European Journal of Operational Research 324 (2025) 941–952

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/eor

Decision support

Randomized strategyproof mechanisms with best of both worlds fairness and efficiency

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ARTICLE INFO

Keywords:
Multi-agent systems
Resource allocation
Mechanism design
Strategyproof
Randomization

ABSTRACT

We study the problem of mechanism design for allocating a set of indivisible items among agents with private preferences on items. We aim to design a mechanism that is strategyproof (in which agents find it optimal to report their true preferences) and ensures a certain level of fairness and efficiency. We first establish that no deterministic mechanism can simultaneously be strategyproof, fair, and efficient for the allocation of indivisible chores. We then introduce randomness to address this impossibility. For allocating indivisible chores, we propose randomized mechanisms that are strategyproof in expectation as well as ex-ante and ex-post (best of both worlds) fair and efficient. For allocating mixed items—where an item may be a good (positive utility) for one agent and a chore (negative utility) for another, we propose randomized mechanisms that are strategyproof in expectation while ensuring fairness and efficiency for two-agent scenarios.

1. Introduction

Resource allocation is a fundamental problem that has garnered significant attention at the intersection of operations research, economics, and computer science (Brandt et al., 2016; Ibaraki & Katoh, 1988; Moulin, 2003). Among different types of resources, indivisible items stand out as a prominently studied model. This model has attracted interest as a metaphor for real-world allocation problems, such as outpatient diagnostic tests scheduling (Zaerpour et al., 2017), school choice (Abdulkadiroğlu & Sönmez, 2003), order dispatching in ride-hailing platform (Qin et al., 2020) and conference paper assignment (Lian et al., 2018).

In most practical allocation problems, individual preferences are private. This necessitates allocation protocols to operate based on announced preferences. Knowing that declared preferences determine the allocation outcome, participants may misreport their preferences to secure more favorable results. False preferences significantly undermine outcomes if the allocation protocol is not designed to address such manipulation. Some recent studies have addressed such strategic behavior in supply chain management (Karaenke et al., 2020), production system (Norde et al., 2016) and project management (Chen & Hall, 2021).

To circumvent the challenge of strategic manipulation, an ideal allocation protocol should incentivize truthful reporting, making it the optimal strategy for individuals. Such allocation protocols are termed *strategyproof* mechanisms. Following the seminal work of Nisan and Ronen (2001), research has focused on strategyproof mechanisms for indivisible items (without money payment[1]) to achieve fair outcomes (Amanatidis et al., 2017, 2016; Bezáková & Dani, 2005; Markakis & Psomas, 2011). Many argue that, in general, strategyproofness prohibits mechanisms from achieving fair or even approximately fair outcomes. In particular, for indivisible goods, Amanatidis et al. (2017) consider the notion of envy-free up to one item[2] (EF1) and show that EF1 and strategyproofness are incompatible even for two agents and five items when agents' valuations are additive. To escape from the strong impossibility result, follow-up studies have explored restricted preference domains, such as the binary margin where the marginal valuation of an item is either zero or one. These studies demonstrate that strategyproofness, fairness and efficiency can be achieved simultaneously under the setting of binary margin (Babaioff et al., 2021; Barman & Verma, 2022; Halpern et al., 2020).

The aforementioned positive results are mainly established in the context of allocating goods, items with positive utilities. However, in practical scenarios, the items to be allocated extend beyond goods to chores, which impose costs on individuals. The chore allocation scenario includes, but is not limited to, workload assignment in manufacturing industry (Pereira & Ritt, 2023), job scheduling (Etesami,

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[1] Money transfer is either infeasible or illegal in practice.
[2] No one envies another after hypothetically eliminating one good from the bundle of the latter.

https://doi.org/10.1016/j.ejor.2025.02.027
Received 8 March 2024; Accepted 19 February 2025
Available online 27 February 2025

2022) and household chores division among family members (Igarashi & Yokoyama, 2023). Moreover, whether an item is a good or a chore is not only subjective but also agent dependent. For example, a pesticide may have a positive utility for a farmer who wants to protect his crops from pests, but a negative utility for a beekeeper who loses his bees due to the toxic chemicals. Such a setting is referred to as *mixed items*. For chores or mixed items, whether strategyproofness, efficiency and fairness can be achieved simultaneously is not well understood. In particular, an approach applied successfully in the setting of goods does not convert to chores (or mixed items) straightforwardly. Halpern et al. (2020) propose a fair, efficient and strategyproof mechanism for indivisible goods and additive valuations with a binary margin. The proposed mechanism relies on the allocation that maximizes the product of individuals' utilities, known as Nash welfare. The allocation with the maximum Nash welfare allocation is EF1 and Pareto optimal for goods (Caragiannis et al., 2019), while no (known) equivalent rule with fairness and efficiency guarantees for chores. This leads to our first research question:

> For allocating chores or mixed items, whether strategyproofness, fairness and efficiency are compatible? In general or in a restricted preference domain?

One way to address these impossibilities is by restricting preference domains. Another approach is to relax deterministic strategyproofness to strategyproofness *in expectation* using lotteries. Informally, a randomized mechanism is considered strategyproof in expectation if every agent's expected utility is maximized when they truthfully reveal their preferences. For allocating indivisible items, achieving performance guarantees before the randomness is realized (ex-ante) along with strategyproofness in expectation is not a hard task. For instance, consider a mechanism that selects an agent uniformly at random and assigns all items to that agent. This mechanism is strategyproof in expectation as reported preferences are ignored. In the allocation of such a mechanism, no agent prefers the bundle of another before realization. However, such a mechanism unlikely guarantees fairness or efficiency after the randomness is realized (ex-post). An ideal randomized mechanism should have both ex-ante and ex-post performance guarantees, also referred to as the "best of both worlds" (Akrami et al., 2023; Aziz, Freeman et al., 2024; Babaioff et al., 2022; Hylland & Zeckhauser, 1979). This then leads to our second research question:

> For allocating chores or mixed items, whether strategyproofness in expectation, best of both worlds fairness and efficiency are compatible? In general or in a restricted preference domain?

1.1. Main results

This paper focuses on fair and efficient allocation mechanisms, addressing (i) *indivisible chores*, where all items are chores for agents, and (ii) *mixed items*, where an item may be a good for one agent but a chore for another. In Section 3, we study deterministic mechanisms and present two impossibility results. We analyze the widely-studied sequential picking mechanism and show it fails to guarantee Pareto optimality. Then we establish an impossibility result for all deterministic mechanisms, which informally states that deterministic strategyproofness, equitability-based fairness notions and Pareto optimality are incompatible, even for additive valuations with a binary margin. Given this strong impossibility result, we relax deterministic strategyproofness to strategyproofness in expectation and concentrate on randomized mechanisms in the subsequent sections.

In Section 4, we consider randomized mechanisms and are concerned with allocation of indivisible chores. We focus on the *k-restricted additive* valuations, where a single item has k inherent values and the item's value for an agent is either from the k inherent values or zero. For 1-restricted additive valuations, we propose randomized mechanisms that are group-strategyproof[3] in expectation and have both ex-ante and ex-post fairness and efficiency guarantees. However, when extended to 2-restricted additive valuation functions, no allocation possesses the ex-ante and ex-post properties of the allocation returned by the proposed mechanisms in the 1-restricted setting. Moving on to Section 5, we consider the setting of mixed items and propose randomized mechanisms for two agents that are strategyproof in expectation. These mechanisms also ensure both ex-ante and ex-post fairness and efficiency.

1.2. Other related works

The exploration of strategyproof mechanisms with guaranteed properties began with Hylland and Zeckhauser (1979). Subsequently, the existence of strategyproof mechanisms with specific properties became a key focus. Gale (1987) raises the question of whether we can find a "nice" mechanism that satisfies strategyproofness, efficiency, and other distributional properties, simultaneously. Zhou (1990) partially resolves Gale's conjecture, proving that no mechanism for assigning n items to n agents can be simultaneously strategyproof, Pareto optimal, and symmetric[4] when $n \geq 3$. Then Pápai (2001b) studies the problem of allocating a single indivisible item to several agents without money transfer and examines possible extra properties that a strategyproof mechanism can have. In another study, Pápai (2001a) considers the scenario where each agent can receive more than one item and presents a characterization of subclass of strategyproof mechanisms. Bezáková and Dani (2005) consider the problem of allocating m items to two agents and show that no strategyproof mechanism can maximize the value of the worse agent. Markakis and Psomas (2011) present a lower bound on the values of agents in the worst-case scenario and also show that no deterministic strategyproof mechanism can achieve a $(2/3)$-approximation of the worst-case bound.

As indicated in the aforementioned studies, pursuing an optimal strategyproof mechanism is sometimes impossible. Procaccia and Tennenholtz (2013) start to consider strategyproof mechanisms that achieve an approximation of the optimal objective when the optimal solution is computationally intractable or when no such mechanism exists. Amanatidis et al. (2016) focus on indivisible goods and present a deterministic strategyproof mechanism that uses ordinal preferences (rankings over items) and returns an allocation with $\Omega(1/m)$-approximation on maximin-share fairness, where m is the number of items. They complement the result by showing that, with ordinal preferences, no deterministic strategyproof mechanism can be better than $(1/2)$-approximation. For allocating indivisible chores, Aziz, Li et al. (2024) present deterministic and randomized mechanisms that take ordinal preferences as inputs and return allocations that guarantee, respectively, $O(\log(m/n))$-approximation and $O(\sqrt{\log n})$-approximation of maximin share, where n is the number of agents.

There are also some studies concerning strategyproofness along with other fairness criteria such as EF1. Amanatidis et al. (2017) demonstrate that no deterministic strategyproof mechanism guarantees EF1, even there are only two agents with additive valuations. To achieve positive results, subsequent studies focus on domains of restricted agent preferences. Halpern et al. (2020) focus on additive valuations with a binary margin and indicate that the rule of maximizing Nash welfare with a lexicographic tie-breaking rule is group strategyproof, EF1 and Pareto optimal. Babaioff et al. (2021) consider submodular and monotone valuation functions with a binary margin and show that the Lorenz dominating[5] rule is strategyproof, EF1 and Pareto optimal. A follow-up work (Barman & Verma, 2022) show that the proposed mechanism in Babaioff et al. (2021) is indeed group strategyproof.

[3] No group of agents has an incentive to misreport.

[4] In a symmetric mechanism, agents with the same reported valuations should receive identical (expected) utility.

[5] For two ascending-ordered vectors $\mathbf{a}$ and $\mathbf{b}$, the Lorenz domination partial order is defined as $\mathbf{a} \succ_{Lorenz} \mathbf{b}$ if for every k, the sum of the first k entries of $\mathbf{a}$ is at least as large as that of $\mathbf{b}$. A Lorenz dominating allocation is the allocation whose valuation vector Lorenz dominates the vector of every other allocation.

1.3. Organization of the paper

Section 2 introduces basic definitions, followed by impossibility results for deterministic mechanisms in Section 3. Sections 4 and 5 detail randomized mechanisms for chores and mixed items, highlighting their key properties. Proofs of lemmas, propositions and theorems are provided either in the main body of the paper or are relegated to the Appendix. We make some concluding remarks in Section 6.

2. Preliminaries

Let $[k] = \{1, \dots, k\}$ represent the set of integers from 1 to k for any positive integer k. A set $E = \{e_1, \dots, e_m\}$ of m indivisible items are to be allocated to a set $N = \{1, \dots, n\}$ of n agents. Each agent i is associated with a valuation function $v_i : 2^E \to \mathbb{R}$. An item e is a good (resp., a chore) for agent i if $v_i(e) \geq 0$ (resp., $v_i(e) \leq 0$). In this paper, we focus on two scenarios: (i) chore allocation, where all items are chores for all agents, and (ii) mixed-item allocation, where an item may be a good for one agent but a chore for another. Agents are rational and can interact strategically towards making a collective decision. Let $\mathbb{A}$ be the set of *outcomes* or *allocations*. An allocation $\mathbf{A} = (A_1, \dots, A_n) \in \mathbb{A}$ is an n-partition of E, that is, $A_i \cap A_j = \emptyset$ for any $i, j \in [n]$ with $i \neq j$ and $\bigcup_{i\in[n]} A_i = E$. Given an allocation $\mathbf{A} = (A_1, \dots, A_n)$, for any $i \in [n]$, A_i represents the set of items or the *bundle* allocated to agent i. Note that in any allocation, no item is left unallocated.

Throughout the paper, for any $i \in [n]$, function v_i represents the true valuation function of agent i, and following the convention of mechanism design literature, v_i is also called the *type* of agent i. Each agent i privately observes his preferences over outcomes in $\mathbb{A}$, which is defined by that agent i knows his valuation function v_i that is not available to others. Denote by V_i the set of all possible types of agent i and by $\mathcal{V} = V_1 \times \cdots \times V_n$ the set of type profiles. A type profile is denoted by $\mathbf{v} = (v_1, \dots, v_n)$. Additionally, each agent i is associated with a utility function $u_i : \mathbb{A} \times V_i \to \mathbb{R}$. Given an outcome $\mathbf{A}$ and type $v_i \in V_i$, utility $u_i(\mathbf{A}, v_i)$ denotes the value or payoffs of agent i under outcome $\mathbf{A}$ when his type is v_i. In this paper, we focus on the setting where agent i does not care about how bundles are allocated to others, i.e., *no externalities*. Hence, the utility of agent i with type v_i under allocation $\mathbf{A} = (A_1, \dots, A_n)$ can also be expressed as $u_i(A_i, v_i)$, equivalent to $v_i(A_i)$.

By the revelation principle, we can focus on *direct* mechanisms in which the set of *actions* available to each agent is equal to the set of types of the agents. The *social choice function* or *mechanism* is a mapping $\mathcal{M} : V_1 \times \cdots \times V_n \to \mathbb{A}$ that assigns every possible action profile an outcome from set $\mathbb{A}$. The mechanism elicits type information from agents by asking them to reveal their true valuations. Each agent i reports a valuation function $\hat{v}_i \in V_i$, which may differ from v_i. The mechanism then outputs an allocation based on the reported profile $\hat{\mathbf{v}} = (\hat{v}_1, \dots, \hat{v}_n)$. Given the combinatorial explosion, it is unrealistic for agents to report their valuations on every subset $T \subseteq E$. Consequently, we assume that valuation function v_i is *additive* for all $i \in [n]$, that is, for any $T \subseteq E$, $v_i(T) = \sum_{e\in T} v_i(\{e\})$. For simple notations, we use $v_i(e)$, instead of $v_i(\{e\})$, to denote agent i's value of item e. The set $\mathbb{A}$ of outcomes, the set N of agents, the set E of items and the type sets $\{V_i\}_{i\in[n]}$ are common knowledge, while type v_i is private to agent i only.

2.1. Fractional and randomized allocations

In this paper, we also write allocations in the form of matrices. With a slight abuse of notation, an allocation or an *allocation matrix* $\mathbf{A} = (a_{j,i})_{j\in[m],i\in[n]}$ represents the fraction $a_{j,i}$ of item e_j allocated to agent i, with $\sum_{i\in[n]} a_{j,i} = 1$ for each $j \in [m]$. A *deterministic* allocation requires $a_{j,i} \in \{0, 1\}$ for all j, i, while in a *fractional* allocation $\mathbf{A}$, $a_{j,i}$ can be any real number in interval $[0, 1]$. A randomized allocation $\widetilde{\mathbf{A}}$ is a probability distribution over deterministic allocations and inherently implements a fractional allocation. More specifically, let the support of $\widetilde{\mathbf{A}}$ be $\mathbf{A}^1, \dots, \mathbf{A}^k \in \{0, 1\}^{m\times n}$ and the probability of being $\mathbf{A}^j$ is p_j for all $j \in [k]$. Then we say that $\widetilde{\mathbf{A}}$ *implements* fractional allocation $\sum_{j\in[k]} p_j \mathbf{A}^j$. Note that for a given fractional allocation, there can be several randomized allocations that implement it.

Given an arbitrary randomized allocation $\widetilde{\mathbf{A}}$, let us suppose it implements fractional allocation $\mathbf{A}' = (a'_{j,i})_{j\in[m],i\in[n]}$. The term $a'_{j,i}$ can be interpreted as the probability of assigning item e_j to agent i. For any $i \in [n]$, the expected value of agent i under $\widetilde{\mathbf{A}}$ is $\sum_{j\in[m]} a'_{j,i} v_i(e_j)$, which is equivalent to the value of agent i under fractional allocation $\mathbf{A}'$.

2.2. Strategyproofness

Let us formally define the notions of (group) strategyproofness and strategyproofness in expectation. We first introduce a common notation used in the mechanism design literature. Given the reported profile $\hat{\mathbf{v}}$ and a set of agents $S \subseteq N$, denote by $\hat{v}_S$ the set of reported valuations of agents S and by $\hat{v}_{-S}$ the set of reported valuations of agents $N \setminus S$, i.e., $\hat{v}_{-S} = \hat{v}_{N\setminus S}$. Similar notations extend to the agent type and the set of agent types.

Definition 2.1 (*SP*). A deterministic mechanism $\mathcal{M}$ is strategyproof if $\forall i \in [n]$, $\forall v_i \in V_i$, $\forall \hat{v}_i \in V_i$, $\forall \hat{v}_{-i} \in V_{-i}$, the following holds:

$$u_i(\mathcal{M}(v_i, \hat{v}_{-i}), v_i) \geq u_i(\mathcal{M}(\hat{v}_i, \hat{v}_{-i}), v_i).$$

A strategyproof (SP) mechanism ensures that truthfully reporting their type is always an optimal strategy for any agent, irrespective of other agents' reports. In other words, no agent can gain additional utility by misreporting compared to the received utility when reporting truthfully.

A mechanism is *randomized* if it returns randomized allocations. Accordingly, we use the notion of strategyproof in expectation (SPIE) for a randomized mechanism under which every agent's expected utility is maximized when reporting truthfully. Formally, we have the following definition.

Definition 2.2 (*SPIE*). A randomized mechanism $\widetilde{\mathcal{M}}$ is strategyproof in expectation if $\forall i \in [n]$, $\forall v_i \in V_i$, $\forall \hat{v}_i \in V_i$, $\forall \hat{v}_{-i} \in V_{-i}$, the following holds:

$$\mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}(v_i,\hat{v}_{-i})}[u_i(\mathbf{A}, v_i)] \geq \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}(\hat{v}_i,\hat{v}_{-i})}[u_i(\mathbf{A}, v_i)].$$

While SP and SPIE mechanisms incentivize truthful reporting by individual agents, they may not prevent strategic manipulation by groups of agents. To address strategic behavior at the group level there is a need for *group-strategyproof* (GSP) and *group-strategyproof in expectation* (GSPIE) mechanisms, in which no group of agents can collude to misreport their valuations in a way that makes at least one member of the group better off without making any of the remaining members worse off.

Definition 2.3 (*GSP*). A deterministic mechanism $\mathcal{M}$ is group-strategyproof if there does not exist a group of agents $S \subseteq N$ and a reported profile $(\hat{v}_S, \hat{v}_{-S})$ such that for any $i \in S$,

$$u_i(\mathcal{M}(\hat{v}_S, \hat{v}_{-S}), v_i) \geq u_i(\mathcal{M}(v_S, \hat{v}_{-S}), v_i),$$

and at least one strict inequality holds.

Definition 2.4 (*GSPIE*). A randomized mechanism $\widetilde{\mathcal{M}}$ is group-strategyproof in expectation if there does not exist a group of agents $S \subseteq N$ and a reported profile $(\hat{v}_S, \hat{v}_{-S})$ such that for any $i \in S$,

$$\mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}(\hat{v}_S,\hat{v}_{-S})}[u_i(\mathbf{A}, v_i)] \geq \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}(v_S,\hat{v}_{-S})}[u_i(\mathbf{A}, v_i)],$$

and at least one strict inequality holds.

2.3. Efficiency, fairness and other distributional properties

We conclude this section by presenting efficiency measurement and fairness criteria that we are concerned with in this study. We begin with Pareto optimal allocations and social welfare functions.

Definition 2.5 (*PO*). An allocation $\mathbf{B} = (B_1, \ldots, B_n)$ Pareto dominates another allocation $\mathbf{A} = (A_1, \ldots, A_n)$ if for any $i \in [n]$, $v_i(B_i) \geq v_i(A_i)$ and at least one inequality is strict. An allocation $\mathbf{A}$ is Pareto optimal if no allocation Pareto dominates it.

Definition 2.6. Given an allocation $\mathbf{A} = (A_1, \ldots, A_n)$, its utilitarian welfare is $\mathsf{UW}(\mathbf{A}) = \sum_{i\in[n]} v_i(A_i)$ and the egalitarian welfare is $\mathsf{EW}(\mathbf{A}) = \min_{i\in[n]} v_i(A_i)$. An allocation $\mathbf{A}$ is a utilitarian welfare maximizer (UWM) and an egalitarian welfare maximizer (EWM) if $\mathbf{A}$ achieves the maximum utilitarian welfare and egalitarian welfare, respectively.

Definition 2.7 (*EF and EF1*). An allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is envy-free (EF) if for any $i, j \in [n]$, $v_i(A_i) \geq v_i(A_j)$. An allocation $\mathbf{A}$ is envy-free up to one item (EF1) if for any $i, j \in [n]$, there exists $e \in A_i \cup A_j$ such that $v_i(A_i \setminus \{e\}) \geq v_i(A_j \setminus \{e\})$.

Informally, for mixed items, an allocation is EF1 if for every agent i, he does not envy agent j after either hypothetically removing a chore (from agent i's view) in his own bundle or removing a good in agent j's bundle.

Definition 2.8 (*EQ and EQ1*). An allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is equitable (EQ) if for any $i, j \in [n]$, $v_i(A_i) = v_j(A_j)$. The allocation $\mathbf{A}$ is equitable up to one item (EQ1) if for any $i, j \in [n]$, there exists $e \in A_i \cup A_j$ such that $v_i(A_i \setminus \{e\}) \geq v_j(A_j \setminus \{e\})$.

The notion of EQ1 requires that the value of agent i must be at least that of agent j when either a chore (from agent i's view) is removed from agent i's bundle or a good is removed from agent j's bundle.

Definition 2.9 (*PROP and PROP1*). An allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is proportional (PROP) if for any $i \in [n]$, $v_i(A_i) \geq \frac{1}{n} v_i(E)$. An allocation $\mathbf{A}$ is proportional up to one item (PROP1) if for any $i \in [n]$, $v_i(A_i) \geq \frac{1}{n} v_i(E)$ or $v_i(A_i \cup \{e\}) \geq \frac{1}{n} v_i(E)$ for some $e \in E \setminus A_i$ or $v_i(A_i \setminus \{e\}) \geq \frac{1}{n} v_i(E)$ for some $e \in A_i$.

In a PROP allocation, every agent i is guaranteed a value of $\frac{1}{n} v_i(E)$, also known as his *fair share*. In a PROP1 allocation, agent i is guaranteed his fair share after either hypothetically receiving an extra good, not assigned to his yet, or eliminating a chore from his bundle. We remark that in the setting of mixed items, when agents are associated with additive valuations, EF allocations are PROP, and EF1 allocations are also PROP1 (Aziz et al., 2022).

3. Impossibility results of deterministic mechanisms

In this section, we focus on deterministic mechanisms. Consider a class of widely-studied mechanisms, called *sequential picking* (Kohler & Chandrasekaran, 1971), where agents are ordered in advance and each agent i picks a number t_i of items in his turn. The advantage of sequential picking is that once $\{t_i\}_{i\in[n]}$ are predetermined or unrelated to reported profiles and $\sum_{i\in[n]} t_i = m$, the corresponding mechanism is strategyproof and outputs a complete allocation. The sequential picking rule has been employed to design truthful mechanisms with performance guarantees regarding share-based fairness notions (Amanatidis et al., 2016; Aziz, Li et al., 2024). However, we find that for indivisible chores, no sequential picking mechanism can always return Pareto optimal outcomes, even when the margin of chores is binary, i.e., $v_i(e_j) \in \{0, -1\}$ for all i and j.

Theorem 3.1. *No sequential picking mechanism for indivisible chores can consistently return Pareto-optimal allocations, even when agents' valuations are binary.*

We next present a stronger impossibility result regarding all deterministic mechanisms in the context of allocating indivisible chores.

Theorem 3.2. *No deterministic mechanism for indivisible chores can be* SP, PO *and* EQ1*, simultaneously, even when there are four chores and two agents with binary additive valuations.*

Proof. For a contradiction, let $\mathcal{M}$ be a deterministic SP, PO and EQ1 mechanism. Consider an instance $\mathcal{I}$ with two agents and a set E of four chores e_1, e_2, e_3, e_4. With the reported profile $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2)$ with $\hat{v}_i(e_j) = -1$ for all i, j, the mechanism $\mathcal{M}$ must allocate two chores to each agent to achieve an EQ1 outcome when agents report truthfully. Without loss of generality, we assume $\mathcal{M}(\hat{\mathbf{v}}) = \mathbf{A}$ with $A_1 = \{e_1, e_2\}$ and $A_2 = \{e_3, e_4\}$. We now consider another reported profile $\hat{\mathbf{v}}' = (\hat{v}'_1, \hat{v}'_2)$ where $\hat{v}'_2(e) = \hat{v}_2(e)$ for all e and $\hat{v}'_1(e_1) = \hat{v}'_1(e_2) = 0$ and $\hat{v}'_1(e_j) = -1$ for $j = 3, 4$. Suppose $\mathcal{M}(\hat{\mathbf{v}}') = \mathbf{A}'$. As $\mathcal{M}$ always returns PO allocations, allocation $\mathbf{A}'$ needs to be PO when $\hat{\mathbf{v}}'$ is true valuations. Thus, e_1, e_2 should be assigned to agent 1 in $\mathbf{A}'$. Furthermore, under the property of EQ1, each agent $i \in [2]$ must receive one chore from the set $\{e_3, e_4\}$ in $\mathbf{A}'$.

Now suppose for each $i \in [2]$, $\hat{v}'_i$ is the type of agent i, i.e., $\hat{v}'_i = v_i$. Then, if both agents report truthfully, $\mathcal{M}$ returns allocation $\mathbf{A}'$ with $v_1(A'_1) = -1$. However, if agent 1 misreports $\hat{v}_1$, then the outcome becomes $\mathcal{M}(\hat{v}_1, \hat{v}'_2) = \mathbf{A}$, and the value of agent 1 in the allocation $\mathbf{A}$ is equal to $v_1(A_1) = 0$. Therefore, agent 1 has incentive to behave strategically, contradicting the strategyproofness of $\mathcal{M}$. □

We remark that the above impossibility result is mainly owing to the fact that deterministic strategyproofness is too strict to be achieved together with Pareto efficiency and fairness. Note that for indivisible chores, EQ1 and PO allocations are guaranteed to exist for additive valuations (Freeman et al., 2020). Given such a strong impossibility result, we relax strategyproofness via randomness.

4. Randomized mechanisms for indivisible chores

Throughout this section, we focus on the setting of indivisible chores, and assume agents have *k-restricted additive* valuation functions, that is, every chore e_j is associated with k inherent values $v^1(e_j), v^2(e_j), \ldots, v^k(e_j)$ with $v^t(e_j) < 0$ for all $t \in [k]$, and for any agent i, his value of e_j is $v_i(e_j) \in \{0, v^1(e_j), v^2(e_j), \ldots, v^k(e_j)\}$. In particular, 1-restricted additive valuation is simply called *restricted additive* valuation where for simple notations, the inherent value of e_j is $v(e_j) < 0$. The restricted additive valuation has been studied in the literature on fair division and has resulted in significant and challenging research questions (Akrami et al., 2022; Asadpour et al., 2012; Bezáková & Dani, 2005). Note that restricted additive functions strictly generalize binary additive functions. Problems that can be solved efficiently in binary additive valuations may become computationally intractable in the restrictive additive domain. For example, in the context of indivisible goods, the egalitarian welfare-maximizing allocation can be computed in polynomial time under binary additive valuations (Barman et al., 2018), while for restricted additive valuations, it is NP-hard to approximate for a factor better than 1/2 (Bezáková & Dani, 2005). As type sets $\{V_i\}_{i\in[n]}$ are common knowledge, the inherent values $\{v^t(e_j)\}$ of e_j is publicly known, which further constrains each agent i's reported value on e_j to satisfy $\hat{v}_i(e_j) \in \{0, v^1(e_j), v^2(e_j), \ldots, v^k(e_j)\}$.

Designing SPIE mechanisms with both ex-ante and ex-post fairness guarantee is not a trivial task. Existing randomized mechanisms such as Random Priority and Probabilistic Serial (Bogomolnaia & Moulin, 2001) cannot guarantee strategyproofness and fairness at the same time when $m > n$. While some straightforward mechanism, such as assigning all items to an agent chosen uniformly at random, satisfies strategyproofness and fairness before realization, its performance guarantee is poor after realization, with all items allocated to a single agent. Therefore, among randomized mechanisms, we focus on those that are strategyproof in expectation and have both ex-ante and ex-post performance guarantees. In the following, we formally define ex-ante and ex-post properties.

Definition 4.1. For a fairness or efficiency notion P, a randomized allocation $\widetilde{\mathbf{A}}$ is ex-ante P if the fractional allocation implemented by $\widetilde{\mathbf{A}}$ satisfies P. It is ex-post P if every deterministic allocation within its support satisfies P.

Definition 4.2. Given the fairness or efficiency notions of P_1 and P_2, a randomized mechanism $\widetilde{\mathcal{M}}$ is ex-ante P_1 and ex-post P_2 if it always returns a randomized allocation that is ex-ante P_1 and ex-post P_2.

For ex-ante and ex-post efficiency, we present the following implication relation.

Proposition 4.3. *An ex-ante* UWM *randomized allocation* $\widetilde{\mathbf{A}}$ *is also ex-ante* PO *and ex-post* UWM. *An ex-ante* PO *randomized allocation* $\widetilde{\mathbf{B}}$ *is also ex-post* PO.

Proof. Suppose that $\widetilde{\mathbf{A}}$ has support $\{\mathbf{A}^1, \dots, \mathbf{A}^k\}$ with probability p_i on deterministic allocation $\mathbf{A}^i$ for all $i \in [k]$. Let $\mathbf{A}'$ be the fractional allocation implemented by $\widetilde{\mathbf{A}}$. If there exists another fractional allocation $\widehat{A}$ that Pareto dominates $\mathbf{A}'$, then $\mathsf{UW}(\widehat{A}) > \mathsf{UW}(\mathbf{A}')$ holds, contradicting that $\widetilde{\mathbf{A}}$ is ex-ante UWM. Hence, allocation $\widetilde{\mathbf{A}}$ is also ex-ante PO. Next, we prove that $\widetilde{\mathbf{A}}$ is also ex-post UWM. For a contradiction, suppose that $\widetilde{\mathbf{A}}$ is not ex-post UWM, then there must exist a deterministic allocation $\mathbf{A}^{k+1}$ such that $\mathsf{UW}(\mathbf{A}^{k+1}) > \mathsf{UW}(\mathbf{A}^i)$ for some $i \in [k]$. Consider the fractional allocation $\mathbf{A}'' = p_i\mathbf{A}^{k+1} + \sum_{j\neq i} p_j\mathbf{A}^j$. One can verify that $\mathsf{UW}(\mathbf{A}'') > \mathsf{UW}(\mathbf{A}')$, contradicting that $\widetilde{\mathbf{A}}$ is ex-ante UWM. Therefore, $\widetilde{\mathbf{A}}$ is also ex-post UWM.

For an arbitrary ex-ante PO randomized allocation $\widetilde{\mathbf{B}}$, let its support be $\{\mathbf{B}^1, \dots, \mathbf{B}^k\}$ with probability p_i on $\mathbf{B}^i$ for all $i \in [k]$. Moreover, let $\mathbf{B}'$ be the fractional allocation implemented by $\widetilde{\mathbf{B}}$. For a contradiction, assume that there exists another deterministic allocation $\mathbf{B}^{k+1}$ that Pareto dominates $\mathbf{B}^i$ for some $i \in [k]$. Then, the fractional allocation $\mathbf{B}'' = p_i\mathbf{B}^{k+1} + \sum_{j\neq i} p_j\mathbf{A}^j$ Pareto dominates $\mathbf{B}'$, a contradiction. Therefore, allocation $\widetilde{\mathbf{B}}$ is ex-post PO. □

In contrast to UWM and PO, not every ex-ante fair solution guarantees ex-post fairness. Let us again consider the mechanism of assigning all items to an agent chosen uniformly at random. It is easy to verify that the resulting allocation is ex-ante EF. However, it provides little fairness guarantee after realization.

The main result of this section is a GSPIE randomized mechanism (Algorithm 1) that provides ex-ante and ex-post fairness and efficiency guarantees for n agents with 1-restricted additive valuation functions. Intuitively, the mechanism first collects the reported type profile and then partitions $[m]$ into Q and $\bar{Q}$ where for any e_j with $j \in Q$, there exists an agent whose reported value is zero on e_j, while for any $e_{j'}$ with $j' \in \bar{Q}$, every agent i reports $\hat{v}_i(e_{j'}) = v(e_{j'})$. Then, for every $j \in Q$, RandChore assigns e_j to an agent chosen uniformly at random with reported value zero on e_j. Items $\bigcup_{j\in\bar{Q}} e_j$ are then assigned to agents in a round-robin fashion based on a permutation σ of $\{1, \dots, n\}$, generated uniformly at random.

Theorem 4.4. *Mechanism* RandChore *is* GSPIE *and satisfies the following:*

- *Fairness guarantee: ex-ante* EF, PROP, EQ *and ex-post* EF1, PROP1, EQ1*;*
- *Efficiency guarantee: ex-ante* PO, UWM, EWM *and ex-post* PO, UWM, 2-*approximation of* EWM.

Algorithm 1 RandChore

1: Collects reported profile $(\hat{v}_1, \dots, \hat{v}_n)$.
2: Let $Q = \{q \in [m] \mid \exists i$ such that $\hat{v}_i(e_q) = 0\}$ and $\bar{Q} = [m] \setminus Q$.
3: For every $q \in Q$, uniformly at random pick an agent reporting zero value on e_q and assign e_q to that agent.
4: Let $\bar{Q} = \{l_1, l_2, \dots, l_k\}$ with inherent values $v(e_{l_1}) \geq \dots \geq v(e_{l_k})$.
5: Uniformly at random generate a permutation σ of $\{1, \dots, n\}$. According to σ, assign the chore with the largest inherent value from the remaining items to an agent each time, until all chores are assigned. If there is a tie on the largest value item, choose e_{l_j} with the smallest index j.

Intuitively, the properties of PO and UWM arise from the fact that in any realized allocation, every chore is allocated to the agent who has the maximum value for it. The properties of EF1, PROP1 and EQ1 are due to the round-robin procedure in Step 5. In the following, we split our proof of Theorem 4.4 into several propositions (Propositions 4.5, 4.9–4.11): beginning with property GSPIE followed by the fairness and efficiency guarantees. For simplicity, we use the notation $\widetilde{\mathcal{M}}^*$ and RandChore interchangeably in this section.

Proposition 4.5. *Mechanism* RandChore *is* GSPIE.

Before proving Proposition 4.5, we first present several lemmas.

Lemma 4.6. *Given the reported profile* $(\hat{v}_1, \dots, \hat{v}_n)$ *and the corresponding* Q *and* $\bar{Q}$*, for any agent* $i \in [n]$*, he receives an expected utility* $\frac{1}{n} v_i(\bigcup_{q\in\bar{Q}} e_q)$ *in Step 5.*

Proof. Fix index i. Since σ is a uniformly random permutation of $\{1, 2, \dots, n\}$, the probability of agent i on position $\sigma(j)$ is $1/n$ for all $j \in [n]$. Based on Step 5, if agent i is in position $\sigma(j)$, he receives a utility $v_i(e_{l_j} \cup e_{l_{n+j}} \cup \dots \cup e_{l_{kn+j}})$, where $k = \lfloor (|\bar{Q}| - j)/n \rfloor$. Thus, his expected utility derived from the assignment of Step 5 is equal to

$$\sum_{j=1}^{n} \frac{1}{n} v_i(\bigcup_{p=0}^{k} e_{l_{pn+j}}) = \frac{1}{n} v_i(\bigcup_{j=1}^{n}\bigcup_{p=0}^{k} e_{l_{pn+j}}) = \frac{1}{n} v_i(\bigcup_{q\in\bar{Q}} e_q),$$

where the first equality transition is due to the additivity of $v_i(\cdot)$. □

For any group S of agents and any agent $i \in S$, if agent i's true valuation on e_j is $v_i(e_j) = 0$, then he cannot gain additional expected utility by misreporting $\hat{v}_i(e_j) = v(e_j)$.

Lemma 4.7. *Given a subset* $S \subseteq N$ *and a reported profile* $(\hat{v}_S, \hat{v}_{-S})$ *with* $\hat{v}_S \neq v_S$*, construct another reported profile* $(\hat{v}'_S, \hat{v}_{-S})$ *as follows: for any* $i \in S$ *and* $e_j \in E$*, if* $v_i(e_j) = 0$*, then set* $\hat{v}'_i(e_j) = 0$*; otherwise,* $\hat{v}'_i(e_j) = \hat{v}_i(e_j)$*. Then for any* $i \in S$*, the expected utility of agent* i *under reported profile* $(\hat{v}'_S, \hat{v}_{-S})$ *is at least that under* $(\hat{v}_S, \hat{v}_{-S})$*,*

$$\mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(\hat{v}'_S,\hat{v}_{-S})}[u_i(\mathbf{A}, v_i)] \geq \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(\hat{v}_S,\hat{v}_{-S})}[u_i(\mathbf{A}, v_i)].$$

Lemma 4.8. *Given a subset* $S \subseteq N$ *and reported valuations* $\hat{v}_{-S}$*, the summation of the expected utilities of agents in* S *is maximized when every agent* $i \in S$ *reports his true valuation.*

Proof sketch. The proof is by contradiction. If some valuation functions $\hat{v}_S$ result in the summation of expected utilities of agents in S being larger than that under v_S, then according to Lemma 4.7, we can further assume that $\hat{v}_S$ contains only one type of misreporting: some agent $j \in S$ reports $\hat{v}_j(e) = 0$ on chore e, while his true valuation is $v_j(e) = v(e) < 0$. Based on Steps 2 and 5, deviating from $v(e)$ to zero can only increase the probability of allocating e to agents in S. Therefore, agents of S have no incentive to manipulate. The formal proof is deferred to Appendix A.2. □

Now we are ready to prove that RandChore is GSPIE.

Proof of Proposition 4.5. For the sake of contradiction, assume that there exists a group of agents $S \subseteq N$ and a reported profile $(\hat{v}_S, \hat{v}_{-S})$ such that for any agent $i \in S$, it holds that

$$\mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(\hat{v}_S,\hat{v}_{-S})}[u_i(\mathbf{A}, v_i)] \geq \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(v_S,\hat{v}_{-S})}[u_i(\mathbf{A}, v_i)],$$

and at least one strict inequality holds. Accordingly, we have the following inequality,

$$\sum_{i\in S} \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(\hat{v}_S,\hat{v}_{-S})}[u_i(\mathbf{A}, v_i)] > \sum_{i\in S} \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(v_S,\hat{v}_{-S})}[u_i(\mathbf{A}, v_i)],$$

which contradicts Lemma 4.8. □

After establishing the group strategyproofness, we proceed to prove that RandChore can ensure both ex-ante and ex-post properties. Specifically, RandChore always returns solutions with ex-ante exact fairness (EF, PROP, and EQ) and ex-post approximate fairness (EF1, PROP1, and EQ1).

Proposition 4.9. *Mechanism* RandChore *is ex-ante* EF, PROP, EQ *and ex-post* EF1, PROP1, *and* EQ1.

We then show that RandChore also guarantees ex-ante and ex-post efficiency.

Proposition 4.10. *Mechanism* RandChore *is ex-ante* PO, UWM, EWM *and ex-post* PO, UWM, *and 2-approximation of* EWM.

Note that the 2-approximation of optimal egalitarian welfare is nearly the (ex-post) limitation of RandChore, especially when the number of agents is large.

Proposition 4.11. *For any $\epsilon > 0$, mechanism* RandChore *is not ex-post $((n-1)/n-\epsilon)$-approximation of* EWM.

Although RandChore is a randomized mechanism, the randomness does not affect its running time. The worst-case running time of Step 3 is $O(m)$ and of Step 5 is $O(mn)$. Thus, the worst-case running time of RandChore is $O(mn)$. Since efficient computability is a critical consideration when designing mechanisms in practice, the running time of $O(mn)$ makes our mechanism easily implementable, which is an advantage of using round-robin in Step 5.

On the other hand, ex-post performance guarantees can be improved at the expense of efficient computability. Below, we consider another mechanism with improved ex-post guarantees, where instead of using round-robin at Step 5, we use another procedure, namely the *Leximin rule*, which first partitions $\bigcup_{q\in\bar{Q}} e_q$ into n bundles $\bar{Q}_1, \dots, \bar{Q}_n$ such that, with respect to $v(\cdot)$, the minimum value of a bundle is maximized and, subject to that, the second minimum value of a bundle is maximized and so on. It then generates a permutation σ of $\{1, \dots, n\}$ uniformly at random and for any $j \in [n]$, allocates $\bar{Q}_j$ to agent $\sigma(j)$.

One may question whether the mechanism with the Leximin rule can be strategyproof. Indeed, if we replace Step 5 of RandChore with another procedure of allocating $\bigcup_{q\in\bar{Q}} e_q$ such that Lemma 4.6 holds, the proofs of Lemmas 4.7 and 4.8 and of Proposition 4.5 carry over to the new mechanism, which then implies GSPIE. We have the following theorem.

Theorem 4.12. *The new mechanism, obtained by replacing Step 5 of* RandChore *with the Leximin rule, satisfies* GSPIE *and the following properties:*

- *Fairness guarantee: ex-ante* EF, PROP, EQ *and ex-post* EFX_, PROPX_, EQX_.
- *Efficiency guarantee: ex-ante* PO, UWM, EWM *and ex-post* PO, UWM, *and* EWM.

Informally, the notions of EFX_, EQX_, and PROPX_ require that envy-freeness, equitability, and proportionality can be achieved, respectively, by hypothetically removing any item with a negative value. These notions are stronger than EF1, EQ1, and PROP1. Their definitions and the proof of Theorem 4.12 are deferred to Appendix A.2. Moreover, the Leximin rule also enables us to improve the 2-approximation EWM to (exact) EWM. On the negative side, as discussed above, these improvements come at the expense of efficient implementation because computing bundles $\bar{Q}_1, \dots, \bar{Q}_n$ in the Leximin rule is NP-hard: computing $\bar{Q}_1, \dots, \bar{Q}_n$ is at least as hard as computing the solution that minimizes the makespan of n identical machines, known to be NP-hard (Garey & Johnson, 1979).

Proposition 4.13. *Computing the bundles $\bar{Q}_1, \dots, \bar{Q}_n$ in the Leximin-rule is NP-hard.*

We conclude this section with a discussion on possible extensions of the proposed mechanisms. As demonstrated, the two mechanisms presented above simultaneously guarantee envy-based and equitability-based fairness criteria, Pareto optimality, and strategyproofness when agents have restricted additive valuation functions. A natural question arises: in a broader preference domain, such as additive valuation functions, can a strategyproof (in expectation) mechanism achieve EF1, EQ1, and PO? Unfortunately, the answer is negative, even without the requirement of strategyproofness. Freeman et al. (2020) showed that for indivisible chores, EF1, EQ1, and PO are incompatible for four agents with additive valuation functions. For completeness, we include their example in Appendix A.2. Note that valuation functions in Freeman et al.'s example are 2-restricted additive, so the proposed mechanisms achieve the best that we can hope for in the k-restricted additive preference domain.

5. Randomized mechanisms for allocating mixed items

In this section, we consider the allocation of mixed items to two agents, where an item can be a good for one agent but a chore for another. Agents are assumed to have *M-restricted additive* valuations, a generalization of restricted additive functions to the setting of mixed items. Namely, each item e_q has two inherent values $\{-c(e_q), v(e_q)\}$ with $c(e_q), v(e_q) > 0$. If e_q is a chore (resp., a good) for an agent, then their valuation of e_q is $-c(e_q)$ (resp., $v(e_q)$). Otherwise, their valuation of e_q is zero. Note that $c(e_q)$ and $v(e_q)$ are not required to be identical. Recall that type sets $\{V_i\}_{i\in[n]}$ are common knowledge, so for any $q \in [m]$, inherent values $-c(e_q)$ and $v(e_q)$ are publicly known. Thus, the reported valuation functions satisfy $\hat{v}_i(e_q) \in \{-c(e_q), 0, v(e_q)\}$ for all $i \in [n]$.

The main result of this section is a randomized SPIE mechanism with ex-ante and ex-post fairness and efficiency guarantees for two agents. We propose the mechanism RandMixed and formally introduce it below (see Algorithm 2). A simple description of RandMixed is as follows. Based on reported profile $(\hat{v}_1, \hat{v}_2)$, partition $[m]$ into four parts $\{Q_0, Q_1, Q_2, Q_3\}$. Assign items in Q_0 one-by-one to one of the two agents uniformly at random. For any $i \in \{1, 2\}$, assign items $\bigcup_{q\in Q_i} e_q$ to agent i and then assign items $\bigcup_{q\in Q_3} e_q$ to agents in a round-robin fashion based on a permutation σ generated uniformly at random.

Theorem 5.1. *Mechanism* RandMixed *is* SPIE, *ex-ante* PO, UWM, EF, PROP *and ex-post* PO, UWM, EF1 *and* PROP1.

The guarantees of ex-ante and ex-post PO and UWM are due to the fact that, in the realized allocations, every item is always assigned to the agent who values it most. The properties of EF1 and PROP1 arise from the round-robin procedure. In the following, we split the proof of Theorem 5.1, and for simplicity, we use $\widetilde{\mathcal{M}}^2$ and RandMixed interchangeably in this section.

Algorithm 2 RandMixed

1: Collect reported profile $(\hat{v}_1, \hat{v}_2)$.
2: Partition $[m] = Q_0 \cup Q_1 \cup Q_2 \cup Q_3$ where $Q_0 = \{q \in [m] \mid \hat{v}_1(e_q) = \hat{v}_2(e_q) = 0\}$, $Q_1 = \{q \in [m] \mid \hat{v}_1(e_q) > \hat{v}_2(e_q)\}$, $Q_2 = \{q \in [m] \mid \hat{v}_1(e_q) < \hat{v}_2(e_q)\}$, and $Q_3 = \{q \in [m] \mid \hat{v}_1(e) = \hat{v}_2(e) \neq 0\}$.
3: For each e_q with $q \in Q_0$, uniformly at random choose an agent and assign e_q to her.
4: For $i = 1, 2$, assign items $\bigcup_{q \in Q_i} e_q$ to agent i.
5: Let σ be a permutation of $\{1, 2\}$ generated at random uniformly. Among the unassigned items, assign the one with the largest inherent value to the two agents in a round-robin fashion based on the permutation σ.

Proposition 5.2. *Given reported profile $(\hat{v}_1, \hat{v}_2)$ and the corresponding partition $\{Q_i\}_{i=0}^3$, each agent $i \in \{1, 2\}$ receives expected utility $\frac{1}{2} v_i(\bigcup_{q \in Q_0 \cup Q_3} e_q) + v_i(\bigcup_{q \in Q_i} e_q)$.*

Proof. Fix $i \in \{1, 2\}$. As items $\bigcup_{q \in Q_{3-i}} e_q$ are allocated to agent $3-i$, the utility of agent i indeed comes from the assignment of $\bigcup_{q \in Q_0 \cup Q_i \cup Q_3} e_q$. Note that e_q with $q \in Q_0$ is assigned to agent i with probability $1/2$, and e_q with $q \in Q_i$ is assigned to agent i with probability one. As for items $\bigcup_{q \in Q_3} e_q$, similar to that in the proof of Lemma 4.6, this part results in an expected utility $\frac{1}{2} v_i(\bigcup_{q \in Q_3} e_q)$ for agent i. Therefore, agent i's expected utility derived from the assignment is $\frac{1}{2} v_i(\bigcup_{q \in Q_0 \cup Q_3} e_q) + v_i(\bigcup_{q \in Q_i} e_q)$. □

Lemma 5.3. *Given agent i and reported profile $(\hat{v}_i^k, \hat{v}_{3-i})$ with $\hat{v}_i^k(e_k) \neq v_i(e_k)$ for some $k \in [m]$, construct another reported profile $(\hat{v}_i, \hat{v}_{3-i})$ where $\hat{v}_i(e_k) = v_i(e_k)$ and $\hat{v}_i(e) = \hat{v}_i^k(e)$ for all $e \neq e_k$. Then, the expected utility of agent i under reported profile $(\hat{v}_i, \hat{v}_{3-i})$ is at least that under $(\hat{v}_i^k, \hat{v}_{3-i})$.*

Lemma 5.3 states that, for any non-truthful reported profile, correcting the reported value of a single item of agent i does not decrease agent i's expected utility. Then, for any reported valuation $\hat{v}_i \neq v_i$, one can start from $\hat{v}_i$ and reach v_i by a sequence of corrections of the reported values for a single item, without decreasing the expected utility.

Proposition 5.4. *Mechanism* RandMixed *is* SPIE.

Proof. Fix $i \in \{1, 2\}$ and $\hat{v}_{3-i}$. Let $\hat{v}_i$ be arbitrary reported valuations that differ from the true valuations v_i on a set $\{e_{p_1}, \dots, e_{p_r}\}$ of $r \in \mathbb{N}^+$ items. First let $\hat{v}_i^0 = \hat{v}_i$ and then for every $l \in [r]$, construct valuation function $\hat{v}_i^l$ as follows:

$$\hat{v}_i^l(e) = \begin{cases} v_i(e_{p_l}) & \text{if } e = e_{p_l}, \\ \hat{v}_i^{l-1}(e) & \text{otherwise.} \end{cases}$$

It is not hard to verify that valuation functions $\hat{v}_i^r$ is identical to v_i. Then, according to Lemma 5.3, we have

$$\mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^2(\hat{v}_i^r, \hat{v}_{3-i})}[u_i(\mathbf{A}, v_i)] \geq \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^2(\hat{v}_i^{r-1}, \hat{v}_{3-i})}[u_i(\mathbf{A}, v_i)] \geq \cdots \geq \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^2(\hat{v}_1^0, \hat{v}_{3-i})}[u_i(\mathbf{A}, v_i)].$$

Since $\hat{v}_i^r = v_i$ and $\hat{v}_i^0 = \hat{v}_i$, the above inequalities become

$$\mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^2(v_i, \hat{v}_{3-i})}[u_i(\mathbf{A}, v_i)] \geq \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^2(\hat{v}_i, \hat{v}_{3-i})}[u_i(\mathbf{A}, v_i)],$$

which completes the proof. □

In the following, we show that RandMixed outputs allocations that guarantee ex-ante and ex-post fairness and efficiency.

Proposition 5.5. *Mechanism* RandMixed *is ex-ante* EF, PROP *and ex-post* EF1, PROP1.

Proposition 5.6. *Mechanism* RandMixed *is ex-ante and ex-post* PO *and* UWM.

Since the randomness does not impact the execution time, the worst-case running time of RandMixed is $O(m)$, which makes the mechanism feasible for efficient implementation in practice. As in Section 4, we can improve the ex-post performance guarantees at the expense of efficient computability. We adapt (and keep the name of) the Leximin rule introduced in Section 4 to the setting of mixed items and two agents: First partition $\bigcup_{q \in Q_3} e_q$ into two bundles $\{Q_{3,1}, Q_{3,2}\}$ such that, with respect to $-c(\cdot), v(\cdot)$, the minimum value of a bundle is maximized and, subject to that, the second minimum value of a bundle is maximized. Then uniformly at random generate a permutation σ of $\{1, 2\}$ and allocates $Q_{3,j}$ to agent $\sigma(j)$ for every j.

Note that if Step 5 of RandMixed is replaced with another procedure of allocating $\bigcup_{q \in Q_3} e_q$ such that Proposition 5.2 holds, the proofs of Lemma 5.3 and Proposition 5.4 carry over, making the new mechanism SPIE.

Theorem 5.7. *The new mechanism, obtained by replacing Step 5 of* RandMixed *with the Leximin rule, satisfies* SPIE *and ex-ante* PO, UWM, EF, PROP, *and ex-post* PO, UWM, EFX_-^+, *and* PROPX_-^+.

The notions of EFX_-^+ and PROPX_-^+, which are formally defined in Appendix A.2, are stronger than EF1 and PROP1, respectively. The proof of Theorem 5.7 is deferred to Appendix A.3. As discussed, the downside of these fairness improvements is that the mechanism is highly unlikely to be efficiently implementable in practice, since computing $Q_{3,1}$ and $Q_{3,2}$ is NP-hard: at least as hard as computing the solution that minimizes the makespan of two identical machines, known to be NP-hard (Garey & Johnson, 1979).

Proposition 5.8. *Computing the bundles $Q_{3,1}$ and $Q_{3,2}$ in the Leximin rule is NP-hard.*

One may observe that the two mechanisms presented above do not provide strong performance guarantee regarding the notion of equitability. We note that this is because (relaxed) equitability and PO are incompatible in the mixed items setting, even without the requirement of strategyproofness. For example, consider an instance with two agents and two mixed items, where agent 1 values each item at 1 and agent 2 values each item at −1. The PO allocation assigns both items to agent 1, violating EQ and EQ1.

6. Conclusions

This paper has explored mechanism design approaches for allocating indivisible items, emphasizing fairness, efficiency, and strategyproofness. We have focused on the settings where (i) all items are chores and (ii) an item can be a good for one agent and a chore for another. If randomization is not allowed, we have showed that strategyproofness, fairness and efficiency are incompatible, and in particular, no deterministic mechanism can be SP, PO and EQ1 simultaneously, even when agents' valuations are binary additive. On the other hand, if randomization is allowed, we have proposed GSPIE mechanisms providing allocations that are efficient and exactly fair ex-ante and approximately fair ex-post, when all items are chores and agents' valuations are 1-restricted additive. We have also studied the model of mixed items and designed a randomized SPIE mechanism with ex-ante and ex-post fairness and efficiency guarantees for two agents with M-restricted additive valuations.

Our findings contribute to the theoretical understanding of strategyproof mechanisms by addressing limitations in deterministic settings and demonstrating the potential of randomized approaches for practical allocation scenarios. Our constructive proofs and analyses provide a solid foundation for designing strategyproof mechanisms in complex allocation settings, offering insights into the mechanisms' ability to align agents' incentives with truthful reporting. Moreover, the proposed mechanisms address key challenges in designing allocation methods that incentivize truthful reporting while balancing fairness and efficiency, showcasing the potential of randomized mechanisms in achieving strategyproofness, fairness and efficiency simultaneously. In practice, our proposed mechanisms can be implemented easily and efficiently in various real-world scenarios where agents' preferences are private but fair and efficient resource allocation is crucial.

Looking forward, a significant open question is whether RandMixed's approach can be extended to scenarios with three or more agents, particularly given the combinatorial complexity and trade-offs between fairness and efficiency. Another key unresolved question is whether similar mechanisms can ensure strategyproofness and robust fairness guarantees in the allocation of indivisible goods. In addition, while Theorem 3.2 demonstrates an impossibility result regarding the notion of EQ1, it remains unknown whether deterministic mechanisms can achieve strategyproofness, Pareto optimality, and other fairness criteria, such as EF1 or PROP1. Nonetheless, our mechanisms pave the way for future advancements in allocation theory, offering promising frameworks for addressing fairness in increasingly complex multi-agent systems.

CRediT authorship contribution statement

Ankang Sun: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Validation, Visualization, Writing – original draft, Writing – review & editing. **Bo Chen:** Conceptualization, Funding acquisition, Project administration, Resources, Supervision, Writing – review & editing.

Appendix

A.1. For Section 3

A.1.1. Proof of Theorem 3.1

Without loss of generality, we assume that agents are ordered $1, \dots, n$ and hence agent 1 is the first to pick. Every sequential picking mechanism can be characterized by a sequence of number $t_1, \dots, t_n$ with $\sum_{i \in [n]} t_i = m$. Moreover, such a sequence is predetermined and is not affected by reported profiles. Fix a sequence $\{t_i\}_{i=1}^n$. If $t_1 < m$, consider an instance $\mathcal{I}_1$. In this instance, the type of agent 1 is $v_1(e) = 0$ for all $e \in E$ and the type of agent $i \geq 2$ is $v_i(e) = -1$ for all $e \in E$. Instance $\mathcal{I}_1$ admits an allocation in which all agents have value 0. However, in the allocation returned by a sequential picking algorithm with $t_1 < m$, there exists at least one agent receiving negative value, and consequently, the returned allocation is not Pareto optimal. Notice that Pareto optimality requires $t_1 = m$, while any sequential picking algorithm with $t_1 = m$ cannot output Pareto optimal allocation for another instance $\mathcal{I}_2$ where the type of agent 1 is $v_1(e) = -1$ for all $e \in E$ and the type of agent $i \geq 2$ is $v_i(e) = 0$ for all $e \in E$. Therefore, no sequential picking mechanism can always return PO allocations for both $\mathcal{I}_1$ and $\mathcal{I}_2$.

A.2. For Section 4

A.2.1. Proof of Lemma 4.7

Let $Q, \bar{Q}$ and $Q', \bar{Q}'$ be the corresponding index sets constructed in Step 2 of Algorithm 1 with reported profiles $(\hat{v}_S, \hat{v}_{-S})$ and $(\hat{v}'_S, \hat{v}_{-S})$, respectively. For every $i \in S$, define $P_i = \{e \in E \mid \hat{v}_i(e) = 0\}$ and $P'_i = \{e \in E \mid \hat{v}'_i(e) = 0\}$. Then, for any $i \in S$ and $e \in P_i$, if $v_i(e) = 0$, by the construction of $\hat{v}'_i$, we have $\hat{v}'_i(e) = 0$, implying $e \in P'_i$. If $v_i(e) \neq 0$, again from the definition of $\hat{v}'_i$, we have $\hat{v}'_i(e) = \hat{v}_i(e) = 0$, implying $e \in P'_i$. Thus, for any $i \in S$ and $e \in P_i$, it follows that $e \in P'_i$, implying $P_i \subseteq P'_i$. Note that reported valuations of agents $N \setminus S$ are identical in these two profiles, then we can claim that $Q \subseteq Q'$, and equivalently, $\bar{Q}' \subseteq \bar{Q}$.

The expected utility of an agent indeed comes from two parts, the assignment of Steps 3 and 5. For any $i \in S$, let $C_i = \{e \in E \mid v_i(e) \neq 0\}$ be the set of items of which the valuation is non-zero for agent i. If $C_i \cap P_i \neq \emptyset$ (resp., $C_i \cap P'_i \neq \emptyset$) for some i, then agent i would receive a negative expected utility from the assignment of $C_i \cap P_i$ (resp., $C_i \cap P'_i$) under the reported profile $(\hat{v}_S, \hat{v}_{-S})$ (resp., $(\hat{v}'_S, \hat{v}_{-S})$). Recall that for any $i \in S$, $P_i \subseteq P'_i$ holds, and thus, $C_i \cap P_i \subseteq C_i \cap P'_i$. For any $i \in S$ and $e \in C_i \cap P'_i$, we have $\hat{v}'_i(e) = 0$, and moreover, by the construction of $\hat{v}'_i$, it holds that $\hat{v}_i(e) = 0$, implying $e \in C_i \cap P_i$. Accordingly, $C_i \cap P'_i \subseteq C_i \cap P_i$ holds for all $i \in S$, and thus, $C_i \cap P_i = C_i \cap P'_i$.

We now analyze the expected utility of agent $i \in S$ caused by the assignment of Steps 3 and 5. For Step 3, note that for any $e \in C_i \cap P_i$, it never happens that $\hat{v}_j(e) = 0$ but $\hat{v}'_j(e) = v(e)$ for some agent $j \in N$. Then, the number of agents reporting zero on e under $(\hat{v}'_S, \hat{v}_{-S})$ is at least that under $(\hat{v}_S, \hat{v}_{-S})$. Since RandChore uniformly at random assigns e to an agent whose reported value is zero, the probability of assigning e to agent i under reported profile $(\hat{v}'_S, \hat{v}_{-S})$ is no greater than that under $(\hat{v}_S, \hat{v}_{-S})$. Hence, the expected utility of agent i caused by the assignment of Step 3 does not decrease when agents S deviate their reporting from $\hat{v}_S$ to $\hat{v}'_S$. As for Step 5, by Lemma 4.6, the expected utility here of agent $i \in S$ under reported profiles $(\hat{v}'_S, \hat{v}_{-S})$ and $(\hat{v}_S, \hat{v}_{-S})$ is equal to $\frac{1}{n} v_i(\bigcup_{q \in \bar{Q}'} e_q)$ and $\frac{1}{n} v_i(\bigcup_{q \in \bar{Q}} e_q)$, respectively. Recall that $\bar{Q}' \subseteq \bar{Q}$, we have $\frac{1}{n} v_i(\bigcup_{q \in \bar{Q}'} e_q) \geq \frac{1}{n} v_i(\bigcup_{q \in \bar{Q}} e_q)$ as chores yield non-positive values. Therefore, for any $i \in S$, agent i' expected utility under $(\hat{v}'_S, \hat{v}_{-S})$ is at least that under $(\hat{v}_S, \hat{v}_{-S})$.

A.2.2. Proof of Lemma 4.8

For the sake of contradiction, assume that there exist valuation functions $\hat{v}_S \neq v_S$ such that

$$\sum_{i \in S} \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(\hat{v}_S, \hat{v}_{-S})}[u_i(\mathbf{A}, v_i)] > \sum_{i \in S} \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(v_S, \hat{v}_{-S})}[u_i(\mathbf{A}, v_i)].$$

Then we consider valuation functions $\hat{v}'_S$ constructed as follows; for any $i \in S$ and $e \in E$, if $v_i(e) = 0$, then set $\hat{v}'_i(e) = 0$; otherwise, $\hat{v}'_i(e) = \hat{v}_i(e)$. By Lemma 4.7, we know $\mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(\hat{v}'_S, \hat{v}_{-S})}[u_i(\mathbf{A}, v_i)] \geq \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(\hat{v}_S, \hat{v}_{-S})}[u_i(\mathbf{A}, v_i)]$ for all $i \in S$. As a consequence, we can without loss of generality assume that for every agent $i \in S$, if $v_i(e) = 0$, then $\hat{v}_i(e) = 0$. In other words, reported valuation functions $\hat{v}_S$ only contain one type of misreporting; that is, some agent $j \in S$ reports $\hat{v}_j(e) = 0$ for chore e, while his true valuation is $v_j(e) = v(e) < 0$.

Let $Q, \bar{Q}$ and $Q^b, \bar{Q}^b$ be respectively the corresponding sets of indices constructed in Step 2 of Algorithm 1 under reported profiles $(v_S, \hat{v}_{-S})$ and $(\hat{v}_S, \hat{v}_{-S})$. For every $i \in S$, define $P_i = \{e \in E \mid v_i(e) = 0\}$ and $P_i^b = \{e \in E \mid \hat{v}_i(e) = 0\}$. Based on the aforementioned assumption, for any $i \in S$ and $e \in P_i$, it holds that $e \in P_i^b$. As reported valuations of agents $N \setminus S$ are consistent, we have $Q \subseteq Q^b$, equivalent to $\bar{Q}^b \subseteq \bar{Q}$.

If $\bar{Q}^b = \bar{Q}$, then we have

$$\begin{aligned}\sum_{i \in S} \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(v_S, \hat{v}_{-S})}[u_i(\mathbf{A}, v_i)] &= \sum_{i \in S} \frac{1}{n} v_i(\bigcup_{q \in \bar{Q}} e_q) \\ &= \sum_{i \in S} \frac{1}{n} v_i(\bigcup_{q \in \bar{Q}^b} e_q) \\ &\geq \sum_{i \in S} \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(\hat{v}_S, \hat{v}_{-S})}[u_i(\mathbf{A}, v_i)],\end{aligned}$$

where the first equality transition is because the assignment of $\bigcup_{q \in Q} e_q$ results in expected utility 0 for every agent $i \in S$ under the reported profile $(v_S, \hat{v}_{-S})$ and the inequality transition is due to that chores allocated in Step 3 yields non-positive expected utility. The above inequality contradicts the construction of $\hat{v}_S$.

If $\bar{Q}^b \subsetneq \bar{Q}$, as agents $N \setminus S$ consistently report $\hat{v}_{-S}$, every item e_q with $q \in \bar{Q} \setminus \bar{Q}^b$ must be allocated with probability one to one or a subset of agents in S under reported profile $(\hat{v}_S, \hat{v}_{-S})$. As the set of indices $\bar{Q}$ corresponds to reported profile $(v_S, \hat{v}_{-S})$, then for every chore e_q with $q \in \bar{Q}$, it holds that $v_i(e_q) = v(e_q)$ for all $i \in S$; note that if $v_i(e_q) = 0$, then $q \in Q$. Hence, under $(v_S, \hat{v}_{-S})$, the expected utility of agents S caused by assignment of Step 3 is equal to 0, and we have the following:

$$\begin{aligned}\sum_{i\in S}\mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(v_S,\hat{v}_{-S})}[u_i(\mathbf{A},v_i)] &= \sum_{i\in S}\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}^b}e_q)+\sum_{i\in S}\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}\setminus\bar{Q}^b}e_q)\\ &= \sum_{i\in S}\frac{1}{n}v(\bigcup_{q\in\bar{Q}^b}e_q)+\sum_{i\in S}\frac{1}{n}v(\bigcup_{q\in\bar{Q}\setminus\bar{Q}^b}e_q)\\ &\geq \sum_{i\in S}\frac{1}{n}v(\bigcup_{q\in\bar{Q}^b}e_q)+\sum_{q\in\bar{Q}\setminus\bar{Q}^b}v(e_q)\\ &= \sum_{i\in S}\mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(\hat{v}_S,\hat{v}_{-S})}[u_i(\mathbf{A},v_i)],\end{aligned}$$

where the second equality transition is due to $v_i(e_q)=v(e_q)$ for all $i\in S$ and $q\in\bar{Q}$ and the last equality transition is owing to the fact that every item e_q with $q\in\bar{Q}\setminus\bar{Q}^b$ is allocated to agents S with probability 1 under reported profile $(\hat{v}_S,\hat{v}_{-S})$; the inequality transition is due to $|S|\leq n$ and chores yielding non-positive value. The above inequality leads to a contradiction, completing the proof.

A.2.3. Proof of Proposition 4.9

Let $Q,\bar{Q}$ be the sets of indices constructed in Step 2 under profile $\mathbf{v}=(v_1,\ldots,v_n)$ and let $\mathbf{A}=(A_1,\ldots,A_n)$ be the returned randomized allocation. Suppose $\mathbf{A}'=(A'_1,\ldots,A'_n)=(a'_{j,i})_{j\in[m],i\in[n]}$ be the corresponding fractional allocation (matrix) implemented by $\mathbf{A}$. As the assignment of Step 3 yields an expected utility of zero for all agents, we then focus on the utility driven by the allocation in Step 5. Since permutation σ is generated uniformly at random, for any agent i and any item e_q with $q\in\bar{Q}$, the probability of assigning e_q to agent i is equal to $1/n$, and accordingly, $a'_{q,i}=1/n$. Then for any $i,j\in[n]$, $v_i(A'_i)=v_i(A'_j)=\frac{1}{n}\cdot\sum_{q\in\bar{Q}}v(e_q)=v_j(A'_j)$ since $v_l(e_q)=v(e_q)$ for all $q\in\bar{Q}$ and all $l\in[n]$. Thus, fractional allocation $\mathbf{A}'$ is EF (and hence PROP) and EQ, and therefore, randomized allocation $\mathbf{A}$ is ex-ante EF (and hence PROP) and EQ.

For the ex-post fairness guarantee, let $\mathbf{A}^*=(A^*_1,\ldots,A^*_n)$ be an arbitrary deterministic allocation in the support of $\mathbf{A}$. We first show that $\mathbf{A}^*$ is EQ1. Note that for any $q\in Q$, chore e_q is allocated to some agent i with $v_i(e_q)=0$ in $\mathbf{A}^*$, then the assignment of $\bigcup_{q\in Q}e_q$ in Step 3 does not affect agents' value. Thus, when proving EQ1, it suffices to consider only the reduced instance with set of items $E=\bigcup_{q\in\bar{Q}}e_q$ and valuation functions $v_i(e)=v(e)<0$ for all $i\in[n],e\in E$. Let σ^* be the permutation in Step 5 corresponding the deterministic allocation $\mathbf{A}^*$. Let the number of items be $m=kn+d$ with $k,d\in\mathbb{N}$ and $0\leq d<n$, and hence, the assignment in Step 5 has k or $k+1$ rounds. Given two agents $i,j\in[n]$, without loss of generality, we assume $\sigma^*(i)<\sigma^*(j)$. If agents i,j receive same number of items in $\mathbf{A}^*$, then in every single round, agent i receives value no less than that of agent j, implying $v_i(A^*_i)\geq v_j(A^*_j)$. As for agent j, the value of the item of j in every round l is at least the value of the item of agent i in round $l+1$. So by eliminating the last chore received by agent j, the value of agent j is at least that of agent i. Thus, allocation $\mathbf{A}^*$ is EQ1. For the case when agent i receives one more item, similarly the value received by agent j in every round $l\leq k$ is at least the value received by agent i in round $l+1$, implying $v_j(A^*_j)\geq v_i(A^*_i)$, i.e., agent j satisfies the property of EQ1. As for agent i, if the last item she receives is removed, she would have a value at least that of agent j. Therefore, allocation $\mathbf{A}^*$ is EQ1.

Next, we prove that $\mathbf{A}^*$ is also EF1. For any pair of agents $i,j\in[n]$, we have

$$\begin{aligned}v_i(A^*_j) &= v_i(A^*_j\cap\bigcup_{q\in\bar{Q}}e_q)+v_i(A^*_j\cap\bigcup_{q\in Q}e_q)\leq v(A^*_j\cap\bigcup_{q\in\bar{Q}}e_q)\\ &= v_j(A^*_j\cap\bigcup_{q\in\bar{Q}}e_q)=v_j(A^*_j),\end{aligned}$$

where the second and the third equality transitions are due to the fact that for any agent l and any $q\in\bar{Q}$, $v_l(e_q)=v(e_q)$ holds. As $\mathbf{A}^*$ is EQ1, the following holds,

$$\max_{e\in A^*_i}v_i(A_i\setminus\{e\})\geq v_j(A^*_j)\geq v_i(A^*_j).$$

Therefore, allocation $\mathbf{A}^*$ is also EF1 (and hence PROP1).

A.2.4. Proof of Proposition 4.10

Let $Q,\bar{Q}$ be the sets of indices constructed in Step 2 under profile $\mathbf{v}=(v_1,\ldots,v_n)$ and let $\mathbf{A}=(A_1,\ldots,A_n)$ be the returned randomized allocation, which implements fractional allocation (matrix) $\mathbf{A}'=(A'_1,\ldots,A'_n)=(a'_{j,i})_{j\in[m],i\in[n]}$. According to the proof of Proposition 4.9, for any $i\in[n]$, $v_i(A'_i)=\frac{1}{n}\sum_{q\in\bar{Q}}v(e_q)$ holds, which implies $\mathsf{UW}(\mathbf{A}')=\sum_{q\in\bar{Q}}v(e_q)$ and $\mathsf{EW}(\mathbf{A}')=\frac{1}{n}\sum_{q\in\bar{Q}}v(e_q)$. For an arbitrary fractional allocation $\mathbf{B}=(B_1,\ldots,B_n)$, it holds that $\mathsf{UW}(\mathbf{B})\leq\sum_{q\in\bar{Q}}v(e_q)$ as $v_i(e_q)=v(e_q)$ for all $i\in[n]$ and all $q\in\bar{Q}$. Moreover, by the pigeonhole principle, $\mathsf{EW}(\mathbf{B})\leq\frac{1}{n}\sum_{q\in\bar{Q}}v(e_q)$ holds. Therefore, the randomized allocation $\mathbf{A}$ is ex-ante UWM and EWM. By Proposition 4.3, allocation $\mathbf{A}$ is also ex-ante PO and ex-post UWM and PO.

It remains to show the ex-post guarantee regarding EWM. Let $\mathbf{A}^*=(A^*_1,\ldots,A^*_n)$ be an arbitrary deterministic allocation in the support of $\mathbf{A}$ and σ^* be the permutation in Step 5 corresponding to $\mathbf{A}^*$. Without loss of generality, assume $\sigma^*(i)=i$ for all $i\in[n]$. Similar to the proof of Proposition 4.9, we can further assume $E=\bigcup_{q\in\bar{Q}}e_q$; note that agents have identical valuation functions for items $\bigcup_{q\in\bar{Q}}e_q$. Let $|E|=kn+d$ with $k,d\in\mathbb{N}$ and $0\leq d<n$, then each agent $i\leq d$ receives $k+1$ items and each agent $i\geq d+1$ receives k items. Moreover, agent d receives the last item in Step 5; note that if $d=0$, we refer to agent n as agent d. We now show $v_d(A^*_d)\leq v_i(A^*_i)$ for all $i\in[n]$. For $i<d$, both agents d and i receive $k+1$ items and in each round, the value of agent i is at least the value of agent d, which implies $v_d(A^*_d)\leq v_i(A^*_i)$. For $i>d$, agent i receives an item in rounds $1,\ldots,k$ and moreover for any $l\in[k]$, the value of agent i in every round l is at least the value of agent d in round $l+1$. Therefore, it holds that $v_d(A^*_d)\leq v_i(A^*_i)$ for all $i\in[n]$ and thus $\mathsf{EW}(\mathbf{A}^*)=v_d(A^*_d)$. Denote by OPT_E and by OPT_U respectively the optimal egalitarian and utilitarian welfare among all deterministic allocations. We then show $v_d(A^*_d\setminus\{e'\})\geq\mathsf{OPT}_E$ where e' is the last item received by agent d. For a contradiction, assume $v_d(A^*_d\setminus\{e'\})<\mathsf{OPT}_E$. By Proposition 4.9, allocation $\mathbf{A}^*$ is EQ1, and thus for any $i\in[n]$, $v_i(A^*_i)\leq\max_{e\in A^*_d}v_d(A^*_d\setminus\{e\})=v_d(A^*_d\setminus\{e'\})$ holds where the equality transition is due to the allocation rule of Step 5. Accordingly, we have an upper bound of the optimal utilitarian welfare as follows,

$$\mathsf{OPT}_U=\mathsf{UW}(\mathbf{A}^*)=\sum_{i\in[n]}v_i(A^*_i)\leq n\cdot v_d(A^*_d\setminus\{e'\})+v_d(e')<n\cdot\mathsf{OPT}_E,$$

where the first equality transition is because $\mathbf{A}^*$ is ex-post UWM and the last inequality is due to our assumption $v_d(A^*_d\setminus\{e'\})<\mathsf{OPT}_E$ and $v_d(e')\leq 0$. However, since agents have identical valuations over $\bigcup_{q\in\bar{Q}}e_q$, it must hold that $\mathsf{OPT}_U\geq n\mathsf{OPT}_E$, contradicting the above inequality. Therefore, the claim $v_d(A^*_d\setminus\{e'\})\geq\mathsf{OPT}_E$ is proved. Additionally, it is not hard to verify $v_i(e')\geq\mathsf{OPT}_E$ for all $i\in[n]$ as agents have identical valuations over $\bigcup_{q\in\bar{Q}}e_q$. Then, the following holds,

$$\mathsf{EW}(\mathbf{A}^*)=v_d(A^*_d)=v_d(A^*_d\setminus\{e'\})+v_d(e')\geq 2\cdot\mathsf{OPT}_E,$$

which completes the proof.

A.2.5. Proof of Proposition 4.11

Let us consider the instance with n agents and a set $E=\{e_1,\ldots,e_{(n-1)n+1}\}$ of $(n-1)n+1$ chores. Agents have identical valuation functions: $v_i(e_j)=v(e_j)$ for all i,j, and inherent values are $v(e_j)=-1$ for all $j\leq(n-1)n$ and $v(e_{n(n-1)+1})=-n$. For any item e, we say e a *β-chore* if $v(e)=-\beta$ for $\beta\in\{1,n\}$. Thus, there are $n(n-1)$ 1-chore and one n-chore. Denote by OPT_E the optimal egalitarian welfare among all deterministic allocations, and naturally, $\mathsf{OPT}_E\leq -n$ as there must be an agent receiving the n-chore. Consider an allocation $\mathbf{B}$ in which every agent $i\in[n-1]$ receives a number n of 1-chore and agent n receives the unique n-chore. One can verify that $\min_{i\in[n]}v_i(B_i)=-n$ and thus $\mathsf{OPT}_E=-n$.

Let σ^* with $\sigma^*(i)=i$ for all $i\in[n]$ be a deterministic permutation in Step 5 and $\mathbf{A}^*=(A^*_1,\ldots,A^*_n)$ be the deterministic allocation returned by $\mathsf{RandChore}$ with respect to σ^*. Accordingly, $\mathbf{A}^*$ is a deterministic allocation in the support of $\widetilde{\mathcal{M}}^*(\mathbf{v})$. By the allocation rule of Step 5, for any $2 \le i \le n$, agent i receives a number $n-1$ of 1-chore and agent 1 receives a number $n-1$ of 1-chore and the unique n-chore. Thus, it holds that $\mathsf{EW}(\mathbf{A}^*) = v_1(A_1^*) = -2n+1$ and therefore, the approximation of $\mathbf{A}^*$ towards the optimal egalitarian welfare is at least $2 - \frac{1}{n}$, approaching 2 as $n \to +\infty$.

A.2.6. Other fairness notions

We define EFX_{-}^{+}, EQX_{-}^{+}, and PROPX_{-}^{+} in the setting of mixed items. These three fairness criteria reduce to EFX_{-}, EQX_{-} and PROPX_{-}, respectively, when all items are chores.

Definition A.1. An allocation $\mathbf{A} = (A_1, \dots, A_n)$ is EFX_{-}^{+} if for any $i, j \in [n]$, at least one of the following holds:

(i) for any $e \in A_i$ with $v_i(e) < 0$, $v_i(A_i \setminus \{e\}) \ge v_i(A_j)$,
(ii) for any $e \in A_j$ with $v_i(e) > 0$, $v_i(A_i) \ge v_i(A_j \setminus \{e\})$.

When all items are chores, the allocation $\mathbf{A}$ is EFX_{-} if property (i) holds.

Definition A.2. An allocation $\mathbf{A} = (A_1, \dots, A_n)$ is EQX_{-}^{+} if for any $i, j \in [n]$, at least one of the following holds:

(i) for any $e \in A_i$ with $v_i(e) < 0$, $v_i(A_i \setminus \{e\}) \ge v_j(A_j)$,
(ii) for any $e \in A_j$ with $v_j(e) > 0$, $v_i(A_i) \ge v_j(A_j \setminus \{e\})$.

When all items are chores, the allocation $\mathbf{A}$ is EQX_{-} if property (i) holds.

Definition A.3. An allocation $\mathbf{A} = (A_1, \dots, A_n)$ is PROPX_{-}^{+} if for any $i \in [n]$, at least one of the following holds:

(i) for any $e \in A_i$ with $v_i(e) < 0$, $v_i(A_i \setminus \{e\}) \ge \frac{1}{n} v_i(E)$,
(ii) for any $e \notin A_i$ with $v_i(e) > 0$, $v_i(A_i \cup \{e\}) \ge \frac{1}{n} v_i(E)$.

When all items are chores, the allocation $\mathbf{A}$ is PROPX_{-} if property (i) holds.

We present implications among these fairness notions.

Proposition A.4. *When all items are chores and $n \ge 2$, EFX_{-} implies PROPX_{-}.*

Proof. Suppose $\mathbf{A} = (A_1, \dots, A_n)$ is an EFX_{-} allocation. Fix agent i. According to the definition of EFX_{-}, for any $j \in [n]$, we have

$$\min_{e \in A_i : v_i(e) < 0} v_i(A_i \setminus \{e\}) \ge v_i(A_j).$$

Then summing up $j \in [n]$ on both sides implies

$$\min_{e \in A_i : v_i(e) < 0} v_i(A_i \setminus \{e\}) \ge \frac{1}{n} v_i(E),$$

equivalent to PROPX_{-}. □

Proposition A.5. *For mixed items and $n = 2$, EFX_{-}^{+} implies PROPX_{-}^{+}, but when $n \ge 3$, the implication does not necessarily hold.*

Proof. Suppose $\mathbf{A} = (A_1, A_2)$ is an EFX_{-}^{+} allocation. Let us focus on agent 1. By the definition of EFX_{-}^{+}, at least one of the following holds: (i) for any $e \in A_1$ with $v_1(e) < 0$, $v_1(A_1 \setminus \{e\}) \ge v_1(A_2)$, and (ii) for any $e \in A_2$ with $v_1(e) > 0$, $v_1(A_1) \ge v_1(A_2 \setminus \{e\})$. If property (i) holds, we have

$$\min_{e \in A_1 : v_1(e) < 0} v_1(A_1 \setminus \{e\}) \ge v_1(A_2) \implies \min_{e \in A_1 : v_1(e) < 0} v_1(A_1 \setminus \{e\}) \ge \frac{1}{2} v_1(E).$$

If property (ii) holds, we have

$$v_1(A_1) \ge \max_{e \in A_2 : v_1(e) > 0} v_1(A_2 \setminus \{e\}), \text{ equivalent to}$$

$$\min_{e \notin A_1 : v_1(e) > 0} v_1(A_1 \cup \{e\}) \ge v_1(A_2),$$

which implies

$$\min_{e \notin A_1 : v_1(e) > 0} v_1(A_1 \cup \{e\}) \ge \frac{1}{2} v_1(E).$$

Therefore, allocation $\mathbf{A}$ is PROPX_{-}^{+}.

When $n \ge 3$, let us consider the following instance with three agents and three items e_1, e_2, e_3. Agents have identical valuations $v'(\cdot)$ with $v'(e_1) = 5$, $v'(e_2) = 1$, and $v'(e_3) = 0$. Consider allocation $\mathbf{A}$ with $A_i = \{e_i\}$ for all i. Since every agent receives one item, $\mathbf{A}$ is EFX_{-}^{+}. However, agent 3 does not satisfy PROPX_{-}^{+}. The instance can be extended to any n-agent instance by adding $n-3$ copies of e_3. □

A.2.7. Proof of Theorem 4.12

Based on the Leximin rule, one can verify that Lemma 4.6 holds. Then Lemmas 4.7 and 4.8 (and hence Proposition 4.5) also hold for the new mechanism, which implies GSPIE. Let Q and $\bar{Q}$ be the sets of indices constructed in Step 2 under profile $\mathbf{v} = (v_1, \dots, v_n)$ and let $\mathbf{A} = (A_1, \dots, A_n)$ be the returned randomized allocation. Suppose $\mathbf{A}' = (A'_1, \dots, A'_n)$ is the fractional allocation implemented by $\mathbf{A}$. It is easy to see that for any $i, j \in [n]$, $v_i(A'_i) = v_i(A'_j) = \frac{1}{n} \sum_{q \in \bar{Q}} v(e_q) = v_j(A'_j)$. Then by arguments similar to those in the proof of Propositions 4.9 and 4.10, the mechanism satisfies ex-ante EF, PROP, EQ, UWM, EWM, PO and ex-post PO and UWM.

For other ex-post guarantees, let $\mathbf{A}^* = (A_1^*, \dots, A_n^*)$ be an arbitrary deterministic allocation in the support of $\mathbf{A}$. That $\mathbf{A}^*$ is EWM follows from the construction of $\bar{Q}_1, \dots, \bar{Q}_n$. Next, we show EQX_{-}. Since the allocation of $\bigcup_{q \in Q} e_q$ does not affect agents' values, it suffices to prove that the partial allocation of $\bigcup_{q \in \bar{Q}} e_q$ satisfies EQX_{-}. With a slight abuse of notation, let $\bar{Q}$ also denote $\bigcup_{q \in \bar{Q}} e_q$. For a contradiction, suppose that there exists i, j such that $\min_{e \in A_i^* \cap \bar{Q}} v_i(A_i^* \cap \bar{Q} \setminus \{e\}) < v_j(A_j^* \cap \bar{Q})$; note that as $v_i(e) < 0$ for all $e \in \bar{Q}$, we drop the requirement of $v_i(e) < 0$. Let $e^* \in \arg\max_{e \in A_i^* \cap \bar{Q}} v_i(e)$ and then we have

$$\min\left(v_i(A_i^* \cap \bar{Q} \setminus \{e^*\}), v_j(A_j^* \cap \bar{Q} \cup \{e^*\})\right) > v_i(A_i^* \cap \bar{Q}).$$

Moreover, $\{A_1^* \cap \bar{Q}, \dots, A_n^* \cap \bar{Q}\}$ is a permutation of $\{\bar{Q}_1, \dots, \bar{Q}_n\}$ and without loss of generality, assume that it is an identity permutation, i.e., $\bar{Q}_i = A_i^* \cap \bar{Q}$ for all i. Then the above inequality indicates that starting from $\{\bar{Q}_1, \dots, \bar{Q}_n\}$, reallocating e^* from $\bar{Q}_i$ to $\bar{Q}_j$ results in another n-partition of $\bar{Q}$ that violates the construction of $\bar{Q}_i$'s. We derive the desired contradiction.

For ex-post EFX_{-}, we have

$$\min_{e \in A_i^* : v_i(e) < 0} v_i(A_i^* \setminus \{e\}) = \min_{e \in A_i^* \cap \bar{Q}} v_i(A_i^* \cap \bar{Q} \setminus \{e\}) \ge v_j(A_j^*) \ge v_i(A_j^*),$$

where the first inequality is due to EQX_{-} and the second is because $v_j(A_j^*) = v_j(A_j^* \cap \bar{Q}) = v_i(A_j^* \cap \bar{Q})$ and $v_i(A_j^* \setminus \bar{Q}) \le 0$. By Proposition A.4, $\mathbf{A}^*$ is also PROPX_{-}.

A.2.8. Example in Freeman et al.

Consider an indivisible chores instance with four agents and eight chores. Agents have 2-restricted additive valuations as shown in the following table.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
$v_1(\cdot)$	−10	−10	−10	−10	−10	−10	−10	−10
$v_2(\cdot)$	−10	−10	−10	−10	−10	−10	−10	−10
$v_3(\cdot)$	−73	−1	−1	−1	−1	−1	−1	−1
$v_4(\cdot)$	−73	−1	−1	−1	−1	−1	−1	−1

For a contradiction, these exists an allocation $\mathbf{A}$ that is EQ1, EF1 and PO. Then by EQ1 and PO, one can verify that both agents 1 and 2 get exactly one chore in $\mathbf{A}$. Then a total of six chores are assigned between agents 3 and 4. Assume agent 3 gets at least three chores. Note that at least one of agents 1 and 2 does not receive e_1 in $\mathbf{A}$, and therefore, agent 3 violates EF1 when comparing to that agent. For the detailed proof, we refer readers to Freeman et al. (2020).

A.3. For Section 5

A.3.1. Proof of Lemma 5.3

Denote by $\{Q_i\}_{i=0}^3$ and $\{Q_i^k\}_{i=0}^3$, respectively, the corresponding indices sets constructed in Step 2 under reported profiles $(\hat{v}_i, \hat{v}_{3-i})$ and $(\hat{v}_i^k, \hat{v}_{3-i})$. Let Δ be the difference between expected utility of agent i under reported profile $(\hat{v}_i, \hat{v}_{3-i})$ and that under $(\hat{v}_i^k, \hat{v}_{3-i})$, i.e.,

$$\Delta = \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^2(\hat{v}_i,\hat{v}_{3-i})}[u_i(\mathbf{A}, v_i)] - \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^2(\hat{v}_i^k,\hat{v}_{3-i})}[u_i(\mathbf{A}, v_i)].$$

In what follows, we prove $\Delta \geq 0$ by carefully checking possible combinations of $\hat{v}_i(e_k)$, $\hat{v}_i^k(e_k)$ and $\hat{v}_{3-i}$.

Case 1: $\hat{v}_i(e_k) = v_i(e_k) = v(e_k)$.

Subcase 1.1: $\hat{v}_i^k(e_k) = 0$ and $\hat{v}_{3-i}(e_k) = v(e_k)$. Under these reported valuations, it holds that $Q_0 = Q_0^k$, $Q_i = Q_i^k$, $Q_{3-i} = Q_{3-i}^k \setminus \{e_k\}$ and $Q_3 = Q_3^k \cup \{e_k\}$. Then, by Proposition 5.2, we have $\Delta = \frac{1}{2}v_i(e_k) = \frac{1}{2}v(e_k) > 0$.

Subcase 1.2: $\hat{v}_i^k(e_k) = 0$ and $\hat{v}_{3-i}(e_k) = 0$. Under these reported valuations, it holds that $Q_0 = Q_0^k \setminus \{e_k\}$, $Q_i = Q_i^k \cup \{e_k\}$, $Q_{3-i} = Q_{3-i}^k$ and $Q_3 = Q_3^k$. Similarly, by Proposition 5.2, we have $\Delta = \frac{1}{2}v_i(e_k) = \frac{1}{2}v(e_k) > 0$.

Subcase 1.3: $\hat{v}_i^k(e_k) = 0$ and $\hat{v}_{3-i}(e_k) = -c(e_k)$. Under these reported valuations, the set of indices Q_t is identical to Q_t^k for all $t = 0, 1, 2, 3$, and therefore, $\Delta = 0$.

Subcase 1.4: $\hat{v}_i^k(e_k) = -c(e_k)$ and $\hat{v}_{3-i}(e_k) = v(e_k)$. Under these reported valuations, the composition of $\{Q_t^k\}_{t=0}^3$ and of $\{Q_t\}_{t=0}^3$ are identical to that of subcase 1.1, and accordingly, we also have $\Delta = \frac{1}{2}v(e_k) > 0$.

Subcase 1.5: $\hat{v}_i^k(e_k) = -c(e_k)$ and $\hat{v}_{3-i}(e_k) = 0$. Under these reported valuations, it holds that $Q_0 = Q_0^k$, $Q_i = Q_i^k \cup \{e_k\}$, $Q_{3-i} = Q_{3-i}^k \setminus \{e_k\}$ and $Q_3 = Q_3^k$. Again, by Proposition 5.2, we have $\Delta = v_i(e_k) = v(e_k) > 0$.

Subcase 1.6: $\hat{v}_i^k(e_k) = -c(e_k)$ and $\hat{v}_{3-i}(e_k) = -c(e_k)$. Under these reported valuations, it holds that $Q_0 = Q_0^k$, $Q_i = Q_i^k \cup \{e_k\}$, $Q_{3-i} = Q_{3-i}^k$ and $Q_3 = Q_3^k \setminus \{e_k\}$. According to Proposition 5.2, we have $\Delta = \frac{1}{2}v_i(e_k) = \frac{1}{2}v(e_k) > 0$.

Case 2: $\hat{v}_i(e_k) = v_i(e_k) = 0$. Note that reported valuations $\hat{v}_i$ and $\hat{v}_i^k$ only differ on e_k. According to RandMixed, deviating from $(\hat{v}_i, \hat{v}_{3-i})$ to $(\hat{v}_i^k, \hat{v}_{3-i})$ only affects indices sets Q_t's and Q_t^k's on whether includes e_k or not. As the valuation of e_k of agent i is 0, one can verify that for Case 2, $\Delta = 0$ holds.

Case 3: $\hat{v}_i(e_k) = v_i(e_k) = -c(e_k)$.

Subcase 3.1: $\hat{v}_i^k(e_k) = v(e_k)$ and $\hat{v}_{3-i}(e_k) = v(e_k)$. Under these reported valuations, it holds that $Q_0 = Q_0^k$, $Q_i = Q_i^k$, $Q_{3-i} = Q_{3-i}^k \cup \{e_k\}$ and $Q_3 = Q_3^k \setminus \{e_k\}$. By Proposition 5.2, we have $\Delta = -\frac{1}{2}v_i(e_k) = \frac{1}{2}c(e_k) > 0$.

Subcase 3.2: $\hat{v}_i^k(e_k) = v(e_k)$ and $\hat{v}_{3-i}(e_k) = 0$. Under these reported valuations, it holds that $Q_0 = Q_0^k$, $Q_i = Q_i^k \setminus \{e_k\}$, $Q_{3-i} = Q_i^k \cup \{e_k\}$ and $Q_3 = Q_3^k$. Then according to Proposition 5.2, we have $\Delta = -v_i(e_k) = c(e_k) > 0$.

Subcase 3.3: $\hat{v}_i^k(e_k) = v(e_k)$ and $\hat{v}_{3-i}(e_k) = -c(e_k)$. Under these reported valuations, it holds that $Q_0 = Q_0^k$, $Q_i = Q_i^k \setminus \{e_k\}$, $Q_{3-i} = Q_{3-i}^k$ and $Q_3 = Q_3^k \cup \{e_k\}$. Similarly, by Proposition 5.2, we have $\Delta = -\frac{1}{2}v_i(e_k) = \frac{1}{2}c(e_k) > 0$.

Subcase 3.4: $\hat{v}_i^k(e_k) = 0$ and $\hat{v}_{3-i}(e_k) = v(e_k)$. Under these reported valuations, we have $Q_t^k = Q_t$ for all $t = 0, 1, 2, 3$, and accordingly, $\Delta = 0$ holds.

Subcase 3.5: $\hat{v}_i^k(e_k) = 0$ and $\hat{v}_{3-i}(e_k) = 0$. Under these reported valuations, it holds that $Q_0 = Q_0^k \setminus \{e_k\}$, $Q_i = Q_i^k$, $Q_{3-i} = Q_{3-i}^k \cup \{e_k\}$ and $Q_3 = Q_3^k$. According to Proposition 5.2, we have $\Delta = -\frac{1}{2}v_i(e_k) = \frac{1}{2}c(e_k) > 0$.

Subcase 3.6: $\hat{v}_i^k(e_k) = 0$ and $\hat{v}_{3-i}(e_k) = -c(e_k)$. Under these reported valuations, it holds that $Q_0 = Q_0^k$, $Q_i = Q_i^k \setminus \{e_k\}$, $Q_{3-i} = Q_{3-i}^k$ and $Q_3 = Q_3^k \cup \{e_k\}$. Again, by Proposition 5.2, we have $\Delta = -\frac{1}{2}v_i(e_k) = \frac{1}{2}c(e_k) > 0$.

A.3.2. Proof of Proposition 5.5

Denote by $\{Q_t\}_{t=0}^3$ the set of indices constructed in Step 2 under profile $\mathbf{v} = (v_1, v_2)$ and by $\mathbf{A} = (A_1, A_2)$ the returned randomized allocation. Suppose $\mathbf{A}' = (A_1', A_2')$ be the fractional allocation (matrix) implemented by $\mathbf{A}$. According to Proposition 5.2, for any agent $i \in [2]$, he receives a value $v_i(A_i') = \frac{1}{2}v_i(\bigcup_{q\in Q_0\cup Q_3} e_q) + v_i(\bigcup_{q\in Q_i} e_q)$ in fractional allocation $\mathbf{A}'$. Moreover by arguments similar to that in the proof of Proposition 5.2, agent i's value of A_{3-i}' is equal to $v_i(A_{3-i}') = \frac{1}{2}v_i(\bigcup_{q\in Q_0\cup Q_3} e_q) + v_i(\bigcup_{q\in Q_{3-i}} e_q)$. Based on the definitions of Q_i and Q_{3-i}, for any $i \in [2]$ and $q \in Q_i$ (resp., $q \in Q_{3-i}$), we have $v_i(e_q) \geq 0$ (resp., $v_i(e_q) \leq 0$). Accordingly, it holds that $v_i(\bigcup_{q\in Q_i} e_q) \geq v_i(\bigcup_{q\in Q_{3-i}} e_q)$, implying $v_i(A_i') \geq v_i(A_{3-i}')$. Thus, fractional allocation $\mathbf{A}'$ is EF (and hence PROP), and therefore, randomized allocation $\mathbf{A}$ is ex-ante EF (and hence PROP).

For the ex-post fairness guarantee, let $\mathbf{A}^* = (A_1^*, A_2^*)$ be an arbitrary deterministic allocation in the support of $\mathbf{A}$. Fix an agent i. As $v_i(e_q) = 0$ for all $q \in Q_0$, we have $v_i(A_i^*) = v_i(A_i^* \cap \bigcup_{q\in Q_i} e_q) + v_i(A_i^* \cap \bigcup_{q\in Q_3} e_q)$ and $v_i(A_{3-i}^*) = v_i(A_{3-i}^* \cap \bigcup_{q\in Q_{3-i}} e_q) + v_i(A_{3-i}^* \cap \bigcup_{q\in Q_3} e_q)$. By the definitions of Q_i and Q_{3-i}, we have $v_i(A_i^* \cap \bigcup_{q\in Q_i} e_q) \geq v_i(A_{3-i}^* \cap \bigcup_{q\in Q_{3-i}} e_q)$. As a consequence, it suffices to show that the partial allocation of items $\bigcup_{q\in Q_3} e_q$ satisfies EF1. Then by arguments similar to that in the proof of Proposition 4.9, it is not hard to verify that the partial allocation upon $\bigcup_{q\in Q_3} e_q$ is EF1, and therefore, allocation $\mathbf{A}$ is ex-post EF1 and PROP1.

A.3.3. Proof of Proposition 5.6

According to Proposition 4.3, it suffices to prove ex-ante UWM. Let $\{Q_t\}_{t=0}^3$ be the set of indices constructed in Step 2 under profile $\mathbf{v} = (v_1, v_2)$. Denote by $\mathbf{A} = (A_1, A_2)$ the returned randomized allocation and by $\mathbf{A}' = (A_1', A_2')$ the fractional allocation implemented by $\mathbf{A}$. As two agents have identical valuations on items $\bigcup_{q\in Q_0\cup Q_3} e_q$, the utilitarian welfare derived from the assignment of $\bigcup_{q\in Q_0\cup Q_3} e_q$ must be identical among all (fractional) allocations. For every item e_q with $q \in Q_1 \cup Q_2$, it is allocated to the agent with a larger value in allocation $\mathbf{A}'$, and therefore, no fractional allocation can achieve a utilitarian welfare larger than $\mathsf{UW}(\mathbf{A}')$.

A.3.4. Proof of Theorem 5.7

Based on the Leximin rule, one can verify that Proposition 5.2 holds for the new mechanism. Then it is easy to check that Lemma 5.3 and Proposition 5.4 hold, which implies SPIE. Denote by $\{Q_t\}_{t=0}^3$ the sets of indices constructed in Step 2 under profile $\mathbf{v} = (v_1, v_2)$ and by $\mathbf{A} = (A_1, A_2)$ the returned randomized allocation. By arguments similar to those in the proofs of Propositions 5.5 and 5.6, one can show that the new mechanism is ex-ante PO, UWM, EF, and PROP, and ex-post PO and UWM.

As for other ex-post guarantees, let $\mathbf{A}^* = (A_1^*, A_2^*)$ be an arbitrary deterministic allocation in the support of $\mathbf{A}$. By Proposition A.5, it suffices to prove EFX_-^+. We focus on agent 1 and, with a slight abuse of notation, let Q_j denote $\bigcup_{q\in Q_j} e_q$ for all j. We first show that when restricted to Q_3, agent 1 satisfies EFX_-^+ in $\mathbf{A}^*$. Suppose not, at least one of the following two cases happens: (i) $\exists e \in A_1^* \cap Q_3$ with $v_1(e) < 0$ such that $v_1(A_1^* \cap Q_3 \setminus \{e\}) < v_1(A_2^* \cap Q_3)$, and (ii) $\exists e \in A_2^*$ with $v_1(e) > 0$ such that $v_1(A_1^* \cap Q_3) < v_1(A_2^* \cap Q_3 \setminus \{e\})$. If case (i) happens, let $e^1 \in \arg\max_{e\in A_1^*\cap Q_3 : v_i(e)<0} v_i(e)$ and we have $v_2(A_2^* \cap Q_3 \cup \{e^1\}) = v_1(A_2^* \cap Q_3 \cup \{e^1\}) > v_1(A_1^* \cap Q_3)$. Then, starting from $\{A_1^* \cap Q_3, A_2^* \cap Q_3\}$, reassigning e^1 from agent 1's bundle to agent 2 results in another 2-partition of Q_3 that contradicts the construction of $Q_{3,1}$ and $Q_{3,2}$. Similarly, if case (ii) happens, one can reassign one item, with positive value for agent 1, from the bundle of agent 2 to agent 1, resulting in another 2-partition of Q_3 that contradicts the construction of $Q_{3,1}$ and $Q_{3,2}$. Thus, restricting to Q_3, the partial allocation is EFX_-^+.

For the complete allocation $\mathbf{A}^* = (A_1^*, A_2^*)$, since $v_1(e) \leq 0$ for all $e \notin A_1^*$ and $v_1(e) \geq 0$ for all $e \in A_1^* \setminus Q_3$, the fact that the partial allocation of Q_3 is EFX_-^+ implies that $\mathbf{A}^*$ is EFX_-^+.

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