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Constrained Stochastic Linear Quadratic Control Under Regime Switching with Controlled Jump Size

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Abstract

In this paper, we examine a stochastic linear-quadratic control problem characterized by regime switching and Poisson jumps. All the coefficients in the problem are random processes adapted to the filtration generated by Brownian motion and Poisson random measure for each given regime. The model incorporates two distinct types of controls: the first is a conventional control that appears in the continuous diffusion component, while the second is an unconventional control, dependent on the variable z, which influences the jump size in the jump diffusion component. Both controls are constrained within general closed cones. By employing the Meyer-Itô formula in conjunction with a generalized squares completion technique, we rigorously and explicitly derive the optimal value and optimal feedback control. These depend on solutions to certain multi-dimensional fully coupled stochastic Riccati equations, which are essentially backward stochastic differential equations with jumps (BSDEJs). We establish the existence of a unique nonnegative solution to the BSDEJs. One of the major tools used in the proof is the newly established comparison theorems for multi-dimensional BSDEJs.

Keywords Linear-quadratic control · Regime switching · Controlled jump size · Fully coupled stochastic Riccati equations · Backward stochastic differential equations with jumps

Mathematics Subject Classification 60H30 · 60J28 · 60J76 · 93E20

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1 Introduction

Since the pioneering work of Wonham [22], stochastic linear-quadratic (LQ) theory has been extensively studied by numerous researchers. For instance, Bismut [2] was the first one who studied stochastic LQ problems with random coefficients. In order to obtain the optimal random feedback control, he formally derived a stochastic Riccati equation (SRE). But he could not solve the SRE in the general case. It is Kohlmann and Tang [12], for the first time, that established the existence and uniqueness of the one-dimensional SRE. Tang [18, 19] made another breakthrough and proved the existence and uniqueness of the matrix valued SRE with uniformly positive control weighting matrix using two different approaches. Sun, Xiong and Yong [17] studied the indefinite stochastic LQ problem with random coefficients. Hu and Zhou [10] solved the stochastic LQ problem with cone control constraint. Zhang, Dong and Meng [25] made a great progress in solving stochastic LQ control and related SRE with jumps with uniformly positive definite control weight by inverse flow technique. Li, Wu and Yu [13] considered the stochastic LQ problem with jumps in the indefinite case. Please refer to Chapter 6 in Yong and Zhou [24] for a systematic account on this subject.

Stochastic LQ problems for Markovian regime switching system were studied in Wen, Li and Xiong [21] and Zhang, Li and Xiong [26] where weak closed-loop solvability, open-loop solvability and closed-loop solvability were established. But the coefficients are assumed to be *deterministic* functions of time t for each given regime i in the above papers, so their SREs are indeed deterministic ordinary differential equations (ODEs). Hu, Shi and Xu [6, 7] formulated cone-constrained stochastic LQ problems with regime switching on finite time horizon and infinite time horizon respectively, in which the coefficients are *stochastic* processes adapted to the filtration generated by the Brownian motion for each give regime i. Due to the randomness of the coefficients, the corresponding SREs in [6, 7] are actually backward stochastic differential equations (BSDEs). Hu, Shi and Xu [8] extended the model [6] to include non-homogeneous terms, but without control constraints. Please note that finding feedback controls for non-homogeneous LQ problems with control constraints seems a formidable challenge, even if all the coefficients are deterministic. In addition to the SREs in [6], a system of linear BSDEs with unbounded coefficients is employed to construct the optimal feedback control in [8]. The main contribution of [8] is to prove the existence of unique solution to this system of linear BSDEs with unbounded coefficients by means of BMO martingales and contraction mapping method.

In this paper, we generalize the LQ problem in [6] to a model in which the coefficients are *stochastic* processes adapted to the filtration generated by a Brownian motion and a Poisson random measure for each give regime i. In addition to a usual control u_1, we introduce a second control $u_2(z)$ depending on the jump size z. The motivations to incorporating the second control are, in insurance area, the optimal reinsurance strategies may depend on the claim size in general, see, e.g., Liu and Ma [14] and Wu, Shen, Zhang and Ding [23]; and in controllability issues for stochastic systems with jump diffusions, a control depending on the jump size is necessary as a consequence of martingale representation theorem of Poisson random measures, see, e.g., Goreac [5] and Song [16]. The application of this kind of stochastic LQ model in a optimal liquidation problem with dark pools can be found in our working paper [3].

The first main contribution of this paper is to provide a pure analysis method (using tools like approximation technique, comparison theorem for multi-dimensional BSDE with jumps (BSDEJs), log transformation, etc) of the existence of a unique solution to the corresponding system of SREs, which is a 2ℓ-dimensional coupled BSDEJs. This is interesting in its own right from the point of view of BSDE theory. Note that although the SREs in [6] are 2ℓ-dimensional, they are partially coupled, that is, the first ℓ equations for $\{P_1^i\}_{i\in\mathcal{M}}$ and the second ℓ equations for $\{P_2^i\}_{i\in\mathcal{M}}$ are totally decoupled. But in our new model, the equation for P_1^i also depends on (P_2^i, Γ_2^i), rendering the 2ℓ-dimensional SREs in our new model are fully coupled. This more complicated phenomenon comes from the fact that, to the best of our knowledge, the optimal state process will probably change its sign at the jump time of the underlying Poisson random measure. Compared with the 2-dimensional SREs in Hu, Shi and Xu [9], here we need to study 2ℓ-dimensional SREs because of the new coupling terms $\sum_j q^{ij}P_1^j$ and $\sum_j q^{ij}P_2^j$. Please note that the uniqueness of the solution to SREs in [9] is obtained by verification arguments which is an indirect approach. The second main contribution is to give a rigorous verification theorem of the optimal value and optimal control, using the unique solution to the corresponding system of SREs, Meyer-Itô's formula, a generalized squares completion technique and some delicate analysis.

The rest part of this paper is organized as follows. In Section 2, we formulate a constrained stochastic LQ control problem with regime switching, controlled jump size and random coefficients. Section 3 is devoted to proving the existence of a unique nonnegative solution to the related 2ℓ-dimensional fully coupled SREs in standard and singular cases. In Section 4, we solve the LQ problem by establishing a rigorous verification theorem.

2 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fixed complete filtered probability space. The filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is generated by the following three independent random sources augmented by all the $\mathbb{P}$-null sets.

- The first random source is a standard n-dimensional Brownian motion $W_t = (W_{1,t}, \ldots, W_{n_1,t})^\top$.
- The second one is an n_2-dimensional Poisson random measure $N = (N_1, \ldots, N_{n_2})^\top$ defined on $\mathbb{R}_+ \times \mathcal{Z}$, where $\mathcal{Z} \subset \mathbb{R}^\ell \setminus \{0\}$ is a nonempty Borel subset of some Euclidean space. For each $k = 1, \ldots, n_2$, N_k posses the same stationary compensator (intensity measure) $\nu(\mathrm{d}z)\mathrm{d}t$ satisfying $\nu(\mathcal{Z}) < \infty$. The compensated Poisson random measure is denoted by $\tilde{N}(\mathrm{d}t, \mathrm{d}z)$.
- The third one is a continuous-time stationary Markov chain α_t valued in a finite state space $\mathcal{M} = \{1, 2, \ldots, \ell\}$ with $\ell \geq 1$. The Markov chain has a generator

$$Q = (q_{ij})_{\ell\times\ell} \text{ with } q_{ij} \geq 0 \text{ for } i \neq j \text{ and } \sum_{j=1}^{\ell} q_{ij} = 0 \text{ for every } i \in \mathcal{M}.$$

Besides the filtration $\mathbb{F}$, we will often use the filtration $\mathbb{F}^{W,N} = \{\mathcal{F}_t^{W,N}, t \geq 0\}$ which is generated by the Brownian motion W and the Poisson random measures N and augmented by all the $\mathbb{P}$-null sets. Throughout the paper, let T denote a fixed positive constant, $\mathcal{P}$ (resp. $\mathcal{P}^{W,N}$) denote the $\mathbb{F}$ (resp. $\mathbb{F}^{W,N}$)-predictable σ-field on $\Omega \times [0,T]$, and $\mathcal{B}(\mathcal{Z})$ denote the Borelian σ-field on $\mathcal{Z}$.

We denote by $\mathbb{R}^\ell$ the set of ℓ-dimensional column vectors, by $\mathbb{R}^\ell_+$ the set of vectors in $\mathbb{R}^\ell$ whose components are nonnegative, by $\mathbb{R}^{\ell\times n}$ the set of $\ell \times n$ real matrices, by $\mathbb{S}^n$ the set of $n \times n$ symmetric real matrices, by $\mathbb{S}^n_+$ the set of $n \times n$ nonnegative definite real matrices, and by 1_n the n-dimensional identity matrix. For any vector Y, we denote Y_i as its i-th component. For any matrix $M = (m_{ij})$, we denote its transpose by $M^\top$, and its norm by $|M| = \sqrt{\sum_{ij} m_{ij}^2}$. If $M \in \mathbb{S}^n$ is positive definite (resp. positive semidefinite), we write $M >$ (resp. $\geq$) 0. We write $A >$ (resp. $\geq$) B if $A, B \in \mathbb{S}^n$ and $A - B >$ (resp. $\geq$) 0. We write the positive and negative parts of $x \in \mathbb{R}$ as $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$ respectively. The elementary inequality $|a^\top b| \leq c|a|^2 + \frac{|b|^2}{2c}$ for any $a, b \in \mathbb{R}^n$, $c > 0$, will be used frequently without claim. Throughout the paper, we use c to denote a suitable positive constant, which is independent of (t, ω, i) and can be different from line to line.

2.1 Notation

We use the following notation throughout the paper:

$$\begin{aligned}
L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}) &= \Big\{\xi : \Omega \to \mathbb{R} \;\Big|\; \xi \text{ is } \mathcal{F}_T\text{-measurable, and essentially bounded}\Big\},\\
L^2_{\mathbb{F}}(0,T;\mathbb{R}) &= \Big\{\phi : [0,T]\times\Omega \to \mathbb{R} \;\Big|\; \phi \text{ is } \mathbb{F}\text{-predictable and } \mathbb{E}\int_0^T |\phi_t|^2 \mathrm{d}t < \infty\Big\},\\
L^\infty_{\mathbb{F}}(0,T;\mathbb{R}) &= \Big\{\phi : [0,T]\times\Omega \to \mathbb{R} \;\Big|\; \phi \text{ is } \mathbb{F}\text{-predictable and essentially bounded}\Big\},\\
L^{2,\nu}(\mathbb{R}) &= \Big\{\phi : \mathcal{Z} \to \mathbb{R} \text{ is measurable with } \|\phi(\cdot)\|_\nu^2 := \int_{\mathcal{Z}} \phi(z)^2 \nu(\mathrm{d}z) < \infty\Big\},\\
L^{\infty,\nu}(\mathbb{R}) &= \Big\{\phi : \mathcal{Z} \to \mathbb{R} \text{ is measurable and } \phi \text{ is bounded } \mathrm{d}\nu\text{-a.e.}\Big\},\\
L^{2,\nu}_{\mathcal{P}}(0,T;\mathbb{R}) &= \Big\{\phi : [0,T]\times\Omega\times\mathcal{Z} \to \mathbb{R} \;\Big|\; \phi \text{ is } \mathcal{P}\otimes\mathcal{B}(\mathcal{Z})\text{-measurable}\\
&\qquad \text{and } \mathbb{E}\int_0^T\int_{\mathcal{Z}} |\phi_t(z)|^2 \nu(\mathrm{d}z)\mathrm{d}t < \infty\Big\},\\
L^\infty_{\mathcal{P}}(0,T;\mathbb{R}) &= \Big\{\phi : [0,T]\times\Omega\times\mathcal{Z} \to \mathbb{R} \;\Big|\; \phi \text{ is } \mathcal{P}\otimes\mathcal{B}(\mathcal{Z})\text{-measurable and essentially bounded}\Big\},\\
S^\infty_{\mathbb{F}}(0,T;\mathbb{R}) &= \Big\{\phi : \Omega\times[0,T] \to \mathbb{R} \;\Big|\; \phi \text{ is càd-làg}, \mathbb{F}\text{-adapted and essentially bounded}\Big\}.
\end{aligned}$$

These definitions are generalized in the obvious way to the cases that $\mathcal{F}$ is replaced by $\mathcal{F}^{W,N}$, $\mathbb{F}$ by $\mathbb{F}^{W,N}$, $\mathcal{P}$ by $\mathcal{P}^{W,N}$ and $\mathbb{R}$ by $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ or $\mathbb{S}^n$. In our argument, t, ω, "almost surely" and "almost everywhere", will be suppressed for simplicity in many circumstances, when no confusion occurs. All the processes and maps considered in this paper, unless otherwise stated, are stochastic, so, for notation simplicity, we will not write their dependence on ω explicitly. Equations and inequalities shall be understood to hold true $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t \otimes \mathrm{d}\nu$-a.e. For a random variable or stochastic process X, we write $X \gg 1$ (resp. $X \ll 1$) if there exists a constant $c > 0$ such that $X \geq c$ (resp. $|X| \leq c$).

Consider the following real-valued linear stochastic differential equation (SDE) with jumps:

$$\begin{cases} \mathrm{d}X_t = \left[A_t^{\alpha_{t-}} X_{t-} + (B_{1,t}^{\alpha_{t-}})^\top u_{1,t} + \int_{\mathcal{Z}} B_{2,t}^{\alpha_{t-}}(z)^\top u_{2,t}(z)\nu(\mathrm{d}z)\right] \mathrm{d}t \\ \qquad\qquad + \left[C_t^{\alpha_{t-}} X_{t-} + D_t^{\alpha_{t-}} u_{1,t}\right]^\top \mathrm{d}W_t \\ \qquad\qquad + \int_{\mathcal{Z}} \left[E_t^{\alpha_{t-}}(z) X_{t-} + F_t^{\alpha_{t-}}(z) u_{2,t}(z)\right]^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \quad t \in [0, T], \\ X_0 = x, \quad \alpha_0 = i_0, \end{cases} \tag{2.1}$$

where $A^i,\ B_1^i,\ C^i,\ D^i$ are all $\mathbb{F}^{W,N}$-predictable processes, and $B_2^i(\cdot),\ E^i(\cdot),\ F^i(\cdot)$ are $\mathcal{P}^{W,N} \otimes \mathcal{B}(\mathcal{Z})$-measure processes of suitable sizes, (u_1, u_2) is the control and $x \in \mathbb{R}$, $i_0 \in \mathcal{M}$ are the known initial values.

Let Π_1, Π_2 be two given closed cones (not necessarily convex) in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively. The class of admissible controls is defined as the set

$$\mathcal{U} := \Big\{(u_1, u_2) \;\Big|\; u_1 \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{m_1}),\ u_{1,t} \in \Pi_1,\ \mathrm{d}\mathbb{P} \otimes \mathrm{d}t\text{-a.e.}, \\ \text{and} \quad u_2 \in L^{2,\nu}_{\mathcal{P}}(0, T; \mathbb{R}^{m_2}),\ u_{2,t} \in \Pi_2,\ \mathrm{d}\mathbb{P} \otimes \mathrm{d}t \otimes \mathrm{d}\nu\text{-a.e.}\Big\}.$$

If $u \equiv (u_1, u_2) \in \mathcal{U}$, then the SDE (2.1) admits a unique strong solution X, and we refer to (X, u) as an admissible pair.

Let us now state our stochastic linear quadratic optimal control problem as follows:

$$\begin{cases} \text{Minimize} & J(u; x, i_0) \\ \text{subject to} & u \in \mathcal{U}, \end{cases} \tag{2.2}$$

where the cost functional J is given as the following quadratic form

$$J(u; x, i_0) := \mathbb{E}\Big\{G^{\alpha_T} X_T^2 + \int_0^T \Big[u_{1,t}^\top R_{1,t}^{\alpha_t} u_{1,t} + Q_t^{\alpha_t} X_t^2 + \int_{\mathcal{Z}} u_{2,t}(z)^\top R_{2,t}^{\alpha_t}(z) u_{2,t}(z)\nu(\mathrm{d}z)\Big]\mathrm{d}t\Big\}. \tag{2.3}$$

The optimal value of the problem is defined as

$$V(x, i_0) = \inf_{u \in \mathcal{U}} J(u; x, i_0).$$

Problem (2.2) is said to be solvable, if there exists a control $u^* \in \mathcal{U}$ such that

$$-\infty < J(u^*; x, i_0) \leq J(u; x, i_0), \quad \forall\, u \in \mathcal{U},$$

in which case, u^* is called an optimal control for problem (2.2) and one has

$$V(x, i_0) = J(u^*; x, i_0).$$

Remark 2.1 By choosing $\Pi_1 = \{0\}$ (resp. $\Pi_2 = \{0\}$), our model covers the case of $R_1^i = 0$, $D^i = 0$, $B_1^i = 0$ (resp. $R_2^i = 0$, $F^i = 0$, $B_2^i = 0$). In particular, our model covers the pure jump (i.e. $(B_1^i, C^i, D^i, R_1^i) = 0$) and pure continuous diffusion (i.e., $(B_2^i, E^i, F^i, R_2^i) = 0$) models.

Throughout this paper, we put the following assumption on the coefficients.

Assumption 2.1 It holds, for every $i \in \mathcal{M}$, that

$$\begin{cases} A^i \in L^\infty_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}),\ B_1^i \in L^\infty_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}^{m_1}),\ B_2^i \in L^\infty_{\mathcal{P}^{W,N}}(0,T;\mathbb{R}^{m_2}),\\ C^i \in L^\infty_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}^{n_1}),\ D^i \in L^\infty_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}^{n_1\times m_1}),\\ E^i \in L^\infty_{\mathcal{P}^{W,N}}(0,T;\mathbb{R}^{n_2}),\ F^i \in L^\infty_{\mathcal{P}^{W,N}}(0,T;\mathbb{R}^{n_2\times m_2}),\\ R_1^i \in L^\infty_{\mathbb{F}^{W,N}}(0,T;\mathbb{S}^{m_1}_+),\ R_2^i \in L^\infty_{\mathcal{P}^{W,N}}(0,T;\mathbb{S}^{m_2}_+),\\ Q^i \in L^\infty_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}_+),\ G^i \in L^\infty_{\mathcal{F}_T^{W,N}}(\Omega;\mathbb{R}_+). \end{cases}$$

Under Assumption 2.1, the cost functional (2.3) is nonnegative, hence problem (2.2) is well-posed.

Besides Assumption 2.1, we need the following hypotheses:

1. $R_1^i \geq \delta 1_{m_1}$.
2. $(D^i)^\top D^i \geq \delta 1_{m_1}$.
3. $R_2^i \geq \delta 1_{m_2}$.
4. $(F^i)^\top F^i \geq \delta 1_{m_2}$.

We will consider the problem under one of following two assumptions.

Assumption 2.2 (Standard case) There exists a constant $\delta > 0$ such that both hypothesizes 1 and 3 hold.

Assumption 2.3 (Singular case) There exists a constant $\delta > 0$ such that $G^i \geq \delta$ and one of the following holds:

Case I. Both hypothesizes 2 and 3 hold;
Case II. Both hypothesizes 2 and 4 hold;
Case III. Both hypothesizes 1 and 4 hold.

Remark 2.2 As is well-known, there is no essential difficulty to consider a more general cost functional comprising cross terms:

$$\begin{aligned} \tilde{J}(u;x,i_0) := \mathbb{E}\Big\{ G^{\alpha_T} X_T^2 + \int_0^T \Big[u_{1,t}^\top R_{1,t}^{\alpha_t} u_{1,t} + 2u_{1,t}^\top S_{1,t}^{\alpha_t} X_t + Q_t^{\alpha_t} X_t^2 \\ + \int_{\mathcal{Z}} \Big(u_{2,t}(z)^\top R_{2,t}^{\alpha_t}(z) u_{2,t}(z) + 2u_{2,t}(z)^\top S_{2,t}^{\alpha_t}(z) X_t \Big) \nu(\mathrm{d}z) \Big] \mathrm{d}t \Big\}. \end{aligned} \tag{2.4}$$

Let us explain this point in the Standard case. Under Assumption 2.2, by regarding

$$(\tilde{u}_1, \tilde{u}_2) := (u_1 - R_1^{-1} S_1 X, u_2 - R_2^{-1} S_2 X)$$

as the new control, LQ problem under the state process (2.1), and cost functional (2.4) could be reduced to one without cross terms:

$$J(\tilde{u}; x, i_0) = \mathbb{E}\Big\{G^{\alpha_T} \tilde{X}_T^2 + \int_0^T \Big[\tilde{u}_{1,t}^\top R_{1,t}^{\alpha_t} \tilde{u}_{1,t} + \tilde{Q}_t^{\alpha_t} \tilde{X}_t^2 + \int_{\mathcal{Z}} \tilde{u}_{2,t}(z)^\top R_{2,t}^{\alpha_t}(z) \tilde{u}_{2,t}(z) \nu(\mathrm{d}z)\Big] \mathrm{d}t\Big\}.$$

subject to the state

$$\begin{cases} \mathrm{d}\tilde{X}_t = \Big[\tilde{A}_t^{\alpha_{t-}} \tilde{X}_{t-} + (B_{1,t}^{\alpha_{t-}})^\top \tilde{u}_{1,t} + \int_{\mathcal{Z}} B_{2,t}^{\alpha_{t-}}(z)^\top \tilde{u}_{2,t}(z) \nu(\mathrm{d}z)\Big] \mathrm{d}t \\ \qquad\qquad + \Big[\tilde{C}_t^{\alpha_{t-}} \tilde{X}_{t-} + D_t^{\alpha_{t-}} \tilde{u}_{1,t}\Big]^\top \mathrm{d}W_t \\ \qquad\qquad + \int_{\mathcal{Z}} \Big[\tilde{E}_t^{\alpha_{t-}}(z) \tilde{X}_{t-} + F_t^{\alpha_{t-}}(z) \tilde{u}_{2,t}(z)\Big]^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \quad t \in [0, T], \\ \tilde{X}_0 = x, \quad \alpha_0 = i_0, \end{cases}$$

where

$$\tilde{A}^i := A^i - (B_1^i)^\top (R_1^i)^{-1} S_1^i - \int_{\mathcal{Z}} (B_2^i)^\top (R_2^i)^{-1} S_2^i \nu(\mathrm{d}z),\ \tilde{C}^i := C^i - D(R_1^i)^{-1} S_1^i,$$

$$\tilde{E}^i := E^i - F^i (R_2^i)^{-1} S_2^i,\ \tilde{Q}^i := Q^i - (S_1^i)^\top (R_1^i)^{-1} S_1^i - \int_{\mathcal{Z}} (S_2^i)^\top (R_2^i)^{-1} S_2^i \nu(\mathrm{d}z)$$

provided $\tilde{Q}^i \geq 0$.

3 Solvability of the Riccati Equations

For any $i \in \mathcal{M}$, $j = 1, 2$, let us denote by E_k^i the *k*-th component of E^i, Γ_{jk}^i the *k*-th component of Γ_j^i and F_k^i the *k*-th row of F^i, $k = 1, \ldots, n_2$. To solve problem (2.2), we need to study the following 2ℓ-dimensional SRE with jumps (the arguments *t* and ω are suppressed):

$$\begin{cases} \mathrm{d}P_1^i = -\Big[(2A^i + |C^i|^2) P_1^i + 2(C^i)^\top \Lambda_1^i + Q^i + H_{11}^{i,*}(P_1^i, \Lambda_1^i) \\ \qquad\qquad + \int_{\mathcal{Z}} H_{12}^{i,*}(z, P_1^i, P_2^i, \Gamma_1^i, \Gamma_2^i) \nu(\mathrm{d}z) + \sum_{j=1}^{\ell} q^{ij} P_1^j\Big] \mathrm{d}t \\ \qquad + (\Lambda_1^i)^\top \mathrm{d}W + \int_{\mathcal{Z}} \Gamma_1^i(z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \\ \mathrm{d}P_2^i = -\Big[(2A^i + |C^i|^2) P_2^i + 2(C^i)^\top \Lambda_2^i + Q^i + H_{21}^{i,*}(P_2^i, \Lambda_2^i) \\ \qquad\qquad + \int_{\mathcal{Z}} H_{22}^{i,*}(z, P_1^i, P_2^i, \Gamma_1^i, \Gamma_2^i) \nu(\mathrm{d}z) + \sum_{j=1}^{\ell} q^{ij} P_2^j\Big] \mathrm{d}t \\ \qquad + (\Lambda_2^i)^\top \mathrm{d}W + \int_{\mathcal{Z}} \Gamma_2^i(z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \\ P_{1,T}^i = G^i,\ P_{2,T}^i = G^i, \quad R_1^i + P_1^i (D^i)^\top D^i > 0,\ R_1^i + P_2^i (D^i)^\top D^i > 0, \quad i \in \mathcal{M}, \end{cases} \tag{3.1}$$

where, for any $(t, v, z, P_1, P_2, \Lambda, \Gamma_1, \Gamma_2) \in [0, T] \times \Pi_1(\text{ or } \Pi_2) \times \mathcal{Z} \times \mathbb{R}_+^2 \times \mathbb{R}^n \times (L^{\infty,\nu}(\mathbb{R}^{n_2})^2$, we define

$$
\begin{aligned}
H_{11}^i(t,v,P_1,\Lambda) &:= v^\top(R_1^i + P_1(D^i)^\top D^i)v + 2(P_1(B_1^i + (D^i)^\top C^i) + (D^i)^\top\Lambda)^\top v,\\
H_{12}^i(t,v,z,P_1,P_2,\Gamma_1,\Gamma_2) &:= v^\top R_2^i v + \sum_{k=1}^{n_2}(P_1+\Gamma_{1k})\big[((1+E_k^i+F_k^i v)^+)^2 - 1\big]\\
&\quad - 2P_1\sum_{k=1}^{n_2}(E_k^i + F_k^i v) + 2P_1(B_2^i)^\top v\\
&\quad + \sum_{k=1}^{n_2}(P_2+\Gamma_{2k})((1+E_k^i+F_k^i v)^-)^2,\\
H_{21}^i(t,v,P_2,\Lambda) &:= v^\top(R_1^i + P_2(D^i)^\top D^i)v - 2(P_2(B_1^i + (D^i)^\top C^i) + (D^i)^\top\Lambda)^\top v,\\
H_{22}^i(t,v,z,P_1,P_2,\Gamma_1,\Gamma_2) &:= v^\top R_2^i v + \sum_{k=1}^{n_2}(P_2+\Gamma_{2k})\big[((-1-E_k^i+F_k^i v)^-)^2 - 1\big]\\
&\quad - 2P_2\sum_{k=1}^{n_2}(E_k^i - F_k^i v) - 2P_2(B_2^i)^\top v\\
&\quad + \sum_{k=1}^{n_2}(P_1+\Gamma_{1k})((-1-E_k^i+F_k^i v)^+)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
H_{11}^{i,*}(t,P_1,\Lambda) &:= \inf_{v\in\Pi_1} H_{11}^i(t,v,P_1,\Lambda),\\
H_{12}^{i,*}(t,z,P_1,P_2,\Gamma_1,\Gamma_2) &:= \inf_{v\in\Pi_2} H_{12}^i(t,v,z,P_1,P_2,\Gamma_1,\Gamma_2),\\
H_{21}^{i,*}(t,P_2,\Lambda) &:= \inf_{v\in\Pi_1} H_{21}^i(t,v,P_2,\Lambda),\\
H_{22}^{i,*}(t,z,P_1,P_2,\Gamma_1,\Gamma_2) &:= \inf_{v\in\Pi_2} H_{22}^i(t,v,z,P_1,P_2,\Gamma_1,\Gamma_2).
\end{aligned}
$$

To shorten notation, we omit the arguments t of $B_1^i, B_2^i, C^i, D^i, E^i, F^i, R_1^i, R_2^i$ in the definitions of $H_{kj}^i, k, j = 1, 2$. Because the generators in (3.1) depend on all P_j^is, hence (3.1) is a system of fully coupled BSDEJs.

Definition 3.1 A vector of stochastic process $(P_j^i, \Lambda_j^i, \Gamma_j^i)_{i\in\mathcal{M},\, j=1,2}$ is called a solution to the BSDEJ (3.1) if it satisfies all the equations and constraints in (3.1), and $(P_j^i, \Lambda_j^i, \Gamma_j^i) \in S^\infty_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}^{n_1}) \times L^{\infty,\nu}_{\mathcal{P}^{W,N}}(0,T;\mathbb{R}^{n_2})$ for all $i \in \mathcal{M}$, $j = 1, 2$. Furthermore, the solution is called nonnegative if $P_j^i \geq 0$, $P_j^i + \Gamma_j^i \geq 0$, and called uniformly positive if $P_j^i \gg 1$ and $P_j^i + \Gamma_j^i \gg 1$, for all $i \in \mathcal{M}$, $j = 1, 2$.

Before giving the proof of main theorem in this section, let us recall the definition of bounded mean oscillation martingales, briefly called BMO martingales. Please refer to Kazamaki [11] for a systematic account on continuous BMO martingales. A process $\int_0^t \phi_s^\top dW_s$ is called a BMO martingale if and only if there exists a constant $c > 0$ such that

$$\mathbb{E}\Big[\int_\tau^T |\phi_s|^2 \mathrm{d}s \;\Big|\; \mathcal{F}_\tau^{W,N}\Big] \le c$$

for all $\mathbb{F}^{W,N}$ stopping times $\tau \le T$. In the uniqueness part of the Theorem 3.1, we will use the following property of BMO martingales: If $\int_0^t \phi_s^\top dW_s$ is a BMO martingale on [0, T], then the Doléans-Dade stochastic exponential $\mathcal{E}\big(\int_0^t \phi_s^\top dW_s\big)$ is a uniformly integrable martingale on [0, T]. We have finer estimates for solutions to (3.1) under Assumption 2.1 shown as below.

Lemma 3.1 *Let Assumption 2.1 hold. If $(P_j^i, \Lambda_j^i, \Gamma_j^i)_{i\in\mathcal{M},\ j=1,2}$ is a solution to (3.1), then $\int_0^\cdot \Lambda_j^i dW$ is a BMO martingale, $i \in \mathcal{M},\ j = 1, 2$.*

Proof Applying Itô's formula to $|P_{1,t}^i|^2$, we get, for any $\mathbb{F}^{W,N}$ stopping time $\tau \le T$,

$$\begin{aligned}
&\mathbb{E}\Big[\int_\tau^T |\Lambda_1^i|^2 \mathrm{d}s \;\Big|\; \mathcal{F}_\tau^{W,N}\Big]\\
&\le |G^i|^2 + \mathbb{E}\Big\{\int_\tau^T 2P_1^i\Big[(2A^i + |C^i|^2)P_1^i + 2(C^i)^\top \Lambda_1^i + Q^i\\
&\qquad + H_{11}^{i,*}(P_1^i, \Lambda_1^i) + \int_{\mathcal{Z}} H_{12}^{i,*}(P_1^i, P_2^i, \Gamma_1^i, \Gamma_2^i)\nu(\mathrm{d}z) + \sum_{j=1}^{\ell} q^{ij} P_1^j\Big]\mathrm{d}s \;\Big|\; \mathcal{F}_\tau^{W,N}\Big\}\\
&\le c + \frac{1}{2}\mathbb{E}\Big[\int_\tau^T |\Lambda_1^i|^2 \mathrm{d}s \;\Big|\; \mathcal{F}_\tau^{W,N}\Big],
\end{aligned}$$

where we used Assumption 2.1, $H_{11}^{i,*} \le 0$, $H_{12}^{i,*}(P_1^i, P_2^i, \Gamma_1^i, \Gamma_2^i) \le H_{12}^i(0, P_1^i, P_2^i, \Gamma_1^i, \Gamma_2^i)$, and the solution $(P_j^i, \Lambda_j^i, \Gamma_j^i)_{i\in\mathcal{M},\ j=1,2}$ is uniformly bounded. Note both sides in the above estimate are finite since $\Lambda_1^i \in L^2_{\mathbb{F}^{W,N}}(0, T; \mathbb{R}^{n_1})$. After rearrangement, we conclude that $\int_0^\cdot \Lambda_1^i \mathrm{d}W$ is a BMO martingale. Likewise, $\int_0^\cdot \Lambda_2^i \mathrm{d}W$ is also a BMO martingale. □

The following comparison theorem for multi-dimensional BSDEJs was firstly established in [9, Theorem 2.2]. We list it here as it plays crucial role in the solvability of the BSDEJ (3.1).

Lemma 3.2 *Suppose, for every $i \in \{1, 2, \ldots, m\}$,*

$$(Y_i, Z_i, \Phi_i), (\overline{Y}_i, \overline{Z}_i, \overline{\Phi}_i) \in S^2_{\mathbb{F}}(0, T; \mathbb{R}) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^n) \times L^{2,\nu}_{\mathcal{P}}(0, T; \mathbb{R}),$$

and they satisfy BSDEJs

$$Y_{i,t} = \xi_i + \int_t^T f_i(s, Y_{s-}, Z_{i,s}, \Phi_s)ds - \int_t^T Z_{i,s}^\top dW_s - \int_t^T \int_{\mathcal{E}} \Phi_{i,s}(e)\widetilde{N}(ds, de),$$

and

$$\overline{Y}_{i,t} = \overline{\xi}_i + \int_t^T \overline{f}_i(s, \overline{Y}_{s-}, \overline{Z}_{i,s}, \overline{\Phi}_s)ds - \int_t^T \overline{Z}_{i,s}^\top dW_s - \int_t^T \int_{\mathcal{E}} \overline{\Phi}_{i,s}(e)\widetilde{N}(ds, de),$$

respectively. Also suppose that, for all $i \in \{1, 2, \dots, m\}$ *and* $s \in [0, T]$*,*
(1) $\xi_i \le \overline{\xi}_i$*;*
(2) *there exists a constant* $c > 0$ *such that*

$$\begin{aligned} & f_i(s, Y_{s-}, Z_{i,s}, \Phi_{1,s}, \cdots, \Phi_{i,s}, \cdots, \Phi_{\ell,s}) \\ & \qquad - f_i(s, Y_{s-}, Z_{i,s}, \Phi_{1,s}, \cdots, \overline{\Phi}_{i,s}, \cdots, \Phi_{\ell,s}) \\ & \le c \int_{\mathcal{E}} (\Phi_{i,s}(e) - \overline{\Phi}_{i,s}(e))^+ \nu(\mathrm{de}) + \int_{\mathcal{E}} |\Phi_{i,s}(e) - \overline{\Phi}_{i,s}(e)| \nu(\mathrm{de}); \end{aligned}$$

(3) *there exists a constant* $c > 0$ *such that*

$$\begin{aligned} & f_i(s, Y_{s-}, Z_{i,s}, \Phi_{1,s}, \cdots, \overline{\Phi}_{i,s}, \cdots, \Phi_{\ell,s}) - f_i(s, \overline{Y}_{s-}, \overline{Z}_{i,s}, \overline{\Phi}_s) \\ & \le c\Big(|Y_{i,s-} - \overline{Y}_{i,s-}| + \sum_{j \ne i} (Y_{j,s-} - \overline{Y}_{j,s-})^+ + |Z_{i,s} - \overline{Z}_{i,s}| \\ & \qquad + \sum_{j \ne i} \int_{\mathcal{E}} (Y_{j,s-} + \Phi_{j,s}(e) - \overline{Y}_{j,s-} - \overline{\Phi}_{j,s}(e))^+ \nu(\mathrm{de}) \Big); \end{aligned}$$

(4) $f_i(\cdot, 0, 0, 0)$ *and* $\overline{f}_i(\cdot, 0, 0, 0) \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$*;*
(5) *both* f_i *and* $\overline{f}_i$ *are Lipschitz in* (y, z, ϕ)*; and*
(6) $f_i(s, \overline{Y}_{s-}, \overline{Z}_{i,s}, \overline{\Phi}_s) \le \overline{f}_i(s, \overline{Y}_{s-}, \overline{Z}_{i,s}, \overline{\Phi}_s)$.
Then $Y_{i,t} \le \overline{Y}_{i,t}, \forall t \in [0, T] = 1, i \in \{1, 2, \dots, m\}$.

Theorem 3.1 *Under Assumptions 2.1 and 2.2 , the BSDEJ (3.1) admits a unique non-negative solution* $(P^i_j, \Lambda^i_j, \Gamma^i_j)_{i \in \mathcal{M},\, j=1,2}$.

Proof (Existence). For each natural number *k*, define maps

$$\begin{aligned} H_{11}^{i,*,k}(P_1, \Lambda) &:= \inf_{v \in \Pi_1, |v| \le k} H_{11}^i(v, P_1, \Lambda), \\ H_{12}^{i,*,k}(z, P_1, P_2, \Gamma_1, \Gamma_2) &:= \inf_{v \in \Pi_2, |v| \le k} H_{12}^i(v, z, P_1, P_2, \Gamma_1, \Gamma_2), \\ H_{21}^{i,*,k}(P_2, \Lambda) &:= \inf_{v \in \Pi_1, |v| \le k} H_{21}^i(v, P_2, \Lambda), \\ H_{22}^{i,*,k}(z, P_1, P_2, \Gamma_1, \Gamma_2) &:= \inf_{v \in \Pi_2, |v| \le k} H_{22}^i(v, z, P_1, P_2, \Gamma_1, \Gamma_2). \end{aligned}$$

Then they are uniformly Lipschitz in $(P_1, P_2, \Lambda, \Gamma_1, \Gamma_2)$ and decreasingly approach to $H_{11}^{i,*}$, $H_{12}^{i,*}$, $H_{21}^{i,*}$, $H_{22}^{i,*}$ and respectively as *k* goes to infinity.

For each *k*, the following BSDE

$$
\begin{cases}
\mathrm{d}P_{1,t}^{i,k} = -\Big[(2A^i + |C^i|^2)P_{1,t-}^{i,k} + 2(C^i)^\top \Lambda_1^{i,k} + Q^i + H_{11}^{i,*,k}(P_1^{i,k}, \Lambda_1^{i,k}) \\
\qquad\qquad + \sum_{j=1}^{\ell} q^{ij} P_1^{j,k} + \int_{\mathcal{Z}} H_{12}^{i,*,k}(P_1^{i,k}, P_2^{i,k}, \Gamma_1^{i,k}, \Gamma_2^{i,k})\nu(\mathrm{d}z)\Big]\mathrm{d}t \\
\qquad + (\Lambda_1^{i,k})^\top \mathrm{d}W + \int_{\mathcal{Z}} \Gamma_1^{i,k}(z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \\
\mathrm{d}P_{2,t}^{i,k} = -\Big[(2A^i + |C^i|^2)P_{2,t-}^{i,k} + 2(C^i)^\top \Lambda_2^{i,k} + Q^i + H_{21}^{i,*,k}(P_2^{i,k}, \Lambda_2^{i,k}) \\
\qquad\qquad + \sum_{j=1}^{\ell} q^{ij} P_2^{j,k} + \int_{\mathcal{Z}} H_{22}^{i,*,k}(P_1^{i,k}, P_2^{i,k}, \Gamma_1^{i,k}, \Gamma_2^{i,k})\nu(\mathrm{d}z)\Big]\mathrm{d}t \\
\qquad + (\Lambda_2^{i,k})^\top \mathrm{d}W + \int_{\mathcal{Z}} \Gamma_2^{i,k}(z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \\
P_{1,T}^{i,k} = G^i, \ P_{2,T}^{i,k} = G^i, \ i \in \mathcal{M},
\end{cases}
\tag{3.2}
$$

is a 2ℓ-dimensional BSDEJ with a Lipschitz generator. According to [20, Lemma 2.4], it admits a unique solution $(P_j^{i,k}, \Lambda_j^{i,k}, \Gamma_j^{i,k})_{i\in\mathcal{M},\, j=1,2}$ such that

$$
(P_j^{i,k}, \Lambda_j^{i,k}, \Gamma_j^{i,k}) \in S^2_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}^{n_1}) \times L^{2,\nu}_{\mathcal{P}^{W,N}}(0,T;\mathbb{R}^{n_2}), \ i \in \mathcal{M}, \ j = 1, 2.
$$

We next show that $(P_1^{i,k}, P_2^{i,k})_{i\in\mathcal{M}}$ are lower and upper bounded. Actually, the following two linear (with bounded coefficients) BSDEJs (see, e.g., [1, Proposition 2.2])

$$
\begin{cases}
\mathrm{d}\overline{P}_{1,t}^{i} = -\Big[(2A^i + |C^i|^2)\overline{P}_{1,t-}^{i} + 2(C^i)^\top \overline{\Lambda}_1^{i} + Q^i + \int_{\mathcal{Z}} H_{12}^{i}(0, \overline{P}_1^{i}, \overline{P}_2^{i}, \overline{\Gamma}_1^{i}, \overline{\Gamma}_2^{i})\nu(\mathrm{d}z) \\
\qquad\qquad + \sum_{j=1}^{\ell} q^{ij} \overline{P}_1^{j}\Big]\mathrm{d}t + (\overline{\Lambda}_1^{i})^\top \mathrm{d}W + \int_{\mathcal{Z}} \overline{\Gamma}_1^{i}(z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \\
\mathrm{d}\overline{P}_{2,t}^{i} = -\Big[(2A^i + |C^i|^2)\overline{P}_{2,t-}^{i} + 2(C^i)^\top \overline{\Lambda}_2^{i} + Q^i + \int_{\mathcal{Z}} H_{22}^{i}(0, \overline{P}_1^{i}, \overline{P}_2^{i}, \overline{\Gamma}_1^{i}, \overline{\Gamma}_2^{i})\nu(\mathrm{d}z) \\
\qquad\qquad + \sum_{j=1}^{\ell} q^{ij} \overline{P}_2^{j}\Big]\mathrm{d}t + (\overline{\Lambda}_2^{i})^\top \mathrm{d}W + \int_{\mathcal{Z}} \overline{\Gamma}_2^{i}(z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \\
\overline{P}_{1,T}^{i} = G^i, \ \overline{P}_{2,T}^{i} = G^i, \ i \in \mathcal{M}.
\end{cases}
\tag{3.3}
$$

and

$$
\begin{cases}
\mathrm{d}\underline{P}_{1,t}^{i} = -\Big[(2A^i + |C^i|^2)\underline{P}_{1,t-}^{i} + 2(C^i)^\top \underline{\Lambda}_1^{i} + \sum_{j=1}^{\ell} q^{ij} \underline{P}_1^{j}\Big]\mathrm{d}t \\
\qquad\qquad + (\underline{\Lambda}_1^{i})^\top \mathrm{d}W + \int_{\mathcal{Z}} \underline{\Gamma}_1^{i}(z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \\
\mathrm{d}\underline{P}_{2,t}^{i} = -\Big[(2A^i + |C^i|^2)\underline{P}_{2,t-}^{i} + 2(C^i)^\top \underline{\Lambda}_2^{i} + \sum_{j=1}^{\ell} q^{ij} \underline{P}_2^{j}\Big]\mathrm{d}t \\
\qquad\qquad + (\underline{\Lambda}_2^{i})^\top \mathrm{d}W + \int_{\mathcal{Z}} \underline{\Gamma}_2^{i}(z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \\
\underline{P}_{1,T}^{i} = 0, \ \underline{P}_{2,T}^{i} = 0, \ i \in \mathcal{M}.
\end{cases}
\tag{3.4}
$$

admit unique uniformly bounded solutions $(\overline{P}_j^i, \overline{\Lambda}_j^i, \overline{\Gamma}_j^i)_{i\in\mathcal{M},\, j=1,2}$ and $(\underline{P}_j^i, \underline{\Lambda}_j^i, \underline{\Gamma}_j^i)_{i\in\mathcal{M},\, j=1,2}$ respectively. Clearly, $(\underline{P}_j^i, \underline{\Lambda}_j^i, \underline{\Gamma}_j^i)_{i\in\mathcal{M},\, j=1,2} = 0$ by uniqueness. According to the definitions of $H_{jj'}^{i,*,k}, \ H_{jj'}^{i}$, we have

$$
\begin{aligned}
H_{11}^{i,*,k}(\overline{P}_1, \overline{\Lambda}_1) &\le H_{11}^{i}(0, \overline{P}_1, \overline{\Lambda}_1) = 0, \\
H_{12}^{i,*,k}(\overline{P}_1, \overline{P}_2, \overline{\Gamma}_1, \overline{\Gamma}_2) &\le H_{12}^{i}(0, \overline{P}_1, \overline{P}_2, \overline{\Gamma}_1, \overline{\Gamma}_2), \\
H_{21}^{i,*,k}(\overline{P}_2, \overline{\Lambda}_2) &\le H_{21}^{i}(0, \overline{P}_2, \overline{\Lambda}_2) = 0, \\
H_{22}^{i,*,k}(\overline{P}_1, \overline{P}_2, \overline{\Gamma}_1, \overline{\Gamma}_2) &\le H_{22}^{i}(0, \overline{P}_1, \overline{P}_2, \overline{\Gamma}_1, \overline{\Gamma}_2).
\end{aligned}
$$

Also, thanks to Assumption 2.1,

$$Q^i + H_{11}^{i,*,k}(\underline{P}_1, \underline{\Lambda}_1) + \int_{\mathcal{Z}} H_{12}^{i,*,k}(\underline{P}_1, \underline{P}_2, \underline{\Gamma}_1, \underline{\Gamma}_2)\nu(\mathrm{d}z) = Q^i \geq 0,$$
$$Q^i + H_{21}^{i,*,k}(\underline{P}_2, \underline{\Lambda}_2) + \int_{\mathcal{Z}} H_{22}^{i,*,k}(\underline{P}_1, \underline{P}_2, \underline{\Gamma}_1, \underline{\Gamma}_2)\nu(\mathrm{d}z) = Q^i \geq 0.$$

We can apply the Lemma 3.2[1] to (3.2) and (3.3), and to (3.2) and (3.4) (actually to $-P_j^{i,k}$ and $-\underline{P}_j^i$), respectively, to get

$$0 = \underline{P}_1^i \leq P_1^{i,k} \leq \overline{P}_1^i, \quad 0 = \underline{P}_2^i \leq P_2^{i,k} \leq \overline{P}_2^i.$$

Applying the same comparison theorem to different ks in (3.2), we get $P_j^{i,k}$ is non-increasing in k, for any $i \in \mathcal{M}, \ j = 1, 2$.

A nonnegative solution to (3.1) can be constructed in much the same way as [1, Theorem 1], [9, Theorem 3.1] and [12, Theorem 2.1] by proving the strong convergence of $(P_j^{i,k}, \Lambda_j^{i,k}, \Gamma_j^{i,k})_{i\in\mathcal{M},\ j=1,2}$ as $k \to \infty$. Details are left to the interested readers. This completes the proof of existence.

(Uniqueness). We now turn to the proof of uniqueness. Suppose $(P_j^i, \Lambda_j^i, \Gamma_j^i)_{i\in\mathcal{M},\ j=1,2}$ and

$(\tilde{P}_j^i, \tilde{\Lambda}_j^i, \tilde{\Gamma}_j^i)_{i\in\mathcal{M},\ j=1,2}$ are two nonnegative solutions of (3.1). Then there exists a constant $M > 0$ such that, for all $i \in \mathcal{M}$, $j = 1, 2$,

$$0 \leq P_j^i, \ \tilde{P}_j^i \leq M.$$

Estimates similar as in [9, Theorem 3.1] yields also that

$$0 \leq P_{j,t-}^i + \Gamma_{j,t}^i, \ \tilde{P}_{j,t-}^i + \tilde{\Gamma}_{j,t}^j \leq M.$$

Let $a > 0$ be a sufficiently small constant such that $R_1^i - a(D^i)^\top D^i > 0$. Write $\varrho = \frac{a}{a+M}$, then $0 < \varrho < 1$. Let

$$(U_j^i, V_j^i, \Phi_{jk}^i) = \Big(\ln(P_j^i + a), \frac{\Lambda_j^i}{P_j^i + a}, \ln\Big(1 + \frac{\Gamma_{jk}^i}{P_{j,t-}^i + a}\Big)\Big),$$
$$(\tilde{U}_j^i, \tilde{V}_j^i, \tilde{\Phi}_{jk}^i) = \Big(\ln(\tilde{P}_j^i + a), \frac{\tilde{\Lambda}_j^i}{\tilde{P}_j^i + a}, \ln\Big(1 + \frac{\tilde{\Gamma}_{jk}^i}{\tilde{P}_{j,t-}^i + a}\Big)\Big)$$

for all i, j, k. Then we have the estimates

$$\varrho \leq e^{\Phi_{jk}^i}, e^{\tilde{\Phi}_{jk}^i} \leq \varrho^{-1}, \quad \frac{e^{U_1^i + \Phi_{1,k}^i} - a}{e^{U_1^i}} = \frac{P_1^i + \Gamma_{1k}^i}{e^{U_1^i}} \geq 0, \quad \frac{e^{U_2^i + \Phi_{2,k}^i} - a}{e^{U_1^i}} = \frac{P_2^i + \Gamma_{2k}^i}{e^{U_1^i}} \geq 0. \quad (3.5)$$

Also,

[1] Conditions (1)-(5) can be obtained in a similar way as [9, Theorem 3.1]

$$
\begin{cases}
\mathrm{d}U_1^i = -\Big[(2A^i + |C^i|^2)(1 - ae^{-U_1^i}) + 2(C^i)^\top V_1^i + Q^i e^{-U_1^i} + \frac{1}{2}|V_1^i|^2 + \sum_{j=1}^{\ell} q^{ij} e^{U_1^j - U_1^i} \\
\qquad + \tilde{H}_{11}^{i,*}(U_1^i, V_1^i) + \int_{\mathcal{Z}} \tilde{H}_{12}^{i,*}(U_1^i, U_2^i, \Phi_1^i, \Phi_2^i)\nu(\mathrm{d}z) + \sum_{k=1}^{n_2} \int_{\mathcal{Z}} (e^{\Phi_{1,k}^i} - \Phi_{1,k}^i - 1)\nu(\mathrm{d}z)\Big]\mathrm{d}t \\
\quad + (V_1^i)^\top \mathrm{d}W + \int_{\mathcal{Z}} \Phi_1^i(z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \\
\mathrm{d}U_2^i = -\Big[(2A^i + |C^i|^2)(1 - ae^{-U_2^i}) + 2(C^i)^\top V_2^i + Q^i e^{-U_2^i} + \frac{1}{2}|V_2^i|^2 + \sum_{j=1}^{\ell} q^{ij} e^{U_2^j - U_2^i} \\
\qquad + \tilde{H}_{21}^{i,*}(U_2^i, V_2^i) + \int_{\mathcal{Z}} \tilde{H}_{22}^{i,*}(U_1^i, U_2^i, \Phi_1^i, \Phi_2^i)\nu(\mathrm{d}z) + \int_{\mathcal{Z}} \sum_{k=1}^{n_2} (e^{\Phi_{2,k}^i} - \Phi_{2,k}^i - 1)\nu(\mathrm{d}z)\Big]\mathrm{d}t \\
\quad + (V_2^i)^\top \mathrm{d}W + \int_{\mathcal{Z}} \Phi_2^i(z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z), \\
U_{1,T}^i = U_{2,T}^i = \ln(G^i + a), \quad i \in \mathcal{M},
\end{cases}
$$

where

$$
\begin{aligned}
\tilde{H}_{11}^i(v, U_1, V_1) &:= v^\top (R_1^i e^{-U_1} + (1 - ae^{-U_1})(D^i)^\top D^i) v \\
&\quad + 2((1 - ae^{-U_1})(B_1^i + (D^i)^\top C^i) + (D^i)^\top V_1)^\top v, \\
\tilde{H}_{12}^i(v, U_1, U_2, \Phi_1, \Phi_2) &:= v^\top R_2^i e^{-U_1} v + \sum_{k=1}^{n_2} \frac{e^{U_1 + \Phi_{1,k}} - a}{e^{U_1}} \Big(((1 + E_k^i + F_k^i v)^+)^2 - 1\Big) \\
&\quad - 2(1 - ae^{-U_1}) \sum_{k=1}^{n_2} (E_k^i + F_k^i v) + 2(1 - ae^{-U_1})(B_2^i)^\top v \\
&\quad + \sum_{k=1}^{n_2} \frac{e^{U_2 + \Phi_{2,k}} - a}{e^{U_1}} ((1 + E_k^i + F_k^i v)^-)^2, \\
\tilde{H}_{21}^i(v, U_2, V_2) &:= v^\top (R_1^i e^{-U_2} + (1 - ae^{-U_2})(D^i)^\top D^i) v \\
&\quad - 2((1 - ae^{-U_2})(B_1^i + (D^i)^\top C^i) + (D^i)^\top V_2)^\top v, \\
\tilde{H}_{22}^i(v, U_1, U_2, \Phi_1, \Phi_2) &:= v^\top R_2^i e^{-U_2} v + \sum_{k=1}^{n_2} \frac{e^{U_2 + \Phi_{2,k}} - a}{e^{U_2}} \Big(((-1 - E_k^i + F_k^i v)^-)^2 - 1\Big) \\
&\quad - 2(1 - ae^{-U_2}) \sum_{k=1}^{n_2} (-E_k^i + F_k^i v) - 2(1 - ae^{-U_2})(B_2^i)^\top v \\
&\quad + \sum_{k=1}^{n_2} \frac{e^{U_1 + \Phi_{1,k}} - a}{e^{U_2}} ((-1 - E_k^i + F_k^i v)^-)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{H}_{11}^{i,*}(U_1, V_1) &:= \inf_{v \in \Pi_1} \tilde{H}_{11}^i(v, U_1, V_1), \\
\tilde{H}_{12}^{i,*}(z, U_1, U_2, \Phi_1, \Phi_2) &:= \inf_{v \in \Pi_2} \tilde{H}_{12}^i(v, z, U_1, U_2, \Phi_1, \Phi_2), \\
\tilde{H}_{21}^{i,*}(U_2, V_2) &:= \inf_{v \in \Pi_1} \tilde{H}_{21}^i(v, U_2, V_2), \\
\tilde{H}_{22}^{i,*}(z, U_1, U_2, \Phi_1, \Phi_2) &:= \inf_{v \in \Pi_2} \tilde{H}_{22}^i(v, z, U_1, U_2, \Phi_1, \Phi_2).
\end{aligned}
$$

Set

$$
\bar{U}_j^i = U_j^i - \tilde{U}_j^i, \ \bar{V}_j^i = V_j^i - \tilde{V}_j^i, \ \bar{\Phi}_j^i = \Phi_j^i - \tilde{\Phi}_j^i, \ i \in \mathcal{M}, \ j = 1, 2.
$$

Then applying Itô's formula to $(\bar{U}_j^i)^2$, we deduce that

$$
\begin{aligned}
&(U_{1,t}^i)^2 + \int_t^T |\bar{V}_1^i|^2 \mathrm{d}s + \int_t^T \int_{\mathcal{Z}} |\bar{\Phi}_1^i|^2 \nu(\mathrm{d}z)\mathrm{d}t \\
&= \int_t^T \Big[L_{11}^i + \int_{\mathcal{Z}} L_{12}^i(z)\nu(\mathrm{d}z)\Big]\mathrm{d}t - \int_t^T 2\bar{U}_1^i(\bar{V}_1^i)^\top \mathrm{d}W \\
&\quad - \sum_{k=1}^{n_2} \int_t^T \int_{\mathcal{Z}} (2\bar{U}_{1,k}^i \bar{\Phi}_{1,k}^i + (\bar{\Phi}_{1,k}^i)^2)\tilde{N}_k(\mathrm{d}t, \mathrm{d}z),
\end{aligned}
$$

and

$$
\begin{aligned}
&(U_{2,t}^i)^2 + \int_t^T |\bar{V}_2^i|^2 ds + \int_t^T \int_{\mathcal{Z}} |\bar{\Phi}_2^i|^2 \nu(\mathrm{d}z)\mathrm{d}t \\
&= \int_t^T \Big[L_{21}^i + \int_{\mathcal{Z}} L_{22}^i(z)\nu(\mathrm{d}z)\Big]\mathrm{d}t - \int_t^T 2\bar{U}_2^i(\bar{V}_2^i)^\top \mathrm{d}W \\
&\quad - \sum_{k=1}^{n_2} \int_t^T \int_{\mathcal{Z}} (2\bar{U}_{2,k}^i \bar{\Phi}_{2,k}^i + (\bar{\Phi}_{2,k}^i)^2)\tilde{N}_k(\mathrm{d}t, \mathrm{d}z),
\end{aligned}
$$

where

$$
\begin{aligned}
L_{11}^i :&= 2\bar{U}_1^i\Big[(Q^i - 2aA^i - a|C^i|^2)(e^{-U_1^i} - e^{-\tilde{U}_1^i}) + 2(C^i)^\top \bar{V}_1^i + \frac{1}{2}(V_1^i + \tilde{V}_1^i)\bar{V}_1^i \\
&\quad + \sum_{j=1}^{\ell} q^{ij}\Big(e^{U_1^i - U_1^i} - e^{\tilde{U}_1^j - \tilde{U}_1^i}\Big) + \tilde{H}_{11}^{i,*}(U_1^i, V_1^i,) - \tilde{H}_{11}^{i,*}(\tilde{U}_1^i, \tilde{V}_1^i)\Big], \\
L_{12}^i(z) :&= 2\bar{U}_1^i\Big[\sum_{k=1}^{n_2}[(e^{\Phi_{1,k}^i} - \Phi_{1,k}^i - 1) - (e^{\tilde{\Phi}_{1,k}^i} - \tilde{\Phi}_{1,k}^i - 1)] \\
&\quad + \tilde{H}_{12}^{i,*}(z, U_1^i, \Phi_1^i, U_2^i, \Phi_2^i) - \tilde{H}_{12}^{i,*}(z, \tilde{U}_1^i, \tilde{\Phi}_1^i, \tilde{U}_2^i, \tilde{\Phi}_2^i)\Big], \\
L_{21}^i :&= 2\bar{U}_2^i\Big[(Q^i - 2aA^i - a|C^i|^2)(e^{-U_2^i} - e^{-\tilde{U}_2^i}) + 2(C^i)^\top \bar{V}_2^i + \frac{1}{2}(V_2^i + \tilde{V}_2^i)\bar{V}_2^i \\
&\quad + \sum_{j=1}^{\ell} q^{ij}\Big(e^{U_2^i - U_2^i} - e^{\tilde{U}_2^j - \tilde{U}_2^i}\Big) + \tilde{H}_{21}^{i,*}(U_2^i, V_2^i,) - \tilde{H}_{21}^{i,*}(\tilde{U}_2^i, \tilde{V}_2^i)\Big], \\
L_{22}^i(z) :&= 2\bar{U}_2^i\Big[\sum_{k=1}^{n_2}[(e^{\Phi_{2,k}^i} - \Phi_{2,k}^i - 1) - (e^{\tilde{\Phi}_{2,k}^i} - \tilde{\Phi}_{2,k}^i - 1)] \\
&\quad + \tilde{H}_{22}^{i,*}(z, U_1^i, \Phi_1^i, U_2^i, \Phi_2^i) - \tilde{H}_{22}^{i,*}(z, \tilde{U}_1^i, \tilde{\Phi}_1^i, \tilde{U}_2^i, \tilde{\Phi}_2^i)\Big].
\end{aligned}
$$

The terms L_{11} and L_{21} can be estimated in much the same way as [6, Theorem 3.5] to get[2]

[2] $R_1^i - a(D^i)^\top D^i > 0$ is required in these estimates.

$$L^i_{11} \leq |\beta^i|(\bar{U}^i_1)^2 + c|\bar{U}^i_1| \sum_{j=1}^{\ell} |\bar{U}^j_1| + c(\beta^i)^\top \bar{U}^i_1 \bar{V}^i_1,$$

$$L^i_{21} \leq |\beta^i|(\bar{U}^i_2)^2 + c|\bar{U}^i_2| \sum_{j=1}^{\ell} |\bar{U}^j_2| + c(\beta^i)^\top \bar{U}^i_2 \bar{V}^i_2,$$

where β^i is some $\mathbb{F}^{W,N}$-predictable process satisfying $|\beta^i| \leq c(1 + |V^i_1| + |\tilde{V}^i_1| + |V^i_2| + |\tilde{V}^i_2|)$ so that $\int_0^\cdot (\beta^i)^\top \mathrm{d}W$ is a BMO martingale noting Lemma 3.1.

On the other hand, from Assumptions 2.1, 2.2 and (3.5), there are positive constants c_1, c_2, c_3 such that

$$\begin{aligned}
&\tilde{H}^i_{12}(v, z, U^i_1, U^i_2, \Phi^i_1, \Phi^i_2) - \tilde{H}^i_{12}(0, z, U^i_1, U^i_2, \Phi^i_1, \Phi^i_2) \\
\geq\ & \frac{\delta}{M+a}|v|^2 - \sum_{k=1}^{n_2} \frac{e^{U^i_1 + \Phi^i_{1,k}} - a}{e^{U_1}} - 2(1 - ae^{-U^i_1}) \sum_{k=1}^{n_2} (E^i_k + F^i_k v) + 2(1 - ae^{-U^i_1})(B^i_2)^\top v \\
& - \Big[\sum_{k=1}^{n_2} \frac{e^{U^i_1 + \Phi^i_{1,k}} - a}{e^{U_1}} \Big(((1 + E^i_k)^+)^2 - 1 \Big) \\
& \quad - 2(1 - ae^{-U^i_1}) \sum_{k=1}^{n_2} E^i_k + \sum_{k=1}^{n_2} \frac{e^{U^i_2 + \Phi^i_{2,k}} - a}{e^{U^i_1}} ((1 + E^i_k)^-)^2 \Big] \\
\geq\ & \frac{\delta}{M+a}|v|^2 - c_2|v| - c_3 > 0,
\end{aligned}$$

if $|v| > c$ with $c > 0$ being sufficiently large. Hence,

$$\tilde{H}^{i,*}_{12}(z, U^i_1, U^i_2, \Phi^i_1, \Phi^i_2) := \inf_{v \in \Pi_2, |v| \leq c} \tilde{H}^i_{12}(v, z, U^i_1, U^i_2, \Phi^i_1, \Phi^i_2). \tag{3.6}$$

Furthermore, noting $U^i_1, \tilde{U}^i_1, U^i_2, \tilde{U}^i_2, \Phi^i_1, \tilde{\Phi}^i_1, \Phi^i_2, \tilde{\Phi}^i_2$ are bounded, we have

$$L^i_{12}(z) \leq c|\bar{U}^i_1|(|\bar{U}^i_1| + |\bar{\Phi}^i_1(z)| + |\bar{U}^i_2| + |\bar{\Phi}^i_2(z)|).$$

Similar arguments applying to $\tilde{H}^{i,*}_{22}(v, z, U^i_1, U^i_2, \Phi^i_1, \Phi^i_2)$ yield that

$$L^i_{22}(z) \leq c|\bar{U}^i_2|(|\bar{U}^i_1| + |\bar{\Phi}^i_1(z)| + |\bar{U}^i_2| + |\bar{\Phi}^i_2(z)|).$$

For each $i \in \mathcal{M}$, introduce the processes

$$J^i_t = \exp\Big(\int_0^t |\beta^i_s| \mathrm{d}s \Big), \ \ N^i_t = \exp\Big(\int_0^t (\beta^i_s)^\top \mathrm{d}W_s - \frac{1}{2} \int_0^t |\beta^i_s|^2 \mathrm{d}s \Big).$$

Itô's formula gives

$$
\begin{aligned}
& J_t^i N_t^i |\bar{U}_{1,t}^i|^2 + \mathbb{E}_t \int_t^T J_s^i N_s^i |\bar{V}_1^i|^2 \mathrm{d}s + \mathbb{E}_t \int_t^T \int_{\mathcal{Z}} J_s^i N_s^i |\bar{\Phi}_1^i|^2 \nu(\mathrm{d}z)\mathrm{d}s \\
\leq\ & \mathbb{E}_t \int_t^T J_s^i N_s^i \Big[c|\bar{U}_1^i||\bar{U}_2^i| + c|\bar{U}_1^i| \sum_{j=1}^{\ell} |\bar{U}_1^j| + c|U_1^i| \int_{\mathcal{Z}} (\bar{\Phi}_1^i(z) + \bar{\Phi}_2^i(z))\nu(\mathrm{d}z) \Big] \mathrm{d}s \\
\leq\ & c\mathbb{E}_t \int_t^T J_s^i N_s^i (|\bar{U}_1^i|^2 + |\bar{U}_2^i|^2 + \sum_{j=1}^{\ell} |\bar{U}_1^j|^2)\mathrm{d}s + \frac{1}{4}\mathbb{E}_t \int_t^T \int_{\mathcal{Z}} J_s^i N_s^i (|\bar{\Phi}_1^i(z)|^2 + |\bar{\Phi}_2^i(z)|^2)\nu(\mathrm{d}z)\mathrm{d}s.
\end{aligned}
$$

Note that N_t^i is a uniformly integrable martingale, thus

$$
\widetilde{W}_t^i := W_t - \int_0^t (\beta_s^i)^\top \mathrm{d}W_s,
$$

is a Brownian motion under the probability $\tilde{\mathbb{P}}^i$ defined by

$$
\frac{d\tilde{\mathbb{P}}^i}{d\mathbb{P}}\Big|_{\mathcal{F}_T^{W,N}} = N_T^i.
$$

We denote by $\widetilde{\mathbb{E}}_t^i$ the conditional expectation with respect to the probability $\tilde{\mathbb{P}}^i$, then

$$
\begin{aligned}
& J_t^i |\bar{U}_{1,t}^i|^2 + \widetilde{\mathbb{E}}_t^i \int_t^T J_s^i |\bar{V}_1^i|^2 \mathrm{d}s + \widetilde{\mathbb{E}}_t^i \int_t^T \int_{\mathcal{Z}} J_s^i |\bar{\Phi}_1^i|^2 \nu(\mathrm{d}z)\mathrm{d}s \\
\leq\ & c\widetilde{\mathbb{E}}_t^i \int_t^T J_s^i (|\bar{U}_1^i|^2 + |\bar{U}_2^i|^2 + \sum_{j=1}^{\ell} |\bar{U}_1^j|^2)\mathrm{d}s + \frac{1}{4}\widetilde{\mathbb{E}}_t^i \int_t^T \int_{\mathcal{Z}} J_s^i (|\bar{\Phi}_1^i|^2 + |\bar{\Phi}_2^i|^2)\nu(\mathrm{d}z)\mathrm{d}s.
\end{aligned} \tag{3.7}
$$

Similarly, we have

$$
\begin{aligned}
& J_t^i |\bar{U}_{2,t}^i|^2 + \widetilde{\mathbb{E}}_t^i \int_t^T J_s^i |\bar{V}_2^i|^2 \mathrm{d}s + \widetilde{\mathbb{E}}_t^i \int_t^T \int_{\mathcal{Z}} J_s^i |\bar{\Phi}_2^i|^2 \nu(\mathrm{d}z)\mathrm{d}s \\
\leq\ & c\widetilde{\mathbb{E}}_t^i \int_t^T J_s^i (|\bar{U}_1^i|^2 + |\bar{U}_2^i|^2 + \sum_{j=1}^{\ell} |\bar{U}_2^j|^2)\mathrm{d}s + \frac{1}{4}\widetilde{\mathbb{E}}_t^i \int_t^T \int_{\mathcal{Z}} J_s^i (|\bar{\Phi}_1^i|^2 + |\bar{\Phi}_2^i|^2)\nu(\mathrm{d}z)\mathrm{d}s.
\end{aligned} \tag{3.8}
$$

Combining the above two inequalities yields

$$
\begin{aligned}
|\bar{U}_{1,t}^i|^2 + |\bar{U}_{2,t}^i|^2 &\leq c\widetilde{\mathbb{E}}_t^i \int_t^T \exp(\int_t^s |\beta_r^i|\mathrm{d}r)(\sum_{j=1}^{\ell} \int_t^T (|\bar{U}_{1,s}^j|^2 + |\bar{U}_{2,s}^j|^2)\mathrm{d}s \\
&\leq c\widetilde{\mathbb{E}}_t^i \Big[\exp(\int_t^T |\beta_r^i|\mathrm{d}r) \sum_{j=1}^{\ell} \int_t^T (|\bar{U}_{1,s}^j|^2 + |\bar{U}_{2,s}^j|^2)\mathrm{d}s \Big] \\
&\leq c\widetilde{\mathbb{E}}_t^i \Big[\exp(\int_t^T |\beta_r^i|\mathrm{d}r) \Big] \sum_{j=1}^{\ell} \int_t^T \Xi_s^j \mathrm{d}s,
\end{aligned}
$$

where

$$\Xi_s^j := \operatorname*{ess\,sup}_{\omega\in\Omega}\Big(|\bar{U}_{1,s}^j|^2 + |\bar{U}_{2,s}^j|^2\Big).$$

According to [8, Lemma 3.4], $\widetilde{\mathbb{E}}_t^i\Big[\exp(\int_t^T|\beta_r^i|\mathrm{d}r)\Big] \le c$, then taking essential supreme on both sides, we deduce that

$$0 \le \sum_{i=1}^{\ell}\Xi_t^i \le c\int_t^T\sum_{i=1}^{\ell}\Xi_s^i\mathrm{d}s.$$

We infer from Gronwall's inequality that $\Xi^i = 0$, so $\bar{U}_1^i = \bar{U}_2^i = 0$, for all $i \in \mathcal{M}$. Consequently, it follows from (3.7) and (3.8) that $\bar{V}_1^i = \bar{V}_2^i = 0$ and $\bar{\Phi}_1^i = \bar{\Phi}_2^i = 0$ for all $i \in \mathcal{M}$. This completes the proof. □

Remark 3.1 In the above proof, Assumption 2.1 alone is sufficient for the existence of a nonnegative solution to (3.1), and Assumption 2.2 is only used in the proof of uniqueness part (will also be used in Lemma 4.2).

Theorem 3.2 *Under Assumptions 2.1 and 2.3 , the BSDEJ (3.1) admits a unique uniformly positive solution* $(P_j^i, \Lambda_j^i, \Gamma_j^i)_{i\in\mathcal{M},\, j=1,2}$.

Proof The proof of the existence is similar to the above Theorem 3.1 and will only be indicated briefly why the solution to (3.1) is uniformly positive.

When both 2 and 3 hold. In this case, there exists constant $c_2 > 0$, such that

$$2A^i + |C^i|^2 - \delta^{-1}|B_1^i + (D^i)^\top C^i|^2 \ge -c_2,\ -\delta^{-1}\int_{\mathcal{Z}}|B_2^i|^2\nu(\mathrm{d}z) \ge -c_2,$$

where δ is the constant in Assumption 2.3. Notice that $(\underline{P}_t, \underline{\Lambda}_t, \underline{\Gamma}_t) = (\frac{1}{(\delta^{-1}+1)e^{c_2(T-t)}-1}, 0, 0)$ solves the following BSDEJ

$$\begin{cases} \mathrm{d}\underline{P} = -(-c_2\underline{P} - c_2\underline{P}^2)\mathrm{d}t + \underline{\Lambda}^\top\mathrm{d}W + \int_{\mathcal{E}}\underline{\Gamma}(e)\widetilde{N}(\mathrm{d}t, \mathrm{d}z), \\ \underline{P}_T = \delta. \end{cases} \tag{3.9}$$

And we have the following inequalities

$$H_{11}^{i,*,k}(\underline{P},\underline{\Lambda}) \geq \inf_{v\in\mathbb{R}^{m_1}} H_{11}^i(v,\underline{P},\underline{\Lambda}) \geq -\delta^{-1}|B_1^i+(D^i)^\top C^i|^2\underline{P},$$

$$\int_{\mathcal{Z}} H_{12}^{i,*,k}(z,\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma})\nu(\mathrm{d}z) \geq \int_{\mathcal{Z}} \inf_{v\in\mathbb{R}^{m_1}} H_{12}^i(v,z,\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma})\nu(\mathrm{d}z) \geq -\delta^{-1}\int_{\mathcal{Z}} |B_2^i|^2\nu(\mathrm{d}z)\underline{P}^2 \geq -c_2\underline{P}^2.$$

Similar results also hold for $H_{21}^{i,*,k}(\underline{P},\underline{\Lambda})$ and $\int_{\mathcal{Z}} H_{22}^{i,*,k}(z,\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma})\nu(\mathrm{d}z)$. Applying Lemma 3.2 to (3.2) and (3.9), we get

$$P_{j,t}^{i,k} \geq \underline{P}_t \geq \frac{1}{(\delta^{-1}+1)e^{c_2 T}-1},\ t\in[0,T],\ i\in\mathcal{M},\ j=1,2.$$

Sending $k\to\infty$ leads to the desired uniformly positive lower bound.

When both 2 and 4 hold. In this case, there exists constant $c_3>0$, such that

$$2A^i+|C^i|^2+\int_{\mathcal{Z}}|E^i|^2\nu(\mathrm{d}z)-\delta^{-1}|B_1^i+(D^i)^\top C^i|^2-\delta^{-1}\int_{\mathcal{Z}}|(F^i)^\top E^i+B_2^i|^2\nu(\mathrm{d}z)\geq -c_3,$$

where δ is the constant in Assumption 2.3. Notice $(\underline{P}_t,\underline{\Lambda}_t,\underline{\Gamma}_t)=(\delta e^{-c_3(T-t)},0,0)$ solves the following BSDEJ

$$\begin{cases} \mathrm{d}\underline{P}=-(-c_3\underline{P})\mathrm{d}t+\underline{\Lambda}^\top\mathrm{d}W+\int_{\mathcal{E}}\underline{\Gamma}(e)\widetilde{N}(\mathrm{d}t,\mathrm{d}z),\\ \underline{P}_T=\delta. \end{cases} \tag{3.10}$$

And we have the following inequalities

$$H_{11}^{i,*,k}(\underline{P},\underline{\Lambda}) \geq \inf_{v\in\mathbb{R}^{m_1}} H_{11}^i(v,\underline{P},\underline{\Lambda}) \geq -\delta^{-1}|B_1^i+(D^i)^\top C^i|^2\underline{P},$$

$$H_{12}^{i,*,k}(z,\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma}) \geq \inf_{v\in\mathbb{R}^{m_1}} H_{12}^i(z,v,\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma}) \geq -\delta^{-1}|(F^i)^\top E^i+B_2^i|^2\underline{P}.$$

Therefore,

$$(2A^i+|C^i|^2)\underline{P}+2(C^i)^\top\underline{\Lambda}+Q^i+\sum_{j=1}^{\ell}q^{ij}\underline{P}+H_{11}^{i,*,k}(\underline{P},\underline{\Lambda})+\int_{\mathcal{Z}}H_{12}^{i,*,k}(\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma})\nu(\mathrm{d}z)$$
$$\geq (2A^i+|C^i|^2)\underline{P}-\delta^{-1}|B_1^i+(D^i)^\top C^i|^2\underline{P}-\int_{\mathcal{Z}}\delta^{-1}|(F^i)^\top E^i+B_2^i|^2\underline{P}\nu(\mathrm{d}z)\geq -c_3\underline{P}.$$

Similar results also hold for $H_{21}^{i,*,k}(\underline{P},\underline{\Lambda})$ and $H_{22}^{i,*,k}(z,\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma})$. Applying Lemma 3.2 to (3.2) and (3.10), we get

$$P_{j,t}^{i,k} \geq \underline{P}_t=\delta e^{-c_3(T-t)}\geq \delta e^{-c_3 T},\ t\in[0,T],\ i\in\mathcal{M},\ j=1,2.$$

Sending $k\to\infty$ leads to the desired uniformly positive lower bound.

When both 1 and 4 hold. In this case, there exists constant $c_4>0$, such that

$$-\delta^{-1}|B_1^i+(D^i)^\top C^i|^2\geq -c_4,\ 2A^i+|C^i|^2+\int_{\mathcal{Z}}|E^i|^2\nu(\mathrm{d}z)-\delta^{-1}\int_{\mathcal{Z}}|(F^i)^\top E^i+B_2^i|^2\nu(\mathrm{d}z)\geq -c_4,$$

where δ is the constant in Assumption 2.3. Notice that $(\underline{P}_t,\underline{\Lambda}_t,\underline{\Gamma}_t)=(\frac{1}{(\delta^{-1}+1)e^{c_4(T-t)}-1},0,0)$ solves the following BSDEJ

$$\begin{cases} \mathrm{d}\underline{P}=-(-c_4\underline{P}^2-c_4\underline{P})\mathrm{d}t+\underline{\Lambda}^\top\mathrm{d}W+\int_{\mathcal{E}}\underline{\Gamma}(e)\widetilde{N}(\mathrm{d}t,\mathrm{d}z),\\ \underline{P}_T=\delta.\end{cases} \tag{3.11}$$

And we have the following inequalities

$$H_{11}^{i,*,k}(\underline{P},\underline{\Lambda})\geq\inf_{v\in\mathbb{R}^{m_1}}H_{11}^i(v,\underline{P},\underline{\Lambda})\geq-\delta^{-1}|B_1^i+(D^i)^\top C^i|^2\underline{P}^2\geq -c_4\underline{P}^2,$$
$$H_{12}^{i,*,k}(z,\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma})\geq\inf_{v\in\mathbb{R}^{m_1}}H_{12}^i(v,z,\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma})\geq|E^i|^2\underline{P}-\delta^{-1}|(F^i)^\top E^i+B_2^i|^2\underline{P}.$$

Therefore

$$\begin{aligned}&\big(2A^i+|C^i|^2\big)\underline{P}+\int_{\mathcal{Z}}H_{12}^{i,*,k}(z,\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma})\nu(\mathrm{d}z)\\ &\geq\Big(2A^i+|C^i|^2+\int_{\mathcal{Z}}|E^i|^2\nu(\mathrm{d}z)-\delta^{-1}\int_{\mathcal{Z}}|(F^i)^\top E^i+B_2^i|^2\nu(\mathrm{d}z)\Big)\underline{P}\\ &\geq -c_4\underline{P}.\end{aligned}$$

Similar results also hold for $H_{21}^{i,*,k}(\underline{P},\underline{\Lambda})$ and $H_{22}^{i,*,k}(z,\underline{P},\underline{P},\underline{\Gamma},\underline{\Gamma})$. Applying Lemma 3.2 to (3.2) and (3.11), we get

$$P_{j,t}^{i,k}\geq\underline{P}_t\geq\frac{1}{(\delta^{-1}+1)e^{c_4T}-1},\ t\in[0,T],\ i\in\mathcal{M},\ j=1,2.$$

Sending $k\to\infty$ leads to the desired uniformly positive lower bound.

As for the uniqueness, one just need to repeat the proof of Theorem 3.1 with $a=0$ which is allowed because the solutions are positive. □

Remark 3.2 *As one referee pointed out, it is more general if both u_1 and u_2 appear before dW_t and $\tilde{N}(dt,dz)$ in (2.1). In this case, we can still heuristically derive the corresponding SRE which will comprise the cross terms between v_1 and v_2. But, it is difficult to establish some estimates in Page 13, thus making it challenging to prove the existence and uniqueness of the solution to the SRE. We will study this problem in our future research.*

4 Solution to the LQ Problem (2.2)

In this subsection we will present an explicit solution to the LQ problem (2.2) in terms of solutions to the BSDEJ (3.1).

When $R_1^i+P(D^i)^\top D^i>0,\ i\in\mathcal{M}$, we define

$$\begin{aligned} \hat{v}^i_{11}(t,P,\Lambda) &:= \operatorname*{argmin}_{v\in\Pi_1} H^i_{11}(t,v,P,\Lambda), \\ \hat{v}^i_{21}(t,P,\Lambda) &:= \operatorname*{argmin}_{v\in\Pi_1} H^i_{21}(t,v,P,\Lambda). \end{aligned} \tag{4.1}$$

When $R^i_2 > 0$ or $(P_j + \Gamma_{jk})(F^i)^\top F^i > 0$, $i \in \mathcal{M},\ j = 1,2,\ k = 1,2,\ldots,n_2$, we define

$$\begin{aligned} \hat{v}^i_{12}(t,z,P_1,P_2,\Gamma_1,\Gamma_2) &:= \operatorname*{argmin}_{v\in\Pi_2} H^i_{12}(t,v,z,P_1,P_2,\Gamma_1,\Gamma_2), \\ \hat{v}^i_{22}(t,z,P_1,P_2,\Gamma_1,\Gamma_2) &:= \operatorname*{argmin}_{v\in\Pi_2} H^i_{22}(t,v,z,P_1,P_2,\Gamma_1,\Gamma_2). \end{aligned} \tag{4.2}$$

Theorem 4.1 *Let Assumptions 2.1, and 2.2 (resp. 2.3) hold. Let $(P^i_j, \Lambda^i_j, \Gamma^i_j) \in S^\infty_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}^{W,N}}(0,T;\mathbb{R}^{n_1}) \times L^{\infty,\nu}_{\mathcal{P}^{W,N}}(0,T;\mathbb{R}^{n_2})$, $i \in \mathcal{M}$, $j = 1,2$ be the nonnegative (resp. uniformly positive) solution to the BSDEJ (3.1). Then the state feedback control $u^* = (u^*_1, u^*_2)$ given by*

$$\begin{cases} u^*_1(t,X,\alpha) = \hat{v}^{\alpha_{t-}}_{11}(t,P^{\alpha_{t-}}_{1,t-},\Lambda^{\alpha_{t-}}_{1,t})X^+_{t-} + \hat{v}^{\alpha_{t-}}_{21}(t,P^{\alpha_{t-}}_{2,t-},\Lambda^{\alpha_{t-}}_{2,t})X^-_{t-}, \\ u^*_2(t,X,\alpha) = \hat{v}^{\alpha_{t-}}_{12}(t,z,P^{\alpha_{t-}}_{1,t-},P^{\alpha_{t-}}_{2,t-},\Gamma^{\alpha_{t-}}_{1,t},\Gamma^{\alpha_{t-}}_{2,t})X^+_{t-} + \hat{v}^{\alpha_{t-}}_{22}(t,z,P^{\alpha_{t-}}_{1,t-},P^{\alpha_{t-}}_{2,t-},\Gamma^{\alpha_{t-}}_{1,t},\Gamma^{\alpha_{t-}}_{2,t})X^-_{t-}, \end{cases} \tag{4.3}$$

is optimal for the LQ problem (2.2). Moreover, the optimal value is

$$V(x,i_0) = P^{i_0}_{1,0}(x^+)^2 + P^{i_0}_{2,0}(x^-)^2.$$

A proof of this theorem is contained in the following two Lemmas. In order to avoid as far as possible unwieldy formulas, we agree to suppress the the superscripts and subscripts of $A, B, C, D, E, F, R, Q, G$. And we will write $v^{\alpha_{t-}}_{ij}(t, P^{\alpha_{t-}}_1, \Lambda^{\alpha_{t-}}_1)$ simply $\hat{v}_{ij}$, $i,j = 1,2$ when no confusion can arise.

Lemma 4.1 *Under the condition of Theorem 4.1, we have*

$$J(u;x,i_0) \geq P^{i_0}_{1,0}(x^+)^2 + P^{i_0}_{2,0}(x^-)^2, \tag{4.4}$$

for any $u \in \mathcal{U}$, and

$$J(u^*;x,i_0) = P^{i_0}_{1,0}(x^+)^2 + P^{i_0}_{2,0}(x^-)^2. \tag{4.5}$$

Proof Noting the convex function $f(x) = (x^+)^2, x \in \mathbb{R}$, admits an absolutely continuous derivative $f'(x) = 2x^+$ such that for any $a, b \in \mathbb{R}$, $f'(b) - f'(a) = \int_a^b f''(s)\mathrm{d}s$, with $f''(x) = 21_{x>0}$. For any $u = (u_1, u_2) \in \mathcal{U}$, applying the Extant Second Derivative Meyer-Itô formula [15, Theorem 71] to $(X^+_t)^2$, we deduce that

$$
\begin{aligned}
\mathrm{d}\,(X_t^+)^2 = \Big[&2X_{t-}^+\Big(AX + B_1^\top u_1 + \int_{\mathcal{Z}} B_2(z)^\top u_2(z)\nu(\mathrm{d}z) - \sum_{k=1}^{n_2}\int_{\mathcal{Z}}(E_k(z)X + F_k(z)u_2(z))^\top \nu(\mathrm{d}z)\Big) \\
&+ 1_{\{X_{t-}>0\}}|CX + Du_1|^2\Big]\mathrm{d}t + 2X^+(CX + Du_1)^\top \mathrm{d}W_t \\
&+ \sum_{k=1}^{n_2}\int_{\mathcal{Z}}\Big[((X + E_k(z)X + F_k(e)u_2(z))^+)^2 - (X^+)^2\Big]N_k(\mathrm{d}t,\mathrm{d}z).
\end{aligned}
$$

The integration by parts formula applied to $P_{1,t}^{\alpha_t}(X_t^+)^2$ yields

$$
\begin{aligned}
\mathrm{d}P_{1,t}^{\alpha_t}(X_t^+)^2 = P_{1,t-}^{\alpha_{t-}}\Big[&2X_{t-}^+\Big(AX + B_1^\top u_1 + \int_{\mathcal{Z}} B_2(z)^\top u_2 \nu(\mathrm{d}z) \\
&- \sum_{k=1}^{n_2}\int_{\mathcal{Z}}(E_k(z)X + F_k(z)u_2(z))\nu(\mathrm{d}z)\Big) + 1_{\{X_{t-}>0\}}|CX + Du_1|^2\Big]\mathrm{d}t \\
&+ 2X^+(CX + Du_1)^\top \Lambda_1^{\alpha_{t-}}\mathrm{d}t \\
&- (X^+)^2\Big[(2A + |C|^2)P_{1,t-}^{\alpha_{t-}} + 2C^\top \Lambda_1^{\alpha_{t-}} + Q + H_{11}^{\alpha_{t-},*}(P_1^{\alpha_{t-}}, \Lambda_1^{\alpha_{t-}}) \\
&\quad + \int_{\mathcal{Z}} H_{12}^{\alpha_{t-},*}(P_1^{\alpha_{t-}}, P_2^{\alpha_{t-}}, \Gamma_1^{\alpha_{t-}}, \Gamma_2^{\alpha_{t-}})\nu(\mathrm{d}z)\Big]\mathrm{d}t \\
&+ \sum_{k=1}^{n_2}\int_{\mathcal{Z}}(P_{1,t-}^{\alpha_{t-}} + \Gamma_{1k,t}^{\alpha_{t-}})\Big[\big((X + E_k(z)X + F_k(e)u_2(z))^+\big)^2 - (X^+)^2\Big]\nu(\mathrm{d}z)\mathrm{d}t \\
&+ \Big[2X^+(CX + Du_1) + (X^+)^2\Lambda_1^{\alpha_{t-}}\Big]^\top \mathrm{d}W \\
&+ \sum_{k=1}^{n_2}\int_{\mathcal{Z}}(P_{1,t-}^{\alpha_{t-}} + \Gamma_{1k,t}^{\alpha_{t-}})\Big[\big((X + E_k(z)X + F_k(e)u_2(z))^+\big)^2 - (X^+)^2\Big]\tilde{N}_k(\mathrm{d}t,\mathrm{d}e) \\
&+ (X^+)^2\int_{\mathcal{Z}}\Gamma_1^{\alpha_{t-}}(z)^\top \tilde{N}(\mathrm{d}t,\mathrm{d}z) + (X^+)^2\sum_{j,j'\in\mathcal{M}}(P_1^j - P_1^{j'})1_{\{\alpha_{t-}=j'\}}\mathrm{d}\tilde{N}^{j'j},
\end{aligned}
$$

where $\{N^{j'j}\}_{j,j'\in\mathcal{M}}$ are independent Poisson processes each with intensity $q^{j'j}$, and $\tilde{N}_t^{j'j} = N_t^{j'j} - q^{j'j}t,\ t \geq 0$ are the corresponding compensated Poisson martingale. Likewise,

$$
\begin{aligned}
\mathrm{d}P_{2,t}^{\alpha_t}(X_t^-)^2 = P_{2,t-}^{\alpha_{t-}}\Big[&- 2X_{t-}^+\Big(AX + B_1^\top u_1 + \int_{\mathcal{Z}} B_2(z)^\top u_2 \nu(\mathrm{d}z) \\
&- \sum_{k=1}^{n_2}\int_{\mathcal{Z}}(E_k(z)X + F_k(z)u_2(z))\nu(\mathrm{d}z)\Big) + 1_{\{X_{t-}\le 0\}}|CX + Du_1|^2\Big]\mathrm{d}t \\
&- 2X^-(CX + Du_1)^\top \Lambda_1^{\alpha_{t-}}\mathrm{d}t \\
&- (X^-)^2\Big[(2A + |C|^2)P_{2,t-}^{\alpha_{t-}} + 2C^\top\Lambda_2^{\alpha_{t-}} + Q + H_{21}^{\alpha_{t-},*}(P_2^{\alpha_{t-}}, \Lambda_2^{\alpha_{t-}}) \\
&\quad + \int_{\mathcal{Z}} H_{22}^{\alpha_{t-},*}(P_1^{\alpha_{t-}}, P_2^{\alpha_{t-}}, \Gamma_1^{\alpha_{t-}}, \Gamma_2^{\alpha_{t-}})\nu(\mathrm{d}z)\Big]\mathrm{d}t \\
&+ \sum_{k=1}^{n_2}\int_{\mathcal{Z}}(P_{2,t-}^{\alpha_{t-}} + \Gamma_{2k,t}^{\alpha_{t-}})\Big[\big((X + E_k(z)X + F_k(e)u_2(z))^-\big)^2 - (X^-)^2\Big]\nu(\mathrm{d}z)\mathrm{d}t \\
&+ \Big[- 2X^-(CX + Du_1) + (X^-)^2\Lambda_1^{\alpha_{t-}}\Big]^\top \mathrm{d}W \\
&+ \sum_{k=1}^{n_2}\int_{\mathcal{Z}}(P_{2,t-}^{\alpha_{t-}} + \Gamma_{2k,t}^{\alpha_{t-}})\Big[\big((X + E_k(z)X + F_k(e)u_2(z))^-\big)^2 - (X^-)^2\Big]\tilde{N}_k(\mathrm{d}t, \mathrm{d}z) \\
&+ (X^-)^2\int_{\mathcal{Z}}\Gamma_2^{\alpha_{t-}}(z)^\top\tilde{N}(\mathrm{d}t, \mathrm{d}z) + (X^-)^2\sum_{j,j'\in\mathcal{M}}(P_2^j - P_2^{j'})1_{\{\alpha_t-=j'\}}\mathrm{d}\tilde{N}^{j'j}.
\end{aligned}
$$

We define, for $n \ge 1$, the following stopping time τ_n:

$$
\tau_n := \inf\{t \ge 0 : |X_t| \ge n\} \wedge T,
$$

with the convention that $\inf\emptyset = \infty$. Obviously, $\tau_n \uparrow T$ a.s. along $n \uparrow \infty$.

Summing the two equations above, taking integration from 0 to τ_n, and then taking expectation, we deduce

$$
\begin{aligned}
&\mathbb{E}\Big[P_{1,\tau_n}^{\alpha_{\tau_n}}(X_{\tau_n}^+)^2 + P_{2,\tau_n}^{\alpha_{\tau_n}}(X_{\tau_n}^-)^2\Big] \\
&\qquad + \mathbb{E}\int_0^{\tau_n}\Big[u_1^\top R_1 u_1 + QX^2 + \int_{\mathcal{Z}}(u_2(z)^\top R_2(z)u_2(z))\nu(\mathrm{d}z)\Big]\mathrm{d}t \\
&= P_{1,0}^{i_0}(x^+)^2 + P_{2,0}^{i_0}(x^-)^2 + \mathbb{E}\int_0^{\tau_n}\Big\{u_1^\top(R_1 + 1_{\{X>0\}}P_1D^\top D + 1_{\{X\le 0\}}P_2D^\top D)u_1 \\
&\quad + 2u_1^\top(P_1B_1 + D^\top(P_1C + \Lambda_1))X^+ - 2u_1^\top(P_2B_1 + D^\top(P_2C + \Lambda_2))X^- \\
&\quad - H_{11}^{\alpha_{t-},*}(P_1, \Lambda_1)(X^+)^2 - H_{21}^{\alpha_{t-},*}(P_2, \Lambda_2)(X^-)^2 \\
&\quad + \int_{\mathcal{Z}}\Big[u_2^\top R_2 u_2 + 2P_1X^+\Big(B_2(z)^\top u_2(z) - \sum_{k=1}^{n_2}(E_k(z)X + F_k(z)u_2(z))\Big) \\
&\qquad - 2P_2X^-\Big(B_2(z)^\top u_2(z) - \sum_{k=1}^{n_2}(E_k(z)X + F_k(z)u_2(z))\Big) \\
&\qquad + \sum_{k=1}^{n_2}(P_1 + \Gamma_{1k})\Big(\big((X + E_k(z)X + F_k(e)u_2(z))^+\big)^2 - (X^+)^2\Big) \\
&\qquad + \sum_{k=1}^{n_2}(P_2 + \Gamma_{2k})\Big(\big((X + E_k(z)X + F_k(e)u_2(z))^-\big)^2 - (X^-)^2\Big) \\
&\qquad - H_{12}^{\alpha_{t-},*}(P_1, P_2, \Gamma_1, \Gamma_2)(X^+)^2 - H_{22}^{\alpha_{t-},*}(P_1, P_2, \Gamma_1, \Gamma_2)(X^-)^2\Big]\nu(\mathrm{d}z)\Big\}\mathrm{d}t.
\end{aligned} \tag{4.6}
$$

We will denote by $\phi(X,u)$the integrand w.r.t. t on the right-hand side of the above equation and show $\phi(X,u)\geq 0,\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\otimes\mathrm{d}\nu$-a.e., for any $u\in\mathcal{U}$.

Indeed, let us define

$$v_t=(v_{1,t},v_{2,t}(z))=\begin{cases}\left(\frac{u_{1,t}}{|X_{t-}|},\frac{u_{2,t}(z)}{|X_{t-}|}\right), & \text{if } |X_{t-}|>0;\\ (0,0), & \text{if } |X_{t-}|=0.\end{cases}$$

It is clear that the above process v is valued in $\Gamma_1\times\Gamma_2$ since Γ_1,Γ_2 are cones. If $X_{t-}>0$, then

$$\begin{aligned}\phi(X,u)&=X^2\Big[v_1^\top(R_1+P_1D^\top D)v_1+2v_1^\top(P_1B_1+D^\top(P_1C+\Lambda_1))-H_{11}^{\alpha_{t-},*}(P_1,\Lambda_1)\Big]\\&\quad+X^2\int_{\mathcal{Z}}\Big[v_2^\top R_2v_2+2P_1B_2(z)^\top v_2(z)-2P_1\sum_{k=1}^{n_2}(E_k(z)X+F_k(z)v_2(z))\\&\quad+\sum_{k=1}^{n_2}(P_1+\Gamma_{1k})\Big(\big((1+E_k(z)+F_k(e)v_2(z))^+\big)^2-1\Big)\\&\quad+\sum_{k=1}^{n_2}(P_2+\Gamma_{2k})\big((1+E_k(z)+F_k(e)v_2(z))^-\big)^2\\&\quad-H_{12}^{\alpha_{t-},*}(P_1,P_2,\Gamma_1,\Gamma_2)\nu(\mathrm{d}z)\Big]\nu(\mathrm{d}z)\geq 0,\end{aligned}$$

from the definitions of $H_{11}^{i,*},H_{12}^{i,*}$. Moreover, the equality holds at

$$u_1^*(t,X,\alpha)=\hat{v}_{11}^{\alpha_{t-}}(t,P_1^{\alpha_{t-}},\Lambda_1^{\alpha_{t-}})X_{t-}^+,\ \ u_2^*(t,X,\alpha)=\hat{v}_{12}^{\alpha_{t-}}(t,P_1^{\alpha_{t-}},P_2^{\alpha_{t-}},\Gamma_1^{\alpha_{t-}},\Gamma_2^{\alpha_{t-}})X_{t-}^+.$$

Next if $X_{t-}<0$, then

$$\begin{aligned}\phi(X,u)&=X^2\Big[v_1^\top(R_1+P_2D^\top D)v_1-2v_1^\top(P_2B_1+D^\top(P_2C+\Lambda_2))-H_{21}^{\alpha_{t-},*}(P_2,\Lambda_2)\Big]\\&\quad+X^2\int_{\mathcal{Z}}\Big[v_2^\top R_2v_2-2P_2B_2(z)^\top v_2(z)-2P_2\sum_{k=1}^{n_2}(E_k(z)-F_k(z)v_2(z))\\&\quad+\sum_{k=1}^{n_2}(P_1+\Gamma_{1k})\big((-1-E_k(z)+F_k(e)v_2(z))^+\big)^2\\&\quad+\sum_{k=1}^{n_2}(P_2+\Gamma_{2k})\Big(\big((-1-E_k(z)+F_k(e)v_2(z))^-\big)^2-1\Big)\\&\quad-H_{22}^{\alpha_{t-},*}(P_1,P_2,\Gamma_1,\Gamma_2)\nu(\mathrm{d}z)\Big]\nu(\mathrm{d}z)\geq 0,\end{aligned}$$

from the definitions of $H_{21}^{i,*},H_{22}^{i,*}$. Moreover, the equality holds at

$$u_1^*(t,X,\alpha)=\hat{v}_{21}^{\alpha_{t-}}(t,P_1^{\alpha_{t-}},\Lambda_1^{\alpha_{t-}})X_{t-}^-,\ \ u_2^*(t,X,\alpha)=\hat{v}_{22}^{\alpha_{t-}}(t,P_1^{\alpha_{t-}},P_2^{\alpha_{t-}},\Gamma_1^{\alpha_{t-}},\Gamma_2^{\alpha_{t-}})X_{t-}^-.$$

Finally, when $X_{t-}=0$, then

$$\phi(X,u) = u_1^\top(R_1+P_2D^\top D)u_1 + \int_{\mathcal{Z}}\Big[u_2^\top R_2u_2 + \sum_{k=1}^{n_2}(P_1+\Gamma_{1k})\big((F_ku_2)^+\big)^2 + \sum_{k=1}^{n_2}(P_2+\Gamma_{2k})\big((F_ku_2)^-\big)^2\Big]\nu(\mathrm{d}z) \geq 0;$$

here the equality holds at $u_1^*=0$, $u_2^*=0$.

The above analysis together with (4.6) shows that

$$\mathbb{E}\Big[P_{1,\tau_n}^{\alpha_{\tau_n}}(X_{\tau_n}^+)^2 + P_{2,\tau_n}^{\alpha_{\tau_n}}(X_{\tau_n}^-)^2\Big] + \mathbb{E}\int_0^{\tau_n}\Big[u_1^\top R_1u_1 + QX^2 + \int_{\mathcal{Z}}u_2(z)^\top R_2(z)u_2(z)\nu(\mathrm{d}z)\Big]\mathrm{d}t \geq P_{1,0}^{i_0}(x^+)^2 + P_{2,0}^{i_0}(x^-)^2.$$

By noting that for any $u\in\mathcal{U}$, the corresponding state process $X\in S_{\mathbb{F}}^2(0,T;\mathbb{R})$. Sending $n\to\infty$, we conclude, from the dominated convergence theorem, that (4.4) holds for any $u\in\mathcal{U}$, where the equality is achieved when u^* is defined by (4.3). □

Lemma 4.2 *Under the condition of Theorem 4.1,* $(u_1^*(t,X,\alpha),u_2^*(t,X,\alpha))\in\mathcal{U}$.

Proof It is clear that $(u_1^*(t,X,\alpha),u_2^*(t,X,\alpha))$ is valued in $\Pi_1\times\Pi_2$. It remains to prove

$$(u_1^*(t,X,\alpha),u_2^*(t,X,\alpha))\in L_{\mathbb{F}}^2(0,T;\mathbb{R}^{m_1})\times L_{\mathcal{P}}^{2,\nu}(0,T;\mathbb{R}^{m_2}).$$

Substituting (4.3) into the state process (2.1), we have

$$\begin{cases} \mathrm{d}X_t = \big[AX + B^\top(\hat{v}_{11}X^+ + \hat{v}_{21}X^-) + \int_{\mathcal{Z}}B(z)^\top(\hat{v}_{12}X^+ + \hat{v}_{22}X^-)\nu(\mathrm{d}z)\big]\,\mathrm{d}t \\ \qquad\qquad + [CX + D(\hat{v}_{11}X^+ + \hat{v}_{21}X^-)]^\top dw_t \\ \qquad\qquad + \int_{\mathcal{Z}}[E(z)X + F(z)(\hat{v}_{12}X^+ + \hat{v}_{22}X^-)]^\top \tilde{N}(\mathrm{d}t,\mathrm{d}z),\ t\in[0,T], \\ X_0 = x,\ \alpha_0 = i_0, \end{cases} \tag{4.7}$$

According to [7, Theorem 3.5], we have $\hat{v}_{11},\hat{v}_{21}\in L_{\mathbb{F}}^2(0,T;\mathbb{R}^{m_1})$. And from (3.6), we know that $\hat{v}_{12},\hat{v}_{22}\in L_{\mathcal{P}}^\infty(0,T;\mathbb{R}^{m_2})$. By the basic theorem of Gal'chuk [4, p.756-757], the SDE (4.7) admits a unique solution, denoted by X^*.

From the proof of Lemma 4.1, we find that, for any stopping time $\iota\leq T$,

$$\begin{aligned} &\mathbb{E}\Big[P_{1,\theta_n\wedge\iota}^{\alpha_{\theta_n}\wedge\iota}((X_{\theta_n\wedge\iota}^*)^+)^2 + P_{2,\theta_n\wedge\iota}^{\alpha_{\theta_n}\wedge\iota}((X_{\theta_n\wedge\iota}^*)^-))^2\Big] \\ &\qquad + \mathbb{E}\int_0^{\theta_n\wedge\iota}\Big[(u_1^*)^\top R_1u_1^* + Q(X^*)^2 + \int_{\mathcal{Z}}(u_2^*(z)^\top R_2(z)u_2^*(z))\nu(\mathrm{d}z)\Big]\mathrm{d}t \\ &= P_{1,0}^{i_0}(x^+)^2 + P_{2,0}^{i_0}(x^-)^2, \end{aligned} \tag{4.8}$$

where

$$\theta_n := \inf\{t\geq 0 : |X_t^*| > n\}\wedge T.$$

Suppose Assumption 2.2 holds. We have, from (4.8),

$$\delta\mathbb{E}\int_0^{\theta_n\wedge T}\Big[|u_1^*|^2+\int_{\mathcal{Z}}|u_2^*(z)|^2\nu(\mathrm{d}z)\Big]\mathrm{d}t\le P_{1,0}^{i_0}(x^+)^2+P_{2,0}^{i_0}(x^-)^2.$$

Letting $n\to\infty$, it follows from the monotone convergence theorem that $(u_1^*,u_2^*)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{m_1})\times L^{2,\nu}_{\mathcal{P}}(0,T;\mathbb{R}^{m_2})$.

Suppose Assumption 2.3 holds. In this case, there exists $c>0$ such that $P_j^i\ge c,\ P_j^i+\Gamma_{jk}^i\ge c,\ i\in\mathcal{M},\ j=1,2,\ k=1,\dots,n_2$. From (4.8), we get

$$c\mathbb{E}[|X^*_{\theta_n\wedge\iota}|^2]\le P_{1,0}^{i_0}(x^+)^2+P_{2,0}^{i_0}(x^-)^2.$$

Letting $n\to\infty$, it follows from Fatou's lemma that

$$\mathbb{E}[|X^*_\iota|^2]\le c,$$

for any stopping time $\iota\le T$. This further implies

$$\mathbb{E}\int_0^T|X_t^*|^2\mathrm{d}t\le cT.$$

Applying Itô formula to $|X_t^*|^2$, yields that

$$\begin{aligned}
&x^2+\mathbb{E}\int_0^{\theta_n\wedge T}|Du_1^*|^2\mathrm{d}t+\mathbb{E}\int_0^{\theta_n\wedge T}\int_{\mathcal{Z}}|Fu_2^*|^2\nu(\mathrm{d}z)\mathrm{d}t\\
&=\mathbb{E}[X^*_{\theta_n\wedge T}]^2-\mathbb{E}\int_0^{\theta_n\wedge T}\Big[(2A+|C|^2+\int_{\mathcal{Z}}|E(z)|^2\nu(\mathrm{d}z))|X_t^*|^2\\
&\quad+2X_t^*(B_1+D^\top C)^\top u_1^*+2X_t^*\int_{\mathcal{Z}}(B_2^\top u_2^*+\sum_{k=1}^{n_2}E_kF_ku_2^*)\nu(\mathrm{d}z)\Big]\mathrm{d}t.
\end{aligned}$$

When 2 and 4 hold. We have

$$\begin{aligned}
&\delta\mathbb{E}\int_0^{\theta_n\wedge T}|u_1^*|^2\mathrm{d}t+\delta\mathbb{E}\int_0^{\theta_n\wedge T}\int_{\mathcal{Z}}|u_2^*|^2\nu(\mathrm{d}z)\mathrm{d}t\\
&\le c+c\mathbb{E}\int_0^{\theta_n\wedge T}\Big[|X_t^*|^2+2|X_t^*||u_1^*|+2|X_t^*|\int_{\mathcal{Z}}|u_2^*|\nu(\mathrm{d}z)\Big]\mathrm{d}t\\
&\le c+c\Big(1+\frac{2}{\delta}+\frac{2\nu(\mathcal{Z})}{\delta}\Big)\mathbb{E}\int_0^{\theta_n\wedge T}|X_t^*|^2\mathrm{d}t+\frac{\delta}{2}\mathbb{E}\int_0^{\theta_n\wedge T}|u_1^*|^2\mathrm{d}t+\frac{\delta}{2}\mathbb{E}\int_0^{\theta_n\wedge T}\int_{\mathcal{Z}}|u_2^*|^2\nu(\mathrm{d}z)\mathrm{d}t.
\end{aligned}$$

After rearrangement, it follows from the monotone convergence theorem that

$$\mathbb{E}\int_0^T |u_1^*|^2 \mathrm{d}t + \mathbb{E}\int_0^T \int_{\mathcal{Z}} |u_2^*|^2 \nu(\mathrm{d}z)\mathrm{d}t \le c.$$

Hence $(u_1^*, u_2^*) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{m_1}) \times L^{2,\nu}_{\mathcal{P}}(0, T; \mathbb{R}^{m_2})$.

When 2 and 3 hold. We get $u_1^* \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{m_1})$ exactly as above. On the other hand, we can get $u_2^* \in L^{2,\nu}_{\mathcal{P}}(0, T; \mathbb{R}^{m_2})$ from (4.8).

The last case in Assumption 2.3 can be handled similarly. □

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Declarations

Conflict of interest The authors have no conflict of interest to declare.

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