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Diffusion limit with optimal convergence rate of classical solutions to the Vlasov-Maxwell-Boltzmann system

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A B S T R A C T

We study the diffusion limit of the classical solution to the Vlasov-Maxwell-Boltzmann (VMB) system with initial data near a global Maxwellian. By introducing a new decomposition of the solution to identify the essential components for generating the initial layer, we prove the convergence and establish the optimal convergence rate of the classical solution to the VMB system to the solution of the Navier-Stokes-Maxwell system based on the spectral analysis.

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Contents

1. Introduction

The Vlasov-Maxwell-Boltzmann (VMB) system is a fundamental model in plasma physics describing the time evolution of dilute charged particles, such as electrons and ions, under the influence of the self-induced Lorentz forces governed by Maxwell equations, cf. [7] for derivation and the physical background. The rescaled two-species VMB system in the incompressible diffusive regime takes the form [2]

$$\begin{cases} \partial_t F_\epsilon^+ + \dfrac{1}{\epsilon} v \cdot \nabla_x F_\epsilon^+ + \dfrac{1}{\epsilon}(\alpha E_\epsilon + \beta v \times B_\epsilon) \cdot \nabla_v F_\epsilon^+ = \dfrac{1}{\epsilon^2} Q(F_\epsilon^+, F_\epsilon^+) + \dfrac{\delta^2}{\epsilon^2} Q(F_\epsilon^+, F_\epsilon^-), \\ \partial_t F_\epsilon^- + \dfrac{1}{\epsilon} v \cdot \nabla_x F_\epsilon^- - \dfrac{1}{\epsilon}(\alpha E_\epsilon + \beta v \times B_\epsilon) \cdot \nabla_v F_\epsilon^- = \dfrac{1}{\epsilon^2} Q(F_\epsilon^-, F_\epsilon^-) + \dfrac{\delta^2}{\epsilon^2} Q(F_\epsilon^-, F_\epsilon^+), \\ \gamma \partial_t E_\epsilon - \nabla_x \times B_\epsilon = -\dfrac{\beta}{\epsilon^2} \displaystyle\int_{\mathbb{R}^3} (F_\epsilon^+ - F_\epsilon^-) v dv, \\ \gamma \partial_t B_\epsilon + \nabla_x \times E_\epsilon = 0, \\ \nabla_x \cdot E_\epsilon = \dfrac{\alpha}{\epsilon^2} \displaystyle\int_{\mathbb{R}^3} (F_\epsilon^+ - F_\epsilon^-) dv, \quad \nabla_x \cdot B_\epsilon = 0, \end{cases} \tag{1.1}$$

where $\epsilon > 0$ is a small parameter proportional to the mean free path, $\delta > 0$ measures the strength of interactions, and α, β, γ have the following physical meanings:

- α measures the electric repulsion according to Gauss law;
- β measures the magnetic induction according to Ampère law;
- γ is the ratio of the bulk velocity to the speed of light.

Notice that the parameters α, β, γ satisfy the relation

$$\beta = \frac{\alpha\gamma}{\epsilon}.$$

In addition, in (1.1), $F_\epsilon^\pm = F_\epsilon^\pm(t,x,v)$ are the density distribution functions of charged particles at $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$, and $E_\epsilon(t,x)$, $B_\epsilon(t,x)$ denote the electro and magnetic fields respectively. Since we study the diffusive limit when ϵ tends to zero, we assume $\epsilon \in (0,1)$ in the following analysis. As usual, $Q(F,G)$ is the Boltzmann collision operator for hard sphere model given by

$$Q(F,G) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v-v_*)\cdot\omega|(F(v')G(v'_*) - F(v)G(v_*))dv_*d\omega, \tag{1.2}$$

where v, v_* are the velocities of gas particles before collision and v', v'_* are the velocities after collision:

$$v' = v - [(v-v_*)\cdot\omega]\omega, \quad v'_* = v_* + [(v-v_*)\cdot\omega]\omega, \quad \omega \in \mathbb{S}^2.$$

That is, the collisions are elastic so that the following conservation of momentum and energy hold

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2. \tag{1.3}$$

In [2], the authors considered different ciritical scalings in the incompressible diffusive regime and studied the diffusive limits of the VMB systems. In this paper, we consider a typical case when $\alpha = \epsilon, \beta = 1, \gamma = 1, \delta = 1$, i.e.,

$$\begin{cases} \partial_t F_\epsilon^+ + \dfrac{1}{\epsilon} v\cdot\nabla_x F_\epsilon^+ + \dfrac{1}{\epsilon}(\epsilon E_\epsilon + v\times B_\epsilon)\cdot\nabla_v F_\epsilon^+ = \dfrac{1}{\epsilon^2}[Q(F_\epsilon^+, F_\epsilon^+) + Q(F_\epsilon^+, F_\epsilon^-)], \\ \partial_t F_\epsilon^- + \dfrac{1}{\epsilon} v\cdot\nabla_x F_\epsilon^- - \dfrac{1}{\epsilon}(\epsilon E_\epsilon + v\times B_\epsilon)\cdot\nabla_v F_\epsilon^- = \dfrac{1}{\epsilon^2}[Q(F_\epsilon^-, F_\epsilon^+) + Q(F_\epsilon^-, F_\epsilon^-)], \\ \partial_t E_\epsilon - \nabla_x \times B_\epsilon = -\dfrac{1}{\epsilon^2}\displaystyle\int_{\mathbb{R}^3} (F_\epsilon^+ - F_\epsilon^-) v\, dv, \\ \partial_t B_\epsilon + \nabla_x \times E_\epsilon = 0, \\ \nabla_x \cdot E_\epsilon = \dfrac{1}{\epsilon}\displaystyle\int_{\mathbb{R}^3} (F_\epsilon^+ - F_\epsilon^-)dv, \quad \nabla_x \cdot B_\epsilon = 0. \end{cases} \tag{1.4}$$

The Vlasov-Maxwell-Boltzmann system has been intensively studied and much important progress has been made in [12,8,15,20,32]. For instance, the global existence of unique strong solution with initial data near the normalized global Maxwellian was obtained in spatial period domain [15] and in three dimensional space [32] for hard sphere collision, and then in [11,10] for general collision kernels with or without angular cut-off assumption. For the long time behaviors, it was shown in [8] that the total energy of the linearized one-species VMB system decays at the rate $(1+t)^{-\frac{3}{8}}$, and in [12] that the total energy of nonlinear two-species VMB system decays at the rate $(1+t)^{-\frac{3}{4}}$. The spectrum structure and the optimal decay rate of the global solution to the VMB systems for both one-spices and two-spices were investigated in [25].

The diffusive limit for two-species VMB system was shown in [2,20,22]. Precisely, the diffusive limits of the renormalized solutions under different scalings were studied in [2]. Moreover, the diffusive limit of the strong solution to the incompressible NSMF system (1.24) with $(\alpha,\beta,\gamma,\delta)=(\epsilon,1,1,1)$ was proved in [22], and to an incompressible NSPF system with $(\alpha,\beta,\gamma,\delta)=(\epsilon,\epsilon,\epsilon,1)$ was proved in [20]. However, the convergence rate and initial layer of the diffusive limits have not been studied in the previous works.

On the other hand, the diffusion limit to the Boltzmann equation is a classical problem with pioneer work by Bardos-Golse-Levermore in [3], and significant progress on the limit of renormalized solutions to Leray solution to Navier-Stokes system in [14]. One effective approach to study the fluid dynamic in the perturbative framework is based on the spectral analysis. For example, Ellis-Pinsky [13] first studied the linear compressible Euler limit of the linear Boltzmann equation and showed the convergence rate outside the initial layer. The initial layer in the fluid limit arises from the incompatibility of the initial data, in particular due to the high oscillation of the eigen-modes in the system. For the Boltzmann equation, Bardos-Ukai [5] firstly studied the incompressible Navier-Stokes limit with the estimation on the initial layer for the linear Boltzmann equation. The analysis in [5] uses an estimate on the semigroup with highly oscillating eigen-modes in [34] which is about the incompressible limit of the compressible Euler equation. Note that the estimation on the initial layer is not optimal in these papers. In contrast to the extensive study on Boltzmann equation [3–5,16,28,30], the VPB system [17,27,35] and the VMB system [2,20,22], the convergence rate of the classical solution to the VMB system (1.4) towards its fluid dynamical limits and the estimation of the initial layer have not been given despite of its importance.

Let $F_\epsilon=F_\epsilon^+ + F_\epsilon^-$ and $G_\epsilon=F_\epsilon^+ - F_\epsilon^-$. Then the system (1.4) becomes to

$$\begin{cases}\partial_t F_\epsilon + \frac{1}{\epsilon} v\cdot\nabla_x F_\epsilon + \frac{1}{\epsilon}(\epsilon E_\epsilon + v\times B_\epsilon)\cdot\nabla_v G_\epsilon = \frac{1}{\epsilon^2}Q(F_\epsilon,F_\epsilon),\\ \partial_t G_\epsilon + \frac{1}{\epsilon} v\cdot\nabla_x G_\epsilon + \frac{1}{\epsilon}(\epsilon E_\epsilon + v\times B_\epsilon)\cdot\nabla_v F_\epsilon = \frac{1}{\epsilon^2}Q(G_\epsilon,F_\epsilon),\\ \partial_t E_\epsilon - \nabla_x\times B_\epsilon = -\frac{1}{\epsilon^2}\int\limits_{\mathbb{R}^3} G_\epsilon v dv,\\ \partial_t B_\epsilon + \nabla_x\times E_\epsilon = 0,\\ \nabla_x\cdot E_\epsilon = \frac{1}{\epsilon}\int\limits_{\mathbb{R}^3} G_\epsilon dv, \quad \nabla_x\cdot B_\epsilon = 0.\end{cases} \tag{1.5}$$

In this paper, we study the diffusion limit of the strong solution to the rescaled VMB system (1.5) with initial data near the equilibrium $(F_*, G_*, E_*, B_*) = (M(v), 0, 0, 0)$, where $M(v)$ is the normalized Maxwellian given by

$$M = M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}^3.$$

Hence, we define the perturbation of $(F_\epsilon, G_\epsilon, E_\epsilon, B_\epsilon)$ as

$$F_\epsilon = M + \epsilon\sqrt{M} f_\epsilon, \quad G_\epsilon = \epsilon\sqrt{M} g_\epsilon.$$

Then Cauchy problem of the VMB system (1.5) for $(f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)$ can be rewritten as

$$\partial_t f_\epsilon + \frac{1}{\epsilon} v \cdot \nabla_x f_\epsilon - \frac{1}{\epsilon^2} L f_\epsilon = H_\epsilon^1, \tag{1.6}$$

$$\partial_t g_\epsilon + \frac{1}{\epsilon} v \cdot \nabla_x g_\epsilon - \frac{1}{\epsilon^2} L_1 g_\epsilon - \frac{1}{\epsilon} v\sqrt{M} \cdot E_\epsilon = H_\epsilon^2, \tag{1.7}$$

$$\partial_t E_\epsilon - \nabla_x \times B_\epsilon = -\frac{1}{\epsilon} \int_{\mathbb{R}^3} g_\epsilon v \sqrt{M} dv, \tag{1.8}$$

$$\partial_t B_\epsilon + \nabla_x \times E_\epsilon = 0, \tag{1.9}$$

$$\nabla_x \cdot E_\epsilon = \int_{\mathbb{R}^3} g_\epsilon \sqrt{M} dv, \quad \nabla_x \cdot B_\epsilon = 0, \tag{1.10}$$

where the nonlinear terms $H_\epsilon^1, H_\epsilon^2$ are defined by

$$H_\epsilon^1 = \frac{1}{2}(v \cdot E_\epsilon) g_\epsilon - \left(E_\epsilon + \frac{1}{\epsilon} v \times B_\epsilon\right) \cdot \nabla_v g_\epsilon + \frac{1}{\epsilon}\Gamma(f_\epsilon, f_\epsilon), \tag{1.11}$$

$$H_\epsilon^2 = \frac{1}{2}(v \cdot E_\epsilon) f_\epsilon - \left(E_\epsilon + \frac{1}{\epsilon} v \times B_\epsilon\right) \cdot \nabla_v f_\epsilon + \frac{1}{\epsilon}\Gamma(g_\epsilon, f_\epsilon). \tag{1.12}$$

The initial condition is given by

$$(f_\epsilon, g_\epsilon)(0, x, v) = (f_0, g_0)(x, v), \quad (E_\epsilon, B_\epsilon)(0, x) = (E_0, B_0)(x), \tag{1.13}$$

which is independent of ϵ. On the other hand, the initial data should satisfy the compatibility conditions

$$\nabla_x \cdot E_0(x) = \int_{\mathbb{R}^3} g_0 \sqrt{M} dv, \quad \nabla_x \cdot B_0(x) = 0. \tag{1.14}$$

In (1.6)–(1.12), the linear operators L, L_1 and the nonlinear operator $\Gamma(f, g)$ are defined by

$$\begin{cases} Lf = \dfrac{1}{\sqrt{M}}[Q(M,\sqrt{M}f) + Q(\sqrt{M}f, M)], \\ L_1 f = \dfrac{1}{\sqrt{M}} Q(\sqrt{M}f, M), \\ \Gamma(f,g) = \dfrac{1}{\sqrt{M}} Q(\sqrt{M}f, \sqrt{M}g). \end{cases} \tag{1.15}$$

As usual, the linearized operators L and L_1 can be written as (cf. [6,1])

$$\begin{cases} (Lf)(v) = (Kf)(v) - \nu(v)f(v), \quad (L_1 f)(v) = (K_1 f)(v) - \nu(v)f(v), \\ (Kf)(v) = \displaystyle\int_{\mathbb{R}^3} k(v,v_*)f(v_*)dv_*, \quad (K_1 f)(v) = \displaystyle\int_{\mathbb{R}^3} k_1(v,v_*)f(v_*)dv_*, \\ \nu(v) = \sqrt{2\pi}\left(e^{-\frac{|v|^2}{2}} + \left(|v| + \dfrac{1}{|v|}\right)\displaystyle\int_0^{|v|} e^{-\frac{|u|^2}{2}} du\right), \\ k(v,v_*) = \dfrac{2}{\sqrt{2\pi}|v-v_*|} e^{-\frac{(|v|^2-|v_*|^2)^2}{8|v-v_*|^2} - \frac{|v-v_*|^2}{8}} - \dfrac{|v-v_*|}{2\sqrt{2\pi}} e^{-\frac{|v|^2+|v_*|^2}{4}}, \\ k_1(v,v_*) = \dfrac{2}{\sqrt{2\pi}|v-v_*|} e^{-\frac{(|v|^2-|v_*|^2)^2}{8|v-v_*|^2} - \frac{|v-v_*|^2}{8}}, \end{cases} \tag{1.16}$$

where $\nu(v)$ is the collision frequency, K and K_1 are self-adjoint compact operators on $L^2(\mathbb{R}^3_v)$ with real symmetric integral kernels $k(v,v_*)$ and $k_1(v,v_*)$. In addition, $\nu(v)$ satisfies

$$\nu_0(1+|v|) \le \nu(v) \le \nu_1(1+|v|). \tag{1.17}$$

The nullspace of the operator L denoted by N_0 is a subspace spanned by the orthonormal basis $\{\chi_j,\ j = 0,1,\cdots,4\}$ given by

$$\chi_0 = \sqrt{M}, \quad \chi_j = v_j\sqrt{M}\ (j=1,2,3), \quad \chi_4 = \frac{(|v|^2-3)\sqrt{M}}{\sqrt{6}}, \tag{1.18}$$

and the null space of the operator L_1 denoted by N_1 is spanned only by $\sqrt{M}$.

Let $L^2(\mathbb{R}^3)$ be a Hilbert space of complex-value functions $f(v)$ on $\mathbb{R}^3$ with the inner product and the norm

$$(f,g) = \int_{\mathbb{R}^3} f(v)\overline{g(v)}dv, \quad \|f\| = \left(\int_{\mathbb{R}^3} |f(v)|^2 dv\right)^{1/2}.$$

And let P_0, P_d be the projection operators from $L^2(\mathbb{R}^3_v)$ to the subspace N_0, N_1 with

$$P_0 f = \sum_{j=0}^{4} (f, \chi_j)\chi_j, \quad P_1 = I - P_0, \tag{1.19}$$

$$P_d f = (f, \sqrt{M})\sqrt{M}, \quad P_r = I - P_d. \tag{1.20}$$

For any $U = (g, X, Y) \in L^2 \times \mathbb{R}^3 \times \mathbb{R}^3$, we denote

$$P_2 U = (P_d g, X, Y), \quad P_3 U = (I - P_2)U = (P_r g, 0, 0). \tag{1.21}$$

By Boltzmann H-theorem, the linearized collision operators L and L_1 are non-positive. Precisely, there is a constant $\mu > 0$ such that

$$(Lf, f) \le -\mu \|P_1 f\|^2, \qquad f \in D(L), \tag{1.22}$$

$$(L_1 f, f) \le -\mu \|P_r f\|^2, \qquad f \in D(L_1), \tag{1.23}$$

where $D(L)$ and $D(L_1)$ are the domains of L and L_1 given by

$$D(L) = D(L_1) = \left\{ f \in L^2(\mathbb{R}^3) \,|\, \nu(v) f \in L^2(\mathbb{R}^3) \right\}.$$

Without the loss of generality, we assume $\nu(0) \ge \nu_0 \ge \mu > 0$.

This paper aims to study the optimal convergence rate of the classical solution $(f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)$ of the system (1.6)–(1.13) to (u_1, u_2, E, B), where $u_1 = n\chi_0 + m\cdot v\chi_0 + q\chi_4$ and $u_2 = \rho\chi_0$ with $(n, m, q, \rho, E, B)(t, x)$ being the solution of the following bipolar incompressible Navier-Stokes-Maxwell-Fourier (NSMF) system:

$$\begin{cases} \nabla_x \cdot m = 0, \quad n + \sqrt{\frac{2}{3}} q = 0, \\ \partial_t m - \kappa_0 \Delta_x m + \nabla_x p = \rho E + j \times B - \nabla_x \cdot (m \otimes m), \\ \partial_t q - \kappa_1 \Delta_x q = -\nabla_x \cdot (qm), \\ \partial_t E - \nabla_x \times B = -j, \quad \nabla_x \cdot E = \rho, \\ \partial_t B + \nabla_x \times E = 0, \quad \nabla_x \cdot B = 0, \\ j = -\eta(\nabla_x \rho - E) + (\rho m - \eta m \times B). \end{cases} \tag{1.24}$$

Here, p is the pressure and the initial data $(n, m, q, \rho, E, B)(0)$ satisfies

$$\begin{cases} m(0) = (f_0, v\chi_0) - \Delta_x^{-1} \nabla_x \mathrm{div}_x (f_0, v\chi_0), \\ n(0) = -\sqrt{\frac{2}{3}} q(0) = \sqrt{\frac{2}{5}} (f_0, \sqrt{\frac{2}{5}}\chi_0 - \sqrt{\frac{3}{5}}\chi_4), \\ \rho(0) = \mathrm{div}_x E_0, \quad E(0) = E_0, \quad B(0) = B_0. \end{cases} \tag{1.25}$$

Correspondingly, the viscosity coefficients κ_0, κ_1, $\eta > 0$ are defined by

$$\begin{cases} \kappa_0 = -(L^{-1}P_1(v_1\chi_2), v_1\chi_2), \quad \kappa_1 = -\dfrac{3}{5}(L^{-1}P_1(v_1\chi_4), v_1\chi_4), \\ \eta = -(L_1^{-1}(v_1\chi_0), v_1\chi_0). \end{cases} \tag{1.26}$$

There have been extensive studies on the existence and solution behavior about the incompressible Navier-Stokes-Maxwell system. Precisely, the existence, uniqueness and an exponential growth estimate of global strong solutions were proved in [29] for a slightly different system. And the existence of global small mild solution was given in [19]. In addition, the authors in [23,21,39] studied the global classical solutions to the two-fluid incompressible Navier-Stokes-Fourier-Maxwell system with Ohm's law with small initial data. Regularity results for the Cauchy problem of the incompressible Navier-Stokes-Maxwell system with Ohm's law in two and three space dimensions were given in [36].

On the other hand, for the compressible Navier-Stokes-Maxwell system, the Green's function to the linearized system with applications were obtained in [9]. The existence and uniqueness of global strong solutions with large initial data and vacuum were given in [18]. And the large-time behavior of solutions to the outflow problem was studied in [37]. Moreover, the authors in [38] studied the large-time asymptotic behavior of solutions to the superposition of a viscous contact wave with two rarefaction waves.

Back to the convergence of VMB to NSMF, the convergence is not uniform near $t = 0$ because of initial layer unless we impose extra assumption on the initial data (f_0, g_0, E_0, B_0):

$$\begin{cases} f_0(x,v) = n_0(x)\chi_0 + m_0(x)\cdot v\chi_0 + q_0(x)\chi_4, \\ g_0(x,v) = \rho_0(x)\chi_0, \quad \nabla_x \cdot E_0(x) = \rho_0(x), \\ \nabla_x \cdot m_0 = 0, \quad n_0 + \sqrt{\tfrac{2}{3}}q_0 = 0, \quad \nabla_x \cdot B_0 = 0. \end{cases} \tag{1.27}$$

In order to estimate the initial layer, we need the following decomposition. For $U = U(x) \in \mathbb{R}^3$, we denote the Helmholtz's decomposition $U = U_{||} + U_\perp$ with

$$U_{||} = \Delta_x^{-1}\nabla_x \mathrm{div}_x U, \quad U_\perp = \Delta_x^{-1}(\nabla_x \times \nabla_x \times U). \tag{1.28}$$

Correspondingly, for $f = f(x,v) \in L^2$, we define the following projections

$$\begin{cases} P_{||}f = (f, v\chi_0)_{||}v\chi_0 + (f, \tilde{h}_1)\tilde{h}_1, \\ P_\perp f = (f, v\chi_0)_\perp v\chi_0 + (f, \tilde{h}_0)\tilde{h}_0 + P_1 f, \end{cases} \tag{1.29}$$

where

$$\tilde{h}_0 = \sqrt{\frac{2}{5}}\chi_0 - \sqrt{\frac{3}{5}}\chi_4, \quad \tilde{h}_1 = \sqrt{\frac{3}{5}}\chi_0 + \sqrt{\frac{2}{5}}\chi_4. \tag{1.30}$$

To state the main results, we need the following notations. First of all, C, c denote some generic constants. For any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$, set

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}, \quad \partial_v^\beta = \partial_{v_1}^{\beta_1}\partial_{v_2}^{\beta_2}\partial_{v_3}^{\beta_3}.$$

The Fourier transform of $h = h(x)$ is denoted by

$$\hat{h}(\xi) = \mathcal{F}h(\xi) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} h(x) e^{-ix\cdot\xi} dx,$$

where and throughout this paper we denote $i = \sqrt{-1}$.

For any $q \in [1, \infty]$, the Sobolev Space $L^q = L_x^q(L_v^2)$ for function $f = f(x, v)$ or $L^q = L_x^q(L_v^2) \times L_x^q \times L_x^q$ for vector $U = (g(x, v), E(x), B(x))$ is defined with the norms

$$\|f\|_{L^q} = \left(\int\limits_{\mathbb{R}^3} \left(\int\limits_{\mathbb{R}^3} |f(x, v)|^2 dv \right)^{q/2} dx \right)^{1/q},$$

$$\|U\|_{L^q} = \left(\int\limits_{\mathbb{R}^3} \left(\int\limits_{\mathbb{R}^3} |g(x, v)|^2 dv \right)^{q/2} dx \right)^{1/q} + \left(\int\limits_{\mathbb{R}^3} (|E(x)|^q + |B(x)|^q) dx \right)^{1/q}.$$

Also for an integer $k \geq 1$ and $q \in [1, \infty]$, the Sobolev Space $H^k = H_x^k(L_v^2)$ (or $H^k = H_x^k(L_v^2) \times H_x^k \times H_x^k$) for a function $f = f(x, v)$ (or a vector $U = (g(x, v), E(x), B(x))$) is defined with the norms

$$\|f\|_{H^k} = \left(\int\limits_{\mathbb{R}^3} (1 + |\xi|^2)^k \int\limits_{\mathbb{R}^3} |\hat{f}|^2 dv d\xi \right)^{1/2},$$

$$\|U\|_{H^k} = \left(\int\limits_{\mathbb{R}^3} (1 + |\xi|^2)^k \left(\int\limits_{\mathbb{R}^3} |\hat{g}|^2 dv + |\hat{E}|^2 + |\hat{B}|^2 \right) d\xi \right)^{1/2};$$

and the Sobolev space $W^{k,q} = W_x^{k,q}(L_v^2)$ (or $W^{k,q} = W_x^{k,q}(L_v^2) \times W_x^{k,q} \times W_x^{k,q}$) is defined with the norms

$$\|f\|_{W^{k,q}} = \sum_{|\alpha| \leq k} \left(\int\limits_{\mathbb{R}^3} \left(\int\limits_{\mathbb{R}^3} |\partial_x^\alpha f|^2 dv \right)^{q/2} dx \right)^{1/q},$$

$$\|U\|_{W^{k,q}} = \sum_{|\alpha| \leq k} \left(\int\limits_{\mathbb{R}^3} \left(\int\limits_{\mathbb{R}^3} |\partial_x^\alpha g|^2 dv + |\partial_x^\alpha E|^2 + |\partial_x^\alpha B|^2 \right)^{q/2} dx \right)^{1/q}.$$

Some weighted Sobolev space D_l^k ($D^k = D_0^k$) will also be used with the norms given by

$$\|f\|_{D^k_l} = \sum_{|\alpha|+|\beta|\le k} \|\nu^l \partial_x^\alpha \partial_v^\beta f\|_{L^2},$$

$$\|U\|_{D^k_l} = \sum_{|\alpha|+|\beta|\le k} \|\nu^l \partial_x^\alpha \partial_v^\beta g\|_{L^2} + \sum_{|\alpha|\le k} (\|\partial_x^\alpha E\|^2_{L^2_x} + \|\partial_x^\alpha B\|^2_{L^2_x})$$

We now state the following two main results in this paper.

Theorem 1.1. *For any $\epsilon \in (0,1)$, there exists a small constant $\delta_0 > 0$ such that if the initial data $U_0 = (f_0, g_0, E_0, B_0)$ satisfy that $\|U_0\|_{D^6_1} + \|U_0\|_{L^1} \le \delta_0$, then the VMB system (1.6)–(1.13) admits a unique global solution $U_\epsilon(t,x,v) = (f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)$ satisfying the following time-decay estimate:*

$$\|U_\epsilon(t)\|_{D^4_1} \le C\delta_0(1+t)^{-\frac{3}{4}}. \tag{1.31}$$

Also, there exists a small constant $\delta_0 > 0$ such that if $\|U_0\|_{H^4} + \|U_0\|_{L^1} \le \delta_0$, then the NSMF system (1.24)–(1.25) admits a unique global solution $\tilde{U}(t,x) = (n, m, q, \rho, E, B)$ satisfying the following time-decay estimate:

$$\|\tilde{U}(t)\|_{H^4_x} \le C\delta_0(1+t)^{-\frac{3}{4}}. \tag{1.32}$$

Theorem 1.2. *There exist small positive constants δ_0 and ϵ_0 such that if the initial data $U_0 = (f_0, g_0, E_0, B_0)$ satisfies that $\|U_0\|_{D^9_1} + \|U_0\|_{L^1} \le \delta_0$, then there exists a unique function $U_1 = (u_1, V_1)$ such that for any $\epsilon \in (0, \epsilon_0)$, the solution $U_\epsilon = (f_\epsilon, V_\epsilon)$ with $V_\epsilon = (g_\epsilon, E_\epsilon, B_\epsilon)$ to the VMB system (1.6)–(1.13) satisfies*

$$\begin{aligned} &\|P_{||} f_\epsilon(t)\|_{W^{2,\infty}} + \|P_\perp f_\epsilon(t) - u_1(t)\|_{H^2} + \|V_\epsilon(t) - V_1(t)\|_{H^2} \\ &\le C\delta_0\Big(\epsilon|\ln\epsilon|^2(1+t)^{-\frac{1}{2}} + \Big(1 + \frac{t}{\epsilon}\Big)^{-1}\Big), \end{aligned} \tag{1.33}$$

where $u_1 = n\chi_0 + m \cdot v\chi_0 + q\chi_4$ and $V_1 = (\rho\chi_0, E, B)$ with $(n, m, q, \rho, E, B)(t,x)$ being the solution to the incompressible NSMF system (1.24)–(1.25).

Moreover, if the initial data U_0 satisfies (1.27) and $\|U_0\|_{H^9} + \|U_0\|_{L^1} \le \delta_0$, then we have

$$\|P_{||} f_\epsilon(t)\|_{W^{2,\infty}} + \|P_\perp f_\epsilon(t) - u_1(t)\|_{H^2} + \|V_\epsilon(t) - V_1(t)\|_{H^2} \le C\delta_0\epsilon|\ln\epsilon|(1+t)^{-\frac{1}{2}}. \tag{1.34}$$

Remark 1.3. By Sobolev Embedding Theorem, we have the following estimates.

(1) Under the first assumption in Theorem 1.2, the solution U_ϵ to the VMB system (1.6)–(1.13) satisfies

$$\|U_\epsilon(t) - U_1(t)\|_{L^\infty} \le C\delta_0\Big(\epsilon|\ln\epsilon|^2(1+t)^{-\frac{1}{2}} + \Big(1 + \frac{t}{\epsilon}\Big)^{-1}\Big). \tag{1.35}$$

(2) Under the second assumption of Theorem 1.2, it holds that

$$\|U_\epsilon(t) - U_1(t)\|_{L^\infty} \le C\delta_0\epsilon|\ln\epsilon|(1+t)^{-\frac{1}{2}}. \tag{1.36}$$

Remark 1.4. Under the first assumption in Theorem 1.2, we have

$$\begin{aligned}&\|f_\epsilon(t) - u_1(t) - U_\epsilon^{osc}(t) - e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}P_1 f_0\|_{L^\infty} + \|V_\epsilon(t) - V_1(t) - e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}P_3 V_0\|_{L^\infty}\\ &\le C\delta_0\epsilon|\ln\epsilon|(1+t)^{-\frac{1}{2}},\end{aligned} \tag{1.37}$$

where $U_\epsilon^{osc}(t) = U_\epsilon^{osc}(t,x,v)$ is the high oscillation part of f_ϵ defined by (3.113). Hence, $U_\epsilon^{osc}(t)$, $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}P_1 f_0$ and $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}P_3 V_0$ are the essential components for generating the initial layer.

Remark 1.5. The global existence of the rescaled VMB system can be obtained by standard energy method in solution space D_k^4 $(k \ge 0)$, namely (cf. Lemma 4.3)

$$\sup_{0\le s<+\infty} \|U_\epsilon(s)\|_{D_k^4} \le C\|U_0\|_{D_k^4}.$$

Note that the VMB system has the property of regularity loss. Precisely, the eigenvalues of the linear VMB operator in high frequency behave like $\mathrm{Re}\lambda_j(\xi,\epsilon) \sim -\frac{1}{\epsilon|\xi|}$ for $\epsilon|\xi| \gg 1$. To obtain the decay rate, we need to compensate the regularity of initial data. That is, when the initial data $U_0 \in D_k^6$ $(k \ge 1)$, the solution satisfies (cf. Lemma 4.4)

$$\|U_\epsilon(t)\|_{D_k^4} \le C\|U_0\|_{D_k^6}(1+t)^{-\frac{3}{4}}.$$

Furthermore, we establish the convergence rate of the diffusion limit by using the decay estimate of the solution and the convergence rate of the semigroup corresponding to its fluid limit. Since the convergence of the fluid limit of the semigroup $e^{\frac{t}{\epsilon}\mathbb{B}_\epsilon}$ in a $L^2 - L^\infty$ norm requires higher regularity (cf. Lemma 3.9), we need higher regularity on initial data to obtain the convergence rate of the diffusion limit. The main reason for requiring the initial data $U_0 \in D_1^9$ is to make sure that the terms $\|(v \times B_\epsilon)\cdot\nabla_v f_\epsilon\|_{H^6}$ and $\|\Gamma(g_\epsilon, f_\epsilon)\|_{H^6}$ decay at the rate $(1+t)^{-\frac{3}{2}}$, as shown in (4.81), so that we can close the priori assumption.

Before the rest of the introduction, we will briefly present the main ideas and the approach of the analysis in the proof. The convergence rates given in Theorem 1.2 of diffusion limit of the VMB system are proved based on the spectral analysis [26] and the ideas inspired by [5,27]. First of all, the solution $U_\epsilon(t) = (f_\epsilon, V_\epsilon)(t)$ with $V_\epsilon(t) = (g_\epsilon, E_\epsilon, B_\epsilon)(t)$ to the VMB system (1.6)–(1.13) can be represented by

$$\begin{cases} f_\epsilon(t) = e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon} f_0 + \int_0^t e^{\frac{t-s}{\epsilon^2}\mathbb{B}_\epsilon} H^1_\epsilon(s) ds, \\ V_\epsilon(t) = e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} V_0 + \int_0^t e^{\frac{t-s}{\epsilon^2}\mathbb{A}_\epsilon} (H^2_\epsilon(s), 0, 0) ds; \end{cases}$$

and the solution $U(t) = (u_1, V_1)(t)$ with $u_1 = n\chi_0 + m \cdot v\chi_0 + q\chi_4$ and $V_1 = (\rho\chi_0, E, B)$ to the NSMF system (1.24)–(1.25) can be represented by

$$\begin{cases} u_1(t) = Y_1(t)P_0 f_0 + \int_0^t Y_1(t-s) H_4(s) ds, \\ V_1(t) = Y_2(t)P_2 V_0 + \int_0^t Y_2(t-s) H_5(s) ds, \end{cases}$$

where $\mathbb{B}_\epsilon$ and $\mathbb{A}_\epsilon$ are the linear Boltzmann and VMB operators defined by (2.3) and (2.4), $Y_1(t)$ and $Y_2(t)$ are two semigroups defined in (3.5) and (3.6), and H_4, H_5 are nonlinear terms given by

$$H_4 = (\rho E + j \times B) \cdot v\chi_0 - \nabla_x \cdot (m \otimes m) \cdot v\chi_0 - \frac{5}{3}\nabla_x \cdot (qm)\chi_4,$$
$$H_5 = -(\nabla_x \cdot (\rho m - \eta m \times B)\chi_0, \rho m - \eta m \times B, 0).$$

The main idea of the proof is to first estimate the convergence rates from $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}$ to $Y_1(t)$ and from $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}$ to $Y_2(t)$ separately by using spectral analysis. Then to obtain the convergence rates from $(f_\epsilon, V_\epsilon)(t)$ to $(u_1, V_1)(t)$ is based on the convergence rates on semigroups and a bootstrap argument. Note that the a priori estimates on the solutions $u_1(t)$ and $V_1(t)$ can be closed in H^2. However, even though L^∞-norm also works for $u_1(t)$, it is not suitable for $V_1(t)$.

Note that the linear Boltzmann operator $\mathbb{B}_\epsilon(\xi)$ given by (2.7) satisfies the scaling transformation $\mathbb{B}_\epsilon(\xi) = \mathbb{B}_1(\epsilon\xi)$. This implies that the eigenvalues $\gamma_j(|\xi|, \epsilon)$ of $\mathbb{B}_\epsilon(\xi)$ depend only on $\epsilon|\xi|$ and satisfy $\gamma_j(|\xi|, \epsilon) = \tilde{\gamma}_j(\epsilon|\xi|)$ for $\epsilon|\xi|$ small, where $\tilde{\gamma}_j(|\xi|)$ are the eigenvalues of $\mathbb{B}_1(\xi)$ in the low frequency regime. Precisely, there exist five eigenvalues $\gamma_j(|\xi|, \epsilon)$ for $\epsilon|\xi|$ small and they satisfy

$$\gamma_j(|\xi|, \epsilon) = i\mu_j\epsilon|\xi| - a_j\epsilon^2|\xi|^2 + O\left(\epsilon^3|\xi|^3\right), \tag{1.38}$$

where $\mu_{\pm 1} = \pm\sqrt{\frac{5}{3}}$, $\mu_k = 0$ $(k = 0, 2, 3)$, and $a_j > 0$ are some constants.

Moreover, we can decompose the semigroup $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon(\xi)}$ into the fluid part and the remainder part, where the remainder part has the decay rate $e^{-\frac{bt}{\epsilon^2}}$ (see Theorem 2.19). Then by applying the expansion (1.38) to the fluid part, we can rewrite $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon(\xi)}$ as

$$P_{||}(e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon(\xi)}\hat{f}_0) = \sum_{j=\pm 1} e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t} P_{0j}\hat{f}_0 + O(\epsilon e^{-c|\xi|^2 t}) + O(e^{-\frac{bt}{\epsilon^2}}),$$
$$P_{\perp}(e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon(\xi)}\hat{f}_0) = \sum_{j=0,2,3} e^{-a_j|\xi|^2 t} P_{0j}\hat{f}_0 + O(\epsilon e^{-c|\xi|^2 t}) + O(e^{-\frac{bt}{\epsilon^2}}),$$

where P_{0j}, $j = -1, 0, 1, 2, 3$ are the first order eigenprojections corresponding to γ_j. Thus, by using the following key estimate:

$$\left\| \mathcal{F}^{-1}\left(e^{\frac{\pm i|\xi|}{\epsilon}t}(1+|\xi|)^{-3}\right)\right\|_{L^\infty_x} \le C\left(\frac{t}{\epsilon}\right)^{-1},$$

we can establish the optimal convergence rate of the semigroup $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}$ to its first and second order fluid limits in a combination of $L^2 - L^\infty$ norm (L^∞ norm for $P_{||}$ part and L^2 norm for $P_\perp$ part) as given in Lemma 3.9.

On the other hand, due to the influence of the electric-magnetic field, the linear VMB operator $\mathbb{A}_\epsilon(\xi)$ given in (2.8) has no scaling property. To study the corresponding eigenvalue problem, we will use a new non-local implicit function theorem to show that there exist five eigenvalues $\lambda_j(|\xi|, \epsilon)$, $j = 0, 1, 2, 3, 4$ of $\mathbb{A}_\epsilon(\xi)$ for $\epsilon(1 + |\xi|)$ being small and they satisfy (see Lemma 2.11):

$$\lambda_0(|\xi|, \epsilon) = \epsilon^2 b_0(|\xi|) + O(\epsilon^4(1 + |\xi|^2)^2), \tag{1.39}$$
$$\lambda_k(|\xi|, \epsilon) = \epsilon^2 b_k(|\xi|) + \begin{cases} O(\epsilon^4|b_k(|\xi|)|), & |\eta^2 - 4|\xi|^2| \ge r_0, \\ O(\epsilon^3), & |\eta^2 - 4|\xi|^2| < r_0, \end{cases} \tag{1.40}$$

where $k = 1, 2, 3, 4$, and $b_j(|\xi|)$ are defined by (2.85).

Moreover, we can decompose the semigroup $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon(\xi)}$ into two fluid parts for low frequency and high frequency and the remainder part, where the remainder part has the decay rate $e^{-\frac{bt}{\epsilon^2}}$ (see Theorem 2.15). Then by applying the expansion (1.39)–(1.40) to the fluid part, we obtain

$$e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon(\xi)}\hat{V}_0 = \sum_{j=0}^{4} e^{b_j(|\xi|)t}\tilde{P}_{0j}\hat{V}_0 + O(\epsilon e^{-\mathrm{Re} b_j t}) + O(e^{-\frac{ct}{\epsilon|\xi|}})1_{\{|\xi| \ge \frac{r_1}{\epsilon}\}} + O(e^{-\frac{bt}{\epsilon^2}}),$$

where $\tilde{P}_{0j}$, $j = 0, 1, 2, 3, 4$ are first order eigenprojections corresponding to λ_j. Thus, we can establish the optimal convergence rate of the semigroup $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}$ to its first and second order fluid limits in L^2 norm as listed in Lemmas 3.3 and 3.5.

By using the estimates on the convergence rates for the fluid limits of the linear Boltzmann and VMB systems, we can prove the convergence and establish the optimal convergence rate of the strong solution (f_ϵ, V_ϵ) to the nonlinear VMB system towards the solution (u_1, V_1) to the NSMF system. Hence, we obtain the precise estimation on the initial layer. To illustrate why the Helmholtz's decomposition (1.29) is necessary, we consider the term

$$\int_0^t Y_1(t-s)(j_\epsilon \times B_\epsilon - j \times B)ds = \int_0^t Y_1(t-s)(\nabla_x(\rho_\epsilon - \rho) \times B_\epsilon)ds + \cdots .$$

If we directly estimate this term in L^∞, we have non-integrability in time because

$$\left\| \int_0^t Y_1(t-s)(\nabla_x(\rho_\epsilon - \rho) \times B_\epsilon)ds \right\|_{W^{2,\infty}} \le \int_0^t (t-s)^{-\frac{5}{4}} \|\rho_\epsilon - \rho\|_{H^2} \|B_\epsilon\|_{H^2} ds.$$

However, if we apply the $L^2 - L^\infty$ by noting that $P_{||}Y_1(t) = 0$, $P_\perp Y_1(t) = Y_1(t)$, then the time integrability holds because

$$\left\| \int_0^t Y_1(t-s)(\nabla_x(\rho_\epsilon - \rho) \times B_\epsilon)ds \right\|_{H^2} \le \int_0^t (t-s)^{-\frac{1}{2}} \|\rho_\epsilon - \rho\|_{H^2} \|B_\epsilon\|_{H^2} ds.$$

The rest of this paper will be organized as follows. In Section 2, we will present the results about the spectrum analysis of the linear operator related to the linearized VMB system. In Section 3, we will establish the first and second order fluid approximations of the solution to the linearized VMB system. In Section 4, we will prove the convergence and establish the convergence rate of the global solution to the original nonlinear VMB system to the solution to the nonlinear NSMF system.

2. Spectral analysis

In this section, we will study the spectral analysis of the linear VMB operator $\mathbb{A}_\epsilon(\xi)$ defined in (2.10). From the system (1.6)–(1.10), we have the following linearized VMB system for f_ϵ and $U_\epsilon = (g_\epsilon, E_\epsilon, B_\epsilon)^T$:

$$\begin{cases} \epsilon^2 \partial_t f_\epsilon = \mathbb{B}_\epsilon f_\epsilon, \quad t > 0, \\ f_\epsilon(0, x, v) = f_0(x, v), \end{cases} \tag{2.1}$$

and

$$\begin{cases} \epsilon^2 \partial_t U_\epsilon = \mathbb{A}_\epsilon U_\epsilon, \quad t > 0, \\ \nabla_x \cdot E_\epsilon = (g_\epsilon, \chi_0), \quad \nabla_x \cdot B_\epsilon = 0, \\ U_\epsilon(0, x, v) = U_0(x, v) = (g_0, E_0, B_0), \end{cases} \tag{2.2}$$

where

$$\mathbb{B}_\epsilon = L - \epsilon v \cdot \nabla_x, \tag{2.3}$$

$$\mathbb{A}_\epsilon = \begin{pmatrix} L_1 - \epsilon v \cdot \nabla_x & \epsilon v \chi_0 & 0 \\ -\epsilon P_m & 0 & \epsilon^2 \nabla_x \times \\ 0 & -\epsilon^2 \nabla_x \times & 0 \end{pmatrix} \tag{2.4}$$

with $P_m h = (h, v\chi_0)$ for any $h \in L^2(\mathbb{R}^3_v)$.

Taking Fourier transform to (2.1) and (2.2) in x yields

$$\begin{cases} \epsilon^2 \partial_t \hat{f}_\epsilon = \mathbb{B}_\epsilon(\xi) \hat{f}_\epsilon, \quad t > 0, \\ \hat{f}_\epsilon(0, \xi, v) = \hat{f}_0(\xi, v), \end{cases} \tag{2.5}$$

and

$$\begin{cases} \epsilon^2 \partial_t \hat{U}_\epsilon = \mathbb{A}_\epsilon(\xi) \hat{U}_\epsilon, \quad t > 0, \\ i\xi \cdot \hat{E}_\epsilon = (\hat{g}_\epsilon, \chi_0), \quad i\xi \cdot \hat{B}_\epsilon = 0, \\ \hat{U}_\epsilon(0, \xi, v) = \hat{U}_0(\xi, v) = (\hat{g}_0, \hat{E}_0, \hat{B}_0), \end{cases} \tag{2.6}$$

where

$$\mathbb{B}_\epsilon(\xi) = L - i\epsilon v \cdot \xi, \tag{2.7}$$

$$\mathbb{A}_\epsilon(\xi) = \begin{pmatrix} L_1 - i\epsilon v \cdot \xi & \epsilon v \chi_0 & 0 \\ -\epsilon P_m & 0 & i\epsilon^2 \xi \times \\ 0 & -i\epsilon^2 \xi \times & 0 \end{pmatrix}. \tag{2.8}$$

By the identity $X = (X \cdot y)y - y \times y \times X$ for any $X \in \mathbb{R}^3$ and $y \in \mathbb{S}^2$, we can transform the system (2.6) to a new system for $\hat{V}_\epsilon = (\hat{g}_\epsilon, \omega \times \hat{E}_\epsilon, \omega \times \hat{B}_\epsilon)$ with $\omega = \xi/|\xi|$ as

$$\begin{cases} \partial_t \hat{V}_\epsilon = \tilde{\mathbb{A}}_\epsilon(\xi) \hat{V}_\epsilon, \quad t > 0, \\ \hat{V}_\epsilon(0, \xi, v) = \hat{V}_0(\xi, v) = (\hat{g}_0, \omega \times \hat{E}_0, \omega \times \hat{B}_0), \end{cases} \tag{2.9}$$

with

$$\tilde{\mathbb{A}}_\epsilon(\xi) = \begin{pmatrix} \tilde{\mathbb{B}}_\epsilon(\xi) & -\epsilon v \chi_0 \cdot \omega \times & 0 \\ -\epsilon \omega \times P_m & 0 & i\epsilon^2 \xi \times \\ 0 & -i\epsilon^2 \xi \times & 0 \end{pmatrix}. \tag{2.10}$$

Here, for $\xi \neq 0$,

$$\tilde{\mathbb{B}}_\epsilon(\xi) = L_1 - i\epsilon v \cdot \xi - i\epsilon \frac{v \cdot \xi}{|\xi|^2} P_d. \tag{2.11}$$

Remark 2.1. The eigenvalues of the operator $\mathbb{A}_\epsilon(\xi)$ are same as those of $\tilde{\mathbb{A}}_\epsilon(\xi)$, and the eigenfunctions of $\mathbb{A}_\epsilon(\xi)$ can be obtained as linear combinations of those for $\tilde{\mathbb{A}}_\epsilon(\xi)$. In fact, let β be an eigenvalue with the corresponding eigenvector denoted by $\mathcal{U} = (\phi, E, B)$

of $\tilde{\mathbb{A}}_\epsilon(\xi)$. Then $U = (\phi, -i\frac{\xi}{|\xi|^2}(\phi, \chi_0) - \frac{\xi}{|\xi|} \times E, -\frac{\xi}{|\xi|} \times B)$ is the corresponding eigenvector with the eigenvalue β for $\mathbb{A}_\epsilon(\xi)$.

2.1. Spectrum structure of $\mathbb{A}_\epsilon(\xi)$

2.1.1. Spectrum structure

As usual, we denote a weighted Hilbert space $L^2_\xi(\mathbb{R}^3)$ for $\xi \neq 0$ as

$$L^2_\xi(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) \,|\, \|f\|_\xi = \sqrt{(f, f)_\xi} < \infty\},$$

with the inner product defined by

$$(f, g)_\xi = (f, g) + \frac{1}{|\xi|^2}(P_d f, P_d g).$$

For any fixed $\xi \neq 0$, we define a subspace of $\mathbb{C}^3$ by

$$\mathbb{C}^3_\xi = \{y \in \mathbb{C}^3 \,|\, y \cdot \xi = 0\}.$$

For any vectors $U = (f, E_1, B_1), V = (g, E_2, B_2) \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3 \times \mathbb{C}^3$, a weighted inner product with the corresponding norm is defined by

$$(U, V)_\xi = (f, g)_\xi + (E_1, E_2) + (B_1, B_2), \quad \|U\|_\xi = \sqrt{(U, U)_\xi}.$$

Another L^2 inner product and norm is denoted by

$$(U, V) = (f, g) + (E_1, E_2) + (B_1, B_2), \quad \|U\| = \sqrt{(U, U)}.$$

Since P_d is a self-adjoint projection operator, it follows that $(P_d f, P_d g) = (P_d f, g) = (f, P_d g)$ and hence

$$(f, g)_\xi = (f, g + \frac{1}{|\xi|^2}P_d g) = (f + \frac{1}{|\xi|^2}P_d f, g). \tag{2.12}$$

By (2.12), we have for any $f, g \in L^2_\xi(\mathbb{R}^3_v) \cap D(\tilde{\mathbb{B}}_\epsilon(\xi))$,

$$(\tilde{\mathbb{B}}_\epsilon(\xi) f, g)_\xi = (\tilde{\mathbb{B}}_\epsilon(\xi) f, g + \frac{1}{|\xi|^2}P_d g) = (f, \tilde{\mathbb{B}}_\epsilon(-\xi) g)_\xi. \tag{2.13}$$

Moreover, $\tilde{\mathbb{B}}_\epsilon(\xi)$ is a dissipate operator in $L^2_\xi(\mathbb{R}^3)$:

$$\mathrm{Re}(\tilde{\mathbb{B}}_\epsilon(\xi) f, f)_\xi = (L_1 f, f) \leq 0. \tag{2.14}$$

Note that $\tilde{\mathbb{B}}_\epsilon(\xi)$ is a linear operator from the space $L^2_\xi(\mathbb{R}^3)$ to itself, and for $y \in \mathbb{C}^3_\xi$,

$$\frac{\xi}{|\xi|} \times \frac{\xi}{|\xi|} \times y = -y. \tag{2.15}$$

Hence, $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ is an invariant subspace of the operator $\tilde{\mathbb{A}}_\epsilon(\xi)$. Thus, $\tilde{\mathbb{A}}_\epsilon(\xi)$ can be regarded as a linear operator on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ and it satisfies for any $U = (g, X, Y) \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ that

$$\mathrm{Re}(\tilde{\mathbb{A}}_\epsilon(\xi)U, U)_\xi = (L_1 g, g) \leq 0. \tag{2.16}$$

Denote the spectrum of the operator A by $\sigma(A)$. The essential spectrum of A, denoted by $\sigma_{ess}(A)$, is the set $\{\lambda \in \mathbb{C} \,|\, \lambda - A \text{ is not a Fredholm operator}\}$ (cf. [24]). The discrete spectrum of A, denoted by $\sigma_d(A)$, is the set $\sigma(A) \setminus \sigma_{ess}(A)$ which consists of all isolated eigenvalues with finite multiplicity. And $\rho(A)$ denotes its resolvent set.

We first have the following lemma about $\tilde{\mathbb{A}}_\epsilon(\xi)$.

Lemma 2.2. *The operator $\tilde{\mathbb{A}}_\epsilon(\xi)$ generates a strongly continuous contraction semigroup on $L^2_\xi(\mathbb{R}^3) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$, which satisfies*

$$\|e^{t\tilde{\mathbb{A}}_\epsilon(\xi)}U\|_\xi \leq \|U\|_\xi, \quad \forall t > 0,\, U \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi.$$

Moreover, $\rho(\tilde{\mathbb{A}}_\epsilon(\xi)) \supset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda > 0\}$.

Proof. We first show that both $\tilde{\mathbb{A}}_\epsilon(\xi)$ and $\tilde{\mathbb{A}}^*_\epsilon(\xi)$ are dissipative operators on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$. By (2.13), for any $U, V \in D(\tilde{\mathbb{B}}_\epsilon(\xi)) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ it holds that

$$(\tilde{\mathbb{A}}_\epsilon(\xi)U, V)_\xi = (U, \tilde{\mathbb{A}}^*_\epsilon(\xi)V)_\xi,$$

where

$$\tilde{\mathbb{A}}^*_\epsilon(\xi) = \begin{pmatrix} \tilde{\mathbb{B}}_\epsilon(-\xi) & \epsilon v\chi_0 \cdot \omega\times & 0 \\ \epsilon\omega \times P_m & 0 & -i\epsilon^2\xi\times \\ 0 & i\epsilon^2\xi\times & 0 \end{pmatrix}. \tag{2.17}$$

By (2.16), both $\tilde{\mathbb{A}}_\epsilon(\xi)$ and $\tilde{\mathbb{A}}^*_\epsilon(\xi)$ are dissipative, namely,

$$\mathrm{Re}(\tilde{\mathbb{A}}_\epsilon(\xi)U, U)_\xi = \mathrm{Re}(\tilde{\mathbb{A}}^*_\epsilon(\xi)U, U)_\xi = (L_1 g, g) \leq 0, \quad \forall U = (g, X, Y).$$

Since $\tilde{\mathbb{A}}_\epsilon(\xi)$ is a densely defined closed operator, it follows from Corollary 4.4 in [31] that the operator $\tilde{\mathbb{A}}_\epsilon(\xi)$ generates a C_0-contraction semigroup on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$. In addition, it holds that $\rho(\tilde{\mathbb{A}}_\epsilon(\xi)) \supset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda > 0\}$. $\square$

Now we denote by T a linear operator on $L^2(\mathbb{R}^3_v)$ or $L^2_\xi(\mathbb{R}^3_v)$ with norms by

$$\|T\| = \sup_{\|f\|=1} \|Tf\|, \quad \|T\|_\xi = \sup_{\|f\|_\xi=1} \|Tf\|_\xi.$$

Obviously,

$$(1+|\xi|^{-2})^{-1/2}\|T\| \le \|T\|_\xi \le (1+|\xi|^{-2})^{1/2}\|T\|. \tag{2.18}$$

Set

$$D_\epsilon(\xi) = -\nu(v) - i\epsilon v\cdot\xi, \quad \mathbb{B}_2(\xi) = \begin{pmatrix} 0 & i\xi\times \\ -i\xi\times & 0 \end{pmatrix}_{6\times 6}. \tag{2.19}$$

Since $\mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ is an invariant subspace of the operator $\mathbb{B}_2(\xi)$, we can regard $\mathbb{B}_2(\xi)$ as an operator on $\mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$. Moreover, the operator $\lambda - \mathbb{B}_2(\xi)$ is invertible on $\mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ for any $\lambda \neq \pm i|\xi|$ and satisfies (cf. [26])

$$\|(\lambda - \mathbb{B}_2(\xi))^{-1}\| = \max_{j=\pm 1} |\lambda - ji|\xi||^{-1}. \tag{2.20}$$

We now study the spectrum of $\tilde{\mathbb{A}}_\epsilon(\xi)$.

Lemma 2.3. *The following statements hold for all $\xi \neq 0$ and $\epsilon \in [0,1)$.*

(1) $\sigma_{ess}(\tilde{\mathbb{A}}_\epsilon(\xi)) \subset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \le -\nu_0\}$ *and* $\sigma(\tilde{\mathbb{A}}_\epsilon(\xi)) \cap \{\lambda \in \mathbb{C} \,|\, -\nu_0 < \mathrm{Re}\lambda \le 0\} \subset \sigma_d(\tilde{\mathbb{A}}_\epsilon(\xi))$.
(2) *If λ is an eigenvalue of $\tilde{\mathbb{A}}_\epsilon(\xi)$, then* $\mathrm{Re}\lambda < 0$ *for any $\epsilon \neq 0$ and $\lambda = 0$ if and only if $\epsilon = 0$.*

Proof. We decompose $\tilde{\mathbb{A}}_\epsilon(\xi)$ into

$$\tilde{\mathbb{A}}_\epsilon(\xi) = G^1_\epsilon(\xi) + G^2_\epsilon(\xi), \tag{2.21}$$

where

$$G^1_\epsilon(\xi) = \begin{pmatrix} D_\epsilon(\xi) & 0 & 0 \\ 0 & 0 & i\epsilon^2\xi\times \\ 0 & -i\epsilon^2\xi\times & 0 \end{pmatrix}, \tag{2.22}$$

$$G^2_\epsilon(\xi) = \begin{pmatrix} K_1 - i\epsilon\frac{v\cdot\xi}{|\xi|^2}P_d & -\epsilon v\chi_0\cdot\omega\times & 0 \\ -\epsilon\omega\times P_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.23}$$

By (2.20) and (1.17), the operator $\lambda - G^1_\epsilon(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$ for $\mathrm{Re}\lambda > -\nu_0$ and $\lambda \neq \pm i\epsilon^2|\xi|$, and it satisfies

$$(\lambda - G^1_\epsilon(\xi))^{-1} = \begin{pmatrix} (\lambda - D_\epsilon(\xi))^{-1} & 0 \\ 0 & (\lambda - \epsilon^2\mathbb{B}_2(\xi))^{-1} \end{pmatrix}_{7\times 7}.$$

Since $G^2_\epsilon(\xi)$ is a compact operator on $L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$ for any fixed $\xi \neq 0$, $\tilde{\mathbb{A}}_\epsilon(\xi)$ is a compact perturbation of $G^1_\epsilon(\xi)$. Hence, it follows from Weyl's Theorem (Theorem 5.35 in [24]) that

$$\sigma_{ess}(\tilde{\mathbb{A}}_\epsilon(\xi)) = \sigma_{ess}(G^1_\epsilon(\xi)) = R(D_\epsilon(\xi)) \subset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \le -\nu_0\}.$$

Thus the spectrum of $\tilde{\mathbb{A}}_\epsilon(\xi)$ in the domain $\mathrm{Re}\lambda > -\nu_0$ consists of discrete eigenvalues with possible accumulation points only on the line $\mathrm{Re}\lambda = -\nu_0$. This and Lemma 2.2 prove the part (1).

We claim that for any $\lambda \in \sigma_d(\tilde{\mathbb{A}}_\epsilon(\xi))$ in the region $\mathrm{Re}\lambda > -\nu_0$, it holds that $\mathrm{Re}\lambda < 0$ for $\epsilon \neq 0$. Indeed, set $\xi = s\omega$ and let $U = (f, E, B) \in L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$ be the eigenvector corresponding to the eigenvalue λ so that

$$\begin{cases} \lambda f = L_1 f - i\epsilon s(v\cdot\omega)(f + \frac{1}{s^2}P_d f) - \epsilon v\chi_0\cdot(\omega\times E), \\ \lambda E = -\epsilon\omega\times(f, v\chi_0) + i\epsilon^2\xi\times B, \\ \lambda B = -i\epsilon^2\xi\times E. \end{cases} \tag{2.24}$$

Taking the inner product $(2.24)_1$ with $f + \frac{1}{s^2}P_d f$, we have

$$\mathrm{Re}\lambda\left(\|f\|^2_\xi + |E|^2 + |B|^2\right) = (L_1 f, f) \le 0,$$

which implies $\mathrm{Re}\lambda \le 0$.

Furthermore, if there exists an eigenvalue λ with $\mathrm{Re}\lambda = 0$, then it follows from the above that $(L_1 f, f) = 0$, namely, $f = C_0\sqrt{M} \in N_1$. Substitute this into (2.24), we obtain

$$\lambda C_0\sqrt{M} = -i\epsilon(v\cdot\omega)\Big(s + \frac{1}{s}\Big)C_0\chi_0 - \epsilon v\chi_0\cdot(\omega\times E),$$

which implies that $C_0 \neq 0$ and $\omega\times E \neq 0$ unless $\epsilon = 0$ and $\lambda = 0$. When $\epsilon \neq 0$, it holds that $C_0 = 0$ and $\omega\times E = 0$ and hence $f = 0$ and $E = 0$. Substitute this into (2.24), we obtain $B \equiv 0$. This is a contradiction and thus it holds $\mathrm{Re}\lambda < 0$ for $\epsilon \neq 0$. This proves the part (2) and then it completes the proof of the lemma. □

We now consider the spectrum and resolvent sets of $\tilde{\mathbb{A}}_\epsilon(\xi)$ for $\epsilon|\xi|$ large. For $\mathrm{Re}\lambda > -\nu_0$ and $\lambda \neq \pm i\epsilon^2|\xi|$, we decompose $\lambda - \tilde{\mathbb{A}}_\epsilon(\xi)$ into

$$\begin{aligned} \lambda - \tilde{\mathbb{A}}_\epsilon(\xi) &= \lambda - G^1_\epsilon(\xi) - G^2_\epsilon(\xi) \\ &= \left(I - G^2_\epsilon(\xi)(\lambda - G^1_\epsilon(\xi))^{-1}\right)\left(\lambda - G^1_\epsilon(\xi)\right), \end{aligned} \tag{2.25}$$

where $G_\epsilon^1(\xi)$ and $G_\epsilon^2(\xi)$ are defined by (2.22) and (2.23) respectively. A direct computation shows that

$$\begin{cases} G_\epsilon^2(\xi)(\lambda - G_\epsilon^1(\xi))^{-1} = \begin{pmatrix} X_\epsilon^1(\lambda,\xi) & X_\epsilon^2(\lambda,\xi) \\ X_\epsilon^3(\lambda,\xi) & 0 \end{pmatrix}_{7\times 7}, \\ X_\epsilon^1(\lambda,\xi) = \left(K_1 - i\epsilon \dfrac{v\cdot\xi}{|\xi|^2} P_d\right)(\lambda - D_\epsilon(\xi))^{-1}, \\ X_\epsilon^2(\lambda,\xi) = (\epsilon v\chi_0 \cdot \omega \times, 0_{1\times 3})_{1\times 6}\,(\lambda - \epsilon^2 \mathbb{B}_2(\xi))^{-1}, \\ X_\epsilon^3(\lambda,\xi) = \begin{pmatrix} -\epsilon\omega \times P_m(\lambda - D_\epsilon(\xi))^{-1} \\ 0_{3\times 1} \end{pmatrix}_{6\times 1}. \end{cases} \tag{2.26}$$

Then, we have the estimates on the terms on the right hand side of (2.26) as follows.

Lemma 2.4. *The following estimates hold.*

(1) *For any $\delta > 0$, if $\mathrm{Re}\lambda \ge -\nu_0 + \delta$, then we have*

$$\|K_1(\lambda - D_\epsilon(\xi))^{-1}\| \le C\delta^{-\frac12}(1+\epsilon|\xi|)^{-\frac12}. \tag{2.27}$$

(2) *For any $\delta > 0$, $\tau_0 > 0$, if $\mathrm{Re}\lambda \ge -\nu_0 + \delta$ and $\epsilon|\xi| \le \tau_0$, then we have*

$$\|K_1(\lambda - D_\epsilon(\xi))^{-1}\| \le C\delta^{-1}(1+\tau_0)^{\frac12}(1+|\mathrm{Im}\lambda|)^{-\frac12}. \tag{2.28}$$

(3) *For any $\delta > 0$, if $\mathrm{Re}\lambda \ge -\nu_0 + \delta$, then we have*

$$\|P_m(\lambda - D_\epsilon(\xi))^{-1}\| \le C\delta^{-\frac12}(1+\epsilon|\xi|)^{-\frac12}, \tag{2.29}$$

$$\|P_m(\lambda - D_\epsilon(\xi))^{-1}\| \le C(\delta^{-1}+1)(1+\epsilon|\xi|)|\lambda|^{-1}. \tag{2.30}$$

(4) *For any $\delta > 0$, if $\mathrm{Re}\lambda \ge -\nu_0 + \delta$, then we have*

$$\|(v\cdot\xi)|\xi|^{-2}P_d(\lambda - D_\epsilon(\xi))^{-1}\| \le C\delta^{-1}|\xi|^{-1}, \tag{2.31}$$

$$\|(v\cdot\xi)|\xi|^{-2}P_d(\lambda - D_\epsilon(\xi))^{-1}\| \le C(\delta^{-1}+1)(|\xi|^{-1}+1)|\lambda|^{-1}. \tag{2.32}$$

Proof. The estimates (2.29)–(2.32) are proved in Lemma 3.5 of [26]. Thus, we only prove (2.27) and (2.28) in the following. Let $\lambda = x + iy$ with $(x,y) \in \mathbb{R}\times\mathbb{R}$. Then

$$\|K_1(\lambda - D_\epsilon(\xi))^{-1} f\|^2$$

$$= \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} k_1(v,u)(\nu(u) + \lambda + i\epsilon u\cdot\xi)^{-1} f(u)du \right|^2 dv$$

$$
\begin{aligned}
&\leq \int\limits_{\mathbb{R}^3} \left(\int\limits_{\mathbb{R}^3} k_1(v,u) \frac{1}{|\nu(u)+\lambda+i\epsilon u\cdot\xi|^2} du \right) \left(\int\limits_{\mathbb{R}^3} k_1(v,u)|f(u)|^2 du \right) dv \\
&\leq C \sup_{v\in\mathbb{R}^3} \int\limits_{\mathbb{R}^3} k_1(v,u) \frac{1}{(\nu(u)+x)^2+(y+\epsilon u\cdot\xi)^2} du \|f\|^2. \qquad (2.33)
\end{aligned}
$$

From (1.16), one has

$$
|k_1(v,u)| \leq C \frac{1}{|\bar{v}-\bar{u}|} e^{-\frac{|v-u|^2}{8}}, \quad \bar{u}=(u_2,u_3). \qquad (2.34)
$$

Let $\mathbb{O}$ be a rotation in $\mathbb{R}^3$ satisfying $\mathbb{O}^T\xi = (|\xi|,0,0)$. By changing variables $v \to \mathbb{O}v$, $u \to \mathbb{O}u$, we obtain from (2.33) and (2.34) that for $\delta = \nu_0 + x > 0$,

$$
\begin{aligned}
&\|K_1(\lambda - D_\epsilon(\xi))^{-1} f\|^2 \\
\leq& C \sup_{v\in\mathbb{R}^3} \int\limits_{\mathbb{R}^3} k_1(v,u) \frac{1}{(\nu(u)+x)^2+(y+\epsilon u_1|\xi|)^2} du \|f\|^2 \\
\leq& C \sup_{v\in\mathbb{R}^3} \int\limits_{\mathbb{R}} \frac{1}{(\nu_0+x)^2+(y+\epsilon u_1|\xi|)^2} du_1 \int\limits_{\mathbb{R}^2} \frac{1}{|\bar{v}-\bar{u}|} e^{-\frac{|\bar{v}-\bar{u}|^2}{8}} d\bar{u} \|f\|^2 \\
\leq& C \frac{1}{\epsilon|\xi|} \int\limits_{\mathbb{R}} \frac{1}{(\nu_0+x)^2+u_1^2} du_1 \|f\|^2 \leq C\delta^{-1}(\epsilon|\xi|)^{-1}\|f\|^2.
\end{aligned}
$$

This gives (2.27).

For (2.28), we first decompose

$$
\begin{aligned}
\|K_1(\lambda - D_\epsilon(\xi))^{-1} f\|^2 \leq& 2 \int\limits_{\mathbb{R}^3} \left| \int\limits_{|u|\leq R} k_1(v,u)(\nu(u)+\lambda+i\epsilon u\cdot\xi)^{-1} f(u) du \right|^2 dv \\
&+ 2 \int\limits_{\mathbb{R}^3} \left| \int\limits_{|u|\geq R} k_1(v,u)(\nu(u)+\lambda+i\epsilon u\cdot\xi)^{-1} f(u) du \right|^2 dv \\
=:& I_1 + I_2. \qquad (2.35)
\end{aligned}
$$

For I_1, it holds that

$$
\begin{aligned}
I_1 \leq& C \sup_{v\in\mathbb{R}^3} \int\limits_{|u|\leq R} k_1(v,u) \frac{1}{(\nu(u)+x)^2+(y+\epsilon u_1|\xi|)^2} du \|f\|^2 \\
\leq& C \sup_{v\in\mathbb{R}^3} \int_{-R}^{R} \frac{1}{(\nu_0+x)^2+(y+\epsilon u_1|\xi|)^2} du_1 \int_{-R}^{R}\int_{-R}^{R} k_1(v,u) d\bar{u} \|f\|^2
\end{aligned}
$$

$$\leq C \int_{-R}^{R} \frac{1}{(\nu_0 + x)^2 + (y + \epsilon u_1 |\xi|)^2} du_1 \|f\|^2.$$

If $\epsilon|\xi| \leq \tau_0$, $|u| \leq R$ and $|y| \geq 2\tau_0 R$, we have

$$|y + \epsilon u_1 |\xi|| \geq |y| - \tau_0 R \geq \frac{|y|}{2}.$$

Thus

$$I_1 \leq C \int_{-R}^{R} \frac{1}{\delta^2 + y^2} du_1 \|f\|^2 = C \frac{1}{\delta^2 + y^2} R \|f\|^2. \tag{2.36}$$

For I_2, since

$$\int_{\mathbb{R}^3} k_1(v,u) du \leq C(1 + |v|)^{-1},$$

we obtain

$$\begin{aligned} I_2 \leq & \int_{\mathbb{R}^3} \left(\int_{|u| \geq R} k_1(v,u) \delta^{-2} du \right) \left(\int_{|u| \geq R} k(v,u) |f(u)|^2 du \right) dv \\ \leq & C\delta^{-2} \int_{|u| \geq R} \left(\int_{\mathbb{R}^3} k_1(v,u) dv \right) |f(u)|^2 du \\ \leq & C\delta^{-2} \int_{|u| \geq R} (1 + |u|)^{-1} |f(u)|^2 du \leq C\delta^{-2} R^{-1} \|f\|^2. \end{aligned} \tag{2.37}$$

By choosing $R = |y| / \max\{2, 2\tau_0\}$, (2.28) follows from (2.35)–(2.37). And this completes the proof of the lemma. □

We now state a lemma from [26].

Lemma 2.5 *([26]). Let K_1, K_4 be the operators on the space X and Y, and K_2, K_3 be the operators $Y \to X$ and $X \to Y$ respectively. Let K be a matrix operator on $X \times Y$ defined by*

$$K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}.$$

If the norms of K_1, K_2, K_3 and K_4 satisfy

$$\|K_1\| < 1, \quad \|K_4\| < 1, \quad \|K_2\|\|K_3\| < (1 - \|K_1\|)(1 - \|K_4\|),$$

then the operator $I + K$ is invertible on $X \times Y$.

By Lemmas 2.4 and 2.5, we have the following lemma about the spectrum structure of the operator $\tilde{\mathbb{A}}_\epsilon(\xi)$ for $\epsilon|\xi|$ being large.

Lemma 2.6. *Fixed $\epsilon \in (0, 1)$. The following statements hold.*

(1) *For any $\delta > 0$, there exists $R_1 = R_1(\delta) > 0$ such that for $\epsilon|\xi| > R_1$,*

$$\sigma(\tilde{\mathbb{A}}_\epsilon(\xi)) \cap \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \geq -\frac{\nu_0}{2}\} \subset \sum_{j=\pm 1} \{\lambda \in \mathbb{C} \,|\, |\lambda - \epsilon^2 ji|\xi|| \leq \epsilon^2 \delta\}. \tag{2.38}$$

(2) *For any $r_1 > r_0 > 0$, there exists $\alpha = \alpha(r_0, r_1) > 0$ such that for $r_0 \leq \epsilon|\xi| \leq r_1$,*

$$\sigma(\tilde{\mathbb{A}}_\epsilon(\xi)) \subset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda < -\alpha\}. \tag{2.39}$$

Proof. We prove (2.38) first. By Lemma 2.4, (2.26), (2.20) and (2.18), there exists $R_1 = R_1(\delta) > 0$ such that for $\mathrm{Re}\lambda \geq -\nu_0/2$, $|\lambda - \epsilon^2 ji|\xi|| > \epsilon^2\delta$ and $\epsilon|\xi| > R_1$,

$$\|X^1_\epsilon(\lambda, \xi)\|_\xi \leq 1/2, \quad \|X^2_\epsilon(\lambda, \xi)\| \leq \epsilon^{-1}\delta^{-1}, \quad \|X^3_\epsilon(\lambda, \xi)\| \leq \epsilon\delta/4.$$

This and Lemma 2.5 imply that the operator $I - G^2_\epsilon(\xi)(\lambda - G^1_\epsilon(\xi))^{-1}$ is invertible on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ and thus $\lambda - \tilde{\mathbb{A}}_\epsilon(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ and satisfies

$$(\lambda - \tilde{\mathbb{A}}_\epsilon(\xi))^{-1} = \left(\lambda - G^1_\epsilon(\xi)\right)^{-1} \left(I - G^2_\epsilon(\xi)(\lambda - G^1_\epsilon(\xi))^{-1}\right)^{-1}.$$

Therefore, it holds that for $\epsilon|\xi| > R_1$,

$$\rho(\tilde{\mathbb{A}}_\epsilon(\xi)) \supset \{\lambda \in \mathbb{C} \,|\, \min_{j=\pm 1} |\lambda - \epsilon^2 ji|\xi|| > \epsilon^2\delta,\ \mathrm{Re}\lambda \geq -\frac{\nu_0}{2}\}, \tag{2.40}$$

which implies (2.38).

Next, we turn to prove (2.39). By Lemma 2.4, (2.20) and (2.18), there exists $y_1 = y_1(r_0, r_1) > 0$ large enough such that for $\mathrm{Re}\lambda \geq -\nu_0/2$, $|\mathrm{Im}\lambda| > y_1$ and $r_0 \leq \epsilon|\xi| \leq r_1$,

$$\|X^1_\epsilon(\lambda, \xi)\|_\xi \leq 1/6, \quad \|X^2_\epsilon(\lambda, \xi)\| \leq 1/6, \quad \|X^3_\epsilon(\lambda, \xi)\| \leq 1/6.$$

This implies that the operator $I - G^2_\epsilon(\xi)(\lambda - G^1_\epsilon(\xi))^{-1}$ is invertible on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$, which together with (2.25) yield that $\lambda - \tilde{\mathbb{A}}_\epsilon(\xi)$ is also invertible on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ when $\mathrm{Re}\lambda \geq -\nu_0/2$, $|\mathrm{Im}\lambda| > y_1$ and $r_0 \leq \epsilon|\xi| \leq r_1$. Hence, for $r_0 \leq \epsilon|\xi| \leq r_1$ we have

$$\sigma(\tilde{\mathbb{A}}_\epsilon(\xi)) \cap \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \geq -\frac{\nu_0}{2}\} \subset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \geq -\frac{\nu_0}{2},\ |\mathrm{Im}\lambda| \leq y_1\}. \tag{2.41}$$

By (2.41), it is sufficient to prove (2.39) holds for $|\mathrm{Im}\lambda| \le y_1$. We prove this by contradiction. If it does not hold, then there exists a sequence of $\{(\xi_n, \lambda_n, U_n)\}$ satisfying $\epsilon|\xi_n| \in [r_0, r_1]$, $U_n = (f_n, E_n, B_n) \in L^2_{\xi_n}(\mathbb{R}^3) \times \mathbb{C}^3_{\xi_n} \times \mathbb{C}^3_{\xi_n}$ with $\|U_n\|_{\xi_n} = 1$, and $\lambda_n U_n = \tilde{\mathbb{A}}_\epsilon(\xi_n) U_n$ with $|\mathrm{Im}\lambda_n| \le y_1$ and $\mathrm{Re}\lambda_n \to 0$ as $n \to \infty$. That is,

$$\begin{cases} \lambda_n f_n = (L_1 - i\epsilon v\cdot\xi_n - i\epsilon \frac{v\cdot\xi_n}{|\xi_n|^2} P_d) f_n - \epsilon v\chi_0 \cdot (\omega_n \times E_n), \\ \lambda_n E_n = -\epsilon \omega_n \times (f_n, v\chi_0) + i\epsilon^2 \xi_n \times B_n, \\ \lambda_n B_n = -i\epsilon^2 \xi_n \times E_n. \end{cases}$$

Rewrite the first equation as

$$(\lambda_n + \nu + i\epsilon v\cdot\xi_n) f_n = K_1 f_n - i\epsilon \frac{v\cdot\xi_n}{|\xi_n|^2} P_d f_n - \epsilon v\chi_0 \cdot (\omega_n \times E_n).$$

Since K_1 is a compact operator on $L^2(\mathbb{R}^3)$, there exists a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ and $g_1 \in L^2(\mathbb{R}^3)$ such that

$$K_1 f_{n_j} \to g_1 \quad \text{as} \quad j \to \infty.$$

By using the fact that $\epsilon|\xi_n| \in [r_0, r_1]$ and $P_d f_n = C_0^n \sqrt{M}$ with $|C_0^n|^2 + |E_n|^2 + |B_n|^2 \le 1$, there exists a subsequence of (still denoted by) $\{(\xi_{n_j}, f_{n_j}, E_{n_j}, B_{n_j})\}$, and (ξ_0, C_0, E_0, B_0) with $\epsilon|\xi_0| \in [r_0, r_1]$ and $|C_0|^2 + |E_0|^2 + |B_0|^2 \le 1$ such that $(\xi_{n_j}, C_0^{n_j}, E_{n_j}, B_{n_j}) \to (\xi_0, C_0, E_0, B_0)$ as $j \to \infty$. In particular,

$$\frac{v\cdot\xi_{n_j}}{|\xi_{n_j}|^2} P_d f_{n_j} \to \frac{v\cdot\xi_0}{|\xi_0|^2} C_0\sqrt{M} =: g_2, \qquad \frac{\xi_{n_j}}{|\xi_{n_j}|} \times E_{n_j} \to \frac{\xi_0}{|\xi_0|} \times E_0 =: Y_0 \ \text{ as } \ j \to \infty.$$

Since $|\mathrm{Im}\lambda_n| \le y_1$ and $\mathrm{Re}\lambda_n \to 0$, we can extract a subsequence of (still denoted by) $\{\lambda_{n_j}\}$ such that $\lambda_{n_j} \to \lambda_0$ with $\mathrm{Re}\lambda_0 = 0$. Then

$$\lim_{j\to\infty} f_{n_j} = \lim_{j\to\infty} \frac{g_1 - \epsilon g_2 - \epsilon(v\cdot Y_0)\chi_0}{\lambda_{n_j} + \nu + i\epsilon(v\cdot\xi_{n_j})} = \frac{g_1 - \epsilon g_2 - \epsilon(v\cdot Y_0)\chi_0}{\lambda_0 + \nu + i\epsilon(v\cdot\xi_0)} =: f_0 \ \text{ in } \ L^2.$$

It follows that $\tilde{\mathbb{A}}_\epsilon(\xi_0) U_0 = \lambda_0 U_0$ with $U_0 = (f_0, E_0, B_0) \in L^2_{\xi_0}(\mathbb{R}^3) \times \mathbb{C}^3_{\xi_0} \times \mathbb{C}^3_{\xi_0}$ and λ_0 is an eigenvalue of $\tilde{\mathbb{A}}_\epsilon(\xi_0)$ with $\mathrm{Re}\lambda_0 = 0$. This contradicts to the fact that $\mathrm{Re}\lambda < 0$ for $\epsilon \ne 0$ as stated in Lemma 2.3. Thus, the proof the lemma is completed. □

We now investigate the spectrum and resolvent sets of $\tilde{\mathbb{A}}_\epsilon(\xi)$ for $\epsilon(1+|\xi|)$ small. Based on macro-micro decomposition, we can split $\tilde{\mathbb{B}}_\epsilon(\xi)$ into

$$\begin{cases} \tilde{\mathbb{B}}_\epsilon(\xi) = Q_\epsilon(\xi) + \epsilon \mathbb{B}_3(\xi), \\ Q_\epsilon(\xi) = L_1 - i\epsilon P_r (v\cdot\xi) P_r, \\ \mathbb{B}_3(\xi) = i P_d (v\cdot\xi) P_r + i(1 + \frac{1}{|\xi|^2}) P_r (v\cdot\xi) P_d. \end{cases} \tag{2.42}$$

Thus, we can decompose $\lambda - \tilde{\mathbb{A}}_\epsilon(\xi)$ into

$$\lambda - \tilde{\mathbb{A}}_\epsilon(\xi) = \lambda - G_\epsilon^3(\xi) - G_\epsilon^4(\xi), \tag{2.43}$$

where

$$G_\epsilon^3(\xi) = \begin{pmatrix} Q_\epsilon(\xi) & 0 & 0 \\ 0 & 0 & i\epsilon^2\xi\times \\ 0 & -i\epsilon^2\xi\times & 0 \end{pmatrix}, \tag{2.44}$$

$$G_\epsilon^4(\xi) = \begin{pmatrix} \epsilon\mathbb{B}_3(\xi) & -\epsilon v\chi_0\cdot\omega\times & 0 \\ -\epsilon\omega\times P_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.45}$$

Lemma 2.7. *Let $\xi \neq 0$ and $Q_\epsilon(\xi)$ defined by (2.42). We have*

(1) *If $\lambda \neq 0$, then*

$$\left\| \lambda^{-1}\left(1 + \frac{1}{|\xi|^2}\right) P_r(v\cdot\xi)P_d \right\|_\xi \le C(|\xi|+1)|\lambda|^{-1}. \tag{2.46}$$

(2) *If $\mathrm{Re}\lambda > -\mu$, then the operator $\lambda P_r - Q_\epsilon(\xi)$ is invertible on $N_1^\perp$ and satisfies*

$$\|(\lambda P_r - Q_\epsilon(\xi))^{-1}\| \le (\mathrm{Re}\lambda + \mu)^{-1}, \tag{2.47}$$

$$\|P_d(v\cdot\xi)(\lambda P_r - Q_\epsilon(\xi))^{-1}P_r\|_\xi \le C(\mathrm{Re}\lambda+\mu)^{-1}(1+|\xi|)[1+(1+\epsilon|\xi|)^{-1}|\lambda|]^{-1}, \tag{2.48}$$

$$\|P_m(\lambda P_r - Q_\epsilon(\xi))^{-1}P_r\| \le C(\mathrm{Re}\lambda+\mu)^{-1}[1+(1+\epsilon|\xi|)^{-1}|\lambda|]^{-1}. \tag{2.49}$$

Proof. The estimates (2.46) and (2.47) are proved in Lemma 3.5 of [25]. By (2.47) and the fact that $\|P_d(v\cdot\xi)P_r f\|_\xi \le C(|\xi|+1)\|P_r f\|$, we have

$$\|P_d(v\cdot\xi)(\lambda P_r - Q_\epsilon(\xi))^{-1}P_r f\|_\xi \le C(|\xi|+1)(\mathrm{Re}\lambda+\mu)^{-1}\|P_r f\|. \tag{2.50}$$

We now decompose the operator $P_d(v\cdot\xi)(\lambda P_r - Q_\epsilon(\xi))^{-1}P_r$ as

$$P_d(v\cdot\xi)(\lambda P_r - Q_\epsilon(\xi))^{-1}P_r = \frac{1}{\lambda}P_d(v\cdot\xi)P_r + \frac{1}{\lambda}P_d(v\cdot\xi)Q_\epsilon(\xi)(\lambda P_r - Q_\epsilon(\xi))^{-1}P_r.$$

This together with (2.47) and the fact that $\|P_d(v\cdot\xi)Q_\epsilon(\xi)f\|_\xi \le C(|\xi|+1)(1+\epsilon|\xi|)\|P_r f\|$ give

$$\|P_d(v\cdot\xi)(\lambda P_r - Q_\epsilon(\xi))^{-1}P_r f\|_\xi \le C(|\xi|+1)(1+\epsilon|\xi|)|\lambda|^{-1}[(\mathrm{Re}\lambda+\mu)^{-1}+1]\|f\|. \tag{2.51}$$

The combination of the two cases (2.50) and (2.51) yields (2.48). (2.49) can be proved similarly. This completes the proof of the lemma. □

Lemma 2.8. *For fixed $\epsilon \in (0,1)$, the following holds.*

(1) *For any $\delta > 0$, there are two constants $r_1 = r_1(\delta)$, $y_1 = y_1(\delta) > 0$ such that for all $|\xi| \neq 0$,*

$$\rho(\tilde{\mathbb{A}}_\epsilon(\xi)) \supset \begin{cases} \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \geq -\dfrac{\nu_0}{2},\ |\lambda \pm \epsilon^2 i|\xi|| \geq \epsilon^2\delta\} \cup \mathbb{C}_+, & \epsilon|\xi| \geq r_1; \\ \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \geq -\dfrac{\mu}{2},\ |\mathrm{Im}\lambda| \geq y_1\} \cup \mathbb{C}_+, & \epsilon|\xi| \leq r_1, \end{cases} \tag{2.52}$$

where $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda > 0\}$.

(2) *For any $\delta > 0$, there exists $r_0 = r_0(\delta) > 0$ such that for $\epsilon(1 + |\xi|) \leq r_0$,*

$$\sigma(\tilde{\mathbb{A}}_\epsilon(\xi)) \cap \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \geq -\frac{\mu}{2}\} \subset \{\lambda \in \mathbb{C} \,|\, |\lambda| \leq \delta\}. \tag{2.53}$$

Proof. By (2.40), there exists $r_1 = r_1(\delta) > 0$ such that the first part of (2.52) holds. Thus, we only need to prove the second part of (2.52). By Lemma 2.7, we have for $\mathrm{Re}\lambda > -\mu$ and $\lambda \neq 0$ that the operator $\lambda - Q_\epsilon(\xi) = \lambda P_d + \lambda P_r - Q_\epsilon(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v)$ and it satisfies

$$(\lambda - Q_\epsilon(\xi))^{-1} = \lambda^{-1} P_d + (\lambda P_r - Q_\epsilon(\xi))^{-1} P_r.$$

Here, we have used the fact that the operator λP_d is orthogonal to $\lambda P_r - Q_\epsilon(\xi)$. Thus, for $\mathrm{Re}\lambda \geq -\mu/2$ and $\lambda \neq 0, \pm\epsilon^2 i|\xi|$, the operator $\lambda - G^3_\epsilon(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ and satisfies

$$(\lambda - G^3_\epsilon(\xi))^{-1} = \begin{pmatrix} \lambda^{-1} P_d + (\lambda P_r - Q_\epsilon(\xi))^{-1} P_r & 0 \\ 0 & (\lambda - \epsilon^2 \mathbb{B}_2(\xi))^{-1} \end{pmatrix}_{7\times 7}.$$

Therefore, we can rewrite (2.43) as

$$\lambda - \tilde{\mathbb{A}}_\epsilon(\xi) = \left(I - G^4_\epsilon(\xi)(\lambda - G^3_\epsilon(\xi))^{-1}\right)\left(\lambda - G^3_\epsilon(\xi)\right),$$

where

$$\begin{cases} G^4_\epsilon(\xi)(\lambda - G^3_\epsilon(\xi))^{-1} = \begin{pmatrix} X^4_\epsilon(\lambda,\xi) & X^2_\epsilon(\lambda,\xi) \\ X^5_\epsilon(\lambda,\xi) & 0 \end{pmatrix}_{7\times 7}, \\ X^4_\epsilon(\lambda,\xi) = i\epsilon P_d (v\cdot\xi)(\lambda P_r - Q_\epsilon(\xi))^{-1} P_r + i\epsilon\lambda^{-1}\left(1 + \dfrac{1}{|\xi|^2}\right) P_r (v\cdot\xi) P_d, \\ X^5_\epsilon(\lambda,\xi) = \begin{pmatrix} -\epsilon\omega \times P_m (\lambda P_r - Q_\epsilon(\xi))^{-1} P_r \\ 0_{3\times 1} \end{pmatrix}_{6\times 1}. \end{cases} \tag{2.54}$$

For $\epsilon|\xi| \leq r_1$, by (2.26), (2.54) and (2.46)–(2.49) we can choose $y_1 = y_1(r_1) > 0$ such that it holds for $\mathrm{Re}\lambda \geq -\mu/2$ and $|\mathrm{Im}\lambda| \geq y_1$ that

$$\|X_\epsilon^4(\lambda,\xi)\|_\xi + \|X_\epsilon^2(\lambda,\xi)\| + \|X_\epsilon^5(\lambda,\xi)\| \leq 1/2. \tag{2.55}$$

This implies that the operator $I - G_\epsilon^4(\xi)(\lambda - G_\epsilon^3(\xi))^{-1}$ is invertible on $L_\xi^2(\mathbb{R}_v^3) \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3$ and thus $\lambda - \tilde{\mathbb{A}}_\epsilon(\xi)$ is invertible on $L_\xi^2(\mathbb{R}_v^3) \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3$ and it satisfies

$$(\lambda - \tilde{\mathbb{A}}_\epsilon(\xi))^{-1} = \left(\lambda - G_\epsilon^3(\xi)\right)^{-1}\left(I - G_\epsilon^4(\xi)(\lambda - G_\epsilon^3(\xi))^{-1}\right)^{-1}.$$

Therefore, $\rho(\tilde{\mathbb{A}}_\epsilon(\xi)) \supset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \geq -\mu/2, |\mathrm{Im}\lambda| \geq y_1\}$ for $\epsilon|\xi| \leq r_1$. This and Lemma 2.2 prove (2.52).

Assume that $|\lambda| > \delta$ and $\mathrm{Re}\lambda \geq -\mu/2$. Then, by (2.46)–(2.49) we can choose $r_0 = r_0(\delta) > 0$ small enough so that (2.55) still holds for $\epsilon(1+|\xi|) \leq r_0$. This implies that the operator $\lambda - \tilde{\mathbb{A}}_\epsilon(\xi)$ is invertible on $L_\xi^2(\mathbb{R}^3) \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3$. Therefore, we have $\rho(\tilde{\mathbb{A}}_\epsilon(\xi)) \supset \{\lambda \in \mathbb{C} \,|\, |\lambda| > \delta, \mathrm{Re}\lambda \geq -\mu/2\}$ for $\epsilon(1+|\xi|) \leq r_0$, which gives (2.53). And this completes the proof of the lemma. □

2.1.2. Eigenvalues in $\epsilon(1+|\xi|) \leq r_0$

Now we prove the existence and establish the asymptotic expansions of the eigenvalues of $\tilde{\mathbb{A}}_\epsilon(\xi)$ for $\epsilon(1+|\xi|)$ being small. In terms of (2.10), the eigenvalue problem $\tilde{\mathbb{A}}_\epsilon(\xi)U = \lambda U$ for $U = (f, X, Y) \in L_\xi^2(\mathbb{R}_v^3) \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3$ can be written as

$$\lambda f = \left(L_1 - i\epsilon v\cdot\xi - i\epsilon\frac{v\cdot\xi}{|\xi|^2}P_d\right)f - \epsilon v\chi_0\cdot(\omega\times X), \tag{2.56}$$

$$\lambda X = -\epsilon\omega\times(f, v\chi_0) + i\epsilon^2\xi\times Y, \tag{2.57}$$

$$\lambda Y = -i\epsilon^2\xi\times X, \quad |\xi| \neq 0.$$

We rewrite f in the form of $f = f_0 + f_1$, where $f_0 = P_d f = C_0\sqrt{M}$ and $f_1 = (I - P_d)f = P_r f$. Then (2.56) gives

$$\lambda f_0 = -i\epsilon P_d(v\cdot\xi)(f_0 + f_1), \tag{2.58}$$

$$\lambda f_1 = L_1 f_1 - i\epsilon P_r(v\cdot\xi)(f_0+f_1) - i\epsilon\frac{v\cdot\xi}{|\xi|^2}f_0 - \epsilon v\chi_0\cdot(\omega\times X). \tag{2.59}$$

By Lemma 2.7 and (2.59), the microscopic part f_1 can be represented by

$$f_1 = i\epsilon R(\lambda,\epsilon\xi)(v\cdot\xi)\left(1 + \frac{1}{|\xi|^2}\right)f_0 + \epsilon R(\lambda,\epsilon\xi)v\chi_0\cdot(\omega\times X), \quad \mathrm{Re}\lambda > -\mu, \tag{2.60}$$

where

$$R(\lambda,\xi) = (L_1 - \lambda - iP_r(v\cdot\xi))^{-1}.$$

Substituting (2.60) into (2.58) and (2.57), we obtain the eigenvalue problem for (λ, C_0, X, Y) as

$$\begin{aligned}\lambda C_0 =&\epsilon^2(1+|\xi|^{-2})(R(\lambda,\epsilon\xi)(v\cdot\xi)\chi_0,(v\cdot\xi)\chi_0)C_0\\&-i\epsilon^2(R(\lambda,\epsilon\xi)v\chi_0\cdot(\omega\times X),(v\cdot\xi)\chi_0),\end{aligned}\tag{2.61}$$

$$\begin{aligned}\lambda X =&-i\epsilon^2\omega\times(1+|\xi|^{-2})(R(\lambda,\epsilon\xi)(v\cdot\xi)\chi_0,v\chi_0)C_0\\&-\epsilon^2\omega\times(R(\lambda,\epsilon\xi)v\chi_0\cdot(\omega\times X),v\chi_0)+i\epsilon^2\xi\times Y,\end{aligned}\tag{2.62}$$

$$\lambda Y =-i\epsilon^2\xi\times X.\tag{2.63}$$

Let $\mathbb{O}$ be a rotation in $\mathbb{R}^3$ satisfying $\mathbb{O}^T\xi=(|\xi|,0,0)$. By changing variable $v\to\mathbb{O}v$ and using the rotational invariance of the operator L_1, we have the following transformation:

$$(R(\lambda,\xi)\chi_i,\chi_j)=\omega_i\omega_j(R(\lambda,se_1)\chi_1,\chi_1)+(\delta_{ij}-\omega_i\omega_j)(R(\lambda,se_1)\chi_2,\chi_2),\tag{2.64}$$

where $e_1=(1,0,0)$, $\xi=s\omega$ with $s=|\xi|$, $\omega\in\mathbb{S}^2$.

Substituting (2.64) into (2.61) and (2.62), we obtain

$$\lambda C_0 =\epsilon^2(1+s^2)(R(\lambda,\epsilon se_1)\chi_1,\chi_1)C_0,\tag{2.65}$$

$$\lambda X =\epsilon^2(R(\lambda,\epsilon se_1)\chi_2,\chi_2)X+i\epsilon^2\xi\times Y.\tag{2.66}$$

Multiplying (2.66) by λ and using (2.63) and (2.15), we obtain

$$(\lambda^2-\epsilon^2(R(\lambda,\epsilon se_1)\chi_2,\chi_2)\lambda+\epsilon^4s^2)X=0.$$

Denote

$$D_0(z,s,\epsilon)=:z-(1+s^2)(R(\epsilon^2z,\epsilon s)\chi_1,\chi_1),\tag{2.67}$$

$$D_1(z,s,\epsilon)=:z^2-(R(\epsilon^2z,\epsilon s)\chi_2,\chi_2)z+s^2.\tag{2.68}$$

The eigenvalues $\lambda=\epsilon^2z$ can be obtained by solving $D_0(z,s,\epsilon)=0$ and $D_1(z,s,\epsilon)=0$. The following two lemmas are about the solutions to the equations $D_0(z,s,\epsilon)=0$ and $D_1(z,s,\epsilon)=0$ respectively.

Lemma 2.9. *There are two constants $r_0,r_1>0$ such that the equation $D_0(z,s,\epsilon)=0$ has a unique solution $z=z_0(s,\epsilon)$: $I\to J_0$ for $I=\{(s,\epsilon)\in\mathbb{R}^2\,|\,\epsilon(1+|s|)\le r_0\}$ and $J_0=\{z\in\mathbb{C}\,|\,|z+\eta(1+s^2)|\le r_1(1+s^2)\}$, which is a C^∞ function of s, ϵ and satisfies*

$$z_0(s,0)=-\eta(1+s^2),\quad \partial_\epsilon z_0(s,0)=0,\tag{2.69}$$

where $\eta>0$ is a constant given by (1.26). In particular, $z_0(s,\epsilon)$ satisfies the following expansion for $\epsilon(1+|s|)\le r_0$:

$$z_0(s,\epsilon) = -\eta(1+s^2) + O(\epsilon^2(1+s^2)^2). \tag{2.70}$$

Proof. By (2.67), the equation

$$D_0(z,s,0) = z + \eta(1+s^2) = 0 \tag{2.71}$$

has a unique solution $b_0(s) = -\eta(1+s^2)$.

For any fixed s and ϵ, we define

$$D(z,s,\epsilon) = (1+s^2)R_{11}(\epsilon^2 z, \epsilon s),$$

where $R_{11}(x,y) = ((L_1 - x - iyP_r v_1)^{-1}\chi_1, \chi_1)$. It is straightforward to check that for any fixed s and ϵ, a solution of $D_0(z,s,\epsilon) = 0$ is a fixed point of $D(z,s,\epsilon)$.

Since $R_{11}(x,y)$ is smooth for $(x,y) \in \mathbb{C} \times \mathbb{R}$ and satisfies

$$\begin{cases} \partial_1 R_{11}(0,0) = (L_1^{-2}\chi_1, \chi_1) > 0, \\ \partial_2 R_{11}(0,0) = i(v_1 L_1^{-1}\chi_1, L^{-1}\chi_1) = 0, \end{cases} \tag{2.72}$$

it follows that

$$\begin{aligned} |D(z,s,\epsilon) - b_0(s)| &= (1+s^2)\Big|R_{11}(\epsilon^2 z, \epsilon s) - R_{11}(0,0)\Big| \\ &\le C(1+s^2)(|\epsilon^2 z| + \epsilon^2 s^2) \le r_1(1+s^2), \\ |D(z_1,s,\epsilon) - D(z_2,s,\epsilon)| &\le C\epsilon^2(1+s^2)|z_1 - z_2| \le \frac{1}{2}|z_1 - z_2|, \end{aligned}$$

for $|z - b_0(s)| \le r_1(1+s^2)$ and $\epsilon(1+|s|) \le r_0$ with $r_0, r_1 > 0$ being sufficiently small.

Hence, by the contraction mapping theorem, there exists a unique fixed point $z_0(s,\epsilon)$: $I \to J_0$ such that $D(z_0(s,\epsilon),s,\epsilon) = z_0(s,\epsilon)$ for $(s,\epsilon) \in I$ and $z_0(s,0) = b_0(s)$. This is equivalent to $D_0(z(s,\epsilon),s,\epsilon) = 0$. Since $D_0(z,s,\epsilon)$ is C^∞ with respect to $z \in J_0$ and $(s,\epsilon) \in I$, it follows that $z_0(s,\epsilon)$ is a C^∞ function with respect to $(s,\epsilon) \in I$. In particular, we obtain

$$\partial_z D_0(z,s,0) = 1, \quad \partial_\epsilon D_0(z,s,0) = -(1+s^2)s\partial_2 R_{11}(0,0) = 0,$$

which gives

$$\partial_\epsilon z_0(s,0) = -\frac{\partial_\epsilon D_0(b_0(s),s,0)}{\partial_z D_0(b_0(s),s,0)} = 0. \tag{2.73}$$

Combining (2.71) and (2.73) yields (2.69).

For (2.70), by (2.72), we can obtain that for $|z - b_0(s)| \le r_1(1+s^2)$ and $\epsilon(1+|s|) \le r_0$,

$$R_{11}(\epsilon^2 z, \epsilon s) = -\eta + O(1)(\epsilon^2|z| + \epsilon^2 s^2).$$

Thus

$$\begin{aligned} z_0(s,\epsilon) &= (1+s^2)R_{11}(\epsilon^2 z_0(s,\epsilon), \epsilon s) \\ &= -\eta(1+s^2) + O(1)[\epsilon^2(|z_0(s,\epsilon)| + s^2)(1+s^2)] \\ &= -\eta(1+s^2) + O(\epsilon^2(1+s^2)^2), \quad \epsilon(1+|s|) \le r_0. \end{aligned}$$

Hence, the proof of the lemma is completed. □

Lemma 2.10. *There are small constants $r_0, r_1 > 0$ such that the equation $D_1(z,s,\epsilon) = 0$ has two continuous solutions $z_j = z_j(s,\epsilon) : I \to J_1$, $j = \pm 1$ for $I = \{(s,\epsilon) \in \mathbb{R}^2 \mid \epsilon(1+|s|) \le r_0\}$ and $J_1 = \{z \in \mathbb{C} \mid |z - b_j(s)| \le r_1 |b_j(s)|\}$. In particular, $z_j(s,\epsilon)$ are C^∞ functions in s, ϵ for $\epsilon(1+|s|) \le r_0$ and $|\eta^2 - 4s^2| \ge r_0$, which satisfy*

$$z_j(s,0) = b_j(s), \quad \partial_\epsilon z_j(s,0) = 0, \quad j = \pm 1, \tag{2.74}$$

where

$$b_j(s) = -\frac{\eta}{2} + \frac{j\sqrt{\eta^2 - 4s^2}}{2}.$$

Moreover, $z_j(s,\epsilon)$ satisfies the following expansion for $\epsilon(1+|s|) \le r_0$:

$$z_j(s,\epsilon) = b_j(s) + \begin{cases} O(\epsilon^2 |b_j(s)|), & |\eta^2 - 4s^2| \ge r_0, \\ O(\epsilon), & |\eta^2 - 4s^2| < r_0. \end{cases} \tag{2.75}$$

In addition, there exists a continuous real function $\vartheta_0(\epsilon) : (-r_0, r_0) \to B(\eta/2, r_1)$ such that $z_1(s,\epsilon) = z_{-1}(s,\epsilon)$ if and only if $(s,\epsilon) = (\vartheta_0(\epsilon), \epsilon)$ and $\vartheta_0(0) = \eta/2$.

Proof. By (2.68) and noting that $\eta = -(L_1^{-1}\chi_2, \chi_2)$, the equation

$$D_1(z,s,0) = z^2 + \eta z + s^2 = 0$$

has two solutions $b_j(s) = -\eta/2 + j\sqrt{\eta^2 - 4s^2}/2$ for $j = \pm 1$.

For any fixed s and ϵ, we define

$$G_j(z,s,\epsilon) = \frac{1}{2}\Big(R_{22}(\epsilon^2 z, \epsilon s) + j\sqrt{R_{22}(\epsilon^2 z, \epsilon s)^2 - 4s^2}\Big), \quad j = -1, 1, \tag{2.76}$$

where $R_{22}(x,y) = ((L_1 - x - iyP_r v_1)^{-1}\chi_2, \chi_2)$. It is straightforward to check that for any fixed s and ϵ, a solution of $D_1(z,s,\epsilon) = 0$ is a fixed point of $G_j(z,s,\epsilon)$. We consider the existence of the solutions of $D_1(z,s,\epsilon) = 0$ for two case: $(s,\epsilon) \in I_1$ and $(s,\epsilon) \in I_2$ with

$$I_1 = \big\{(s,\epsilon)\,|\,\epsilon(1+|s|) \le r_0, |\eta^2-4s^2| \ge r_0\big\},$$
$$I_2 = \big\{(s,\epsilon)\,|\,\epsilon(1+|s|) \le r_0, |\eta^2-4s^2| \le r_0\big\}.$$

First, we will study the existence of the solutions of $D_1(z,s,\epsilon)=0$ for $(s,\epsilon)\in I_1$. Since $R_{22}(x,y)$ is smooth for $(x,y)\in\mathbb{C}\times\mathbb{R}$ and satisfies

$$\begin{cases}\partial_1 R_{22}(0,0) = (L_1^{-2}\chi_2,\chi_2) > 0,\\ \partial_2 R_{22}(0,0) = i(v_1 L_1^{-1}\chi_2, L_1^{-1}\chi_2) = 0,\end{cases} \tag{2.77}$$

it follows that for $\epsilon^2|z|\le r_0$ and $\epsilon|s|\le r_0$ with $r_0\ll 1$,

$$R_{22}(\epsilon^2 z,\epsilon s) = -\eta + O(1)(\epsilon^2|z| + \epsilon^2 s^2). \tag{2.78}$$

Thus, it holds that for $|z-b_j(s)|\le r_1|b_j(s)|$, $\epsilon(1+|s|)\le r_0$ and $|\eta^2-4s^2|\ge r_0$ with $r_0,r_1\ll 1$,

$$|R_{22}(\epsilon^2 z,\epsilon s)^2 - 4s^2| \ge |\eta^2-4s^2| - C\epsilon^2|z| - C\epsilon^2|s|^2 \ge \frac{1}{2}r_0. \tag{2.79}$$

From (2.76)–(2.79), we obtain that for $|z-b_j(s)|\le r_1|b_j(s)|$, $\epsilon(1+|s|)\le r_0$ and $|\eta^2-4s^2|\ge r_0$ with $r_0,r_1\ll 1$,

$$\begin{aligned}|G_j(z,s,\epsilon)-b_j(s)| &\le \frac{1}{2}\big|R_{22}(\epsilon^2 z,\epsilon s) - R_{22}(0,0)\big|\\ &\quad + \frac{|R_{22}(\epsilon^2 z,\epsilon s)^2 - R_{22}(0,0)^2|}{2|\sqrt{R_{22}(\epsilon^2 z,\epsilon s)^2-4s^2}| + 2|\sqrt{R_{22}(0,0)^2-4s^2}|}\\ &\le C\epsilon^2(|z|+s^2) \le r_1|b_j(s)|,\\ |G_j(z_1,s,\epsilon)-G_j(z_2,s,\epsilon)| &\le \frac{1}{2}\big|R_{22}(\epsilon^2 z_1,\epsilon s) - R_{22}(\epsilon^2 z_2,\epsilon s)\big|\\ &\quad + \frac{|R_{22}(\epsilon^2 z,\epsilon s)^2 - R_{22}(\epsilon^2 z_1,\epsilon s)^2|}{2|\sqrt{R_{22}(\epsilon^2 z,\epsilon s)^2-4s^2}| + 2|\sqrt{R_{22}(\epsilon^2 z_1,\epsilon s)^2-4s^2}|}\\ &\le C\epsilon^2|z_1-z_2| \le \frac{1}{2}|z_1-z_2|.\end{aligned}$$

Hence by contraction mapping theorem, there exists a unique fixed point $z_j(s,\epsilon): I_1\to J_1$, $j=\pm 1$ such that $G_j(z_j(s,\epsilon),s,\epsilon) = z_j(s,\epsilon)$ for $(s,\epsilon)\in I_1$ and $z_j(s,0)=b_j(s)$. This is equivalent to $D_1(z_j(s,\epsilon),s,\epsilon)=0$. Since $G_j(z,s,\epsilon)$ is C^∞ with respect to $z\in J_1$ and $(s,\epsilon)\in I_1$, it follows that $z_j(s,\epsilon)$ is a C^∞ function with respect to $(s,\epsilon)\in I_1$. In particular, it holds that

$$\partial_z D_1(z,s,0) = 2z-\eta, \quad \partial_\epsilon D_1(z,s,0) = 0,$$

which leads to

$$\partial_\epsilon z_j(s,0) = \frac{\partial_\epsilon D_1(b_j(s),s,0)}{\partial_z D_1(b_j(s),s,0)} = 0, \quad 2|s| \neq \eta.$$

Thus, we obtain (2.74). By (2.78), we obtain that for $(s,\epsilon) \in I_1$,

$$\begin{aligned} 2z_j(s,\epsilon) &= R_{22}(\epsilon^2 z_j(s,\epsilon),\epsilon s) + j\sqrt{R_{22}(\epsilon^2 z_j(s,\epsilon),\epsilon s)^2 - 4s^2} \\ &= -\eta + O(1)(\epsilon^2|z_j(s,\epsilon)| + \epsilon^2 s^2) + j\sqrt{\eta^2 - 4s^2} \\ &\quad + O(1)(\epsilon^2|z_j(s,\epsilon)| + \epsilon^2 s^2)/\sqrt{\eta^2-4s^2} \\ &= 2b_j(s) + O\left(\frac{\epsilon^2(|b_j(s)| + s^2)}{\sqrt{\eta^2-4s^2}}\right), \quad j=-1,1, \end{aligned} \tag{2.80}$$

which gives (2.75) for $|\eta^2 - 4s^2| \geq r_0$.

Next, we will study the existence of the solutions of $D_1(z,s,\epsilon) = 0$ for $(s,\epsilon) \in I_2$. For any $z, z_1, z_2 \in J_1$ and $(s,\epsilon) \in I_2$, we obtain

$$\begin{aligned} |G_j(z,s,\epsilon) - b_j(s)| &\leq \frac{1}{2}\big|R_{22}(\epsilon^2 z,\epsilon s) - R_{22}(0,0)\big| \\ &\quad + \frac{1}{2}\sqrt{|R_{22}(\epsilon^2 z,\epsilon s)^2 - R_{22}(0,0)^2|} \\ &\leq C\epsilon\sqrt{|z|} + C\epsilon|s|, \end{aligned} \tag{2.81}$$

$$\begin{aligned} |G_j(z_1,s,\epsilon) - G_j(z_2,s,\epsilon)| &\leq \frac{1}{2}\big|R_{22}(\epsilon^2 z_1,\epsilon s) - R_{22}(\epsilon^2 z_2,\epsilon s)\big| \\ &\quad + \frac{1}{2}\sqrt{|R_{22}(\epsilon^2 z_1,\epsilon s)^2 - R_{22}(\epsilon^2 z_2,\epsilon s)^2|} \\ &\leq C\epsilon\sqrt{|z_1 - z_2|}, \end{aligned} \tag{2.82}$$

where we have used the inequality $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$. This implies that for any fixed $(s,\epsilon) \in I_2$, $G_j(z,s,\epsilon)$ is a continuous mapping in $z \in J_1$. Hence by Brouwer fixed point theorem, there exists at least a fixed point $z_j(s,\epsilon) : I_2 \to J_1$, $j = \pm 1$ such that $G_j(z_j(s,\epsilon),s,\epsilon) = z_j(s,\epsilon)$ for $(s,\epsilon) \in I_2$ and $z_j(s,0) = b_j(s)$. This is equivalent to $D_1(z_j(s,\epsilon),s,\epsilon) = 0$. Moreover, by (2.81) and (2.82), $z_j(s,\epsilon)$ is a continuous function with respect to $(s,\epsilon) \in I_2$ and satisfies

$$2z_j(s,\epsilon) = 2b_j(s) + O(1)(\epsilon\sqrt{|z_j(s,\epsilon)|} + \epsilon|s|) = 2b_j(s) + O(\epsilon), \quad j=-1,1,$$

which gives (2.75) for $|\eta^2 - 4s^2| \leq r_0$.

We claim that there are at most two different solutions $z_j = z_j(s,\epsilon)$, $j = \pm 1$ of the equation $D_1(z,s,\epsilon) = 0$ for $(s,\epsilon) \in I_2$. Indeed, it is easy to check that $D_1(z,s,\epsilon)$ is C^∞ with respect to $(z,s,\epsilon) \in \mathbb{C}^2 \times (-1,1)$ and satisfies

$$D_1(-\eta/2,\eta/2,0) = 0, \quad \partial_s D_1(-\eta/2,\eta/2,0) = \eta.$$

Thus by the implicit function theorem, there exists a unique C^∞ function $s(z,\epsilon):(z,\epsilon)\in B(-\eta/2,r_0)\times(-r_0,r_0)\to B(\eta/2,r_1)$ such that $D_1(z,s(z,\epsilon),\epsilon)=0$ and $s(-\eta/2,0)=\eta/2$, where $B(x,r)$ is a ball in $\mathbb{C}$ given by

$$B(x,r)=\{y\in\mathbb{C}\,|\,|y-x|<r\},\quad (x,r)\in\mathbb{C}\times\mathbb{R}_+.$$

In particular, it holds that

$$\partial_z D_1(-\eta/2,\eta/2,0)=0,\quad \partial_{zz}D_1(-\eta/2,\eta/2,0)=2,$$

which gives

$$\partial_z s(-\eta/2,0)=0,\quad \partial_{zz}s(-\eta/2,0)=-2/\eta.$$

Moreover, if $s(z,\epsilon)$ is real, then z must be a solution of $D_1(z,s,\epsilon)=0$.

If there are three different solution $z_j=z_j(s,\epsilon)$, $j=1,2,3$ of $D_1(z_j,s,\epsilon)=0$ for $(s,\epsilon)\in I_2$, then there exists a point $(s,\epsilon)\in B(\eta/2,r_0)\times(-r_0,r_0)$ such that $z_1(s,\epsilon)\neq z_2(s,\epsilon)\neq z_3(s,\epsilon)\in B(-\eta/2,r_0)$. Thus, there exists a function $s(z,\epsilon)$ defined as above satisfies that $s(z_1,\epsilon)=s(z_2,\epsilon)=s(z_3,\epsilon)$. By mean value theorem, there exist $z_4,z_5\in B(-\eta/2,r_0)$ such that $\partial_z s(z_4,\epsilon)=\partial_z s(z_5,\epsilon)$, and then there exist $z_6\in B(-\eta/2,r_0)$ such that $\partial_{zz}s(z_6,\epsilon)=0$. This is a contradiction to $\partial_{zz}s(-\eta/2,0)=-2/\eta$. Thus, the equation $D_1(z,s,\epsilon)=0$ admits exactly two continuous solutions $z_j(s,\epsilon):I_2\to J_1$ for $j=\pm1$.

Furthermore, it is straightforward to check that $z_1(s,\epsilon)=z_{-1}(s,\epsilon)$ if and only if

$$z_{\pm1}=\frac{1}{2}R_{22}(\epsilon^2 z_{\pm1},\epsilon s),\quad R_{22}(\epsilon^2 z_{\pm1},\epsilon s)^2-4s^2=0,$$

which is equivalent to

$$z_{\pm1}=-s,\quad R_{22}(-\epsilon^2 s,\epsilon s)+2s=0,\quad s>0.$$

Denote $D_2(s,\epsilon)=R_{22}(-\epsilon^2 s,\epsilon s)+2s$ with $(s,\epsilon)\in\mathbb{C}\times(-1,1)$. It holds that

$$D_2(\eta/2,0)=0,\quad \partial_s D_2(\eta/2,0)=2.$$

By the implicit function theorem, there exist small constants $r_0,r_1>0$ such that the equation $D_2(s,\epsilon)=0$ has a unique solution $s=\vartheta_0(\epsilon):(-r_0,r_0)\to B(\eta/2,r_1)$. Since $\overline{D_2(s,\epsilon)}=D_2(\overline{s},\epsilon)$, it follows that $\vartheta_0(\epsilon)=\overline{\vartheta_0}(\epsilon)$. Thus, the solutions $z_1(s,\epsilon)=z_{-1}(s,\epsilon)$ if and only if $s=\vartheta_0(\epsilon)$ and $\epsilon\in(-r_0,r_0)$. This proves the lemma. □

With Lemmas 2.9 and 2.10, we have the following lemma about the eigenvalue $\lambda_j(|\xi|,\epsilon)$ and the corresponding eigenfunction $\mathcal{U}_j(\xi,\epsilon)$ of the operator $\tilde{\mathbb{A}}_\epsilon(\xi)$ for $\epsilon(1+|\xi|)\le r_0$.

Lemma 2.11. (1) *There exists a small constant* $r_0 > 0$ *such that* $\sigma(\tilde{\mathbb{A}}_\epsilon(\xi)) \cap \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda > -\mu/2\}$ *consists of five points* $\{\lambda_j(s,\epsilon),\ j = 0,1,2,3,4\}$ *for* $\epsilon(1+|s|) \le r_0$ *and* $s = |\xi|$. *The eigenvalues* $\lambda_j(s,\epsilon)$ *are* C^∞ *functions of* s *and* ϵ, *and admit the following asymptotic expansions for* $\epsilon(1+|s|) \le r_0$:

$$\lambda_0(s,\epsilon) = \epsilon^2 b_0(s) + O(\epsilon^4(1+s^2)^2), \tag{2.83}$$

$$\lambda_k(s,\epsilon) = \epsilon^2 b_k(s) + \begin{cases} O(\epsilon^4|b_k(s)|), & |\eta^2 - 4s^2| \ge r_0, \\ O(\epsilon^3), & |\eta^2 - 4s^2| < r_0, \end{cases} \tag{2.84}$$

where $k = 1,2,3,4$, $\lambda_1 = \lambda_2$ *and* $\lambda_3 = \lambda_4$, *and*

$$\begin{cases} b_0 = -\eta(1+s^2), \quad b_1 = b_2 = -\dfrac{\eta}{2} - \dfrac{\sqrt{\eta^2 - 4s^2}}{2}, \\ b_3 = b_4 = -\dfrac{\eta}{2} + \dfrac{\sqrt{\eta^2 - 4s^2}}{2}. \end{cases} \tag{2.85}$$

Moreover, $\lambda_1(s,\epsilon) = \lambda_3(s,\epsilon)$ *if and only if* $(s,\epsilon) = (\vartheta_0(\epsilon),\epsilon)$ *with* $\vartheta_0(\epsilon)$ *being a real continuous function given in Lemma 2.10.*

(2) *The eigenfunctions* $\mathcal{U}_j(\xi,\epsilon) = (u_j(\xi,\epsilon), X_j(\xi,\epsilon), Y_j(\xi,\epsilon))$, $j = 0,1,2,3,4$ *are* C^∞ *in* s *and* ϵ, *and satisfy for* $\epsilon(1+|s|) \le r_0$ *and* $(s,\epsilon) \ne (\vartheta_0(\epsilon),\epsilon)$:

$$\begin{cases} (\mathcal{U}_i, \mathcal{U}_j^*)_\xi =: (u_i, \overline{u_j})_\xi - (X_i, \overline{X_j}) - (Y_i, \overline{Y_j}) = \delta_{ij}, \quad 0 \le i,j \le 4, \\ u_0 = P_d u_0 + P_r u_0, \quad (X_0, Y_0) \equiv (0,0), \\ P_d u_0 = \dfrac{s}{\sqrt{1+s^2}} \left[1 + O(\epsilon^2(1+s^2))\right] \chi_0, \\ P_r u_0 = i\epsilon\sqrt{1+s^2} L_1^{-1}(v\cdot\omega)\chi_0 + O(\epsilon^2(1+s^2)), \\ u_k = \dfrac{\epsilon\lambda_k\Theta_k}{\sqrt{\lambda_1\lambda_3 - \lambda_k^2}} \left[L_1^{-1}(v\cdot e_k)\chi_0 + O(\epsilon(1+|s|))\right], \\ (X_k, Y_k) = \dfrac{\lambda_k\Theta_k}{\sqrt{\lambda_1\lambda_3 - \lambda_k^2}} \left(\omega \times e_k, \dfrac{i\epsilon^2 s e_k}{\lambda_k}\right), \quad k = 1,2,3,4, \end{cases} \tag{2.86}$$

where $\mathcal{U}_j^* = (\overline{u_j}, -\overline{X_j}, -\overline{Y_j})$, *and* e_k, $k = 1,2,3,4$ *are normal vectors satisfying* $e_1 = e_3$, $e_2 = e_4$, *and* $e_k \cdot \omega = e_1 \cdot e_2 = 0$, *and*

$$\Theta_k = 1 + \begin{cases} O(\epsilon^2), & |\eta^2 - 4s^2| \ge r_0, \\ O(\epsilon), & |\eta^2 - 4s^2| < r_0. \end{cases} \tag{2.87}$$

Proof. The eigenvalue $\lambda_j(s,\epsilon)$ and the eigenfunction $\mathcal{U}_j(\xi,\epsilon)$ for $j = 0,1,2,3,4$ can be constructed as follows. For $j = 0$, we take $\lambda_0 = \epsilon^2 z_0(s,\epsilon)$ with $z_0(s,\epsilon)$ being the solution of the equation $D_0(z,s,\epsilon) = 0$ given in Lemma 2.9, and choose $X = Y = 0$ in

(2.61)–(2.63). And the corresponding eigenfunction $\mathcal{U}_0(\xi,\epsilon) = (u_0(\xi,\epsilon), X_0(\xi,\epsilon), Y_0(\xi,\epsilon))$ is defined by

$$\begin{cases} u_0(\xi,\epsilon) = a_0(s,\epsilon)\chi_0 + i\epsilon\left(s+\dfrac{1}{s}\right)a_0(s,\epsilon)(L_1-\lambda_0-i\epsilon sP_r(v\cdot\omega))^{-1}(v\cdot\omega)\chi_0, \\ X_0(\xi,\epsilon) = Y_0(\xi,\epsilon) \equiv 0, \end{cases} \tag{2.88}$$

where $a_0(s,\epsilon)$ is a complex value function determined later.

For $j = 1,2,3,4$, we take $\lambda_1 = \lambda_2 = \epsilon^2 z_{-1}(s,\epsilon)$ and $\lambda_3 = \lambda_4 = \epsilon^2 z_1(s,\epsilon)$ with $z_{\pm1}(s,\epsilon)$ being the solution of the equation $D_1(z,s,\epsilon) = 0$ given in Lemma 2.10, and choose $C_0 = 0$ in (2.61)–(2.63). Then we define the corresponding eigenfunction $\mathcal{U}_j(\xi,\epsilon) = (u_j(\xi,\epsilon), X_j(\xi,\epsilon), Y_j(\xi,\epsilon))$ by

$$\begin{cases} u_j(\xi,\epsilon) = \epsilon a_j(s,\epsilon)(L_1-\lambda_j-i\epsilon sP_r(v\cdot\omega))^{-1}(v\cdot e_j)\chi_0, \\ X_j(\xi,\epsilon) = a_j(s,\epsilon)\omega\times e_j, \quad Y_j(\xi,\epsilon) = i\dfrac{\epsilon^2 s a_j(s,\epsilon)}{\lambda_j(s,\epsilon)}e_j, \end{cases} \tag{2.89}$$

where e_j, $j = 1,2,3,4$ are normal vectors satisfying $e_1 = e_3$, $e_2 = e_4$ and $e_k\cdot\omega = e_1\cdot e_2 = 0$, and $a_j(s,\epsilon)$ is a complex value function determined later. It's easy to verify that $(\mathcal{U}_1,\mathcal{U}_2^*)_\xi = (\mathcal{U}_3,\mathcal{U}_4^*)_\xi = 0$, where $\mathcal{U}_j^* = (\overline{u_j}, -\overline{X_j}, -\overline{Y_j})$ is the eigenvector of $\tilde{\mathbb{A}}_\epsilon^*(\xi)$ corresponding to the eigenvalue $\overline{\lambda_j(s,\epsilon)}$.

Rewrite the eigenvalue problem as

$$\tilde{\mathbb{A}}_\epsilon(\xi)\mathcal{U}_j(\xi,\epsilon) = \lambda_j(s,\epsilon)\mathcal{U}_j(\xi,\epsilon), \quad j = 0,1,2,3,4.$$

Taking the inner product $(\cdot,\cdot)_\xi$ of the above equation with $\mathcal{U}_k^*(\xi,\epsilon)$ and using the facts that

$$(\tilde{\mathbb{A}}_\epsilon(\xi)U,V)_\xi = (U,\tilde{\mathbb{A}}_\epsilon^*(\xi)V)_\xi, \quad U,V\in D(\tilde{\mathbb{A}}_\epsilon(\xi)),$$
$$\tilde{\mathbb{A}}_\epsilon^*(\xi)\mathcal{U}_j^*(\xi,\epsilon) = \overline{\lambda_j(s,\epsilon)}\mathcal{U}_j^*(\xi,\epsilon),$$

we have

$$(\lambda_j(s,\epsilon)-\lambda_k(s,\epsilon))(\mathcal{U}_j(\xi,\epsilon),\mathcal{U}_k^*(\xi,\epsilon))_\xi = 0, \quad 0\le j,k\le 4.$$

For $\epsilon(1+|s|)\le r_0$ and $(s,\epsilon)\ne(\vartheta_0(\epsilon),\epsilon)$, we have $\lambda_j(s,\epsilon)\ne\lambda_k(s,\epsilon)$ for $j,k\in\{0,1,3\}$ and $j\ne k$. Thus,

$$(\mathcal{U}_j(\xi,\epsilon),\mathcal{U}_k^*(\xi,\epsilon))_\xi = 0, \quad 0\le j\ne k\le 4.$$

We can normalize $\mathcal{U}_j(\xi,\epsilon)$ by taking

$$\big(\mathcal{U}_j(\xi,\epsilon),\mathcal{U}_j^*(s,\epsilon)\big)_\xi = 1, \quad 0\le j\le 4.$$

The coefficients $a_j(s,\epsilon)$, $0 \le j \le 4$ satisfy the normalization conditions $(\mathcal{U}_j, \mathcal{U}_j^*)_\xi = 1$ as

$$a_0(s,\epsilon)^2\left(1 + \frac{1}{s^2} + \epsilon^2\left(s + \frac{1}{s}\right)^2 D_0(s,\epsilon)\right) = 1, \tag{2.90}$$

$$a_k(s,\epsilon)^2\left(-1 + \frac{\epsilon^4 s^2}{\lambda_k(s,\epsilon)^2} + \epsilon^2 D_{kk}(s,\epsilon)\right) = 1, \quad k = 1,2,3,4, \tag{2.91}$$

where

$$D_0(s,\epsilon) = (R(\lambda_0, \epsilon s e_1)\chi_1, R(\overline{\lambda_0}, -\epsilon s e_1)\chi_1),$$
$$D_{jk}(s,\epsilon) = (R(\lambda_j, \epsilon s e_1)\chi_2, R(\overline{\lambda_k}, -\epsilon s e_1)\chi_2), \quad j,k = 1,2,3,4.$$

Substituting (2.83) into (2.90), we obtain

$$\begin{aligned} a_0(s,\epsilon) &= \frac{s}{\sqrt{1+s^2}}\Big(1 + \epsilon^2(1+s^2)D_0(s,\epsilon)\Big)^{-\frac{1}{2}} \\ &= \frac{s}{\sqrt{1+s^2}}\left[1 + O(\epsilon^2(1+s^2))\right]. \end{aligned} \tag{2.92}$$

Due to the fact that $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$, we have $a_1 = a_2$ and $a_3 = a_4$. Note that

$$(\mathcal{U}_1(\xi,\epsilon), \mathcal{U}_3^*(s,\epsilon))_\xi = 0 \Longrightarrow -1 + \frac{\epsilon^4 s^2}{\lambda_1\lambda_3} + \epsilon^2 D_{13}(s,\epsilon) = 0.$$

Thus, it follows that for $\lambda_1 \ne \lambda_3$,

$$\begin{aligned} -1 + \frac{\epsilon^4 s^2}{\lambda_1^2} + \epsilon^2 D_{11} &= \frac{\epsilon^4 s^2}{\lambda_1^2} - \frac{\epsilon^4 s^2}{\lambda_1\lambda_3} + \epsilon^2(D_{11} - D_{13}) \\ &= \frac{\epsilon^4 s^2}{\lambda_1^2\lambda_3}(\lambda_3 - \lambda_1) + \epsilon^2 D_{113}(\lambda_3 - \lambda_1) \\ &= \frac{\lambda_3 - \lambda_1}{\lambda_1}\left(\frac{\epsilon^4 s^2}{\lambda_1\lambda_3} + \epsilon^2\lambda_1 D_{113}\right), \end{aligned} \tag{2.93}$$

where

$$D_{ijk}(s,\epsilon) = (R(\lambda_i, \epsilon s)\chi_2, R(\overline{\lambda_j}, -\epsilon s)R(\overline{\lambda_k}, -\epsilon s)\chi_2).$$

Similarly,

$$-1 + \frac{\epsilon^4 s^2}{\lambda_3^2} + \epsilon^2 D_{33} = \frac{\lambda_1 - \lambda_3}{\lambda_3}\left(\frac{\epsilon^4 s^2}{\lambda_1\lambda_3} + \epsilon^2\lambda_3 D_{313}\right). \tag{2.94}$$

Thus, it follows from (2.91), (2.93) and (2.94) that

$$a_k(s,\epsilon) = \frac{\lambda_k}{\sqrt{\lambda_1\lambda_3 - \lambda_k^2}} \left(\frac{\epsilon^4 s^2}{\lambda_1\lambda_3} + \epsilon^2 \lambda_k D_{k13} \right)^{-\frac{1}{2}}, \quad k = 1, 3. \tag{2.95}$$

Let

$$\Theta_k = \left(\frac{\epsilon^4 s^2}{\lambda_1\lambda_3} + \epsilon^2 \lambda_k D_{k13} \right)^{-\frac{1}{2}}, \quad k = 1, 2, 3, 4. \tag{2.96}$$

By (2.84), it holds for $|\eta^2 - 4s^2| \geq r_0$ that

$$\begin{aligned}
\Theta_k &= \left(\frac{s^2}{(b_1 + O(\epsilon^2|b_1|))(b_3 + O(\epsilon^2|b_3|))} + \epsilon^4 b_k D_{k13} \right)^{-\frac{1}{2}} \\
&= \left(1 + \frac{1}{b_1} O(\epsilon^2|b_1|) + \frac{1}{b_3} O(\epsilon^2|b_3|) + O(\epsilon^4|b_k|) \right)^{-\frac{1}{2}} \\
&= 1 + O(\epsilon^2),
\end{aligned} \tag{2.97}$$

and for $|\eta^2 - 4s^2| \leq r_0$ it holds that

$$\begin{aligned}
\Theta_k &= \left(\frac{s^2}{(b_1 + O(\epsilon))(b_3 + O(\epsilon))} + \epsilon^4 b_k D_{k13} \right)^{-\frac{1}{2}} \\
&= \left(1 + \frac{1}{b_1} O(\epsilon) + \frac{1}{b_3} O(\epsilon) + O(\epsilon^4|b_k|) \right)^{-\frac{1}{2}} \\
&= 1 + O(\epsilon).
\end{aligned} \tag{2.98}$$

By combining (2.88), (2.89), (2.92), (2.97) and (2.98), and using the fact that

$$\begin{aligned}
R(\lambda_j, \epsilon s) &= L_1^{-1} + \lambda_j R(\lambda_j, \epsilon s) L_1^{-1} + i\epsilon s R(\lambda_j, \epsilon s)(v \cdot \omega) L_1^{-1} \\
&= L_1^{-1} + O(\epsilon^2|b_k| + \epsilon|s|),
\end{aligned}$$

we obtain the expansion of $\mathcal{U}_j(\xi,\epsilon)$ given in (2.86). This completes the proof of the lemma. □

2.1.3. Eigenvalues in $\epsilon|\xi| \geq r_1$

We now turn to study the asymptotic expansions of the eigenvalues and eigenvectors in the high frequency regime. Firstly, recalling the eigenvalue problem

$$\lambda f = \tilde{\mathbb{B}}_\epsilon(\xi) f - \epsilon v\chi_0 \cdot (\omega \times X), \tag{2.99}$$

$$\lambda X = -\epsilon\omega \times (f, v\chi_0) + i\epsilon^2 \xi \times Y, \tag{2.100}$$

$$\lambda Y = -i\epsilon^2 \xi \times X, \quad |\xi| \neq 0. \tag{2.101}$$

By Lemma 2.4, there exists a large constant $R_0 > 0$ such that the operator $\lambda - \tilde{\mathbb{B}}_\epsilon(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3)$ for $\mathrm{Re}\lambda \geq -\nu_0/2$ and $\epsilon|\xi| > R_0$. Then it follows from (2.99) that

$$f = \epsilon(\tilde{\mathbb{B}}_\epsilon(\xi) - \lambda)^{-1} v\chi_0 \cdot (\omega \times X), \quad \epsilon|\xi| > R_0. \tag{2.102}$$

By a similar argument as (2.64), it holds that

$$\begin{aligned}\left((\tilde{\mathbb{B}}_\epsilon(\xi) - \lambda)^{-1}\chi_i, \chi_j\right) =& \omega_i\omega_j \left((\tilde{\mathbb{B}}_\epsilon(se_1) - \lambda)^{-1}\chi_1, \chi_1\right) \\ &+ (\delta_{ij} - \omega_i\omega_j)\left((\tilde{\mathbb{B}}_\epsilon(se_1) - \lambda)^{-1}\chi_2, \chi_2\right),\end{aligned} \tag{2.103}$$

where $e_1 = (1, 0, 0)$, $s = |\xi|$ and $\omega = \xi/|\xi|$. Substituting (2.102) and (2.103) into (2.100) gives

$$\lambda X = \epsilon^2((\tilde{\mathbb{B}}_\epsilon(se_1) - \lambda)^{-1}\chi_2, \chi_2)X + i\epsilon^2\xi \times Y, \quad \epsilon s > R_0. \tag{2.104}$$

By multiplying (2.104) by λ and using (2.101), we obtain

$$\left(\lambda^2 - \epsilon^2((\tilde{\mathbb{B}}_\epsilon(se_1) - \lambda)^{-1}\chi_2, \chi_2)\lambda + \epsilon^4 s^2\right) X = 0, \quad \epsilon s > R_0.$$

Denote

$$D_2(z, s, \epsilon) = z^2 - ((\tilde{\mathbb{B}}_\epsilon(se_1) - \epsilon^2 z)^{-1}\chi_2, \chi_2)z + s^2, \quad \epsilon s > R_0. \tag{2.105}$$

The eigenvalues $\lambda = \epsilon^2 z$ can be obtained by solving $D(z, s, \epsilon) = 0$. Firstly, similar to Lemma 2.4, we can prove the following lemma.

Lemma 2.12. *For $\epsilon \in (0, 1)$ and any $\delta > 0$, if $\mathrm{Re}\lambda \geq -\nu_0 + \delta$, then*

$$\|(\lambda - D_\epsilon(\xi))^{-1} K_1\| \leq C\delta^{-\frac{1}{2}}(1 + \epsilon|\xi|)^{-\frac{1}{2}}, \tag{2.106}$$

$$\|(\lambda - D_\epsilon(\xi))^{-1}\chi_j\| \leq C\delta^{-\frac{1}{2}}(1 + \epsilon|\xi|)^{-\frac{1}{2}}, \tag{2.107}$$

where $j = 0, 1, 2, 3, 4$. Furthermore, there exists a sufficiently large $R_0 > 0$ such that $\lambda - \tilde{\mathbb{B}}_\epsilon(\xi)$ is invertible for $\mathrm{Re}\lambda \geq -\nu_0 + \delta$ and $\epsilon|\xi| > R_0$, and

$$\|(\lambda - \tilde{\mathbb{B}}_\epsilon(\xi))^{-1}\chi_j\| \leq C\delta^{-\frac{1}{2}}(1 + \epsilon|\xi|)^{-\frac{1}{2}}. \tag{2.108}$$

We now study the equation (2.105) as follows.

Lemma 2.13. *There exists a large constant $r_1 > 0$ such that the equation $D_2(z, s, \epsilon) = 0$ has two solutions $z_j(s, \epsilon) = jis + y_j(s, \epsilon)$, $j = \pm 1$ for $\epsilon s > r_1$, where $y_j(s, \epsilon)$ is a C^∞ function in s and ϵ for $\epsilon s > r_1$ satisfying*

$$\frac{C_1}{\epsilon s} \leq -\mathrm{Re} y_j(s, \epsilon) \leq \frac{C_2}{\epsilon s}, \quad |\mathrm{Im} y_j(s, \epsilon)| \leq C_3 \frac{\ln \epsilon s}{\epsilon s}, \tag{2.109}$$

where $C_1, C_2, C_3 > 0$ are some constants.

Proof. For any fixed (s, ϵ) satisfying $\epsilon s > R_0$, we define a function of z as

$$G_j(z, s, \epsilon) = \frac{1}{2}\Big(R_0(z, s, \epsilon) + j\sqrt{R_0(z, s, \epsilon)^2 - 4s^2}\Big), \quad j = \pm 1, \ \ \epsilon|s| > R_0,$$

where $R_0(z, s, \epsilon) = ((\tilde{\mathbb{B}}_\epsilon(se_1) - \epsilon^2 z)^{-1}\chi_2, \chi_2)$. It is straightforward to check that a solution of $D_2(z, s, \epsilon) = 0$ for any fixed s, ϵ is a fixed point of $G_j(z, s, \epsilon)$.

By (2.108), when $R_1 > 0$ is large enough and $\delta > 0$ is small enough, it holds for $\epsilon s > R_1$ and $z, z_1, z_2 \in B(jis, \delta)$ that

$$|G_j(z, s, \epsilon) - jis| \le \frac{1}{2}|R_0(z, s, \epsilon)| + \frac{|R_0(z, s, \epsilon)^2|}{2|\sqrt{R_0(z, s, \epsilon)^2 - 4s^2} + 2is|} \le \delta, \tag{2.110}$$

$$\begin{aligned} |G_j(z_1, s, \epsilon) - G_j(z_2, s, \epsilon)| &\le \frac{1}{2}|R_0(z_1, s, \epsilon) - R_0(z_2, s, \epsilon)| \\ &\quad + \frac{|R_0(z_1, s, \epsilon)^2 - R_0(z_2, s, \epsilon)^2|}{2|\sqrt{R_0(z_1, s, \epsilon)^2 - 4s^2}| + 2|\sqrt{R_0(z_2, s, \epsilon)^2 - 4s^2}|} \\ &\le C\epsilon^2|z_1 - z_2|\left(1 + \frac{1}{s}\right) \le \frac{1}{2}|z_1 - z_2|. \end{aligned}$$

Thus, by the contraction mapping theorem, there is a unique function $z_j(s, \epsilon)$ satisfying that $z_j(s, \epsilon) = G_j(z, s, \epsilon)$, namely, $z_j(s, \epsilon)$ is the solution of $D_2(z, s, \epsilon) = 0$. Set $y_j(s, \epsilon) = z_j(s, \epsilon) - jis$. By (2.110) and (2.108), we have

$$|y_j(s, \epsilon)| \le C|R_0(z_j, s, \epsilon)|\left(1 + \frac{1}{s}\right) \le C|\epsilon s|^{-\frac{1}{2}} \to 0, \quad \epsilon|s| \to \infty. \tag{2.111}$$

We now turn to prove (2.109). We obtain from (2.76) that

$$y_j(s, \epsilon) = \frac{1}{2}R_j(y_j, s, \epsilon) + \frac{1}{2}\frac{jR_j(y_j, s, \epsilon)^2}{\sqrt{R_j(y_j, s, \epsilon)^2 - 4s^2} + 2is} =: I_1 + I_2, \tag{2.112}$$

where

$$R_j(y_j, s, \epsilon) = R_0(z_j, s, \epsilon) = ((\tilde{\mathbb{B}}_\epsilon(se_1) - \epsilon^2 jis - \epsilon^2 y_j)^{-1}\chi_2, \chi_2).$$

First, we estimate I_1. For this, we decompose

$$-\Big(L_1 - i(v_1 + j\epsilon)\epsilon s - i\epsilon\frac{v_1}{s}P_d - \epsilon^2 y_j\Big)^{-1} = X_j(s, \epsilon) + Z_j(s, \epsilon), \tag{2.113}$$

where

$$\begin{cases} X_j(s,\epsilon) = (\nu(v) + i(v_1 + j\epsilon)\epsilon s)^{-1}, \\ Z_j(s,\epsilon) = (I - Y_j(s,\epsilon))^{-1} Y_j(s,\epsilon) X_j(s,\epsilon), \\ Y_j(s,\epsilon) = X_j(s,\epsilon)\left(K_1 - i\epsilon \frac{v_1}{s} P_d - \epsilon^2 y_j\right). \end{cases} \tag{2.114}$$

By (2.113), we divide I_1 into

$$I_1 = -\left(X_j(s,\epsilon)\chi_2, \chi_2\right) - \left(Z_j(s,\epsilon)\chi_2, \chi_2\right) =: I_3 + I_4. \tag{2.115}$$

It holds that

$$\begin{aligned} I_3 = & -\int_{\mathbb{R}^3} \frac{\nu}{\nu^2 + (v_1 \pm \epsilon)^2 \epsilon^2 s^2} v_2^2 M dv - i \int_{\mathbb{R}^3} \frac{(v_1 \pm \epsilon)\epsilon s}{\nu^2 + (v_1 \pm \epsilon)^2 \epsilon^2 s^2} v_2^2 M dv \\ = & -\mathrm{Re} I_3 - i \mathrm{Im} I_3. \end{aligned} \tag{2.116}$$

By changing variable $(u_1, u_2, u_3) = ((v_1 \pm \epsilon)\epsilon s, v_2, v_3)$, we obtain for $\epsilon s > 1$ that

$$\begin{aligned} \mathrm{Re} I_3 \leq & C \int_{\mathbb{R}^3} \frac{1}{\nu_0^2 + (v_1 \pm \epsilon)^2 \epsilon^2 s^2} e^{-\frac{|v|^2}{4}} dv \\ \leq & \frac{C}{\epsilon s} \int_{\mathbb{R}^3} \frac{1}{\nu_0^2 + u_1^2} e^{-\frac{1}{4}(\frac{u_1}{\epsilon s} \mp \epsilon)^2} e^{-\frac{u_2^2 + u_3^2}{4}} du \\ \leq & \frac{C}{\epsilon s} \int_{\mathbb{R}^3} \frac{1}{\nu_0^2 + u_1^2} e^{-\frac{u_1^2}{4}} du_1 \leq \frac{C_1}{\epsilon s}, \end{aligned} \tag{2.117}$$

$$\begin{aligned} \mathrm{Re} I_3 \geq & \int_{\mathbb{R}^3} \frac{\nu_0}{\nu_1^2(1 + |v|^2) + (v_1 \pm \epsilon)^2 \epsilon^2 s^2} v_2^2 M dv \\ \geq & \frac{1}{\epsilon s} \int_{\mathbb{R}^3} \frac{\nu_0}{\nu_1^2(1 + (\frac{u_1}{\epsilon s} \mp \epsilon)^2 + u_2^2 + u_3^2) + u_1^2} u_2^2 e^{-\frac{1}{2}(\frac{u_1}{\epsilon s} \mp \epsilon)^2} e^{-\frac{u_2^2 + u_3^2}{2}} du \\ \geq & \frac{C}{\epsilon s} \int_{\mathbb{R}^3} \frac{1}{\nu_1^2(3 + u_1^2 + u_2^2 + u_3^2) + u_1^2} u_2^2 e^{-\frac{u_1^2 + u_2^2 + u_3^2}{2}} du \geq \frac{C_2}{\epsilon s}, \end{aligned} \tag{2.118}$$

and

$$\begin{aligned} |\mathrm{Im} I_3| \leq & \frac{C}{\epsilon s} \int_{\mathbb{R}^3} \frac{|u_1|}{\nu_0^2 + u_1^2} e^{-\frac{1}{2}(\frac{u_1}{s} \mp \epsilon)^2} e^{-\frac{u_2^2 + u_3^2}{2}} du \\ \leq & \frac{C}{\epsilon s} \int_0^{\epsilon s} \frac{u_1}{\nu_0^2 + u_1^2} du_1 + \frac{C}{\epsilon^2 s^2} \int_{\epsilon s}^{\infty} e^{-\frac{1}{2}(\frac{u_1}{\epsilon s} \mp \epsilon)^2} du_1 \\ \leq & C_3 \frac{\ln \epsilon s}{\epsilon s}. \end{aligned} \tag{2.119}$$

We now consider I_4. By changing variable $v_2 \to -v_2$, we obtain that $(X_j(s,\epsilon)\chi_2, \chi_0) = 0$. This together with (2.114) implies that

$$Z_j(s,\epsilon)\chi_2 = (I - Y_j)^{-1} X_j (K_1 - \epsilon^2 y_j) X_j \chi_2.$$

Since

$$|k_1(v,u)| \le C \frac{1}{|\bar{v} - \bar{u}|} e^{-\frac{|v-u|^2}{8}}, \quad \bar{u} = (u_2, u_3),$$

we can obtain from (2.117) and (2.119) that

$$\begin{aligned}|K_1 X_{\pm 1}\chi_2| &\le C \int_{\mathbb{R}} e^{-\frac{|v_1-u_1|^2}{8}} \frac{\nu + |(u_1 \pm \epsilon)\epsilon s|}{\nu_0^2 + |(u_1 \pm \epsilon)\epsilon s|^2} e^{-\frac{u_1^2}{4}} du_1 \int_{\mathbb{R}^2} \frac{1}{|\bar{v} - \bar{u}|} e^{-\frac{|\bar{v}-\bar{u}|^2}{8}} e^{-\frac{|\bar{u}|^2}{4}} d\bar{u} \\ &\le C e^{-\frac{|v|^2}{8}} \int_{\mathbb{R}} \frac{1 + |(u_1 \pm \epsilon)\epsilon s|}{\nu_0^2 + |(u_1 \pm \epsilon)\epsilon s|^2} e^{-\frac{u_1^2}{8}} du_1 \le C \frac{\ln \epsilon s}{\epsilon s} e^{-\frac{|v|^2}{8}}.\end{aligned}$$

This and (2.117) lead to

$$\|X_{\pm 1} K_1 X_{\pm 1}\chi_2\|^2 \le C \frac{\ln^2 \epsilon s}{\epsilon^2 s^2} \int_{\mathbb{R}^3} \frac{1}{\nu_0^2 + (v_1 \pm \epsilon)^2 \epsilon^2 s^2} e^{-\frac{|v|^2}{4}} dv \le C \frac{\ln^2 \epsilon s}{|\epsilon s|^3}. \tag{2.120}$$

By (2.111) and Lemma 2.12, it holds that for $\epsilon|s| > R_1$,

$$\|(I - Y_j(s,\epsilon))^{-1}\| \le 2, \quad j = \pm 1. \tag{2.121}$$

Thus, it follows from (2.111), (2.117), (2.120) and (2.121) that

$$\begin{aligned}|I_4| &\le \big|\big(X_j(K_1 - \epsilon^2 y_j) X_j \chi_2, (I - Y_j^*)^{-1}\chi_2\big)\big| \\ &\le C(\|X_j K_1 X_j \chi_2\| + \epsilon^2 |y_j| \|X_j^2 \chi_2\|) \le C|\epsilon s|^{-\frac{3}{2}} \ln|\epsilon s|.\end{aligned} \tag{2.122}$$

Next, we estimate I_2 as follows:

$$|I_2| \le \frac{C}{s} |R_j(y_j, s, \epsilon)|^2 \le \frac{C}{s} |I_1|^2 \le C \frac{\ln^2 \epsilon s}{\epsilon^2 s^3}. \tag{2.123}$$

Combining (2.115)–(2.119), (2.122) and (2.123), we obtain (2.109). The proof of the lemma is then completed. □

With Lemma 2.13, we have the following lemma about the eigenvalues $\beta_j(|\xi|,\epsilon)$ and the corresponding eigenvectors $\mathcal{V}_j(\xi,\epsilon)$ of the operator $\tilde{\mathbb{A}}_\epsilon(\xi)$ for $\epsilon|\xi| \ge r_1$.

Lemma 2.14. (1) *There exists a constant $r_1 > 0$ such that the spectrum $\sigma(\tilde{\mathbb{A}}_\epsilon(\xi)) \cap \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda > -\mu/2\}$ consists of four eigenvalues $\{\beta_j(s,\epsilon),\ j = 1,2,3,4\}$ for $\epsilon s > r_1$ and $s = |\xi|$. In particular, the eigenvalues $\beta_j(s,\epsilon)$ are C^∞ functions in s and ϵ and satisfy the following expansion for $\epsilon s > r_1$:*

$$\begin{cases} \beta_1(s,\epsilon) = \beta_2(s,\epsilon) = -\epsilon^2 is + \epsilon^2\zeta_{-1}(s,\epsilon), \\ \beta_3(s,\epsilon) = \beta_4(s,\epsilon) = \epsilon^2 is + \epsilon^2\zeta_1(s,\epsilon), \end{cases} \tag{2.124}$$

where $\zeta_{\pm1}(s,\epsilon)$ is a C^∞ function in s and ϵ for $\epsilon s > r_1$ satisfying

$$\frac{C_1}{\epsilon s} \le -\mathrm{Re}\zeta_{\pm1}(s,\epsilon) \le \frac{C_2}{\epsilon s}, \quad |\mathrm{Im}\zeta_{\pm1}(s,\epsilon)| \le C_3\frac{\ln \epsilon s}{\epsilon s}, \tag{2.125}$$

with positive constants C_1, C_2 and C_3.

(2) *The eigenvectors $\mathcal{V}_j(\xi,\epsilon) = (w_j(\xi,\epsilon), X_j(\xi,\epsilon), Y_j(\xi,\epsilon))$, $j = 1,2,3,4$ are C^∞ in s and ϵ, and satisfy for $\epsilon s > r_1$:*

$$\begin{cases} (\mathcal{V}_i, \mathcal{V}_j^*) = (w_i, \overline{w_j}) - (X_i, \overline{X_j}) - (Y_i, \overline{Y_j}) = \delta_{ij}, \quad 1 \le i,j \le 4, \\ w_j(\xi,\epsilon) = \epsilon c_j(s,\epsilon)\left(\beta_j(s,\epsilon) - L_1 + i\epsilon s(v\cdot\omega) + i\epsilon\frac{v\cdot\omega}{s}P_d\right)^{-1}(v\cdot e_j)\chi_0, \\ X_j(\xi,\epsilon) = c_j(s,\epsilon)\omega\times e_j, \quad Y_j(\xi,\epsilon) = \dfrac{i\epsilon^2 s c_j(s,\epsilon)}{\beta_j(s,\epsilon)}e_j, \end{cases} \tag{2.126}$$

where $\mathcal{V}_j^* = (\overline{w_j}, -\overline{X_j}, -\overline{Y_j})$, and e_k, $k = 1,2,3,4$ are normal vectors satisfying $e_1 = e_3$, $e_2 = e_4$, and $e_k \cdot \omega = e_1 \cdot e_2 = 0$, and $c_j(s,\epsilon)$ are C^∞ functions of s and ϵ for $\epsilon s > r_1$ satisfying

$$c_j(s,\epsilon) = i\frac{1}{\sqrt{2}} + \epsilon^2 O\left(\frac{1}{\epsilon s}\right) + \epsilon O\left(\frac{\ln \epsilon s}{\epsilon^2 s^2}\right). \tag{2.127}$$

Proof. The eigenvalues $\beta_j(s,\epsilon)$ and the eigenvectors $\mathcal{V}_j(\xi,\epsilon)$ for $j = 1,2,3,4$ can be constructed as follows. We take $\beta_1 = \beta_3 = z_{-1}(s,\epsilon)$ and $\beta_2 = \beta_4 = z_1(s,\epsilon)$ to be the solution of the equation $D_2(z,s,\epsilon) = 0$ defined in Lemma 2.13. The corresponding eigenvectors $\mathcal{V}_j(\xi,\epsilon) = (w_j(\xi,\epsilon), X_j(\xi,\epsilon), Y_j(\xi,\epsilon))$, $1 \le j \le 4$ are given by

$$\begin{cases} w_j(\xi,\epsilon) = \epsilon c_j(s,\epsilon)\left(\beta_j(s,\epsilon) - L_1 + i\epsilon s(v\cdot\omega) + i\epsilon\frac{v\cdot\omega}{s}P_d\right)^{-1}(v\cdot e_j\chi_0) \\ X_j(\xi,\epsilon) = c_j(s,\epsilon)\omega\times e_j, \quad Y_j(\xi,\epsilon) = \dfrac{i\epsilon^2 s c_j(s,\epsilon)}{\beta_j(s,\epsilon)}e_j, \end{cases}$$

where e_j, $j = 1,2,3,4$ are normal vectors satisfying $e_1 = e_3$, $e_2 = e_4$, and $e_j \cdot \omega = e_1 \cdot e_2 = 0$. It is straightforward to check that $(\mathcal{V}_1, \mathcal{V}_2^*) = (\mathcal{V}_3, \mathcal{V}_4^*) = 0$, where $\mathcal{V}_j^* = (\overline{w_j}, -\overline{X_j}, -\overline{Y_j})$ is the eigenvector of $\tilde{\mathbb{A}}_\epsilon^*(\xi)$ corresponding to the eigenvalue $\overline{\beta_j}$.

Rewrite the eigenvalue problem as

$$\tilde{\mathbb{A}}_\epsilon(\xi)\mathcal{V}_j(\xi,\epsilon) = \beta_j(s,\epsilon)\mathcal{V}_j(\xi,\epsilon), \quad j = 1,2,3,4.$$

By taking the inner product $(\cdot,\cdot)_\xi$ of above equation with $\mathcal{V}_j^*(s,\epsilon)$, and by using the fact that

$$(\tilde{\mathbb{A}}_\epsilon(\xi)U,V)_\xi = (U,\tilde{\mathbb{A}}_\epsilon^*(\xi)V)_\xi, \quad U,V \in D(\tilde{\mathbb{A}}_\epsilon(\xi)),$$
$$\tilde{\mathbb{A}}_\epsilon^*(\xi)\mathcal{V}_j^*(\xi,\epsilon) = \overline{\beta_j(s,\epsilon)}\mathcal{V}_j^*(\xi,\epsilon),$$

we have

$$(\beta_i(s,\epsilon) - \beta_j(s,\epsilon))\left(\mathcal{V}_i(\xi,\epsilon),\mathcal{V}_j^*(\xi,\epsilon)\right)_\xi = 0, \quad 1 \le i,j \le 4.$$

Since $\beta_k(s,\epsilon) \neq \beta_j(s,\epsilon)$ for $k = 1,3$, $j = 2,4$ and $P_d w_j(\xi,\epsilon) = 0$, we have the orthogonal relation

$$\left(\mathcal{V}_k(\xi,\epsilon),\mathcal{V}_j^*(\xi,\epsilon)\right)_\xi = \left(\mathcal{V}_k(\xi,\epsilon),\mathcal{V}_j^*(\xi,\epsilon)\right) = 0, \quad 1 \le k \neq j \le 4.$$

By normalization, we have

$$\left(\mathcal{V}_j(\xi,\epsilon),\mathcal{V}_j^*(\xi,\epsilon)\right) = 1, \quad j = 1,2,3,4.$$

Precisely, the coefficient $c_j(s,\epsilon)$ is determined by the normalization condition:

$$c_j(s,\epsilon)^2\left(-1 + \frac{\epsilon^4 s^2}{\beta_j(s,\epsilon)^2} + \epsilon^2 D_j(s,\epsilon)\right) = 1, \quad j = 1,2,3,4,$$

where $D_j(s,\epsilon) = ((\tilde{\mathbb{B}}_\epsilon(se_1) - \beta_j)^{-1}\chi_2, (\tilde{\mathbb{B}}_\epsilon(-se_1) - \overline{\beta_j})^{-1}\chi_2)$. Since

$$\frac{\beta_j^2(s,\epsilon)}{\epsilon^4 s^2} = \left(i + O\left(\frac{\ln \epsilon s}{\epsilon s^2}\right)\right)^2 = -1 + O\left(\frac{\epsilon \ln \epsilon s}{\epsilon^2 s^2}\right),$$
$$D_j(s,\epsilon) = O(1)\|(\tilde{\mathbb{B}}_\epsilon(se_1) - \beta_j)^{-1}\chi_2\|^2 = O\left(\frac{1}{\epsilon s}\right),$$

it follows that

$$c_j^2(s,\epsilon) = -\left(2 - \left(\frac{s^2}{\beta_j^2(s,\epsilon)} + 1\right) - \epsilon^2 D_j(s,\epsilon)\right)^{-1}$$
$$= -\frac{1}{2} + O\left(\frac{\epsilon^2}{\epsilon s}\right) + O\left(\frac{\epsilon \ln \epsilon s}{\epsilon^2 s^2}\right).$$

Thus, we obtain (2.126) and (2.127) so that the proof of the theorem is completed. □

With Lemmas 2.6, 2.8, 2.11 and 2.14, similar to Theorem 3.4 in [26], we have the following decomposition of the semigroup $e^{\frac{t}{\epsilon^2}\tilde{\mathbb{A}}_\epsilon(\xi)}$.

Theorem 2.15. *The semigroup $e^{\frac{t}{\epsilon^2}\tilde{\mathbb{A}}_\epsilon(\xi)}$ with $\xi \neq 0$ can be decomposed into*

$$e^{\frac{t}{\epsilon^2}\tilde{\mathbb{A}}_\epsilon(\xi)}U = S_1(t,\xi,\epsilon)U + S_2(t,\xi,\epsilon)U + S_3(t,\xi,\epsilon)U, \quad \forall U \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi, \tag{2.128}$$

where

$$S_1(t,\xi,\epsilon)U = \sum_{j=0}^{4} e^{\frac{t}{\epsilon^2}\lambda_j(|\xi|,\epsilon)} \big(U, \mathcal{U}_j^*(\xi,\epsilon)\big)_\xi \mathcal{U}_j(\xi,\epsilon) 1_{\{\epsilon(1+|\xi|)\le r_0\}}, \tag{2.129}$$

$$S_2(t,\xi,\epsilon)U = \sum_{k=1}^{4} e^{\frac{t}{\epsilon^2}\beta_k(|\xi|,\epsilon)} \left(U, \mathcal{V}_k^*(\xi,\epsilon)\right) \mathcal{V}_k(\xi,\epsilon) 1_{\{\epsilon|\xi|\ge r_1\}}, \tag{2.130}$$

with $(\lambda_j(|\xi|,\epsilon), \mathcal{U}_j(\xi,\epsilon))$ and $(\beta_k(|\xi|,\epsilon), \mathcal{V}_k(\xi,\epsilon))$ being the eigenvalue and eigenvector of the operator $\tilde{\mathbb{A}}_\epsilon(\xi)$ for $\epsilon(1+|\xi|) \le r_0$ and $\epsilon|\xi| \ge r_1$ respectively. And $S_3(t,\xi,\epsilon)U$ satisfies

$$\|S_3(t,\xi,\epsilon)U\|_\xi \le Ce^{-\frac{bt}{\epsilon^2}}\|U\|_\xi, \tag{2.131}$$

where the two constants $b > 0$ and $C > 0$ are independent of ξ and ϵ.

2.2. Spectral structure of $\mathbb{B}_\epsilon(\xi)$

The spectrum structure of $\mathbb{B}_\epsilon(\xi)$ is now well known. We just list the results in the following to be self-contained.

Lemma 2.16 *([13]). The operator $\mathbb{B}_\epsilon(\xi)$ generates a strongly continuous contraction semigroup on $L^2(\mathbb{R}^3_v)$, which satisfies*

$$\|e^{t\mathbb{B}_\epsilon(\xi)}f\| \le \|f\|, \quad \forall\, t > 0,\ f \in L^2(\mathbb{R}^3_v).$$

Lemma 2.17 *([5,33]). The following statements hold.*

(1) *For any $\delta > 0$ and all $\xi \in \mathbb{R}^3$, there exists $y_1 = y_1(\delta) > 0$ such that*

$$\rho(\mathbb{B}_\epsilon(\xi)) \supset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \ge -\nu_0 + \delta,\ |\mathrm{Im}\lambda| \ge y_1\} \cup \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda > 0\}.$$

(2) *For any $r_1 > 0$, there exists a constant $\alpha = \alpha(r_1) > 0$ such that for $\epsilon|\xi| \ge r_1$,*

$$\sigma(\mathbb{B}_\epsilon(\xi)) \subset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda < -\alpha\}.$$

Theorem 2.18 *([5,33]).* (1) *There exists a constant $r_0 > 0$ such that for $\epsilon|\xi| \le r_0$,*

$$\sigma(\mathbb{B}_\epsilon(\xi)) \cap \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \ge -\frac{\nu_0}{2}\} = \{\gamma_j(|\xi|, \epsilon),\, j = -1, 0, 1, 2, 3\}.$$

In particular, the eigenvalues $\gamma_j(|\xi|, \epsilon)$, $j = -1, 0, 1, 2, 3$ are analytic functions of $\epsilon|\xi|$ and satisfy the following expansion for $\epsilon|\xi| \le r_0$:

$$\gamma_j(|\xi|, \epsilon) = i\mu_j \epsilon|\xi| - a_j \epsilon^2|\xi|^2 + O\left(\epsilon^3|\xi|^3\right), \tag{2.132}$$

where

$$\begin{cases} \mu_{\pm 1} = \pm\sqrt{\frac{5}{3}}, \quad \mu_k = 0, \quad k = 0, 2, 3, \\ a_j = -(L^{-1}P_1(v \cdot \omega)h_j, (v \cdot \omega)h_j) > 0, \\ h_0(\xi) = \sqrt{\frac{2}{5}}\chi_0 - \sqrt{\frac{3}{5}}\chi_4, \\ h_{\pm 1}(\xi) = \sqrt{\frac{3}{10}}\chi_0 \mp \frac{\sqrt{2}}{2}(v \cdot \omega)\chi_0 + \sqrt{\frac{1}{5}}\chi_4, \\ h_k(\xi) = (v \cdot W^k)\chi_0, \quad k = 2, 3, \end{cases} \tag{2.133}$$

and W^j $(j = 2, 3)$ are orthonormal vectors satisfying $W^j \cdot \omega = 0$.

(2) *The corresponding eigenfunctions $\psi_j(\xi, \epsilon) = \psi_j(\epsilon|\xi|, \omega)$, $j = -1, 0, 1, 2, 3$ satisfy*

$$\begin{cases} (\psi_i(\xi, \epsilon), \overline{\psi_j(\xi, \epsilon)}) = \delta_{ij}, \quad -1 \le i, j \le 3, \\ \psi_j(\xi, \epsilon) = P_0\psi_0(\xi, \epsilon) + P_1\psi_0(\xi, \epsilon), \\ P_0\psi_j(\xi, \epsilon) = h_j(\xi) + O(\epsilon|\xi|), \\ P_1\psi_j(\xi, \epsilon) = i\epsilon|\xi| L^{-1}P_1(v \cdot \omega)h_j(\xi) + O(\epsilon^2|\xi|^2). \end{cases} \tag{2.134}$$

Theorem 2.19 *([5,33]). The semigroup $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon(\xi)}$ with $\xi \in \mathbb{R}^3$ satisfies*

$$e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon(\xi)} f = S_4(t, \xi, \epsilon)f + S_5(t, \xi, \epsilon)f, \quad \forall\, f \in L^2(\mathbb{R}^3_v), \tag{2.135}$$

where

$$S_4(t, \xi, \epsilon)f = \sum_{j=-1}^{3} e^{\frac{t}{\epsilon^2}\gamma_j(|\xi|, \epsilon)} \left(f, \overline{\psi_j(\xi, \epsilon)}\right) \psi_j(\xi, \epsilon) 1_{\{\epsilon|\xi| \le r_0\}},$$

and $S_5(t, \xi, \epsilon)f$ satisfies

$$\|S_5(t, \xi, \epsilon)f\| \le Ce^{-\frac{bt}{\epsilon^2}}\|f\| \tag{2.136}$$

with two constants $b > 0$ and $C > 0$ independent of ξ and ϵ.

3. Fluid approximations

In this section, we will present the first and second order of fluid approximations to the semigroups $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}$ and $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}$ that is a key step to show the convergence of the solution of the VMB system to the solution of the NSMF system.

For any $U_0 = (g_0, E_0, B_0) \in H^l$, set

$$e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon(\xi)}\hat{U}_0 = \left(g, -\frac{i\xi}{|\xi|^2}(g, \chi_0) - \frac{\xi}{|\xi|} \times X, -\frac{\xi}{|\xi|} \times Y\right), \tag{3.1}$$

where

$$e^{\frac{t}{\epsilon^2}\tilde{\mathbb{A}}_\epsilon(\xi)}\hat{V}_0 = (g, X, Y) \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi,$$
$$\hat{V}_0 = \left(\hat{g}_0, \frac{\xi}{|\xi|} \times \hat{E}_0, \frac{\xi}{|\xi|} \times \hat{B}_0\right).$$

For any $f_0 \in H^l$ and any $U_0 = (g_0, E_0, B_0) \in H^l$, set

$$\begin{cases} e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon} f_0 = (\mathcal{F}^{-1} e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon(\xi)}\mathcal{F}) f_0, \\ e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} U_0 = (\mathcal{F}^{-1} e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon(\xi)}\mathcal{F}) U_0. \end{cases} \tag{3.2}$$

Then $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon} f_0$ and $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} U_0$ are the solutions of the systems (2.1) and (2.2) respectively. By Lemmas 2.2 and 2.16, it holds that

$$\|e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon} f_0\|_{H^l} = \int_{\mathbb{R}^3} (1+|\xi|^2)^l \|e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon(\xi)}\hat{f}_0\|^2 d\xi \le \int_{\mathbb{R}^3} (1+|\xi|^2)^l \|\hat{f}_0\|^2 d\xi = \|f_0\|_{H^l},$$
$$\|e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} U_0\|_{H^l} = \int_{\mathbb{R}^3} (1+|\xi|^2)^l \|e^{\frac{t}{\epsilon^2}\tilde{\mathbb{A}}_\epsilon(\xi)}\hat{V}_0\|^2_\xi d\xi \le \int_{\mathbb{R}^3} (1+|\xi|^2)^l \|\hat{V}_0\|^2_\xi d\xi = \|U_0\|_{H^l},$$

where we have used $\|\hat{V}_0\|^2_\xi = \|\hat{U}_0\|^2$.

3.1. Semigroup of the linear NSMF system

In this subsection, we will study the solution to the linear bipolar NSMF system. Firstly, we consider the following linearized bipolar NSMF system to (1.24) for $U_1 = (n, m, q)$ and $U_2 = (\rho, E, B)$:

$$\begin{cases} \nabla_x \cdot m = 0, \quad n + \sqrt{\frac{2}{3}} q = 0, \\ \partial_t m - \kappa_0 \Delta_x m + \nabla_x p = G_1, \\ \partial_t q - \kappa_1 \Delta_x q = \frac{3}{5} G_2, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \partial_t E - \nabla_x \times B = \eta(\nabla_x \rho - E) + G_3, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla_x \cdot E = \rho, \quad \nabla_x \cdot B = 0, \end{cases} \tag{3.4}$$

where G_1, $G_3 \in \mathbb{R}^3$ and $G_2 \in \mathbb{R}$ are given functions, p is the pressure satisfying $p = \Delta_x^{-1}\mathrm{div}_x G_1$, and the initial data $(n, m, q)(0)$ and $(\rho, E, B)(0)$ satisfy (1.25).

For any $\hat{f}_0 = \hat{f}_0(\xi, v) \in N_0$ and $\hat{V}_0 = (\hat{\rho}_0(\xi)\chi_0, \hat{E}_0(\xi), \hat{B}_0(\xi)) \in N_1 \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3$, set

$$Y_1(t,\xi)\hat{f}_0 = \sum_{j=0,2,3} e^{-a_j|\xi|^2 t} \left(\hat{f}_0, h_j(\xi)\right) h_j(\xi), \tag{3.5}$$

$$\tilde{Y}_2(t,\xi)\hat{V}_0 = \sum_{j=0}^{4} e^{b_j(|\xi|)t} \left(\hat{V}_0, \overline{\mathcal{X}_j(\xi)}\right)_\xi \mathcal{X}_j(\xi), \quad |\xi| \neq \frac{\eta}{2}, \tag{3.6}$$

where $b_k(|\xi|)$ $(k = 0,1,2,3,4)$ and $(a_j,\ h_j(\xi))$ $(j = 0,2,3)$ are defined by (2.85) and (2.133) respectively, and $\mathcal{X}_j(\xi)$, $j = 0,1,2,3,4$ are given by

$$\begin{cases} \mathcal{X}_0(\xi) = \left(\dfrac{|\xi|}{\sqrt{1+|\xi|^2}}\chi_0, 0, 0\right), \\ \mathcal{X}_k(\xi) = \dfrac{b_k}{\sqrt{b_k^2 - |\xi|^2}} \left(0, \dfrac{\xi}{|\xi|} \times e_k, \dfrac{i|\xi|e_k}{b_k}\right), \quad k = 1,2,3,4. \end{cases} \tag{3.7}$$

Here, e_k, $k = 1,2,3,4$ are normal vectors satisfying $e_1 = e_3$, $e_2 = e_4$, and $e_k \cdot \omega = e_1 \cdot e_2 = 0$.

For any $\hat{U}_0 = (\hat{\rho}_0\chi_0, \hat{E}_0, \hat{B}_0) \in N_1 \times \mathbb{C}^3 \times \mathbb{C}^3$ with $\hat{\rho}_0 = i\xi \cdot \hat{E}_0$, set

$$Y_2(t,\xi)\hat{U}_0 = \left(\hat{\rho}\chi_0, -\frac{i\xi}{|\xi|^2}\hat{\rho} - \frac{\xi}{|\xi|} \times \hat{X}, -\frac{\xi}{|\xi|} \times \hat{Y}\right), \tag{3.8}$$

with

$$\tilde{Y}_2(t,\xi)\hat{V}_0 = (\hat{\rho}\chi_0, \hat{X}, \hat{Y}) \in L_\xi^2(\mathbb{R}_v^3) \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3,$$
$$\hat{V}_0 = \left(\hat{\rho}_0\chi_0, \frac{\xi}{|\xi|} \times \hat{E}_0, \frac{\xi}{|\xi|} \times \hat{B}_0\right).$$

Denote

$$\begin{cases} Y_1(t)f_0 = (\mathcal{F}^{-1}Y_1(t,\xi)\mathcal{F})f_0, \\ Y_2(t)U_0 = (\mathcal{F}^{-1}Y_2(t,\xi)\mathcal{F})U_0. \end{cases} \tag{3.9}$$

It is straightforward to check that

$$\begin{cases} f = P_{||}f + P_{\perp}f, \quad P_{||}Y_1(t) = 0, \quad P_{\perp}Y_1(t) = Y_1(t), \\ \|P_{||}f\|^2 = |(f, v\chi_0)_{||}|^2 + |(f, \tilde{h}_1)|^2, \\ \|P_{\perp}f\|^2 = |(f, v\chi_0)_{\perp}|^2 + |(f, \tilde{h}_0)|^2 + \|P_1 f\|^2. \end{cases} \tag{3.10}$$

Then, we can represent the solutions to the NS system (3.3) and NSM type system (3.4) by the semigroups $Y_1(t)$ and $Y_2(t)$ respectively.

Lemma 3.1. *For any $f_0 \in L^2$, $U_0 = (g_0, E_0, B_0) \in L^2$ and $G_j \in L^1_t(L^2_x)$, $j = 1, 2, 3$, we define*

$$u(t, x, v) = Y_1(t)P_0 f_0 + \int_0^t Y_1(t - s)H_1(s)ds, \tag{3.11}$$

$$U(t, x, v) = Y_2(t)P_2 U_0 + \int_0^t Y_2(t - s)H_2(s)ds, \tag{3.12}$$

where

$$H_1(t, x, v) = G_1(t, x) \cdot v\chi_0 + G_2(t, x)\chi_4,$$
$$H_2(t, x, v) = (\nabla_x \cdot G_3(t, x)\chi_0, G_3, 0).$$

Let $(n, m, q) = ((u, \chi_0), (u, v\chi_0), (u, \chi_4))$ and $U = (\rho\chi_0, E, B)$. Then $(n, m, q)(t, x) \in L^\infty_t(L^2_x)$ and $(\rho, E, B)(t, x) \in L^\infty_t(L^2_x)$ are the unique global solutions to the linear NS system (3.3) and NSM type system (3.4) with the initial data $(n, m, q)(0)$ and $(\rho, E, B)(0)$ satisfying (1.25).

Proof. To show (3.11), by taking Fourier transform to (3.3), we have

$$i\xi \cdot \hat{m} = 0, \quad \hat{n} + \sqrt{\frac{2}{3}}\hat{q} = 0, \tag{3.13}$$

$$\partial_t \hat{m} - \kappa_0|\xi|^2\hat{m} + i\xi\hat{p}_1 = \hat{G}_1, \tag{3.14}$$

$$\partial_t \hat{q} - \kappa_1|\xi|^2\hat{q} = \frac{3}{5}\hat{G}_2, \tag{3.15}$$

where the initial data $(\hat{n}, \hat{m}, \hat{q})(0)$ satisfies

$$\hat{m}(0) = (P_0\hat{f}_0, v\chi_0)_{\perp}, \quad \sqrt{\frac{3}{2}}\hat{n}(0) = -\hat{q}(0) = \sqrt{\frac{3}{5}}\Big(P_0\hat{f}_0, \sqrt{\frac{2}{5}}\chi_0 - \sqrt{\frac{3}{5}}\chi_4\Big).$$

Then, it follows from (3.15) and (3.13) that

$$\hat{q}(t, \xi) = e^{-\kappa_1|\xi|^2 t}\hat{q}(0) + \int_0^t e^{-\kappa_1|\xi|^2(t-s)}\hat{G}_2(s)ds$$

$$
\begin{aligned}
&= e^{-\kappa_1|\xi|^2 t}\left(P_0\hat{f}_0, h_0(\xi)\right)(h_0(\xi),\chi_4) \\
&\quad + \int_0^t e^{-\kappa_1|\xi|^2(t-s)}\left(\hat{H}_1(s), h_0(\xi)\right)(h_0(\xi),\chi_4)ds,
\end{aligned} \tag{3.16}
$$

and

$$
\begin{aligned}
\hat{n}(t,\xi) &= e^{-\kappa_1|\xi|^2 t}\left(P_0\hat{f}_0, h_0(\xi)\right)(h_0(\xi),\chi_0) \\
&\quad + \int_0^t e^{-\kappa_1|\xi|^2(t-s)}\left(\hat{H}_1(s), h_0(\xi)\right)(h_0(\xi),\chi_0)ds.
\end{aligned} \tag{3.17}
$$

By (3.13) and (3.14) and noting that $\hat{m} = \hat{m}_\perp$, we have

$$
\begin{aligned}
\hat{m}(t,\xi) &= e^{-\kappa_0|\xi|^2 t}\hat{m}(0) + \int_0^t e^{-\kappa_0|\xi|^2(t-s)}\hat{G}_1(s)_\perp ds \\
&= \sum_{j=2,3} e^{-\kappa_0|\xi|^2 t}\left(P_0\hat{f}_0, h_j(\xi)\right)(h_j(\xi), v\chi_0) \\
&\quad + \sum_{j=2,3}\int_0^t e^{-\kappa_0|\xi|^2(t-s)}\left(\hat{H}_1(s), h_j(\xi)\right)(h_j(\xi), v\chi_0)ds.
\end{aligned} \tag{3.18}
$$

Noting that $\kappa_0 = a_2 = a_3$, $\kappa_1 = a_0$, $(h_0(\xi), v\chi_0) = 0$ and $(h_j(\xi),\chi_0) = (h_j(\xi),\chi_4) = 0$, $j = 2,3$, we obtain (3.11) by using (3.16)–(3.18).

Next, we prove (3.12) as follows. Taking Fourier transform to (3.4) gives the system for $(\hat{\rho}, \hat{E}, \hat{B})$:

$$
\partial_t\hat{\rho} + \eta(1+|\xi|^2)\hat{\rho} = i\xi\cdot\hat{G}_3, \tag{3.19}
$$

$$
\partial_t\hat{E} = i\xi\times B + \eta(i\xi\hat{\rho} - \hat{E}) + \hat{G}_3, \tag{3.20}
$$

$$
\partial_t\hat{B} = -i\xi\times\hat{E}, \tag{3.21}
$$

$$
i\xi\cdot\hat{E} = \hat{\rho}, \quad i\xi\cdot\hat{B} = 0,
$$

where the initial data $(\hat{\rho}, \hat{E}, \hat{B})(0)$ satisfies

$$
\hat{\rho}(0) = i\xi\cdot\hat{E}_0, \quad \hat{E}(0) = \hat{E}_0, \quad \hat{B}(0) = \hat{B}_0.
$$

Taking $\omega\times$ to (3.20) and (3.21) yields

$$
\partial_t(\omega\times\hat{E}) = i\xi\times(\omega\times\hat{B}) - \eta(\omega\times\hat{E}) + \omega\times\hat{G}_3, \tag{3.22}
$$

$$
\partial_t(\omega\times\hat{B}) = -i\xi\times(\omega\times\hat{E}). \tag{3.23}
$$

Let $\hat{V} = (\hat{\rho}\chi_0, \omega \times \hat{E}, \omega \times \hat{B})^T \in L^2_\xi(\mathbb{R}^3) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$. Then, the system (3.19), (3.22) and (3.23) can be written as

$$\partial_t \hat{V} = A(\xi)\hat{V} + \hat{H}_3,$$

where $\hat{H}_3 = (i\xi \cdot \hat{G}_3\chi_0, \omega \times \hat{G}_3, 0)^T$, and

$$A(\xi) = \begin{pmatrix} -\eta(1+|\xi|^2) & 0 & 0 \\ 0 & -\eta & i\xi\times \\ 0 & -i\xi\times & 0 \end{pmatrix}.$$

It is straightforward to check that $A^*(\xi) = \overline{A(\xi)}$ and $A(\xi)$ admits five eigenvalues $b_j(|\xi|)$ given by (2.85) with eigenfunctions $\mathcal{X}_j(\xi)$ given by (3.7). Note that $b_1(|\xi|) = b_3(|\xi|) = -\eta/2$ when $|\xi| = \eta/2$, and $\mathcal{X}_j(\xi)$ satisfy the orthonormal relation $(\mathcal{X}_i(\xi), \overline{\mathcal{X}_j(\xi)})_\xi = \delta_{ij}$ for $|\xi| \neq \eta/2$. Thus

$$\begin{aligned} \hat{V}(t,\xi) = &\sum_{j=0}^{4} e^{b_j(|\xi|)t} \left(\hat{V}(0), \overline{\mathcal{X}_j(\xi)}\right)_\xi \mathcal{X}_j(\xi) \\ &+ \sum_{j=0}^{4} \int_0^t e^{b_j(|\xi|)(t-s)} \left(\hat{H}_3(s), \overline{\mathcal{X}_j(\xi)}\right)_\xi \mathcal{X}_j(\xi) ds, \quad |\xi| \neq \frac{\eta}{2}. \end{aligned}$$

This proves (3.12) and completes the proof of the lemma. □

3.2. Fluid approximation of $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}$

The following lemma will be used to study the fluid dynamical approximations of the semigroups $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}$ and $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}$.

Lemma 3.2. *For any $V_0 \in N_1 \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ and $f_0 \in N_0$, we have*

$$\|S_3(t,\xi,\epsilon)V_0\|_\xi \le C\left(\epsilon(1+|\xi|)1_{\{\epsilon(1+|\xi|)\le r_0\}} + 1_{\{\epsilon(1+|\xi|)\ge r_0\}}\right) e^{-\frac{bt}{\epsilon^2}} \|V_0\|_\xi, \tag{3.24}$$

$$\|S_5(t,\xi,\epsilon)f_0\| \le C\left(\epsilon|\xi|1_{\{\epsilon|\xi|\le r_0\}} + 1_{\{\epsilon|\xi|\ge r_0\}}\right) e^{-\frac{bt}{\epsilon^2}} \|f_0\|, \tag{3.25}$$

where $S_3(t,\xi,\epsilon)$ and $S_5(t,\xi,\epsilon)$ are given in Theorems 2.15 and 2.19 respectively.

Proof. For any $V \in L^2_\xi(\mathbb{R}^3) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$, define a projection $P_\epsilon(\xi)$ by

$$P_\epsilon(\xi)V = \sum_{j=0}^{4} \left(V, \mathcal{U}_j^*(\xi,\epsilon)\right)_\xi \mathcal{U}_j(\xi,\epsilon), \quad \epsilon(1+|\xi|) \le r_0,$$

where $\mathcal{U}_j(\xi,\epsilon)$, $j=0,1,2,3,4$ are the eigenfunctions of $\tilde{\mathbb{A}}_\epsilon(\xi)$ for $\epsilon(1+|\xi|)\le r_0$ given in (2.86).

By Theorem 2.15, we claim that

$$S_1(t,\xi,\epsilon)=e^{\frac{t}{\epsilon^2}\tilde{\mathbb{A}}_\epsilon(\xi)}P_\epsilon(\xi)1_{\{\epsilon(1+|\xi|)\le r_0\}}. \tag{3.26}$$

Indeed, it follows from semigroup theory that for $\kappa>0$,

$$\begin{aligned}
e^{\frac{t}{\epsilon^2}\tilde{\mathbb{A}}_\epsilon(\xi)}P_\epsilon(\xi)V =&\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\frac{\lambda t}{\epsilon^2}}(\lambda-\tilde{\mathbb{A}}_\epsilon(\xi))^{-1}P_\epsilon(\xi)V d\lambda\\
=&\frac{1}{2\pi i}\sum_{j=0}^{4}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\frac{\lambda t}{\epsilon^2}}(\lambda-\lambda_j(|\xi|,\epsilon))^{-1}\left(V,\mathcal{U}_j^*\right)_\xi\mathcal{U}_j d\lambda\\
=&\sum_{j=0}^{4}e^{\frac{t}{\epsilon^2}\lambda_j(|\xi|,\epsilon)}\left(V,\mathcal{U}_j^*\right)_\xi\mathcal{U}_j=S_1(t,\xi,\epsilon)V.
\end{aligned}$$

Thus, by Theorem 2.15 we can decompose $S_3(t,\xi,\epsilon)$ into

$$S_3(t,\xi,\epsilon)=S_{31}(t,\xi,\epsilon)+S_{32}(t,\xi,\epsilon), \tag{3.27}$$

where

$$\begin{cases} S_{31}(t,\xi,\epsilon)=S(t,\xi,\epsilon)(I-P_\epsilon(\xi))1_{\{\epsilon(1+|\xi|)\le r_0\}},\\ S_{32}(t,\xi,\epsilon)=S_3(t,\xi,\epsilon)1_{\{\epsilon(1+|\xi|)\ge r_0\}}.\end{cases}$$

Moreover, $S_{3k}(t,\xi,\epsilon)$, $k=1,2$ satisfy

$$\|S_{3k}(t,\xi,\epsilon)V\|_\xi\le Ce^{-\frac{bt}{\epsilon^2}}\|V\|_\xi,\quad k=1,2. \tag{3.28}$$

For any $V_0\in N_1\times\mathbb{C}_\xi^3\times\mathbb{C}_\xi^3$, we can obtain by (3.38) and (3.44) that

$$\|V_0-P_\epsilon(\xi)V_0\|_\xi=\|\tilde{Y}_2(0,\xi)V_0-S_1(0,\xi,\epsilon)V_0\|_\xi\le C\epsilon\|V_0\|_\xi.$$

The above estimate and the fact $S_{31}(t,\xi,\epsilon)=S_{31}(t,\xi,\epsilon)(I-P_\epsilon(\xi))$ imply that

$$\|S_{31}(t,\xi,\epsilon)V_0\|_\xi\le C\epsilon(1+|\xi|)1_{\{\epsilon(1+|\xi|)\le r_0\}}e^{-\frac{bt}{\epsilon^2}}\|V_0\|_\xi. \tag{3.29}$$

By combining (3.27)–(3.29), we obtain (3.24). (3.25) can be proved similarly. And this completes the proof of the lemma. □

The following lemma gives the first order fluid approximation of the semigroup $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}$.

Lemma 3.3. *For any $\epsilon \ll 1$, any integer $k, m \geq 0$ and $U_0 = (g_0, E_0, B_0) \in L^2 \cap L^1$, it holds that*

$$\left\| e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} U_0 - Y_2(t)P_2U_0 \right\|_{H^k} \leq C\left(\epsilon(1+t)^{-\frac{3}{4}} + e^{-\frac{bt}{\epsilon^2}}\right)(\|U_0\|_{H^{k+1}} + \|U_0\|_{L^1}) + C\epsilon^m(1+t)^{-m}\|\nabla_x^m U_0\|_{H^k}, \tag{3.30}$$

where $Y_2(t)$ is defined by (3.8), $P_2U_0 = (P_d g_0, E_0, B_0)$, and $b > 0$ is a constant given by (2.131). Moreover, if $U_0 = (g_0, E_0, B_0) \in L^2 \cap L^1$ satisfying $P_r g_0 = 0$, then

$$\left\| e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} U_0 - Y_2(t)P_2U_0 \right\|_{H^k} \leq C\epsilon(1+t)^{-\frac{3}{4}}(\|U_0\|_{H^{k+1}} + \|U_0\|_{L^1}) + C\epsilon^m(1+t)^{-m}\|\nabla_x^m U_0\|_{H^k}. \tag{3.31}$$

Proof. For brevity, we only prove the case when $k = 0$ because the proof for $k > 0$ is similar. By (2.128) and by taking $\epsilon \leq r_0/2$ with $r_0 > 0$ given in Lemma 2.11, we have

$$\begin{aligned}
\left\| e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} U_0 - Y_2(t)P_2U_0 \right\|_{L^2}^2 &= \int_{\mathbb{R}^3} \left\| e^{\frac{t}{\epsilon^2}\tilde{\mathbb{A}}_\epsilon(\xi)} \hat{V}_0 - \tilde{Y}_2(t,\xi)P_2\hat{V}_0 \right\|_\xi^2 d\xi \\
&\leq 4 \int_{1+|\xi|\leq \frac{r_0}{\epsilon}} \left\| S_1(t,\xi,\epsilon)\hat{V}_0 - \tilde{Y}_2(t,\xi)P_2\hat{V}_0 \right\|_\xi^2 d\xi \\
&\quad + 4 \int_{|\xi|\geq \frac{r_1}{\epsilon}} \left\| S_2(t,\xi,\epsilon)\hat{V}_0 \right\|_\xi^2 d\xi + 4\int_{\mathbb{R}^3} \left\| S_3(t,\xi,\epsilon)\hat{V}_0 \right\|_\xi^2 d\xi \\
&\quad + 4 \int_{1+|\xi|\geq \frac{r_0}{\epsilon}} \left\| \tilde{Y}_2(t,\xi)P_2\hat{V}_0 \right\|_\xi^2 d\xi \\
&=: I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{3.32}$$

where $\hat{V}_0 = (\hat{g}_0, \omega \times \hat{E}_0, \omega \times \hat{B}_0)$.

First, we decompose I_1 into

$$\begin{aligned}
I_1 &= 4\left(\int_{1+|\xi|\leq \frac{r_0}{\epsilon}, |\eta^2-4|\xi|^2|\geq r_0} + \int_{|\eta^2-4|\xi|^2|\leq r_0} \right) \left\| S_1(t,\xi,\epsilon)\hat{V}_0 - \tilde{Y}_2(t,\xi)P_2\hat{V}_0 \right\|_\xi^2 d\xi \\
&=: I_{11} + I_{12}.
\end{aligned} \tag{3.33}$$

We estimate I_{11} and I_{12} separately as follows. By Lemma 2.11, it holds that for $|\eta^2 - 4|\xi|^2| \geq r_0$ and $\epsilon(1+|\xi|) \leq r_0$ with $r_0 \ll 1$,

$$\frac{1}{\epsilon^2}\lambda_k = b_k(1 + O(\epsilon^2)), \qquad \frac{\lambda_k \Theta_k}{\sqrt{\lambda_1\lambda_3 - \lambda_k^2}} = \frac{ib_k}{\sqrt{b_k^2 - |\xi|^2}}(1 + O(\epsilon^2)), \quad k = 1,2,3,4. \tag{3.34}$$

Thus, we can obtain by (3.34) and (2.129) that for $\epsilon(1+|\xi|)\le r_0$ and $|\eta^2-4|\xi|^2|\ge r_0$,

$$\begin{aligned}S_1(t,\xi,\epsilon)\hat{V}_0 =&e^{b_0(|\xi|)t+O(\epsilon^2(1+|\xi|^2)^2)t}\left[\left(\hat{V}_0,\overline{\mathcal{X}_0(\xi)}\right)_\xi\mathcal{X}_0(\xi)+T_0(\xi,\epsilon)\right]\\&+\sum_{k=1}^4 e^{b_k(|\xi|)t+O(\epsilon^2|b_k(|\xi|)|)t}\left[\left(\hat{V}_0,\overline{\mathcal{X}_k(\xi)}\right)\mathcal{X}_k(\xi)+T_k(\xi,\epsilon)\right],\end{aligned}\tag{3.35}$$

where

$$\begin{cases}\|T_0(\xi,\epsilon)\|_\xi=O(1)\|\mathcal{U}_0-\mathcal{X}_0\|_\xi\|\hat{V}_0\|_\xi=O(\epsilon\sqrt{1+|\xi|^2})\|\hat{U}_0\|,\\ \|T_k(\xi,\epsilon)\|_\xi=O(1)\|\mathcal{U}_k-\mathcal{X}_k\|\|\hat{V}_0\|=O(\epsilon)\|\hat{U}_0\|,\quad k=1,2,3,4.\end{cases}\tag{3.36}$$

Note that

$$\begin{cases}\mathrm{Re}b_1(|\xi|)\le -c_1|\xi|^2,\ \ |\xi|\le r_0;\quad \mathrm{Re}b_1(|\xi|)\le -c_2,\ \ |\xi|\ge r_0,\\ \mathrm{Re}b_3(|\xi|)\le -\eta/2,\ \ \xi\in\mathbb{R}^3;\quad b_1(|\xi|)=b_2(|\xi|),\ \ b_3(|\xi|)=b_4(|\xi|),\end{cases}\tag{3.37}$$

where c_1,c_2 are two positive constants. Thus, we have

$$\begin{aligned}I_{11}\le& C\int_{1+|\xi|\le\frac{r_0}{\epsilon}} e^{b_0t}\left[r_0^2\epsilon^2(1+|\xi|^2)^3t^2+\epsilon^2(1+|\xi|^2)\right]\|\hat{U}_0\|^2d\xi\\&+C\sum_{j=1}^4\int_{1+|\xi|\le\frac{r_0}{\epsilon}} e^{\mathrm{Re}b_jt}\left(\epsilon^4|b_j|^2t^2+\epsilon^2\right)\|\hat{U}_0\|^2d\xi\\ \le& C\int_{|\xi|\le r_0}\epsilon^2e^{-c_1|\xi|^2t}(1+\epsilon^2|\xi|^4t^2)\|\hat{U}_0\|^2d\xi\\&+C\int_{1+|\xi|\le\frac{r_0}{\epsilon}}\epsilon^2e^{-c_2t}(1+|\xi|^2)\|\hat{U}_0\|^2d\xi\\ \le& C\epsilon^2\left(\sup_{|\xi|\le r_0}\|\hat{U}_0\|^2\int_{|\xi|\le r_0}e^{-c_1|\xi|^2t}d\xi+e^{-c_2t}\int_{\mathbb{R}^3}(1+|\xi|^2)\|\hat{U}_0\|^2d\xi\right)\\ \le& C\epsilon^2\left[(1+t)^{-3/2}\|U_0\|_{L^1}^2+e^{-c_2t}\|U_0\|_{H^1}^2\right],\end{aligned}\tag{3.38}$$

where we have used

$$\sup_{|\xi|\le r_0}\|\hat{U}_0\|^2\le C\|g_0\|^2_{L^2_v(L^1_x)}+C\|(E_0,B_0)\|^2_{L^1_x}\le C\|U_0\|^2_{L^1}.$$

Note that when $|\eta^2-4|\xi|^2|\to 0$, it holds that $\lambda_1\to\lambda_3$ and $b_1\to b_3$. This implies that $\|\mathcal{X}_k\|_\xi\to\infty$ and $\|\mathcal{U}_k\|_\xi\to\infty$ for $k=1,2,3,4$. In this situation, the expansions

(3.35)–(3.36) do not hold. To overcome this difficulty, we rewrite $S_1(t,\xi,\epsilon)$ and $\tilde{Y}_2(t,\xi)$ in other forms. Indeed, it follows from (2.129) and (2.86) that

$$S_1(t,\xi,\epsilon)\hat{V}_0 = e^{\frac{t}{\epsilon^2}\lambda_0}(\hat{V}_0,\mathcal{U}_0^*)_\xi\mathcal{U}_0 + e^{\frac{t}{\epsilon^2}\lambda_1}\frac{\lambda_1\Theta_1^2}{\lambda_3-\lambda_1}\left[(\hat{V}_0,\tilde{\mathcal{U}}_1^*)\tilde{\mathcal{U}}_1 + (\hat{V}_0,\tilde{\mathcal{U}}_2^*)\tilde{\mathcal{U}}_2\right]$$
$$+ e^{\frac{t}{\epsilon^2}\lambda_3}\frac{\lambda_3\Theta_3^2}{\lambda_1-\lambda_3}\left[(\hat{V}_0,\tilde{\mathcal{U}}_3^*)\tilde{\mathcal{U}}_3 + (\hat{V}_0,\tilde{\mathcal{U}}_4^*)\tilde{\mathcal{U}}_4\right],$$

where Θ_j, $j=1,2,3,4$ are defined by (2.96), and

$$\tilde{\mathcal{U}}_k = \frac{\sqrt{\lambda_1\lambda_3-\lambda_k^2}}{\lambda_k\Theta_k}\mathcal{U}_k = \left(\epsilon R(\lambda_k,\epsilon\xi)(v\cdot e_k)\chi_0, \omega\times e_k, \frac{ise_k}{z_k}\right)$$

with $R(\lambda,\epsilon\xi) = (L_1-\lambda-i\epsilon P_r(v\cdot\xi))^{-1}$. Thus, we can rewrite $S_1(t,\xi,\epsilon)$ for $|\eta^2-4|\xi|^2|<r_0$ as

$$S_1(t,\xi,\epsilon)\hat{V}_0 = e^{z_0t}\tilde{V}_0 + e^{z_1t}\Theta_1^2(\tilde{V}_1+\tilde{V}_2) + (e^{z_1t}-e^{z_3t})\frac{z_3}{z_3-z_1}\Theta_3^2(\tilde{V}_1+\tilde{V}_2)$$
$$+ e^{z_3t}\frac{z_3}{z_3-z_1}\Theta_3^2\left[(\tilde{V}_1-\tilde{V}_3)+(\tilde{V}_2-\tilde{V}_4)\right]$$
$$+ e^{z_1t}\frac{z_3}{z_3-z_1}(\Theta_1^2-\Theta_3^2)(\tilde{V}_1+\tilde{V}_2),$$

where $z_j = \epsilon^{-2}\lambda_j$, and

$$\tilde{V}_0 = (\hat{V}_0,\mathcal{U}_0^*)_\xi\mathcal{U}_0, \quad \tilde{V}_k = (\hat{V}_0,\tilde{\mathcal{U}}_k^*)\tilde{\mathcal{U}}_k, \quad k=1,2,3,4.$$

Note that

$$(e^{z_1t}-e^{z_3t})\frac{z_3}{z_3-z_1} = -z_3e^{z_3t}\int_0^t e^{\tau(z_1-z_3)}d\tau,$$
$$e^{z_3t}\frac{z_3}{z_3-z_1}\left[(\tilde{V}_1-\tilde{V}_3)+(\tilde{V}_2-\tilde{V}_4)\right] = z_3e^{z_3t}(\tilde{V}_{13}+\tilde{V}_{24}),$$
$$e^{z_1t}\frac{z_3}{z_3-z_1}(\Theta_1^2-\Theta_3^2) = O(1)\epsilon^4z_3e^{z_1t},$$

where

$$\begin{cases}\tilde{V}_{jk} = (\hat{V}_0,\tilde{\mathcal{U}}_{jk}^*)\tilde{\mathcal{U}}_j + (\hat{V}_0,\tilde{\mathcal{U}}_k^*)\tilde{\mathcal{U}}_{jk}, \quad j,k=1,2,3,4,\\ \tilde{\mathcal{U}}_{jk} = \left(\epsilon^3R(\lambda_j,\epsilon\xi)R(\lambda_k,\epsilon\xi)(v\cdot e_j)\chi_0, 0, \dfrac{ise_j}{z_jz_k}\right).\end{cases} \tag{3.39}$$

We have

$$
\begin{aligned}
S_1(t,\xi,\epsilon)\hat{V}_0 = {}& e^{z_0 t}\tilde{V}_0 + e^{z_1 t}\Theta_1^2(\tilde{V}_1+\tilde{V}_2) + z_3 e^{z_3 t}\Theta_3^2(\tilde{V}_{13}+\tilde{V}_{24}) \\
& - z_3 e^{z_3 t}\int_0^t e^{\tau(z_1-z_3)}d\tau\,\Theta_3^2(\tilde{V}_1+\tilde{V}_2) + O(1)\epsilon^4 z_3 e^{z_1 t}(\tilde{V}_1+\tilde{V}_2).
\end{aligned}
\tag{3.40}
$$

Similarly, we rewrite $\tilde{Y}_2(t,\xi)$ as

$$
\begin{aligned}
\tilde{Y}_2(t,\xi)P_2\hat{V}_0 = {}& e^{b_0 t}\tilde{W}_0 + e^{b_1 t}(\tilde{W}_1+\tilde{W}_2) - b_3 e^{b_3 t}\int_0^t e^{\tau(b_1-b_3)}d\tau(\tilde{W}_1+\tilde{W}_2) \\
& + b_3 e^{b_3 t}(\tilde{W}_{13}+\tilde{W}_{24}),
\end{aligned}
\tag{3.41}
$$

where

$$
\begin{cases}
\tilde{W}_0 = (P_2\hat{V}_0,\overline{\mathcal{X}_0})_\xi\mathcal{X}_0, \quad \tilde{W}_k = (P_2\hat{V}_0,\overline{X_k})X_k, \\
\tilde{W}_{jk} = (P_2\hat{V}_0,\overline{X_{jk}})X_j + (P_2\hat{V}_0,\overline{X_k})X_{jk}, \quad j,k=1,2,3,4, \\
X_k = \left(0,\omega\times e_k,\dfrac{ise_k}{b_k}\right), \quad X_{jk} = \left(0,0,\dfrac{ise_j}{b_jb_k}\right).
\end{cases}
\tag{3.42}
$$

Since it follows from Lemma 2.11 that for $|\eta^2-4|\xi|^2|\le r_0$,

$$
\begin{cases}
z_0 = b_0 + O(\epsilon^2), \quad z_k = b_k + O(\epsilon), \quad \Theta_k = 1+O(\epsilon), \\
|\tilde{V}_k-\tilde{W}_k| + |\tilde{V}_{jk}-\tilde{W}_{jk}| = O(\epsilon)\|\hat{U}_0\|, \quad j,k=1,2,3,4,
\end{cases}
\tag{3.43}
$$

we obtain by (3.40) and (3.41) that

$$
\begin{aligned}
I_{12} \le{}& C\int_{|\eta^2-4|\xi|^2|\le r_0} e^{-\frac{\eta}{2}t}\left(|z_0-b_0|^2 + \|\mathcal{U}_0-\mathcal{X}_0\|_\xi^2\right)\|\hat{U}_0\|^2 d\xi \\
& + C\sum_{k=1,3}\int_{|\eta^2-4|\xi|^2|\le r_0} e^{-\frac{\eta}{2}t}\left(\epsilon^2 + |z_k-b_k|^2 + |\Theta_k^2-1|^2\right)\|\hat{U}_0\|^2 d\xi \\
\le{}& C\epsilon^2 e^{-\frac{\eta}{2}t}\|U_0\|_{L^2}^2.
\end{aligned}
\tag{3.44}
$$

Thus, it follows from (3.38) and (3.44) that

$$
I_1 \le C\epsilon^2\left(e^{-\frac{\eta}{2}t}\|U_0\|_{L^2}^2 + (1+t)^{-\frac{3}{2}}\|U_0\|_{L^1}^2\right). \tag{3.45}
$$

By (2.130) and Lemma 2.14, we have

$$
S_2(t,\xi,\epsilon)\hat{V}_0 = \sum_{k=1}^4 e^{\frac{t}{\epsilon^2}\beta_k(|\xi|,\epsilon)}\left(\hat{V}_0,\mathcal{V}_k^*(\xi,\epsilon)\right)\mathcal{V}_k(\xi,\epsilon), \quad \epsilon|\xi|\ge r_1,
$$

which gives

$$I_2 = 4 \int_{|\xi|\ge \frac{r_1}{\epsilon}} \left\| S_2(t,\xi,\epsilon)\hat{V}_0 \right\|_\xi^2 d\xi \le C \int_{|\xi|\ge \frac{r_1}{\epsilon}} e^{-\frac{ct}{\epsilon|\xi|}} \|\hat{V}_0\|^2 d\xi$$

$$\le C \sup_{|\xi|\ge \frac{r_1}{\epsilon}} \frac{1}{|\xi|^{2m}} e^{-\frac{ct}{\epsilon|\xi|}} \int_{|\xi|\ge \frac{r_1}{\epsilon}} |\xi|^{2m} \|\hat{U}_0\|^2 d\xi \le C\epsilon^{2m}(1+t)^{-2m} \|\nabla_x^m U_0\|_{L^2}^2. \tag{3.46}$$

By (2.131), we have

$$I_3 = 4 \int_{\mathbb{R}^3} \left\| S_3(t,\xi,\epsilon)\hat{V}_0 \right\|_\xi^2 d\xi \le C \int_{\mathbb{R}^3} e^{-2\frac{bt}{\epsilon^2}} \|\hat{V}_0\|_\xi^2 d\xi \le C e^{-2\frac{bt}{\epsilon^2}} \|U_0\|_{L^2}^2. \tag{3.47}$$

For I_4, it holds that

$$I_4 = 4 \int_{1+|\xi|\ge \frac{r_0}{\epsilon}} \left\| \tilde{Y}_2(t,\xi) P_2 \hat{V}_0 \right\|_\xi^2 d\xi \le C \int_{1+|\xi|\ge \frac{r_0}{\epsilon}} e^{-\eta t} \|\hat{V}_0\|_\xi^2 d\xi$$

$$\le C \frac{\epsilon^2}{r_0^2} e^{-\eta t} \int_{1+|\xi|\ge \frac{r_0}{\epsilon}} (1+|\xi|)^2 \|\hat{U}_0\|^2 d\xi \le C\epsilon^2 e^{-\eta t} \|U_0\|_{H^1}^2. \tag{3.48}$$

Therefore, it follows from (3.32) and (3.45)–(3.48) that

$$\left\| e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} U_0 - Y_2(t) P_2 U_0 \right\|_{L^2}^2 \le C \left(\epsilon^2 (1+t)^{-\frac{3}{2}} + e^{-2\frac{bt}{\epsilon^2}} \right) (\|U_0\|_{H^1}^2 + \|U_0\|_{L^1}^2). \tag{3.49}$$

We now turn to (3.31). Since $\hat{V}_0 \in N_1 \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3$, by Lemma 3.2 we have

$$I_3 \le C \int_{1+|\xi|\le \frac{r_0}{\epsilon}} \epsilon^2 (1+|\xi|^2) e^{-2\frac{bt}{\epsilon^2}} \|\hat{V}_0\|_\xi^2 d\xi + C \int_{1+|\xi|\ge \frac{r_0}{\epsilon}} e^{-2\frac{bt}{\epsilon^2}} \|\hat{V}_0\|_\xi^2 d\xi$$

$$\le C\epsilon^2 e^{-2\frac{bt}{\epsilon^2}} \left(\int_{1+|\xi|\le \frac{r_0}{\epsilon}} (1+|\xi|^2) \|\hat{U}_0\|^2 d\xi + \int_{1+|\xi|\ge \frac{r_0}{\epsilon}} |\xi|^2 \|\hat{U}_0\|^2 d\xi \right)$$

$$\le C\epsilon^2 e^{-2\frac{bt}{\epsilon^2}} \|U_0\|_{H^1}^2. \tag{3.50}$$

Thus, by (3.45), (3.46), (3.48) and (3.50) we obtain (3.31) for $k = 0$. And this completes the proof of the lemma. □

Remark 3.4. From Lemma 3.3, we have

$$\left\| e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} P_2 U_0 - Y_1(t) P_2 U_0 \right\|_{L^2} \le C\epsilon (1+t)^{-\frac{3}{4}} \left(\|U_0\|_{H^1} + \|U_0\|_{L^1} \right). \tag{3.51}$$

The following lemma gives the second order fluid approximation of the semigroup $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}$.

Lemma 3.5. *For any $\epsilon \ll 1$, any integer $k, m \geq 0$ and $U_0 = (g_0, 0, 0) \in H^{k+2} \cap L^1$ satisfying $P_d g_0 = 0$, we have*

$$\left\| \frac{1}{\epsilon} e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} U_0 - Y_2(t) Z_0 \right\|_{H^k} \leq C \left(\epsilon (1+t)^{-\frac{3}{4}} + \frac{1}{\epsilon} e^{-\frac{bt}{\epsilon^2}} \right) (\|U_0\|_{H^{k+2}} + \|U_0\|_{L^1}) + C\epsilon^m (1+t)^{-m} \|\nabla_x^m U_0\|_{H^k}, \tag{3.52}$$

where $Y_2(t)$ is defined in (3.8), $Z_0 = (P_d(v \cdot \nabla_x L_1^{-1} g_0), (v L_1^{-1} g_0, \chi_0), 0)$, and $b > 0$ is a constant given by (2.131).

Proof. Again we only prove the case when $k = 0$ because the proof for $k > 0$ is similar. By (2.128) and by taking $\epsilon \leq r_0/2$ with $r_0 > 0$ given in Lemma 2.11, we have

$$\begin{aligned} \left\| \frac{1}{\epsilon} e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon} U_0 - Y_2(t) Z_0 \right\|_{L^2}^2 \leq 4 \int_{1+|\xi| \leq \frac{r_0}{\epsilon}} \left\| \frac{1}{\epsilon} S_1(t,\xi,\epsilon) \hat{U}_0 - \tilde{Y}_2(t,\xi) \hat{Z}_1 \right\|_\xi^2 d\xi \\ + 4 \int_{|\xi| \geq \frac{r_1}{\epsilon}} \left\| \frac{1}{\epsilon} S_2(t,\xi,\epsilon) \hat{U}_0 \right\|_\xi^2 d\xi + 4 \int_{\mathbb{R}^3} \left\| \frac{1}{\epsilon} S_3(t,\xi,\epsilon) \hat{U}_0 \right\|_\xi^2 d\xi \\ + 4 \int_{1+|\xi| \geq \frac{r_0}{\epsilon}} \left\| \tilde{Y}_2(t,\xi) \hat{Z}_1 \right\|_\xi^2 d\xi \\ =: I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{3.53}$$

where $\hat{Z}_1 = (iP_d(v \cdot \xi L_1^{-1} \hat{g}_0), \omega \times (v L_1^{-1} \hat{g}_0, \chi_0), 0)$.

Firstly, we decompose I_1 into

$$\begin{aligned} I_1 &= 4 \left(\int_{1+|\xi| \leq \frac{r_0}{\epsilon}, |\eta^2 - 4|\xi|^2| \geq r_0} + \int_{|\eta^2 - 4|\xi|^2| \leq r_0} \right) \left\| \frac{1}{\epsilon} S_1(t,\xi,\epsilon) \hat{V}_0 - \tilde{Y}_2(t,\xi) \hat{Z}_1 \right\|_\xi^2 d\xi \\ &=: I_{11} + I_{12}. \end{aligned} \tag{3.54}$$

By Lemma 2.11 and Theorem 2.15, it holds for any $U_0 = (g_0, 0, 0)$ with $P_d g_0 = 0$ that

$$S_1(t,\xi,\epsilon) \hat{U}_0 = \sum_{j=0}^{4} e^{\frac{t}{\epsilon^2}\lambda_j(|\xi|,\epsilon)} \epsilon \big(\hat{g}_0, \tilde{u}_j(\xi,\epsilon) \big) \mathcal{U}_j(\xi,\epsilon), \tag{3.55}$$

where

$$\begin{cases}\tilde{u}_0=\epsilon^{-1}P_r\overline{u_0}=i\sqrt{1+s^2}\left[1+O(\epsilon^2(1+s^2))\right]R(\overline{\lambda_0},-\epsilon\xi)(v\cdot\omega)\chi_0,\\ \tilde{u}_k=\epsilon^{-1}P_r\overline{u_k}=\dfrac{\lambda_k\Theta_k}{\sqrt{\lambda_1\lambda_3-\lambda_k^2}}R(\overline{\lambda_k},-\epsilon\xi)(v\cdot e_k)\chi_0,\quad k=1,2,3,4\end{cases}$$

with $R(\lambda,\epsilon\xi)=(L_1-\lambda-i\epsilon P_r(v\cdot\xi))^{-1}$.

Thus, we can obtain by (3.55) and (3.34) that for $\epsilon(1+|\xi|)\le r_0$ and $|\eta^2-4|\xi|^2|\ge r_0$,

$$\begin{aligned}\frac{1}{\epsilon}S_1(t,\xi,\epsilon)\hat{U}_0&=e^{\frac{t}{\epsilon^2}\lambda_0(|\xi|,\epsilon)}\left[i\sqrt{1+|\xi|^2}(v\cdot\omega L_1^{-1}\hat{g}_0,\chi_0)\mathcal{X}_0(\xi)+R_0(\xi,\epsilon)\right]\\&\quad+\sum_{k=1}^4 e^{\frac{t}{\epsilon^2}\lambda_k(|\xi|,\epsilon)}\left[\frac{b_k}{\sqrt{|\xi|^2-b_k^2}}(v\cdot e_kL_1^{-1}\hat{g}_0,\chi_0)\mathcal{X}_k(\xi)+R_k(\xi,\epsilon)\right]\\&=e^{b_0(|\xi|)t+O(\epsilon^2(1+|\xi|^2)^2)t}\left[\left(\hat{Z}_1,\overline{\mathcal{X}_0(\xi)}\right)_\xi\mathcal{X}_0(\xi)+R_0(\xi,\epsilon)\right]\\&\quad+\sum_{k=1}^4 e^{b_k(|\xi|)t+O(\epsilon^2b_k(|\xi|))t}\left[\left(\hat{Z}_1,\overline{\mathcal{X}_k(\xi)}\right)\mathcal{X}_k(\xi)+R_k(\xi,\epsilon)\right],\end{aligned}\tag{3.56}$$

where

$$\begin{cases}\|R_0(\xi,\epsilon)\|_\xi=O(\epsilon(1+|\xi|^2))\|\hat{U}_0\|,\\ \|R_k(\xi,\epsilon)\|_\xi=O(\epsilon\sqrt{1+|\xi|^2})\|\hat{U}_0\|,\quad k=1,2,3,4\end{cases}$$

Thus, it follows form (3.37) and (3.56) that

$$\begin{aligned}I_{11}&\le C\int\limits_{1+|\xi|\le\frac{r_0}{\epsilon}}e^{b_0t}\left[r_0^2\epsilon^2(1+|\xi|^2)^4t^2+\epsilon^2(1+|\xi|^2)^2\right]\|\hat{U}_0\|^2d\xi\\&\quad+C\sum_{j=1}^4\int\limits_{1+|\xi|\le\frac{r_0}{\epsilon}}e^{\mathrm{Re}b_jt}\left(\epsilon^4|b_j|^2t^2+\epsilon^2\right)(1+|\xi|^2)\|\hat{U}_0\|^2d\xi\\&\le C\int\limits_{|\xi|\le r_0}\epsilon^2e^{-c_1|\xi|^2t}\|\hat{U}_0\|^2d\xi+C\int\limits_{1+|\xi|\le\frac{r_0}{\epsilon}}\epsilon^2e^{-c_2t}(1+|\xi|^2)^2\|\hat{U}_0\|^2d\xi\\&\le C\epsilon^2\left[(1+t)^{-3/2}\|U_0\|_{L^1}^2+e^{-c_2t}\|U_0\|_{H^2}^2\right].\end{aligned}\tag{3.57}$$

For $|\eta^2-4|\xi|^2|<r_0$, we rewrite $\frac{1}{\epsilon}S_1(t,\xi,\epsilon)$ as

$$\begin{aligned}\frac{1}{\epsilon}S_1(t,\xi,\epsilon)\hat{U}_0&=e^{z_0t}\tilde{V}_0+e^{z_1t}\Theta_1^2(\tilde{V}_1+\tilde{V}_2)+z_3e^{z_3t}(\tilde{V}_{13}+\tilde{V}_{24})\\&\quad-z_3e^{z_3t}\int_0^t e^{\tau(z_1-z_3)}d\tau\Theta_3^2(\tilde{V}_1+\tilde{V}_2)+O(1)\epsilon^4z_3e^{z_1t}(\tilde{V}_1+\tilde{V}_2),\end{aligned}\tag{3.58}$$

where

$$\begin{cases} \tilde{V}_0 = (\hat{g}_0, \tilde{u}_0)\mathcal{U}_0, \quad \tilde{V}_k = (\hat{g}_0, \tilde{u}_k)\tilde{\mathcal{U}}_k, \\ \tilde{V}_{jk} = (\hat{g}_0, \tilde{u}_{jk})\tilde{\mathcal{U}}_j + (\hat{g}_0, \tilde{u}_k)\tilde{\mathcal{U}}_{jk}, \quad j,k = 1,2,3,4, \\ \tilde{u}_{jk} = \epsilon^2 R(\overline{\lambda_j}, -\epsilon\xi) R(\overline{\lambda_k}, -\epsilon\xi)(v \cdot e_j)\chi_0 \end{cases}$$

with $\tilde{\mathcal{U}}_k$, $\tilde{\mathcal{U}}_{jk}$ defined by (3.39). Similarly, we rewrite $\tilde{Y}_2(t,\xi)$ as

$$\tilde{Y}_2(t,\xi)\hat{Z}_1 = e^{b_0 t}\tilde{W}_0 + e^{b_1 t}(\tilde{W}_1 + \tilde{W}_2) - b_3 e^{b_3 t}\int_0^t e^{\tau(b_1 - b_3)}d\tau(\tilde{W}_1 + \tilde{W}_2) \\ + b_3 e^{b_3 t}(\tilde{W}_{13} + \tilde{W}_{24}), \tag{3.59}$$

where

$$\begin{cases} \tilde{W}_0 = (\hat{Z}_1, \overline{\mathcal{X}_0})_\xi \mathcal{X}_0, \quad \tilde{W}_k = (\hat{Z}_1, \overline{X_k})X_k, \\ \tilde{W}_{jk} = (\hat{Z}_1, X_{jk})X_j + (\hat{Z}_1, X_k)X_{jk}, \quad j,k = 1,2,3,4 \end{cases}$$

with X_k, X_{jk} defined by (3.42).

Thus, it follows from (3.43), (3.58) and (3.59) that

$$I_{12} \le C \int_{|\eta^2 - 4|\xi|^2| \le r_0} e^{-\frac{\eta}{2}t}\epsilon^2 \|\hat{U}_0\|^2 d\xi \le C\epsilon^2 e^{-\frac{\eta}{2}t}\|U_0\|_{L^2}^2. \tag{3.60}$$

By combining (3.56), (3.57) and (3.60), we obtain

$$I_1 \le C\epsilon^2 (1+t)^{-\frac{3}{2}}(\|U_0\|_{H^2}^2 + \|U_0\|_{L^1}^2). \tag{3.61}$$

By (2.130) and Lemma 2.14, it holds for $U_0 = (g_0, 0, 0)$ with $P_d g_0 = 0$ that

$$S_2(t,\xi,\epsilon)\hat{U}_0 = \sum_{k=1}^4 e^{\frac{t}{\epsilon^2}\beta_k(|\xi|,\epsilon)}\left(\hat{g}_0, \overline{w_k(\xi,\epsilon)}\right)\mathcal{V}_k(\xi,\epsilon), \quad \epsilon|\xi| \ge r_1,$$

which gives

$$I_2 \le C \int_{|\xi| \ge \frac{r_1}{\epsilon}} \frac{1}{\epsilon|\xi|} e^{-\frac{ct}{\epsilon|\xi|}} \|\hat{U}_0\|^2 d\xi \le C\frac{1}{r_1} \sup_{|\xi| \ge \frac{r_1}{\epsilon}} \frac{1}{|\xi|^{2m}} e^{-\frac{ct}{\epsilon|\xi|}} \int_{|\xi| \ge \frac{r_1}{\epsilon}} |\xi|^{2m} \|\hat{U}_0\|^2 d\xi \\ \le C\epsilon^{2m}(1+t)^{-2m}\|\nabla_x^m U_0\|_{L^2}^2. \tag{3.62}$$

By (2.131) and $P_d g_0 = 0$, we have

$$I_3 \le C\int_{\mathbb{R}^3} \frac{1}{\epsilon^2} e^{-2\frac{bt}{\epsilon^2}} \|\hat{U}_0\|^2 d\xi \le C\frac{1}{\epsilon^2} e^{-2\frac{bt}{\epsilon^2}} \|U_0\|_{L^2}^2. \tag{3.63}$$

For I_4, it holds that

$$I_4 \le C \int\limits_{1+|\xi|\ge \frac{r_0}{\epsilon}} e^{-\eta t}\|\hat{Z}_1\|^2 d\xi \le C\frac{\epsilon^2}{r_0^2}e^{-\eta t}\int\limits_{1+|\xi|\ge \frac{r_0}{\epsilon}} (1+|\xi|)^4\|\hat{g}_0\|^2 d\xi$$
$$\le C\epsilon^2 e^{-\eta t}\|U_0\|_{H^2}^2. \tag{3.64}$$

By combining (3.61), (3.62), (3.63) and (3.64), we obtain (3.52) for $k = 0$. And this completes the proof of the lemma. □

In the following lemma, we will present the time decay rates of the semigroup $e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}$.

Lemma 3.6. *For any $\epsilon \ll 1$, $\alpha \in \mathbb{N}^3$, any integer $k \ge 0$ and $U_0 = (g_0, E_0, B_0)$, we have*

$$\|\partial_x^\alpha P_2 e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}U_0\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{m}{2}}\left(\|\partial_x^{\alpha'}U_0\|_{L^1} + \|\partial_x^\alpha U_0\|_{L^2}\right)$$
$$+ C\epsilon^k(1+t)^{-k}\|\nabla_x^{|\alpha|+k}U_0\|_{L^2}, \tag{3.65}$$
$$\|\partial_x^\alpha P_3 e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}U_0\|_{L^2} \le C\left(\epsilon(1+t)^{-\frac{5}{4}-\frac{m}{2}} + e^{-\frac{bt}{\epsilon^2}}\right)\left(\|\partial_x^{\alpha'}U_0\|_{L^1} + \|\partial_x^\alpha U_0\|_{H^1}\right)$$
$$+ C\epsilon^{k+1}(1+t)^{-k}\|\nabla_x^{|\alpha|+k}U_0\|_{L^2}, \tag{3.66}$$

where P_2, P_3 are defined by (1.21), $\alpha' \le \alpha$, $m = |\alpha - \alpha'|$, and $b > 0$ is a constant given by (2.131).

Moreover, if $U_0 = (g_0, 0, 0)$ satisfies that $P_d g_0 = 0$, then

$$\|\partial_x^\alpha P_2 e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}U_0\|_{L^2} \le C\left(\epsilon(1+t)^{-\frac{5}{4}-\frac{m}{2}} + e^{-\frac{bt}{\epsilon^2}}\right)\left(\|\partial_x^{\alpha'}U_0\|_{L^1} + \|\partial_x^\alpha U_0\|_{H^1}\right)$$
$$+ C\epsilon^{k+1}(1+t)^{-k}\|\nabla_x^{|\alpha|+k}U_0\|_{L^2}, \tag{3.67}$$
$$\|\partial_x^\alpha P_3 e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}U_0\|_{L^2} \le C\left(\epsilon^2(1+t)^{-\frac{5}{4}-\frac{m}{2}} + e^{-\frac{bt}{\epsilon^2}}\right)\left(\|\partial_x^{\alpha'}U_0\|_{L^1} + \|\partial_x^\alpha U_0\|_{H^2}\right)$$
$$+ C\epsilon^{k+2}(1+t)^{-k}\|\nabla_x^{|\alpha|+k}U_0\|_{L^2}. \tag{3.68}$$

Proof. By Theorem 2.19, we have for $j = 2, 3$ that

$$\left\|\partial_x^\alpha P_j e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}U_0\right\|_{L^2}^2 = \int\limits_{\mathbb{R}^3}\left\|\xi^\alpha P_j e^{\frac{t}{\epsilon^2}\tilde{\mathbb{A}}_\epsilon(\xi)}\hat{V}_0\right\|_\xi^2 d\xi$$
$$\le \int\limits_{1+|\xi|\le \frac{r_0}{\epsilon}}\left\|\xi^\alpha P_j S_1(t,\xi,\epsilon)\hat{V}_0\right\|_\xi^2 d\xi + \int\limits_{|\xi|\ge \frac{r_1}{\epsilon}}\left\|\xi^\alpha S_2(t,\xi,\epsilon)\hat{V}_0\right\|_\xi^2 d\xi$$
$$+ \int\limits_{\mathbb{R}^3}\left\|\xi^\alpha S_3(t,\xi,\epsilon)\hat{V}_0\right\|_\xi^2 d\xi, \tag{3.69}$$

where $\hat{V}_0 = (\hat{g}_0, \omega \times \hat{E}_0, \omega \times \hat{B}_0)$. By noting $\|\hat{V}_0\|_\xi^2 = \|\hat{U}_0\|^2$, we can estimate the third term on the right hand side of (3.69) as follows:

$$\int_{\mathbb{R}^3} (\xi^\alpha)^2 \|S_3(t,\xi,\epsilon)\hat{V}_0\|_\xi^2 d\xi \le Ce^{-2\frac{bt}{\epsilon^2}} \int_{\mathbb{R}^3} (\xi^\alpha)^2 \|\hat{V}_0\|_\xi^2 d\xi \le Ce^{-2\frac{bt}{\epsilon^2}} \|\partial_x^\alpha U_0\|_{L^2}^2. \tag{3.70}$$

By (3.35) and (3.40), we have

$$\begin{aligned}
&\int_{1+|\xi|\le \frac{r_0}{\epsilon}} \|\xi^\alpha P_2 S_1(t,\xi,\epsilon)\hat{V}_0\|_\xi^2 d\xi \\
&\le C \int_{|\xi|\le r_0} e^{-c_1|\xi|^2 t} (\xi^\alpha)^2 \|\hat{V}_0\|^2 d\xi + C \int_{1+|\xi|\le \frac{r_0}{\epsilon}} e^{-c_2 t} (\xi^\alpha)^2 \|\hat{V}_0\|^2 d\xi \\
&\quad + C\epsilon^2 \int_{1+|\xi|\le \frac{r_0}{\epsilon}} e^{-\frac{\eta}{2}(1+|\xi|^2)t} (\xi^\alpha)^2 \|\hat{V}_0\|_\xi^2 d\xi \\
&\le C \sup_{|\xi|\le r_0} \|(\xi)^{\alpha'} \hat{U}_0\|^2 \int_{|\xi|\le r_0} e^{-c_1|\xi|^2 t} |\xi|^{2|\alpha-\alpha'|} d\xi \\
&\quad + Ce^{-c_2 t} \int_{|\xi|\le \frac{r_0}{\epsilon}} (\xi^\alpha)^2 \|\hat{U}_0\|^2 d\xi \\
&\le C(1+t)^{-\frac{3}{2}-m} \left(\|\partial_x^{\alpha'} U_0\|_{L^1}^2 + \|\partial_x^\alpha U_0\|_{L^2}^2 \right),
\end{aligned} \tag{3.71}$$

where $\alpha' \le \alpha$, $m = |\alpha - \alpha'|$, and $c_1, c_2 > 0$ are some generic constants. Combining (3.69)–(3.71) and (3.46) gives (3.65).

By (3.56) and (3.58), it holds that for $V_0 = (g_0, 0, 0)$ with $P_d g_0 = 0$,

$$\begin{aligned}
&\int_{1+|\xi|\le \frac{r_0}{\epsilon}} \|\xi^\alpha P_2 S_1(t,\xi,\epsilon)\hat{V}_0\|^2 d\xi \\
&\le C\epsilon^2 \int_{|\xi|\le r_0} e^{-c_1|\xi|^2 t} (\xi^\alpha)^2 |\xi|^2 \|\hat{V}_0\|^2 d\xi + C\epsilon^2 \int_{1+|\xi|\le \frac{r_0}{\epsilon}} e^{-c_2 t} (\xi^\alpha)^2 \|\hat{V}_0\|^2 d\xi \\
&\quad + C\epsilon^2 \int_{1+|\xi|\le \frac{r_0}{\epsilon}} e^{-\frac{\eta}{2}(1+|\xi|^2)t} (\xi^\alpha)^2 (1+|\xi|^2) \|\hat{V}_0\|^2 d\xi \\
&\le C\epsilon^2 (1+t)^{-\frac{5}{2}-m} \left(\|\partial_x^{\alpha'} U_0\|_{L^1}^2 + \|\partial_x^\alpha U_0\|_{H^1}^2 \right).
\end{aligned} \tag{3.72}$$

Combining (3.69), (3.72) and (3.62) yields (3.67). (3.66) and (3.68) can be proved similarly. And this completes the proof of the lemma. □

Lemma 3.7. *For any $1 \le q \le 2$, $\alpha \in \mathbb{N}^3$, and any $U_0 = (\rho_0\chi_0, E_0, B_0)$ with $\rho_0 = \nabla_x \cdot E_0$, we have*

$$\|\partial_x^\alpha Y_2(t)U_0\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{m}{2}} \left(\|\partial_x^{\alpha'} U_0\|_{L^q} + \|\partial_x^\alpha U_0\|_{L^2}\right), \tag{3.73}$$

where $\alpha' \le \alpha$ and $m = |\alpha - \alpha'|$. Moreover, if $U_0 = (\nabla_x \cdot E_0\chi_0, E_0, 0)$, then we have

$$\|\partial_x^\alpha Y_2(t)U_0\|_{L^2} \le C\left((1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{m+1}{2}} + t^{-\frac{1}{2}}e^{-\frac{\eta}{2}t}\right)\left(\|\partial_x^{\alpha'} E_0\|_{L^q} + \|\partial_x^\alpha E_0\|_{L^2}\right). \tag{3.74}$$

Proof. By (3.8), we have

$$\|\partial_x^\alpha Y_2(t)U_0\|_{L^2}^2 = \int_{\mathbb{R}^3} \left\|\xi^\alpha \tilde{Y}_2(t,\xi)\hat{V}_0\right\|_\xi^2 d\xi, \tag{3.75}$$

where $\hat{V}_0 = (\hat{\rho}_0\chi_0, \omega \times \hat{E}_0, \omega \times \hat{B}_0)$. Since

$$\|\mathcal{X}_0(\xi)\|_\xi^2 = 1, \quad \|\mathcal{X}_k(\xi)\|_\xi^2 \le \frac{|b_k|^2 + |\xi|^2}{|b_k^2 - |\xi|^2|}, \ \ k = 1,2,3,4,$$

it follows that for $|\eta^2 - 4|\xi|^2| \ge r_0$ with $r_0 \ll 1$,

$$\begin{aligned}
&\int_{|\eta^2-4|\xi|^2|\ge r_0} \|\xi^\alpha \tilde{Y}_2(t,\xi)\hat{V}_0\|_\xi^2 d\xi \\
&\le C\sum_{j=0}^4 \int_{|\eta^2-4|\xi|^2|\ge r_0} (\xi^\alpha)^2 e^{2\mathrm{Re}b_j(|\xi|)t}\|\hat{V}_0\|_\xi^2 \|\mathcal{X}_j(\xi)\|_\xi^4 d\xi \\
&\le C\int_{|\xi|\le r_0} e^{-c_1|\xi|^2 t}(\xi^\alpha)^2\|\hat{U}_0\|^2 d\xi + C\int_{\mathbb{R}^3} e^{-c_2 t}(\xi^\alpha)^2\|\hat{U}_0\|^2 d\xi \\
&\le C\left(\int_{|\xi|\le r_0} |\xi|^{2pm}e^{-c_1 p|\xi|^2 t}d\xi\right)^{1/p}\left(\int_{|\xi|\le r_0} \|\xi^{\alpha'}\hat{U}_0\|^{2p'}d\xi\right)^{1/p'} \\
&\quad + C\int_{\mathbb{R}^3} e^{-c_2 t}(\xi^\alpha)^2\|\hat{U}_0\|^2 d\xi \\
&\le C(1+t)^{-\frac{3}{2}(\frac{2}{q}-1)-m}\left(\|\partial_x^{\alpha'}U_0\|_{L^q}^2 + \|\partial_x^\alpha U_0\|_{L^2}^2\right),
\end{aligned} \tag{3.76}$$

where $1/q + 1/p' = 1$, $1/(2p') + 1/q = 1$, $\alpha' \le \alpha$, $m = |\alpha - \alpha'|$ and $c_1, c_2 > 0$ are constants.

For $|\eta^2-4|\xi|^2|\le r_0$, it follows from (3.41) that

$$\int_{|\eta^2-4|\xi|^2|\le r_0}\|\xi^\alpha\tilde{Y}_2(t,\xi)\hat{V}_0\|_\xi^2d\xi\le C\int_{|\eta^2-4|\xi|^2|\le r_0}(\xi^\alpha)^2e^{-\frac{\eta}{2}t}\|\hat{V}_0\|_\xi^2d\xi$$
$$\le Ce^{-\frac{\eta}{2}t}\|\partial_x^\alpha U_0\|_{L^2}^2. \tag{3.77}$$

Combining (3.75) and (3.76)–(3.77) gives (3.73).

For $\hat{V}_0=(i(\hat{E}_0\cdot\xi)\chi_0,\omega\times\hat{E}_0,0)$, we obtain

$$\begin{aligned}\tilde{Y}_2(t,\xi)\hat{V}_0&=e^{-\eta(1+|\xi|^2)t}(i(\hat{E}_0\cdot\xi)\chi_0,0,0)\\&\quad+e^{b_1t}\frac{b_1}{b_3-b_1}\sum_{k=1}^{2}(\omega\times\hat{E}_0,\omega\times e_k)(0,\omega\times e_k,\frac{i|\xi|e_k}{b_1})\\&\quad+e^{b_3t}\frac{b_3}{b_1-b_3}\sum_{k=3}^{4}(\omega\times\hat{E}_0,\omega\times e_k)(0,\omega\times e_k,\frac{i|\xi|e_k}{b_3}).\end{aligned}$$

Thus, it follows that for $|\eta^2-4|\xi|^2|\ge r_0$ with $r_0\ll 1$,

$$\begin{aligned}&\int_{|\eta^2-4|\xi|^2|\ge r_0}\|\xi^\alpha\tilde{Y}_2(t,\xi)\hat{V}_0\|_\xi^2d\xi\\&\le C\int_{\mathbb{R}^3}e^{-2\eta(1+|\xi|^2)t}(\xi^\alpha)^2(1+|\xi|^2)|\hat{E}_0|^2d\xi\\&\quad+C\int_{|\xi|\le r_0}e^{-c_1|\xi|^2t}(\xi^\alpha)^2|\xi|^2|\hat{E}_0|^2d\xi+C\int_{\mathbb{R}^3}e^{-c_2t}(\xi^\alpha)^2|\hat{E}_0|^2d\xi\\&\le C\left((1+t)^{-\frac{3}{2}(\frac{2}{q}-1)-m-1}+t^{-1}e^{-\eta t}\right)\left(\|\partial_x^{\alpha'}E_0\|_{L^q}^2+\|\partial_x^\alpha E_0\|_{L^2}^2\right).\end{aligned} \tag{3.78}$$

For $|\eta^2-4|\xi|^2|\le r_0$, it follows from (3.41) that

$$\int_{|\eta^2-4|\xi|^2|\le r_0}\|\xi^\alpha\tilde{Y}_2(t,\xi)\hat{V}_0\|_\xi^2d\xi\le Ce^{-\frac{\eta}{2}t}\|\partial_x^\alpha E_0\|_{L^2}^2. \tag{3.79}$$

Combining (3.75) and (3.78)–(3.79) yields (3.74). And this completes the proof of the lemma. □

3.3. Fluid approximation of $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}$

The following preliminary lemma is for the study of the fluid dynamic approximation of the semigroup $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}$.

Lemma 3.8. *For any function $\phi(r)$ satisfying $|\phi^{(k)}(r)| \le C(1+|r|)^{-2-k-\delta}$ for any $\delta > 0$ and $k = 0, 1$, we have*

$$\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{i\vartheta|\xi|}\alpha(\omega)\phi(|\xi|)d\xi\right| \le C|\vartheta|^{-1},$$

where $\alpha(\omega)$ is a smooth function for $\omega = \xi/|\xi| \in \mathbb{S}^2$.

Proof. Firstly, note that

$$\int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{i\vartheta|\xi|}\alpha(\omega)\phi(|\xi|)d\xi = \int_0^\infty e^{i\vartheta r} g(x,r)\phi(r)r^2 dr,$$

where

$$g(x,r) = \int_{\mathbb{S}^2} e^{irx\cdot\omega}\alpha(\omega)d\omega.$$

For any function $\alpha(\omega) \in C^\infty(\mathbb{S}^2)$, by change of variable $\omega \to O_x\omega$, where $O_x = (a_{ij})_{3\times 3}$ is an orthogonal matrix satisfying $O_x^T x = (0, 0, |x|)$, we obtain

$$\begin{aligned} g(x,r) &= \int_0^{2\pi} d\varphi \int_0^\pi e^{i|x|r\cos\theta}\sin\theta\alpha(O_x\omega)d\theta \\ &= -\frac{1}{i|x|r}\int_0^{2\pi} d\varphi \int_0^\pi \alpha(O_x\omega)de^{i|x|r\cos\theta} \\ &= -\frac{1}{i|x|r}\int_0^{2\pi} \alpha(O_x\omega)e^{ir|x|\cos\theta}\Big|_0^\pi d\varphi \\ &\quad + \frac{1}{i|x|r}\int_0^{2\pi} d\varphi \int_0^\pi e^{ir|x|\cos\theta}\nabla\alpha(O_x\omega)\cdot O_x\partial_\theta\omega d\theta, \end{aligned}$$

which gives

$$|g(x,r)| \le C(1+|x|r)^{-1}.$$

Similarly, we can prove

$$|\partial_r g(x,r)| = \left|\int_0^{2\pi} d\varphi \int_0^\pi i|x|\cos\theta e^{i|x|r\cos\theta}\sin\theta\alpha(O_x\omega)d\theta\right|$$

$$\leq C|x|(1+|x|r)^{-1}.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{i\vartheta|\xi|}\alpha(\omega)\phi(|\xi|)d\xi &= \frac{1}{i\vartheta}\int_0^\infty g(x,r)\phi(r)r^2de^{i\vartheta r}\\ &= \frac{1}{i\vartheta}g(x,r)\phi(r)r^2e^{i\vartheta r}\Big|_0^\infty - \frac{1}{i\vartheta}\int_0^\infty e^{i\vartheta r}\partial_r g(x,r)\phi(r)r^2dr\\ &\quad - \frac{1}{i\vartheta}\int_0^\infty e^{i\vartheta r}g(x,r)(\phi'(r)r^2+2\phi(r)r)dr,\end{aligned}$$

which yields

$$\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{i\vartheta|\xi|}\alpha(\omega)\phi(|\xi|)d\xi\right| \leq C|\vartheta|^{-1}.$$

And this completes the proof of the lemma. □

Lemma 3.9. (1) *For any $\epsilon \ll 1$, any integer $k \geq 0$ and $f_0 \in L^2$, it holds that*

$$\left\|P_{||}e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0\right\|_{W^{k,\infty}} \leq C\left(\epsilon(1+t)^{-\frac{5}{2}} + \left(1+\frac{t}{\epsilon}\right)^{-1}\right)(\|f_0\|_{H^{k+3}} + \|f_0\|_{W^{k+3,1}}), \tag{3.80}$$

$$\left\|P_{\perp}e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0 - Y_1(t)P_0f_0\right\|_{H^k} \leq C\left(\epsilon(1+t)^{-\frac{5}{4}} + e^{-\frac{bt}{\epsilon^2}}\right)(\|f_0\|_{H^{k+1}} + \|f_0\|_{L^1}), \tag{3.81}$$

where $Y_1(t)$ is defined in (3.9), and $b > 0$ is a constant given by (2.136). Moreover, if f_0 satisfies (1.27), i.e., $P_{||}f_0 = P_1f_0 = 0$, then

$$\left\|P_{||}e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0\right\|_{W^{k,\infty}} \leq C\epsilon(1+t)^{-\frac{5}{2}}\left(\|f_0\|_{H^{k+3}} + \|f_0\|_{L^1}\right), \tag{3.82}$$

$$\left\|P_{\perp}e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0 - Y_1(t)P_0f_0\right\|_{H^k} \leq C\epsilon(1+t)^{-\frac{5}{4}}\left(\|f_0\|_{H^{k+1}} + \|f_0\|_{L^1}\right). \tag{3.83}$$

(2) *For any $\epsilon \ll 1$, any integer $k \geq 0$ and $f_0 \in L^2$ satisfying $P_0f_0 = 0$, it holds that*

$$\left\|\frac{1}{\epsilon}P_{||}e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0\right\|_{W^{k,\infty}} \leq C\left(\epsilon(1+t)^{-\frac{7}{2}} + \left(1+\frac{t}{\epsilon}\right)^{-1} + \frac{1}{\epsilon}e^{-\frac{bt}{\epsilon^2}}\right)\left(\|f_0\|_{H^{k+4}} + \|f_0\|_{W^{k+4,1}}\right), \tag{3.84}$$

$$\left\|\frac{1}{\epsilon}P_{\perp}e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0 - Y_1(t)Z_2\right\|_{H^k} \leq C\left(\epsilon(1+t)^{-\frac{7}{4}} + \frac{1}{\epsilon}e^{-\frac{bt}{\epsilon^2}}\right)\left(\|f_0\|_{H^{k+2}} + \|f_0\|_{L^1}\right), \tag{3.85}$$

where $Z_2 = P_0(v\cdot\nabla_x L^{-1}f_0)$.

Proof. Again we only prove the case when $k = 0$ because the proof for $k > 0$ is similar. By (2.135) and by taking $\epsilon \le r_0$ with $r_0 > 0$ given in Theorem 2.18, we have

$$
\begin{aligned}
e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon} f_0 - Y_1(t)P_0 f_0 &= \int_{\mathbb{R}^3} e^{ix\cdot\xi}\left(e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon(\xi)}\hat{f}_0 - Y_1(t,\xi)P_0\hat{f}_0\right)d\xi \\
&= \int_{|\xi|\le\frac{r_0}{\epsilon}} e^{ix\cdot\xi}\left(S_4(t,\xi,\epsilon)\hat{f}_0 - Y_1(t,\xi)P_0\hat{f}_0\right)d\xi \\
&\quad + \int_{\mathbb{R}^3} e^{ix\cdot\xi} S_5(t,\xi,\epsilon)\hat{f}_0 d\xi + \int_{|\xi|\ge\frac{r_0}{\epsilon}} e^{ix\cdot\xi} Y_1(t,\xi)P_0\hat{f}_0 d\xi \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.86}
$$

We estimate I_j, $j = 1,2,3$ one by one as follows. Since

$$
S_4(t,\xi,\epsilon)\hat{f}_0 = \sum_{j=0}^{4} e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t + O(\epsilon|\xi|^3)t}\left[\left(P_0\hat{f}_0, h_j\right)h_j + O(\epsilon|\xi|)\right],
$$

it follows that

$$
\begin{aligned}
I_1 = &\sum_{j=-1}^{3}\int_{|\xi|\le\frac{r_0}{\epsilon}} e^{ix\cdot\xi}\Big\{e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t + O(\epsilon|\xi|^3)t}\left[\left(P_0\hat{f}_0, h_j\right)h_j + O(\epsilon|\xi|)\right] \\
&\quad - e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t}\left(P_0\hat{f}_0, h_j\right)h_j\Big\}d\xi \\
&+ \sum_{j=\pm1}\int_{|\xi|\le\frac{r_0}{\epsilon}} e^{ix\cdot\xi} e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t}\left(P_0\hat{f}_0, h_j\right)h_j d\xi \\
=:& I_{11} + I_{12}.
\end{aligned} \tag{3.87}
$$

For I_{11}, it holds that

$$
\begin{aligned}
\|P_{||}I_{11}\|_{L^\infty} &\le C\epsilon\int_{|\xi|\le\frac{r_0}{\epsilon}} e^{-c|\xi|^2 t}\left(|\xi|^3 t\|P_0\hat{f}_0\| + |\xi|\|\hat{f}_0\|\right)d\xi \\
&\le C\epsilon\sup_{|\xi|\le1}\|\hat{f}_0\|\int_{|\xi|\le1} e^{-c|\xi|^2 t}(|\xi|^2 t + 1)|\xi| d\xi \\
&\quad + C\epsilon\Big(\int_{|\xi|>1} e^{-c|\xi|^2 t}\frac{(1+|\xi|^2 t)^2}{|\xi|^4}d\xi\Big)^{1/2}\Big(\int_{|\xi|>1}|\xi|^6\|\hat{f}_0\|^2 d\xi\Big)^{1/2} \\
&\le C\epsilon(1+t)^{-\frac{5}{2}}\left(\|f_0\|_{H^3} + \|f_0\|_{L^1}\right),
\end{aligned} \tag{3.88}
$$

$$
\begin{aligned}
\|P_\perp I_{11}\|_{L^2} &\le C\epsilon\Big(\int_{|\xi|\le\frac{r_0}{\epsilon}} e^{-2c|\xi|^2 t}\left(|\xi|^6 t^2\|P_0\hat f_0\|^2+|\xi|^2\|\hat f_0\|^2\right)d\xi\Big)^{1/2}\\
&\le C\epsilon\sup_{|\xi|\le 1}\|\hat f_0\|\Big(\int_{|\xi|\le 1} e^{-c|\xi|^2 t}|\xi|^2 d\xi\Big)^{1/2}+C\epsilon\Big(\int_{|\xi|>1} e^{-c|\xi|^2 t}|\xi|^2\|\hat f_0\|^2 d\xi\Big)^{1/2}\\
&\le C\epsilon(1+t)^{-\frac{5}{4}}\left(\|f_0\|_{H^1}+\|f_0\|_{L^1}\right).
\end{aligned}\tag{3.89}
$$

To estimate I_{12}, we first note that

$$
P_{||}I_{12}=I_{12},\quad P_\perp I_{12}=0. \tag{3.90}
$$

It is straightforward to verify that

$$
\|P_{||}I_{12}\|_{L^\infty}\le C\int_{|\xi|\le\frac{r_0}{\epsilon}} e^{-c|\xi|^2 t}\|P_0\hat f_0\|d\xi\le C\|P_0 f_0\|_{H^2}. \tag{3.91}
$$

Set

$$
\begin{aligned}
I_{12}&=\sum_{j=\pm1}\left(\int_{\mathbb{R}^3}-\int_{|\xi|\ge\frac{r_0}{\epsilon}}\right)e^{ix\cdot\xi}e^{\frac{\mu_j|\xi|}{\epsilon}t-a_j|\xi|^2 t}\left(P_0\hat f_0,h_j\right)h_j d\xi\\
&=: I_{13}+I_{14}.
\end{aligned}\tag{3.92}
$$

For I_{14}, it holds that

$$
\begin{aligned}
\|P_{||}I_{14}\|_{L^\infty}&\le C\int_{|\xi|\ge\frac{r_0}{\epsilon}} e^{-c|\xi|^2 t}\|P_0\hat f_0\|d\xi\\
&\le C\left(\int_{|\xi|\ge\frac{r_0}{\epsilon}} e^{-\frac{2cr_0^2}{\epsilon^2}t}\frac{1}{(1+|\xi|^2)^2}d\xi\right)^{1/2}\left(\int_{|\xi|\le\frac{r_0}{\epsilon}}(1+|\xi|^2)^2\|\hat f_0\|^2 d\xi\right)^{1/2}\\
&\le Ce^{-\frac{cr_0^2}{\epsilon^2}t}\|f_0\|_{H^2}.
\end{aligned}\tag{3.93}
$$

Note that

$$
(I_{13},v\chi_0)_{||}=-\sum_{j=\pm1}\frac12\int_{\mathbb{R}^3}e^{ix\cdot\xi}e^{\frac{i\mu_j|\xi|}{\epsilon}t-a_j|\xi|^2 t}\left[j\left(\sqrt{\frac35}\hat n_0+\sqrt{\frac25}\hat q_0\right)-(\hat m_0\cdot\omega)\right]\omega d\xi,
$$

$$
(I_{13},\tilde h_1)=\sum_{j=\pm1}\frac12\int_{\mathbb{R}^3}e^{ix\cdot\xi}e^{\frac{i\mu_j|\xi|}{\epsilon}t-a_j|\xi|^2 t}\left[\left(\sqrt{\frac35}\hat n_0+\sqrt{\frac25}\hat q_0\right)-j(\hat m_0\cdot\omega)\right]d\xi,
$$

where

$$(n_0, m_0, q_0) = ((f_0, \chi_0), (f_0, v\chi_0), (f_0, \chi_4)).$$

Set

$$\begin{cases} G_{jk}(t,x) = \int\limits_{\mathbb{R}^3} e^{ix\cdot\xi} e^{\frac{i\mu_j|\xi|}{\epsilon}t}(1+|\xi|)^{-3}\alpha_k(\omega)d\xi, \quad k=0,1,2, \\ H_j(t,x) = \int\limits_{\mathbb{R}^3} e^{ix\cdot\xi} e^{-a_j|\xi|^2 t} d\xi = Ct^{-\frac{3}{2}} e^{-\frac{|x|^2}{4a_j t}}, \\ \alpha_0(\omega) = 1, \quad \alpha_1(\omega) = \omega, \quad \alpha_2(\omega) = \omega \otimes \omega. \end{cases}$$

Then by Lemma 3.8, we have

$$\begin{aligned} \|P_{||} I_{13}\|_{L^\infty} &\le \|(I_{13}, v\chi_0)_{||}\|_{L^\infty_x} + \|(I_{13}, \tilde{h}_1)\|_{L^\infty_x} \\ &\le C \sum_{j=\pm 1} \sum_{k=0}^{2} \|G_{jk}(t)\|_{L^\infty_x} \|H_j(t)\|_{L^1_x} \|(n_0 + \sqrt{\frac{2}{3}} q_0, m_0)\|_{W^{3,1}_x} \\ &\le C \left(\frac{t}{\epsilon}\right)^{-1} \|P_0 f_0\|_{W^{3,1}}. \end{aligned} \tag{3.94}$$

By combining (3.91)–(3.94), we obtain

$$\|P_{||} I_{12}\|_{L^\infty} \le C \left(1 + \frac{t}{\epsilon}\right)^{-1} \left(\|P_0 f_0\|_{H^2} + \|P_0 f_0\|_{W^{3,1}}\right). \tag{3.95}$$

Thus, it follows from (3.87)–(3.90) and (3.95) that

$$\|P_{||} I_1\|_{L^\infty} \le C\Big(\epsilon(1+t)^{-\frac{5}{2}} + \left(1 + \frac{t}{\epsilon}\right)^{-1}\Big) \left(\|f_0\|_{H^3} + \|f_0\|_{W^{3,1}}\right), \tag{3.96}$$

$$\|P_\perp I_1\|_{L^2} \le C\epsilon(1+t)^{-\frac{5}{4}} \left(\|f_0\|_{H^1} + \|f_0\|_{L^1}\right). \tag{3.97}$$

I_2 and I_3 can be estimated directly as follows.

$$\|I_2\|_{L^2}^2 \le \int\limits_{\mathbb{R}^3} \|S_5(t,\xi,\epsilon)\hat{f}_0\|^2 d\xi \le C \int\limits_{\mathbb{R}^3} e^{-\frac{2b}{\epsilon^2}t} \|\hat{f}_0\|^2 d\xi \le Ce^{-\frac{2b}{\epsilon^2}t} \|f_0\|_{L^2}^2, \tag{3.98}$$

$$\begin{aligned} \|I_3\|_{L^2}^2 &\le C \int\limits_{|\xi| \ge \frac{r_0}{\epsilon}} e^{-2c|\xi|^2 t} \|P_0 \hat{f}_0\|^2 d\xi \le C\epsilon^2 \int\limits_{|\xi| \ge \frac{r_0}{\epsilon}} e^{-\frac{2cr_0^2}{\epsilon^2}t} |\xi|^2 \|P_0 \hat{f}_0\|^2 d\xi \\ &\le C\epsilon^2 e^{-\frac{2cr_0^2}{\epsilon^2}t} \|P_0 f_0\|_{H^1}^2. \end{aligned} \tag{3.99}$$

By (3.96), (3.97), (3.98) and (3.99), and the Sobolev embedding theorem, we obtain (3.80) and (3.81).

We now turn to (3.83). If f_0 satisfies (1.27), we have

$$\left(P_0\hat{f}_0, h_j(\xi)\right) = 0, \quad j = -1, 1,$$

which implies that $I_{12} = 0$. By Lemma 3.2, we have

$$\begin{aligned}\|I_2\|_{L^2}^2 &\le C \int\limits_{|\xi|\le\frac{r_0}{\epsilon}} \epsilon^2|\xi|^2 e^{-\frac{2bt}{\epsilon^2}}\|\hat{f}_0\|^2 d\xi + C \int\limits_{|\xi|\ge\frac{r_0}{\epsilon}} e^{-\frac{2bt}{\epsilon^2}}\|\hat{f}_0\|^2 d\xi \\ &\le C\epsilon^2 e^{-\frac{2bt}{\epsilon^2}}\|f_0\|_{H^1}^2.\end{aligned}$$

Thus, (3.82) and (3.83) hold.

Finally, we prove (3.84) and (3.85). By (2.135), we have

$$\begin{aligned}\frac{1}{\epsilon}e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0 - Y_1(t)Z_2 &= \int\limits_{|\xi|\le\frac{r_0}{\epsilon}} e^{ix\cdot\xi}\left(\frac{1}{\epsilon}S_4(t,\xi,\epsilon)\hat{f}_0 - Y_1(t,\xi)(iv\cdot\xi L^{-1}\hat{f}_0)\right)d\xi \\ &\quad + \int\limits_{\mathbb{R}^3}\frac{1}{\epsilon}e^{ix\cdot\xi}S_5(t,\xi,\epsilon)\hat{f}_0 d\xi + \int\limits_{|\xi|\ge\frac{r_0}{\epsilon}} e^{ix\cdot\xi}Y_1(t,\xi)(iv\cdot\xi L^{-1}\hat{f}_0)d\xi \\ &=: I_4 + I_5 + I_6. \end{aligned} \tag{3.100}$$

For any $f_0 \in L^2$ satisfying $P_0 f_0 = 0$, we obtain

$$S_4(t,\xi,\epsilon)\hat{f}_0 = \epsilon\sum_{j=0}^{4} e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t + O(\epsilon^3|\xi|^3)t}\left[\left(i(v\cdot\xi)L^{-1}\hat{f}_0, h_j\right)h_j + O(\epsilon|\xi|^2)\right]. \tag{3.101}$$

Thus, it follows that

$$\begin{aligned} I_4 &= \sum_{j=-1}^{3}\int\limits_{|\xi|\le\frac{r_0}{\epsilon}} e^{ix\cdot\xi}\Big\{e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t + O(\epsilon|\xi|^3)t}\left[\left(i(v\cdot\xi)L^{-1}\hat{f}_0, h_j\right)h_j + O(\epsilon|\xi|^2)\right] \\ &\qquad - e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t}\left(i(v\cdot\xi)L^{-1}\hat{f}_0, h_j\right)h_j\Big\}d\xi \\ &\quad + \sum_{j=\pm1}\int\limits_{|\xi|\le\frac{r_0}{\epsilon}} e^{ix\cdot\xi}e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t}\left(i(v\cdot\xi)L^{-1}\hat{f}_0, h_j\right)h_j d\xi \\ &=: I_{41} + I_{42}. \end{aligned} \tag{3.102}$$

For I_{41}, it holds that

$$\begin{aligned}\|P_{||}I_{41}\|_{L^\infty} &\le C\epsilon \int\limits_{|\xi|\le\frac{r_0}{\epsilon}} e^{-c|\xi|^2 t}\left(|\xi|^4 t\|P_0\hat{f}_0\| + |\xi|^2\|\hat{f}_0\|\right)d\xi \\ &\le C\epsilon(1+t)^{-\frac{7}{2}}\left(\|f_0\|_{H^4} + \|f_0\|_{L^1}\right), \qquad (3.103)\\ \|P_\perp I_{41}\|_{L^2} &\le C\epsilon\Bigg(\int\limits_{|\xi|\le\frac{r_0}{\epsilon}} e^{-2c|\xi|^2 t}\left(|\xi|^8 t^2\|P_0\hat{f}_0\|^2 + |\xi|^4\|\hat{f}_0\|^2\right)d\xi\Bigg)^{1/2} \\ &\le C\epsilon(1+t)^{-\frac{7}{4}}\left(\|f_0\|_{H^2} + \|f_0\|_{L^1}\right). \qquad (3.104)\end{aligned}$$

Denote

$$\begin{aligned}I_{42} &= \sum_{j=\pm1}\Bigg(\int\limits_{\mathbb{R}^3} - \int\limits_{|\xi|\ge\frac{r_0}{\epsilon}}\Bigg) e^{ix\cdot\xi} e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t}\left(i(v\cdot\xi)L^{-1}\hat{f}_0, h_j\right)h_j d\xi \\ &=: I_{43} + I_{44}. \qquad (3.105)\end{aligned}$$

For I_{44}, it holds that

$$\|P_{||}I_{44}\|_{L^\infty} \le C\int\limits_{|\xi|\ge\frac{r_0}{\epsilon}} e^{-c|\xi|^2 t}\|P_0(v\cdot\xi L^{-1}\hat{f}_0)\|d\xi \le Ce^{-\frac{cr_0^2}{\epsilon^2}t}\|f_0\|_{H^3}. \qquad (3.106)$$

Note that

$$\begin{aligned}(I_{43}, v\chi_0)_{||} &= -\sum_{j=\pm1}\frac{1}{2}\int\limits_{\mathbb{R}^3} e^{ix\cdot\xi} e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t}\left[j\sqrt{\frac{2}{5}}\hat{F}_4 - (\hat{F}_1\cdot\omega)\right]\omega, \\ (I_{43}, \tilde{h}_1) &= \sum_{j=\pm1}\frac{1}{2}\int\limits_{\mathbb{R}^3} e^{ix\cdot\xi} e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t}\left[\sqrt{\frac{2}{5}}\hat{F}_4 - j(\hat{F}_1\cdot\omega)\right],\end{aligned}$$

where

$$(F_1, F_4) = ((v\cdot\nabla_x L^{-1}f_0, v\chi_0), (v\cdot\nabla_x L^{-1}f_0, \chi_4)).$$

Hence, by Lemma 3.8, we have

$$\begin{aligned}\|P_{||}I_{43}\|_{L^\infty} &\le C\sum_{j=\pm1}\sum_{k=0}^{2}\|G_{jk}(t)\|_{L_x^\infty}\|H_j(t)\|_{L_x^1}\|(F_1, F_4)\|_{W_x^{3,1}} \\ &\le C\left(\frac{t}{\epsilon}\right)^{-1}\|f_0\|_{W^{4,1}}. \qquad (3.107)\end{aligned}$$

Thus, it follows from (3.102)–(3.104) and (3.105)–(3.107) that

$$\|P_{||}I_4\|_{L^\infty} \le C\left(\epsilon(1+t)^{-\frac{5}{2}} + \left(1+\frac{t}{\epsilon}\right)^{-1}\right)(\|f_0\|_{H^4} + \|f_0\|_{W^{4,1}}), \tag{3.108}$$

$$\|P_\perp I_4\|_{L^2} \le C\epsilon(1+t)^{-\frac{5}{4}}(\|f_0\|_{H^2} + \|f_0\|_{L^1}). \tag{3.109}$$

Finally, I_5 and I_6 can be estimated directly as follows.

$$\|I_5\|_{L^2}^2 \le \int_{\mathbb{R}^3} \frac{1}{\epsilon^2}\|S_5(t,\xi,\epsilon)\hat{f}_0\|^2 d\xi \le C\int_{\mathbb{R}^3} \frac{1}{\epsilon^2} e^{-\frac{2b}{\epsilon^2}t}\|\hat{f}_0\|^2 d\xi \le C\frac{1}{\epsilon^2}e^{-\frac{2b}{\epsilon^2}t}\|f_0\|_{L^2}^2, \tag{3.110}$$

$$\begin{aligned}\|I_6\|_{L^2}^2 &\le C\int_{|\xi|\ge\frac{r_0}{\epsilon}} e^{-2c|\xi|^2 t}\|P_0(v\cdot\xi L^{-1}\hat{f}_0)\|^2 d\xi \le C\epsilon^2 \int_{|\xi|\ge\frac{r_0}{\epsilon}} e^{-\frac{2cr_0^2}{\epsilon^2}t}|\xi|^4\|\hat{f}_0\|^2 d\xi \\ &\le C\epsilon^2 e^{-\frac{2cr_0^2}{\epsilon^2}t}\|f_0\|_{H^2}^2. \end{aligned}\tag{3.111}$$

By (3.108), (3.109), (3.110) and (3.111), and the Sobolev embedding theorem, we obtain (3.84) and (3.85). And this completes the proof of the lemma. □

Remark 3.10. From Lemma 3.9, we have

$$\left\|e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}P_0 f_0 - Y_1(t)P_0 f_0 - U_\epsilon^{osc}(t)\right\|_{L^2} \le C\epsilon(1+t)^{-\frac{5}{4}}(\|f_0\|_{H^1} + \|f_0\|_{L^1}), \tag{3.112}$$

where $U_\epsilon^{osc}(t) = U_\epsilon^{osc}(t,x,v)$ is the high oscillation part of $e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0$ defined by

$$U_\epsilon^{osc}(t) = \sum_{j=\pm1}\int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{\frac{i\mu_j|\xi|}{\epsilon}t - a_j|\xi|^2 t}\left(P_0\hat{f}_0, h_j\right)h_j d\xi. \tag{3.113}$$

Lemma 3.11. *For any $\epsilon\in(0,1)$, $\alpha\in\mathbb{N}^3$ and any $f_0\in L^2$, we have*

$$\|P_0\partial_x^\alpha e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{m}{2}}(\|\partial_x^\alpha f_0\|_{L^2} + \|\partial_x^{\alpha'}f_0\|_{L^1}), \tag{3.114}$$

$$\|P_1\partial_x^\alpha e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0\|_{L^2} \le C\left(\epsilon(1+t)^{-\frac{5}{4}-\frac{m}{2}} + e^{-\frac{bt}{\epsilon^2}}\right)(\|\partial_x^\alpha f_0\|_{H^1} + \|\partial_x^{\alpha'}f_0\|_{L^1}), \tag{3.115}$$

where $\alpha'\le\alpha$, $m=|\alpha-\alpha'|$, and $b>0$ is a constant given by (2.136). Moreover, if $P_0 f_0 = 0$, then

$$\|P_0\partial_x^\alpha e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0\|_{L^2} \le C\left(\epsilon(1+t)^{-\frac{5}{4}-\frac{m}{2}} + e^{-\frac{bt}{\epsilon^2}}\right)(\|\partial_x^\alpha f_0\|_{H^1} + \|\partial_x^{\alpha'}f_0\|_{L^1}), \tag{3.116}$$

$$\|P_1\partial_x^\alpha e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0\|_{L^2} \le C\left(\epsilon^2(1+t)^{-\frac{7}{4}-\frac{m}{2}} + e^{-\frac{bt}{\epsilon^2}}\right)(\|\partial_x^\alpha f_0\|_{H^2} + \|\partial_x^{\alpha'}f_0\|_{L^1}). \tag{3.117}$$

Proof. By (2.135), for $j=0,1$ we have

$$\left\|P_j\partial_x^\alpha e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0\right\|_{L^2}^2 = \int_{\mathbb{R}^3}\left\|P_j\xi^\alpha e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon(\xi)}\hat{f}_0\right\|^2 d\xi$$

$$\leq \int\limits_{|\xi|\leq \frac{r_0}{\epsilon}} \left\|\xi^\alpha P_j S_4(t,\xi,\epsilon)\hat{f}_0\right\|^2 d\xi + \int\limits_{\mathbb{R}^3} \left\|\xi^\alpha S_5(t,\xi,\epsilon)\hat{f}_0\right\|^2 d\xi. \quad (3.118)$$

By (2.136), we can estimate the second term on the right hand side of (3.118) as follows.

$$\int\limits_{\mathbb{R}^3} (\xi^\alpha)^2 \|S_5(t,\xi,\epsilon)\hat{f}_0\|^2 d\xi \leq Ce^{-2\frac{bt}{\epsilon^2}} \int\limits_{\mathbb{R}^3} (\xi^\alpha)^2 \|\hat{f}_0\|^2 d\xi \leq Ce^{-2\frac{bt}{\epsilon^2}} \|\partial_x^\alpha f_0\|_{L^2}^2. \quad (3.119)$$

By Theorems 2.19 and 2.18, we have

$$\begin{aligned} \int\limits_{|\xi|\leq \frac{r_0}{\epsilon}} \|\xi^\alpha P_0 S_4(t,\xi,\epsilon)\hat{f}_0\|^2 d\xi &\leq C \int\limits_{|\xi|\leq \frac{r_0}{\epsilon}} e^{-c|\xi|^2 t} (\xi^\alpha)^2 \|\hat{f}_0\|^2 d\xi \\ &\leq C(1+t)^{-\frac{3}{2}-m} \left(\|\partial_x^\alpha f_0\|_{L^2}^2 + \|\partial_x^{\alpha'} f_0\|_{L^1}^2\right), \quad (3.120) \\ \int\limits_{|\xi|\leq \frac{r_0}{\epsilon}} \|\xi^\alpha P_1 S_4(t,\xi,\epsilon)\hat{f}_0\|^2 d\xi &\leq C \int\limits_{|\xi|\leq \frac{r_0}{\epsilon}} e^{-c|\xi|^2 t} \epsilon^2 |\xi|^2 (\xi^\alpha)^2 \|\hat{f}_0\|^2 d\xi \\ &\leq C\epsilon^2 (1+t)^{-\frac{5}{2}-m} \left(\|\partial_x^\alpha f_0\|_{H^1}^2 + \|\partial_x^{\alpha'} f_0\|_{L^1}^2\right), \quad (3.121) \end{aligned}$$

where $\alpha' \leq \alpha$ and $m = |\alpha - \alpha'|$. Combining (3.118)–(3.121) yields (3.114) and (3.115).

Moreover, it holds that for $P_0 f_0 = 0$,

$$\begin{aligned} \int\limits_{|\xi|\leq \frac{r_0}{\epsilon}} \|\xi^\alpha P_0 S_4(t,\xi,\epsilon)\hat{f}_0\|^2 d\xi &\leq C \int\limits_{|\xi|\leq \frac{r_0}{\epsilon}} e^{-c|\xi|^2 t} \epsilon^2 |\xi|^2 (\xi^\alpha)^2 \|\hat{f}_0\|^2 d\xi \\ &\leq C\epsilon^2 (1+t)^{-\frac{5}{2}-m} \left(\|\partial_x^\alpha f_0\|_{H^1}^2 + \|\partial_x^{\alpha'} f_0\|_{L^1}^2\right), \quad (3.122) \\ \int\limits_{|\xi|\leq \frac{r_0}{\epsilon}} \|\xi^\alpha P_1 S_4(t,\xi,\epsilon)\hat{f}_0\|^2 d\xi &\leq C \int\limits_{|\xi|\leq \frac{r_0}{\epsilon}} e^{-c|\xi|^2 t} \epsilon^4 |\xi|^4 (\xi^\alpha)^2 \|\hat{f}_0\|^2 d\xi \\ &\leq C\epsilon^4 (1+t)^{-\frac{7}{2}-m} \left(\|\partial_x^\alpha f_0\|_{H^2}^2 + \|\partial_x^{\alpha'} f_0\|_{L^1}^2\right). \quad (3.123) \end{aligned}$$

Combining (3.118), (3.122) and (3.123) gives (3.116)–(3.117). The proof of the lemma is completed. □

Lemma 3.12. *For any $1 \leq q \leq 2$, $\alpha \in \mathbb{N}^3$ and any $u_0 \in N_0$, we have*

$$\|\partial_x^\alpha Y_1(t)u_0\|_{L^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{m}{2}} \|\partial_x^{\alpha'} u_0\|_{L^q} + Ct^{-\frac{k}{2}} e^{-ct} \|\partial_x^{\alpha''} u_0\|_{L^2}, \quad (3.124)$$

where $\alpha', \alpha'' \leq \alpha$, $m = |\alpha - \alpha'|$ and $k = |\alpha - \alpha''|$.

Proof. By (3.5), we have

$$\begin{aligned}
\|\partial_x^\alpha Y_1(t)u_0\|_{L^2}^2 &\le C\left(\int\limits_{|\xi|\le 1}+\int\limits_{|\xi|\ge 1}\right)(\xi^\alpha)^2 e^{-2c|\xi|^2 t}\|\hat{u}_0\|^2 d\xi \\
&\le C\left(\int\limits_{|\xi|\le 1}|\xi|^{2pm}e^{-2cp|\xi|^2 t}d\xi\right)^{1/p}\left(\int\limits_{|\xi|\le 1}\|\xi^{\alpha'}\hat{u}_0\|^{2p'}d\xi\right)^{1/p'} \\
&\quad+\sup_{|\xi|\ge 1}\left(|\xi|^{2k}e^{-2c|\xi|^2 t}\right)\int\limits_{|\xi|\ge 1}(\xi^{\alpha''})^2\|\hat{u}_0\|^2 d\xi \\
&\le C(1+t)^{-\frac{3}{2}(\frac{2}{q}-1)-m}\|\partial_x^{\alpha'}u_0\|_{L^q}^2+Ct^{-k}e^{-2ct}\|\partial_x^{\alpha''}u_0\|_{L^2}^2,
\end{aligned}$$

where $1/q+1/p'=1$, $1/(2p')+1/q=1$, $\alpha'\le\alpha$ and $m=|\alpha-\alpha'|$. This completes the proof of the lemma. □

4. Diffusion limit

In this section, we will study the diffusion limit of the nonlinear VMB system (1.6)–(1.10) based on the fluid approximations of the linear VMB system given in Section 3.

Since the operators $\mathbb{A}_\epsilon$ and $\mathbb{B}_\epsilon$ generate contraction semigroups in H^k, the solution $U_\epsilon(t)=(f_\epsilon,V_\epsilon)(t)$ with $V_\epsilon(t)=(g_\epsilon,E_\epsilon,B_\epsilon)(t)$ to the VMB system (1.6)–(1.13) can be represented by

$$f_\epsilon(t)=e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0+\int_0^t e^{\frac{t-s}{\epsilon^2}\mathbb{B}_\epsilon}\left(G_1(s)+\frac{1}{\epsilon}G_2(s)\right)ds, \tag{4.1}$$

$$V_\epsilon(t)=e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}V_0+\int_0^t e^{\frac{t-s}{\epsilon^2}\mathbb{A}_\epsilon}\left(G_3(s)+\frac{1}{\epsilon}G_4(s)\right)ds, \tag{4.2}$$

where $V_0=(g_0,E_0,B_0)$, and the nonlinear terms G_k, $k=1,2,3,4$ are given by

$$\begin{aligned}
G_1&=\frac{1}{2}v\cdot E_\epsilon g_\epsilon-E_\epsilon\cdot\nabla_v g_\epsilon-\frac{1}{\epsilon}P_0(v\times B_\epsilon)\cdot\nabla_v P_r g_\epsilon,\\
G_2&=-P_1(v\times B_\epsilon)\cdot\nabla_v P_r g_\epsilon+\Gamma(f_\epsilon,f_\epsilon),\\
G_3&=(G_{31},0,0),\quad G_{31}=\frac{1}{2}v\cdot E_\epsilon f_\epsilon-E_\epsilon\cdot\nabla_v f_\epsilon,\\
G_4&=(G_{41},0,0),\quad G_{41}=-(v\times B_\epsilon)\cdot\nabla_v f_\epsilon+\Gamma(g_\epsilon,f_\epsilon).
\end{aligned}$$

Also, the solution $U(t) = (u_1, V_1)(t)$ with $u_1 = n\chi_0 + m \cdot v\chi_0 + q\chi_4$ and $V_1 = (\rho\chi_0, E, B)$ to the NSMF system (1.24)–(1.25) can be represented by

$$u_1(t) = Y_1(t)P_0 f_0 + \int_0^t Y_1(t-s)(H_1(s) + \nabla_x \cdot H_2(s))ds, \tag{4.3}$$

$$V_1(t) = Y_2(t)P_2 V_0 + \int_0^t Y_2(t-s)H_3(s)ds, \tag{4.4}$$

where

$$H_1 = (\rho E + j \times B) \cdot v\chi_0, \quad H_2 = -(m \otimes m) \cdot v\chi_0 - \frac{5}{3}(qm)\chi_4,$$
$$H_3 = -(\nabla_x \cdot H_4\chi_0, H_4, 0), \quad H_4 = \rho m - \eta m \times B.$$

4.1. Energy estimate

We first derive some energy estimates. Let $N \geq 1$ be a positive integer and $U_\epsilon = (f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)$, and

$$E_{N,k}(U_\epsilon) = \sum_{|\alpha|+|\beta|\leq N} \|\nu^k \partial_x^\alpha \partial_v^\beta (f_\epsilon, g_\epsilon)\|_{L^2}^2 + \sum_{|\alpha|\leq N} \|\partial_x^\alpha (E_\epsilon, B_\epsilon)\|_{L_x^2}^2, \tag{4.5}$$

$$\begin{aligned} D_{N,k}(U_\epsilon) = &\sum_{|\alpha|+|\beta|\leq N} \frac{1}{\epsilon^2}\|\nu^{\frac{1}{2}+k} \partial_x^\alpha \partial_v^\beta (P_1 f_\epsilon, P_r g_\epsilon)\|_{L^2}^2 + \sum_{1\leq|\alpha|\leq N-1} \|\partial_x^\alpha (E_\epsilon, B_\epsilon)\|_{L_x^2}^2 \\ &+ \sum_{|\alpha|\leq N-1} \|\partial_x^\alpha \nabla_x (P_0 f_\epsilon, P_d g_\epsilon)\|_{L^2}^2 + \|E_\epsilon\|_{L_x^2}^2, \end{aligned} \tag{4.6}$$

for $k \geq 0$. For brevity, we denote $E_N(U_\epsilon) = E_{N,0}(U_\epsilon)$ and $D_N(U_\epsilon) = D_{N,0}(U_\epsilon)$.

Firstly, by taking the inner product of χ_j $(j = 0, 1, 2, 3, 4)$ and (1.6), we have a compressible Euler-Maxwell type system

$$\partial_t n_\epsilon + \frac{1}{\epsilon}\mathrm{div}_x m_\epsilon = 0, \tag{4.7}$$

$$\partial_t m_\epsilon + \frac{1}{\epsilon}\nabla_x n_\epsilon + \frac{1}{\epsilon}\sqrt{\frac{2}{3}}\nabla_x q_\epsilon = \rho_\epsilon E_\epsilon + \frac{1}{\epsilon}u_\epsilon \times B_\epsilon - \frac{1}{\epsilon}(v \cdot \nabla_x(P_1 f_\epsilon), v\chi_0), \tag{4.8}$$

$$\partial_t q_\epsilon + \frac{1}{\epsilon}\sqrt{\frac{2}{3}}\mathrm{div}_x m_\epsilon = \sqrt{\frac{2}{3}}E_\epsilon \cdot u_\epsilon - \frac{1}{\epsilon}(v \cdot \nabla_x(P_1 f_\epsilon), \chi_4), \tag{4.9}$$

where

$$(n_\epsilon, m_\epsilon, q_\epsilon) = ((f_\epsilon, \chi_0), (f_\epsilon, v\chi_0), (f_\epsilon, \chi_4)), \quad (\rho_\epsilon, u_\epsilon) = ((g_\epsilon, \chi_0), (g_\epsilon, v\chi_0)). \tag{4.10}$$

Taking the microscopic projection P_1 on (1.6) gives

$$\partial_t(P_1 f_\epsilon) + \frac{1}{\epsilon}P_1(v\cdot\nabla_x P_1 f_\epsilon) - \frac{1}{\epsilon^2}L(P_1 f_\epsilon) = -\frac{1}{\epsilon}P_1(v\cdot\nabla_x P_0 f_\epsilon) + P_1 G_1 - \frac{1}{\epsilon}P_1 G_2. \tag{4.11}$$

By (4.11), we can express the microscopic part $P_1 f_\epsilon$ as

$$\frac{1}{\epsilon}P_1 f_\epsilon = L^{-1}[\epsilon\partial_t(P_1 f_\epsilon) + P_1(v\cdot\nabla_x P_1 f_\epsilon) - \epsilon P_1 G_1 - P_1 G_2] + L^{-1}P_1(v\cdot\nabla_x P_0 f_\epsilon). \tag{4.12}$$

By substituting (4.12) into (4.7)–(4.9), we obtain a compressible Navier-Stokes-Maxwell type system

$$\partial_t n_\epsilon + \frac{1}{\epsilon}\mathrm{div}_x m_\epsilon = 0, \tag{4.13}$$

$$\begin{aligned}
&\partial_t m_\epsilon + \epsilon\partial_t R_1 + \frac{1}{\epsilon}\nabla_x n_\epsilon + \frac{1}{\epsilon}\sqrt{\frac{2}{3}}\nabla_x q_\epsilon \\
&= \kappa_0\left(\Delta_x m_\epsilon + \frac{1}{3}\nabla_x \mathrm{div}_x m_\epsilon\right) + \rho_\epsilon E_\epsilon + \frac{1}{\epsilon}u_\epsilon \times B_\epsilon + R_3,
\end{aligned} \tag{4.14}$$

$$\partial_t q_\epsilon + \epsilon\partial_t R_2 + \frac{1}{\epsilon}\sqrt{\frac{2}{3}}\mathrm{div}_x m_\epsilon = \kappa_1\Delta_x q_\epsilon + \sqrt{\frac{2}{3}}E_\epsilon\cdot u_\epsilon + R_4, \tag{4.15}$$

where the remainder terms R_1, R_2, R_3, R_4 are given by

$$\begin{aligned}
&R_1 = (v\cdot\nabla_x L^{-1}(P_1 f_\epsilon), v\chi_0), \quad R_2 = (v\cdot\nabla_x L^{-1}(P_1 f_\epsilon), \chi_4), \\
&R_3 = (v\cdot\nabla_x L^{-1}[P_1(v\cdot\nabla_x P_1 f_\epsilon) - \epsilon P_1 G_1 - P_1 G_2], v\chi_0), \\
&R_4 = (v\cdot\nabla_x L^{-1}[P_1(v\cdot\nabla_x P_1 f_\epsilon) - \epsilon P_1 G_1 - P_1 G_2], \chi_4).
\end{aligned}$$

By taking the inner product between $\sqrt{M}$ and (1.7), we obtain

$$\partial_t \rho_\epsilon + \frac{1}{\epsilon}\mathrm{div}_x u_\epsilon = 0. \tag{4.16}$$

Taking the microscopic projection P_r on (1.7) gives

$$\begin{aligned}
&\partial_t(P_r g_\epsilon) + \frac{1}{\epsilon}P_r(v\cdot\nabla_x P_r g_\epsilon) - \frac{1}{\epsilon}v\sqrt{M}\cdot E_\epsilon - \frac{1}{\epsilon^2}L_1(P_r g_\epsilon) \\
&= -\frac{1}{\epsilon}(v\cdot\nabla_x\rho_\epsilon)\sqrt{M} + G_{31} + \frac{1}{\epsilon}G_{41}.
\end{aligned} \tag{4.17}$$

By (4.17), we can express the microscopic part $P_r g_\epsilon$ as

$$\frac{1}{\epsilon}P_r g_\epsilon = L_1^{-1}[\epsilon\partial_t(P_r g_\epsilon) + P_r(v\cdot\nabla_x P_r g_\epsilon) - \epsilon G_{31} - G_{41}] - L_1^{-1}[v\chi_0\cdot(E_\epsilon - \nabla_x\rho_\epsilon)]. \tag{4.18}$$

Substituting (4.18) into (4.16) and (1.8) gives

$$\partial_t\rho_\epsilon + \epsilon\partial_t \mathrm{div}_x R_5 = -\eta\rho_\epsilon + \eta\Delta_x\rho_\epsilon - \mathrm{div}_x R_6, \tag{4.19}$$

$$\partial_t E_\epsilon + \epsilon\partial_t R_5 = \nabla_x\times B_\epsilon + \eta\nabla_x\rho_\epsilon - \eta E_\epsilon + R_6, \tag{4.20}$$

$$\partial_t B_\epsilon = -\nabla_x\times E_\epsilon, \tag{4.21}$$

where the remainder terms R_5, R_6 are defined by

$$R_5 = (L_1^{-1}P_r g_\epsilon, v\chi_0), \quad R_6 = (L_1^{-1}(P_r(v\cdot\nabla_x P_r g_\epsilon) - \epsilon G_{31} - G_{41}), v\chi_0).$$

Similar to [12,26,32], we have the existence and the energy estimate for the solution $U_\epsilon = (f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)$ to the VMB system (1.6)–(1.13).

Lemma 4.1 *(Macroscopic dissipation). Given $N \geq 4$. Let $(n_\epsilon, m_\epsilon, q_\epsilon)$ and $(\rho_\epsilon, E_\epsilon, B_\epsilon)$ be the strong solutions to (4.13)–(4.15) and (4.19)–(4.21) respectively. Then, there are two constants $s_0, s_1 > 0$ such that for any $\epsilon \in (0,1)$,*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leq N-1} s_0\left(\|\partial_x^\alpha(n_\epsilon, m_\epsilon, q_\epsilon)\|_{L_x^2}^2 + 2\epsilon\int_{\mathbb{R}^3}\partial_x^\alpha R_1\partial_x^\alpha m_\epsilon dx + 2\epsilon\int_{\mathbb{R}^3}\partial_x^\alpha R_2\partial_x^\alpha q_\epsilon dx\right)\\
&+\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leq N-1}4\epsilon\int_{\mathbb{R}^3}\partial_x^\alpha m_\epsilon\partial_x^\alpha\nabla_x n_\epsilon dx + \sum_{|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x(n_\epsilon, m_\epsilon, q_\epsilon)\|_{L_x^2}^2\\
&\leq C\sqrt{E_N(U_\epsilon)}D_N(U_\epsilon) + C\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x P_1 f_\epsilon\|_{L^2}^2,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leq N-1} s_1\left(\|\partial_x^\alpha(\rho_\epsilon, E_\epsilon, B_\epsilon)\|_{L_x^2}^2 + 2\epsilon\int_{\mathbb{R}^3}\partial_x^\alpha \mathrm{div}_x R_5\partial_x^\alpha\rho_\epsilon dx + 2\epsilon\int_{\mathbb{R}^3}\partial_x^\alpha R_5\partial_x^\alpha E_\epsilon dx\right)\\
&-\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leq N-2}4\int_{\mathbb{R}^3}\partial_x^\alpha E_\epsilon\partial_x^\alpha(\nabla_x\times B_\epsilon)dx\\
&+\sum_{|\alpha|\leq N-1}(\|\partial_x^\alpha\rho_\epsilon\|_{L_x^2}^2 + \|\partial_x^\alpha\nabla_x\rho_\epsilon\|_{L_x^2}^2 + \|\partial_x^\alpha E_\epsilon\|_{L_x^2}^2) + \sum_{1\leq|\alpha|\leq N-1}\|\partial_x^\alpha B_\epsilon\|_{L_x^2}^2\\
&\leq CE_N(U_\epsilon)D_N(U_\epsilon) + \frac{C}{\epsilon^2}\sum_{|\alpha|\leq N}\|\partial_x^\alpha P_r g_\epsilon\|_{L^2}^2.
\end{aligned}$$

Lemma 4.2 *(Microscopic dissipation). Given $N \geq 4$. Let $U_\epsilon = (f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)$ be a strong solution to VMB system (1.6)–(1.13). Then, there are constants $p_k > 0$, $1 \leq k \leq N$ and $\mu_1 > 0$ such that for any $\epsilon \in (0,1)$,*

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\le N}(\|\partial_x^\alpha(f_\epsilon,g_\epsilon)\|_{L^2}^2+\|\partial_x^\alpha(E_\epsilon,B_\epsilon)\|_{L_x^2}^2)+\frac{\mu_1}{\epsilon^2}\sum_{|\alpha|\le N}\|\nu^{\frac{1}{2}}\partial_x^\alpha(P_1f_\epsilon,P_rg_\epsilon)\|_{L^2}^2\\
&\le C\sqrt{E_N(U_\epsilon)}D_N(U_\epsilon),\\
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le k\le N}p_k\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_v^\beta(P_1f_\epsilon,P_rg_\epsilon)\|_{L^2}^2+\frac{\mu_1}{\epsilon^2}\sum_{\substack{|\beta|\ge1\\|\alpha|+|\beta|\le N}}\|\nu^{\frac{1}{2}}\partial_x^\alpha\partial_v^\beta(P_1f_\epsilon,P_rg_\epsilon)\|_{L^2}^2\\
&\le C\sum_{|\alpha|\le N-1}(\|\partial_x^\alpha\nabla_x(f_\epsilon,g_\epsilon)\|_{L^2}^2+\|\partial_x^\alpha E_\epsilon\|_{L_x^2}^2)+C\sqrt{E_N(U_\epsilon)}D_N(U_\epsilon).
\end{aligned}
$$

Lemma 4.3. *Let $N\ge4$. For any $\epsilon\in(0,1)$, there exists a small constant $\delta_0>0$ and an energy functional $\mathcal{E}_N(U_\epsilon)\sim E_N(U_\epsilon)$ such that if the initial data $U_0=(f_0,g_0,E_0,B_0)$ satisfies $E_N(U_0)\le\delta_0^2$, then the system (1.6)–(1.13) admits a unique global solution $U_\epsilon=(f_\epsilon,g_\epsilon,E_\epsilon,B_\epsilon)$ satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_N(U_\epsilon(t))+D_N(U_\epsilon(t))\le0. \tag{4.22}$$

Moreover, there exists an energy functional $\mathcal{E}_{N,1}(U_\epsilon)\sim E_{N,1}(U_\epsilon)$ such that if the initial data U_0 satisfies $E_{N,1}(U_0)\le\delta_0^2$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{N,1}(U_\epsilon(t))+D_{N,1}(U_\epsilon(t))\le0. \tag{4.23}$$

With Lemmas 3.6, 3.11 and 4.3, we have the optimal time decay rate of $U_\epsilon=(f_\epsilon,g_\epsilon,E_\epsilon,B_\epsilon)$ stated in the following lemma.

Lemma 4.4. *Let $N\ge4$. For any $\epsilon\in(0,1)$, there exists a small constant $\delta_0>0$ such that if the initial data $U_0=(f_0,g_0,E_0,B_0)$ satisfies that $E_{N+2,1}(U_0)+\|U_0\|_{L^1}^2\le\delta_0^2$, then the solution $U_\epsilon(t)=(f_\epsilon,g_\epsilon,E_\epsilon,B_\epsilon)$ to the system (1.6)–(1.13) has the following time-decay rate estimates:*

$$\|(f_\epsilon,g_\epsilon)\|_{D_1^N}+\|(E_\epsilon,B_\epsilon)\|_{H_x^N}\le C\delta_0(1+t)^{-\frac{3}{4}}. \tag{4.24}$$

In particular, we have

$$\|P_1f_\epsilon(t)\|_{H^{N-2}}\le C\delta_0\left(\epsilon(1+t)^{-\frac{3}{4}}+e^{-\frac{bt}{4\epsilon^2}}\right), \tag{4.25}$$

$$\|P_rg_\epsilon(t)\|_{H^{N-1}}\le C\delta_0\left(\epsilon(1+t)^{-\frac{3}{4}}+e^{-\frac{bt}{4\epsilon^2}}\right), \tag{4.26}$$

where $b>0$ is a constant.

Proof. Define

$$Q_\epsilon(t)=\sup_{0\le s\le t}\left\{(1+s)^{\frac{3}{4}}E_{N,1}(U_\epsilon(s))^{\frac{1}{2}}+\|P_rg_\epsilon(s)\|_{H^{N-1}}\left(\epsilon(1+s)^{-\frac{3}{4}}+e^{-\frac{bs}{4\epsilon^2}}\right)^{-1}\right\}.$$

We claim that

$$Q_\epsilon(t) \le C\delta_0. \tag{4.27}$$

It is straightforward to check that the estimates (4.24) and (4.26) follow from (4.27).

Since $P_0 G_2 = 0$ and $P_0(v \times B_\epsilon) \cdot \nabla_v P_r g_\epsilon = (u_\epsilon \times B_\epsilon) \cdot v\chi_0$, it follows from Lemma 3.11 and (4.1) that

$$\begin{aligned}
\|P_0 f_\epsilon(t)\|_{L^2} \le{}& C(1+t)^{-\frac{3}{4}} \left(\|f_0\|_{L^2} + \|f_0\|_{L^1}\right) \\
&+ C\int_0^t (1+t-s)^{-\frac{3}{4}} \left(\|G_1(s)\|_{L^2} + \|G_1(s)\|_{L^1}\right) ds \\
&+ C\int_0^t \left((1+t-s)^{-\frac{5}{4}} + \frac{1}{\epsilon} e^{-\frac{b(t-s)}{\epsilon^2}}\right) \\
&\times \left(\|G_2(s)\|_{H^1} + \|G_2(s)\|_{L^1}\right) ds \\
\le{}& C\delta_0 (1+t)^{-\frac{3}{4}} + CQ_\epsilon(t)^2 (1+t)^{-\frac{3}{4}},
\end{aligned} \tag{4.28}$$

where we have used

$$\|G_1(s)\|_{H^{N-1}} + \|G_1(s)\|_{L^1} \le CQ_\epsilon(t)^2 \left[(1+s)^{-\frac{3}{2}} + \frac{1}{\epsilon}(1+s)^{-\frac{3}{4}} e^{-\frac{bs}{4\epsilon^2}}\right], \tag{4.29}$$

$$\|G_2(s)\|_{H^{N-1}} + \|G_2(s)\|_{L^1} \le CQ_\epsilon(t)^2 (1+s)^{-\frac{3}{2}}. \tag{4.30}$$

Let $V_\epsilon = (g_\epsilon, E_\epsilon, B_\epsilon)$ and $V_0 = (g_0, E_0, B_0)$. Since $P_2 G_{31} = P_2 G_{41} = 0$, it follows from Lemma 3.6 and (4.2) that

$$\begin{aligned}
\|V_\epsilon(t)\|_{L^2} \le{}& C(1+t)^{-\frac{3}{4}} \left(\|V_0\|_{L^2} + \|V_0\|_{L^1} + \|\nabla_x V_0\|_{L^2}\right) \\
&+ C\sum_{k=3}^{4} \int_0^t \left((1+t-s)^{-\frac{5}{4}} + \frac{1}{\epsilon} e^{-\frac{b(t-s)}{\epsilon^2}}\right) \\
&\times \left(\|G_k(s)\|_{H^1} + \|G_k(s)\|_{L^1} + \|\nabla_x G_k(s)\|_{L^2}\right) ds \\
\le{}& C\delta_0 (1+t)^{-\frac{3}{4}} + CQ_\epsilon(t)^2 (1+t)^{-\frac{3}{4}},
\end{aligned} \tag{4.31}$$

where we have used

$$\|G_k(s)\|_{H^{N-1}} + \|G_k(s)\|_{L^1} \le CQ_\epsilon(t)^2 (1+s)^{-\frac{3}{2}}, \quad k = 3, 4. \tag{4.32}$$

Let $1 < l < 2$ and $n \ge 4$. Multiplying (4.23) by $(1+t)^l$ and then taking time integration over $[0, t]$ gives

$$
\begin{aligned}
&(1+t)^l E_{n,1}(U_\epsilon)(t) + \int_0^t (1+s)^l D_{n,1}(U_\epsilon)(s)ds \\
&\le CE_{n,1}(U_0) + Cl\int_0^t (1+s)^{l-1} E_{n,1}(U_\epsilon)(s)ds \\
&\le CE_{n,1}(U_0) + Cl\int_0^t (1+s)^{l-1} D_{n+1,1}(U_\epsilon)(s)ds \\
&\quad + Cl\int_0^t (1+s)^{l-1}(\|P_0 f_\epsilon(s)\|_{L^2}^2 + \|B_\epsilon(s)\|_{L_x^2}^2)ds,
\end{aligned} \tag{4.33}
$$

where we have used

$$
E_{n,1}(U_\epsilon) \le CD_{n+1,1}(U_\epsilon) + C(\|P_0 f_\epsilon\|_{L^2}^2 + \|B_\epsilon\|_{L_x^2}^2).
$$

Similarly, we can obtain the estimate for $n+1$ as follows.

$$
\begin{aligned}
&(1+t)^{l-1} E_{n+1,1}(U_\epsilon)(t) + \int_0^t (1+s)^{l-1} D_{n+1,1}(U_\epsilon)(s)ds \\
&\le CE_{n+1,1}(U_0) + C(l-1)\int_0^t (1+s)^{l-2}(\|P_0 f_\epsilon(s)\|_{L^2}^2 + \|B_\epsilon(s)\|_{L_x^2}^2)ds \\
&\quad + C(l-1)\int_0^t (1+s)^{l-2} D_{n+2,1}(U_\epsilon)ds.
\end{aligned} \tag{4.34}
$$

And it follows from (4.23) that

$$
E_{n+2,1}(U_\epsilon)(t) + \int_0^t D_{n+2,1}(U_\epsilon)(s)ds \le CE_{n+2,1}(U_0). \tag{4.35}
$$

By (4.33)–(4.35), we obtain

$$
\begin{aligned}
&(1+t)^l E_{n,1}(U_\epsilon)(t) + \int_0^t (1+s)^l D_{n,1}(U_\epsilon)(s)ds \\
&\le CE_{n+2,1}(U_0) + C\int_0^t (1+s)^{l-1}(\|P_0 f_\epsilon(s)\|_{L^2}^2 + \|B_\epsilon(s)\|_{L_x^2}^2)ds
\end{aligned}
$$

for $1 < l < 2$ and $n \geq 4$. Taking $l = 3/2 + \epsilon$ for a fixed constant $0 < \epsilon < 1/2$ yields

$$(1+t)^{\frac{3}{2}+\epsilon}E_{n,1}(U_\epsilon)(t) + \int_0^t (1+s)^{\frac{3}{2}+\epsilon}D_{n,1}(U_\epsilon)(s)ds$$
$$\leq CE_{n+2,1}(U_0) + C(\delta_0 + Q_\epsilon^2(t))^2 \int_0^t (1+s)^{\frac{1}{2}+\epsilon}(1+s)^{-\frac{3}{2}}ds$$
$$\leq CE_{n+2,1}(U_0) + C(1+t)^\epsilon(\delta_0 + Q_\epsilon^2(t))^2.$$

This gives

$$E_{n,1}(U_\epsilon)(t) \leq C(1+t)^{-\frac{3}{2}}(\delta_0 + Q_\epsilon^2(t))^2. \tag{4.36}$$

Next, it follows from Lemma 3.6 and (4.23) that

$$\|P_r g_\epsilon(t)\|_{H^{N-1}} \leq C\left(\epsilon(1+t)^{-\frac{5}{4}} + e^{-\frac{bt}{\epsilon^2}}\right)\left(\|V_0\|_{H^N} + \|V_0\|_{L^1} + \|\nabla_x^2 V_0\|_{H^{N-1}}\right)$$
$$+ C\sum_{k=3}^4 \int_0^t \left(\epsilon(1+t-s)^{-\frac{5}{4}} + \frac{1}{\epsilon}e^{-\frac{b(t-s)}{\epsilon^2}}\right)$$
$$\times \left(\|G_k(s)\|_{H^{N+1}} + \|G_k(s)\|_{L^1} + \|\nabla_x^2 G_k(s)\|_{H^{N-1}}\right)ds$$
$$\leq C(\delta_0 + \delta_0 Q_\epsilon(t) + Q_\epsilon(t)^2)\left(\epsilon(1+t)^{-\frac{3}{4}} + e^{-\frac{bt}{4\epsilon^2}}\right), \tag{4.37}$$

where we had used

$$\|G_3(s)\|_{H^{N+1}} \leq C\sum_{|\alpha'|\leq 2}\|\partial_x^{\alpha'}E_\epsilon\|_{L^\infty}(\|\langle v\rangle\partial_x^{\alpha-\alpha'}f_\epsilon\|_{L^2} + \|\nabla_v\partial_x^{\alpha-\alpha'}f_\epsilon\|_{L^2})$$
$$+ C\sum_{3\leq|\alpha'|\leq N+1}\|\partial_x^{\alpha'}E_\epsilon\|_{L^2}(\|\langle v\rangle\partial_x^{\alpha-\alpha'}f_\epsilon\|_{L^\infty} + \|\nabla_v\partial_x^{\alpha-\alpha'}f_\epsilon\|_{L^\infty})$$
$$\leq CE_{N,1}(U)^{\frac{1}{2}}E_{N+2,1}(U)^{\frac{1}{2}} \leq C(1+t)^{-\frac{3}{4}}\delta_0 Q_\epsilon(t). \tag{4.38}$$

Note that (4.38) also holds for G_2, G_4. Then, by from Lemma 3.11 and (4.29), (4.38), we have

$$\|P_1 f_\epsilon(t)\|_{H^{N-2}} \leq C\left(\epsilon(1+t)^{-\frac{5}{4}} + e^{-\frac{bt}{\epsilon^2}}\right)(\|f_0\|_{H^{N-1}} + \|f_0\|_{L^1})$$
$$+ C\int_0^t \left(\epsilon(1+t-s)^{-\frac{5}{4}} + e^{-\frac{b(t-s)}{\epsilon^2}}\right)\|G_1(s)\|_{H^{N-1}\cap L^1}ds$$

$$
\begin{aligned}
&+C\int_0^t \left(\epsilon(1+t-s)^{-\frac{7}{4}}+\frac{1}{\epsilon}e^{-\frac{b(t-s)}{\epsilon^2}}\right)\|G_2(s)\|_{H^N\cap L^1}ds\\
&\le C(\delta_0+\delta_0 Q_\epsilon(t)+Q_\epsilon(t)^2)\left(\epsilon(1+t)^{-\frac{3}{4}}+e^{-\frac{bt}{4\epsilon^2}}\right). \qquad (4.39)
\end{aligned}
$$

Combining (4.36) and (4.37) gives

$$
Q_\epsilon(t)\le C(\delta_0+\delta_0 Q_\epsilon(t)+Q_\epsilon(t)^2),
$$

which shows (4.27) provided $\delta_0>0$ being sufficiently small. Finally, (4.25) follows from (4.39). The proof of the lemma is then completed. □

By (4.3), (4.4) and Lemma 3.12, we will prove the following lemma.

Lemma 4.5. *Let $N\ge 2$. There exists a small constant $\delta_0>0$ such that if $\|U_0\|_{H^N}+\|U_0\|_{L^1}\le\delta_0$, then the NSMF system (1.24)–(1.25) admits a unique global solution $\tilde U(t,x)=(n,m,q,\rho,E,B)(t,x)$ satisfying*

$$
\|\tilde U(t)\|_{H^N_x}\le C\delta_0(1+t)^{-\frac{3}{4}}. \qquad (4.40)
$$

Proof. Define

$$
Q(t)=\sup_{0\le s\le t}\left\{(1+s)^{3/4}\left(\|u_1(s)\|_{H^N}+\|V_1(s)\|_{H^N}\right)\right\},
$$

where u_1 and V_1 are defined by (4.3) and (4.4) respectively.

Then, it follows from Lemmas 3.7 and 3.12 that

$$
\begin{aligned}
\|u_1(t)\|_{H^N}&\le C(1+t)^{-\frac{3}{4}}(\|P_0f_0\|_{H^N}+\|P_0f_0\|_{L^1})\\
&\quad+C\sum_{k=1}^2\int_0^t\left((1+t-s)^{-\frac{3}{4}}+(t-s)^{-\frac{1}{2}}e^{-c(t-s)}\right)\\
&\qquad\times(\|H_k(s)\|_{H^N}+\|H_k(s)\|_{L^1})ds\\
&\le C\delta_0(1+t)^{-\frac{3}{4}}+CQ(t)^2(1+t)^{-\frac{3}{4}}, \qquad (4.41)\\
\|V_1(t)\|_{H^N}&\le C(1+t)^{-\frac{3}{4}}(\|P_2V_0\|_{H^N}+\|P_2V_0\|_{L^1})\\
&\quad+C\int_0^t\left((1+t-s)^{-\frac{3}{4}}+(t-s)^{-\frac{1}{2}}e^{-c(t-s)}\right)\\
&\qquad\times(\|H_4(s)\|_{H^N}+\|H_4(s)\|_{L^1})ds\\
&\le C\delta_0(1+t)^{-\frac{3}{4}}+CQ(t)^2(1+t)^{-\frac{3}{4}}. \qquad (4.42)
\end{aligned}
$$

By (4.41) and (4.42), we can obtain

$$Q(t) \le C\delta_0,$$

provided $\delta_0 > 0$ being sufficiently small. This proves (4.40). The existence of the solution can be proved by the contraction mapping theorem with the details omitted. Then the proof of the lemma is completed. □

Note that Theorem 1.1 follows directly from Lemmas 4.4 and 4.5.

4.2. Optimal convergence rate

In this section, we will complete the proof of Theorem 1.2 about the optimal convergence rate of the diffusion limit.

Lemma 4.6 *([27]). For any $i, j = 1, 2, 3$, it holds that*

$$\Gamma_*(v_i\chi_0, v_j\chi_0) = -\frac{1}{2}LP_1(v_iv_j\chi_0), \tag{4.43}$$

$$\Gamma_*(v_i\chi_0, |v|^2\chi_0) = -\frac{1}{2}LP_1(v_i|v|^2\chi_0), \tag{4.44}$$

$$\Gamma_*(|v|^2\chi_0, |v|^2\chi_0) = -\frac{1}{2}LP_1(|v|^4\chi_0), \tag{4.45}$$

where

$$\Gamma_*(f,g) = \frac{1}{2}[\Gamma(f,g) + \Gamma(g,f)].$$

Lemma 4.7. *For any $j = 1, 2, 3$, it holds that*

$$\Gamma(\chi_0, v_j\chi_0) = -L_1(v_j\chi_0), \tag{4.46}$$

$$\Gamma(\chi_0, |v|^2\chi_0) = -L_1(|v|^2\chi_0). \tag{4.47}$$

Proof. Let $u = v_*$, $u' = v'_*$. By (1.3), it holds that

$$v_j + u_j = v'_j + u'_j, \quad j = 1, 2, 3, \quad |v|^2 + |u|^2 = |v'|^2 + |u'|^2.$$

Thus

$$\begin{aligned}\Gamma(\chi_0, v_j\chi_0) =& \sqrt{M}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v - v_*)\cdot\omega|(u'_j - u_j)M(u)d\omega du \\ =& -\sqrt{M}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v - v_*)\cdot\omega|(v'_j - v_j)M(u)d\omega du\end{aligned}$$

$$
\begin{aligned}
&= - L_1(v_j\chi_0), \\
\Gamma(\chi_0, |v|^2\chi_0) &= \sqrt{M} \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{S}^2} |(v-v_*)\cdot\omega|(|u'|^2 - |u|^2)M(u)d\omega du \\
&= - \sqrt{M} \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{S}^2} |(v-v_*)\cdot\omega|(|v'|^2 - |v|^2)M(u)d\omega du \\
&= - L_1(|v|^2\chi_0),
\end{aligned}
$$

which proves the lemma. □

Proof of Theorem 1.2. First, for (1.33), set

$$
\begin{aligned}
\Lambda_\epsilon(t) = \sup_{0\le s\le t} \left(\epsilon|\ln\epsilon|^2(1+s)^{-\frac{1}{2}} + \left(1+\frac{s}{\epsilon}\right)^{-1}\right)^{-1} \\
\times \Big(\|P_{||}f_\epsilon(s)\|_{W^{2,\infty}} + \|P_\perp(f_\epsilon - u_1)(s)\|_{H^2} + \|V_\epsilon(s) - V_1(s)\|_{H^2}\Big),
\end{aligned}
\tag{4.48}
$$

where $P_{||}$ and $P_\perp$ are defined by (1.29). Note that

$$
\begin{aligned}
\|P_{||}f_\epsilon\| &\sim |(m_\epsilon)_{||}| + \left|n_\epsilon + \sqrt{\frac{2}{3}}q_\epsilon\right|, \\
\|P_\perp(f_\epsilon - u_1)\| &\sim |(m_\epsilon)_\perp - m| + \left|n_\epsilon - \sqrt{\frac{3}{2}}q_\epsilon - n + \sqrt{\frac{3}{2}}q\right| + \|P_1 f_\epsilon\|.
\end{aligned}
$$

We claim that

$$
\Lambda_\epsilon(t) \le C\delta_0, \quad \forall\, t > 0. \tag{4.49}
$$

It is straightforward to check that the estimate (1.33) follows from (4.49).

By (4.1) and (4.2), we have

$$
\begin{aligned}
V_\epsilon(t) - V_1(t) = &\left(e^{\frac{t}{\epsilon^2}\mathbb{A}_\epsilon}U_0 - Y_2(t)P_2U_0\right) + \int_0^t e^{\frac{t-s}{\epsilon^2}\mathbb{A}_\epsilon}G_3(s)ds \\
&+ \int_0^t \left(\frac{1}{\epsilon}e^{\frac{t-s}{\epsilon^2}\mathbb{A}_\epsilon}G_4(s) - Y_2(t-s)H_3(s)\right)ds \\
=: &\, I_1 + I_2 + I_3,
\end{aligned}
\tag{4.50}
$$

$$
f_\epsilon(t) - u_1(t) = \left(e^{\frac{t}{\epsilon^2}\mathbb{B}_\epsilon}f_0 - Y_1(t)P_0f_0\right) + \int_0^t \left(e^{\frac{t-s}{\epsilon^2}\mathbb{B}_\epsilon}G_1(s) - Y_1(t-s)H_1(s)\right)ds
$$

$$
\begin{aligned}
&+\int_0^t \left(\frac{1}{\epsilon}e^{\frac{t-s}{\epsilon^2}\mathbb{B}_\epsilon}G_2(s)-Y_1(t-s)\mathrm{div}_x H_2(s)\right)ds\\
&=: I_4+I_5+I_6. \qquad (4.51)
\end{aligned}
$$

By (3.30), (3.80) and (3.81), we can estimate I_1 and I_4 as follows.

$$
\|I_1\|_{H^2}\le C\delta_0\left(\epsilon(1+t)^{-\frac{3}{4}}+e^{-\frac{bt}{\epsilon^2}}\right), \qquad (4.52)
$$

$$
\|P_{||}I_4\|_{W^{2,\infty}}\le C\delta_0\left(\epsilon(1+t)^{-\frac{5}{2}}+\left(1+\frac{t}{\epsilon}\right)^{-1}\right), \qquad (4.53)
$$

$$
\|P_\perp I_4\|_{H^2}\le C\delta_0\left(\epsilon(1+t)^{-\frac{5}{4}}+e^{-\frac{bt}{\epsilon^2}}\right). \qquad (4.54)
$$

By (3.30), (4.32) and noting that $P_2G_3=0$, we have

$$
\begin{aligned}
\|I_2\|_{H^2}&\le C\int_0^t\left(\epsilon(1+t-s)^{-\frac{3}{4}}+e^{-\frac{b(t-s)}{\epsilon^2}}\right)\left(\|G_3\|_{H^3}+\|G_3\|_{L^1}\right)ds\\
&\quad +C\int_0^t \epsilon^2(1+t-s)^{-2}\|\nabla_x^2G_3\|_{H^2}ds\\
&\le C\delta_0^2\int_0^t\left(\epsilon(1+t-s)^{-\frac{3}{4}}+e^{-\frac{b(t-s)}{\epsilon^2}}\right)(1+s)^{-\frac{3}{2}}ds\\
&\le C\delta_0^2\epsilon(1+t)^{-\frac{3}{4}}, \qquad (4.55)
\end{aligned}
$$

where we had used (refer to (4.32))

$$
\|G_3(s)\|_{H^4}+\|G_3(s)\|_{L^1}\le C\|U_\epsilon(s)\|^2_{D^5_1}\le C\|U_0\|^2_{D^7_1\cap L^1}(1+s)^{-\frac{3}{2}}.
$$

To estimate I_3, we decompose

$$
\begin{aligned}
I_3&=\int_0^t\left(\frac{1}{\epsilon}e^{\frac{t-s}{\epsilon^2}\mathbb{A}_\epsilon}G_4(s)-Y_2(t-s)Z_0(s)\right)ds\\
&\quad+\int_0^t\Big(Y_2(t-s)Z_0(s)-Y_2(t-s)H_3(s)\Big)ds\\
&=: I_{31}+I_{32}, \qquad (4.56)
\end{aligned}
$$

where $Z_0=(P_d(v\cdot\nabla_xL_1^{-1}G_{41}),(L_1^{-1}G_{41},v\chi_0),0)$. Thus, it follows from (3.52) and (4.32) that

$$
\begin{aligned}
\|I_{31}\|_{H^2} \leq & \, C \int_0^t \left(\epsilon(1+t-s)^{-\frac{3}{4}} + \frac{1}{\epsilon} e^{-\frac{b(t-s)}{\epsilon^2}} \right) (\|G_4\|_{H^4} + \|G_4\|_{L^1})\, ds \\
& + C \int_0^t \epsilon^2 (1+t-s)^{-2} \|\nabla_x^2 G_4\|_{H^2} ds \\
& \leq C\delta_0^2 \int_0^t \left(\epsilon(1+t-s)^{-\frac{3}{4}} + \frac{1}{\epsilon} e^{-\frac{b(t-s)}{\epsilon^2}} \right) (1+s)^{-\frac{3}{2}} ds \\
& \leq C\delta_0^2 \epsilon (1+t)^{-\frac{3}{4}}. \qquad (4.57)
\end{aligned}
$$

By Lemma 4.7, we obtain

$$
\begin{aligned}
(L_1^{-1} G_{41}, v\chi_0) &= (-L_1^{-1}[(v\times B_\epsilon)\cdot \nabla_v P_0 f_\epsilon] + L_1^{-1}\Gamma(P_d g_\epsilon, P_0 f_\epsilon) + L_1^{-1}\tilde{R}, v\chi_0) \\
&= (-L_1^{-1}[(v\times B_\epsilon)\cdot m_\epsilon \chi_0] - \rho_\epsilon (m_\epsilon \cdot v)\chi_0, v\chi_0) + (L_1^{-1}\tilde{R}, v\chi_0) \\
&= \eta(m_\epsilon \times B_\epsilon) - \rho_\epsilon m_\epsilon + (L_1^{-1}\tilde{R}, v\chi_0), \qquad (4.58)
\end{aligned}
$$

where

$$
\tilde{R} = -(v\times B_\epsilon)\cdot \nabla_v P_1 f_\epsilon + \Gamma(P_r g_\epsilon, f_\epsilon) + \Gamma(P_d g_\epsilon, P_1 f_\epsilon). \qquad (4.59)
$$

Thus, by noting that $m_\epsilon - m = (m_\epsilon)_{||} + (m_\epsilon)_\perp - m$ and by using (4.58), (4.24)–(4.26) and (3.74), we have

$$
\begin{aligned}
\|I_{32}\|_{H^2} \leq & \, C \int_0^t \left[(1+t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}} e^{-\frac{\eta}{2}(t-s)} \right] \\
& \times \|(L_1^{-1} G_{41}, v\chi_0) + (\rho m - \eta m \times B)\|_{H_x^2} ds \\
\leq & \, C \int_0^t (t-s)^{-\frac{1}{2}} \Big[\|m\|_{H_x^2} (\|\rho_\epsilon - \rho\|_{H_x^2} + \|B_\epsilon - B\|_{H_x^2}) \\
& + (\|(m_\epsilon)_{||}\|_{W_x^{2,\infty}} + \|(m_\epsilon)_\perp - m\|_{H_x^2})(\|\rho_\epsilon\|_{H_x^2} + \|B_\epsilon\|_{H_x^2}) \\
& + \|U_\epsilon\|_{H^2} (\|P_1 f_\epsilon\|_{H^2} + \|P_r g_\epsilon\|_{H^2}) \Big] ds \\
\leq & \, C\delta_0 \Lambda_\epsilon(t) \int_0^t (t-s)^{-\frac{1}{2}} \left(\epsilon |\ln\epsilon|^2 (1+s)^{-\frac{1}{2}} + \left(1 + \frac{s}{\epsilon}\right)^{-1} \right) (1+s)^{-\frac{3}{4}} ds \\
& + C\delta_0^2 \int_0^t (t-s)^{-\frac{1}{2}} \left(\epsilon(1+s)^{-\frac{3}{4}} + e^{-\frac{bs}{4\epsilon^2}} \right) (1+s)^{-\frac{3}{4}} ds. \qquad (4.60)
\end{aligned}
$$

Denote

$$J_0 = \int_0^t (t-s)^{-\frac{1}{2}} \left(1 + \frac{s}{\epsilon}\right)^{-1} (1+s)^{-\frac{3}{4}} ds, \tag{4.61}$$

$$J_1 = \int_0^t (t-s)^{-\frac{1}{2}} e^{-\frac{bs}{4\epsilon^2}} (1+s)^{-\frac{3}{4}} ds. \tag{4.62}$$

For J_0, it holds that

$$\begin{aligned} J_0 &= \left(\int_0^{t/2} + \int_{t/2}^t \right) (t-s)^{-\frac{1}{2}} \left(1 + \frac{s}{\epsilon}\right)^{-1} (1+s)^{-\frac{3}{4}} ds \\ &\le Ct^{-\frac{1}{2}} \int_0^{t/2} \left(1 + \frac{s}{\epsilon}\right)^{-1} (1+s)^{-\frac{3}{4}} ds \\ &\quad + C\left(1 + \frac{t}{\epsilon}\right)^{-1} (1+t)^{-\frac{3}{4}} \int_{t/2}^t (t-s)^{-\frac{1}{2}} ds \\ &\le C\epsilon |\ln \epsilon|^2 (1+t)^{-\frac{1}{2}} + C\left(1 + \frac{t}{\epsilon}\right)^{-1}, \end{aligned} \tag{4.63}$$

where we have used

$$\begin{aligned} & t^{-\frac{1}{2}} \int_0^{t/2} \left(1 + \frac{s}{\epsilon}\right)^{-1} (1+s)^{-\frac{3}{4}} ds \\ \le & \begin{cases} t^{-\frac{1}{2}} \int_0^t ds \le \sqrt{t} \le C\sqrt{\epsilon}(1+\frac{t}{\epsilon})^{-1}, & t \le \epsilon, \\ t^{-\frac{1}{2}} \int_0^1 (1+\frac{s}{\epsilon})^{-1} ds \le Ct^{-\frac{1}{2}} \epsilon |\ln \epsilon| \le C\epsilon |\ln \epsilon|^2 + C(1+\frac{t}{\epsilon})^{-1}, & \epsilon < t \le 1, \\ t^{-\frac{1}{2}} \left(\int_0^1 (1+\frac{s}{\epsilon})^{-1} ds + \epsilon \int_1^t s^{-\frac{7}{4}} ds \right) \le C\epsilon |\ln \epsilon| (1+t)^{-\frac{1}{2}}, & t > 1. \end{cases} \end{aligned}$$

For J_1, it holds that

$$\begin{aligned} J_1 &= \left(\int_0^{t/2} + \int_{t/2}^t \right) (t-s)^{-\frac{1}{2}} e^{-\frac{bs}{4\epsilon^2}} (1+s)^{-\frac{3}{4}} ds \\ &\le Ct^{-\frac{1}{2}} \int_0^{t/2} e^{-\frac{bs}{4\epsilon^2}} ds + Ce^{-\frac{bt}{4\epsilon^2}} (1+t)^{-\frac{3}{4}} \int_{t/2}^t (t-s)^{-\frac{1}{2}} ds \\ &\le C\epsilon^2 t^{-\frac{1}{2}} (1 - e^{-\frac{bt}{4\epsilon^2}}) + Ct^{\frac{1}{2}} e^{-\frac{bt}{4\epsilon^2}} (1+t)^{-\frac{3}{4}} \end{aligned}$$

$$\leq C\epsilon(1+t)^{-\frac{1}{2}}, \tag{4.64}$$

where we have used

$$t^{-\frac{1}{2}}(1-e^{-\frac{bt}{4\epsilon^2}}) \leq \begin{cases} C\epsilon^{-1}, & t \leq 1, \\ C(1+t)^{-\frac{1}{2}}, & t > 1. \end{cases}$$

Thus, it follows from (4.60)–(4.64) that

$$\|I_{32}\|_{H^2} \leq C(\delta_0^2 + \delta_0\Lambda_\epsilon(t))\left(\epsilon|\ln\epsilon|^2(1+t)^{-\frac{1}{2}} + \left(1+\frac{t}{\epsilon}\right)^{-1}\right). \tag{4.65}$$

By combining (4.52), (4.55), (4.56), (4.57) and (4.65), we obtain

$$\|V_\epsilon(t) - V_1(t)\|_{H^2} \leq C(\delta_0 + \delta_0^2 + \delta_0\Lambda_\epsilon(t))\left(\epsilon|\ln\epsilon|^2(1+t)^{-\frac{1}{2}} + \left(1+\frac{t}{\epsilon}\right)^{-1}\right). \tag{4.66}$$

To estimate I_5, we decompose

$$\begin{aligned} I_5 &= \int_0^t \left(e^{\frac{t-s}{\epsilon^2}\mathbb{B}_\epsilon} G_1(s) - Y_1(t-s)P_0G_1(s)\right) ds \\ &\quad + \int_0^t \Big(Y_1(t-s)P_0G_1(s) - Y_1(t-s)H_1(s)\Big) ds \\ &=: I_{51} + I_{52}. \end{aligned} \tag{4.67}$$

By Lemma 3.9, (4.29), and noting that $P_{||}Y_1(t) = 0$, $P_\perp Y_1(t) = Y_1(t)$, we obtain

$$\begin{aligned} \|P_{||}I_{51}\|_{W^{2,\infty}} &\leq C\int_0^t \left(\epsilon(1+t-s)^{-\frac{3}{2}} + \left(1+\frac{t-s}{\epsilon}\right)^{-1}\right)(\|G_1\|_{H^5} + \|G_1\|_{W^{5,1}})\, ds \\ &\leq C\delta_0^2 \int_0^t \left(\epsilon(1+t-s)^{-\frac{3}{2}} + \left(1+\frac{t-s}{\epsilon}\right)^{-1}\right)\left((1+s)^{-\frac{3}{2}} + \frac{1}{\epsilon}e^{-\frac{bs}{4\epsilon^2}}\right) ds \\ &\leq C\delta_0^2\epsilon|\ln\epsilon|(1+t)^{-\frac{3}{4}}, \end{aligned} \tag{4.68}$$

$$\begin{aligned} \|P_\perp I_{51}\|_{H^2} &\leq C\int_0^t \left(\epsilon(1+t-s)^{-\frac{3}{4}} + e^{-\frac{b(t-s)}{\epsilon^2}}\right)(\|G_1\|_{H^3} + \|G_1\|_{L^1})\, ds \\ &\leq C\delta_0^2 \int_0^t \left(\epsilon(1+t-s)^{-\frac{3}{4}} + e^{-\frac{b(t-s)}{\epsilon^2}}\right)\left((1+s)^{-\frac{3}{2}} + \frac{1}{\epsilon}e^{-\frac{bs}{4\epsilon^2}}\right) ds \\ &\leq C\delta_0^2\epsilon(1+t)^{-\frac{3}{4}}, \end{aligned} \tag{4.69}$$

where we have used

$$
\begin{aligned}
\int_0^t \left(1+\frac{t-s}{\epsilon}\right)^{-1}(1+s)^{-\frac{3}{2}}ds &\le \int_0^t \left(1+\frac{t-s}{\epsilon}\right)^{-1} ds \\
&= \epsilon \ln\left(1+\frac{t}{\epsilon}\right) \le \epsilon|\ln\epsilon|, \quad t\le 1; \\
\int_0^t \left(1+\frac{t-s}{\epsilon}\right)^{-1}(1+s)^{-\frac{3}{2}}ds &= \left(\int_0^{t/2} + \int_{t/2}^t\right)\left(1+\frac{t-s}{\epsilon}\right)^{-1}(1+s)^{-\frac{3}{2}}ds \\
&\le C\left(1+\frac{t}{\epsilon}\right)^{-1} + C(1+t)^{-\frac{3}{2}}\epsilon\ln\left(1+\frac{t}{\epsilon}\right) \\
&\le C\epsilon|\ln\epsilon|(1+t)^{-1}, \quad t>1.
\end{aligned}
$$

Since

$$
P_0G_1 = (\rho_\epsilon E_\epsilon)\cdot v\chi_0 + \sqrt{\frac{2}{3}}(u_\epsilon\cdot E_\epsilon)\chi_4 + \frac{1}{\epsilon}(u_\epsilon\times B_\epsilon)\cdot v\chi_0,
$$

it follows from (4.67) that

$$
\begin{aligned}
I_{52} &= \int_0^t Y_1(t-s)\Big((\rho_\epsilon E_\epsilon - \rho E)\cdot v\chi_0 + \sqrt{\frac{2}{3}}(u_\epsilon\cdot E_\epsilon)\chi_4\Big)ds \\
&\quad + \int_0^t Y_1(t-s)\left(\frac{1}{\epsilon}(u_\epsilon\times B_\epsilon) - j\times B\right)\cdot v\chi_0 ds \\
&=: J_{51}+J_{52}.
\end{aligned} \tag{4.70}
$$

For J_{51}, it follows from Lemmas 3.12 and 4.4 that

$$
\begin{aligned}
\|J_{51}\|_{H^2} &\le C\int_0^t (1+t-s)^{-\frac{3}{4}}\Big(\|\rho_\epsilon E_\epsilon-\rho E\|_{L^1_x\cap H^2_x} + \|u_\epsilon\cdot E_\epsilon\|_{L^1_x\cap H^2_x}\Big)ds \\
&\le C\delta_0\Lambda_\epsilon(t)\int_0^t (1+t-s)^{-\frac{3}{4}}\left(\epsilon|\ln\epsilon|^2(1+s)^{-\frac{1}{2}} + \left(1+\frac{s}{\epsilon}\right)^{-1}\right)(1+s)^{-\frac{3}{4}}ds \\
&\quad + C\delta_0^2\int_0^t (1+t-s)^{-\frac{3}{4}}\left(\epsilon(1+s)^{-\frac{3}{4}} + e^{-\frac{bs}{\epsilon^2}}\right)(1+s)^{-\frac{3}{4}}ds \\
&\le C(\delta_0^2+\delta_0\Lambda_\epsilon(t))\epsilon|\ln\epsilon|^2(1+t)^{-\frac{3}{4}}.
\end{aligned} \tag{4.71}
$$

By (4.18) and (4.58), we have

$$\begin{aligned}\frac{1}{\epsilon}u_\epsilon &= (L_1^{-1}[\epsilon\partial_t(P_r g_\epsilon) + P_r(v\cdot\nabla_x P_r g_\epsilon) - \epsilon G_{31} - G_{41}], v\chi_0) + \eta(E_\epsilon - \nabla_x\rho_\epsilon)\\ &= \epsilon\partial_t(L_1^{-1}(P_r g_\epsilon), v\chi_0) + (L_1^{-1}P_r(v\cdot\nabla_x P_r g_\epsilon - \epsilon G_{31} - \tilde{R}), v\chi_0)\\ &\quad + \eta(E_\epsilon - \nabla_x\rho_\epsilon) + (\rho_\epsilon m_\epsilon - \eta m_\epsilon \times B_\epsilon),\end{aligned}$$

which leads to

$$\begin{aligned}\frac{1}{\epsilon}(u_\epsilon \times B_\epsilon) &= \epsilon\partial_t[(L_1^{-1}(P_r g_\epsilon), v\chi_0)\times B_\epsilon] + \epsilon(L_1^{-1}(P_r g_\epsilon), v\chi_0)\times(\nabla_x \times E_\epsilon)\\ &\quad + (L_1^{-1}P_r(v\cdot\nabla_x P_r g_\epsilon - \epsilon G_{31} - \tilde{R}), v\chi_0)\times B_\epsilon + j_\epsilon \times B_\epsilon.\end{aligned}$$

Thus

$$\begin{aligned}J_{52} &= \int_0^t Y_1(t-s)[j_\epsilon \times B_\epsilon - j\times B]\cdot v\chi_0 ds\\ &\quad + \epsilon\int_0^t Y_1(t-s)\partial_s[(L_1^{-1}(P_r g_\epsilon), v\chi_0)\times B_\epsilon]\cdot v\chi_0 ds\\ &\quad + \epsilon\int_0^t Y_1(t-s)[(L_1^{-1}(P_r g_\epsilon), v\chi_0)\times(\nabla_x\times E_\epsilon)]\cdot v\chi_0 ds\\ &\quad + \int_0^t Y_1(t-s)[(L_1^{-1}P_r(v\cdot\nabla_x P_r g_\epsilon - \epsilon G_{31} - \tilde{R}), v\chi_0)\times B_\epsilon]\cdot v\chi_0 ds\\ &=: J_{52}^1 + J_{52}^2 + J_{52}^3 + J_{52}^4. \end{aligned} \tag{4.72}$$

We estimate J_{52}^i, $i = 1,2,3,4$ as follows. By Lemma 3.12, we obtain

$$\begin{aligned}\|J_{52}^1\|_{H^2} &\le C\int_0^t \|Y_1(t-s)\|\|(j_\epsilon\times B_\epsilon - j\times B)\|_{H_x^1\cap L_x^1} ds\\ &\quad + C\int_0^t \|\nabla_x Y_1(t-s)\|\|\nabla_x(j_\epsilon\times B_\epsilon - j\times B)\|_{L_x^2\cap L_x^1} ds\\ &\le C\delta_0\Lambda_\epsilon(t)\int_0^t \Big((1+t-s)^{-\frac{3}{4}} + (t-s)^{-\frac{1}{2}}e^{-c(t-s)}\Big)\\ &\qquad \times\left(\epsilon|\ln\epsilon|^2(1+s)^{-\frac{1}{2}} + \left(1+\frac{s}{\epsilon}\right)^{-1}\right)(1+s)^{-\frac{3}{4}} ds\end{aligned}$$

$$\leq C\delta_0\Lambda_\epsilon(t)\left(\epsilon|\ln\epsilon|^2(1+t)^{-\frac{3}{4}}+\left(1+\frac{t}{\epsilon}\right)^{-1}\right), \tag{4.73}$$

$$\begin{aligned}\|J_{52}^3\|_{H^2} &\leq C\epsilon\int_0^t(1+t-s)^{-\frac{3}{4}}\|(L_1^{-1}(P_rg_\epsilon),v\chi_0)\times(\nabla_x\times E_\epsilon)\|_{L_x^1\cap H_x^2}ds\\ &\leq C\epsilon\int_0^t(1+t-s)^{-\frac{3}{4}}\|P_rg_\epsilon\|_{H^2}\|\nabla_x\times E_\epsilon\|_{H_x^2}ds\\ &\leq C\delta_0^2\epsilon(1+t)^{-\frac{3}{4}},\end{aligned} \tag{4.74}$$

and

$$\begin{aligned}\|J_{52}^4\|_{H^2} &\leq C\int_0^t(1+t-s)^{-\frac{3}{4}}\Big[(\|\nabla_xP_rg_\epsilon\|_{H^2}+\epsilon\|E_\epsilon\|_{H_x^2}\|f_\epsilon\|_{H^2})\|B_\epsilon\|_{H_x^2}\\ &\qquad+(\|(\rho_\epsilon,B_\epsilon)\|_{H_x^2}\|P_1f_\epsilon\|_{H^2}+\|P_rg_\epsilon\|_{H^2}\|f_\epsilon\|_{H^2})\|B_\epsilon\|_{H_x^2}\Big]ds\\ &\leq C\delta_0^2\int_0^t(1+t-s)^{-\frac{3}{4}}\left(\epsilon(1+s)^{-\frac{3}{4}}+e^{-\frac{bs}{\epsilon^2}}\right)(1+s)^{-\frac{3}{4}}ds\\ &\leq C\delta_0^2\epsilon(1+t)^{-\frac{3}{2}}.\end{aligned} \tag{4.75}$$

Moreover, it holds that

$$\begin{aligned}\|J_{52}^2\|_{H^2} &= \epsilon\Big\|[(L_1^{-1}(P_rg_\epsilon),v\chi_0)\times B_\epsilon]\cdot v\chi_0-Y_1(t)[(L_1^{-1}(P_rg_0),v\chi_0)\times B_0]\cdot v\chi_0\\ &\qquad+\int_0^t\partial_tY_1(t-s)[(L_1^{-1}(P_rg_\epsilon),v\chi_0)\times B_\epsilon]\cdot v\chi_0ds\Big\|_{H^2}\\ &\leq C\delta_0^2\epsilon(1+t)^{-\frac{3}{4}}+C\epsilon\int_0^t(1+t-s)^{-\frac{7}{4}}\|P_rg_\epsilon\|_{H^4}\|B_\epsilon\|_{H_x^4}ds\\ &\leq C\delta_0^2\epsilon(1+t)^{-\frac{3}{2}},\end{aligned} \tag{4.76}$$

where we have used

$$\|\partial_tY_1(t)f_0\|_{L^2}^2\leq\int_{\mathbb{R}^3}\Big|\sum_{j=0,2,3}a_j|\xi|^2e^{-a_j|\xi|^2t}\Big|^2\|\hat f_0\|^2d\xi\leq C(1+t)^{-\frac{7}{2}}\|f_0\|_{L^1\cap H^2}^2.$$

Thus, it follows from (4.70)–(4.76) that

$$\|I_{52}\|_{H^2}\leq C(\delta_0^2+\delta_0\Lambda_\epsilon(t))\left(\epsilon|\ln\epsilon|^2(1+t)^{-\frac{1}{2}}+\left(1+\frac{t}{\epsilon}\right)^{-1}\right). \tag{4.77}$$

To estimate I_3, we decompose

$$\begin{aligned} I_6 &= \int_0^t \left(\frac{1}{\epsilon} e^{\frac{t-s}{\epsilon^2}\mathbb{B}_\epsilon} G_2 - Y_1(t-s)P_0(v\cdot\nabla_x L^{-1}G_2)\right) ds \\ &\quad + \int_0^t \Big(Y_1(t-s)P_0(v\cdot\nabla_x L^{-1}G_2) - Y_1(t-s)\mathrm{div}_x H_2\Big) ds \\ &=: I_{61} + I_{62}. \end{aligned} \tag{4.78}$$

By (3.52), (4.30), and noting that $P_0 G_2 = 0$, we have

$$\begin{aligned} \|P_{||} I_{61}\|_{W^{2,\infty}} &\le C\int_0^t \left(\epsilon(1+t-s)^{-\frac{3}{2}} + \left(1+\frac{t-s}{\epsilon}\right)^{-1} + \frac{1}{\epsilon} e^{-\frac{b(t-s)}{\epsilon^2}}\right) \\ &\qquad \times (\|G_2\|_{H^6} + \|G_2\|_{W^{6,1}})\, ds \\ &\le C\delta_0^2 \int_0^t \left(\epsilon(1+t-s)^{-\frac{3}{2}} + \left(1+\frac{t-s}{\epsilon}\right)^{-1} + \frac{1}{\epsilon} e^{-\frac{b(t-s)}{\epsilon^2}}\right)(1+s)^{-\frac{3}{2}} ds \\ &\le C\delta_0^2 \epsilon |\ln\epsilon| (1+t)^{-1}, \end{aligned} \tag{4.79}$$

$$\begin{aligned} \|P_\perp I_{61}\|_{H^2} &\le C\int_0^t \left(\epsilon(1+t-s)^{-\frac{3}{4}} + \frac{1}{\epsilon} e^{-\frac{b(t-s)}{\epsilon^2}}\right) (\|G_2\|_{H^4} + \|G_2\|_{L^1})\, ds \\ &\le C\delta_0^2 \int_0^t \left(\epsilon(1+t-s)^{-\frac{3}{4}} + \frac{1}{\epsilon} e^{-\frac{b(t-s)}{\epsilon^2}}\right) (1+s)^{-\frac{3}{2}} ds \\ &\le C\delta_0^2 \epsilon (1+t)^{-\frac{3}{4}}, \end{aligned} \tag{4.80}$$

where we had used (refer to (4.30))

$$\|G_2(s)\|_{H^6} + \|G_2(s)\|_{W^{6,1}} \le C\|U_\epsilon(s)\|_{D_1^7}^2 \le C\|U_0\|_{D_1^9\cap L^1}^2 (1+s)^{-\frac{3}{2}}. \tag{4.81}$$

To estimate I_{32}, we decompose

$$\begin{aligned} P_0(v\cdot\nabla_x L^{-1}G_2) &= P_0(v\cdot\nabla_x L^{-1}\Gamma_*(P_0 f_\epsilon, P_0 f_\epsilon)) + 2P_0(v\cdot\nabla_x L^{-1}\Gamma_*(P_0 f_\epsilon, P_1 f_\epsilon)) \\ &\quad + P_0(v\cdot\nabla_x L^{-1}\Gamma_*(P_1 f_\epsilon, P_1 f_\epsilon)) + P_0(v\cdot\nabla_x L^{-1}(v\times B_\epsilon)\cdot\nabla_v P_r g_\epsilon) \\ &=: J_1 + J_2 + J_3 + J_4, \end{aligned}$$

where $2\Gamma_*(f,g) = \Gamma(f,g) + \Gamma(g,f)$. By Lemma 4.6, we can obtain (cf. [27])

$$J_1 = -\sum_{i,j=1}^{3} \partial_i (m_\epsilon^i m_\epsilon^j) v_j \chi_0 + \frac{1}{3} \sum_{i,j=1}^{3} \partial_j (m_\epsilon^i)^2 v_j \chi_0 - \frac{5}{3} \sum_{j=1}^{3} \partial_j (m_\epsilon^j q_\epsilon) \chi_4.$$

This and the fact that $Y_1(t)P_0 f = Y_1(t)P_\perp f$ and $(\nabla_x |m_\epsilon|^2)_\perp = 0$ give

$$Y_1(t)J_1 = -Y_1(t)\mathrm{div}_x \left[(m_\epsilon \otimes m_\epsilon) \cdot v\chi_0 + \frac{5}{3}(q_\epsilon m_\epsilon)\chi_4 \right] =: Y_1(t)\mathrm{div}_x J_4. \tag{4.82}$$

Note that

$$|m_\epsilon - m| + |q_\epsilon - q| \le C(\|P_\perp(f_\epsilon - u_1)\| + \|P_{||} f_\epsilon\|).$$

Thus, by (3.124), (4.24), (4.25) and (4.82), we have

$$\begin{aligned}
\|I_{62}\|_{H^2} \le& \int_0^t \|Y_1(t-s)\mathrm{div}_x (J_4 - H_2)\|_{H^2}\, ds + \sum_{k=2}^{4} \int_0^t \|Y_1(t-s) J_k\|_{H^2}\, ds \\
\le& C\int_0^t (t-s)^{-\frac{1}{2}} \|J_4 - H_2\|_{H^2} ds + \sum_{k=2}^{4} C \int_0^t (1+t-s)^{-\frac{3}{4}} \|J_k\|_{L^1 \cap H^2} ds \\
\le& C\int_0^t (t-s)^{-\frac{1}{2}} (\|P_\perp(f_\epsilon - u_1)\|_{H^2} + \|P_{||} f_\epsilon\|_{W^{2,\infty}}) \|(f_\epsilon, u_1)\|_{H^2} ds \\
&+ C\int_0^t (1+t-s)^{-\frac{3}{4}} \left(\|P_1 f_\epsilon\|_{H^3} \|f_\epsilon\|_{H^3} + \|B_\epsilon\|_{H^3_x} \|P_r g_\epsilon\|_{H^3} \right) ds \\
\le& C\delta_0 \Lambda_\epsilon(t) \int_0^t (t-s)^{-\frac{1}{2}} \left(\epsilon |\ln \epsilon|^2 (1+s)^{-\frac{1}{2}} + \left(1 + \frac{s}{\epsilon}\right)^{-1} \right) (1+s)^{-\frac{3}{4}} ds \\
&+ C\delta_0^2 \int_0^t (1+t-s)^{-\frac{3}{4}} \left(\epsilon (1+s)^{-\frac{3}{4}} + e^{-\frac{bs}{\epsilon^2}} \right) (1+s)^{-\frac{3}{4}} ds \\
\le& C(\delta_0^2 + \delta_0 \Lambda_\epsilon(t)) \left(\epsilon |\ln \epsilon|^2 (1+t)^{-\frac{1}{2}} + \left(1 + \frac{t}{\epsilon}\right)^{-1} \right).
\end{aligned} \tag{4.83}$$

Therefore, it follows from (4.67)–(4.70), (4.79), (4.80) and (4.83) that

$$\begin{aligned}
&\|P_{||} f_\epsilon\|_{W^{2,\infty}} + \|P_\perp (f_\epsilon - u_1)\|_{H^2} \\
&\le C\left(\delta_0 + \delta_0^2 + \delta_0 \Lambda_\epsilon(t)\right) \left(\epsilon |\ln \epsilon|^2 (1+t)^{-\frac{1}{2}} + \left(1 + \frac{t}{\epsilon}\right)^{-1} \right).
\end{aligned} \tag{4.84}$$

By combining (4.66) and (4.84), we obtain

$$\Lambda_\epsilon(t) \leq C\left(\delta_0 + \delta_0^2 + \delta_0 \Lambda_\epsilon(t)\right).$$

By taking $\delta_0 > 0$ small enough, we obtain (4.49) which gives (1.33).

Next, we prove (1.34) as follows. Set

$$\begin{aligned}\Omega_\epsilon(t) = \sup_{0\leq s\leq t}(\epsilon|\ln\epsilon|)^{-1}(1+s)^{\frac{1}{2}}\big(&\|P_{||}f_\epsilon(s)\|_{W^{2,\infty}} + \|P_\perp(f_\epsilon - u_1)(s)\|_{H^2}\\ &+ \|(V_\epsilon - V_1)(s)\|_{H^2}\big).\end{aligned}$$

By (3.31), (3.82) and (3.83), I_1 and I_4 are estimated by

$$\|I_1\|_{H^2} + \|P_{||}I_4\|_{W^{2,\infty}} + \|P_\perp I_4\|_{H^2} \leq C\delta_0\epsilon(1+t)^{-\frac{3}{4}}. \tag{4.85}$$

By (3.74) and (4.58), we have

$$\begin{aligned}\|I_{32}\|_{H^2} \leq& C\int_0^t (t-s)^{-\frac{1}{2}}\|(L_1^{-1}G_{41}, v\chi_0) + (\rho m - \eta m\times B)\|_{H_x^2}ds\\ \leq& C(\delta_0^2 + \delta_0\Omega_\epsilon(t))\int_0^t (t-s)^{-\frac{1}{2}}\left(\epsilon|\ln\epsilon|(1+s)^{-\frac{1}{2}} + e^{-\frac{bs}{4\epsilon^2}}\right)(1+s)^{-\frac{3}{4}}ds\\ \leq& C(\delta_0^2 + \delta_0\Omega_\epsilon(t))\epsilon|\ln\epsilon|(1+t)^{-\frac{1}{2}}.\end{aligned} \tag{4.86}$$

By combining (4.85), (4.55), (4.56), (4.57) and (4.86), we obtain

$$\|V_\epsilon(t) - V_1(t)\|_{H^2} \leq C\left(\delta_0 + \delta_0^2 + \delta_0\Omega_\epsilon(t)\right)\epsilon|\ln\epsilon|(1+t)^{-\frac{1}{2}}. \tag{4.87}$$

Similarly,

$$\|P_{||}f_\epsilon\|_{W^{2,\infty}} + \|P_\perp(f_\epsilon - u_1)\|_{H^2} \leq C\left(\delta_0 + \delta_0^2 + \delta_0\Omega_\epsilon(t)\right)\epsilon|\ln\epsilon|(1+t)^{-\frac{1}{2}}. \tag{4.88}$$

By (4.87) and (4.88), we obtain

$$\Omega_\epsilon(t) \leq C\delta_0 + C\delta_0^2 + C\delta_0\Omega_\epsilon(t).$$

By taking $\delta_0 > 0$ small enough, we obtain (1.34). And this completes the proof of the theorem. □

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Data availability

No data was used for the research described in the article.

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