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Mental Accounting in Allocating Capacity

Meng Li
Department of Decision and Information Sciences, C. T. Bauer College of Business, University of Houston
mli@bauer.uh.edu

Yan Liu*
Department of Logistics and Maritime Studies, Faculty of Business, The Hong Kong Polytechnic University
yan.y.liu@polyu.edu.hk

Problem definition: This study investigates a seller's allocation of a limited resource to sequentially arriving customers when the seller is influenced by two types of mental accounting bias: prospective accounting (overestimating future revenue) and behavioral discounting (underestimating future revenue). **Methodology/results:** We establish structural properties on how mental accounting affects capacity allocation decisions and performance. Interestingly, while additional capacity consistently benefits the seller, the same does not hold true for additional demands. That is, an additional class of demand can hurt the seller, depending on the type of mental accounting. This is true even if the additional demand class has a higher reservation price than existing ones. **Managerial implications:** This result highlights the importance for companies to address and mitigate biases in decision-makers before embarking on market expansion initiatives through promotions and advertising campaigns.

Key words: revenue management, mental accounting, capacity allocation, behavioral bias

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All organizations, from General Motors down to single person households, have explicit and/or implicit accounting systems. The accounting system often influences decisions in unexpected ways.

—Thaler (1985).

1. Introduction

Mental accounting refers to the cognitive processes individuals and households use to organize, evaluate, and track their financial activities (Thaler 1985, 1999). It plays a pivotal role in deciphering the complex psychology behind decision-making, offering a clear framework for understanding various phenomena across fields such as consumer behavior (Thaler 1985, Heath and Soll 1996), finance (Barberis and Huang 2001), accounting (Burgstahler and Dichev 1997) and operations management (Becker-Peth et al. 2013). The practical implications of mental accounting are observable in real-world contexts. For example, Lungeanu and Weber (2021) highlight its impact on CEOs' resource allocation decisions, emphasizing its crucial influence on corporate decision-making.

In this paper, we explore the impact of mental accounting on decision-making and its influence on expected revenues in capacity allocation. This analysis is crucial, as many companies frequently encounter challenges in effectively managing their capacity. For example, consider an independent hotel that offers discounts to incentivize early bookings. The hotel may provide a steep discount for guests who book several months in advance, a moderate discount for bookings made closer to the stay date, and standard pricing for last-minute reservations. To maximize revenue, the hotel must carefully determine how many rooms to allocate at each price tier. Overcommitting too many rooms to early-booking discounts could leave it unable to meet demand from higher-paying last-minute guests, while underallocating could result in unsold inventory during off-peak times, especially if last-minute demand fails to materialize. Similarly, an upstream factory that manufactures winter coats for retail partners faces seasonal demand and limited production capacity. Early in the season, the factory may receive orders from discount retailers seeking low-cost inventory. However, fulfilling too many of these early orders could exhaust capacity or raw materials, leaving the factory unable to serve premium retailers who may place smaller but more profitable orders later in the season. The factory must therefore decide how to balance short-term fulfillment against the strategic reservation of capacity for higher-margin opportunities, all while considering uncertainties in future demand.

Furthermore, in this capacity allocation setting, studies consistently show significant deviations from theoretically optimal decisions, with mental accounting behavior. For instance, despite normative frameworks such as Littlewood's law in a key two-class setting of revenue management (Littlewood 1972, Talluri et al. 2005), where sellers allocate limited resources between two customer classes with different pricing and uncertain demand, and lower-priced customers arriving before higher-priced ones, research has demonstrated that mental accounting biases participants in controlled experiments (Kocabıyıkoğlu et al. 2018). Specifically, they value revenue differently based on whether it was generated by early low-end customers or later high-end customers.

In response to this, this paper incorporates mental accounting bias into the two-class revenue management model, including two forms of mental accounting: behavioral discounting, where the seller undervalues revenue from later-arriving customers, and prospective accounting, where future revenue is factored into the capacity allocation for earlier-arriving customers. We show that sellers exhibit "over-protecting" or "under-protecting" behavior toward high-end customers when they have prospective accounting and behavioral accounting, respectively. We extend our analysis to a multi-class scenario where customers arrive sequentially, each class representing higher revenue potential. In this more complex setting, incorporating mental accounting reveals patterns similar to those observed in the two-class case: behavioral discounting leads to under-protection of future demand, while prospective accounting results in over-protection. Consequently, we establish that (i) capacity allocation simplifies to setting protection levels for each demand class, (ii) these protection levels are independent of overall capacity, and (iii) the seller's revenue increases with capacity but decreases with bias. We further explore the impact of introducing additional premium demand (customers with higher reservation prices than existing ones) and additional basic demand (customers with lower reservation prices). Although it seems intuitive that more demand would lead to higher revenue, we find that such additional demand may actually harm the seller, depending on the type of mental accounting.

Under behavioral discounting, while additional premium demand benefits the seller, additional basic demand may not. Specifically, with the additional basic demand, the seller becomes less inclined to reserve inventory for future demand, exacerbating their under-protecting bias under behavioral discounting. This additional basic demand can hurt the seller if the associated loss outweighs the benefits of additional demand. For this effect to occur, the bias level must be sufficiently high, because if the bias is low, the seller's capacity allocation decisions remain closely aligned with optimal protection levels, minimizing distortion. Moreover, capacity must be moderate relative to total demand. If capacity is too high, resulting in unsold capacity, additional demand helps absorb the capacity, boosting revenue. However, if capacity is too low and demand greatly exceeds availability, additional demand does not impact revenue, as the limited capacity cannot satisfy even the original demand.

Under prospective accounting, we find that additional premium demand can paradoxically harm the seller. This occurs because the seller overestimates future revenue, leading to excessive protection of future demand. This over-protection bias is exacerbated when higher protection levels are required due to the introduction of premium demand, ultimately disadvantaging the seller under two conditions: (i) when the bias level is moderate, neither too small nor too large,[1] and (ii) when capacity is insufficient relative to demand, forcing the seller to allocate capacity (that would otherwise be sold to low-end classes) for premium customers at the expense of overall performance. This contrasts with the behavioral discounting case, where the seller tends to under-protect future demands, and the higher protection levels driven by premium demand help mitigate the under-protection bias, ultimately benefiting the seller.

Furthermore, we find that additional premium demand might be less valuable than additional basic demand under prospective accounting because, although the premium demand can be harmful, the additional basic demand is always beneficial for the seller. This is because introducing additional basic demand effectively mitigates the seller's over-protective behavior against selling

[1] Note that if the bias is too large, the seller who extremely overvalues the future revenue may withhold product from low-end classes, resulting in unsold capacity. In this case, the introduction of premium demand can help assimilate the unsold capacity and benefit the seller. This is in contrast to the behavioral discounting case where the negative result occurs when the bias is sufficiently large.

to earlier-arriving customers, ultimately benefiting the seller. This contrasts with the behavioral discounting case, where the additional basic demand reinforces, rather than mitigates, the seller's bias towards being overly willing to sell to earlier-arriving customers, ultimately hurting the seller.

Overall, this paper shows that additional demand might harm the seller, while additional capacity always benefits the seller. Moreover, the impact of extra demand hinges on how it interacts with, offsets or amplifies, the seller's existing behavioral bias, rather than on the inherent value of the additional demand itself.

2. Literature Review and Our Contribution

Our study focuses on a seller allocating capacity, incorporating the concept of mental accounting. As such, our work is related to two literature streams: capacity allocation and mental accounting.

Capacity Allocation. Our research is related to the extensive body of work on capacity allocation, a facet of quantity-based revenue management starting from the seminal work of Littlewood (1972). This stream of literature often divides customers into distinct classes, each characterized by unique traits (Zhang and Cooper 2005, Van Ryzin and Vulcano 2008, Cao et al. 2022). Given that deriving analytically tractable optimal allocation decisions can be elusive, this field primarily focuses on establishing structural properties for optimal policies. These properties are valuable not only for managerial insights but also for enabling efficient computation of optimal strategies.

The literature generally presupposes a rational seller striving to maximize expected revenue. However, this does not consistently mirror real scenarios and experimental observations. For example, Belobaba's seminal work (1987b) on capacity allocation reveals that even in prominent contexts like airline yield management, crucial decision elements remain reliant on human judgment rather than systematic analysis; Cooper et al. (2006) also note airlines frequently make capacity allocation errors. In controlled experiments, Bearden et al. (2008) consider a seller managing a fixed capacity over a season, and the seller must decide whether to accept or reject an arriving price offer to purchase a unit of the product. They find that participants can wrongly accept or reject an offer. In a similar design, Bendoly (2011) incorporates a decision support system to measure stress levels via physiological markers. He finds that high capacity levels left at the end of the booking horizon and the number of simultaneous tasks increase stress and induce decision errors. Bendoly (2013) conducts similar experiments with hotel employees and finds that different levels of feedback influence revenue performance. Kocabıyıkoğlu et al. (2015) study the two-class capacity allocation problem and a closely related newsvendor problem, and find the behavior in these two mathematically equivalent models does not align in the laboratory. Cesaret (2015) examines the seller behavior in the two-class capacity allocation model with arbitrary arrivals, and finds that participants often accept too many low-class customers. Cleophas and Schüetze (2024) study a setting with stationary and nonstationary demand, and they observe that subjects might not be able to accommodate a non-stationary demand.

Our work complements this literature by incorporating mental accounting, a prevalent cognitive bias, into the seller's decision-making to explore its theoretical implications within the context of capacity allocation. Accordingly, we establish structural properties on how mental accounting affects capacity allocation decisions: the seller's decision simplifies to establishing a sequence of nested protection levels for future demands, which deviate from the optimal. This echoes the literature showing that sellers frequently deviate from strict optimality while still employing decision policies that mirror the optimal capacity allocation approach (Bearden et al. 2008, Cesaret 2015). Moreover, we find that such a deviation of the protection levels may affect the value of demand substantially, depending on the seller's bias type.

Mental Accounting. Previous research has extensively examined mental accounting and its various forms to better understand and explain decision-making behaviors. Thaler (1985) and Heath and Soll (1996) incorporate mental accounting into utility functions, providing insights into behavioral anomalies. Building on this, Prelec and Loewenstein (1998) present a seminal dynamic model of mental accounting, emphasizing the role of prospective accounting. This model highlights the forward-looking nature of human cognition, where individuals prioritize present and future payments during consumption, with past payments having minimal psychological impact. Likewise, during payment transactions, the psychological discomfort associated with parting with money is tempered primarily by the anticipation of future consumption, rather than being influenced by past outlays. As a result, individuals often prefer prepayment, aligning the discomfort of paying with the expected pleasure of future consumption.

In operations management, Becker-Peth et al. (2013) employ mental accounting to expound upon newsvendor order decisions. Their study delineates between income derived from selling products to consumers and income generated through product returns to suppliers, effectively accounting for the source of income. Chen et al. (2013) propose the existence of two distinct mental accounting paradigms that impact the ordering behavior of newsvendors: time discounting and prospective accounting. The former entails a preference for receiving benefits sooner, while the latter involves comprehensive consideration of future transactions, coupled with a relative discounting of past transactions. In a mathematically equivant problem, namley two class capacity allocation problem, Kocabıyıkoğlu et al. (2015) find that participants' choices diverge markedly from the optimal outcomes in controlled experiments, and mental accounting can explain such divergence

(Kocabıyıkoğlu et al. 2018). Accordingly, we extend the application of both time discounting and prospective accounting to capacity allocation. We find that while greater capacity consistently favors the seller, an increase in demand can inadvertently yield detrimental effects on the seller. This is particularly significant, given that effective customer management often forms a pivotal aspect of a seller's operations decisions.

3. Mental Accounting

Consider the classical two-class model: a seller with a capacity C, and two customer classes with associated reservation prices p_1 and p_2 ($\leq p_1$) and demands D_1 and D_2. Here, D_1 and D_2 are random variables and independent of each other, and the cumulative distribution function of D_1 is denoted by $F(\cdot)$. The demand D_2 arrives earlier than demand D_1. The seller decides how much the second demand class to accept (i.e., the sales u for low-end customers) right after observing the low-end demand d_2 but before the realization of the high-end demand:

$$\max_{y\in[0,C]} \; p_2 u(d_2,C,y) + p_1 \mathsf{E}\Big[D_1 \wedge \big(C - u(d_2,C,y)\big)\Big], \tag{1}$$

where

$$u(d,C,y) := (C-y)^+ \wedge d, \tag{2}$$

for any $d, C, y \geq 0$, $x^+ = \max\{x,0\}$, and $x \wedge y = \min\{x,y\}$. The seller's problem can be formulated as a problem of deciding the *protection level* y for high-end customers such that the seller sells to low-end customers only if the capacity C exceeds the protection level. Accordingly, $(C-y)^+$ represents the maximum capacity the seller is willing to sell to low-end customers. The optimal solution for (1) is represented as $y^* = \bar{F}^{-1}(\beta)$, where $\beta := p_2/p_1$ and $\bar{F}(\cdot) = 1 - F(\cdot)$.

We now incorporate the notion of mental accounting into this classical capacity allocation framework. In particular, the seller displays cognitive bias by valuing revenue from different customer segments (high-end vs. low-end customers) unequally, and behaves as if solving

$$\max_{y\in[0,C]} \; p_2 u(d_2,C,y) + \eta \cdot p_1 \mathsf{E}\Big[D_1 \wedge \big(C - u(d_2,C,y)\big)\Big], \tag{3}$$

where η is the mental accounting parameter. In other words, with mental accounting, the seller behaves as though it were maximizing (3) in lieu of maximizing (1). Note that $\eta > 1$ aligns with the concept of prospective accounting as described by Prelec and Loewenstein (1998), in which decision-makers anchor their evaluations to anticipated future outcomes. In this context, when the seller collects revenue from low-end consumers, they simultaneously factor in the expectation of future revenue, represented by $p_1 \mathsf{E}[D_1 \wedge (C - u(d_2,C,y))]$. This results in an overvaluation of revenue from high-end consumers, particularly those arriving later in the sequence. Simultaneously, this can translate into a tendency to undervalue revenue from early-arriving low-end consumers, thus exhibiting a form of bias consistent with prospective accounting. In another case when $\eta < 1$, it resembles another well-established mental accounting principle: behavioral discounting. Kocabıyıkoğlu et al. (2018) highlight the prevalence of these mental accounting dimensions among participants in capacity allocation experiments; they use data from capacity allocation experiments and estimate that the average level η of prospective accounting exhibited by participants can range from 1.43 to 2.4, whereas the average level of behavioral discounting can range from 0.59 to 0.9.

Let $\hat{y}(\eta)$ denote the solution of (3). Then, the seller's ensued profit is

$$\hat{\pi}(\eta) := p_2 u(d_2, C, \hat{y}(\eta)) + p_1 \mathsf{E}\Big[D_1 \wedge \big(C - u(d_2, C, \hat{y}(\eta))\big)\Big]. \tag{4}$$

Lemma 1. *a) For a seller defined by the mental accounting parameter η, the protection level $\hat{y}(\eta)$ increases in η.*

b) Moreover, the seller's resulting revenue $\hat{\pi}(\eta)$ decreases in η when $\eta > 1$ and increases in η when $\eta \le 1$.

Lemma 1a indicates that a seller's protection level is increasing of its value of the mental accounting parameter η. Intuitively, the larger η, the seller values the revenue from high-end customers more, and thus reserves more products for high-end customers. Hence, the protection level increases in η. Basically, under the behavioral discounting condition ($\eta < 1$), the biased seller's decision on the protection level is lower than the normative level (i.e., *under-protecting* the high-end customers), while under the prospective accounting condition ($\eta > 1$), the biased seller's decision on the protection level is higher than the normative level (i.e., *over-protecting* the high-end customers). This behavior is observed in laboratory settings simulating a seller's capacity allocation decision (Kocabıyıkoğlu et al. 2018, 2015) and showing that the subjects in the experiments systematically over or under protect the high-end consumers. Moreover, the larger the bias magnitude $|\eta - 1|$, the further the protection level deviates from the true optimal, and thus the lower the seller's revenue. Accordingly, Lemma 1b indicates that the seller's revenue decreases in the level of prospective accounting ($\eta > 1$) and increases in the level of behavioral accounting ($\eta < 1$).

We next extend the two-class setting to the general scenario involving n (≥ 2) classes, pioneered by Belobaba (1987a,b). A seller, armed with a capacity C (≥ 0), serves n classes of customers, each associated with a random demand D_i and a reservation price p_i, where $1 \le i \le n$. The classes are ordered such that

$$p_1 \ge p_2 \ge \cdots \ge p_n. \tag{5}$$

This sequence manifests as class n customers entering the initial stage (stage n), followed by class $n-1$ customers in the subsequent stage (stage $n-1$), and eventually culminating with class 1 customers in the final stage (stage 1). This staged progression mirrors the increasing revenue, wherein class n arrives foremost and class 1 arrives last. Such an arrival process is quite common in airline seat allocation problems (Belobaba 1987a,b, Brumelle and McGill 1993). Note that a one-to-one correspondence exists between stages and classes within this n-class framework. Consequently, the value of n serves a dual purpose, representing both the index of classes and the index of stages. Similar to the two-class case, at each stage with remaining capacity x, the seller makes sales decisions based on demand realization. This is akin to determining a protection level y, where the demand is accepted if the remaining capacity surpasses y, and otherwise rejected. Accordingly, given the capacity level x_i and demand realization d_i at stage i, the Bellman equation is

$$V_i(x_i) = \max_{0 \le y_{i-1} \le x_i} p_i u(d_i, x_i, y_{i-1}) + \mathsf{E}\Big[V_{i-1}\big(x_i - u(d_i, x_i, y_{i-1})\big)\Big], \quad \text{for } 1 \le i \le n \text{ and } x_i \le C, \tag{6}$$

where $u(d_i, x_i, y_{i-1})$, defined in (2), represents the sales at stage i, with the boundary condition $V_0(x) = 0$ for any $x \le C$. The right-hand side of (6) comprises two components: the revenue from class i demand (first term) and the value-to-go after selling to class i demand (second term). Ultimately, the seller's decisions revolve around determining a sequence of protection levels $y_{n-1},\ y_{n-2},\ \cdots,\ y_1$, wherein y_i is reserved for class i ($1 \le i \le n-1$) and subsequent classes.

We now incorporate mental accounting into this n-class model framework. For $i = n, n-1, \cdots, 1$, the seller behaves as if solving

$$U_i(x_i) = \max_{0 \le y_{i-1} \le x_i} p_i u(d_i, x_i, y_{i-1}) + \eta_{i-1}\mathsf{E}\Big[U_{i-1}\big(x_i - u(d_i, x_i, y_{i-1})\big)\Big], \tag{7}$$

with the boundary condition $U_0(x) = 0$ for any $x \le C$. Here, η_{i-1} is the mental accounting factor for class $i-1$, and $\boldsymbol{\eta} := \{\eta_{n-1}, \eta_{n-2}, \cdots, \eta_1\}$. The formulation (7) is rooted in the common practice within marketing and economics, wherein the seller's objective function is additively separable over money and time (Prelec and Loewenstein 1998, Kőszegi and Rabin 2006). We also align with the literature (O'Donoghue and Rabin 1999) by assuming that the seller possesses perfect foresight of her future mental accounting behavior. In this paper, we focus on the case of behaviorial discounting where $\boldsymbol{\eta} < 1$ (Section 4) and the case of prospective accounting where $\boldsymbol{\eta} > 1$ (Section 5).

Without loss of generality, we assume that the initial inventory C is always greater than y_{n-1}. Later, we show that the protection level decisions $\{y_i : i = n-1, \cdots, 1\}$ are independent of the initial capacity C. Then, consider an instance where there exists an $i \in \{n-1, n-2, \cdots, 1\}$ such that $y_i > C \ge y_{i-1}$. In this scenario, the seller abstains from selling any product to classes n,

$n-1, \cdots, i+1$. As a result, the original n-class problem reduces to an i-class problem where class i customers arrive first. One can check that if $y_{n-1} \le x_n$, then $y_{i-1} \le x_i$ holds for any $i \in \{n-1, n-2, \cdots, 1\}$, and thus the constraint $0 \le y_{i-1} \le x_i$ in (7) can reduce to $y_{i-1} \ge 0$. We next characterize the solution of the above dynamic program and the biased marginal value of capacity $\Delta U_i(x) := \mathsf{E}_{D_i}[U_i(x) - U_i(x-1)]$.

Proposition 1. *a) Let* $\{\hat{y}_j(\boldsymbol{\eta}) : 1 \le j \le n-1\}$ *denote the solution to dynamic programming equations* (7)*. Then, the optimal protection levels for* $j \in \{1, 2, \cdots, n-1\}$ *are jointly determined by* $n-1$ *probability equations:*

$$\mathrm{P}\Big(D_1 \ge \hat{y}_1(\boldsymbol{\eta}), D_1 + D_2 \ge \hat{y}_2(\boldsymbol{\eta}), \cdots, D_1 + \cdots + D_j \ge \hat{y}_j(\boldsymbol{\eta})\Big) = \frac{p_{j+1}}{\eta_j \eta_{j-1} \cdots \eta_1 p_1}. \tag{8}$$

b) Moreover, for each $i \in \{1, 2, \cdots, n-1\}$, $\hat{y}_i(\boldsymbol{\eta})$ *increases in* η_i.

c) Furthermore, $\Delta U_i(x)$ *is positive and decreasing in* x *for any* $i \in \{1, 2, \cdots, n-1\}$.

Proposition 1a shows that, with mental accounting, the seller's capacity allocation decision can be characterized by a series of protection levels which are jointly decided by $n-1$ equations as specified in (8). Proposition 1b generalizes the essence of Lemma 1a to encompass situations involving multiple customer classes. Specifically, it indicates that the protection level $\hat{y}_i(\boldsymbol{\eta})$ for each class i increases in η_i. As η_i grows, the seller allocates more resources to class i and later classes, prioritizing revenue accumulation from those classes. Proposition 1c demonstrates that the biased marginal value of capacity is positive and decreasing in the capacity level, i.e., the value function is increasing and concave in the capacity level. For insights, note that

$$\begin{aligned}
\Delta U_i(x) = p_i\ & \mathrm{P}(D_i \ge x - \hat{y}_{i-1}(\boldsymbol{\eta})) \\
& + \eta_{i-1} p_{i-1}\ \mathrm{P}(D_i < x - \hat{y}_{i-1}(\boldsymbol{\eta}), D_i + D_{i-1} \ge x - \hat{y}_{i-2}(\boldsymbol{\eta})) \\
& + \quad \cdots \\
& + \eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1\ \mathrm{P}\Big(D_i < x - \hat{y}_{i-1}(\boldsymbol{\eta}), \cdots, \sum_{j=2}^{i} D_j < x - \hat{y}_1(\boldsymbol{\eta}), \sum_{j=1}^{i} D_j \ge x\Big).
\end{aligned} \tag{9}$$

Here, $\hat{y}_{i-1}$ is the optimal protection level for class $i-1$ and later classes, so the maximum amount of products that can be sold to class i is $x - \hat{y}_{i-1}$. If $D_i \ge x - \hat{y}_{i-1}$, the seller sells this unit of product to class i with a unit revenue p_i, corresponding to the first term in (9). If $D_i < x - \hat{y}_{i-1}$ but the total demand of classes i and $i-1$ is greater than $x - \hat{y}_{i-2}$ (the maximum amount of products that can be sold to classes i and $i-1$), then the seller does not sell this unit of product to class i but sells to class $i-1$ with a unit revenue p_{i-1}, corresponding to the second term in (9). We continue in this fashion and, if the cumulative demand at each stage j (for $2 \le j \le i$) is smaller than the maximum amount of products that are allowed to sell by that stage, but the total demand from all classes (class i to class 1) exceeds the capacity x, then the seller sells to class 1 with a unit revenue p_1, corresponding to the last term in (9). At its core, capacity allocation involves the trade-off between selling a product at a lower price and deferring the sale in anticipation of higher future revenue, while facing the risk of unsold inventory. This trade-off is mathematically akin to the trade-off in managing prices. Both strategies, i.e., limiting supply and raising prices, can impact sales. Empirical evidence and controlled experiments consistently reveal that sellers frequently deviate from strict optimality while still employing decision policies that mirror the optimal approach (Bearden et al. 2008, Cesaret 2015).

Given the protection levels $\{\hat{y}_i(\boldsymbol{\eta}) : i = 1, 2, \cdots, n-1\}$ characterized in Proposition 1, the seller's ensuing revenue is

$$\hat{V}_i(x_i) = \mathsf{E}\Big[p_i \min\{D_i, x_i - \hat{y}_{i-1}(\boldsymbol{\eta})\} + \hat{V}_{i-1}\big(x_i - \min\{D_i, x_i - \hat{y}_{i-1}(\boldsymbol{\eta})\}\big)\Big], \quad \text{for } 1 \le i \le n, \tag{10}$$

with the boundary condition $\hat{V}_0(x) = 0$ for any $x \le C$. That is, although the seller behaves as if solving equations (7), her ensuing revenue is described by (10).

PROPOSITION 2. *a) For any $i \in \{1, 2, \cdots, n\}$, $\hat{V}_i(x)$ decreases in η_{i-1} in the prospective accounting case and increases in η_{i-1} in the behavioral discounting case.*
b) Moreover, $\hat{V}_i(x)$ is increasing in x for any $i \in \{1, 2, \cdots, n\}$.

In line with Lemma 1b, Proposition 2a demonstrates that, within the multi-class framework, the seller's profit diminishes as the bias intensifies. This outcome stems from the fact that a greater bias, manifested through either prospective accounting or behavioral discounting, leads to more pronounced deviations from the true optimal allocation decision as $|\eta_{i-1} - 1|$ rises. Additionally, Proposition 2b establishes that the biased seller's revenue escalates in relation to the remaining capacity. This implies that the marginal value of capacity of the biased seller

$$\begin{aligned}
\Delta\hat{V}_i(x) := {} & \hat{V}_i(x) - \hat{V}_i(x-1) \\
= {} & p_i \ \mathrm{P}\left(D_i \ge x - \hat{y}_{i-1}(\boldsymbol{\eta})\right) \\
& + \ p_{i-1} \ \mathrm{P}\left(D_i < x - \hat{y}_{i-1}(\boldsymbol{\eta}), \ D_i + D_{i-1} \ge x - \hat{y}_{i-2}(\boldsymbol{\eta})\right) \\
& + \quad \cdots \\
& + \ p_1 \ \mathrm{P}\Big(D_i < x - \hat{y}_{i-1}(\boldsymbol{\eta}), \ \cdots, \ \sum_{j=2}^{i} D_j < x - \hat{y}_1(\boldsymbol{\eta}), \ \sum_{j=1}^{i} D_j \ge x\Big)
\end{aligned} \tag{11}$$

is non-negative. That is, the seller's supply or capacity retains its inherent value, irrespective of the cognitive disposition of the seller. Rather, we next show that the value of demand might not remain when the seller is biased.

4. Behavioral Discounting

In the context of behavioral discounting, we examine the impact of additional demand that is either (i) basic, with a lower reservation price compared to existing demand, or (ii) premium, with a higher reservation price compared to existing demand.

4.1 Additional Basic Demand

When the additional demand is basic, there are $n+1$-classes of customers with prices

$$p_1 > p_2 > \cdots > p_n > p_{n+1}. \tag{12}$$

In this scenario, compared to the *baseline case* described by (5), the prices p_i and demands D_i for $1 \le i \le n$ remain unchanged. However, there is an additional customer segment characterized by a lower reservation price p_{n+1} than that of the existing customers, along with a positive demand D_{n+1}. Then, the biased seller described by parameters $\{\eta_n, \eta_{n-1}, \cdots, \eta_1\}$ behaves as if solving (7) for $i = n+1, n, \cdots, 1$. Accordingly, the protection levels in the additional demand case are identical to the baseline case for all demand classes from stage $n-1$ to stage 1, owing to the determination rule (8) in Proposition 1.

Given an initial capacity level x and the protection level $\hat{y}_n(\boldsymbol{\eta})$ for class n (and the remaining classes), the total expected revenue in the additional demand case is

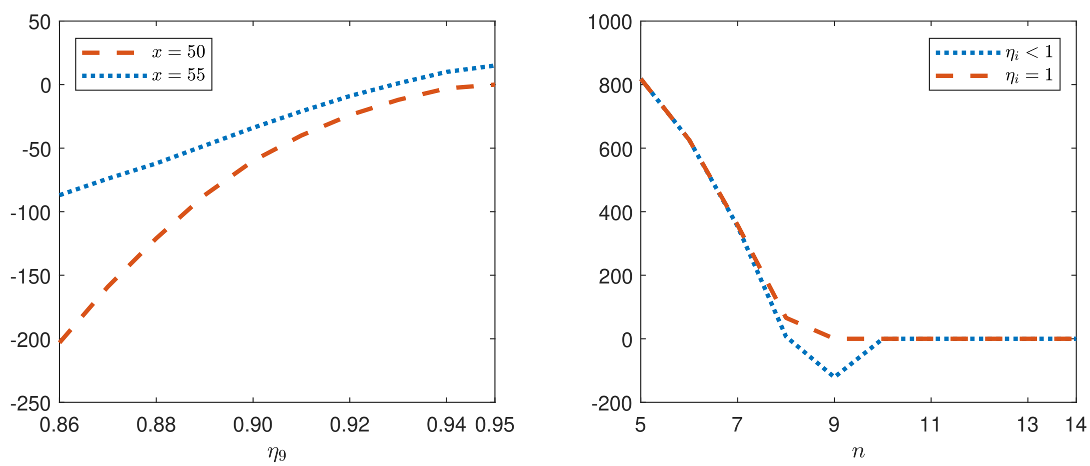
$$\hat{V}_{n+1}(x) = \mathsf{E}\Big[p_{n+1}\min\{D_{n+1}, x - \hat{y}_n(\boldsymbol{\eta})\} + \hat{V}_n\big(x - \min\{D_{n+1}, x - \hat{y}_n(\boldsymbol{\eta})\}\big)\Big], \tag{13}$$

where $\hat{V}_n(x)$ is the revenue in the baseline case. Intuition suggests that the seller gets higher revenue when there is an additional class of demand. This is indeed the case for the unbiased case, where the seller endowed with $n+1$ classes of demand can at least mimic the decisions with n classes. However, we next show this is not always true for the biased seller with behavioral discounting.

Proposition 3. *In the presence of behavioral discounting, there exist parameters such that* $\eta_n \Delta U_n(x) < p_{n+1} < \min\{\eta_n \Delta U_n(x-1), \Delta\hat{V}_n(x)\}$, *under which the biased seller earns less revenue with an additional basic demand—that is,* $\hat{V}_{n+1}(x) < \hat{V}_n(x)$.

Proposition 3 reveals that additional demand can harm the seller. This is noteworthy because, at a minimum, the seller could simply decline to serve the extra demand class, resulting in identical revenue to the baseline case. In other words, if no product is sold to class $n+1$, there would be no difference in revenue between the two cases. Yet, ironically, Proposition 3 demonstrates that by accommodating the additional demand class, the seller can actually earn less. For insights, in the presence of behavioral discounting, a biased seller might choose to accept class $n+1$ demand if the immediate unit revenue p_{n+1} exceeds the perceived (expected) marginal value of capacity $\eta_n \Delta U_n(x)$, where $\Delta U_n(x)$ is defined by (9). However, this perceived marginal value is lower than the actual marginal value of capacity defined in (11), i.e., $\eta_n \Delta U_n(x) < \Delta \hat{V}_n(x)$ due to $\eta_i < 1$ for $1 \le i \le n$. Therefore, $\eta_n \Delta U_n(x) < p_{n+1} < \Delta \hat{V}_n(x)$ might hold, i.e., the immediate unit revenue p_{n+1} is higher than the perceived marginal value of capacity, but lower than the actual marginal value of capacity. As a result, the decision to sell to the additional demand class proves detrimental, to the point that it outweighs the immediate revenue gained from that class, ultimately harming the seller.

We also provide insights into the conditions under which the result in Proposition 3 might occur. First, the bias level should not be too small. As the bias approaches zero, the seller's capacity allocation aligns closely with the optimal levels. As a result, the seller's revenue increases due to the extra demand. However, when the bias is high, it causes significant deviations from the optimal protection levels, and consequently, a negative impact on the value of additional demand emerges. As per Figure 1a, the demand value, $\hat{V}_{n+1}(x) - \hat{V}_n(x)$, is negative when the bias is high (the discounting parameter is small), and becomes positive as the discounting parameter approaches to 1.



(a) $n = 9$ (b) $x = 50$

Figure 1 The demand value $\hat{V}_{n+1}(x) - \hat{V}_n(x)$ under behavioral discounting: For $1 \le i \le n+1$, D_i are independent and identically distributed uniform variables on $[0, 10]$. Moreover, $p_1 = 500$ and $p_{i+1} = 0.8 p_i$. In (a), $\{\eta_9, \eta_8, \ldots, \eta_1\}$ is increasing, where $\eta_{i-1} = \eta_i + 0.005$, and η_9 varies from 0.86 to 0.95. In (b), we use the same increasing sequence for the values of $\{\eta_i : 1 \le i \le 15\}$: $\{\eta_{15}, \eta_{14}, \ldots, \eta_1\} = \{0.855, 0.86, \ldots, 0.925\}$ for the line $\eta_i < 1$, and $\{\eta_{15}, \eta_{14}, \ldots, \eta_1\} = \{1, 1, \ldots, 1\}$ for the line $\eta_i = 1$.

Second, capacity should be balanced relative to demand. When capacity significantly exceeds demand, any additional demand becomes valuable as it absorbs the unused capacity. As shown in Figure 1a, at $x = 50$, the lack of additional capacity limits the ability to meet extra demand, leading to a lower demand value; at $x = 55$, the extra capacity allows for greater demand fulfillment, increasing the demand value. However, when capacity is too small, additional demand does not contribute to revenue since it cannot even meet the original demand. Figure 1b depicts the value of demand under different counts of demand classes (represented by the parameter n) while maintaining a fixed capacity level of 50. It shows that when the seller is biased ($\eta_i < 1$ for all i), the value of demand declines from positive to negative as the number of demand classes increases. Initially, with fewer demand classes, capacity exceeds demand, allowing each additional class to generate profit, resulting in a positive demand value. Yet, as the number of demand classes n rises, total demand approaches capacity, reducing the demand value. When $n = 9$, the demand value turns negative, consistent with Figure 1a. As the number of demand classes continues to increase such that $n \geq 10$, the demand value starts to rise again, eventually stabilizing at zero. This is because the protection level reaches the capacity limit, preventing the seller from fulfilling any orders from lower demand classes. Consequently, the demand value remains at zero. It is worth noting that, when the seller is unbiased ($\eta_i = 1$ for all i) in Figure 1b, the overall trend is similar, but the demand value never falls below zero.

4.2 Additional Premium Demand

When the additional demand is premium, the demand has the following reservation prices:

$$p_0 > p_1 > p_2 > \cdots > p_n. \tag{14}$$

Compared to the baseline case defined by (5), this case introduces an extra demand with the highest reservation price. We use $\{y^i : i = n-1, \cdots, 1, 0\}$ to denote the protection levels in the premium demand case. Given the capacity level x_i and demand realization d_i at stage i, the Bellman equation for the above premium demand case is

$$V^i(x_i) = \max_{0 \leq y^{i-1} \leq x_i} p_i u(d_i, x_i, y^{i-1}) + \mathsf{E}\left[V^{i-1}\left(x_i - u(d_i, x_i, y^{i-1})\right)\right], \quad \text{for } i \geq 1 \text{ and } x_i \leq C,$$

where $u(d_i, x_i, y^{i-1})$, as defined in (2), represents the sales at stage i. In contrast to the baseline case where the boundary condition is $V_0(\cdot) = 0$, the boundary condition now is the revenue generated by the premium demand $V^0(x_0) := p_0 \mathsf{E}[\min\{D_0, x_0\}]$ for any $x_0 \leq C$. Therefore, $V_n(x) \leq V^n(x)$ for any x, meaning that the premium demand is always beneficial for the unbiased seller. This is because the extra demand class offers both an extra sales opportunity and a revenue source with the highest selling price possible.

We now incorporate the notation of mental accounting into the premium demand case defined by (14). Then, at stage i, the seller behaves as if solving

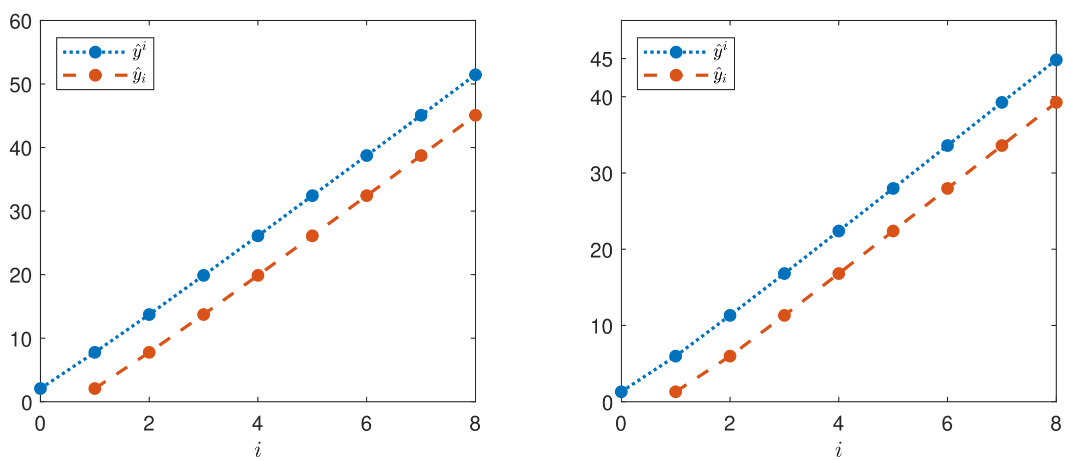
$$U^i(x_i) = \max_{0 \le y^{i-1} \le x_i} p_i u(d_i, x_i, y^{i-1}) + \eta_{i-1} \mathsf{E}\left[U^{i-1}\left(x_i - u(d_i, x_i, y^{i-1})\right)\right]$$

for $i = \{n, n-1, \cdots, 1\}$, and $U^0(x_0) = V^0(x_0)$. One can obtain the protection levels $\{\hat{y}^i(\boldsymbol{\eta}) : i = 0, 1, \cdots, n-1\}$ by solving

$$\mathrm{P}\Big(D_0 \ge \hat{y}^0(\boldsymbol{\eta}), D_0 + D_1 \ge \hat{y}^1(\boldsymbol{\eta}), \cdots, D_0 + \cdots + D_j \ge \hat{y}^j(\boldsymbol{\eta})\Big) = \frac{p_{j+1}}{\eta_j \eta_{j-1} \cdots \eta_1 \eta_0 p_0}. \tag{15}$$

The seller may allocate products for the premium demand, leading to distinct protection levels compared to the baseline case. Next, we compare these protection levels between the two cases.

Lemma 2. *Given $\boldsymbol{\eta}$, the protection levels in the premium demand case are always higher than the baseline case, i.e., $\hat{y}^i(\boldsymbol{\eta}) \ge \hat{y}_i(\boldsymbol{\eta})$ for $i = 1, 2, \cdots, n-1$.*



(a) Prospective Accounting (b) Behavioral Discounting

Figure 2 The protection levels $\hat{y}^i(\boldsymbol{\eta})$ and $\hat{y}_i(\boldsymbol{\eta})$: For $0 \le i \le 9$, D_i are i.i.d. uniform variables on $[0, 10]$. In (a), $\eta_8 = 1.18$, $\eta_{i-1} = \eta_i + 0.005$, $p_0 = 510$, and $p_{i+1} = 29/30 p_i$. In (b), $\eta_8 = 0.88$, $\eta_{i-1} = \eta_i + 0.005$, $p_0 = 510$, and $p_{i+1} = 0.8 p_i$.

Intuitively, the seller needs to reserve a certain amount of products for the premium demand which arrives at the latest with the highest reservation price. Consequently, the seller would reserve more products when deciding protection levels for all other classes, i.e., the protection level at each stage is higher in the premium demand case than in the baseline case; see Figure 2 for an illustration. In this figure, the protection level gap remains consistently stable across stages (classes) between the two cases: $\hat{y}^i - \hat{y}_i \approx \hat{y}^{i-1} - \hat{y}_{i-1}$. That is, the seller in the premium demand case allocates the same amount of capacity to each class i (where $1 \leq i \leq n-1$) as that in the baseline case, implying that the capacity allocated for premium demand is drawn from what was originally allocated to class n customers without premium demand.

Given the protection levels $\{\hat{y}^i(\boldsymbol{\eta}) : i = 0, 1, \cdots, n-1\}$, the seller's resulting revenue is

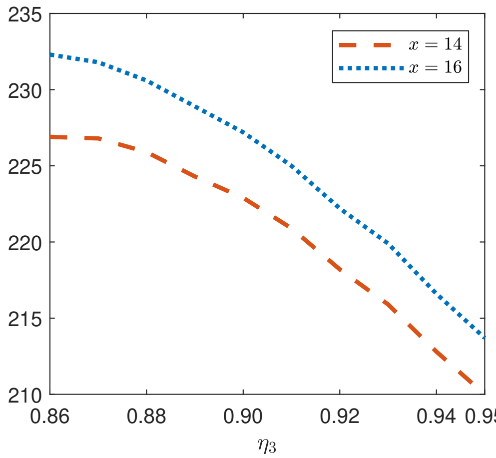
$$\hat{V}^i(x_i) := \mathsf{E}\Big[p_i \min\{D_i, x_i - \hat{y}^{i-1}(\boldsymbol{\eta})\} + \hat{V}^{i-1}\left(x_i - \min\{D_i, x_i - \hat{y}^{i-1}(\boldsymbol{\eta})\}\right)\Big], \tag{16}$$

for $i = n, n-1, \cdots, 1$, with the boundary condition $\hat{V}^0(x_0) = V^0(x_0)$. Next, we compare the revenue $\hat{V}^n(x)$ with the baseline case $\hat{V}_n(x)$ defined in (10).

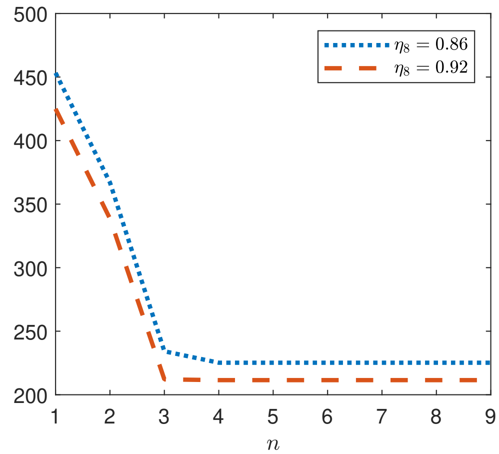
Lemma 3. *In the presence of behavioral discounting, $\hat{V}^n(x) \geq \hat{V}_n(x)$ always holds for any x.*

Unlike the basic demand with a lower reservation price, the premium demand consistently benefits the seller. Under behavioral discounting, the biased seller typically under-protects future demand. However, in the presence of premium demand, protection levels, representing the quantity of product withheld for future sales, are generally higher than in the baseline case; see Lemma 2. This suggests that the biased seller is more likely to delay sales and reserve products for future premium demand, mitigating the impact of behavioral discounting on protection levels. As a result, the premium demand scenario consistently yields higher revenue for the biased seller under behavioral discounting, as shown in Figure 3. In addition, we make two observations in Figure 3. First, the value of premium demand increases as the seller's bias increases. For insights, as the seller becomes more biased and applies greater discounts to future demand, their willingness to allocate products to premium demand diminishes. This can benefit the seller, as reserving fewer/no products for premium demand results in higher revenue compared to the baseline case. Second, the value of premium demand will not decline to zero as the number of demand class n rises. This is because the protection level $\hat{y}^0$ is fixed, independent of the number of classes n. Hence, the actual value of the products allocated to class 0 is fixed.

In sum, under behavioral discounting, the additional demand with a lower reservation price may hurt the seller, whereas the premium demand with a higher reservation price always benefits the seller. Note that under behavioral discounting, the seller tends to under-protect future demands. When the additional demand with the lowest price arrives first, the seller is less willing to allocate products to the future demand. This effect aligns with and exacerbates the seller's under-protection bias, ultimately hurting the seller's revenue. Conversely, when the premium demand with the highest price arrives the latest, the seller is more willing to allocate products to the future demand. This effect is opposite to the behavioral discounting bias. As a result, the higher protection level induced by the premium demand helps alleviate the seller's under-protection bias, eventually improving the



(a) $n = 4$



(b) $x = 14$

Figure 3 The value of premium demand $\hat{V}^n(x) - \hat{V}_n(x)$ under behavioral discounting: For $1 \le i \le n$, D_i are i.i.d. uniform variables on $[0, 10]$, and $D_0 = \frac{p_1}{\eta_0 p_0} X + (1 - \frac{p_1}{\eta_0 p_0}) Y$ where $X \sim N(1, \epsilon^2)$, and $Y \sim N(0, \epsilon^2)$ for a significantly low ϵ. Moreover, $p_0 = 510$, and $p_{i+1} = 0.8 p_i$. In (a), $\{\eta_3, \eta_2, \cdots, \eta_0\}$ is increasing, where $\eta_{i-1} = \eta_i + 0.005$, and η_3 varies from 0.86 to 0.95. In (b), $\{\eta_8, \eta_7, \cdots, \eta_0\} = \{0.86, 0.865, \cdots, 0.9\}$ for the line $\eta_8 = 0.86$, and $\{\eta_8, \eta_7, \cdots, \eta_0\} = \{0.92, 0.925, \cdots, 0.96\}$ for the line $\eta_8 = 0.92$.

seller's overall revenue. In essence, the impact of the additional basic and premium demands hinges on how they interact with, offset or amplify, the seller's existing behavioral biases. The premium demand has a beneficial effect by counteracting the under-protection bias, whereas the additional low-price demand reinforces and worsens that bias.

5. Prospective Accounting

Section 4 demonstrates that under behavioral discounting, the additional premium demand always benefits the seller, whereas the additional basic demand may not. Next, we explore whether this positive effect of additional premium demand on the seller still holds when it exhibits prospective accounting.

5.1 Additional Premium Demand

Proposition 4. *In the presence of prospective accounting, there exist parameters under which the biased seller earns less revenue with premium demand—that is,* $\hat{V}^n(x) < \hat{V}_n(x)$.

Proposition 4 shows that the premium demand can hurt the seller under prospective accounting, i.e., the disadvantage attributed to more demand remains valid even if the additional demand is premium. Under certain conditions, the seller reserves one unit of product for class 0 customers ($\hat{y}^0 = 1$), and the difference between protection levels in the premium demand case and the baseline case remains constant such that $\hat{y}^i - \hat{y}_i = 1$ for each $i \in \{1, 2, \cdots, n-1\}$; see Figure 2. This implies that the single unit reserved for class 0 is effectively taken from the capacity originally allocated to class n customers in the absence of premium demand. Consequently, the revenue difference between the premium demand case and the baseline case hinges on the revenue generated by this single unit of capacity. In the premium demand case, the expected revenue of this unit of capacity is

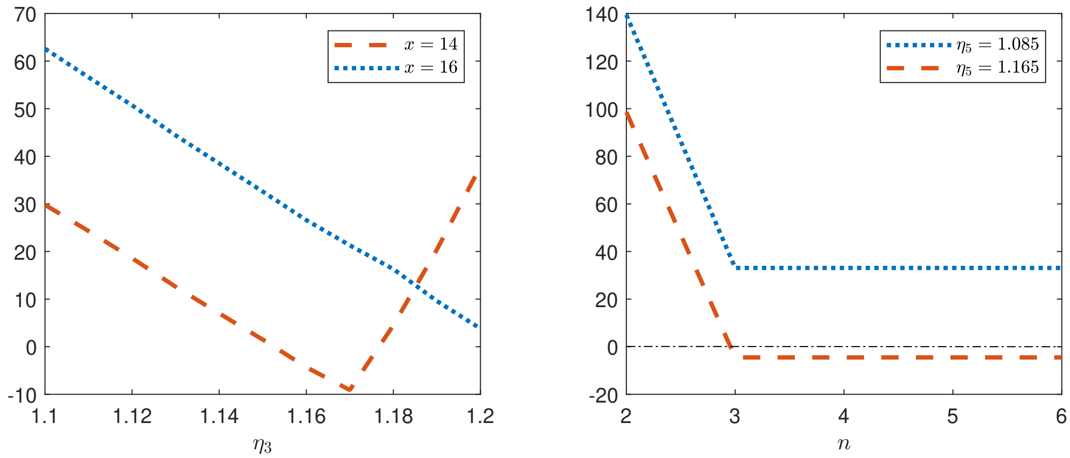
$$p_0 \mathrm{P}(D_0 \geq \hat{y}^0) = p_0 \frac{p_1}{\eta_0 p_0} = \frac{p_1}{\eta_0}. \tag{17}$$

On the other hand, in the baseline case, this unit of product may be sold to class n, class $n-1$, $\cdots$, or class 1, the expected revenue of which is equal to the marginal value of capacity $\Delta \hat{V}_n(x)$, as specified in (11). As long as $\sum_{i=1}^{n} D_i \geq x$, this product can be sold for certain, and the revenue is a weighted average of prices $\{p_i : i = 1, 2, \cdots, n\}$. It can be greater than $\frac{p_1}{\eta_0}$ when n is small and η_0 is sufficiently large. As a result, the premium demand case can lead to a lower revenue than the baseline case.

For insights, in the presence of premium demand, the protection levels are generally higher compared to the baseline case without premium demand. When the seller exhibits a prospective accounting bias and overvalues the revenue from future demand classes, they tend to over-protect those future demands. Thus, the higher protection levels driven by the premium demand exacerbate the seller's over-protection bias, ultimately hurting the seller. This contrasts with the behavioral discounting case, where the seller tends to under-protect future demands, and the higher protection levels driven by premium demand help mitigate the under-protection bias, ultimately benefiting the seller.

We also provide insights into the conditions under which the result in Proposition 4 might occur. First, the bias level should be moderate, neither too small nor too large; see Figure 4a for an illustration. When the bias level is sufficiently small, the seller approaches rational behavior, and the protection levels align with the true optimal levels. In this case, introducing premium demand would increase the seller's revenue as expected. However, when the bias level becomes very large where the seller overvalues the revenue from the high-end classes and withholds product from the low-end classes, the protection levels become highly distorted and result in unsold capacity. In such cases, the introduction of premium demand can help assimilate the unsold capacity and benefit the seller. This is in contrast to the behavioral discounting case where the result in Proposition 3 occurs when the bias level is sufficiently large.

Second, the capacity should not be too large relative to the aggregated demand. If the capacity is too large, the inventory surpasses the demand, and any additional demand (including premium



(a) $n=4$

(b) $x=14$

Figure 4 The value of premium demand $\hat{V}^n(x)-\hat{V}_n(x)$ under prospective accounting: For $1\le i\le n$, D_i are i.i.d. uniform variables on $[0,10]$, and $D_0=\frac{p_1}{\eta_0 p_0}X+(1-\frac{p_1}{\eta_0 p_0})Y$ where $X\sim N(1,\epsilon^2)$ and $Y\sim N(0,\epsilon^2)$ for a significantly low ϵ. Moreover, $p_0=510$, and $p_{i+1}=29/30p_i$. In (a), $\{\eta_3,\eta_2,\eta_1,\eta_0\}$ is increasing, where $\eta_{i-1}=\eta_i+0.005$, and η_3 varies from 1.1 to 1.2. In (b), we adopt the same increasing sequences: $\{\eta_5,\eta_4,\cdots,\eta_0\}=\{1.085,1.09,\cdots,1.11\}$ for the line $\eta_5=1.085$, and $\{\eta_5,\eta_4,\cdots,\eta_0\}=\{1.165,1.17,\cdots,1.19\}$ for the line $\eta_5=1.165$.

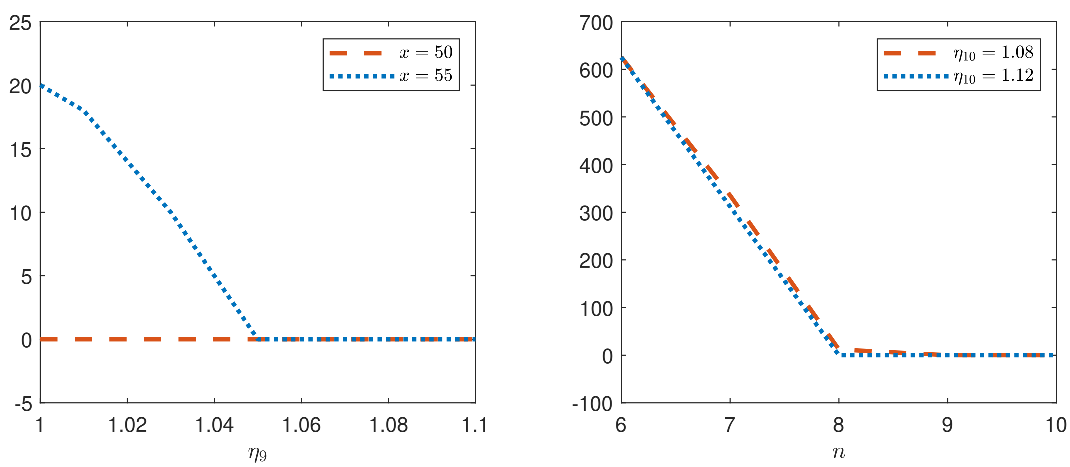
demand) helps assimilate the excess inventory, thereby benefiting the seller. If the capacity is too small, recall that in the behavioral discounting case, any additional demand has no impact on the seller's revenue as the finite capacity cannot even fulfill the original demand. In contrast, in the prospective accounting case, the allocation to premium demand remains fixed according to (15), regardless of changes in capacity or demand count. Therefore, even if the demand is far beyond the capacity, the allocation to premium demand can harm the seller's revenue because the revenue reduction due to the allocation remains fixed; see Figure 4b for an illustration. In Figure 4b, the protection level $\hat{y}^0$ is fixed at 1, independent of the number of classes n. Therefore, the actual expected revenue of the product allocated to class 0 is fixed, as shown in (17). On the other hand, because $\hat{y}_3\ge 14$, due to the limited inventory, the seller does not sell any product to class 4, class 5, and any inferior classes as n increases. This means that the expected revenue of that product (allocated to class 0 in the premium demand case) in the baseline case, as specified in (11), is also fixed. Comparing (17) with (11) leads to a fixed negative value of the premium demand.

5.2 Additional Basic Demand

So far, we have shown that the additional premium demand is not necessarily a plus for the seller. Then, it seems less surprising that the additional basic demand might hurt the seller under some conditions (as shown in Proposition 3), given that the premium demand has a higher reservation price than the basic demand. However, Lemma 4 below shows that the additional basic demand is always beneficial for the seller under prospective accounting, even though it seems less valuable than the premium demand.

LEMMA 4. *In the presence of prospective accounting,* $\hat{V}_{n+1}(x) \geq \hat{V}_n(x)$ *always holds for any* x*.*

For insights, with the additional basic demand, the seller becomes more inclined to sell to these customers when they arrive. Under prospective accounting, the seller overvalues the revenue from future demand and is therefore less willing to sell to earlier-arrived customers. This effect is opposite to the direct impact of introducing the new customer class. As a result, the additional demand essentially counteracts the seller's bias towards being less willing to sell to earlier-arrived customers, eventually benefiting the seller's overall performance. This contrasts with the behavioral discounting case, where the seller discounts the future demand and is thus more willing to sell to earlier-arrived customers, which amplifies the direct impact of introducing the new customer class. Consequently, under behavioral discounting, the additional demand reinforces, rather than mitigates, the seller's bias towards being overly willing to sell to earlier-arrived customers, ultimately hurting the seller's revenue.



(a) $n = 9$ (b) $x = 50$

Figure 5 The demand value $\hat{V}_{n+1}(x) - \hat{V}_n(x)$ under prospective accounting: For $1 \leq i \leq n+1$, D_i are independent and identically distributed uniform variables on $[0, 10]$. Moreover, $p_1 = 500$ and $p_{i+1} = 0.8p_i$. In (a), $\{\eta_9, \eta_8, \cdots, \eta_1\}$ is increasing, where $\eta_{i-1} = \eta_i + 0.005$, and η_9 varies from 1 to 1.1. In (b), $\{\eta_{10}, \eta_9, \cdots, \eta_1\} = \{1.08, 1.085, \cdots, 1.125\}$ for the line $\eta_{10} = 1.08$, and $\{\eta_{10}, \eta_9, \cdots, \eta_1\} = \{1.12, 1.125, \cdots, 1.165\}$ for the line $\eta_{10} = 1.12$.

Figure 5 illustrates the demand value $\hat{V}_{n+1}(x) - \hat{V}_n(x)$ under prospective accounting. As per this figure, $\hat{V}_{n+1}(x) - \hat{V}_n(x)$ is higher when $x = 55$ than $x = 50$ because the scarcity in capacity can limit the value of additional demand. Moreover, $\hat{V}_{n+1}(x) - \hat{V}_n(x) = 0$ when η_9 is sufficiently high, where the seller overvalues the future demands, so that it might not sell to the additional customers. Finally, $\hat{V}_{n+1}(x) - \hat{V}_n(x)$ decreases with n, i.e., the value of additional demand diminishes to zero as existing demands increase.

To summarize, under the prospective accounting bias, where the seller tends to over-protect future demands, the additional demand with a lower reservation price always benefits the seller, whereas the premium demand with a higher reservation price may hurt the seller. When the additional demand with the lowest price arrives first, the seller is less willing to allocate products to the future demand. This effect is opposite to and alleviates the seller's over-protection bias, ultimately improving the seller's revenue. Conversely, when the premium demand with the highest price arrives at the latest, the seller is more willing to allocate products to future demand. This effect aligns with and exacerbates the seller's over-protection bias under prospective accounting, eventually leading to a negative impact on the seller's overall revenue. Similar to the scenario under behavioral discounting, the impact of the additional and premium demands hinges on how they interact with, offset or amplify, the seller's existing bias. The additional low-price demand has a beneficial effect by counteracting the over-protection bias, whereas the premium demand reinforces and worsens that bias. The key distinction is that under prospective accounting, the seller's tendency is to over-protect future demands, in contrast to the under-protection bias observed under behavioral discounting.

6. Conclusion

In this paper, we investigate a capacity allocation problem in which the manager exhibits mental accounting bias: prospective accounting and behavioral discounting. In prospective accounting, biased sellers consistently overvalue future demand and allocate excessive capacity to accommodate it. On the other hand, in behavioral discounting, biased sellers undervalue future demand and allocate insufficient capacity. Contrary to expectations, additional demand does not always translate into higher earnings. The outcome depends on the type of mental accounting and the nature of the demand, sometimes resulting in lower revenue despite the presence of extra demand.

One of the key challenges in operations management is effectively balancing supply and demand. Companies must ensure the availability of resources while stimulating demand for their products or services. For example, an independent hotel might invest in expanding its facilities by adding more rooms or enhancing amenities. Alternatively, it might allocate significant resources to promotions and advertising campaigns aimed at expanding its market and attracting new customer segments. Our research suggests that companies should prioritize capacity expansion, as it consistently increases revenue regardless of the cognitive biases of decision-makers. However, if a company invests resources solely in attracting additional customers, our findings indicate that even successful efforts may not be beneficial. Specifically, attracting more customers can potentially reduce a company's revenue while increasing promotional and advertising costs. This paradox arises because the seller's cognitive biases significantly influence how demand impacts profitability. In cases of behavioral discounting, biased sellers may not benefit from increased demand from low-end customers. Conversely, with prospective accounting biases, targeting high-end customers can lead to negative consequences. In a similar vein, attracting more premium demand is not necessarily better than attracting more basic demand.

Given these findings, it is crucial for companies to address and mitigate the cognitive biases of their decision-makers before implementing demand management strategies. By proactively de-biasing decision-making processes, companies can make more informed and rational choices when targeting new markets. Optimizing the allocation of resources and capacity in this way allows them to maximize outcomes and enhance the overall effectiveness of their demand management efforts. In summary, companies should not only focus on increasing demand and managing supply but also consider the cognitive biases of their decision-makers. By implementing de-biasing strategies, they can optimize their expansion plans and improve their overall success in new markets.

With limited analytical exploration of managerial behavior, several research directions can usefully be pursued in the future. First, in our model, earlier consumers consistently have lower reservation prices compared to later arrivals. A valuable extension would be to relax this assumption, allowing for the possibility that later arrivals may not necessarily have higher reservation prices. Second, we acknowledge that other behavioral biases, such as regret, may also exist. Therefore, similar to Long and Wu (2024), future research could explore a more general model incorporating different functional forms to capture both regret and mental accounting. This is interesting because regret fundamentally differs from mental accounting in several ways—for example, the rule used to determine optimal protection levels in the two-class model does not generalize to a multi-class setting when regret is taken into account. Third, the seller is a price-taker. It would be interesting to consider a price-setter who must decide prices to sell capacity. Fourth, our analytical study could be extended to experimental or empirical contexts. Future research can follow approaches provided in the literature to test different types of mental accounting and then accordingly determine their impacts on capacity allocation.

Acknowledgement

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Appendix: Proofs

Proof of Lemma 1. a) If $d_2 < C - y$, the objective function of (3) $p_2 d_2 + \eta p_1 \mathsf{E}[\min\{D_1, C - d_2\}]$ is independent of y. If $d_2 \geq C - y$, then the first-order-condition of (3) is $p_2 - \eta p_1 \bar{F}(y) = 0 \Rightarrow \bar{F}(\hat{y}(\eta)) = \frac{\beta}{\eta}$. Accordingly, $\hat{y}(\eta)$ increases in η.
b) If $d_2 < C - \hat{y}$, then $\hat{\pi}(\eta) = p_2 d_2 + p_1 \mathsf{E}[\min\{D_1, C - d_2\}]$ is independent of $\hat{y}$. If $d_2 \geq C - \hat{y}$, $\hat{\pi}(\eta) = p_2(C - \hat{y}) + p_1 \mathsf{E}[\min\{D_1, \hat{y}\}]$ and its derivative with respect to $\hat{y}$ is $-p_2 + p_1 \bar{F}(\hat{y}) = -p_2 + p_1 \frac{p_2}{\eta p_1} = p_2(1/\eta - 1)$. Therefore, $\hat{\pi}(\eta)$ increases in $\hat{y}$ when $\eta \leq 1$ and decreases in $\hat{y}$ when $\eta > 1$. According to Lemma 1a, $\hat{y}(\eta)$ increases in η, so $\hat{\pi}(\eta)$ increases in η when $\eta \leq 1$ and decreases in η when $\eta > 1$. ■
Proof of Proposition 1. Part (a): We show this result by induction. When $i = 1$, $\bar{F}(\hat{y}_1) = \frac{p_2}{\eta_1 p_1}$. Suppose

$$\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}) = \frac{p_i}{\eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1} \tag{18}$$

holds. Let $h(x) = p_{i+1}\min\{d_{i+1}, x - \hat{y}_i\} + \eta_i \mathsf{E}[U_i(x - \min\{d_{i+1}, x - \hat{y}_i\})]$. If $d_{i+1} < x - \hat{y}_i$, $h(x) = p_{i+1}d_{i+1} + \eta_i \mathsf{E}[U_i(x - d_{i+1})]$ is independent of $\hat{y}_i$. Hereafter, we assume $d_{i+1} \geq x - \hat{y}_i$. Then, $h(x) = p_{i+1}(x - \hat{y}_i) + \eta_i \mathsf{E}[U_i(\hat{y}_i)]$, and $\frac{dh(x)}{d\hat{y}_i} = -p_{i+1} + \eta_i \frac{d\mathsf{E}[U_i(\hat{y}_i)]}{d\hat{y}_i} = -p_{i+1} + \eta_i p_i \mathrm{P}(D_i \geq \hat{y}_i - \hat{y}_{i-1}) + \eta_i\eta_{i-1}p_{i-1}\mathrm{P}(D_i < \hat{y}_i - \hat{y}_{i-1}, D_{i-1} + D_i \geq \hat{y}_i - \hat{y}_{i-2}) + \eta_i\eta_{i-1}\eta_{i-2}p_{i-2}\mathrm{P}(D_i < \hat{y}_i - \hat{y}_{i-1}, D_{i-1} + D_i < \hat{y}_i - \hat{y}_{i-2}, D_{i-2} + D_{i-1} + D_i \geq \hat{y}_i - \hat{y}_{i-3}) + \cdots + \eta_i\eta_{i-1}\cdots\eta_1 p_1 \mathrm{P}(D_i < \hat{y}_i - \hat{y}_{i-1}, D_{i-1} + D_i < \hat{y}_i - \hat{y}_{i-2}, \cdots, D_2 + \cdots + D_i < \hat{y}_i - \hat{y}_1, D_1 + D_2 + \cdots + D_i \geq \hat{y}_i) = 0$, where the second equality holds because of Lemma 6 (in online appendix). Moving p_{i+1} to the right hand side and dividing $\eta_i\eta_{i-1}\cdots\eta_1 p_1$ on both sides yields

$$\begin{aligned}
&\frac{p_i}{\eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1}\mathrm{P}(D_i \geq \hat{y}_i - \hat{y}_{i-1}) + \frac{p_{i-1}}{\eta_{i-2}\eta_{i-3}\cdots\eta_1 p_1}\mathrm{P}(D_i < \hat{y}_i - \hat{y}_{i-1}, D_{i-1} + D_i \geq \hat{y}_i - \hat{y}_{i-2}) \\
&+ \frac{p_{i-2}}{\eta_{i-3}\eta_{i-4}\cdots\eta_1 p_1}\mathrm{P}(D_i < \hat{y}_i - \hat{y}_{i-1}, D_{i-1} + D_i < \hat{y}_i - \hat{y}_{i-2}, D_{i-2} + D_{i-1} + D_i \geq \hat{y}_i - \hat{y}_{i-3}) + \cdots \\
&+ \mathrm{P}(D_i < \hat{y}_i - \hat{y}_{i-1}, D_{i-1} + D_i < \hat{y}_i - \hat{y}_{i-2}, \cdots, D_2 + \cdots + D_i < \hat{y}_i - \hat{y}_1, D_1 + D_2 + \cdots + D_i \geq \hat{y}_i) \\
&= \frac{p_{i+1}}{\eta_i\eta_{i-1}\cdots\eta_1 p_1}.
\end{aligned} \tag{19}$$

In addition, as (18) holds, Lemma 5 (in online appendix) implies that

$$\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_1 + \cdots + D_i \geq \hat{y}_i)$$

$$=\frac{p_i}{\eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1}\mathrm{P}(D_i \geq \hat{y}_i - \hat{y}_{i-1}) + \frac{p_{i-1}}{\eta_{i-2}\eta_{i-3}\cdots\eta_1 p_1}\mathrm{P}(D_i < \hat{y}_i - \hat{y}_{i-1}, D_i + D_{i-1} \geq \hat{y}_i - \hat{y}_{i-2}) + \cdots$$
$$+ \frac{p_2}{\eta_1 p_1}\mathrm{P}(D_i < \hat{y}_i - \hat{y}_{i-1}, \cdots, D_i + D_{i-1} + \cdots + D_3 < \hat{y}_i - \hat{y}_2, D_i + D_{i-1} + \cdots + D_2 \geq \hat{y}_i - \hat{y}_1)$$
$$+ \mathrm{P}(D_i < \hat{y}_i - \hat{y}_{i-1}, \cdots, D_i + D_{i-1} + \cdots + D_2 < \hat{y}_i - \hat{y}_1, D_i + D_{i-1} + \cdots + D_1 \geq \hat{y}_i). \tag{20}$$

Combining (19) and (20) gives Equation (8).

Part (b): Let $f(\eta_{i-1}, y_{i-1}) = p_i \min\{d_i, x_i - y_{i-1}\} + \eta_{i-1}\mathsf{E}[U_{i-1}(x_i - \min\{d_i, x_i - y_{i-1}\})]$. Then,

$$\frac{\partial f}{\partial \eta_{i-1}} = \mathsf{E}\Big[U_{i-1}(x_i - \min\{d_i, x_i - y_{i-1}\})\Big],$$
$$\frac{\partial^2 f}{\partial \eta_{i-1} \cdot \partial y_{i-1}} = \frac{\partial \mathsf{E}\Big[U_{i-1}(x_i - \min\{d_i, x_i - y_{i-1}\})\Big]}{\partial y_{i-1}} = \frac{d\mathsf{E}[U_{i-1}(x_{i-1})]}{dx_{i-1}} \cdot \frac{\partial x_{i-1}}{\partial y_{i-1}}.$$

Note that $x_{i-1} = x_i - \min\{d_i, x_i - y_{i-1}\}$ weakly increases in y_{i-1}, so $\frac{\partial x_{i-1}}{\partial y_{i-1}} \geq 0$. Moreover, $\frac{d\mathsf{E}[U_{i-1}(x_{i-1})]}{dx_{i-1}} \geq 0$ by Lemma 6. Therefore, $\frac{\partial^2 f}{\partial \eta_{i-1} \cdot \partial y_{i-1}} \geq 0$, i.e., $f(\eta_{i-1}, y_{i-1})$ is submodular with respect to η_{i-1} and y_{i-1}. As a result, $\hat{y}_{i-1}$ increases in η_{i-1}.

Part (c): Lemma 6 shows that $\frac{d\mathsf{E}[U_i(x_i)]}{dx_i} \geq 0$, which implies that $\Delta U_i(x) \geq 0$. Next, we show that $\Delta U_i(x)$ decreases in x. It suffices to show $\frac{d\mathsf{E}[U_i(x)]}{dx}$ decreases in x. First, $\frac{d\mathsf{E}[U_1(x)]}{dx} = p_1 P(D_1 \geq x)$, which decreases in x. Suppose $\frac{d\mathsf{E}[U_{i-1}(x)]}{dx}$ decreases in x, and then we expect to show $\frac{d\mathsf{E}[U_i(x)]}{dx}$ decreases in x. Observe that

$$\mathsf{E}[U_i(x)] = \mathsf{E}_{D_i}\Big[p_i \min\{D_i, x - \hat{y}_{i-1}\} + \eta_{i-1}\mathsf{E}_{D_{i-1}}[U_{i-1}(x - \min\{D_i, x - \hat{y}_{i-1}\})]\Big].$$

Taking derivative with respect to x yields

$$\frac{d\mathsf{E}[U_i(x)]}{dx} = p_i P(D_i \geq x - \hat{y}_{i-1}) + \int_0^{x-\hat{y}_{i-1}} \eta_{i-1}\frac{d\mathsf{E}[U_{i-1}(x - d_i)]}{dx} dF_i(d_i).$$

Observe that the term p_i is independent of x, while the term $\eta_{i-1}\frac{d\mathsf{E}[U_{i-1}(x-d_i)]}{dx}$ in the integral decreases in x by our supposition. Therefore, to show $\frac{d\mathsf{E}[U_i(x)]}{dx}$ decreases in x, it suffices to show $p_i \geq \eta_{i-1}\frac{d\mathsf{E}[U_{i-1}(x-d_i)]}{dx}$. One can check $p_i - \eta_{i-1}\frac{d\mathsf{E}[U_{i-1}(x-d_i)]}{dx} \geq p_i - \eta_{i-1}\frac{p_i}{\eta_{i-1}} = 0$, where the last inequality holds because of Lemma 7 (in online appendix). This establishes that $\Delta U_i(x)$ decreases in x. ■

Proof of Proposition 2. Part (a): It suffices to investigate how $\hat{V}_i(x)$ changes with $\hat{y}_{i-1}$, because $\hat{y}_{i-1}$ increases in η_{i-1} for each $i \in \{1, 2, \cdots, n\}$ by Proposition 1b. From $\hat{V}_{i+1}(x_{i+1}) = \mathsf{E}\Big[p_{i+1}\min\{D_{i+1}, x_{i+1} - \hat{y}_i\} + \hat{V}_i(x_{i+1} - \min\{D_{i+1}, x_{i+1} - \hat{y}_i\})\Big]$, we have

$$\frac{d\hat{V}_{i+1}(x_{i+1})}{d\hat{y}_i} = \mathrm{P}(D_{i+1} \geq x_{i+1} - \hat{y}_i)\Big\{-p_{i+1} + \frac{d\hat{V}_i(\hat{y}_i)}{d\hat{y}_i}\Big\}$$

$$
\begin{aligned}
=&\mathrm{P}(D_{i+1}\geq x_{i+1}-\hat{y}_i)\Big\{-p_{i+1}+p_i\mathrm{P}(D_i\geq \hat{y}_i-\hat{y}_{i-1})\\
&+p_{i-1}\mathrm{P}(D_i<\hat{y}_i-\hat{y}_{i-1},D_i+D_{i-1}\geq \hat{y}_i-\hat{y}_{i-2})+\cdots \qquad (21)\\
&+p_1\mathrm{P}(D_i<\hat{y}_i-\hat{y}_{i-1},\cdots,D_i+D_{i-1}+\cdots+D_2<\hat{y}_i-\hat{y}_1,D_i+D_{i-1}+\cdots+D_1\geq \hat{y}_i)\Big\},
\end{aligned}
$$

where the last equality holds by Lemma 8 (in online appendix).

At optimality, the opportunity cost of the $\hat{y}_i$th inventory is equal to the immediate revenue p_{i+1} by selling that inventory to class $i+1$ customers directly; that is,

$$
\frac{d\mathsf{E}[U_{i+1}(x_{i+1})]}{d\hat{y}_i}=\mathrm{P}(D_{i+1}\geq x_{i+1}-\hat{y}_i)\Big\{-p_{i+1}+\eta_i\frac{d\mathsf{E}[U_i(\hat{y}_i)]}{d\hat{y}_i}\Big\}=0.
$$

Therefore,

$$
\begin{aligned}
p_{i+1}=\eta_i\frac{d\mathsf{E}[U_i(\hat{y}_i)]}{d\hat{y}_i}=&\eta_ip_i\mathrm{P}(D_i\geq \hat{y}_i-\hat{y}_{i-1})+\eta_i\eta_{i-1}p_{i-1}\mathrm{P}(D_i<\hat{y}_i-\hat{y}_{i-1},D_i+D_{i-1}\geq \hat{y}_i-\hat{y}_{i-2})+\cdots\\
&+\eta_i\eta_{i-1}\cdots\eta_1p_1\mathrm{P}(D_i<\hat{y}_i-\hat{y}_{i-1},\cdots,D_i+D_{i-1}+\cdots+D_2<\hat{y}_i-\hat{y}_1,D_i+D_{i-1}+\cdots+D_1\geq \hat{y}_i),
\end{aligned}
$$

where the last equality holds because of Lemma 6. Plugging p_{i+1} into (21) yields

$$
\begin{aligned}
&\frac{d\hat{V}_{i+1}(x_{i+1})}{d\hat{y}_i}=\mathrm{P}(D_{i+1}\geq x_{i+1}-\hat{y}_i)\\
&\Big\{(1-\eta_i)p_i\mathrm{P}(D_i\geq \hat{y}_i-\hat{y}_{i-1})+(1-\eta_i\eta_{i-1})p_{i-1}\mathrm{P}(D_i<\hat{y}_i-\hat{y}_{i-1},D_i+D_{i-1}\geq \hat{y}_i-\hat{y}_{i-2})+\cdots\\
&\quad+(1-\eta_i\eta_{i-1}\cdots\eta_1)p_1\mathrm{P}(D_i<\hat{y}_i-\hat{y}_{i-1},\cdots,D_i+D_{i-1}+\cdots+D_2<\hat{y}_i-\hat{y}_1,D_i+D_{i-1}+\cdots+D_1\geq \hat{y}_i)\Big\},
\end{aligned}
$$

which is positive if $\eta_i<1$ for each i and negative if $\eta_i>1$ for each i. As a result, the seller's resulting revenue $\hat{V}_{i+1}(x_{i+1})$ increases in $\hat{y}_i$ in the behavioral discounting case and decreases in $\hat{y}_i$ in the prospective accounting case. This completes the proof of Part (a).

Part (b): We prove this result by induction. It is easy to see that $\hat{V}_1(x)=\mathsf{E}[p_1\min\{D_1,x\}]$ increases in x. Suppose $\hat{V}_{i-1}(x)$ increases in x, and then we would like to show that $\hat{V}_i(x)$ increases in x. According to (10), $\hat{V}_i(x)=\mathsf{E}\Big[p_i\min\{D_i,x-\hat{y}_{i-1}\}+\hat{V}_{i-1}(x-\min\{D_i,x-\hat{y}_{i-1}\})\Big]$. As $x-\min\{D_i,x-\hat{y}_{i-1}\}$ weakly increases in x and $\hat{V}_{i-1}(x)$ increases in x, the second term in the square brackets increases in x. Note that the first term also weakly increases in x, so it follows immediately that $\hat{V}_i(x)$ increases in x. This completes the proof of Part (b). ■

Proof of Proposition 3. We first provide the conditions under which $\eta_n\Delta U_n(x)<p_{n+1}<\min\{\eta_n\Delta U_n(x-1),\Delta\hat{V}_n(x)\}$ holds: The parameters x, p_1, p_{n+1}, $\boldsymbol{\eta}$, $D_1,D_2,\cdots,D_n$ satisfy

$$
\mathrm{P}(D_1\geq \hat{y}_1,D_1+D_2\geq \hat{y}_2,\cdots,D_1+\cdots+D_{n-1}\geq \hat{y}_{n-1},D_1+\cdots+D_n\geq x-\epsilon)=\frac{p_{n+1}}{\eta_n\eta_{n-1}\cdots\eta_1p_1}, \qquad (22)
$$

$$
\mathrm{P}(D_1\geq \hat{y}_1,D_1+D_2\geq \hat{y}_2,\cdots,D_1+\cdots+D_{n-1}\geq \hat{y}_{n-1},D_1+\cdots+D_n\geq x)\geq\frac{p_{n+1}}{\eta_{n-1}\eta_{n-2}\cdots\eta_1p_1}, \qquad (23)
$$

where $0 < \epsilon \leq 1$. Equation (22) implies that $\hat{y}_n = x - \epsilon$, which means that only $0 < \epsilon \leq 1$ unit of product is allowed to sell to class $n+1$ in the additional demand case. Although the left hand side probability in (23) is smaller than the left hand side probability in (22) because $x > x - \epsilon$, (23) implies that the left hand side probability cannot be too small. There always exist multiple sets of parameters that satisfy these two properties. Below, we show (22) and (23) imply $\eta_n \Delta U_n(x) < p_{n+1} < \min\{\eta_n \Delta U_n(x-1), \Delta \hat{V}_n(x)\}$.

First, by Lemma 5, we have

$$\begin{aligned}
&\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{n-1} \geq \hat{y}_{n-1}, D_1 + \cdots + D_n \geq x) \\
&= \frac{p_n}{\eta_{n-1}\eta_{n-2}\cdots\eta_1 p_1}\mathrm{P}(D_n \geq x - \hat{y}_{n-1}) + \frac{p_{n-1}}{\eta_{n-2}\eta_{n-3}\cdots\eta_1 p_1}\mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} \geq x - \hat{y}_{n-2}) + \cdots \\
&+ \frac{p_2}{\eta_1 p_1}\mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} < x - \hat{y}_{n-2}, \cdots, D_n + \cdots + D_3 < x - \hat{y}_2, D_n + \cdots + D_2 \geq x - \hat{y}_1) \\
&+ \mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} < x - \hat{y}_{n-2}, \cdots, D_n + \cdots + D_2 < x - \hat{y}_1, D_n + \cdots + D_1 \geq x). \qquad (24)
\end{aligned}$$

On the other hand, by (23), we have

$$\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{n-1} \geq \hat{y}_{n-1}, D_1 + \cdots + D_n \geq x) \geq \frac{p_{n+1}}{\eta_{n-1}\eta_{n-2}\cdots\eta_1 p_1}.$$

Therefore,

$$\begin{aligned}
&p_{n+1} \leq p_n \mathrm{P}(D_n \geq x - \hat{y}_{n-1}) + \eta_{n-1} p_{n-1} \mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} \geq x - \hat{y}_{n-2}) + \cdots \\
&+ \eta_{n-1}\eta_{n-2}\cdots\eta_2 p_2 \mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} < x - \hat{y}_{n-2}, \cdots, D_n + \cdots + D_3 < x - \hat{y}_2, D_n + \cdots + D_2 \geq x - \hat{y}_1) \\
&+ \eta_{n-1}\eta_{n-2}\cdots\eta_1 p_1 \mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} < x - \hat{y}_{n-2}, \cdots, D_n + \cdots + D_2 < x - \hat{y}_1, D_n + \cdots + D_1 \geq x) \\
&< p_n \mathrm{P}(D_n \geq x - \hat{y}_{n-1}) + p_{n-1} \mathrm{P}(D_n < x - \hat{y}_{n-1}, D_{n-1} + D_n \geq x - \hat{y}_{n-2}) \\
&+ p_{n-2} \mathrm{P}(D_n < x - \hat{y}_{n-1}, D_{n-1} + D_n < x - \hat{y}_{n-2}, D_{n-2} + D_{n-1} + D_n \geq x - \hat{y}_{n-3}) + \cdots \\
&+ p_1 \mathrm{P}(D_n < x - \hat{y}_{n-1}, D_{n-1} + D_n < x - \hat{y}_{n-2}, \cdots, D_2 + \cdots + D_n < x - \hat{y}_1, D_1 + D_2 + \cdots + D_n \geq x) \\
&= \Delta \hat{V}_n(x), \qquad (25)
\end{aligned}$$

where the last equality holds because of Lemma 8.

Second, (22) implies that

$$\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{n-1} \geq \hat{y}_{n-1}, D_1 + \cdots + D_n \geq x) < \frac{p_{n+1}}{\eta_n \eta_{n-1}\cdots\eta_1 p_1}. \qquad (26)$$

Combining (26) and (24) gives

$$p_{n+1} > \eta_n p_n \mathrm{P}(D_n \geq x - \hat{y}_{n-1}) + \eta_n \eta_{n-1} p_{n-1} \mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} \geq x - \hat{y}_{n-2}) + \cdots$$

$$+\eta_n\eta_{n-1}\cdots\eta_2 p_2 \mathrm{P}(D_n < x-\hat{y}_{n-1},\cdots,D_n+\cdots+D_3 < x-\hat{y}_2, D_n+\cdots+D_2 \geq x-\hat{y}_1)$$
$$+\eta_n\eta_{n-1}\cdots\eta_1 p_1 \mathrm{P}(D_n < x-\hat{y}_{n-1},\cdots,D_n+\cdots+D_2 < x-\hat{y}_1, D_n+\cdots+D_1 \geq x)$$
$$=\eta_n\Delta U_n(x), \tag{27}$$

where the above equality holds because of Lemma 6.

Third, (22) implies that

$$\mathrm{P}(D_1 \geq \hat{y}_1, D_1+D_2 \geq \hat{y}_2,\cdots,D_1+\cdots+D_{n-1} \geq \hat{y}_{n-1}, D_1+\cdots+D_n \geq x-1) > \frac{p_{n+1}}{\eta_n\eta_{n-1}\cdots\eta_1 p_1}. \tag{28}$$

By Lemma 5, we have

$$\begin{aligned}
&\mathrm{P}(D_1 \geq \hat{y}_1, D_1+D_2 \geq \hat{y}_2,\cdots,D_1+\cdots+D_{n-1} \geq \hat{y}_{n-1}, D_1+\cdots+D_n \geq x-1)\\
&=\frac{p_n}{\eta_{n-1}\eta_{n-2}\cdots\eta_1 p_1}\mathrm{P}(D_n \geq x-1-\hat{y}_{n-1})+\frac{p_{n-1}}{\eta_{n-2}\eta_{n-3}\cdots\eta_1 p_1}\mathrm{P}(D_n < x-1-\hat{y}_{n-1}, D_n+D_{n-1} \geq x-1-\hat{y}_{n-2})+\cdots\\
&+\frac{p_2}{\eta_1 p_1}\mathrm{P}(D_n < x-1-\hat{y}_{n-1}, D_n+D_{n-1} < x-1-\hat{y}_{n-2},\cdots,D_n+\cdots+D_3 < x-1-\hat{y}_2, D_n+\cdots+D_2 \geq x-1-\hat{y}_1)\\
&+\mathrm{P}(D_n < x-1-\hat{y}_{n-1}, D_n+D_{n-1} < x-1-\hat{y}_{n-2},\cdots,D_n+\cdots+D_2 < x-1-\hat{y}_1, D_n+\cdots+D_1 \geq x-1).
\end{aligned} \tag{29}$$

Combining (28) and (29) gives

$$\begin{aligned}
p_{n+1} <&\eta_n p_n \mathrm{P}(D_n \geq x-1-\hat{y}_{n-1})+\eta_n\eta_{n-1}p_{n-1}\mathrm{P}(D_n < x-1-\hat{y}_{n-1}, D_n+D_{n-1} \geq x-1-\hat{y}_{n-2})+\cdots\\
&+\eta_n\eta_{n-1}\cdots\eta_2 p_2 \mathrm{P}(D_n < x-1-\hat{y}_{n-1},\cdots,D_n+\cdots+D_3 < x-1-\hat{y}_2, D_n+\cdots+D_2 \geq x-1-\hat{y}_1)\\
&+\eta_n\eta_{n-1}\cdots\eta_1 p_1 \mathrm{P}(D_n < x-1-\hat{y}_{n-1},\cdots,D_n+\cdots+D_2 < x-1-\hat{y}_1, D_n+\cdots+D_1 \geq x-1)\\
=&\eta_n\Delta U_n(x-1).
\end{aligned} \tag{30}$$

Now, we show that under condition $\eta_n\Delta U_n(x) < p_{n+1} < \min\{\eta_n\Delta U_n(x-1), \Delta\hat{V}_n(x)\}$, $\hat{V}_{n+1}(x) < \hat{V}_n(x)$ holds immediately. From $\mathsf{E}[U_{n+1}(x)] = \mathsf{E}[p_{n+1}\min\{D_{n+1}, x-\hat{y}_n\}+\eta_n\mathsf{E}[U_n(x-\min\{D_{n+1}, x-\hat{y}_n\})]]$, we have

$$\frac{d\mathsf{E}[U_{n+1}(x)]}{d\hat{y}_n} = \mathrm{P}(D_{n+1} \geq x-\hat{y}_n)\Big\{-p_{n+1}+\eta_n\frac{d\mathsf{E}[U_n(\hat{y}_n)]}{d\hat{y}_n}\Big\} = 0,$$

which implies that $p_{n+1} = \eta_n\Delta U_n(\hat{y}_n)$. Since $\eta_n\Delta U_n(x-1) \geq p_{n+1} > \eta_n\Delta U_n(x)$, we have $\Delta U_n(x-1) \geq \Delta U_n(\hat{y}_n) > \Delta U_n(x)$. Note that $\Delta U_n(x)$ decreases in x by Proposition 1c, so $x-1 \leq \hat{y}_n < x$. That is, the seller allocates ϵ unit of product to class $n+1$ customers, where $0 < \epsilon \leq 1$. Also, from

$$\begin{aligned}
\hat{V}_{n+1}(x) =&\mathsf{E}\Big[p_{n+1}\min\{D_{n+1}, x-\hat{y}_n\}+\hat{V}_n(x-\min\{D_{n+1}, x-\hat{y}_n\})\Big]\\
=&\mathsf{E}\Big[p_{n+1}\min\{D_{n+1}, \epsilon\}+\hat{V}_n(x-\min\{D_{n+1}, \epsilon\})\Big]
\end{aligned}$$

$$= \int_0^{\epsilon} \Big\{ p_{n+1} d_{n+1} + \hat{V}_n(x - d_{n+1}) \Big\} dF_{n+1}(d_{n+1}) + \int_{\epsilon}^{\infty} \Big\{ p_{n+1}\epsilon + \hat{V}_n(x-\epsilon) \Big\} dF_{n+1}(d_{n+1}),$$

we have

$$\begin{aligned}
&\hat{V}_{n+1}(x) - \hat{V}_n(x) \\
=&\int_0^{\epsilon} \Big\{ p_{n+1} d_{n+1} + \hat{V}_n(x - d_{n+1}) \Big\} dF_{n+1}(d_{n+1}) + \int_{\epsilon}^{\infty} \Big\{ p_{n+1}\epsilon + \hat{V}_n(x-\epsilon) \Big\} dF_{n+1}(d_{n+1}) \\
&- \int_0^{\epsilon} \hat{V}_n(x) dF_{n+1}(d_{n+1}) - \int_{\epsilon}^{\infty} \hat{V}_n(x) dF_{n+1}(d_{n+1}) \\
=&\int_0^{\epsilon} \Big\{ p_{n+1} d_{n+1} + \hat{V}_n(x - d_{n+1}) - \hat{V}_n(x) \Big\} dF_{n+1}(d_{n+1}) + \int_{\epsilon}^{\infty} \Big\{ p_{n+1}\epsilon + \hat{V}_n(x-\epsilon) - \hat{V}_n(x) \Big\} dF_{n+1}(d_{n+1}).
\end{aligned}$$

Note that when $0 < \epsilon \leq 1$,

$$p_{n+1}\epsilon + \hat{V}_n(x-\epsilon) - \hat{V}_n(x) = \epsilon \Big\{ p_{n+1} - \frac{\hat{V}_n(x) - \hat{V}_n(x-\epsilon)}{\epsilon} \Big\} = \epsilon \Big\{ p_{n+1} - \Delta \hat{V}_n(x) \Big\} < 0.$$

Similarly, one can show $p_{n+1} d_{n+1} + \hat{V}_n(x - d_{n+1}) - \hat{V}_n(x) < 0$. Therefore, $\hat{V}_{n+1}(x) < \hat{V}_n(x)$. ■

Proof of Lemma 2. From $\mathrm{P}(D_0 \geq \hat{y}^0) = \frac{p_1}{\eta_0 p_0}$ and $\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_i \geq \hat{y}_i) = \frac{p_{i+1}}{\eta_i \eta_{i-1} \cdots \eta_1 p_1}$, we have $\mathrm{P}(D_0 \geq \hat{y}^0, D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_i \geq \hat{y}_i) = \frac{p_{i+1}}{\eta_i \eta_{i-1} \cdots \eta_0 p_0}$. Therefore, for each $i \in \{1, 2, \cdots, n-1\}$,

$$\begin{aligned}
&\mathrm{P}(D_0 \geq \hat{y}^0, D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_i \geq \hat{y}_i) \\
=&\mathrm{P}(D_0 \geq \hat{y}^0, D_0 + D_1 \geq \hat{y}^1, \cdots, D_0 + D_1 + \cdots + D_i \geq \hat{y}^i).
\end{aligned} \tag{31}$$

Now, we show $\hat{y}^i \geq \hat{y}_i$ by induction. When $i = 1$, (31) reduces to

$$\mathrm{P}(D_0 \geq \hat{y}^0, D_1 \geq \hat{y}_1) = \mathrm{P}(D_0 \geq \hat{y}^0, D_0 + D_1 \geq \hat{y}^1). \tag{32}$$

Suppose $\hat{y}^1 < \hat{y}_1$ for a contradiction. Then $\mathrm{P}(D_0 \geq \hat{y}^0, D_1 \geq \hat{y}_1) = \mathrm{P}(D_1 \geq \hat{y}_1 | D_0 \geq \hat{y}^0)\mathrm{P}(D_0 \geq \hat{y}^0) < \mathrm{P}(D_0 + D_1 \geq \hat{y}^1 | D_0 \geq \hat{y}^0)\mathrm{P}(D_0 \geq \hat{y}^0) = \mathrm{P}(D_0 \geq \hat{y}^0, D_0 + D_1 \geq \hat{y}^1)$, contradicting (32), where the above inequality holds because $\hat{y}^1 < \hat{y}_1$. Suppose $\hat{y}^{i-1} \geq \hat{y}_{i-1}$ holds, and we would like to show $\hat{y}^i \geq \hat{y}_i$. Suppose $\hat{y}^i < \hat{y}_i$ for a contradiction. Note that

$$\begin{aligned}
&\mathrm{P}(D_1 + \cdots + D_i \geq \hat{y}_i | D_0 \geq \hat{y}^0, D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}) \\
=&\mathrm{P}(D_1 + \cdots + D_i \geq \hat{y}_i | D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}), \\
&\mathrm{P}(D_0 + D_1 + \cdots + D_i \geq \hat{y}^i | D_0 \geq \hat{y}^0, D_0 + D_1 \geq \hat{y}^1, \cdots, D_0 + D_1 + \cdots + D_{i-1} \geq \hat{y}^{i-1}) \\
=&\mathrm{P}(D_0 + D_1 + \cdots + D_i \geq \hat{y}^i | D_0 + D_1 + \cdots + D_{i-1} \geq \hat{y}^{i-1}).
\end{aligned}$$

Because $\hat{y}^{i-1} \geq \hat{y}_{i-1}$ and $\hat{y}^i < \hat{y}_i$, it follows immediately that

$$\mathrm{P}(D_1 + \cdots + D_i \geq \hat{y}_i | D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}) < \mathrm{P}(D_0 + D_1 + \cdots + D_i \geq \hat{y}^i | D_0 + D_1 + \cdots + D_{i-1} \geq \hat{y}^{i-1}).$$

Therefore,

$$\begin{aligned}&\mathrm{P}(D_1+\cdots+D_i\geq \hat{y}_i|D_0\geq \hat{y}^0, D_1\geq \hat{y}_1, D_1+D_2\geq \hat{y}_2,\cdots, D_1+\cdots+D_{i-1}\geq \hat{y}_{i-1})\\ <&\mathrm{P}(D_0+D_1+\cdots+D_i\geq \hat{y}^i|D_0\geq \hat{y}^0, D_0+D_1\geq \hat{y}^1,\cdots, D_0+D_1+\cdots+D_{i-1}\geq \hat{y}^{i-1}).\end{aligned} \tag{33}$$

One can check

$$\begin{aligned}&\mathrm{P}(D_0\geq \hat{y}^0, D_1\geq \hat{y}_1, D_1+D_2\geq \hat{y}_2,\cdots, D_1+\cdots+D_i\geq \hat{y}_i)\\ =&\mathrm{P}(D_1+\cdots+D_i\geq \hat{y}_i|D_0\geq \hat{y}^0, D_1\geq \hat{y}_1, D_1+D_2\geq \hat{y}_2,\cdots, D_1+\cdots+D_{i-1}\geq \hat{y}_{i-1})\\ &\quad\cdot \mathrm{P}(D_0\geq \hat{y}^0, D_1\geq \hat{y}_1, D_1+D_2\geq \hat{y}_2,\cdots, D_1+\cdots+D_{i-1}\geq \hat{y}_{i-1})\\ <&\mathrm{P}(D_0+D_1+\cdots+D_i\geq \hat{y}^i|D_0\geq \hat{y}^0, D_0+D_1\geq \hat{y}^1,\cdots, D_0+D_1+\cdots+D_{i-1}\geq \hat{y}^{i-1})\\ &\quad\cdot \mathrm{P}(D_0\geq \hat{y}^0, D_0+D_1\geq \hat{y}^1,\cdots, D_0+D_1+\cdots+D_{i-1}\geq \hat{y}^{i-1})\\ =&\mathrm{P}(D_0\geq \hat{y}^0, D_0+D_1\geq \hat{y}^1,\cdots, D_0+D_1+\cdots+D_i\geq \hat{y}^i),\end{aligned}$$

contradicting (31), where the above inequality holds because of (33) and $\mathrm{P}(D_0\geq \hat{y}^0, D_1\geq \hat{y}_1, D_1+D_2\geq \hat{y}_2,\cdots, D_1+\cdots+D_{i-1}\geq \hat{y}_{i-1}) = \mathrm{P}(D_0\geq \hat{y}^0, D_0+D_1\geq \hat{y}^1,\cdots, D_0+D_1+\cdots+D_{i-1}\geq \hat{y}^{i-1})$. This establishes that $\hat{y}^i\geq \hat{y}_i$ for each $i\in\{1,2,\cdots,n-1\}$. ■

Proof of Lemma 3. By Lemma 2, $\hat{y}^i\geq \hat{y}_i$ for each $i=1,\cdots,n-1$, implying that the seller reserves more products for each class i (and the remaining classes) due to the introduction of class 0 customers. Below, we show such an increase always brings a higher profit to the seller.

In the premium demand case, suppose the seller is in the beginning of stage i when class i customers arrive. If the seller sells a product to class i, then an immediate profit of p_i will be accrued. Otherwise, the expected revenue of this product is as follows:

$$\begin{aligned}&\Delta\hat{V}_{i-1}(\hat{y}^{i-1})\\ =&p_{i-1}\mathrm{P}(D_{i-1}\geq \hat{y}^{i-1}-\hat{y}^{i-2}) + p_{i-2}\mathrm{P}(D_{i-1}<\hat{y}^{i-1}-\hat{y}^{i-2}, D_{i-1}+D_{i-2}\geq \hat{y}^{i-1}-\hat{y}^{i-3})+\cdots\\ &+p_1\mathrm{P}(D_{i-1}<\hat{y}^{i-1}-\hat{y}^{i-2},\cdots, D_{i-1}+D_{i-2}+\cdots+D_2<\hat{y}^{i-1}-\hat{y}^1, D_{i-1}+D_{i-2}+\cdots+D_1\geq \hat{y}^{i-1}-\hat{y}^0)\\ &+p_0\mathrm{P}(D_{i-1}<\hat{y}^{i-1}-\hat{y}^{i-2},\cdots, D_{i-1}+D_{i-2}+\cdots+D_1<\hat{y}^{i-1}-\hat{y}^0, D_{i-1}+D_{i-2}+\cdots+D_0\geq \hat{y}^{i-1}).\end{aligned} \tag{34}$$

By Lemma 5, one can check

$$\begin{aligned}&\mathrm{P}(D_0\geq \hat{y}^0, D_0+D_1\geq \hat{y}^1,\cdots, D_0+\cdots+D_{i-2}\geq \hat{y}^{i-2}, D_0+\cdots+D_{i-1}\geq \hat{y}^{i-1})\\ =&\frac{p_{i-1}}{\eta_{i-2}\eta_{i-3}\cdots\eta_0 p_0}\mathrm{P}(D_{i-1}\geq \hat{y}^{i-1}-\hat{y}^{i-2}) + \frac{p_{i-2}}{\eta_{i-3}\eta_{i-4}\cdots\eta_0 p_0}\mathrm{P}(D_{i-1}<\hat{y}^{i-1}-\hat{y}^{i-2}, D_{i-1}+D_{i-2}\geq \hat{y}^{i-1}-\hat{y}^{i-3})\end{aligned}$$

$$+\cdots$$
$$+\frac{p_1}{\eta_0 p_0}\mathrm{P}(D_{i-1}<\hat{y}^{i-1}-\hat{y}^{i-2},\cdots,D_{i-1}+\cdots+D_2<\hat{y}^{i-1}-\hat{y}^{1},D_{i-1}+\cdots+D_1\geq\hat{y}^{i-1}-\hat{y}^{0})$$
$$+\mathrm{P}(D_{i-1}<\hat{y}^{i-1}-\hat{y}^{i-2},\cdots,D_{i-1}+D_{i-2}+\cdots+D_1<\hat{y}^{i-1}-\hat{y}^{0},D_{i-1}+D_{i-2}+\cdots+D_0\geq\hat{y}^{i-1}).$$

On the other hand, by the protection level determination rule, we have

$$\mathrm{P}(D_0\geq\hat{y}^0,D_0+D_1\geq\hat{y}^1,\cdots,D_0+\cdots+D_{i-2}\geq\hat{y}^{i-2},D_0+\cdots+D_{i-1}\geq\hat{y}^{i-1})=\frac{p_i}{\eta_{i-1}\eta_{i-2}\cdots\eta_0 p_0}.$$

Therefore,

$$\begin{aligned}p_i=&\eta_{i-1}p_{i-1}\mathrm{P}(D_{i-1}\geq\hat{y}^{i-1}-\hat{y}^{i-2})+\eta_{i-1}\eta_{i-2}p_{i-2}\mathrm{P}(D_{i-1}<\hat{y}^{i-1}-\hat{y}^{i-2},D_{i-1}+D_{i-2}\geq\hat{y}^{i-1}-\hat{y}^{i-3})\\&+\cdots\\&+\eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1\mathrm{P}(D_{i-1}<\hat{y}^{i-1}-\hat{y}^{i-2},\cdots,D_{i-1}+\cdots+D_2<\hat{y}^{i-1}-\hat{y}^{1},D_{i-1}+\cdots+D_1\geq\hat{y}^{i-1}-\hat{y}^{0})\\&+\eta_{i-1}\eta_{i-2}\cdots\eta_0 p_0\mathrm{P}(D_{i-1}<\hat{y}^{i-1}-\hat{y}^{i-2},\cdots,D_{i-1}+\cdots+D_1<\hat{y}^{i-1}-\hat{y}^{0},D_{i-1}+\cdots+D_0\geq\hat{y}^{i-1}).\end{aligned}\tag{35}$$

Comparing (35) with (34) establishes $\Delta\hat{V}_{i-1}(\hat{y}^{i-1})\geq p_i$. This completes the proof. ■

Proof of Proposition 4. We first provide the conditions under which $\hat{V}^n(x)<\hat{V}_n(x)$ holds:

(i) $D_0=\frac{p_1}{\eta_0 p_0}X+[1-\frac{p_1}{\eta_0 p_0}]Y$, where $X\sim N(1,\epsilon^2)$, $Y\sim N(0,\epsilon^2)$, and $\epsilon\to 0$.

(ii) $\sum_{i=1}^n D_i\geq x$;

(iii) $\frac{p_1}{\eta_0}\leq p_n$.

One can verify that as $\epsilon\to 0$, $\mathrm{P}(D_0=1)\to\frac{p_1}{\eta_0 p_0}$, and $\mathrm{P}(D_0=0)\to 1-\frac{p_1}{\eta_0 p_0}$. As a result, $\hat{y}^0\to 1$. Moreover, because $\mathrm{P}(D_1\geq\hat{y}_1)=\frac{p_2}{\eta_1 p_1}$, $\mathrm{P}(D_0=1,D_1\geq\hat{y}_1)\approx\frac{p_1}{\eta_0 p_0}\cdot\frac{p_2}{\eta_1 p_1}=\frac{p_2}{\eta_1\eta_0 p_0}=\mathrm{P}(D_0\geq\hat{y}^0,D_0+D_1\geq\hat{y}^1)$. Therefore, $\hat{y}^1\to\hat{y}_1+1$. Similarly, as $\epsilon\to 0$, $\mathrm{P}(D_0\geq 1,D_0+D_1\geq\hat{y}_1+1,D_0+D_1+D_2\geq\hat{y}_2+1)\approx\mathrm{P}(D_0=1,D_1\geq\hat{y}_1,D_1+D_2\geq\hat{y}_2)=\mathrm{P}(D_0=1)\mathrm{P}(D_1\geq\hat{y}_1,D_1+D_2\geq\hat{y}_2)\approx\frac{p_1}{\eta_0 p_0}\cdot\frac{p_3}{\eta_2\eta_1 p_1}=\frac{p_3}{\eta_2\eta_1\eta_0 p_0}=\mathrm{P}(D_0\geq\hat{y}^0,D_0+D_1\geq\hat{y}^1,D_0+D_1+D_2\geq\hat{y}^2)$, which implies that $\hat{y}^2\to\hat{y}_2+1$. Continuing in this fashion, one can show that as $\epsilon\to 0$, $\hat{y}^i\to\hat{y}_i+1$, for $i=1,2,\cdots,n-1$. That is, the product allocated to class 0 in the premium demand case is originally allocated to class n, which could be sold to class n, class $n-1$, $\cdots$, or class 1 in the baseline case. Hence, the revenue difference between $\hat{V}_n(x)$ and $\hat{V}^n(x)$ depends on this single unit of product only.

The expected profit of this product in the baseline case is

$$\begin{aligned}\Delta\hat{V}_n(x)=&p_n\mathrm{P}(D_n\geq x-\hat{y}_{n-1})+p_{n-1}\mathrm{P}(D_n<x-\hat{y}_{n-1},D_n+D_{n-1}\geq x-\hat{y}_{n-2})+\cdots\\&+p_1\mathrm{P}(D_n<x-\hat{y}_{n-1},\cdots,D_n+D_{n-1}+\cdots+D_2<x-\hat{y}_1,D_n+D_{n-1}+\cdots+D_2+D_1\geq x),\end{aligned}$$

and the total profit $\hat{V}_n(x)=\hat{V}_n(x)-\hat{V}_n(x-1)+\hat{V}_n(x-1)=\Delta\hat{V}_n(x)+\hat{V}_n(x-1)$. In the premium demand case, the expected value of the product allocated to class 0 is $p_0\mathrm{P}(D_0\geq 1)\approx p_0\frac{p_1}{\eta_0 p_0}=\frac{p_1}{\eta_0}$, and the total profit can be written as $\hat{V}^n(x) = \hat{V}_n(x-1) + p_0 \mathrm{P}(D_0 \geq 1) \approx \hat{V}_n(x-1) + \frac{p_1}{\eta_0}$. One may wonder whether it is possible for the products except this single unit reserved for class 0 to be sold to class 0. Note that $\sum_{i=1}^n D_i \geq x$, so the products except the single unit are guaranteed to sell out at the end of stage 1. Hence, there is no chance for them to be sold to class 0.

Therefore, $\hat{V}^n(x) - \hat{V}_n(x) \approx \frac{p_1}{\eta_0} - \Delta\hat{V}_n(x)$. Note that $D_n + D_{n-1} + \cdots + D_2 + D_1 \geq x$, so

$$\begin{aligned} &\mathrm{P}(D_n \geq x - \hat{y}_{n-1}) + \mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} \geq x - \hat{y}_{n-2}) + \cdots \\ &+ \mathrm{P}(D_n < x - \hat{y}_{n-1}, \cdots, D_n + D_{n-1} + \cdots + D_2 < x - \hat{y}_1, D_n + D_{n-1} + \cdots + D_2 + D_1 \geq x) = 1. \end{aligned}$$

Therefore, $\Delta\hat{V}_n(x)$ is a weighted average of $p_1, p_2, \cdots, p_n$, i.e., $p_n < \Delta\hat{V}_n(x) < p_1$. Because $\frac{p_1}{\eta_0} \leq p_n$ (which is possible, especially when η_0 is large and n is small), it follows immediately that $\frac{p_1}{\eta_0} < \Delta\hat{V}_n(x)$, indicating that the profit generated by this single unit of product in the premium demand case is smaller than that in the baseline case. Hence, $\hat{V}^n(x) < \hat{V}_n(x)$. This completes the proof. ■

Proof of Lemma 4. The derivative of $\hat{V}_{n+1}(x) = \mathsf{E}\Big[p_{n+1}\min\{D_{n+1}, x - \hat{y}_n\} + \hat{V}_n(x - \min\{D_{n+1}, x - \hat{y}_n\})\Big]$ with respect to $\hat{y}_n$ is

$$\begin{aligned} &\mathrm{P}(D_{n+1} \geq x - \hat{y}_n)\Big\{-p_{n+1} + \frac{d\hat{V}_n(\hat{y}_n)}{d\hat{y}_n}\Big\} \\ =&\mathrm{P}(D_{n+1} \geq x - \hat{y}_n)\Big\{-p_{n+1} + p_n \mathrm{P}(D_n \geq \hat{y}_n - \hat{y}_{n-1}) + p_{n-1}\mathrm{P}(D_n < \hat{y}_n - \hat{y}_{n-1}, D_n + D_{n-1} \geq \hat{y}_n - \hat{y}_{n-2}) \\ &+ \cdots + p_1 \mathrm{P}(D_n < \hat{y}_n - \hat{y}_{n-1}, \cdots, D_n + D_{n-1} + \cdots + D_2 < \hat{y}_n - \hat{y}_1, D_n + D_{n-1} + \cdots + D_1 \geq \hat{y}_n)\Big\}, \end{aligned} \tag{36}$$

where the last equality holds by Lemma 8. On the one hand, by Lemma 5, we have

$$\begin{aligned} &\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{n-1} \geq \hat{y}_{n-1}, D_1 + \cdots + D_n \geq x) \\ =&\frac{p_n}{\eta_{n-1}\eta_{n-2}\cdots\eta_1 p_1}\mathrm{P}(D_n \geq x - \hat{y}_{n-1}) + \frac{p_{n-1}}{\eta_{n-2}\eta_{n-3}\cdots\eta_1 p_1}\mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} \geq x - \hat{y}_{n-2}) + \cdots \\ &+ \frac{p_2}{\eta_1 p_1}\mathrm{P}(D_n < x - \hat{y}_{n-1}, \cdots, D_n + \cdots + D_3 < x - \hat{y}_2, D_n + \cdots + D_2 \geq x - \hat{y}_1) \\ &+ \mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} < x - \hat{y}_{n-2}, \cdots, D_n + \cdots + D_2 < x - \hat{y}_1, D_n + \cdots + D_1 \geq x). \end{aligned}$$

On the other hand, by the protection level determination rule, for any $x \geq \hat{y}_n$,

$$\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{n-1} \geq \hat{y}_{n-1}, D_1 + \cdots + D_n \geq x) \leq \frac{p_{n+1}}{\eta_n \eta_{n-1}\cdots\eta_1 p_1}.$$

Therefore,

$$\begin{aligned} p_{n+1} \geq& \eta_n p_n \mathrm{P}(D_n \geq x - \hat{y}_{n-1}) + \eta_n\eta_{n-1}p_{n-1}\mathrm{P}(D_n < x - \hat{y}_{n-1}, D_n + D_{n-1} \geq x - \hat{y}_{n-2}) + \cdots \\ &+ \eta_n\eta_{n-1}\cdots\eta_2 p_2 \mathrm{P}(D_n < x - \hat{y}_{n-1}, \cdots, D_n + \cdots + D_3 < x - \hat{y}_2, D_n + \cdots + D_2 \geq x - \hat{y}_1) \end{aligned}$$

$$+\eta_n\eta_{n-1}\cdots\eta_1 p_1 \mathrm{P}(D_n < x-\hat{y}_{n-1},\cdots,D_n+\cdots+D_2 < x-\hat{y}_1, D_n+\cdots+D_1 \geq x). \quad (37)$$

Putting (37) back to (36) gives $\frac{d\hat{V}_{n+1}(x)}{d\hat{y}_n} < 0$; that is, the resulting revenue $\hat{V}_{n+1}(x)$ decreases in $\hat{y}_n$. Since $\hat{y}_n \leq x$, $\hat{V}_{n+1}(x)$ takes the minimum when $\hat{y}_n = x$. Note that $\hat{y}_n = x$ means that the seller does not sell any product to class $n+1$ customers in the additional demand case, at which $\hat{V}_{n+1}(x) = \hat{V}_n(x)$. Therefore, $\hat{V}_{n+1}(x) \geq \hat{V}_n(x)$. This completes the proof. ■

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Online Appendix to "Mental Accounting in Allocating Capacity"

Meng Li, Yan Liu

This document contains the intermediate results for the proofs in the paper.

Lemma 5. *For a fixed i, if*

$$\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_j \geq \hat{y}_j) = \frac{p_{j+1}}{\eta_j \eta_{j-1} \cdots \eta_1 p_1}$$

holds for any $1 \leq j \leq i-1$, then the following probability equation holds:

$$\begin{aligned}
&\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_1 + \cdots + D_i \geq x) \\
=&\frac{p_i}{\eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1}\mathrm{P}(D_i \geq x - \hat{y}_{i-1}) + \frac{p_{i-1}}{\eta_{i-2}\eta_{i-3}\cdots\eta_1 p_1}\mathrm{P}(D_i < x - \hat{y}_{i-1}, D_i + D_{i-1} \geq x - \hat{y}_{i-2}) + \cdots \\
&+ \frac{p_2}{\eta_1 p_1}\mathrm{P}(D_i < x - \hat{y}_{i-1}, \cdots, D_i + D_{i-1} + \cdots + D_3 < x - \hat{y}_2, D_i + D_{i-1} + \cdots + D_2 \geq x - \hat{y}_1) \\
&+ \mathrm{P}(D_i < x - \hat{y}_{i-1}, \cdots, D_i + D_{i-1} + \cdots + D_2 < x - \hat{y}_1, D_i + D_{i-1} + \cdots + D_1 \geq x).
\end{aligned}$$

Proof of Lemma 5. One can check

$$\begin{aligned}
&\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_1 + \cdots + D_i \geq x) \\
=&\mathrm{P}(D_1 + \cdots + D_i \geq x, D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}) \\
=&\mathrm{P}(D_1 + \cdots + D_i \geq x, D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_i \geq x - \hat{y}_{i-1}) \\
&+ \mathrm{P}(D_1 + \cdots + D_i \geq x, D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_i < x - \hat{y}_{i-1}) \\
=&\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_i \geq x - \hat{y}_{i-1}) \\
&+ \mathrm{P}(D_1 + \cdots + D_i \geq x, D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_i < x - \hat{y}_{i-1}) \\
=&\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1})\mathrm{P}(D_i \geq x - \hat{y}_{i-1}) \\
&+ \mathrm{P}(D_1 + \cdots + D_i \geq x, D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_i < x - \hat{y}_{i-1}) \\
=&\frac{p_i}{\eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1}\mathrm{P}(D_i \geq x - \hat{y}_{i-1}) \\
&+ \mathrm{P}(D_1 + \cdots + D_i \geq x, D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_i < x - \hat{y}_{i-1}),
\end{aligned}$$

where the last equality holds because of our supposition.

Similarly,

$$\mathrm{P}(D_1 + \cdots + D_i \geq x, D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_i < x - \hat{y}_{i-1})$$

$$
\begin{aligned}
=&\mathrm{P}(D_1+\cdots+D_i\ge x, D_1\ge\hat{y}_1,\cdots,D_1+\cdots+D_{i-1}\ge\hat{y}_{i-1}, D_i<x-\hat{y}_{i-1}, D_i+D_{i-1}\ge x-\hat{y}_{i-2})\\
&+\mathrm{P}(D_1+\cdots+D_i\ge x, D_1\ge\hat{y}_1,\cdots,D_1+\cdots+D_{i-1}\ge\hat{y}_{i-1}, D_i<x-\hat{y}_{i-1}, D_i+D_{i-1}< x-\hat{y}_{i-2})\\
=&\mathrm{P}(D_1\ge\hat{y}_1,\cdots,D_1+\cdots+D_{i-2}\ge\hat{y}_{i-2}, D_i<x-\hat{y}_{i-1}, D_i+D_{i-1}\ge x-\hat{y}_{i-2})\\
&+\mathrm{P}(D_1+\cdots+D_i\ge x, D_1\ge\hat{y}_1,\cdots,D_1+\cdots+D_{i-1}\ge\hat{y}_{i-1}, D_i<x-\hat{y}_{i-1}, D_i+D_{i-1}< x-\hat{y}_{i-2})\\
=&\mathrm{P}(D_1\ge\hat{y}_1,\cdots,D_1+\cdots+D_{i-2}\ge\hat{y}_{i-2})\mathrm{P}(D_i<x-\hat{y}_{i-1}, D_i+D_{i-1}\ge x-\hat{y}_{i-2})\\
&+\mathrm{P}(D_1+\cdots+D_i\ge x, D_1\ge\hat{y}_1,\cdots,D_1+\cdots+D_{i-1}\ge\hat{y}_{i-1}, D_i<x-\hat{y}_{i-1}, D_i+D_{i-1}< x-\hat{y}_{i-2})\\
=&\frac{p_{i-1}}{\eta_{i-2}\eta_{i-3}\cdots\eta_1 p_1}\mathrm{P}(D_i<x-\hat{y}_{i-1}, D_i+D_{i-1}\ge x-\hat{y}_{i-2})\\
&+\mathrm{P}(D_1+\cdots+D_i\ge x, D_1\ge\hat{y}_1,\cdots,D_1+\cdots+D_{i-1}\ge\hat{y}_{i-1}, D_i<x-\hat{y}_{i-1}, D_i+D_{i-1}< x-\hat{y}_{i-2}),
\end{aligned}
$$

where the second equality holds because

$$
\left.\begin{aligned} D_1+\cdots+D_{i-2}\ge\hat{y}_{i-2}\\ D_i+D_{i-1}\ge x-\hat{y}_{i-2}\end{aligned}\right\}\Rightarrow D_1+\cdots+D_i\ge x,
$$

and

$$
\left.\begin{aligned} D_1+\cdots+D_i\ge x\\ D_i< x-\hat{y}_{i-1}\end{aligned}\right\}\Rightarrow D_1+\cdots+D_{i-1}\ge \hat{y}_{i-1}.
$$

Continuing in this fashion, we obtain

$$
\begin{aligned}
&\mathrm{P}(D_1\ge\hat{y}_1, D_1+D_2\ge\hat{y}_2,\cdots,D_1+\cdots+D_{i-1}\ge\hat{y}_{i-1}, D_1+\cdots+D_i\ge x)\\
=&\frac{p_i}{\eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1}\mathrm{P}(D_i\ge x-\hat{y}_{i-1})+\frac{p_{i-1}}{\eta_{i-2}\eta_{i-3}\cdots\eta_1 p_1}\mathrm{P}(D_i<x-\hat{y}_{i-1}, D_i+D_{i-1}\ge x-\hat{y}_{i-2})+\cdots\\
&+\frac{p_2}{\eta_1 p_1}\mathrm{P}(D_i<x-\hat{y}_{i-1},\cdots,D_i+D_{i-1}+\cdots+D_3<x-\hat{y}_2, D_i+D_{i-1}+\cdots+D_2\ge x-\hat{y}_1)\\
&+\mathrm{P}(D_i<x-\hat{y}_{i-1},\cdots,D_i+D_{i-1}+\cdots+D_2<x-\hat{y}_1, D_i+D_{i-1}+\cdots+D_1\ge x).
\end{aligned}
$$

This completes the proof. ■

Lemma 6. *Taking derivative of* $\mathsf{E}[U_i(x_i)]$ *with respect to* x_i *yields*

$$
\begin{aligned}
&\frac{d\mathsf{E}[U_i(x_i)]}{dx_i}\\
=&p_i\mathrm{P}(D_i\ge x_i-\hat{y}_{i-1})+\eta_{i-1}p_{i-1}\mathrm{P}(D_i<x_i-\hat{y}_{i-1}, D_i+D_{i-1}\ge x_i-\hat{y}_{i-2})\\
&+\eta_{i-1}\eta_{i-2}p_{i-2}\mathrm{P}(D_i<x_i-\hat{y}_{i-1}, D_i+D_{i-1}<x_i-\hat{y}_{i-2}, D_i+D_{i-1}+D_{i-2}\ge x_i-\hat{y}_{i-3})\\
&+\cdots\\
&+\eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1\mathrm{P}(D_i<x_i-\hat{y}_{i-1}, D_i+D_{i-1}<x_i-\hat{y}_{i-2},\cdots,D_i+\cdots+D_2<x_i-\hat{y}_1, D_i+\cdots+D_1\ge x_i).
\end{aligned}
$$

Proof of Lemma 6. One can check

$$\mathsf{E}[U_i(x_i)] = \mathsf{E}_{D_i}\Big[\max_{0\le y_{i-1}\le x_i} p_i \min\{D_i, x_i - y_{i-1}\} + \eta_{i-1}\mathsf{E}_{D_{i-1}}[U_{i-1}(x_i - \min\{D_i, x_i - y_{i-1}\})]\Big]$$
$$= \mathsf{E}_{D_i}\Big[p_i \min\{D_i, x_i - \hat{y}_{i-1}\} + \eta_{i-1}\mathsf{E}_{D_{i-1}}[U_{i-1}(x_i - \min\{D_i, x_i - \hat{y}_{i-1}\})]\Big],$$

where the last equality holds because we restrict our attention to the problem with a sufficient initial inventory.

Taking derivative with respect to x_i yields

$$p_i \mathrm{P}(D_i \ge x_i - \hat{y}_{i-1}) + \eta_{i-1}\int_0^{x_i - \hat{y}_{i-1}} \frac{d\mathsf{E}[U_{i-1}(x_i - d_i)]}{dx_i} dF_i(d_i)$$
$$= p_i \mathrm{P}(D_i \ge x_i - \hat{y}_{i-1}) + \eta_{i-1}\int_0^{x_i - \hat{y}_{i-1}} \Big\{ p_{i-1}\mathrm{P}(D_{i-1} \ge x_i - d_i - \hat{y}_{i-2})$$
$$+ \eta_{i-2}\int_0^{x_i - d_i - \hat{y}_{i-2}} \frac{d\mathsf{E}[U_{i-2}(x_i - d_i - d_{i-1})]}{dx_i} dF_{i-1}(d_{i-1})\Big\} dF_i(d_i)$$
$$= p_i \mathrm{P}(D_i \ge x_i - \hat{y}_{i-1}) + \eta_{i-1}\int_0^{x_i - \hat{y}_{i-1}} p_{i-1}\mathrm{P}(D_{i-1} \ge x_i - d_i - \hat{y}_{i-2}) dF_i(d_i)$$
$$+ \eta_{i-1}\eta_{i-2}\int_0^{x_i - \hat{y}_{i-1}}\int_0^{x_i - d_i - \hat{y}_{i-2}} \frac{d\mathsf{E}[U_{i-2}(x_i - d_i - d_{i-1})]}{dx_i} dF_{i-1}(d_{i-1}) dF_i(d_i)$$
$$= p_i \mathrm{P}(D_i \ge x_i - \hat{y}_{i-1}) + \eta_{i-1}p_{i-1}\mathrm{P}(D_i < x_i - \hat{y}_{i-1}, D_i + D_{i-1} \ge x_i - \hat{y}_{i-2})$$
$$+ \eta_{i-1}\eta_{i-2}\int_0^{x_i - \hat{y}_{i-1}}\int_0^{x_i - d_i - \hat{y}_{i-2}} \frac{d\mathsf{E}[U_{i-2}(x_i - d_i - d_{i-1})]}{dx_i} dF_{i-1}(d_{i-1}) dF_i(d_i)$$
$$= p_i \mathrm{P}(D_i \ge x_i - \hat{y}_{i-1}) + \eta_{i-1}p_{i-1}\mathrm{P}(D_i < x_i - \hat{y}_{i-1}, D_i + D_{i-1} \ge x_i - \hat{y}_{i-2})$$
$$+ \eta_{i-1}\eta_{i-2}\int_0^{x_i - \hat{y}_{i-1}}\int_0^{x_i - d_i - \hat{y}_{i-2}} \Big\{ p_{i-2}\mathrm{P}(D_{i-2} \ge x_i - d_i - d_{i-1} - \hat{y}_{i-3})$$
$$+ \eta_{i-3}\int_0^{x_i - d_i - d_{i-1} - \hat{y}_{i-3}} \frac{d\mathsf{E}[U_{i-3}(x_i - d_i - d_{i-1} - d_{i-2})]}{dx_i} dF_{i-2}(d_{i-2})\Big\} dF_{i-1}(d_{i-1}) dF_i(d_i)$$
$$= p_i \mathrm{P}(D_i \ge x_i - \hat{y}_{i-1}) + \eta_{i-1}p_{i-1}\mathrm{P}(D_i < x_i - \hat{y}_{i-1}, D_i + D_{i-1} \ge x_i - \hat{y}_{i-2})$$
$$+ \eta_{i-1}\eta_{i-2}p_{i-2}\mathrm{P}(D_i < x_i - \hat{y}_{i-1}, D_i + D_{i-1} < x_i - \hat{y}_{i-2}, D_i + D_{i-1} + D_{i-2} \ge x_i - \hat{y}_{i-3})$$
$$+ \eta_{i-1}\eta_{i-2}\eta_{i-3}\int_0^{x_i - \hat{y}_{i-1}}\int_0^{x_i - d_i - \hat{y}_{i-2}}\int_0^{x_i - d_i - d_{i-1} - \hat{y}_{i-3}} \frac{d\mathsf{E}[U_{i-3}(x_i - d_i - d_{i-1} - d_{i-2})]}{dx_i} dF_{i-2}(d_{i-2}) dF_{i-1}(d_{i-1}) dF_i(d_i)$$
$$= \cdots$$
$$= p_i \mathrm{P}(D_i \ge x_i - \hat{y}_{i-1}) + \eta_{i-1}p_{i-1}\mathrm{P}(D_i < x_i - \hat{y}_{i-1}, D_i + D_{i-1} \ge x_i - \hat{y}_{i-2})$$
$$+ \eta_{i-1}\eta_{i-2}p_{i-2}\mathrm{P}(D_i < x_i - \hat{y}_{i-1}, D_i + D_{i-1} < x_i - \hat{y}_{i-2}, D_i + D_{i-1} + D_{i-2} \ge x_i - \hat{y}_{i-3})$$
$$+ \cdots$$
$$+ \eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1 \mathrm{P}(D_i < x_i - \hat{y}_{i-1}, D_i + D_{i-1} < x_i - \hat{y}_{i-2}, \cdots, D_i + \cdots + D_2 < x_i - \hat{y}_1, D_i + \cdots + D_1 \ge x_i).$$

■

Lemma 7. *For any* $i \in \{1, 2, \cdots, n-1\}$ *and any* $x \geq \hat{y}_i$,

$$\frac{d\mathsf{E}[U_i(x)]}{dx} \leq \frac{p_{i+1}}{\eta_i},$$

where the strict equality holds when $x = \hat{y}_i$.

Proof of Lemma 7. By Lemma 6, we have

$$\begin{aligned}
&\frac{d\mathsf{E}[U_i(x)]}{dx} \\
=&p_i \mathrm{P}(D_i \geq x - \hat{y}_{i-1}) + \eta_{i-1}p_{i-1}\mathrm{P}(D_i < x - \hat{y}_{i-1}, D_{i-1} + D_i \geq x - \hat{y}_{i-2}) \\
&+ \eta_{i-1}\eta_{i-2}p_{i-2}\mathrm{P}(D_i < x - \hat{y}_{i-1}, D_{i-1} + D_i < x - \hat{y}_{i-2}, D_{i-2} + D_{i-1} + D_i \geq x - \hat{y}_{i-3}) + \cdots \\
&+ \eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1 \mathrm{P}(D_i < x - \hat{y}_{i-1}, D_{i-1} + D_i < x - \hat{y}_{i-2}, \cdots, D_2 + \cdots + D_i < x - \hat{y}_1, D_1 + D_2 + \cdots + D_i \geq x).
\end{aligned} \tag{38}$$

On the one hand, according to (8), for any $x \geq \hat{y}_i$,

$$\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_1 + \cdots + D_i \geq x) \leq \frac{p_{i+1}}{\eta_i\eta_{i-1}\cdots\eta_1 p_1}, \tag{39}$$

where the strict equality holds when $x = \hat{y}_i$. On the other hand, by Lemma 5, we have

$$\begin{aligned}
&\mathrm{P}(D_1 \geq \hat{y}_1, D_1 + D_2 \geq \hat{y}_2, \cdots, D_1 + \cdots + D_{i-1} \geq \hat{y}_{i-1}, D_1 + \cdots + D_i \geq x) \\
=&\frac{p_i}{\eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1}\mathrm{P}(D_i \geq x - \hat{y}_{i-1}) + \frac{p_{i-1}}{\eta_{i-2}\eta_{i-3}\cdots\eta_1 p_1}\mathrm{P}(D_i < x - \hat{y}_{i-1}, D_i + D_{i-1} \geq x - \hat{y}_{i-2}) + \cdots \\
&+ \frac{p_2}{\eta_1 p_1}\mathrm{P}(D_i < x - \hat{y}_{i-1}, \cdots, D_i + D_{i-1} + \cdots + D_3 < x - \hat{y}_2, D_i + D_{i-1} + \cdots + D_2 \geq x - \hat{y}_1) \\
&+ \mathrm{P}(D_i < x - \hat{y}_{i-1}, \cdots, D_i + D_{i-1} + \cdots + D_2 < x - \hat{y}_1, D_i + D_{i-1} + \cdots + D_1 \geq x).
\end{aligned} \tag{40}$$

Combining (39) and (40) yields

$$\begin{aligned}
&p_i \mathrm{P}(D_i \geq x - \hat{y}_{i-1}) + \eta_{i-1}p_{i-1}\mathrm{P}(D_i < x - \hat{y}_{i-1}, D_{i-1} + D_i \geq x - \hat{y}_{i-2}) + \cdots \\
&+ \eta_{i-1}\eta_{i-2}\cdots\eta_1 p_1 \mathrm{P}(D_i < x - \hat{y}_{i-1}, D_{i-1} + D_i < x - \hat{y}_{i-2}, \cdots, D_2 + \cdots + D_i < x - \hat{y}_1, D_1 + D_2 + \cdots + D_i \geq x) \\
\leq&\frac{p_{i+1}}{\eta_i},
\end{aligned}$$

where the strict equality holds when $x = \hat{y}_i$. It follows (38) that $\frac{d\mathsf{E}[U_i(x)]}{dx} \leq \frac{p_{i+1}}{\eta_i}$ holds immediately. ■

Lemma 8. *Taking derivative of* $\hat{V}_i(x_i)$ *with respect to* x_i *yields*

$$\begin{aligned}
\frac{d\hat{V}_i(x_i)}{dx_i} =& p_i \mathrm{P}(D_i \geq x_i - \hat{y}_{i-1}) + p_{i-1}\mathrm{P}(D_i < x_i - \hat{y}_{i-1}, D_i + D_{i-1} \geq x_i - \hat{y}_{i-2}) \\
&+ p_{i-2}\mathrm{P}(D_i < x_i - \hat{y}_{i-1}, D_i + D_{i-1} < x_i - \hat{y}_{i-2}, D_i + D_{i-1} + D_{i-2} \geq x_i - \hat{y}_{i-3}) \\
&+ \cdots \\
&+ p_1 \mathrm{P}(D_i < x_i - \hat{y}_{i-1}, D_i + D_{i-1} < x_i - \hat{y}_{i-2}, \cdots, D_i + \cdots + D_2 < x_i - \hat{y}_1, D_i + \cdots + D_1 \geq x_i).
\end{aligned}$$

Proof of Lemma 8. Recall that

$$\hat{V}_i(x_i) = \mathsf{E}\Big[p_i \min\{D_i, x_i - \hat{y}_{i-1}\} + \hat{V}_{i-1}(x_i - \min\{D_i, x_i - \hat{y}_{i-1}\})\Big].$$

Taking derivative with respect to x_i follows the same process as that in Lemma 6 and thus the detail is omitted. ■