidimensional BSDEJs for the first time. We achieve the goal by directly analyzing $((\delta Y_t)^+)^2$ with the aid of the Meyer–Itô formula and utilizing a tricky elementary inequality (Lemma 2.1) that works for $\gamma \geq -1$. Note one cannot expect to extend the results to the case $\gamma < -1$ since counterexamples do exist in this case; see [2, Remark 2.7].

With the help of the new comparison theorems for multidimensional BSDEJs, we can construct solutions to the two-dimensional coupled SREJ in both standard and singular cases. We then apply the result to solve a cone-constrained stochastic LQ problem with jumps and obtain the efficient portfolio for an MV problem with jumps. It is worth pointing out that even without the cone-constraint, MV problems with jumps have not been investigated thoroughly. Lim [24] studied such a problem, but he assumed all the coefficients are predictable with respect to (w.r.t.) the Brownian filtration, and rendered the corresponding SRE exactly the same as that in the model without jumps. On the other hand, Zhang, Dong, and Meng [40] examined stochastic LQ problems with jumps, but they assumed the control weight in the running cost is uniformly positive so that their result cannot solve the corresponding MV problem where the control weight is 0. We will not only solve the MV problem with jumps, but also incorporate a convex cone-constraint, especially covering the famous no-shorting constraints. By adding a cone-constraint, the associated SREJ becomes a fully coupled two-dimensional BSDEJ, thus causing notably nontrivial difficulty in its solvability.

The rest of this paper is organized as follows. Section 2 is devoted to proving two comparison theorems for multidimensional BSDEJs. In section 3, we study a cone-constrained stochastic LQ control problem with jumps and prove the existence and uniqueness of solution to the associated SREJ. In section 4, we solve a cone-constrained MV problem.

2. Comparison theorems for multidimensional BSDEJs. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete filtered probability space equipped with a standard n-dimensional Brownian motion $W_t = (W_{1,t}, \dots, W_{n,t})^\top$. Let $\mathcal{E} \subseteq \mathbb{R}^\ell \setminus \{0\}$ be a nonempty Borel subset of the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$ and let $N(\mathrm{d}t, \mathrm{d}e)$ be a Poisson random measure on $\mathbb{R}_+ \times \mathcal{E}$ induced by a stationary Poisson point process which is independent of the Brownian motion $W_\cdot$ and whose stationary compensator (intensity measure) $\nu(\mathrm{d}e)\,\mathrm{d}t$ satisfies $\nu(\mathcal{E}) < \infty$. We let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the filtration generated by the

Brownian motion W and the Poisson random measure $N(\mathrm{d}t, \mathrm{d}e)$ and augmented by all $\mathbb{P}$-null sets, i.e., $\mathcal{F}_t := \sigma\big(W_s, N([0,s]\times A) : s\in[0,t], A\in\mathcal{B}(\mathcal{E})\big)\bigvee\mathcal{N}$, where $\mathcal{B}(\mathcal{E})$ is the Borel σ-algebra of $\mathcal{E}$ and $\mathcal{N}$ is the totality of all the $\mathbb{P}$-null sets of $\mathcal{F}$. We use an increasing sequence $\{T_n\}_{n\in\mathbb{N}}$ to denote the jump times of the underlying Poisson point process. The compensated Poisson random measure is denoted by $\widetilde{N}(\mathrm{d}t, \mathrm{d}e)$. For ease of notation, we only consider one-dimensional Poisson random measure, although the results of this paper can be generalized to the multidimensional case without essential difficulties. Let T be a fixed positive constant and $\mathcal{P}$ be the $\mathbb{F}$-predictable σ-field on $\Omega\times[0,T]$. We will denote by $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$ the conditional expectation w.r.t. $\mathcal{F}_t$.

We denote by $\mathbb{R}^\ell_+$ the set of vectors in $\mathbb{R}^\ell$ whose components are nonnegative, by $\mathbb{R}^{\ell\times n}$ the set of $\ell\times n$ real matrices, and by $\mathbb{S}^n$ the set of symmetric $n\times n$ real matrices. Therefore, $\mathbb{R}^\ell\equiv\mathbb{R}^{\ell\times 1}$. For any vector Y, we denote Y_i as its ith component. For any matrix $M=(m_{ij})$, we denote its transpose by $M^\top$ and its norm by $|M| = \sqrt{\sum_{ij} m_{ij}^2}$. If $M\in\mathbb{S}^n$ is positive definite (resp., positive semidefinite), we write $M>$ (resp., $\geq$) 0. We write $A>$ (resp., $\geq$) B if $A,B\in\mathbb{S}^n$ and $A-B>$ (resp., $\geq$) 0. We use the standard notation $x^+=\max\{x,0\}$ and $x^-=\max\{-x,0\}$ for $x\in\mathbb{R}$ and define a set $\mathcal{M}=\{1,2,\ldots,\ell\}$. We will use the elementary inequality $|a^\top b|\leq\frac{c}{2}|a|^2+\frac{|b|^2}{2c}$ for any $a,b\in\mathbb{R}^n, c>0$ frequently throughout the paper without claim.

We use the following spaces throughout the paper:

$$
\begin{aligned}
L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}) &= \Big\{\xi:\Omega\to\mathbb{R}\,\Big|\,\xi \text{ is } \mathcal{F}_T\text{-measurable, and } \mathbb{E}|\xi|^2<\infty\Big\},\\
L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}) &= \Big\{\xi:\Omega\to\mathbb{R}\,\Big|\,\xi \text{ is } \mathcal{F}_T\text{-measurable, and essentially bounded}\Big\},\\
L^2_{\mathbb{F}}(0,T;\mathbb{R}) &= \Big\{\phi:\Omega\times[0,T]\to\mathbb{R}\,\Big|\,\phi \text{ is } \mathcal{P}\text{-measurable and } \mathbb{E}\textstyle\int_0^T|\phi_t|^2\,\mathrm{d}t<\infty\Big\},\\
L^\infty_{\mathbb{F}}(0,T;\mathbb{R}) &= \Big\{\phi:\Omega\times[0,T]\to\mathbb{R}\,\Big|\,\phi \text{ is } \mathcal{P}\text{-measurable and essentially bounded}\Big\},\\
L^{2,\nu}(\mathbb{R}) &= \Big\{\phi:\mathcal{E}\to\mathbb{R}\,\Big|\,\phi \text{ is } \mathcal{B}(\mathcal{E})\text{-measurable and } ||\phi||^2_\nu:=\textstyle\int_{\mathcal{E}}|\phi(e)|^2\nu(\mathrm{d}e)<\infty\Big\},\\
L^{\infty,\nu}(\mathbb{R}) &= \Big\{\phi:\mathcal{E}\to\mathbb{R}\,\Big|\,\phi \text{ is } \mathcal{B}(\mathcal{E})\text{-measurable and essentially bounded w.r.t. } \mathrm{d}\nu\Big\},\\
L^{2,\nu}_{\mathcal{P}}(0,T;\mathbb{R}) &= \Big\{\phi:\Omega\times[0,T]\times\mathcal{E}\to\mathbb{R}\,\Big|\,\phi \text{ is } \mathcal{P}\otimes\mathcal{B}(\mathcal{E})\text{-measurable}\\
&\qquad \text{and } \mathbb{E}\textstyle\int_0^T\int_{\mathcal{E}}|\phi_t(e)|^2\nu(\mathrm{d}e)\,\mathrm{d}t<\infty\Big\},\\
L^{\infty,\nu}_{\mathcal{P}}(0,T;\mathbb{R}) &= \Big\{\phi:\Omega\times[0,T]\times\mathcal{E}\to\mathbb{R}\,\Big|\,\phi \text{ is } \mathcal{P}\otimes\mathcal{B}(\mathcal{E})\text{-measurable and}\\
&\qquad \text{essentially bounded w.r.t. } \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\otimes\mathrm{d}\nu\Big\},\\
S^2_{\mathbb{F}}(0,T;\mathbb{R}) &= \Big\{\phi:\Omega\times[0,T]\to\mathbb{R}\,\Big|\,(\phi_t)_{0\leq t\leq T} \text{ is càd-làg, } \mathbb{F}\text{-adapted}\\
&\qquad \text{and } \mathbb{E}\sup\nolimits_{0\leq t\leq T}|\phi_t|^2<\infty\Big\},\\
S^\infty_{\mathbb{F}}(0,T;\mathbb{R}) &= \Big\{\phi:\Omega\times[0,T]\to\mathbb{R}\,\Big|\,(\phi_t)_{0\leq t\leq T} \text{ is càd-làg, } \mathbb{F}\text{-adapted}\\
&\qquad \text{and essentially bounded}\Big\}.
\end{aligned}
$$

These definitions are generalized in the obvious way to the cases that $\mathbb{R}$ is replaced by $\mathbb{R}^n$, $\mathbb{R}^{n\times\ell}$, or $\mathbb{S}^n$. Arguments s, t, and ω, or statements "almost surely" (a.s.) and "almost everywhere" (a.e.), may be suppressed for simplicity in many circumstances when no confusion occurs. We shall use c to represent a generic positive constant which can be different from line to line. All the equations and inequalities in subsequent analysis shall be understood in the sense that $\mathrm{d}\mathbb{P}$-a.s. or $\mathrm{d}\nu$-a.e. or $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$-a.e. or $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t\otimes\mathrm{d}\nu$-a.e., etc.

In this paper, any ℓ-dimensional BSDEJ (on $[0,T]$) is characterized by a pair (ξ,f), in which $\xi:\Omega\to\mathbb{R}^\ell$ is called the terminal value which is an $\mathcal{F}_T$-measurable random vector, and $f:\Omega\times[0,T]\times\mathbb{R}^\ell\times\mathbb{R}^{n\times\ell}\times L^{2,\nu}(\mathbb{R}^\ell)\to\mathbb{R}^\ell$ is called the generator which is a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^\ell)\otimes\mathcal{B}(\mathbb{R}^{n\times\ell})\otimes\mathcal{B}(L^{2,\nu}(\mathbb{R}^\ell))$-measurable process. We say a generator f is Lipschitz in (y,z,ϕ) if there exist constant $c>0$ such that

$$|f(\omega,t,y,z,\phi)-f(\omega,t,\overline{y},\overline{z},\overline{\phi})|\le c(|y-\overline{y}|+|z-\overline{z}|+||\phi-\overline{\phi}||_\nu)\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.e.}$$

holds for all (y,z,ϕ), $(\overline{y},\overline{z},\overline{\phi})\in\mathbb{R}^\ell\times\mathbb{R}^{n\times\ell}\times L^{2,\nu}(\mathbb{R}^\ell)$. The case that f is Lipschitz only in (y,z) or y can be defined similarly. We call the BSDEJ ℓ-dimensional as its state process is $\mathbb{R}^\ell$-valued. We often rewrite it in its component form for ease of presentation.

2.1. Comparison theorem for bounded processes. We first prove a comparison theorem where the state processes are essentially bounded.

THEOREM 2.1. *Suppose, for every $i\in\mathcal{M}$,*

$$(Y_i,Z_i,\Phi_i),(\overline{Y}_i,\overline{Z}_i,\overline{\Phi}_i)\in S^\infty_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times L^{2,\nu}_{\mathcal{P}}(0,T;\mathbb{R}),$$

and they satisfy BSDEJs

$$\begin{aligned}Y_{i,t}=\xi_i&+\int_t^T f_i(s,Y_{s-},Z_{i,s},\Phi_s)\,\mathrm{d}s\\ &-\int_t^T Z_{i,s}^\top\,\mathrm{d}W_s-\int_t^T\int_{\mathcal{E}}\Phi_{i,s}(e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e)\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}\text{-}a.e.,\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\overline{Y}_{i,t}=\overline{\xi}_i&+\int_t^T \overline{f}_i(s,\overline{Y}_{s-},\overline{Z}_{i,s},\overline{\Phi}_s)\,\mathrm{d}s\\ &-\int_t^T \overline{Z}_{i,s}^\top\,\mathrm{d}W_s-\int_t^T\int_{\mathcal{E}}\overline{\Phi}_{i,s}(e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e)\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-}a.e.\end{aligned}\tag{2.2}$$

Also suppose that, for all $i\in\mathcal{M}$ and $s\in[0,T]$,

H(1) $\xi_i\le\overline{\xi}_i$;

H(2) *there exists a constant $c>0$ such that*

$$\begin{aligned}&f_i(s,Y_{s-},Z_{i,s},\Phi_{1,s},\dots,\Phi_{i,s},\dots,\Phi_{\ell,s})\\ &\quad-f_i(s,Y_{s-},Z_{i,s},\Phi_{1,s},\dots,\overline{\Phi}_{i,s},\dots,\Phi_{\ell,s})\\ &\quad\le c\int_{\mathcal{E}}(\Phi_{i,s}(e)-\overline{\Phi}_{i,s}(e))^+\nu(\mathrm{d}e)+\int_{\mathcal{E}}|\Phi_{i,s}(e)-\overline{\Phi}_{i,s}(e)|\nu(\mathrm{d}e);\end{aligned}$$

H(3) *there exists a constant $c>0$ such that*

$$\begin{aligned}&f_i(s,Y_{s-},Z_{i,s},\Phi_{1,s},\dots,\overline{\Phi}_{i,s},\dots,\Phi_{\ell,s})-f_i(s,\overline{Y}_{s-},\overline{Z}_{i,s},\overline{\Phi}_s)\\ &\quad\le c\Big(|Y_{i,s-}-\overline{Y}_{i,s-}|+\sum_{j\ne i}(Y_{j,s-}-\overline{Y}_{j,s-})^++|Z_{i,s}-\overline{Z}_{i,s}|\\ &\qquad+\sum_{j\ne i}\int_{\mathcal{E}}(Y_{j,s-}+\Phi_{j,s}(e)-\overline{Y}_{j,s-}-\overline{\Phi}_{j,s}(e))^+\nu(\mathrm{d}e)\Big);\ \textit{and}\end{aligned}$$

H(4) $f_i(s,\overline{Y}_{s-},\overline{Z}_{i,s},\overline{\Phi}_s)\le\overline{f}_i(s,\overline{Y}_{s-},\overline{Z}_{i,s},\overline{\Phi}_s)$.

Then $\mathbb{P}\{Y_{i,t} \le \overline{Y}_{i,t} \forall t \in [0,T]\} = 1$ *for all* $i \in \mathcal{M}$.

To prove this theorem, we need the following critical elementary result.

LEMMA 2.1. *For all* $(x, y) \in \mathbb{R} \times \mathbb{R}$ *and* $c \ge -1$, *we have*

$$[(x+y)^+]^2 - (x^+)^2 - 2(1+c)x^+y \ge -(c^2 \vee 1)(x^+)^2.$$

Proof. There are three cases:

- If $x \le 0$, then

$$[(x+y)^+]^2 - (x^+)^2 - 2(1+c)x^+y = [(x+y)^+]^2 \ge 0 = -(c^2 \vee 1)(x^+)^2.$$

- If $y \le 0$, then, since $c \ge -1$,

$$\begin{aligned}[(x+y)^+]^2 - (x^+)^2 - 2(1+c)x^+y &\ge [(x+y)^+]^2 - (x^+)^2 \ge -(x^+)^2 \\ &\ge -(c^2 \vee 1)(x^+)^2.\end{aligned}$$

- If $x \ge 0$ and $y \ge 0$, then

$$\begin{aligned}[(x+y)^+]^2 - (x^+)^2 - 2(1+c)x^+y &= y^2 - 2cxy = (y-cx)^2 - c^2x^2 \\ &\ge -(c^2 \vee 1)x^2 = -(c^2 \vee 1)(x^+)^2.\end{aligned}$$

The proof is complete. ∎

Proof of Theorem 2.1. For $t \in [0,T]$ and $i \in \mathcal{M}$, set

$$\delta Y_{i,t} = Y_{i,t} - \overline{Y}_{i,t}, \ \delta Z_{i,t} = Z_{i,t} - \overline{Z}_{i,t}, \ \delta\Phi_{i,t} = \Phi_{i,t} - \overline{\Phi}_{i,t}.$$

Applying the Meyer–Itô formula [32, Chapter IV, Theorem 70] to $(\delta Y_{i,t})^+$, we get

$$\begin{aligned}\mathrm{d}(\delta Y_{i,t})^+ &= -\mathbf{1}_{\{\delta Y_{i,t-}>0\}}[f_i(t, Y_{t-}, Z_{i,t}, \Phi_t) - \overline{f}_i(t, \overline{Y}_{t-}, \overline{Z}_{i,t}, \overline{\Phi}_t)]\,\mathrm{d}t \\ &+ \int_{\mathcal{E}} [(\delta Y_{i,t-} + \delta\Phi_{i,t}(e))^+ - (\delta Y^i_{i,t-})^+ - \mathbf{1}_{\{\delta Y_{i,t-}>0\}}\delta\Phi_{i,t}(e)]\nu(\mathrm{d}e)\,\mathrm{d}t + \frac{1}{2}\,\mathrm{d}L_{i,t} \\ &+ \mathbf{1}_{\{\delta Y_{i,t-}>0\}}\delta Z_{i,t}^\top\,\mathrm{d}W_t + \int_{\mathcal{E}} [(\delta Y_{i,t-} + \delta\Phi_{i,t}(e))^+ - (\delta Y_{i,t-})^+]\widetilde{N}(\mathrm{d}t, \mathrm{d}e),\end{aligned}$$

where $L_{i,t}$ is the local time of $\delta Y_{i,t}$ at 0. Since $\delta Y_{i,t-}\,\mathrm{d}L_{i,t} = 0$, applying Itô's formula to $((\delta Y_{i,t})^+)^2$ yields

$$\begin{aligned}\mathrm{d}((\delta Y_{i,t})^+)^2 = &-2(\delta Y_{i,t-})^+[f_i(t, Y_{t-}, Z_{i,t}, \Phi_t) - \overline{f}_i(t, \overline{Y}_{t-}, \overline{Z}_{i,t}, \overline{\Phi}_t)]\,\mathrm{d}t \\ &+ \mathbf{1}_{\{\delta Y_{i,t-}>0\}}|\delta Z_{i,t}|^2\,\mathrm{d}t + \int_{\mathcal{E}} [((\delta Y_{i,t-} + \delta\Phi_{i,t}(e))^+)^2 - ((\delta Y_{i,t-})^+)^2 \\ &- 2(\delta Y_{i,t-})^+\delta\Phi_{i,t}(e)]\nu(\mathrm{d}e)\,\mathrm{d}t + 2(\delta Y_{i,t-})^+\delta Z_{i,t}^\top\,\mathrm{d}W_t \\ &+ \int_{\mathcal{E}} [((\delta Y_{i,t-} + \delta\Phi_{i,t}(e))^+)^2 - ((\delta Y_{i,t-})^+)^2]\widetilde{N}(\mathrm{d}t, \mathrm{d}e).\end{aligned} \tag{2.3}$$

Using the condition H(4) and inserting two zero-sum terms, we get

$$\begin{aligned}&f_i(t, Y_{t-}, Z_{i,t}, \Phi_t) - \overline{f}_i(t, \overline{Y}_{t-}, \overline{Z}_{i,t}, \overline{\Phi}_t) \\ &\quad \le f_i(t, Y_{t-}, Z_{i,t}, \Phi_t) - f_i(t, \overline{Y}_{t-}, \overline{Z}_{i,t}, \overline{\Phi}_t) \\ &\quad = [f_i(t, Y_{t-}, Z_{i,t}, \Phi_t) - f_i(t, Y_{t-}, Z_{i,t}, \Phi_{1,t}, \ldots, \overline{\Phi}_{i,t}, \ldots, \Phi_{\ell,t})] \\ &\qquad + [f_i(t, Y_{t-}, Z_{i,t}, \Phi_{1,t}, \ldots, \overline{\Phi}_{i,t}, \ldots, \Phi_{\ell,t}) - f_i(t, \overline{Y}_{t-}, \overline{Z}_{i,t}, \overline{\Phi}_t)].\end{aligned}$$

By the condition H(2), the first difference on the right-hand side (RHS) in the above is upper bounded by

$$\int_{\mathcal{E}} \gamma_{i,t}(e)\delta\Phi_{i,t}(e)\nu(\mathrm{d}e),$$

where

$$\gamma_{i,t}(e) = \begin{cases} c & \text{if } \delta\Phi_{i,t}(e) \geq 0; \\ -1 & \text{if } \delta\Phi_{i,t}(e) < 0. \end{cases}$$

By the condition H(3), the second difference on the RHS is upper bounded respectively by

$$c\left(|\delta Y_{i,t-}| + \sum_{j\neq i}(\delta Y_{j,t-})^+ + |\delta Z_{i,t}| + \sum_{j\neq i}\int_{\mathcal{E}}(\delta Y_{j,t-} + \delta\Phi_{j,t}(e))^+\nu(\mathrm{d}e)\right).$$

Using these estimates and $\nu(\mathcal{E}) < \infty$, we deduce that

$$\begin{aligned} &2(\delta Y_{i,t-})^+[f_i(t, Y_{t-}, Z_{i,t}, \Phi_t) - \overline{f}_i(t, \overline{Y}_{t-}, \overline{Z}_{i,t}, \overline{\Phi}_t)] \\ &\quad \leq c\sum_{i=1}^{\ell}((\delta Y_{i,t-})^+)^2 + \mathbf{1}_{\{\delta Y_{i,t-}>0\}}|\delta Z_{i,t}|^2 \\ &\qquad + \mathbf{1}_{\{\delta Y_{i,t-}>0\}}\sum_{j\neq i}\int_{\mathcal{E}}((\delta Y_{j,t-} + \delta\Phi_{j,t}(e))^+)^2\nu(\mathrm{d}e) \\ &\qquad + 2(\delta Y_{i,t-})^+\int_{\mathcal{E}}\gamma_{i,t}(e)\delta\Phi_{i,t}(e)\nu(\mathrm{d}e). \end{aligned} \tag{2.4}$$

Integrating from t to T in (2.3), taking conditional expectation, and using (2.4), we obtain

$$\begin{aligned} &((\delta Y_{i,t})^+)^2 \\ &\quad \leq \mathbb{E}_t\int_t^T \left(c\sum_{i=1}^{\ell}((\delta Y_{i,s-})^+)^2 + \mathbf{1}_{\{\delta Y_{i,s-}>0\}}\sum_{j\neq i}\int_{\mathcal{E}}((\delta Y_{j,s-} + \delta\Phi_{j,s}(e))^+)^2\nu(\mathrm{d}e)\right)\mathrm{d}s \\ &\qquad - \mathbb{E}_t\int_t^T\int_{\mathcal{E}}\Big[((\delta Y_{i,s-} + \delta\Phi_{i,s}(e))^+)^2 - ((\delta Y_{i,s-})^+)^2 \\ &\qquad - 2(1+\gamma_{i,s}(e))(\delta Y_{i,s-})^+\delta\Phi_{i,s}(e)\Big]\nu(\mathrm{d}e)\,\mathrm{d}s. \end{aligned}$$

Because $\gamma_i \in L^{\infty,\nu}_{\mathcal{P}}(0,T;\mathbb{R})$ and $\gamma_i \geq -1$, it follows from Lemma 2.1 that

$$\begin{aligned} &-[((\delta Y_{i,s-} + \delta\Phi_{i,s}(e))^+)^2 - ((\delta Y_{i,s-})^+)^2 - 2(1+\gamma_{i,s}(e))(\delta Y_{i,s-})^+\delta\Phi_{i,s}(e)] \\ &\quad \leq (\gamma_{i,s}(e)^2 \vee 1)((\delta Y_{i,s-})^+)^2 \leq c((\delta Y_{i,s-})^+)^2. \end{aligned} \tag{2.5}$$

Combining the above estimates and using $\nu(\mathcal{E}) < \infty$, we obtain

$$\begin{aligned} ((\delta Y_{i,t})^+)^2 &\leq c\mathbb{E}_t\int_t^T\sum_{i=1}^{\ell}((\delta Y_{i,s-})^+)^2\,\mathrm{d}s \\ &\quad + \sum_{j\neq i}\mathbb{E}_t\int_t^T\int_{\mathcal{E}}((\delta Y_{j,s-} + \delta\Phi_{j,s}(e))^+)^2\nu(\mathrm{d}e)\,\mathrm{d}s, \end{aligned} \tag{2.6}$$

where the constant c is independent of t, T, and i.

Note that

$$\begin{aligned}
&\mathbb{E}_t \int_t^T \int_{\mathcal{E}} ((\delta Y_{j,s-} + \delta\Phi_{j,s}(e))^+)^2 \nu(\mathrm{d}e)\,\mathrm{d}s \\
&= \mathbb{E}_t \int_t^T \int_{\mathcal{E}} ((\delta Y_{j,s-} + \delta\Phi_{j,s}(e))^+)^2 N(\mathrm{d}s, \mathrm{d}e) \\
&= \mathbb{E}_t \left[\sum_{n\in\mathbb{N},\ t<T_n\le T} ((\delta Y_{j,T_n-} + \delta\Phi_{j,T_n}(\Delta U_{T_n}))^+)^2 \right] \\
&= \mathbb{E}_t \left[\sum_{n\in\mathbb{N},\ t<T_n\le T} ((\delta Y_{j,T_n})^+)^2 \right], \qquad (2.7)
\end{aligned}$$

where $U_t := \int_0^t \int_{\mathcal{E}} eN(\mathrm{d}s, \mathrm{d}e)$, and $\Delta U_{T_n} := U_{T_n} - U_{T_n-}$ (recalling that $\{T_n\}_{n\in\mathbb{N}}$ denotes jump times of underlying Poisson point process). Substituting (2.7) into (2.6) yields

$$((\delta Y_{i,t})^+)^2 \le c\mathbb{E}_t \int_t^T \sum_{i=1}^{\ell} ((\delta Y_{i,s-})^+)^2\,\mathrm{d}s + \sum_{j\ne i} \mathbb{E}_t \left[\sum_{n\in\mathbb{N},\ t<T_n\le T} ((\delta Y_{j,T_n})^+)^2 \right].$$

Since the jumps of δY are countable, we can replace $\delta Y_{i,s-}$ by $\delta Y_{i,s}$ in the above integral to get

$$((\delta Y_{i,t})^+)^2 \le c\mathbb{E}_t \int_t^T \sum_{i=1}^{\ell} ((\delta Y_{i,s})^+)^2\,\mathrm{d}s + \sum_{j\ne i} \mathbb{E}_t \left[\sum_{n\in\mathbb{N},\ t<T_n\le T} ((\delta Y_{j,T_n})^+)^2 \right]. \qquad (2.8)$$

For any constant $h \in (0, T]$, set

$$M(h) := \operatorname*{ess\,sup}_{(t,i)\in[T-h,T]\times\mathcal{M}} ((\delta Y_{i,t})^+)^2,$$

which is finite since δY is bounded. For any $t \in [T-h, T]$, we obtain from (2.8) that

$$\begin{aligned}
((\delta Y_{i,t})^+)^2 &\le c \int_t^T \sum_{i=1}^{\ell} M(h)\,\mathrm{d}s + \sum_{j\ne i} \mathbb{E}_t \left[\sum_{n\in\mathbb{N},\ t<T_n\le T} M(h) \right] \\
&= c\ell M(h)(T-t) + M(h) \sum_{j\ne i} \mathbb{E}_t \int_t^T \int_{\mathcal{E}} 1N(\mathrm{d}s, \mathrm{d}e) \\
&= c\ell M(h)(T-t) + M(h)(\ell-1)\nu(\mathcal{E})(T-t) \\
&\le (c\ell + (\ell-1)\nu(\mathcal{E}))M(h)h.
\end{aligned}$$

Taking essential supreme over $(t,i) \in [T-h, T] \times \mathcal{M}$ on both sides leads to

$$M(h) \le (c\ell + (\ell-1)\nu(\mathcal{E}))M(h)h. \qquad (2.9)$$

Set $h = \min\{1/(c\ell + (\ell-1)\nu(\mathcal{E}) + 1), T\}$ from now on. It then follows from above that $M(h) = 0$, thus $\delta Y_{i,t} \le 0$ for all $t \in [T-h, T]$. Similarly, using $\delta Y_{i,T-h} \le 0$ and repeating the above argument on $[0 \vee (T-2h), T-h]$, one can get $\delta Y_{i,t} \le 0$ for all $t \in [0 \vee (T-2h), T-h]$. Repeating this procedure, the desired comparison result follows. ∎

Remark 2.1. If the inequalities in the conditions H(1) and H(4) are reversed, then so is the conclusion.

Remark 2.2. It is not hard to see that the condition H(2) is equivalent to that there exists a process $\gamma_i \in L^{\infty,\nu}_{\mathcal{P}}(0,T;\mathbb{R})$ with $\gamma_i \geq -1$ such that

$$\begin{aligned} &f_i(s, Y_{s-}, Z_{i,s}, \Phi_{1,s}, \ldots, \Phi_{i,s}, \ldots, \Phi_{\ell,s}) \\ &\quad - f_i(s, Y_{s-}, Z_{i,s}, \Phi_{1,s}, \ldots, \overline{\Phi}_{i,s}, \ldots, \Phi_{\ell,s}) \\ &\leq \int_{\mathcal{E}} \gamma_{i,s}(e)(\Phi_{i,s}(e) - \overline{\Phi}_{i,s}(e))\nu(\mathrm{d}e). \end{aligned}$$

Most existing comparison theorems for BSDEJs require the condition $\gamma > -1$ or even stronger $\gamma > -1 + \varepsilon$ in order to utilize the Girsanov theorem; see, e.g., Barles, Buckdahn, and Pardoux [2] and Royer [34]. Our requirement, namely $\gamma \geq -1$, is the same as Quenez and Sulem's [33]. But all these existing comparison theorems work for one-dimensional BSDEJs only.

Remark 2.3. The condition H(3) holds if, for every $K > 0$, there exists a constant $c > 0$ (depending on K) such that

$$\begin{aligned} &f_i(s, y, z, \phi) - f_i(s, \overline{y}, \overline{z}, \overline{\phi}) \\ &\quad \leq c\left(|y_i - \overline{y}_i| + \sum_{j \neq i} (y_j - \overline{y}_j)^+ + |z - \overline{z}| + \sum_{j \neq i} \int_{\mathcal{E}} (y_j - \overline{y}_j + \phi_j(e) - \overline{\phi}_j(e))^+ \nu(\mathrm{d}e) \right) \end{aligned}$$

holds for all (y, z, ϕ) and $(\overline{y}, \overline{z}, \overline{\phi}) \in \mathbb{R}^\ell \times \mathbb{R}^n \times L^{2,\nu}(\mathbb{R}^\ell)$ satisfying $\phi_i \equiv \overline{\phi}_i$ and $|y| + |\overline{y}| \leq K$. Since $|y| + |\overline{y}| \leq K$, it is a locally Lipschitz condition w.r.t. y. The condition implies f_i is nondecreasing w.r.t. y_j and ϕ_j for every $j \neq i$.

Remark 2.4. The condition H(3) holds if
(1) $f_i(s, y, z, \phi)$ is Lipschitz only in (y, z);
(2) there exists a constant $c > 0$ such that

$$\begin{aligned} &f_i(s, Y_{s-}, Z_{i,s}, \Phi_{1,s}, \ldots, \Phi_{i-1,s}, \overline{\Phi}_{i,s}, \Phi_{i+1,s}, \ldots, \Phi_{\ell,s}) \\ &\quad - f_i(s, \overline{Y}_{1,s-}, \ldots, \overline{Y}_{i-1,s-}, Y_{i,s-}, \overline{Y}_{i+1,s-}, , \ldots, \overline{Y}_{\ell,s-}, Z_{i,s-}, \overline{\Phi}_{s-}) \\ &\leq c\left(\sum_{j \neq i} (Y_{j,s-} - \overline{Y}_{j,s-})^+ + \sum_{j \neq i} \int_{\mathcal{E}} (Y_{j,s-} + \Phi_{j,s}(e) - \overline{Y}_{j,s-} - \overline{\Phi}_{j,s}(e))^+ \nu(\mathrm{d}e) \right). \end{aligned}$$

Remark 2.5. In (2.5) the condition $\gamma \in L^{\infty,\nu}_{\mathcal{P}}(0,T;\mathbb{R}^\ell)$ can be replaced by the following weaker one: there exist constants $0 < h, \varepsilon < 1$ such that

$$\operatorname*{ess\,sup}_{t \in [0,T]} \mathbb{E}_t \int_t^{T \wedge (t+h)} \int_{\mathcal{E}} |\gamma_s(e)|^2 \nu(\mathrm{d}e)\, \mathrm{d}s \leq 1 - \varepsilon.$$

This condition is satisfied, for instance, when $\int_{\mathcal{E}} |\gamma_\cdot(e)|^2 \nu(\mathrm{d}e) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R})$. Indeed, the above condition implies, for $t \in [T-h, T]$,

$$\begin{aligned} \mathbb{E}_t \int_t^T ((\delta Y_{i,s-})^+)^2 \int_{\mathcal{E}} (\gamma_{i,s}(e)^2 \vee 1)\nu(\mathrm{d}e)\, \mathrm{d}s &\leq M(h)\, \mathbb{E}_t \int_t^T \int_{\mathcal{E}} (\gamma_{i,s}(e)^2 + 1)\nu(\mathrm{d}e)\, \mathrm{d}s \\ &\leq M(h)(1 - \varepsilon + h\nu(\mathcal{E})) \leq (1 - \varepsilon/2)M(h), \end{aligned}$$

by choosing h small enough. This together with (2.3) and (2.4) will lead to an estimate similar to (2.9) in the above proof.

2.2. Comparison theorem for square integrable processes. Theorem 2.1 requires the state processes to be bounded, which may be too restrictive for applications. The following result relaxes this assumption to square integrable processes, but we have to, in addition, assume that both f and $\overline{f}$ are globally Lipschitz.

THEOREM 2.2. *We shall use the same notation as in Theorem* 2.1. *Suppose, for all* $i\in\mathcal{M}$,

$$(Y_i,Z_i,\Phi_i),(\overline{Y}_i,\overline{Z}_i,\overline{\Phi}_i)\in S^2_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times L^{2,\nu}_{\mathcal{P}}(0,T;\mathbb{R}),$$

and they satisfy the BSDEJs (2.1) *and* (2.2). *Also suppose that, for every* $i\in\mathcal{M}$,
(1) *the conditions* H(1), H(2), H(3), *and* H(4) *hold;*
(2) $f_i(\cdot,0,0,0)$ *and* $\overline{f}_i(\cdot,0,0,0)\in L^2_{\mathbb{F}}(0,T;\mathbb{R})$*;*
(3) *both* f_i *and* $\overline{f}_i$ *are Lipschitz in* (y,z,ϕ).
Then $Y_i\le\overline{Y}_i$ *for all* $i\in\mathcal{M}$.

Proof. For each $m\ge1$ and $i\in\mathcal{M}$, we denote

$$\xi^m_i=\xi_i\mathbf{1}_{|\xi|+|\overline{\xi}|\le m},\ \ f^m_i(t,y,z,\phi)=f_i(t,y,z,\phi)\mathbf{1}_{|f(t,0,0,0)|+|\overline{f}(t,0,0,0)|\le m},$$
$$\overline{\xi}^m_i=\overline{\xi}_i\mathbf{1}_{|\xi|+|\overline{\xi}|\le m},\ \ \overline{f}^m_i(t,y,z,\phi)=\overline{f}_i(t,y,z,\phi)\mathbf{1}_{|f(t,0,0,0)|+|\overline{f}(t,0,0,0)|\le m}.$$

Note that ξ^m_i, $\overline{\xi}^m_i$, $f^m_i(\cdot,0,0,0)$, and $\overline{f}^m_i(\cdot,0,0,0)$ are bounded by m and the generators $f^m=(f^m_1,\dots,f^m_\ell)$ and $\overline{f}^m=(\overline{f}^m_1,\dots,\overline{f}^m_\ell)$ are both Lipschitz in (y,z,ϕ) with the same Lipschitz constant as f and $\overline{f}$. It then follows from [38, Lemma 2.4] or [2, Theorem 2.1, Proposition 2.2] that the BSDEJs

$$\begin{aligned}Y^m_{i,t}&=\xi^m_i+\int_t^T f^m_i(s,Y^m_{s-},Z^m_{i,s},\Phi^m_s)\,\mathrm{d}s\\&\quad-\int_t^T(Z^m_{i,s})^\top\,\mathrm{d}W_s-\int_t^T\int_{\mathcal{E}}\Phi^m_{i,s}(e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e)\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.e.},\ i\in\mathcal{M},\end{aligned}$$

and

$$\begin{aligned}\overline{Y}^m_{i,t}&=\overline{\xi}^m_i+\int_t^T \overline{f}^m_i(s,\overline{Y}^m_{s-},\overline{Z}^m_{i,s},\overline{\Phi}^m_s)\,\mathrm{d}s\\&\quad-\int_t^T(\overline{Z}^m_{i,s})^\top\,\mathrm{d}W_s-\int_t^T\int_{\mathcal{E}}\overline{\Phi}^m_{i,s}(e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e)\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.e.},\ i\in\mathcal{M},\end{aligned}$$

admit unique solutions (Y^m,Z^m,Φ^m) and $(\overline{Y}^m,\overline{Z}^m,\overline{\Phi}^m)$, respectively, such that

$$\begin{aligned}&(Y^m_i,Z^m_i,\Phi^m_i),\ (\overline{Y}^m_i,\overline{Z}^m_i,\overline{\Phi}^m_i)\in S^2_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\\&\quad\times L^{2,\nu}_{\mathcal{P}}(0,T;\mathbb{R})\ \text{ for all } i\in\mathcal{M}.\end{aligned}$$

We temporally suppose that

$$Y^m_i,\ \overline{Y}^m_i\in S^\infty_{\mathbb{F}}(0,T;\mathbb{R})\ \text{ for all } i\in\mathcal{M}. \tag{2.10}$$

Then applying Theorem 2.1 leads to

$$Y^m_i\le\overline{Y}^m_i\ \text{for all } i\in\mathcal{M}. \tag{2.11}$$

From [2, Proposition 2.2], we know there is constant $c>0$ independent of m such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t-Y_t^m|^2\right]\le c\mathbb{E}\left[|\xi-\xi^m|^2+\int_0^T|f(t,Y_t,Z_t,\Phi_t)-f^m(t,Y_t,Z_t,\Phi_t)|^2\,\mathrm{d}t\right],$$
$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}_t-\overline{Y}_t^m|^2\right]\le c\mathbb{E}\left[|\overline{\xi}-\overline{\xi}^m|^2+\int_0^T|\overline{f}(t,\overline{Y}_t,\overline{Z}_t,\overline{\Phi}_t)-\overline{f}^m(t,\overline{Y}_t,\overline{Z}_t,\overline{\Phi}_t)|^2\,\mathrm{d}t\right].$$

These estimates together with the definitions of $\xi^m,\overline{\xi}^m,f^m,\overline{f}^m$ and the dominated convergence theorem lead to

$$\lim_{m\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|Y_t-Y_t^m|^2+\sup_{0\le t\le T}|\overline{Y}_t-\overline{Y}_t^m|^2\right]=0.$$

Applying the elementary inequalities $(x^+)^2\le 2(y^+)^2+2(x-y)^2$, $(x+y)^2\le 2x^2+2y^2$ for x, $y\in\mathbb{R}$ and (2.11), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\sum_{i=1}^{\ell}[(Y_{i,t}-\overline{Y}_{i,t})^+]^2\right]\\
&\quad\le\mathbb{E}\left[2\sup_{0\le t\le T}\sum_{i=1}^{\ell}[(Y_{i,t}^m-\overline{Y}_{i,t}^m)^+]^2+2\sup_{0\le t\le T}\sum_{i=1}^{\ell}(Y_{i,t}-Y_{i,t}^m+\overline{Y}_{i,t}^m-\overline{Y}_{i,t})^2\right]\\
&\quad=\mathbb{E}\left[2\sup_{0\le t\le T}\sum_{i=1}^{\ell}(Y_{i,t}-Y_{i,t}^m+\overline{Y}_{i,t}^m-\overline{Y}_{i,t})^2\right]\\
&\quad\le\mathbb{E}\left[4\sup_{0\le t\le T}\sum_{i=1}^{\ell}(Y_{i,t}-Y_{i,t}^m)^2+4\sup_{0\le t\le T}\sum_{i=1}^{\ell}(\overline{Y}_{i,t}^m-\overline{Y}_{i,t})^2\right].
\end{aligned}$$

Sending $m\to\infty$ in the above, we get the desired result $Y_i\le\overline{Y}_i$ for all $i\in\mathcal{M}$.

It remains to establish (2.10). To this end, let $\beta>0$ be a large constant to be chosen later. Applying Itô's formula to $e^{\beta t}(Y_{i,t}^m)^2$, for each $i\in\mathcal{M}$, yields

$$\begin{aligned}
&e^{\beta t}(Y_{i,t}^m)^2+\mathbb{E}_t\int_t^T e^{\beta s}\Big(\beta(Y_{i,s}^m)^2+|Z_{i,s}^m|^2+||\Phi_{i,s}^m||_\nu^2\Big)\,\mathrm{d}s\\
&\quad=\mathbb{E}_t[e^{\beta T}(\xi_i^m)^2]+\mathbb{E}_t\int_t^T 2e^{\beta s}Y_{i,s-}^m f_i^m(s,Y_{s-}^m,Z_{i,s}^m,\Phi_s^m)\,\mathrm{d}s\\
&\quad\le m^2e^{\beta T}+\mathbb{E}_t\int_t^T 2e^{\beta s}|Y_{s-}^m||f_i^m(s,Y_{s-}^m,Z_{i,s}^m,\Phi_s^m)-f_i^m(s,0,0,0)|\,\mathrm{d}s\\
&\qquad+\mathbb{E}_t\int_t^T 2e^{\beta s}|Y_{s-}^m||f_i^m(s,0,0,0)|\,\mathrm{d}s\\
&\quad\le m^2e^{\beta T}+\mathbb{E}_t\int_t^T 2e^{\beta s}|Y_{s-}^m|c\Big(|Y_{s-}^m|+|Z_{i,s}^m|+||\Phi_s^m||_\nu\Big)\,\mathrm{d}s\\
&\qquad+\mathbb{E}_t\int_t^T e^{\beta s}|Y_{s-}^m|^2+\mathbb{E}_t\int_t^T e^{\beta s}|f_i^m(s,0,0,0)|^2\,\mathrm{d}s\\
&\quad\le m^2(1+T)e^{\beta T}+\mathbb{E}_t\int_t^T e^{\beta s}\Big(c|Y_{s-}^m|^2+|Z_{i,s}^m|^2+\frac{1}{\ell}||\Phi_s^m||_\nu^2\Big)\,\mathrm{d}s,
\end{aligned}$$

where the last constant c does not depend on t, β, and i. Canceling the common terms involving $|Z_{i,s}^m|^2$, we get

$$
\begin{aligned}
&e^{\beta t}(Y^m_{i,t})^2+\mathbb{E}_t\int_t^T e^{\beta s}\Big(\beta(Y^m_{i,s})^2+||\Phi^m_{i,s}||^2_\nu\Big)\,\mathrm{d}s\\
&\quad\le m^2(1+T)e^{\beta T}+\mathbb{E}_t\int_t^T e^{\beta s}\Big(c|Y^m_{s-}|^2+\frac{1}{\ell}||\Phi^m_s||^2_\nu\Big)\,\mathrm{d}s.
\end{aligned}
$$

Summing i from 1 to ℓ gives

$$
\begin{aligned}
&e^{\beta t}|Y^m_t|^2+\mathbb{E}_t\int_t^T e^{\beta s}\Big(\beta|Y^m_s|^2+||\Phi^m_s||^2_\nu\Big)\,\mathrm{d}s\\
&\quad\le \ell m^2(1+T)e^{\beta T}+\mathbb{E}_t\int_t^T e^{\beta s}\Big(c\ell|Y^m_{s-}|^2+||\Phi^m_s||^2_\nu\Big)\,\mathrm{d}s\\
&\quad= \ell m^2(1+T)e^{\beta T}+\mathbb{E}_t\int_t^T e^{\beta s}\Big(c\ell|Y^m_s|^2+||\Phi^m_s||^2_\nu\Big)\,\mathrm{d}s,
\end{aligned}
$$

where the last equation is due to the fact that the jumps of Y are countable. By setting $\beta=c\ell$ and canceling the common integrals in the above estimate, we obtain $Y^m\in S^\infty_{\mathbb{F}}(0,T;\mathbb{R}^\ell)$. The assertion for $(\overline{Y}^m_i,\overline{Z}^m_i,\overline{\Phi}^m_i)$ in (2.10) can be similarly proved. This completes the proof. □

3. A stochastic LQ control problem with jumps and the related two-dimensional BSDEJ.

3.1. Cone-constrained stochastic LQ control with jumps. Consider the following $\mathbb{R}$-valued linear stochastic differential equation (SDE):

$$
\begin{cases}
\mathrm{d}X_t=\left[A_tX_{t-}+B_t^\top u_t\right]\mathrm{d}t+\left[C_tX_{t-}+D_tu_t\right]^\top\mathrm{d}W_t\\
\qquad\quad+\int_{\mathcal{E}}\left[E_t(e)X_{t-}+F_t(e)^\top u_t\right]\widetilde{N}(\mathrm{d}t,\mathrm{d}e),\ t\in[0,T],\\
X_0=x,
\end{cases}
\tag{3.1}
$$

where A, B, C, D are all $\mathcal{P}$-measurable processes, $E(\cdot)$, $F(\cdot)$ are $\mathcal{P}\otimes\mathcal{B}(\mathcal{E})$-measurable stochastic processes of suitable size, and $x\in\mathbb{R}$ is known.

Let Π be a given closed cone in $\mathbb{R}^m$, so if $u\in\Pi$, then $\lambda u\in\Pi$ for all $\lambda\ge 0$. It is used to represent the constraint set for controls. The class of admissible controls is defined as the set

$$
\mathcal{U}:=\Big\{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)\ \Big|\ u_t\in\Pi,\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.e.}\Big\}.
$$

If $u\in\mathcal{U}$ and A, B, C, D, E, F are bounded, then (3.1) admits a unique solution X. And we call (X,u) an admissible pair.

The cone-constrained stochastic LQ problem is stated as follows:

$$
\begin{cases}
\text{Minimize} & J(x,u)\\
\text{subject to} & (X,u)\text{ is admissible for (3.1)},
\end{cases}
\tag{3.2}
$$

where the cost functional is given as the following quadratic form:

$$
J(x,u):=\mathbb{E}\left[\int_0^T\Big(Q_tX_t^2+u_t^\top R_tu_t+2X_tS_t^\top u_t\Big)\,\mathrm{d}t+GX_T^2\right].
\tag{3.3}
$$

The associated value function is defined as

$$
V(x):=\inf_{u\in\mathcal{U}}J(x,u).
$$

Problem (3.2) is said to be solvable (at x) if there exists a control $u^*\in\mathcal{U}$ such that

$$-\infty < J(x,u^*)\le J(x,u)\quad \text{for all } u\in\mathcal{U},$$

in which case u^* is called an optimal control for problem (3.2), and the optimal value is

$$V(x)=J(x,u^*).$$

Our aim is to solve problem (3.2).

We put the following assumptions on the coefficients in this section.

Assumption 3.1 (bounded coefficients). It holds that

$$\begin{cases} A\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}),\ B\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^m),\ C\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^n),\\ D\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times m}),\ E\in L^{\infty,\nu}_{\mathcal{P}}(0,T;\mathbb{R}),\ F\in L^{\infty,\nu}_{\mathcal{P}}(0,T;\mathbb{R}^m),\\ Q\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}_+),\ R\in L^\infty_{\mathbb{F}}(0,T;\mathbb{S}^m),\ S\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^m),\ G\in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}_+). \end{cases}$$

Assumption 3.2 (standard case). It holds that $\left(\begin{smallmatrix} R & S\\ S^\top & Q\end{smallmatrix}\right)\ge 0$, and there exists a constant $\delta>0$ such that $R\ge\delta\mathbf{1}_m$, where $\mathbf{1}_m$ denotes the m-dimensional identity matrix.

Assumption 3.3 (singular case). It holds that $\left(\begin{smallmatrix} R & S\\ S^\top & Q\end{smallmatrix}\right)\ge 0$ and there exists a constant $\delta>0$ such that $G\ge\delta$ and $D^\top D+\int_{\mathcal{E}}F(e)F(e)^\top\nu(\mathrm{d}e)\ge\delta\mathbf{1}_m$.

3.2. Coupled SRE with jumps. Nowadays, it is well-known that solutions to stochastic LQ problems depend heavily on the solvability of the related SREs. We now introduce the associated SRE for our problem (3.2).[1]

For $(\omega,t,v,P_i,\Lambda,\Gamma_i)\in\Omega\times[0,T]\times\Pi\times\mathbb{R}\times\mathbb{R}^m\times L^{\infty,\nu}(\mathbb{R})$, $i=1,2$, define the following mappings:

$$\begin{aligned} H_1(\omega,t,v,P_1,P_2,\Lambda,\Gamma_1,\Gamma_2) :=\ & v^\top(R+P_1D^\top D)v+2(P_1(B+D^\top C)+D^\top\Lambda+S)^\top v\\ &+\int_{\mathcal{E}}\Big[(P_1+\Gamma_1(e))\Big(((1+E+F^\top v)^+)^2-1\Big)\\ &-2P_1(E+F^\top v)\\ &+(P_2+\Gamma_2(e))((1+E+F^\top v)^-)^2\Big]\nu(\mathrm{d}e),\\ H_2(\omega,t,v,P_1,P_2,\Lambda,\Gamma_1,\Gamma_2) :=\ & v^\top(R+P_2D^\top D)v-2(P_2(B+D^\top C)+D^\top\Lambda+S)^\top v\\ &+\int_{\mathcal{E}}\Big[(P_2+\Gamma_2(e))\Big(((-1-E+F^\top v)^-)^2-1\Big)\\ &+2P_2(-E+F^\top v)\\ &+(P_1+\Gamma_1(e))\Gamma((-1-E+F^\top v)^+)^2\Big]\nu(\mathrm{d}e), \end{aligned}$$

and set

$$H_1^*(\omega,t,P_1,P_2,\Lambda,\Gamma_1,\Gamma_2):=\inf_{v\in\Pi}H_1(\omega,t,v,P_1,P_2,\Lambda,\Gamma_1,\Gamma_2), \tag{3.4}$$

$$H_2^*(\omega,t,P_1,P_2,\Lambda,\Gamma_1,\Gamma_2):=\inf_{v\in\Pi}H_2(\omega,t,v,P_1,P_2,\Lambda,\Gamma_1,\Gamma_2). \tag{3.5}$$

[1]Please see Appendix A in our arXiv version [15] for a heuristic derivation of the SRE. See also Dong [7] for a special SRE with single jump that stems from the theory of filtration enlargement.

Remark 3.1. Similar to [17, Remark 3.1], $H_1^*(\omega,t,P_1,P_2,\Lambda,\Gamma_1,\Gamma_2)$ and $H_2^*(\omega,t,P_1,P_2,\Lambda,\Gamma_1,\Gamma_2)$ have finite values if $R+P_1D^\top D>0,\ P_1+\Gamma_1\geq 0,\ R+P_2D^\top D>0$, and $P_2+\Gamma_2\geq 0$.

The associated SRE for our problem (3.2) is given as follows:

(3.6)
$$\begin{cases}\mathrm{d}P_{1,t}=-\Big[(2A+C^\top C)P_{1,t-}+2C^\top\Lambda_{1,t}+Q+H_1^*(t,P_{1,t-},P_{2,t-},\Lambda_{1,t},\Gamma_{1,t},\Gamma_{2,t})\Big]\,\mathrm{d}t\\ \qquad\qquad +\Lambda_{1,t}^\top\,\mathrm{d}W+\int_{\mathcal{E}}\Gamma_{1,t}(e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e),\\ \mathrm{d}P_{2,t}=-\Big[(2A+C^\top C)P_{2,t-}+2C^\top\Lambda_{2,t}+Q+H_2^*(t,P_{1,t-},P_{2,t-},\Lambda_{2,t},\Gamma_{1,t},\Gamma_{2,t})\Big]\,\mathrm{d}t\\ \qquad\qquad +\Lambda_{2,t}^\top\,\mathrm{d}W+\int_{\mathcal{E}}\Gamma_{2,t}(e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e),\\ P_{1,T}=G,\ \ P_{2,T}=G,\\ R+P_{1,t}D^\top D>0,\ \ P_{1,t-}+\Gamma_{1,t}\geq 0,\ \ R+P_{2,t}D^\top D>0,\ \ P_{2,t-}+\Gamma_{2,t}\geq 0.\end{cases}$$

This is a new two-dimensional coupled nonlinear BSDEJ.

Remark 3.2. Hu and Zhou [17] studied a cone-constrained LQ problem without jumps; the associated SREs [17, equations(3.5) and (3.6)] are decoupled, so that one can solve P_1 and P_2 separately. As is well-known, P_1 and P_2 correspond to the optimal value with positive and negative initial state. When there is no jump in the model, the optimal state process does not change sign, so that only one of P_1 and P_2 is involved. Therefore, they are decoupled.

Things become notably different in models with jumps. Because of jumps, the sign of the optimal state process may switch between positive and negative values, so P_1 and P_2 are coupled together and one cannot treat them separately. So our SRE (3.6) is actually a system of coupled BSDEJs whose solvability is far from trivial compared to the decoupled BSDEJs in [17, equations (3.5) and (3.6)].

If all the coefficients in Assumption 3.1 are predictable w.r.t. the Brownian filtration, then $\Gamma_1=\Gamma_2=0$ and the SRE becomes a two-dimensional coupled BSDE without jumps. Even without jumps, the BSDE is still new and cannot be covered by existing results on multidimensional BSDEs; see, e.g., Fan, Hu, and Tang [11] and Hu and Tang [16].

Remark 3.3. If Π is symmetric, namely, $-v\in\Pi$ whenever $v\in\Pi$, then

$$H_1^*(\omega,t,P_1,P_2,\Lambda,\Gamma_1,\Gamma_2)=H_2^*(\omega,t,P_2,P_1,\Lambda,\Gamma_2,\Gamma_1),$$

and (3.6) will degenerate to one equation since $(P_1,\Lambda_1,\Gamma_1)=(P_2,\Lambda_2,\Gamma_2)$. In particular, if there is no control constraint, that is, $\Pi=\mathbb{R}^m$, then both H_1^* and H_2^* are equal to

$$\begin{aligned}&\int_{\mathcal{E}}((P+\Gamma(e))E^2+2\Gamma(e)E)\nu(\mathrm{d}e)+\bigg(P(B+D^\top C)+D^\top\Lambda+S\\&\quad+\int_{\mathcal{E}}((P+\Gamma(e))E+\Gamma(e))F\nu(\mathrm{d}e)\bigg)^\top\\&\quad\times\bigg(R+PD^\top D+\int_{\mathcal{E}}(P+\Gamma(e))FF^\top\nu(\mathrm{d}e)\bigg)^{-1}\\&\quad\times\bigg(P(B+D^\top C)+D^\top\Lambda+S+\int_{\mathcal{E}}((P+\Gamma(e))E+\Gamma(e))F\nu(\mathrm{d}e)\bigg).\end{aligned}$$

Under $\Pi = \mathbb{R}^m$, Zhang, Dong, and Meng [40] addressed the solvability of the matrix-valued SREJ under the assumption $R \geq \delta \mathbf{1}_m$ and $S \equiv 0$. By contrast, we will solve the BSDEJ (3.6) in both standard and singular cases for general cone Π.

Definition 3.1. *A stochastic process* $(P_1, P_2, \Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2) \in S^\infty_{\mathbb{F}}(0,T;\mathbb{R}^2) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{2m}) \times L^{\infty,\nu}_{\mathcal{P}}(0,T;\mathbb{R}^2)$ *is called a solution to the BSDEJ* (3.6) *if it satisfies the first and second equations of* (3.6) *as well as the third (the terminal conditions) and fourth constraints of* (3.6). *A solution is called nonnegative if* $P_{i,t} \geq 0$ for all $t \in [0,T]$ *and called uniformly positive if* $P_{i,t} \geq c$ for all $t \in [0,T]$ *for some deterministic constant* $c > 0$, $i = 1, 2$.

3.3. Existence of solution to the BSDEJ (3.6). Dong [7] constructed a solution to an SRE with single jump using two recursive systems of BSDEs driven only by Brownian motions. His decomposition approach is tailor-made in the filtration enlargement framework, hence fails in the Poisson random measure model which accommodates probably countable jumps.

Czichowsky and Schweizer [5] characterized the optimal value process of a cone-constrained MV problem in terms of a coupled system of BSDEs [5, equation(4.18)] in a semimartingale model. They claimed in [5, Remark 4.8] that "Due to the coupling term coming from $\mathfrak{h}$, the BSDE system (4.18) is very complicated. It has a nonlinear non-Lipschitz generator plus a generator with jumps, so that finding a solution by general BSDE techniques seems a formidable challenge." We now respond to this formidable challenge in the Wiener–Poisson world by providing a proof of the existence of solution to (3.6) by pure BSDE techniques.

Theorem 3.1 (existence in standard case). *Suppose Assumptions* 3.1 *and* 3.2 *hold; then the BSDEJ* (3.6) *admits a nonnegative solution* $(P_1, P_2, \Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2)$.

Proof. For $k = 1, 2, \ldots$, define maps

$$H^k_1(t, P_1, P_2, \Lambda_1, \Gamma_1, \Gamma_2) := \inf_{v \in \Pi, |v| \leq k} H_1(t, v, P_1, P_2, \Lambda_1, \Gamma_1, \Gamma_2), \tag{3.7}$$

$$H^k_2(t, P_1, P_2, \Lambda_2, \Gamma_1, \Gamma_2) := \inf_{v \in \Pi, |v| \leq k} H_2(t, v, P_1, P_2, \Lambda_2, \Gamma_1, \Gamma_2). \tag{3.8}$$

Then they are uniformly Lipschitz in $(P_1, P_2, \Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2)$ and decreasingly approach to $H^*_1(\omega, t, P_1, P_2, \Lambda_1, \Gamma_1, \Gamma_2)$ and $H^*_2(\omega, t, P_1, P_2, \Lambda_2, \Gamma_1, \Gamma_2)$, respectively, as k goes to infinity.

For each k, the BSDE

(3.9)
$$\begin{cases} \mathrm{d}P^k_{1,t} = -\Big[(2A + C^\top C)P^k_{1,t-} + 2C^\top \Lambda^k_{1,t} + Q + H^k_1(t, P^k_{1,t-}, P^k_{2,t-}, \Lambda^k_{1,t}, \Gamma^k_{1,t}, \Gamma^k_{2,t})\Big]\,\mathrm{d}t \\ \qquad\qquad + (\Lambda^k_{1,t})^\top\,\mathrm{d}W + \int_{\mathcal{E}} \Gamma^k_{1,t}(e)\widetilde{N}(\mathrm{d}t, \mathrm{d}e), \\ \mathrm{d}P^k_{2,t} = -\Big[(2A + C^\top C)P^k_{2,t-} + 2C^\top \Lambda^k_{2,t} + Q + H^k_2(t, P^k_{1,t-}, P^k_{2,t-}, \Lambda^k_{2,t}, \Gamma^k_{1,t}, \Gamma^k_{2,t})\Big]\,\mathrm{d}t \\ \qquad\qquad + (\Lambda^k_{2,t})^\top\,\mathrm{d}W + \int_{\mathcal{E}} \Gamma^k_{2,t}(e)\widetilde{N}(\mathrm{d}t, \mathrm{d}e), \\ P^k_{1,T} = G, \ P^k_{2,T} = G, \end{cases}$$

is a two-dimensional BSDEJ with a Lipschitz generator, so by [38, Lemma 2.4], it admits a unique solution $(P^k_1, P^k_2, \Lambda^k_1, \Lambda^k_2, \Gamma^k_1, \Gamma^k_2)$ such that

$$(P^k_i, \Lambda^k_i, \Gamma^k_i) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n) \times L^{2,\nu}_{\mathcal{P}}(0,T;\mathbb{R}), \ i = 1, 2.$$

From the definition of H_1^k, we have

$$\begin{aligned}
&H_1^k(t,P_1,P_2,\Lambda_1,\Gamma_1,\Gamma_2)-H_1^k(t,P_1,P_2',\Lambda_1,\Gamma_1,\Gamma_2')\\
&\quad\le \sup_{v\in\Pi,|v|\le k}\int_{\mathcal{E}}((1+E+F^\top v)^-)^2(P_2+\Gamma_2(e)-P_2'-\Gamma_2'(e))\nu(\mathrm{d}e)\\
&\quad\le c_k\int_{\mathcal{E}}(P_2+\Gamma_2(e)-P_2'-\Gamma_2'(e))^+\nu(\mathrm{d}e),
\end{aligned}$$

and

$$\begin{aligned}
&H_1^k(t,P_1,P_2,\Lambda_1,\Gamma_1,\Gamma_2)-H_1^k(t,P_1,P_2,\Lambda_1,\Gamma_1',\Gamma_2)\\
&\quad\le \sup_{v\in\Pi,|v|\le k}\int_{\mathcal{E}}(\Gamma_1(e)-\Gamma_1'(e))\Big(((1+E+F^\top v)^+)^2-1\Big)\nu(\mathrm{d}e)\\
&\quad\le \sup_{v\in\Pi,|v|\le k}\int_{\mathcal{E}}(\Gamma_1(e)-\Gamma_1'(e))((1+E+F^\top v)^+)^2\nu(\mathrm{d}e)+\int_{\mathcal{E}}|\Gamma_1(e)-\Gamma_1'(e)|\nu(\mathrm{d}e)\\
&\quad\le c_k\int_{\mathcal{E}}(\Gamma_1(e)-\Gamma_1'(e))^+\nu(\mathrm{d}e)+\int_{\mathcal{E}}|\Gamma_1(e)-\Gamma_1'(e)|\nu(\mathrm{d}e),
\end{aligned}$$

where $c_k<\infty$ is defined as

$$c_k=\operatorname*{ess\,sup}_{v\in\Pi,|v|\le k}|1+E+F^\top v|^2,\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\otimes\mathrm{d}\nu\text{−a.e.}$$

Similar estimates for H_2^k can be established. Hence according to Theorem 2.2, P_i^k is decreasing in k for $i=1,2$.

Next, we will show that the sequence $\{P_i^k\}_{k=1,2,\dots}$ is nonnegative and uniformly bounded from above for $i=1,2$.

From Assumption 3.1, there exists a constant $c>0$ such that

$$2A+C^\top C+\int_{\mathcal{E}}E(e)^2\nu(\mathrm{d}e)\le c,\ Q\le c,\ G\le c.$$

It is easy to check that $(\overline{P}_t,\overline{\Lambda}_t,\overline{\Gamma}_t)=((c+1)e^{c(T-t)}-1,0,0)$ is the unique solution to the one-dimensional BSDEJ

$$\begin{cases}\mathrm{d}\overline{P}_t=-(c\overline{P}_t+c)\,\mathrm{d}t+\overline{\Lambda}_t^\top\,\mathrm{d}W_t+\int_{\mathcal{E}}\overline{\Gamma}_t(e)\tilde{N}(\mathrm{d}t,\mathrm{d}e),\\ \overline{P}_T=c.\end{cases}\tag{3.10}$$

By the definition of H_1^k, we have

$$\begin{aligned}
H_1^k(t,\overline{P}_t,\overline{P}_t,0,0,0)&\le H_1(t,0,\overline{P}_t,\overline{P}_t,0,0,0)\\
&=\overline{P}_t\int_{\mathcal{E}}E_t^2(e)\nu(\mathrm{d}e),
\end{aligned}$$

so

$$\begin{aligned}
&(2A+C^\top C)\overline{P}_t+2C^\top\overline{\Lambda}_t+Q+H_1^k(t,\overline{P}_t,\overline{P}_t,\overline{\Lambda}_t,\overline{\Gamma}_t,\overline{\Gamma}_t)\\
&\quad=(2A+C^\top C)\overline{P}_t+Q+H_1^k(t,\overline{P}_t,\overline{P}_t,0,0,0)\\
&\quad\le c\overline{P}_t+c.
\end{aligned}$$

Similarly, we have

$$(2A+C^\top C)\overline{P}_t+2C^\top\overline{\Lambda}_t+Q+H_2^k(t,\overline{P}_t,\overline{P}_t,\overline{\Lambda}_t,\overline{\Gamma}_t,\overline{\Gamma}_t)\le c\overline{P}_t+c.$$

With the above two inequalities in hand, applying Theorem 2.2 to BSDEJs (3.9) and (3.10), we have for $i=1,2,\ k=1,2,\dots$

$$P^k_{i,t}\le \overline{P}_{i,t}\le M, \tag{3.11}$$

where $M:=(c+1)e^{cT}-1$.

On the other hand, notice that $(\underline{P}_{1,t},\underline{\Lambda}_{1,t},\underline{\Gamma}_{1,t})=(\underline{P}_{2,t},\underline{\Lambda}_{2,t},\underline{\Gamma}_{2,t}):=(0,0,0)$ satisfies

$$\begin{cases}\mathrm{d}\underline{P}=\underline{\Lambda}^\top\,\mathrm{d}W+\int_{\mathcal{E}}\underline{\Gamma}(e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e),\\ \underline{P}_T=0,\end{cases}$$

and

$$\begin{aligned}&(2A+C^\top C)\underline{P}_1+2C^\top\underline{\Lambda}_1+Q+H^k_1(t,\underline{P}_1,\underline{P}_2,\underline{\Lambda}_1,\underline{\Gamma}_1,\underline{\Gamma}_2)\\ &\quad\ge Q+\inf_{v\in\mathbb{R}^m}(v^\top Rv+2S^\top v)=Q-S^\top R^{-1}S\ge 0,\end{aligned}$$

thanks to Assumption 3.2. Hence, by Theorem 2.2 again,

$$P^k_{i,t}\ge \underline{P}_t=0,\ i=1,2,\ k=1,2,\dots. \tag{3.12}$$

Notice, for $i=1,2$,

$$\begin{aligned}\mathbb{E}\int_0^T\int_{\mathcal{E}}\mathbf{1}_{\{P^k_{i,t-}+\Gamma^k_{i,t}(e)<0\}}\nu(\mathrm{d}e)\,\mathrm{d}t&=\mathbb{E}\int_0^T\int_{\mathcal{E}}\mathbf{1}_{\{P^k_{i,t-}+\Gamma^k_{i,t}(e)<0\}}N(\mathrm{d}t,\mathrm{d}e)\\ &=\mathbb{E}\left[\sum_{n\in\mathbb{N},T_n\le T}\mathbf{1}_{\{P^k_{i,T_n-}+\Gamma^k_{i,T_n}(\Delta U_{T_n})<0\}}\right]\\ &=\mathbb{E}\left[\sum_{n\in\mathbb{N},T_n\le T}\mathbf{1}_{\{P^k_{i,T_n}<0\}}\right]=0,\end{aligned}$$

where $U_t=\int_0^t\int_{\mathcal{E}}eN(\mathrm{d}s,\mathrm{d}e)$ and $\Delta U_{T_n}=U_{T_n}-U_{T_n-}$, hence,

$$P^k_{i,t-}+\Gamma^k_{i,t}\ge 0. \tag{3.13}$$

Similarly, we can establish

$$P^k_{i,t-}+\Gamma^k_{i,t}\le M. \tag{3.14}$$

Now we obtain

$$-M\le -P^k_{i,t-}\le\Gamma^k_{i,t}\le M-P^k_{i,t-}\le M.$$

Hence, Γ^k_i, $k=1,2,\dots$, are uniformly bounded by M and thus belong to $L^{\infty,\nu}_{\mathcal{P}}(0,T;\mathbb{R})$.

Since P^k_i is decreasing w.r.t. k, we can define $P_{i,t}:=\lim_{k\to\infty}P^k_{i,t},\ i=1,2$. Combining (3.11) and (3.12), it follows that

$$0\le P_{i,t}\le M,\ i=1,2,\ t\in[0,T].$$

Applying Itô's formula to $(P^k_{1,t})^2$, we deduce that

$$\begin{cases}\mathrm{d}(P^k_{1,t})^2=\Big\{-2P^k_1\Big[(2A+C^\top C)P^k_{1,t-}+2C^\top\Lambda^k_{1,t}+Q\\ \qquad\qquad +H^k_1(t,P^k_{1,t-},P^k_{2,t-},\Lambda^k_{1,t},\Gamma^k_{1,t},\Gamma^k_{2,t})\Big]+|\Lambda^k_1|^2+\int_{\mathcal{E}}\Gamma^k_1(e)^2\nu(\mathrm{d}e)\Big\}\,\mathrm{d}t\\ \qquad\qquad +2P^k_1(\Lambda^k_1)^\top\,\mathrm{d}W+\int_{\mathcal{E}}[(P^k_{1,t-}+\Gamma^k_{1,t}(e))^2-(P^k_{1,t-})^2]\widetilde{N}(\mathrm{d}t,\mathrm{d}e),\\ (P^k_{1,T})^2=G^2.\end{cases}$$

Since $0 \le P_i^k, P_i^k + \Gamma_i^k \le M, i = 1, 2$, and

$$H_1^k \le \int_{\mathcal{E}} \Big[(P_1 + \Gamma_1(e))\big(((1+E)^+)^2 - 1\big) - 2P_1 E + (P_2 + \Gamma_2(e))((1+E)^-)^2\Big]\nu(\mathrm{d}e) \le c,$$

by taking expectation on both sides in the above and integrating over $[0,T]$, we have

$$(P_{1,0}^k)^2 + \frac{1}{2}\mathbb{E}\int_0^T |\Lambda_1^k|^2\,\mathrm{d}s + \mathbb{E}\int_0^T \int_{\mathcal{E}} \Gamma_1^k(e)^2 \nu(\mathrm{d}e)\,\mathrm{d}s \le c, \tag{3.15}$$

where $c > 0$ is a constant independent of k. Therefore, the sequence $(\Lambda_1^k, \Gamma_1^k)$, $k = 1, 2, \ldots$, is bounded in $L^2_{\mathbb{F}}(0,T;\mathbb{R}^n) \times L^{2,\nu}_{\mathcal{P}}(0,T;\mathbb{R})$, and thus we can extract a subsequence (which is still denoted by $(\Lambda_1^k, \Gamma_1^k)$) converging in the weak sense to some $(\Lambda_1, \Gamma_1) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n) \times L^{2,\nu}_{\mathcal{P}}(0,T;\mathbb{R})$. Similar considerations applying to $(P_{2,t}^k)^2$ yield some $(\Lambda_2, \Gamma_2) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n) \times L^{2,\nu}_{\mathcal{P}}(0,T;\mathbb{R})$ which is the weak limit of $(\Lambda_2^k, \Gamma_2^k)$.

Following Kobylanski's argument [21, Proposition 2.4] (see also Antonelli and Mancini [1, Theorem 1] and Kohlmann and Tang [20, Theorem 2.1]), we can establish the following strong convergence.[2]

LEMMA 3.1. *It holds that*

$$\lim_{k\to\infty} \mathbb{E}\int_0^T |\Lambda_i^k - \Lambda_i|^2\,\mathrm{d}t = 0, \ \lim_{k\to\infty} \mathbb{E}\int_0^T \int_{\mathcal{E}} |\Gamma_i^k - \Gamma_i|^2 \nu(\mathrm{d}e)\,\mathrm{d}t = 0, \ i = 1, 2. \tag{3.16}$$

Furthermore, $(P_1, P_2, \Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2)$ *is a nonnegative solution to the BSDEJ* (3.6)*.*

This completes the proof. ∎

THEOREM 3.2 (existence in singular case). *Suppose Assumptions* 3.1 *and* 3.3 *hold; then the BSDEJ* (3.6) *admits a uniformly positive solution* $(P_1, P_2, \Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2)$.

Proof. Similar to the proof of Theorem 3.1, one can show the existence of a nonnegative solution $(P_1, P_2, \Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2)$ to the BSDEJ (3.6), so we omit the details. We only give a sketch on how to find a uniformly positive lower bound for such a solution.

Under Assumptions 3.1 and 3.3, there exists constant $c_2 > 0$ such that

$$2A + C^\top C + \int_{\mathcal{E}} E(e)^2 \nu(\mathrm{d}e) - \delta^{-1}\Big|B + D^\top C \pm \int_{\mathcal{E}} E(e)F(e)\nu(\mathrm{d}e)\Big|^2 \ge -c_2,$$

where δ is the constant in Assumption 3.3.

Notice that $(\underline{P}_t, \underline{\Lambda}_t, \underline{\Gamma}_t) = (\delta e^{-c_2(T-t)}, 0, 0)$ is the unique solution to the one-dimensional BSDEJ

$$\begin{cases} \mathrm{d}\underline{P} = -(-c_2\underline{P})\,\mathrm{d}t + \underline{\Lambda}^\top\,\mathrm{d}W + \int_{\mathcal{E}} \underline{\Gamma}(e)\widetilde{N}(\mathrm{d}t, \mathrm{d}e), \\ \underline{P}_T = \delta. \end{cases} \tag{3.17}$$

[2] Please refer to Appendix B in our arXiv version [15] for the proof.

By the definitions of H_1^k, we have

$$\begin{aligned}
&H_1^k(t,\underline{P},\underline{P},0,0,0)\\
&\quad\geq \inf_{v\in\mathbb{R}^m} H_1(t,v,\underline{P},\underline{P},0,0,0)\\
&\quad\geq \inf_{v\in\mathbb{R}^m}\Big[v^\top Rv+2S^\top v\Big]+\underline{P}\int_{\mathcal{E}}E(e)^2\nu(\mathrm{d}e)\\
&\qquad+\underline{P}\inf_{v\in\mathbb{R}^m}\Bigg[v^\top(D^\top D+\int_{\mathcal{E}}F(e)F(e)^\top\nu(\mathrm{d}e))v+2\Big(B+D^\top C+\int_{\mathcal{E}}E(e)F(e)\nu(\mathrm{d}e)\Big)^\top v\Bigg]\\
&\quad\geq -Q+\underline{P}\Bigg[\int_{\mathcal{E}}E(e)^2\nu(\mathrm{d}e)-\delta^{-1}\Big|B+D^\top C+\int_{\mathcal{E}}E(e)F(e)\nu(\mathrm{d}e)\Big|^2\Bigg],
\end{aligned}$$

where we used

$$\begin{pmatrix} R & S\\ S^\top & Q\end{pmatrix}\geq 0,\quad D^\top D+\int_{\mathcal{E}}F(e)F(e)^\top\nu(\mathrm{d}e)\geq\delta\mathbf{1}_m$$

in the last inequality. So

$$\begin{aligned}
&(2A+C^\top C)\underline{P}_t+2C^\top\underline{\Lambda}_t+Q+H_1^k(t,\underline{P}_t,\underline{P}_t,\underline{\Lambda}_t,\underline{\Gamma}_t,\underline{\Gamma}_t)\\
&\quad=(2A+C^\top C)\underline{P}_t+Q+H_1^k(t,\underline{P}_t,\underline{P}_t,0,0,0)\\
&\quad\geq -c_2\underline{P}_t.
\end{aligned}$$

Similarly, we have

$$(2A+C^\top C)\underline{P}_t+2C^\top\underline{\Lambda}_t+Q+H_2^k(t,\underline{P}_t,\underline{P}_t,\underline{\Lambda}_t,\underline{\Gamma}_t,\underline{\Gamma}_t)\geq -c_2\underline{P}_t.$$

Then applying Theorem 2.2 to BSDEJs (3.9) and (3.17), we have for $i=1,2,\ k=1,2,\ldots,$

$$P_{i,t}^k\geq\underline{P}_t=\delta e^{-c_2(T-t)}\geq\delta e^{-c_2T},\ t\in[0,T],$$

which leads to the desired lower bound. ∎

3.4. Solution to the LQ problem (3.2). In this subsection we will present an explicit solution to the LQ problem (3.2) in terms of solutions to the BSDEJ (3.6).

For $P_i>0$, $\Lambda_i\in\mathbb{R}^n$, $\Gamma_i\in L^{2,\nu}(\mathbb{R})$ satisfying $R+P_{i,t}D^\top D>0,\ P_{i,t-}+\Gamma_{i,t}\geq 0,\ i=1,2$, we define

$$\begin{aligned}
&\hat{v}_1(t,P_1,P_2,\Lambda_1,\Gamma_1,\Gamma_2)=\operatorname*{argmin}_{v\in\Pi}H_1(t,v,P_1,P_2,\Lambda_1,\Gamma_1,\Gamma_2),\\
&\hat{v}_2(t,P_1,P_2,\Lambda_2,\Gamma_1,\Gamma_2)=\operatorname*{argmin}_{v\in\Pi}H_2(t,v,P_1,P_2,\Lambda_2,\Gamma_1,\Gamma_2).
\end{aligned}\tag{3.18}$$

Theorem 3.3. *Let* $(P_i,\Lambda_i,\Gamma_i)\in S_{\mathbb{F}}^\infty(0,T;\mathbb{R})\times L_{\mathbb{F}}^2(0,T;\mathbb{R}^m)\times L_{\mathcal{P}}^{\infty,\nu}(0,T;\mathbb{R}),\ i=1,2$, *be a nonnegative (in the standard case) or uniformly positive (in the singular case) solution to the BSDEJ* (3.6). *Then the state feedback control*

$$u^*(t,X)=\hat{v}_1(t,P_{1,t-},P_{2,t-},\Lambda_{1,t},\Gamma_{1,t},\Gamma_{2,t})X_{t-}^+ +\hat{v}_2(t,P_{1,t-},P_{2,t-},\Lambda_{1,t},\Gamma_{1,t},\Gamma_{2,t})X_{t-}^-$$

is optimal for the LQ problem (3.2). *Moreover, the optimal value is*

$$V(x)=P_{1,0}(x^+)^2+P_{2,0}(x^-)^2.$$

As a by-product of Theorem 3.3, we have the following uniqueness of the solution for BSDEJ (3.6).

THEOREM 3.4. *Suppose Assumptions* 3.1 *and* 3.2 *(resp., Assumptions* 3.1 *and* 3.3*) hold; then the BSDEJ* (3.6) *admits at most one nonnegative (resp., uniformly positive) solution.*

The proofs of Theorems 3.3 and 3.4 are standard and thus omitted here; please see [17, Theorems 5.1, 5.2] or [40, Theorems 5.1, 5.2] for the standard verification argument. It seems a challenging task to prove Theorem 3.4 by pure BSDE techniques.

4. Application to mean-variance portfolio selection problem. Consider a financial market consisting of a risk-free asset (the money market instrument or bond) whose price is S_0 and m risky securities (the stocks) whose prices are $S_1,\dots,S_m$. And assume $m\le n$, i.e., the number of risky securities is no more than the dimension of the Brownian motion. The asset prices S_k, $k=0,1,\dots,m$, are driven by SDEs,

$$\begin{cases}\mathrm{d}S_{0,t}=r_tS_{0,t}\,\mathrm{d}t,\\ S_{0,0}=s_0,\end{cases}$$

and

$$\begin{cases}\mathrm{d}S_{k,t}=S_{k,t}\left((\mu_{k,t}+r_t)\,\mathrm{d}t+\sum_{j=1}^{n}\sigma_{kj,t}\,\mathrm{d}W_{j,t}+\int_{\mathcal{E}}F_{k,t}(e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e)\right),\\ S_{k,0}=s_k,\end{cases}$$

where, for every $k=1,\dots,m$, r is the interest rate process, and μ_k, $\sigma_k:=(\sigma_{k1},\dots,\sigma_{kn})$, and F_k are the mean excess return rate process and volatility rate process of the kth risky security.

Define the vectors $\mu=(\mu_1,\dots,\mu_m)^\top$, $F=(F_1,\dots,F_m)^\top$ and matrix

$$\sigma=\begin{pmatrix}\sigma_1\\ \vdots\\ \sigma_m\end{pmatrix}\equiv(\sigma_{kj})_{m\times n}\ \text{for each } i\in\mathcal{M}.$$

We shall assume, in this section, as follows.

Assumption 4.1. The interest rate r is a bounded deterministic measurable function of t,

$$\mu\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^m),\ \sigma\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{m\times n}),\ F\in L^{\nu,\infty}_{\mathcal{P}}(0,T;\mathbb{R}^m),$$

and there exists a constant $\delta>0$ such that $\sigma\sigma^\top+\int_{\mathcal{E}}F(e)F(e)^\top\nu(\mathrm{d}e)\ge\delta\mathbf{1}_m$ for all $t\in[0,T]$.

A small investor, whose actions cannot affect the asset prices, will decide at every time $t\in[0,T]$ the amount $\pi_{j,t}$ of his wealth to invest in the jth risky asset, $j=1,\dots,m$. The vector process $\pi:=(\pi_1,\dots,\pi_m)^\top$ is called a portfolio of the investor. Then the investor's self-financing wealth process X corresponding to a portfolio π is the unique strong solution of the SDE:

$$\begin{cases}\mathrm{d}X_t=[r_tX_{t-}+\pi_t^\top\mu_t]\,\mathrm{d}t+\pi_t^\top\sigma_t\,\mathrm{d}W_t+\int_{\mathcal{E}}\pi_t^\top F_t(e)\widetilde{N}(\mathrm{d}t,\mathrm{d}e),\\ X_0=x.\end{cases}\tag{4.1}$$

The admissible portfolio set is defined as

$$\mathcal{U}=\left\{\pi\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)\,\middle|\,\pi_t\in\Pi\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.e.}\right\},$$

where $\Pi \in \mathbb{R}^m$ is a given closed convex cone. For instance, $\Pi = \mathbb{R}^m$ means there is no trading constraint, while $\Pi = \mathbb{R}^m_+$ means shorting is not allowed in the market. For any $\pi \in \mathcal{U}$, the SDE (4.1) has a unique strong solution. Different from the previous sections, in this section we request the constraint set Π to be convex in order to apply the dual approach below.

For a given expectation level $z \geq xe^{\int_0^T r_s \, \mathrm{d}s}$, the investor's MV problem is to

$$
\text{(4.2)} \qquad \begin{aligned} &\text{Minimize} \quad \operatorname{Var}(X_T) \equiv \mathbb{E}[X_T^2 - z^2], \\ &\text{s.t.} \quad \begin{cases} \mathbb{E}[X_T] = z, \\ \pi \in \mathcal{U}. \end{cases} \end{aligned}
$$

Remark 4.1. Lim [24] studied an MV problem with jumps without portfolio constraints, i.e., $\Pi = \mathbb{R}^m$. In his model, all the coefficients in (4.1) are assumed to be predictable w.r.t. the Brownian motion filtration, so no jump term has entered into his SRE, which is exactly the same one as in the model without jumps.

We shall say that the MV problem (4.2) is feasible for a given level $z \geq xe^{\int_0^T r_s \, \mathrm{d}s}$ if there is a portfolio $\pi \in \mathcal{U}$ which satisfies the target constraint $\mathbb{E}[X_T] = z$. An optimal portfolio to (4.2) is called an efficient portfolio corresponding to z and the corresponding $(\sqrt{\operatorname{Var}(X_T)}, z)$ is called an efficient point. The set of all efficient points, with $z \geq xe^{\int_0^T r_s \, \mathrm{d}s}$, is called the efficient frontier.

Define the dual cone of Π as

$$
\widehat{\Pi} := \left\{ y \in \mathbb{R}^m \,\middle|\, x^\top y \leq 0 \text{ for all } x \in \Pi \right\}.
$$

The following result gives an equivalent condition for the feasibility of (4.2). The proof is exactly the same as [13, Theorem 5.3], so we omit it.

THEOREM 4.1 (feasibility). *Under Assumption* 4.1, *the MV problem* (4.2) *is feasible for any* $z \geq xe^{\int_0^T r_t \, \mathrm{d}t}$ *if and only if*

$$
\text{(4.3)} \qquad \int_0^T \mathbb{P}(\mu_t \notin \widehat{\Pi}) \, \mathrm{d}t > 0.
$$

For the rest of this section, we will always assume (4.3) holds.

The way to solve (4.2) is rather clear nowadays. To deal with the constraint $\mathbb{E}[X_T] = z$, we introduce a Lagrange multiplier $-2\lambda \in \mathbb{R}$ and obtain the following *relaxed* optimization problem:

$$
\text{(4.4)} \qquad \begin{aligned} &\inf \quad J(x, \pi, \lambda; z) = \mathbb{E}[(X_T - \lambda)^2] - (\lambda - z)^2, \\ &\text{s.t.} \quad \pi \in \mathcal{U}. \end{aligned}
$$

Denote its optimal value as

$$
V(x, \lambda; z) = \inf_{\pi \in \mathcal{U}} J(x, \pi, \lambda; z).
$$

According to the Lagrange duality theorem (see Luenberger [25])

$$
\text{(4.5)} \qquad \inf_{\pi \in \mathcal{U}, \mathbb{E}[X_T] = z} \operatorname{Var}(X_T) = \sup_{\lambda \in \mathbb{R}} V(x, \lambda; z).
$$

So we can solve the problem (4.4) by a two-step procedure: First determine $V(x, \lambda; z)$ for every λ, and then try to find a λ^* to maximize $\lambda \mapsto V(x, \lambda; z)$.

The relaxed problem (4.4) is a special stochastic LQ problem (3.2) studied in section 3, where

$$A = r,\ B = \mu,\ C = 0,\ D = \sigma^\top,\ E = 0,\ Q = 0,\ R = 0,\ S = 0,\ G = 1. \tag{4.6}$$

The associated BSDEJ (3.6) becomes

$$\begin{cases} \mathrm{d}P_{1,t} = -\Big[2rP_{1,t-} + H_1^*(t, P_{1,t-}, P_{2,t-}, \Lambda_{1,t}, \Gamma_{1,t}, \Gamma_{2,t})\Big]\,\mathrm{d}t + \Lambda_{1,t}^\top\,\mathrm{d}W \\ \qquad\qquad + \displaystyle\int_{\mathcal{E}} \Gamma_{1,t}(e)\widetilde{N}(\mathrm{d}t, \mathrm{d}e), \\ \mathrm{d}P_{2,t} = -\Big[2rP_{2,t-} + H_2^*(t, P_{1,t-}, P_{2,t-}, \Lambda_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})\Big]\,\mathrm{d}t + \Lambda_{2,t}^\top\,\mathrm{d}W \\ \qquad\qquad + \displaystyle\int_{\mathcal{E}} \Gamma_{2,t}(e)\widetilde{N}(\mathrm{d}t, \mathrm{d}e), \\ P_{1,T} = 1,\ P_{2,T} = 1, \end{cases} \tag{4.7}$$

where H_1^*, H_2^*, $\hat{v}_1$, $\hat{v}_2$ are defined as in (3.4), (3.5), and (3.18) with coefficients given in (4.6):

$$\begin{aligned} H_1(t, v, P_1, P_2, \Lambda_1, \Gamma_1, \Gamma_2) &= P_1 v^\top \sigma\sigma^\top v + 2(P_1\mu + \sigma\Lambda_1)^\top v \\ &\quad + \int_{\mathcal{E}} \Big[(P_1 + \Gamma_1(e))\Big(((1 + F^\top v)^+)^2 - 1\Big) - 2P_1 F^\top v \\ &\quad + (P_2 + \Gamma_2(e))((1 + F^\top v)^-)^2\Big]\nu(\mathrm{d}e), \\ H_2(t, v, P_1, P_2, \Lambda_2, \Gamma_1, \Gamma_2) &= P_2 v^\top \sigma\sigma^\top v - 2(P_2\mu + \sigma\Lambda_2)^\top v \\ &\quad + \int_{\mathcal{E}} \Big[(P_2 + \Gamma_2(e))\Big(((-1 + F^\top v)^-)^2 - 1\Big) + 2P_2 F^\top v \\ &\quad + (P_1 + \Gamma_1(e))((-1 + F^\top v)^+)^2\Big]\nu(\mathrm{d}e). \end{aligned}$$

Clearly, Theorems 3.2 and 3.4 can be applied to the BSDEJ (4.7) to ensure that it admits a unique uniformly positive solution $(P_1, P_2, \Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2)$. Accordingly, Theorem 3.3 leads to the following solution to the relaxed problem (4.4).

THEOREM 4.2. *Let* $(P_1, P_2, \Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2)$ *be the unique uniformly positive solution to* (4.7). *Then the state feedback control given by*

$$\begin{aligned} \pi^*(t, X) &= \hat{v}_1(t, P_1, P_2, \Lambda_1, \Gamma_1, \Gamma_2)\Big(X_{t-} - \lambda e^{-\int_t^T r_s\,\mathrm{d}s}\Big)^+ \\ &\quad + \hat{v}_2(t, P_1, P_2, \Lambda_2, \Gamma_1, \Gamma_2)\Big(X_{t-} - \lambda e^{-\int_t^T r_s\,\mathrm{d}s}\Big)^- \end{aligned} \tag{4.8}$$

is optimal for the LQ problem (3.2). *Moreover, the optimal value is*

$$V(x, \lambda; z) = P_{1,0}\Big[\Big(x - \lambda e^{-\int_0^T r_s\,\mathrm{d}s}\Big)^+\Big]^2 + P_{2,0}\Big[\Big(x - \lambda e^{-\int_0^T r_s\,\mathrm{d}s}\Big)^-\Big]^2 - (\lambda - z)^2.$$

This resolves the first step problem. To solve the second step problem, i.e., to maximize $\lambda \mapsto V(x, \lambda; z)$, the following result is critical.

LEMMA 4.1. *Assume Assumption* 4.1 *and condition* (4.3) *hold. Then*

$$P_{1,0}e^{-2\int_0^T r_s\,\mathrm{d}s} - 1 \le 0,\ P_{2,0}e^{-2\int_0^T r_s\,\mathrm{d}s} - 1 < 0. \tag{4.9}$$

Proof. Applying Itô's formula to $P_{2,t}e^{-2\int_t^T r_s\,\mathrm{d}s}$ on $[0,T]$, we have

$$1 - P_{2,0}e^{-2\int_0^T r_s\,\mathrm{d}s} = -\mathbb{E}\int_0^T e^{-2\int_t^T r_s\,\mathrm{d}s} H_2^*(t, P_{2,t-}, \Lambda_{2,t}, \Gamma_{2,t}, P_{1,t-}, \Gamma_{1,t})\,\mathrm{d}t. \tag{4.10}$$

Since $H_2^*(t, P_{2,t-}, \Lambda_{2,t}, \Gamma_{2,t}, P_{1,t-}, \Gamma_{1,t}) \le 0$ by its very definition, it follows $P_{2,0}e^{-2\int_0^T r_s\,\mathrm{d}s} - 1 \le 0$. Similarly, we can prove that $P_{1,0}e^{-2\int_0^T r_s\,\mathrm{d}s} - 1 \le 0$.

It remains to prove the strict inequality $P_{2,0}e^{-2\int_0^T r_s\,\mathrm{d}s} - 1 < 0$. Suppose, on the contrary, $P_{2,0}e^{-2\int_0^T r_s\,\mathrm{d}s} - 1 = 0$. It then follows from (4.10) that

$$H_2^*(t, P_{1,t-}, P_{2,t-}, \Lambda_{2,t}, \Gamma_{1,t}, \Gamma_{2,t}) = 0 \ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.e.} \tag{4.11}$$

Thus we deduce, from the uniqueness (Theorem 3.4) of solution to the BSDE (4.7), that $P_{2,t} = e^{2\int_t^T r_s\,\mathrm{d}s}$, $\Lambda_{2,t} = 0$, and $\Gamma_{2,t} = 0$.

On the other hand,

$$((1 - F^\top v)^+)^2 + 2F^\top v - 1 \le (1 - F^\top v)^2 + 2F^\top v - 1 = |F^\top v|^2. \tag{4.12}$$

Since $F \in L_{\mathcal{P}}^{\nu,\infty}(0,T;\mathbb{R}^m)$, there exists $c_1 > 0$ such that $|F(e)| \le c_1$ for almost all $e \in \mathcal{E}$. Hence

$$1 - F^\top v \ge 0 \text{ if } v \in \Pi \text{ and } \ |v| \le c_1^{-1}. \tag{4.13}$$

Combining (4.12) and (4.13), we have

$$\begin{aligned}
H_2^*(t, P_1, P_2, 0, \Gamma_1, 0) &= \inf_{v\in\Pi}\Bigg[P_2 v^\top\sigma\sigma^\top v - 2P_2\mu^\top v + \int_{\mathcal{E}}\Bigg[P_2\Big(((1 - F^\top v)^+)^2 + 2F^\top v - 1\Big) \\
&\qquad + (P_1 + \Gamma_1(e))((1 - F^\top v)^-)^2\Bigg]\nu(\mathrm{d}e)\Bigg] \\
&\le \inf_{v\in\Pi, |v|\le c_1^{-1}}\Bigg[P_2 v^\top\Bigg(\sigma\sigma^\top + \int_{\mathcal{E}} FF^\top\nu(\mathrm{d}e)\Bigg)v - 2P_2\mu^\top v\Bigg] \\
&\le P_2 \inf_{v\in\Pi, |v|\le c_1^{-1}}\Big(c|v|^2 - 2\mu^\top v\Big).
\end{aligned} \tag{4.14}$$

Note (4.3) implies that there exists a $\mathcal{P}$-measurable set $\mathcal{O} \subseteq \Omega\times[0,T]$ satisfying $\int_0^T\int_\Omega \mathbf{1}_{(\omega,t)\in\mathcal{O}}\mathbb{P}(\mathrm{d}\omega)\,\mathrm{d}t > 0$ such that $\mu \notin \widehat{\Pi}$ on $\mathcal{O}$. For each $(\omega,t)\in\mathcal{O}$, there exists $v_0(\omega,t)\in\Pi$ such that $\mu_t^\top(\omega)v_0(\omega,t) > 0$. Clearly, $v_0(\omega,t) \ne 0$, so we can set $v_1(\omega,t) = \varepsilon(\omega,t)v_0(\omega,t)/|v_0(\omega,t)|$ with

$$\varepsilon(\omega,t) = \frac{1}{2}\min\Big\{c_1^{-1}, \frac{2\mu_t^\top(\omega)v_0(\omega,t)}{c|v_0(\omega,t)|}\Big\}.$$

Note $0 < \varepsilon(\omega,t) < c_1^{-1}$. Since Π is a cone and $v_0(\omega,t)\in\Pi$, we have $v_1(\omega,t)\in\Pi$ and $|v_1(\omega,t)| < c_1^{-1}$. Hence,

$$\begin{aligned}
\inf_{v\in\Pi, |v|\le c_1^{-1}}\Big(c|v|^2 - 2\mu_t^\top(\omega)v\Big) &\le c|v_1(\omega,t)|^2 - 2\mu_t^\top(\omega)v_1(\omega,t) \\
&= c\varepsilon(\omega,t)^2 - 2\varepsilon(\omega,t)\frac{\mu_t^\top(\omega)v_0(\omega,t)}{|v_0(\omega,t)|} \\
&= c\varepsilon(\omega,t)\Big(\varepsilon(\omega,t) - \frac{2\mu_t^\top(\omega)v_0(\omega,t)}{c|v_0(\omega,t)|}\Big) < 0.
\end{aligned}$$

This together with (4.14) and $P_2 > 0$ implies

$$H_2^*(\omega, t, P_1, P_2, 0, \Gamma_1, 0) < 0 \text{ for each } (\omega, t) \in \mathcal{O}.$$

Since $\mathcal{O}$ has a positive measure under $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$, it contradicts (4.11). Therefore, $P_{2,0}e^{-2\int_0^T r_s\,\mathrm{d}s} - 1 < 0$. ∎

To find a λ^* to maximize $\lambda \mapsto V(x, \lambda; z)$, we do some tedious calculation (using (4.9)) and obtain

$$\max_{\lambda} V(x, \lambda; z) = V(x, \lambda^*; z) = \frac{P_{2,0}}{1 - P_{2,0}e^{-2\int_0^T r_s\,\mathrm{d}s}} \Big(x - ze^{-\int_0^T r_s\,\mathrm{d}s}\Big)^2,$$

where

$$\lambda^* = \frac{z - xP_{2,0}e^{-\int_0^T r_s\,\mathrm{d}s}}{1 - P_{2,0}e^{-2\int_0^T r_s\,\mathrm{d}s}}.$$

The above analysis boils down to the following solution to the MV problem (4.2).

THEOREM 4.3. *Let* $(P_1, \Lambda_1, \Gamma_1, P_2, \Lambda_2, \Gamma_2)$ *be the unique uniformly positive solution to* (4.7). *Then the state feedback portfolio given by*

$$\begin{aligned} \pi^*(t, X) = {} & \hat{v}_1(t, P_1, P_2, \Lambda_1, \Gamma_1, \Gamma_2)\Big(X_{t-} - \lambda^* e^{-\int_t^T r_s\,\mathrm{d}s}\Big)^+ \\ & + \hat{v}_2(t, P_1, P_2, \Lambda_1, \Gamma_2, \Gamma_2)\Big(X_{t-} - \lambda^* e^{-\int_t^T r_s\,\mathrm{d}s}\Big)^- \end{aligned} \tag{4.15}$$

is optimal to the MV problem (4.2). *Moreover, the efficient frontier is determined by*

$$\mathrm{Var}(X_T) = \frac{P_{2,0}e^{-2\int_0^T r_s\,\mathrm{d}s}}{1 - P_{2,0}e^{-2\int_0^T r_s\,\mathrm{d}s}} \Big(\mathbb{E}[X_T] - xe^{\int_0^T r_s\,\mathrm{d}s}\Big)^2,$$

where $\mathbb{E}[X_T] \geq xe^{\int_0^T r_s\,\mathrm{d}s}$.

Remark 4.2. In the constrained MV model without jumps studied in Hu and Zhou [17], the efficient portfolio only takes the second term on the RHS of (4.15), i.e., the optimal wealth X_t will never exceed $\lambda^* e^{-\int_t^T r_s\,\mathrm{d}s}$ on $[0, T]$, and it only depends on (P_2, Λ_2) as $\hat{v}_2$ does. But in our cone-constrained MV problem with jumps, the optimal wealth X_t will probably cross the bliss point $\lambda^* e^{-\int_t^T r_s\,\mathrm{d}s}$.

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