The following publication X. Wang, B. Wang, L. Teng and Y. Wu, 'Prospect Theory-Based Portfolio Selection Using Multiple Fuzzy Reference Intervals,' in IEEE Transactions on Systems, Man, and Cybernetics: Systems, vol. 55, no. 10, pp. 7100-7114, Oct. 2025 is available at https://doi.org/10.1109/TSMC.2025.3578997.

Prospect theory-based portfolio selection using multiple fuzzy reference intervals

Xianhe Wang, Bo Wang, Long Teng, Yaoxin Wu

***Abstract*—Portfolio selection stands as a paramount concern within the realm of decision-making and management engineering. However, owing to the inherent intricacies of capital markets and the presence of irrational investor behaviors, the attainment of pre-defined investment objectives by investors remains a formidable challenge. In order to comprehensively depict investor behavior patterns and to provide investment guidance in highly uncertain and volatile markets, this study introduces a novel fuzzy model for representing prospect theory and based on this, develops a novel portfolio selection optimisation framework. Additionally, a new particle swarm optimization consists of adaptive and cooperative strategy is proposed to find the optimal solution of this model. The effectiveness of this model is validated through two case study utilizing real-market data, while the efficiency of the solution algorithm is confirmed through a test fitness functions-based case study.**

***Index Terms*—Fuzzy portfolio selection, optimal uncertainty intervals, prospect theory, expected utility theory, mean-absolute deviation.**

I. Introduction

THE core issue of portfolio selection is how to make the best possible investment strategy among a basket of securities, so that investors can maximize the return or minimize the risk. In the past decades, we saw an accelerated evolution in portfolio theory since it was first proposed by Markowitz in 1952 [1]. In Markowitz's mean-variance model, the investment return is expressed as the expected value of a random variable and the risk is evaluated by average squared deviation of the random variable from its expected value [2]. Following the above pioneer work, a series of effective risk measurements have been proposed. For examples, Markowitz et al. [3] developed the mean-semivariance model based on the consideration that an investor care more about the part where the return is lower than the expected value. Konno and Yamazaki [4] introduced the mean-absolute deviation model as an alternative to the mean-variance model. Experimental results show that mean-absolute deviation generates similar portfolio to that of mean-variance within a fraction of time required to solve [4]. Other risk measurements in portfolio selection include, but not limited to entropy [5], Value-at-Risk [6], conditional Value-at-Risk [7] and expected shortfall with loss aversion [8].

The above models requires sufficient historical data to calculate the statistical indexes of random variables. However there are numerous non-probabilistic elements that may influence financial markets. Various empirical studies have highlighted the limitations of probabilistic methods in capturing the uncertainty inherent in financial markets [9]. Fuzzy set theory serves as an effective tool for describing uncertain environments characterized by vagueness and ambiguity, or other forms of fuzziness [10], [11], which are prevalent not only in financial markets but also in the decision-making behaviors of financial managers. Based on the above considerations, fuzzy set theory has been characterized as an effective approach to depict security returns, and a number of fuzzy portfolio selection models have been built [12]–[19].

Despite the tremendous advances recently achieved, most of the existing works were carried out based on the expected utility theory which views investors as rational beings to maximize the expected wealth. Unfortunately, behavioral finance developed in the 1980s posits that the cognitive biases of investors have great influence on the decision making process [20]. Luo [21] examined the impact of sentiment on asset prices during periods of high and low risks and ambiguity across countries. And Gao et al. [22] focus on how the stock price daily momentum is driven by the investor trading behavior in Chinese market. Prospect theory, the representative behavioral theory, employs cognitive psychological features to incorporate irrational human behavior into economic decision making [23]. Building on this, many researchers have embarked on portfolio selection rooted in prospect theory [15], [18], [24]. These endeavors have significantly propelled research in the fields of prospect theory and portfolio selection. Bi et al. [25] investigate a behavioral mean-variance portfolio selection problem in continuous time. Gan et al. [26] review the application of prospect theory in the field of power system economic decisions. Zhou et al. [27] develop a decision-making approach to help managers to select optimal portfolio in which the group contains several experts' personal evaluations. These endeavors have significantly propelled research in the fields of prospect theory and portfolio selection. Ferro et al. [28] propose a new parameterized quantum theory based model with rank dependent theory and prospect theory to describe human risky choices, which takes data on choices between pairs of lotteries as the validation example. Yadav et

This work is supported by the Research Committee, Department of Industrial and Systems Engineering, The Research Institute of Advanced Manufacturing of the Hong Kong Polytechnic University, Hong Kong Special Administrative Region (Project code: BD9Z, CD4Q, UAMU, W22B). We also thank the financial support from Department of Industrial and Systems Engineering, HK PolyU. (*Corresponding author: Long Teng*)

Xianhe Wang and Long Teng are with the Department of Industrial and Systems Engineering, Research Institute for Advanced Manufacturing, The Hong Kong Polytechnic University, Hung Hom, Hong Kong, China (e-mail: thisiswangxianhe@gmail.com, eric-long.teng@polyu.edu.hk).

Bo Wang is with the School of Management and Engineering, Nanjing University, Nanjing 210093, China (e-mail:bowangsme@nju.edu.cn).

Yaoxin Wu is with the department of Information Systems, Faculty of Industrial Engineering and Innovation Sciences, Eindhoven University of Technology, 5612 AZ Eindhoven, The Netehrlands (e-mail: wyxacc@hotmail.com)

al. [29] discussed investors' attitudes' effects and influences in the multiperiod portfolio selection framework. Gao et al. [30] propose a prospect theory framework with multi-attribute decision making for modeling travel behavior. Grant et al. [31] models the decision to 'cash-out' a bet under the prospect theory framework.

Although the combination of prospect theory and fuzzy portfolio selection is not new, the related works could be further extended considering the following three aspects: First, the key property of prospect theory is reference dependable. Investors perceive gains and losses with respect to a reference point (e.g. the status quo). In this regard, the prospect value function can be divided into a gain domain and a loss domain [27]. Although prospect theory is widely accepted, some studies [32] question its accuracy in fitting investors' investment paradigms in complex real-world decision-making environments. For instance, Chapman et al. [33] observed that while investors indeed exhibit loss aversion, some choose high-risk investment targets when faced with opportunities for high returns, a phenomenon that prospect theory cannot explain. Other researchers have sought the inclusion of reference points besides the status quo, and researchers have explained choice data by including multiple reference points within the value function [34]. Comparing with classical prospect theory, the use of multiple reference points divides the prospect value function into more specific domains, e.g. the domain of return rate lower than a minimum requirement, the domain of return rate higher than a goal and the domain of return rate between the minimum requirement and goal [35]. In this way, the multiple reference points-based prospect theory is able to address investors' psychological mechanisms in a more detailed manner, thus providing more comprehensive decision support for investors especially those who are more sensitive to risk-taking and goal-seeking.

Second, most of the existing studies assumed that the numerical value of reference point is exactly known. This is reasonable since investors have their specific target returns towards an investment, or they can directly set the value according to the risk-free rate. However, for some investors, the reference point could be ambiguous and hard to determine due to the variation of psychological processes and the complicated nature of financial market. In this regard, the uncertainty of reference points could be considered in the prospect theory-based portfolio selection. More importantly, for a given distribution of security return, the setting of different reference points can certainty has an impact on the portfolio selection. Therefore, it is of interest to co-optimize the reference point values with portfolio selection, so that not only the exact reference point can be determined, but also an optimal investment decision can be made.

Finally, as the most influential theory for describing human behavior, prospect theory has been used to explain the impact of psychological behavior of decision-makers. Although prospect theory shows certain strong aspects than expected utility theory in portfolio selection, the latter is still widely adopted by a number of investors. In addition, Pfiffelmann et al. [36] compared the assets allocation generated by prospect theory based behavioral portfolio model [37] and mean-variance model. Their simulations were run using U.S. stock price from the Center for Research in Security Prices databases for the 1995-2011 period. They show that behavioral portfolio model optimal portfolios and the mean-variance efficient coincide in 70% of cases. Therefore, they draw the conclusion that even if the prospect optimal portfolio is often located on the mean-variance (the representative expected utility) frontier, it will not be chosen by typical expected utility investors since it is associated with an extremely low degree of risk aversion. As a result, expected utility investors with usual levels of risk aversion would not invest in the prospect optimal portfolio. Therefore, it could be meaningful to incorporate the merits of expected utility into the prospect theory based fuzzy portfolio selection, so that providing expected utility investors with trusted decision support.

Based on the above analysis, this study introduces a prospect theory based portfolio selection model under fuzzy uncertainty. First, for each forecasted membership function of security returns, two reference points i.e., the minimum requirement and goal are used to divide the prospect value function into three domains, the curvature parameters and related coefficients of which are set differently. The detailed knowledge of the prospect value function is given in **Sec**. III-B. Second, the uncertainty of reference points is addressed in the proposed model, and we try to find effective portfolio by optimizing the numerical values of the minimum requirement and goal together with capital allocations. The detailed motivation of performing the above optimization is provided in **Sec**. III-C. Third, the model is established to incorporate expected utility into prospect theory based fuzzy portfolio selection. Especially, mean-absolute deviation is converted to a constraint of the proposed model, thus avoiding the assignment of extreme reference points. The significance of the proposed model is explained in **Sec**. III-D.

Clearly, the fuzzy portfolio selection model proposed in this study presents a complex optimization challenge that demands sophisticated quantitative analysis for its resolution. To address this, we introduce an adaptive cooperative particle swarm optimization (ACPSO) algorithm tailored to tackle the problems in this study. We anticipate that this algorithm will overcome issues related to local convergence and premature, often encountered in traditional PSO and its variant algorithms.

The main contributions of this study can be summarized as follows: (1) A new fuzzy model for representing prospect theory in this study improves existing models to a multiple reference points case, thus addressing investors' psychological mechanisms in a more detailed manner. For investors, the proposed portfolio selection model is more aligned with their decision-making paradigms than a single reference point model, as the former considers both return and risk preferences. On the other hand, having reference points in the form of uncertain intervals is also consistent with investors' decision-making paradigm in an capital market characterized by uncertainty and volatility. (2) The fuzzy model considers the uncertainty of reference points and establishes an investment portfolio selection framework based on this model. Compared to portfolio selection models with a single reference point, which can lead to risk concentration, this study achieves a balance between risk and return by introducing two reference points. The use of uncertain reference points makes the model more robust, allowing it better handle uncertainty and risk in investments. (3) The proposed prospect theory based fuzzy portfolio selection model mixes together the advantages of expected utility and prospect theory, thus improving the practical significant of prospect theory to typical expected utility investors. (4) A novel PSO algorithm is proposed to achieve faster and more accurate solutions.

The paper is organized as follows: In **Sec**. II, some preliminary notions of fuzzy set theory such as the expected value and absolute deviation are introduced. In **Sec**. III, the motivation and validity of this research are explained. In **Sec**. IV, the mathematical model of the prospect theory based fuzzy portfolio selection model is established, which can be solved by the algorithm designed in **Sec**. V. A test function based computational study is provided to demonstrate the efficiency of the proposed solution algorithm and two real market data-based case study is then provided to demonstrate the application of the proposed approach in **Sec**. VI. Finally, we conclude the paper in **Sec**. VII.

II. Preliminary notions

In this section, we briefly review the definitions of expected value and absolute deviation under fuzzy environment, which will be used in subsequent sections.

Definition 2.1. *Let ξ be a fuzzy variable with membership function μ_ξ and r is a real number, the credibility function of an event $\xi \le r$ is expressed as:*

$$\mathrm{Cr}\{\xi \le r\} = \frac{1}{2}\left[\mathrm{POS}\{\xi \le r\} + \mathrm{NEC}\{\xi \le r\}\right], \quad (1)$$

where POS *and* NEC *express the possibility and necessity measurements , and:*

$$\mathrm{POS}\{\xi \le r\} = \sup_{t \le r} \mu_\xi(t), \quad (2)$$

$$\mathrm{NEC}\{\xi \le r\} = 1 - \sup_{t > r} \mu_\xi(t). \quad (3)$$

As a self-dual set function, we have $\mathrm{Cr}\{\xi \le r\} = 1 - \mathrm{Cr}\{\xi > r\}$.

Definition 2.2. *For a fuzzy variable ξ, its expected value is calculated as:*

$$\mathbf{E}[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \le r\}\mathrm{d}r. \quad (4)$$

Definition 2.3. *Let ξ be a fuzzy variable with finite expected value e. Then its absolute deviation is defined as:*

$$\mathbf{A}[\xi] = \mathbf{E}[|\xi - e|]. \quad (5)$$

And for a portfolio decision including n securities, its absolute deviation can be calculated as :

$$\mathbf{A}\left[\sum_{i=1}^{n} x_i\xi_i\right] = \mathbf{E}\left[\left|\sum_{i=1}^{n} x_i\xi_i - \mathbf{E}\left[\sum_{i=1}^{n} x_i\xi_i\right]\right|\right], \quad (6)$$

where ξ_i represents the fuzzy return of security i and x_i is the proportion of total fund invested in security i.

As mentioned before, mean-absolute deviation generates similar portfolio to that of mean-variance within a fraction of time required to solve. Therefore, in this paper, mean-absolute deviation is used to incorporate excepted utility into prospect theory based fuzzy portfolio selection.

III. Motivations

In this section, we first illustrate the motivations of using the minimum requirement and goal as the reference points in prospect theory. Then the corresponding prospect value function is constructed and the significance of involving reference points uncertainties is clarified. Finally, the method to incorporate excepted utility into prospect theory based fuzzy portfolio selection is provided.

A. Prospect theory

In the framework of classical prospect theory [38], the prospect value function has three principles to satisfy: (1) Reference dependence. Investors perceive gains and losses by comparison with a given reference point. (2) Loss aversion. Investors are more sensitive to losses than to gains. (3) Diminishing sensitivity. Investors have a disposition to be risk-averse in the domain of gains and risk-seeking in the domain of losses. Based on the above principles, the prospect value function can be represented as:

$$v(r_i) = \begin{cases} (r_i - \overline{rp_i})^\alpha, & r_i \ge \overline{rp}, \\ -\lambda(\overline{rp} - r_i)^\beta, & r_i < \overline{rp}. \end{cases} \quad (7)$$

Suppose that ξ_i represents the fuzzy return of security i, then $v(r_i)$ in **Eq**. (7) expresses the prospect value of any real value r_i in ξ_i. $\overline{rp}$ denotes the value of the reference point, λ is the loss aversion ratio, α and β indicate the curvature parameters for gains and losses respectively.

B. The using of two reference points

With respect to a single reference point, the prospect value in classical prospect theory can be divided into a gain domain and a loss domain, as shown in **Fig**. 1-(a), in which the reference point is set as 0.05. However, researchers have doubted that the using of one reference point may not sufficiently describe investors' psychological mechanisms [35], especially for those investors who are sensitive to both risk-taking and goal-seeking. More specifically, according to the discussion in [39], investors' psychological mechanisms can be rank ordered as failure ($r_i < \mathtt{MR}$, $\mathtt{MR}$ is the value of the minimum requirement), success ($r_i > \mathtt{G}$, $\mathtt{G}$ is the value of the goal) and status quo ($\mathtt{MR} \le r_i \le \mathtt{G}$). Therefore, in this study, we consider the minimum requirement and goal as the two reference points which divide the prospect value function into three domains, as shown in **Fig**. 1-(b). Accordingly, the classical prospect value function is revised to **Eq**. (8) for investors who are sensitive to both risk-taking and goal-seeking.

$$v(r_i) = \begin{cases} (r_i - \mathtt{MR})^\alpha + \eta(r_i - \mathtt{G})^\gamma, & r_i > \mathtt{G}, \\ (r_i - \mathtt{MR})^\alpha, & \mathtt{MR} \le r_i \le \mathtt{G}, \\ -\lambda(\mathtt{MR} - r_i)^\beta, & r_i < \mathtt{MR}, \end{cases} \quad (8)$$

where r_i is any realization of ξ_i. α, β and γ represent the curvature parameters for status quo, failure and success respectively. λ and η are the loss aversion ratio and seeking pride ratio, and it is assumed that $\lambda \geq \eta \geq 1$ which follows the rank order mentioned above. Moreover, the prospect value of the success domain includes two parts: $(r_i - \mathtt{MR})^\beta$ indicates the prospect value of reaching the minimum requirement, while $\eta(r_i - \mathtt{G})^\alpha$ reflects an additional satisfaction of achieving the goal. Generally, **Eq**. (8) can be changed without difficulty to accommodate investors with different rank orders.

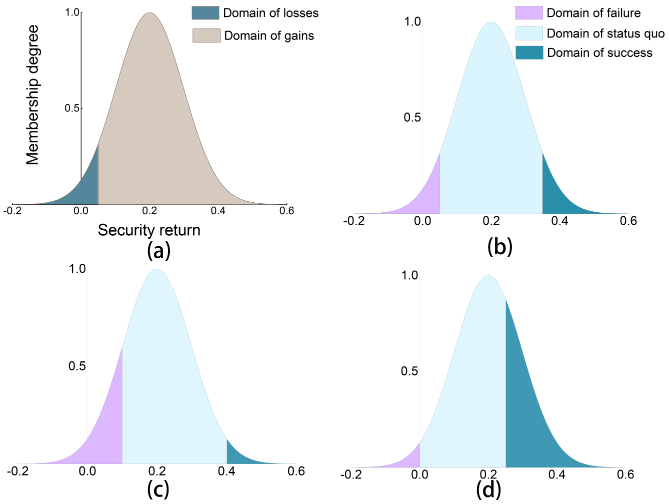


Fig. 1: Motivations of using two reference points, involving reference points uncertainties and incorporating expected utility theory into prospect theory-based fuzzy portfolio selection.

Fig. (2) depicts how the expected utility theory, the classical prospect theory and the prospect theory with two reference points used in this study quantitatively describe the relationship between investment returns and their utility to the investor. As can be seen from the figure, expected utility theory is a linear map which considers the effects produced by gains and losses to be equal in magnitude. In this demonstration figure, we set $\alpha = \beta = 0.88$, $\lambda = 2.25$ in **Eq. (7)**, which follows the findings in [40]. The reference point is set as 0. Prospect theory considers the effects by losses to be more significant than expected utility theory. The differences between prospect theory and expected theory when the return is negative is significantly smaller than the difference when the the return exceeds the reference point. In this demonstration figure, the minimum requirement is 0, the same as the value of the reference point, and the goal is 0.12. The prospect theory-variant used in this study assumes that the impact produced is the same as prospect theory until the investment return reaches the goal. However, after the return exceeds the goal, the return will have a more significant impact on the investor due to the return chasing behavior of the investor.

C. The consideration of reference points uncertainties

In this research, we take the minimum requirement and goal as imprecise values for two reasons. First, due to the variation of psychological process and the complicated nature of financial market, it could be difficult for some investors to assign exact minimum requirement and goal. For example,

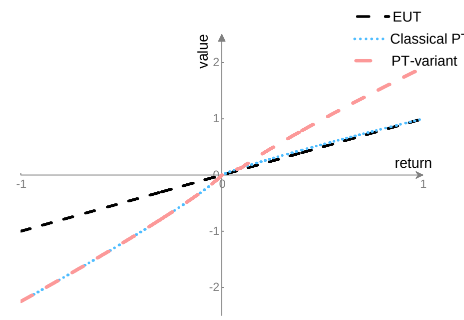


Fig. 2: Value functions of expected utility theory (EUT), classical prospect theory (PT) and prospect theory with two uncertain reference points (PT-variant)

these values might be expressed with ambiguity such as the minimum requirement is about 8% or the goal is within [20%, 25%]. Therefore, it is reasonable to consider the ambiguity of reference points. Second, for a given membership distribution of security return, the assignment of varied the minimum requirement and the goal leads to different prospect values. For example, if we take **Eq**. (8) as the prospect value function, then the prospect value of condition that the minimum requirement is 0.05 and the goal is 0.35 is larger than that of the minimum requirement is 0.1 and the goal is 0.4, as shown in **Fig**. 1-(b) and **Fig**. 1-(c). Therefore, the prospect value is sensitive to reference points, and investors with varied minimum requirement and goal could have totally different evaluations upon the same distribution of security return, which certainty affects the result of portfolio optimization. In view of this, it is also meaningful to investigate the impact of varied reference points on portfolio selection.

D. The incorporation of expected utility theory

Finally, the motivation of incorporating expected utility into prospect theory based fuzzy portfolio selection is explained. In some cases, an investor may set small values to both minimum requirement and goal, e.g. **Fig**. 1-(d) depicts the membership degree when the minimum requirement and goal are 0 and 0.25. According to **Eq**. (8), it can be found that the overall prospect value of **Fig**. 1-(d) is obviously larger than that of **Fig**. 1-(b) and **Fig**. 1-(c). Essentially, for a given distribution of security return, the smaller minimum requirement and goal are, the larger the prospect value will be, and vice versa. Nevertheless, the behavior of obtaining large prospect values via setting small reference points is unacceptable for rational investors, especially those expected utility investors who aim to maximize the possibility of obtaining high expected wealth because the distribution of security return is fixed and does not change with reference points. Generally, it is an important concern for typical expected utility investors to give such a confidence interval, so that the chance of obtaining high expected wealth can be maximized.

Therefore, aims at improving the practical significant of prospect theory, this study incorporates the merits of expected utility into prospect theory based fuzzy portfolio selection by using mean-absolute deviation. Specifically, the cumulative return and absolute deviation of the portfolio is combined to the following inequality, which will be applied as a constraint of the proposed model.

$$\mathbf{E}\left[\sum_{i=1}^{n} x_i\xi_i\right] - \rho\mathbf{A}\left[\sum_{i=1}^{n} x_i\xi_i\right] \geq L, \tag{9}$$

where ξ_i in constraint (9) represents the return rate variable of security i, $\mathbf{E}[\cdot]$ and $\mathbf{A}[\cdot]$ indicate the cumulative return and absolute deviation of the portfolio.

When ξ_i follows normal distribution, the mean-absolute deviation model is equivalent to the mean-variance model in the same form. And when ξ_i does not follow normal distribution, the difference between the two models can be considered negligible. In **Eq**. (9), absolute deviation measures the risk level of the portfolio selection scheme, while ρ is a non-negative constant specified by the investor, suggesting the investor's tolerance for risk. L is another constant specified by the investor, which express the investor's expectation of the risk return profile of the portfolio selection scheme. As discussed later in **Sec**. IV, constraint represented by **Eq**. (9) could avoid the setting of extreme minimum requirement and goal, thus improving the prospect theory portfolio selection.

IV. MATHEMATICAL MODELING

In this section, we present the mathematical formulation of the prospect theory based fuzzy portfolio selection model proposed in this study and discuss its impact. This model aims to maximize the overall prospect value subjects to a given mean-absolute deviation level, as shown in **Eq**. (10).

$$\begin{cases} max \sum_{i=1}^{n} x_i v(\xi_i) = \sum_{i=1}^{n} x_i \left[v(\xi_{i,f}) + v(\xi_{i,sq}) + v(\xi_{i,s})\right] \\ s.t.\ \mathbf{E}\left[\sum_{i=1}^{n} x_i\xi_i\right] - \rho\mathbf{A}\left[\sum_{i=1}^{n} x_i\xi_i\right] \geq L_1 \\ \quad \rho, x_i \geq 0, i = 0, 1, ..., n \\ \quad \sum_{i=1}^{n} x_i = 1, i = 0, 1, ..., n. \end{cases} \tag{10}$$

$$\begin{cases} max\mathbf{E}\left[\sum_{i=1}^{n} x_i\xi_i\right] - \rho\mathbf{A}\left[\sum_{i=1}^{n} x_i\xi_{i,sq}\right] \\ s.t. \sum_{i=1}^{n} x_i v(\xi_i) = \sum_{i=1}^{n} x_i \left[v(\xi_{i,f}) + v(\xi_{i,sq}) + v(\xi_{i,s})\right] \geq L_2 \\ \quad \rho, x_i \geq 0, i = 0, 1, ..., n \\ \quad \sum_{i=1}^{n} x_i = 1, i = 0, 1, ..., n. \end{cases} \tag{11}$$

In **Eq**. (10), $v(\xi_{i,f})$, $v(\xi_{i,sq})$ and $v(\xi_{i,s})$ represent the prospect values of the failure, status quo and success domains of each security respectively, which can be calculated by **Eq**. (12). $x_i \geq 0$ means that short sale is not allowed. Meanwhile, the structure of **Eq**. (10) can be changed easily to maximize $\mathbf{E}[\cdot] - \rho\mathbf{A}[\cdot]$ subjects to a predefined level of prospect value, as shown in **Eq**. (11).

$$\begin{cases} v(\xi_{i,f}) = \int_{-\infty}^{\mathtt{MR}} -\lambda(\mathtt{MR} - r_i)^{\gamma} \mu_{\xi_i}(r_i)\mathrm{d}r_i. \\ v(\xi_{i,sq}) = \int_{\mathtt{MR}}^{\mathtt{G}} (r_i - \mathtt{MR})^{\beta} \mu_{\xi_i}(r_i)\mathrm{d}r_i. \\ v(\xi_{i,s}) = \int_{\mathtt{G}}^{+\infty} \left[(r_i - \mathtt{MR})^{\beta} + \eta(r_i - \mathtt{G})^{\alpha}\right] \mu_{\xi_i}(r_i)\mathrm{d}r_i. \end{cases} \tag{12}$$

Before giving the solution algorithm, we briefly discuss some properties of the two reference points. **Tab**. (I) lists the impact from the fluctuations of the minimum requirement (when the goal is fixed) and the goal (when the minimum requirement is fixed) on the prospect value as well as the cumulative return and absolute deviation, where '↑' and '↓' express the increasing and decreasing of the corresponding values. '-' means that the change of reference points has no influence on the values.

TABLE I: Impact from fluctuations of the minimum requirement and goal.

	MR ↓	MR ↑	G ↓	G ↑
$V(\xi_{i,f})$	↑	↓	-	-
$v(\xi_{i,sq})$	↑	↓	↓	↑
$v(\xi_{i,s})$	↑	↓	↑	↓
$v(\xi_i)$	↑	↓	↑	↓
$\mathbf{E}\left[\sum_{i=1}^{n} x_i\xi_i\right]$	↑	↓	↓	↑
$\mathbf{A}[\sum_{i=1}^{n} x_i\xi_i]$	↑	↓	↓	↑

Tab. (I) illustrates that, as a non-negative value, the decreasing of minimum requirement improves the overall prospect value ($v(\xi_i)$) and the cumulative return ($\mathbf{E}[\cdot]$), but increases the investment risk taken since the absolute deviation ($\mathbf{A}[\cdot]$) has been increased. By contrast, the rising of minimum requirement reduces the absolute deviation, which however leads to a lower prospect value and cumulative return. On the goal side, the decreasing of the goal improves the overall prospect value and absolute deviation, but reduces the cumulative return. By comparison, the increasing of the goal produces a higher cumulative return, which however deteriorates the prospect value as well as the absolute deviation.

Based on the above analysis, the following preliminary conclusions can be arrived at: firstly, the decreasing of both minimum requirement and goal results in a higher overall prospect value; secondly, the decreasing of the minimum requirement and the increasing of the goal produces a higher cumulative return; finally, the increasing of the minimum requirement and the decreasing of the goal leads to a lower absolute deviation. Therefore, in view of the above contradictions, it is meaningful to co-optimize the reference points with capital allocations.

In **Eq**. (10), the mean-absolute deviation function incorporates investment return and risk indexes simultaneously. We briefly discuss the effects of this function before introducing the solution algorithm. To streamline the discussion, we suppose the investment scheme was made from two independent security. Security's return rates are denoted by two symmetric triangular fuzzy variable $\xi_1 = (a_1, b_1, c_1)$ and $\xi_2 = (a_2, b_2, c_2)$. Assuming that the expected return and risks of ξ_2 are higher than those of ξ_1. which means

$a_2 < a_1 < 0 < b_1 < b_2 < c_1 < c_2$. Suppose the investment proportion for ξ_1 is x_1, and for ξ_2 is x_2, $x_1 = 1 - x_2$.

Since ξ_1 and ξ_2 are symmetric, then the expected return of ξ_1 and ξ_2 are b_1 and b_2. Assuming the expected return of the investment scheme is b_0. Since ξ_1 and ξ_2 are independent, then $b_0 = \mathbf{E}\left[\sum_{i=1}^{n} x_i\xi_i\right] = x_1\xi_1 + x_2\xi_2$. It's obvious $b_1 \le b_0 \le b_2$. The absolute deviation of the portfolio could be calculated as :

$$\mathbf{A}[\sum_{i=1}^{2} x_i\xi_i] = \mathbf{E}[\sum_{i=1}^{2} x_i \mid \xi_i - b_0 \mid] = \sum_{i=1}^{2} x_i\mathbf{E}[\mid \xi_i - b_0 \mid] \quad (13)$$

In order to calculate the expected value of $\mid \xi_i - b_0 \mid$, the membership function $\mu_1(t)$ of ξ_1 was divided into two parts $\mu_{1L}(t)$ and $\mu_{1R}(t)$ by b_0. Similar to this, $\mu_2(t)$ was divided into $\mu_{2L}(t)$ and $\mu_{2R}(t)$.

$$\mu_{1L}(t) = \begin{cases} \frac{c_1 - b_1}{t + c_1 - b_0}, & 0 \le t \le b_0 - b_1, \\ \frac{b_1 - a_1}{b_0 - a_1 - t}, & b_0 - b_1 < t \le b_0 - a_1. \end{cases} \quad (14)$$

$$\mu_{1R}(t) = \frac{c_1 - b_1}{c_1 - b_0 - t},\ 0 < t \le c_1 - b_1. \quad (15)$$

$$\mu_{2L}(t) = \frac{b_2 - a_2}{b_0 - a_2 - t},\ 0 \le t \le b_0 - a_2. \quad (16)$$

$$\mu_{2R}(t) = \begin{cases} \dfrac{b_2 - a_2}{t + b_0 - a_2},\ 0 \le t \le b_2 - b_0, \\ \dfrac{c_2 - b_2}{c_2 - b_0 - t},\ b_2 - b_0 < t \le c_2 - b_0. \end{cases} \quad (17)$$

Based on the **Definitions 2.1**, we could derive the $POS(\cdot)$, $NEC(\cdot)$ and $Cr(\cdot)$ of event $\mid \xi_1 - b_0 \mid \le r$ and $\mid \xi_2 - b_0 \mid \le r$, as presented in **Eqs**. (18) $\sim$ (29):

$$POS_{1L} = \begin{cases} \dfrac{r + c_1 - b_0}{c_1 - b_1},\ 0 \le r \le b_0 - b_1, \\ 1,\ b_0 - b_1 < r \le b_0 - a_1. \end{cases} \quad (18)$$

$$POS_{1R} = \frac{c_1 - b_0}{c_1 - b_1},\ 0 < t \le c_1 - b_1. \quad (19)$$

$$NEC_{1L} = \begin{cases} 0,\ r \le b_0 - b_1, \\ \dfrac{r + b_1 - a_0}{b_1 - a_1},\ b_0 - b_1 < t \le b_0 - a_1. \end{cases} \quad (20)$$

$$NEC_{1R} = \frac{r + b_0 - b_1}{c_1 - b_1}. \quad (21)$$

$$Cr_{1L} = \begin{cases} \dfrac{1}{2}\dfrac{r + c + 1 - b_0}{c_1 - b_1},\ 0 \le r \le b_0 - b_1, \\ \dfrac{1}{2} + \dfrac{1}{2}\dfrac{r + b_1 - b_0}{b_1 - a_1},\ b_0 - b_1 < r \le b_0 - a_1. \end{cases} \quad (22)$$

$$Cr_{1R} = \begin{cases} \dfrac{1}{2}\dfrac{c_1 - b_1 + r}{c_1 - b_1},\ 0 < r \le c_1 - b_0, \\ \dfrac{1}{2} + \dfrac{1}{2}\dfrac{c_1 - b_0}{c_1 - b_1},\ c_1 - b_0 < r \le c_1 - b_1. \end{cases} \quad (23)$$

$$POS_{2L} = \frac{b_0 - a_2}{b_2 - a_2},\ 0 \le r \le b_0 - a_2. \quad (24)$$

$$NEC_{2L} = \frac{b_2 - b_0 + r}{b_2 - a_2},\ 0 \le r \le b_0 - a_2. \quad (25)$$

$$POS_{2R} = \begin{cases} \dfrac{r + b_0 - a_2}{b_2 - a_2},\ 0 \le r \le b_2 - b_0, \\ 1, b_2 - b_0 < r \le c_2 - b_0. \end{cases} \quad (26)$$

$$NEC_{2R} = \begin{cases} 0,\ 0 < r \le b_2 - b_0, \\ \dfrac{r + c_2 - b_2}{c_2 - b_2},\ b_2 - b_0 < r \le c_2 - b_0 \end{cases} \quad (27)$$

$$Cr_{2L} = \frac{1}{2} + \frac{1}{2}\frac{r}{b_2 - a_2},\ 0 \le r \le b_0 - a_2. \quad (28)$$

$$Cr_{2R} = \begin{cases} \dfrac{1}{2}\dfrac{r + b_0 - a_2}{b_2 - a_2},\ 0 \le r \le b_2 - b_0, \\ 1 - \dfrac{1}{2}\dfrac{c_2 - b_0 - r}{c_2 - b_2},\ b_2 - b_0 < r \le c_2 - b_0. \end{cases} \quad (29)$$

Then the expected value of $\mid \xi_1 - b_0 \mid$ and $\mid \xi_2 - b_0 \mid$ could be calculated since the credibility function is self-dual and by the **Definitions 2.2**,

$$\begin{aligned} \mathbf{E}[|\xi_1 - b_0|] =& (b_0 - b_1) + \frac{(b_0 - b_1)^2}{2(c_1 - b_1)} + \frac{1}{4}(b_1 - a_1) \\ &+ \frac{1}{2}(c_1 - b_0) - \frac{(c_1 - b_0)^2}{4(c_1 - b_1)}, \end{aligned} \quad (30)$$

$$\mathbf{E}[|\xi_2 - b_0|] = \frac{1}{4}(2b_0 - a_2 - 2b_2 + c_2). \quad (31)$$

Assuming a special condition $b_1 = b_2$ which means these two security holds the same expected return. Then it's obvious $b_0 = b_1 = b_2$. Since ξ_1 and ξ_2 are symmetric, we could assume $c_1 - b_1 = b_1 - a_1 = \Delta_1$, $c_2 - b_2 = b_2 - a_2 = \Delta_2$, $\Delta_2 \ge \Delta_1$. Then the absolute deviation of the portfolio could be denoted as follow:

$$\mathbf{A}[\sum_{i=1}^{2} \xi_i] = \frac{1}{2}(x_1\Delta_1 + x_2\Delta_2). \quad (32)$$

Then the mean-absolute deviation of this portfolio could be denoted as follow:

$$\mathbf{E}[\sum_{i=1}^{2} \xi_i] - \rho\mathbf{A}[\sum_{i=1}^{2} \xi_i] = b_0 - \frac{1}{2}\rho(x_1\Delta_1 + x_2\Delta_2). \quad (33)$$

Since ρ is a predefined constant, b_0 is also a constant in this example, and $\Delta_2 \ge \Delta_1$. Obviously this function will maximize x_1 while minimize x_2, which means increase the investment proportion of the first security while decrease the other. The first security is more conservative than the second one, and it may yield lower potential returns ($c_1 \le c_2$) and involve less risk ($a_1 \ge a_2$) compared to the second one. Thus we can draw the conclusion that the mean-absolute deviation function is a risk-sensitive function, which will drive the model to be conservative.

V. Solution algorithm

In this section, we introduce fuzzy simulation and ACPSO proposed in this study and apply them to obtain the optimal solution for the complex optimization problems encountered in this research.

A. Fuzzy simulation

In **Sec**. II, we appoint the definitions of the expected value, prospect value and absolute deviation of a fuzzy variable, elucidating the methodologies employed for their calculation when applied to individual fuzzy variables. But it is imperative to acknowledge that, given the inter-dependencies among securities within portfolio, the conventional methods outlined above cease to be applicable. To resolve this inherent challenge, Liu and Iwamura [41], as well as Liu and Liu [42] proffer a discertization technique termed "fuzzy simulation". In this study, we employ the fuzzy simulation approach to approximate the expected value, prospect value and absolute deviation, effectively circumventing the constraints posed by correlated securities within the portfolio scheme. The procedures of fuzzy simulation were summarized as below.

The portfolio scheme is represented by $(x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n)$. κ_i^{L} and κ_i^{U} are the lower and upper supports, respectively of continuous fuzzy variable ξ_i. ξ_i is divided into l parts and then its membership function is approximated by a discrete fuzzy vector ς_i^r which could be calculated by:

$$\varsigma_i^r = \kappa_i^{\mathrm{L}} + \frac{l}{r}(\kappa_i^{\mathrm{U}} - \kappa_i^{\mathrm{L}}) \tag{34}$$

$l, r \in \mathbb{Z}$, $0 \leq r \leq l$.

Then the fuzzy portfolio selection scheme could be simulated by these discrete vectors. On the basis of the approximated membership function of the scheme and combined with preliminary knowledge about fuzzy sets theory, the credibility function could be could be calculated. Finally, the expected value, prospect value and absolute deviation of the fuzzy portfolio selection scheme could be obtained by approximation. It should be noted that the larger of the l value, the more precise result could we obtain. You may refer to [43] for more information.

B. Adaptive cooperative particle swarm algorithm

PSO was inspired by swarm behaviors in flocks of birds. Suppose N particles are adopted to find the optimal solution of a D-dimensional optimization problem and each particle in the swarm is regarded as a member in flocks. Each particle contains a D-dimensional position vector which is a feasible solution of the optimization problem and a D-dimensional velocity vector, e.g., particle i contains position vector $P_i = [p_{i,1}, p_{i,2}, \cdots, p_{i,D}]$ and velocity vector $V_i = [v_{i,1}, v_{i,2}, \cdots, v_{i,D}]$.

PSO stores the best position of each particle ever experienced named as personal best, e.g., personal best position of particle i is $Pbest_i = [pb_{i,1}, pb_{i,2}, \cdots, pb_{i,D}]$ and its fitness value is $Pvalue_i$. The best position among all personal best position is named as global best position $Gbest = [g_1, g_2, \cdots, g_D]$, and its fitness value is $Gvalue$. Position and velocity vectors of all particles are randomly initialized and updated by iteration rules in **Eq**. (35).

$$\begin{aligned} &v_{i,k} \leftarrow \omega \cdot v_{i,k} + C_1 \cdot r_1 \cdot (pb_{i,k} - p_{i,k}) + C_2 \cdot r_2 \cdot (g_k - p_{i,k}), \\ &p_{i,k} \leftarrow p_{i,k} + v_{i,k}. \end{aligned} \tag{35}$$

Eq. (35) is the k-th dimensional velocity and position updating rules of particle i. The inertia weight ω is commonly set around 0.5, learning rates C_1 and C_2 are commonly set to 2, r_1 and r_2 are two random seeds within the range of $(0, 1)$.

Wang et al. [18] proposed an improved cooperative PSO (ICPSO) algorithm to deal with the issue "curse of dimensionality" of PSO while facing large-scale optimization problems. They conducted a computational experiment on 8 test functions against other heuristic algorithms, including PSO [44], improved multi-objective PSO [12], adaptive granularity learning distributed PSO [45]. Based on the presented comprehensive experimental outcomes, ICPSO outperforms all other comparison algorithms across all chosen test fitness functions. The ICPSO algorithm demonstrates remarkable capabilities in swiftly converging and identifying precise and reliable optimization solutions. Experiment results also indicate that when the problem dimension of the test function increased to approximately 1000, ICPSO faced challenges in converging to the optimal solution.

In order to address the issue of ICPSO's inability to accurately converge to the optimal solution when facing high-dimensional problems, we introduce the adaptive strategy into ICPSO. This modified version is referred to ACPSO, which is proposed in this research. ACPSO is proposed by combining time-varying acceleration coefficient strategy and cooperative strategy. Specifically, the time-varying acceleration coefficient strategy refers to adopting deep deterministic policy gradient (DDPG) [46] to determine the optimal serial values of C_1 and C_2 in the iteration process. DDPG has demonstrated its capacity to acquire competitive policies for all tasks in our repertoire, leveraging low-dimensional observations while maintaining uniform hyper-parameters and network architecture across the board. Notably, across several instances, DDPG has consistently exhibited its ability to acquire proficient policies for various tasks while upholding constancy in hyper-parameter settings and network structure [47]. Therefore, it is precisely based on the success of DDPG in finding optimal policy over continuous action spaces, this study employ it to search for the optimal serial values of C_1 and C_2. The cooperative strategy refers to dividing the raw particle swarm into several sub-swarms. Each sub-swarm iterates independent and identically while information exchanging still exists among sub-swarms.

Suppose that the objective of the ACPSO algorithm is to find the minimum of a fitness function denoted by $\mathcal{F}(\cdot)$. The swarm comprises N particles and is partitioned into M sub-swarms, with each sub-swarm containing N/M particles. It should be noted that if $N\%M \neq 0$ (% represents the modulo operation), then the first through $(N-1)_{th}$ sub-swarm will contain $(N - N\%M)/(M-1)$ particles, while the last sub-swarm will contain $N\%M$ particles.

ACPSO stores the best position ever achieved by each sub-swarm, denoted as $Sbest_m = (sb_{m,1}, sb_{m,1}, \cdots, sb_{m,D})$ and its fitness value is $Svalue_m$, where $m = 1, 2 \cdots, M$. This position corresponds to the minimum fitness value among all the particles that have been a part of the m_{th} sub-swarm.

Suppose ACPSO undergoes T iterations to obtain the optimal solution In the t_{th} iteration, if particle i in the m_{th}

sub-swarm is different from $Sbest_m$, its velocity and position can be updated as follows:

$$v_{i,k} \leftarrow \omega \cdot v_{i,k} + C_1^{(t)} \cdot (pb_{i,k} - p_{i,k}) + C_2^{(t)} \cdot (sb_k - p_{i,k}),$$
$$p_{i,k} \leftarrow p_{i,k} + v_{i,k}. \tag{36}$$

And if particle i is same with $Sbest_m$, its velocity and position can be updated as follows:

$$\begin{aligned} v_{i,k} &\leftarrow \tau * v_{i,k} + g_k \\ p_{i,k} &\leftarrow v_{i,k} + p_{i,k} \end{aligned} \tag{37}$$

It should be noted that the values of C_1 and C_2 in ACPSO are determined by DDPG and form a list whose length equals to the iteration times T. C_1^t and C_2^t are the learning rates in the t_{th} iteration. The procedures of adopting DDPG to determine the optimal serial of C_1 and C_2 are listed in **Alg.** (1).

The motivation behind adopting DDPG to determine the learning rates C_1 and C_2 over time is to balance the exploration and exploitation abilities of PSO based on the current iteration status. The input parameter for **Alg.** (1) include a benchmark fitness value function $F_b(\cdot)$, searching space dimension D_b, particle number N_b and iteration time T_b. The optimal serial of values for C_1 and C_2 are stored in the output variable a_t.

Alg. (2) lists the main procedures of ACPSO. The input parameter for **Alg.** (2) includes the objective fitness value function $F_o(\cdot)$, searing space dimension D_o, particle number N_o, sub-swarm number M, acceleration coefficient in **Eq.** (37), optimal serial of values for C_1 and C_2 determined by **Alg.** (1), iteration time T_o and constraints. The optimal fitness value $Gvalue$ and position $Gbest$ are stored in the output variable which is the optimal solution to the objective fitness value function.

The cooperative strategy, and the velocity and position updating strategy which refers to **Eq**. (37), has been proven effective in ICPSO, and ACPSO introduces time-varying acceleration strategy based on this. The novelty of ACPSO lies in the optimal serial values of C_1 and C_2 based on the characteristic of ICSPO using DDPG, rather than relying on manually specified adaptive strategy as in other PSO algorithms [48].

VI. Numerical Examples

In this section, we will use one case to test the effectiveness of the ACPSO algorithm and another case to test the effectiveness of the proposed PT-FPS.

A. Case of ACPSO on test functions

In this section, we conduct a computational experiment to test the effectiveness of adopting adaptive strategy into PSO. In this experiment, 8 fitness functions [49] were selected as test optimization problems, as listed in **Tab**. (II). Functions f_1, f_2, f_4-f_6 are unimodal with distinct structures. Function f_3 is a noisy sextic function. Function f_7 is a step function which has one minimum and is discontinuous. Function f_8 is dimension sensitive quartic function.

The first step of this experiment is to find the optimal serial values of learning rates C_1 and C_2 by **Alg.** (1). The benchmark

Algorithm 1 Updating learning rates by DDPG.

Input: $F_b(\cdot)$, D_b, N_b, T_b

Randomly initialize actor and critic network $\mu(s|\theta^\mu)$, $Q(s,a|\theta^Q)$ with weights θ^μ, θ^Q
Initialize target network μ' and Q' with weights $\theta^{\mu'} \leftarrow \theta^\mu, \theta^{Q'} \leftarrow \theta^Q$
Initialize replay buffer D_b
Initialize OU noise process $\mathcal{M}$
for episode$=1$ to E, **do**
  for $i=1$ to N_b, **do**
    Randomly generate position P_i and velocity V_i
    $Pvalue_i \leftarrow F(P_i)$
    $Pbest_i \leftarrow P_i$
  end for
  $Gvalue \leftarrow min\{Pvalue_i, i=1,2,\cdots,N_b\}$
  $Gbest \leftarrow \underset{Pbest_i}{argmin}\{F_b(Pbest_i), i=1,2,\cdots,N_b\}$
  for $t=1$ to T_b, **do**
    State $s_t \leftarrow Pbest$
    Action $a_t \leftarrow \mu(s_t|\theta^\mu) + \mathcal{M}_t$
    for $j=1$ to N_b, **do**
      Iterate V_j and P_j by rules in **Eq**. (38)

$$\begin{aligned} v_{j,k} &= \omega \cdot v_{j,k} + C_1 \cdot (pb_{j,k} - p_{j,k}) \\ &\quad + C_2 \cdot (g_k - p_{j,k}), \\ p_{j,k} &= p_{j,k} + v_{j,k}. \end{aligned} \tag{38}$$

      if $F_b(P_j) < Pvalue_j$ **then**
        $Pbest_j = P_j$
        $Pvalue_j = F_b(P_j)$
      end if
    end for
    $Gvalue \leftarrow min\{Pvalue_j, i=1,2,\cdots,N_b\}$
    $Gbest \leftarrow \underset{Pbest_i}{argmin}\{F_b(Pbest_i), i=1,\cdots,N_b\}$
    Reward $r_t \leftarrow -\Delta(Gvalue + \frac{1}{N}\sum_{i=1}^{N_b} Pvalue_i)$, and scale r to $[-0.1, 0.1]$
    $s_{t+1} \leftarrow Pbest$
    Store experience (s_t, a_t, r_t, s_{t+1}) into D_b
    Randomly sample a minibatch of n experiences (s_i, a_i, r_i, s_{i+1}) from D_b
    $y_i \leftarrow r_i + \gamma Q'(s_{i+1}, \mu'(s_{i+1}|\theta^{\mu'})|\theta^{Q'})$
    Update the critic network by minimizing the loss:

$$L = \frac{1}{n}\sum_i (y_i - Q(s_i, ai|\theta^Q))^2 \tag{39}$$

    Update the actor network by using the sampled policy gradient:

$$\nabla_{\theta^\mu} J \approx \frac{1}{N_b}\sum_i^{N_b} \nabla_a Q(s,a|\theta^Q)|_{s=s_i,a=\mu(s_i)} \nabla_{\theta^\mu}\mu(s|\theta^\mu)|_{s_i} \tag{40}$$

    Update the target networks delayed and softly:

$$\begin{aligned} \theta^{Q'} &\leftarrow \tau\theta^Q + (1-\tau)\theta^{Q'} \\ \theta^{\mu'} &\leftarrow \tau\theta^\mu + (1-\tau)\theta^{\mu'} \end{aligned} \tag{41}$$

  end for
end for

Output: a_t

Algorithm 2 Adaptive Cooperative Particle Swarm Optimization algorithm

Input: $F_o(\cdot)$, N_o, M, D_o, τ, C_1, C_2, T_o, constraints
for $i = 1$ to N **do**
  repeat
    generate random P_i
  until P_i satisfies all the constraints
  $Pvalue_i = J(P_i)$
  $Pbest_i = P_i$
end for
for $m = 1$ to M **do**
  $Svalue_m = \min\{Pvalue_i, i \text{ in } m_{\text{th}} \text{ sub} - \text{swarm}\}$
  $Sbest_m = \underset{Pbest_i}{\arg\min}\{J(Pbest_i), i \text{ in } m_{\text{th}} \text{ sub} - \text{swarm}\}$
end for
$Gvalue = \min\{Pvalue_i, i = 1, 2, \cdots N_o\}$
$Gbest = \underset{Pbest_i}{\arg\min}(J(Pbest_i), i = 1, 2, \cdots N_o)$
for $t = 1$ to T_o **do**
  for $i = 1$ to N_o **do**
    if P_i and $Sbest_m$ are different, **then**
      repeat
        Iterate P_i using **Eq.** (36)
      until P_i satisfies all the constraints
    else
      repeat
        Iterate P_i using **Eq.** (37)
      until P_i satisfies all the constraints
    end if
    if $F_o(P_i) < Pvalue_i$ **then**
      $Pbest_i = P_i$
      $Pvalue_i = F_o(P_i)$
    end if
    if $F_o(P_i) < Svalue_m$ **then**
      $Sbest_m = P_i$
      $Svalue_m = F_o(P_i)$
    end if
    if $F_o(P_i) < Gvalue$ **then**
      $Gbest = P_i$
      $Gvalue = F_o(P_i)$
    end if
  end for
end for
Output: $Gvalue$, $Gbest$

TABLE II: Test fitness functions

$f_1(x) = \sum_{i=1}^{D} x_i^2$	$f_2(x) = (\sum_{i=1}^{D} x_i^2)^2$
$f_3(x) = \sum_{i=1}^{D} x_i^6(2 + \sin\frac{1}{x_i})$	$f_4(x) = -20\exp(-0.2\sqrt{\frac{1}{D}\sum_{i=1}^{D} x_i^2})$
$f_5(x) = \sum_{i=1}^{D} \lvert x_i\rvert + \prod_{i=1}^{D} \lvert x_i\rvert$	$f_6(x) = \sum_{i=1}^{D}\sum_{j=1}^{i} x_j^2$
$f_7(x) = \sum_{i=1}^{D} (\lfloor x_i + 0.5\rfloor)^2$	$f_8(x) = \sum_{i=1}^{D} i x_i^4$

function for **Alg.** (1) was selected as $f_1(x) = \sum_{i=1}^{D} x_i^2$, $D = 100$. The iteration time T_b was set as 1000, it should be noted that the iteration time T_o in **Alg.** (2) and in ICPSO were also set as 1000. After executing **Alg.** (1) 1000 times, the mean values of optimal serial C_1 and C_2 were depicted in **Fig.** (3).

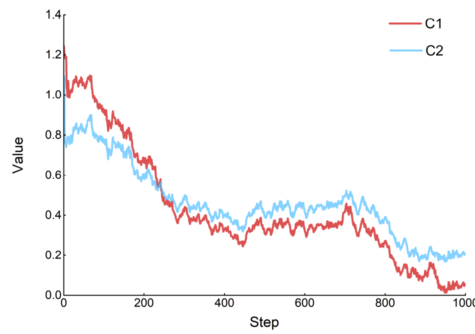


Fig. 3: Optimal serial values of learning rates.

It could be observed from **Fig.** (3) that learning rates C_1 and C_2 decrease in the iteration process. In previous research, it has also been demonstrated that employing larger learning rates during the early stages of iteration in PSO can enhance the algorithm's exploration capability, while using smaller learning rates in the later stages of iteration can strengthen its exploitation capability. The optimal serial values of C_1 and C_2 are contained in the input parameters of **Alg.** (2).

The setting of parameter values significantly impacts this experimental results. Therefore, to ensure the fairness in the experiments, the common parameters of ICPSO and ACPSO have been set to the same values, and certain unique parameters have also been set to the optimal performance values of the respective algorithms. All parameter values are listed in **Tab.** (III).

TABLE III: Parameter settings for computational study.

Parameter	Description	Value
τ	Velocity coefficient in **Eq.**(37)	0.5
ω	Inertia weight	0.5
N	Population size	1000
T_o	Iteration time	1000
M	Sub-swarm number	10
D	Problem dimension	1000
Ss	Searching space	$[-10, 10]^{1000}*$
C_1	Learning weight in ICPSO	2
C_2	Learning weight in ICPSO	2

*:Except for function f_3 and f_8 whose searching space is $[-1.28, 1.28]^{60}$

To assess the performance of ICPSO and ACPSO, each algorithm will solve the minimum value of each selected test fitness function 100 times, and the experiments results were recorded in **Tab.** (IV) and **Tab.** (V). It should be noted the theoretical minimal value for the first to 8th selected test fitness function are all 0.

From the **Tab.** (IV), it can be observed that ICPSO achieves results very close to the theoretical optimal value of 0 for most of the test functions, demonstrating the effectiveness of ICPSO in solving optimization problems. But for the 4th test function, the result obtained in the worst scenario is almost considered unusable. Additionally the results obtained for the

TABLE IV: Test results of ICPSO.

No.	Best	Worst	Median	Mean	Std*
1	3.62E-37	2.58E-15	6.78E-28	2.90E-16	8.11E-16
2	7.46E-77	1.33E-38	1.31E-60	1.13E-39	3.57E-39
3	1.23E-116	6.93E-68	1.38E-97	6.97E-69	2.19E-68
4	4.44E-16	6.81	4.00E-15	0.68	2.15
5	7.36E-21	2.96E-10	1.74E-14	3.16E-11	9.31E-11
6	497.54	498.96	498.79	498.55	0.52
7	0	0	0	0	0
8	4.29E-88	5.36E-59	5.20E-66	5.36E-60	1.69E-59

Std: Standard deviation

TABLE V: Test results of ACPSO.

No.	Best	Worst	Median	Mean	Std
1	4.88E-49	3.87E-41	1.08E-44	4.4E-42	1.22E-41
2	1.26E-97	7.45E-84	3.30E-91	7.50E-85	2.35E-84
3	3.99E-155	6.42E-136	9.40E-145	6.42E-137	2.03E-136
4	7.33E-9	4.50E-5	3.56E-7	1.71E-5	2.144E-5
5	4.27E-25	1.30E-20	8.82E-23	1.69E-21	4.25E-21
6	8.94E-49	3.71E-40	5.29E-45	3.72E-41	1.17E-40
7	0	0	0	0	0
8	1.50E-90	4.71E-84	2.25E-86	6.62E-85	1.49E-84

6th test function deviate significantly from the theoretical optimal value, making them also unusable. These limitations highlight the constraint of ICPSO.

The obtained results show low standard deviations, indicating the stability of ICPSO. We speculate that this may be attributed to the algorithm's structure and a relatively large population size of the particle swarm.

It can be observed from **Tab.** (V) that ACPSO achieves results very close to the theoretical value 0 for all test functions and scenarios. These results are considered acceptable solutions. All solutions obtained by ACPSO exhibit higher accuracy compared to ICPSO. From the most critical metric, the median value, we can conclude that ACPSO consistently outperforms ICPSO across all test functions. For certain test functions, such as the 3th, where the results of ICPSO is 6.37E-69 and the result of ACPSO is 6.42E-137, although the difference is small, it is still significant. As a general-purpose solution algorithm applicable to many optimization problems, obtaining solutions with higher precision is valuable and meaningful in many certain contexts, also for PSO research [50], such as feature selection [51], wireless communications [52] and image processing [53]. Therefore, it can be considered that ACPSO is a more accurate and efficient optimization algorithm than ICPSO. Utilizing DDPG to obtain the optimal learning rates serial values within the PSO proves to be an effective strategy for enhancing the algorithm's solving performance.

Another important metric to evaluate the performance of the algorithm is the convergence speed. To test the convergence performance of these algorithms, we recorded the average number of iterations required for the algorithm to reach the precision levels ($100,\ 1, 10^{-2},\ 10^{-4},\ 10^{-6}$) for all solutions obtained. These results are presented in **Tab.** (VI) to **Tab.** (IX) below.

From **Tab.** (VI) and **Tab.** (VII), it can be observed that ICPSO take less iteration time to convergence to low precision results (100, 1) compared to ACPSO. But in most cases, ACPSO converges to higher-precision results (10^{-6}) in less iteration time than ICPSO. This phenomenon may be attributed to maintaining higher learning rates, which prevents ICPSO's exploratory nature from being weakened but also hinders the exploitation process, making it difficult for the algorithm to achieve higher precision results.

TABLE VI: Convergence speed of ICPSO.

No.	100	1	10^{-2}	10^{-4}	10^{-6}
1	187	239	383	466	558
2	203	224	350	358	415
3	140	185	203	227	297
4	1	175	NA	NA	NA
5	27	136	257	513	613
6	NA	NA	NA	NA	NA
7	178	255	255	255	255
8	167	170	205	249	258

TABLE VII: Convergence speed of ACPSO.

No.	100	1	10^{-2}	10^{-4}	10^{-6}
1	325	344	368	384	408
2	337	345	362	369	378
3	12	284	296	303	311
4	1	639	686	745	NA
5	273	339	382	414	449
6	311	335	361	383	399
7	330	350	350	350	350
8	350	358	366	379	393

From **Tab**. (VIII) and **Tab**. (IX), regarding the standard deviation of convergence speed, we can observe that ACPSO outperforms ICPSO on most test functions. This indicates that ACPSO is able to find the optimal solution more quickly and more stably in most cases. However, in the case of the 5th test function and 3rd test function converging to 100, ACPSO performs worse than ICPSO. We believe this is due to the inherent randomness of the PSO algorithm.

In conclusion, from this experiment it can be inferred that, in the iterative process of PSO, the use of higher learning rates in the early stage can enhance the exploratory capabilities of the algorithm and accelerate convergence. On the other hand, utilizing lower learning rates in the later stage can enhance the algorithm's exploitation capabilities and improve the precision of the solutions obtained.

This experiment has demonstrated that using DDPG can yield optimal learning rate serial values, which in turn enables the ACPSO to converge faster to higher precision solutions. Considering the complexity of the mathematical model established in this study and the scale of the variables used, we believe that the use of ACPSO for model optimization is reliable. Therefore, in the subsequent experiments, we will continue using ACPSO for the optimization tasks.

TABLE VIII: Standard deviation of ICPSO convergence speed.

No.	100	1	10^{-2}	10^{-4}	10^{-6}
1	51.307	69.779	107.444	103.224	113.657
2	49.191	64.192	82.592	86.793	87.654
3	35.129	55.644	56.447	74.038	71.105
4	0	NA	NA	NA	NA
5	18.139	35.548	50.658	82.591	116.061
6	NA	NA	NA	NA	NA
7	43.129	59.672	59.672	59.672	59.672
8	42.441	34.452	37.813	44.187	42.499

TABLE IX: Standard deviation of ACPSO convergence speed.

No.	100	1	10^{-2}	10^{-4}	10^{-6}
1	13.209	9.853	9.727	10.352	12.977
2	12.251	10.176	7.997	8.146	9.166
3	113.754	10.143	10.653	10.364	8.848
4	0	28.873	43.378	49.799	23.081
5	220.164	204.637	190.612	178.979	168.963
6	12.345	13.060	13.501	12.139	10.956
7	18.990	18.969	18.969	18.969	18.969
8	11.766	10.719	11.528	11.960	11.898

B. Ablation study on real market data

Problem description: In this section, we will use a case study based on real market data to discuss the possible effectiveness of the proposed prospect theory based fuzzy portfolio selection model. This case discusses impacts of the following settings on portfolio selection: the impact of setting two reference points compared to setting one reference point, the impact of uncertain intervals compared to fixed values for the reference points, and the impact of different values of the parameter ρ, L_1 and L_2 in **Eq**. (10) and **Eq**. (11). In this case, considering a portfolio selection problem includes 10 stocks selected from Shanghai Stock Exchange (SSE) 50 Index. Experts carefully examined the publicly financial reports of these companies, along with historical stock prices and other relevant economic data, to provide price prediction for these stocks over the next three months in the future, i.e. from October 1, 2022 to December 31, 2022.

The stock selection was based on the following criteria to ensure rigor, representativeness and comparability [54], [55]. 1) Industry representativeness. Stocks from different industries to ensure diversification and avoid sector-specific systematic risk. 2) Market capitalization. Stocks with at least medium capitalization to ensure stability. 3) Liquidity. Stocks with well trading volumes to make the portfolio model practically tradable. 4) Data availability and completeness. Stocks must have complete historical data for proper modeling and validation.

The detailed price prediction processes are listed as follow [56]. Step 1. Collect historical data for the selected stocks over the past 12 months. Step 2. Experts were invited to predict expected returns for the given time period. Step 3. Each expert provides return predictions to the selected stocks in the form of triangular fuzzy variable $\xi = (a, b, c)$ where a is the minimum possible return, b is the most likely return and c is the maximum possible return. Step 4. Aggregating all expert assessments into a singular fuzzy variable for each stocks.

The return on each security is predicted in the form of a triangular fuzzy variable for two main reasons. The first is triangular fuzzy variable could effectively balance between simplicity and expressiveness. And the other is the triangular structure is consistency with the proposed double reference point prospect theory framework, in which (a, b, c) correspond to the minimum requirement, expected return and goal, respectively.

Supposing the closing price of security i at Sept.30, 2022 is p_{ci}, the predicted price during the selected time window is represented as fuzzy variable p_i, the dividend is d_i. So the return rate of security i from Oct.1, 2022 to Dec.31, 2022 is modeled as fuzzy variable $\xi_i = \frac{p_i+d_i}{p_{ci}}$. **Tab.** (X) lists the selected stocks and their corresponding predicted return rates.

TABLE X: Return rate of selected stocks.

No.	Code	Return Rate	No.	Code	Return Rate
1	600000	(0.85, 1.03, 1.15)	2	600048	(0.83, 1.02, 1.22)
3	600050	(0.77, 1.22, 1.75)	4	600276	(0.72, 1.02, 1.13)
5	600309	(0.89, 1.07, 1.29)	6	600703	(0.91, 1.07, 1.17)
7	600809	(0.87, 1.12, 1.31)	8	601012	(0.84, 1.02, 1.14)
9	601668	(0.91, 1.02, 1.09)	10	603986	(0.87, 1.05, 1.21)

In order to discover the impact of setting two reference points compared to setting one reference point, we propose a one reference point fuzzy portfolio selection model, shown in **Eq**. (42), as the comparison model to the mathematical model described in **Eq**. (10). Also the one reference point fuzzy portfolio selection model shown in **Eq**. (43) is the comparison model to the model described in **Eq**. (11). The prospect values in **Eqs**. (42) and (43) are calculated by **Eq**. (7). Moreover, there are some parameter values needed to be specified in advance, which are listed in **Tab**. (XI), before running the real-market case study.

$$\begin{cases} max\sum_{i=1}^{n} x_i v(\xi_i) \\ s.t.\mathbf{E}\left[\sum_{i=1}^{n} x_i\xi_i\right] - \rho\mathbf{A}\left[\sum_{i=1}^{n} x_i\xi_i\right] \geq L_1 \\ \qquad \rho, x_i \geq 0, i = 0, 1, ..., n \\ \qquad \sum_{i=1}^{n} x_i = 1, i = 0, 1, ..., n. \end{cases} \tag{42}$$

$$\begin{cases} max\mathbf{E}\left[\sum_{i=1}^{n} x_i\xi_i\right] - \rho\mathbf{A}\left[\sum_{i=1}^{n} x_i\xi_i\right] \\ s.t.\sum_{i=1}^{n} x_i v(\xi_i) \geq L_2 \\ \qquad \rho, x_i \geq 0, i = 0, 1, ..., n \\ \qquad \sum_{i=1}^{n} x_i = 1, i = 0, 1, ..., n. \end{cases} \tag{43}$$

TABLE XI: Specified parameter values for case study.

Parameter	Description	Value
α	Curvature parameter for status quo	0.88
β	Curvature parameter for fail	0.88
γ	Curvature parameter for success	0.88
λ	Loss aversion ratio	2.25
η	Seeking pride ratio	2.25
ρ	Coefficient in **Eqs**. (10), (11), (42) and (43)	0.5
L_1	Constraint in **Eqs**. (10) and (42)	1.1
L_2	Constraint in **Eqs**. (11) and (43)	1.1

We specify $\alpha = 0.88$, $\beta = 0.88$ and $\lambda = 2.25$ according to Tversky and Kahneman's [40] findings. Setting $\gamma = 0.88$ and $\eta = 2.25$ is done in order to maintain consistency between the psychological mechanism of investors under two reference points and classical prospect theory, and to maximize the alignment with real investor decision-making scenarios. Setting the value of ρ to be 0.5, L_1 and L_2 to be 1.1 is done in order to ensure feasible solutions for the various investment portfolio models involved in this case.

In order to discover the impact of uncertain intervals compared to fixed values for the reference points, we introduce uncertainty reference points and fixed reference points to fuzzy portfolio selection models in this study. Therefore, there are 8 fuzzy portfolio selection models will be discussed in this case, which are two uncertainty reference points model in **Eq**. (10) named as TUP, two uncertainty reference points model in **Eq**. (11) named as TUE, two fixed reference points model in **Eq**. (10) named as TFP, two fixed reference points model in **Eq**. (11) named as TFE, one uncertainty reference points model in **Eq**. (42) named as OUP, one uncertainty reference points model in **Eq**. (43) named as OUE, one fixed reference points model in **Eq**. (42) named as OFP, one fixed reference point model in **Eq**. (43) named as OFE.

The assignment of reference points should comply with the real investment paradigm, therefore we set minimum requirement as non-loss and the goal as the average return of the selected stocks within the investment period. Therefore the uncertain interval for minimum requirement is specified as a fuzzy triangular variable $(0.985, 1.001, 1.017)$, the fixed value for minimum requirement is specified as 1.000, the uncertain interval for the goal is specified as a fuzzy triangular variable $(1.015, 1.021, 1.032)$ and the fixed value for the goal is specified as 1.021.

The optimal solutions for each model were obtained using ACPSO, which represents the stock position results of each model within the investment period, as listed in **Tab**. (XII).

TABLE XII: Stock position in ablation study.

	1	2	3	4	5	6	7	8	9	10
TUP	0	0	0.767	0	0	0	0.233	0	0	0
TUE	0	0	0.154	0	0.234	0.357	0.255	0	0	0
TFP	0	0	0.645	0	0	0	0.355	0	0	0
TFE	0	0	0.132	0	0.257	0.418	0.193	0	0	0
OUP	0	0	0.133	0	0	0	0.867	0	0	0
OUE	0	0	0	0	0	0	1	0	0	0
OFP	0	0	0.127	0	0	0	0.873	0	0	0
OFE	0	0	0	0	0	0	1	0	0	0

It can be seen from **Tab**. (XII) that when using two reference points, using the prospect values as the objective function results in a relatively concentrated stock distribution in the investment portfolio. It can also be observed that using two reference points leads to a more diversified investment portfolio compared to using one reference point. Additionally, having uncertain intervals or fixed values for reference points may cause fluctuations in the specific allocations of selected stocks in the investment portfolio. All these stock position results were tested in the real market to obtain their corresponding investment performances. Their performances were assessed by six widely adopted metrics whose definitions and descriptions are listed below. The metric values of these fuzzy portfolio selection models are listed in **Tab**. (XIV). Meanwhile, the metric values of SSE 50 Index are also listed, which serves as an indicator of the market. Within **Tab**. (XIV), the best performances are highlighted in boldface for better illustration.

TABLE XIII: Overview of selected metrics.

Type	Metric	Abbr.	Descriptions
Return	Cumulative wealth	Cw	The most essential metric
	Annual yield	Ay	A popular evaluation
Risk	Maximum drawdown	Md	Downside risk metric
	Volatility	Vol	Comprehensive risk metric
RAROC*	Sharpe ratio	Sr	Evaluates return and risk
	Calmar ratio	Cr	Widely adopted metric

RAROC: risk adjusted return on capital.

From **Tab**. (XIV), it could be observed that all models experienced growth during the selected investment period. This is because the four stocks being invested in also experienced growth, which demonstrates the effectiveness of stock return prediction in **Tab**. (X).

TABLE XIV: Performances under selected metrics in ablation study.

Metric	Return		Risk		RAROC	
	Cw	Ay	Md	Vol	Sr	Cr
SSE50	1.035	0.158	0.112	0.232	0.679	1.406
TUP	**1.311**	**1.245**	0.106	0.243	**5.123**	**11.745**
TUE	1.108	0.434	0.158	**0.148**	2.932	2.747
TFP	1.258	1.032	**0.089**	0.227	4.546	11.596
TFE	1.088	0.352	0.156	0.131	2.687	2.256
OUP	1.241	0.963	0.220	0.321	3.010	4.377
OUE	1.226	0.904	0.251	0.410	2.205	3.602
OFP	1.240	0.960	0.219	0.330	2.909	4.384
OFE	1.226	0.904	0.251	0.410	2.205	3.602

From the risk metrics, MD and Vol, in **Tab**. (XIV), this case suggests that the risk associated with using a single reference point in portfolio selection may be higher than that of using two reference points. This suggests that incorporating two reference points in portfolio selection can effectively reduce investment risk. In **Sec**. IV, we have demonstrated that the objective function in **Eq**. (11) drives the portfolio selection model to be risk-sensitive , which also results in model TUE performing exceptionally well under the risk metric Vol.

Introducing reference point goal into portfolio selection enhances the model's inclination toward pursuing higher returns. Therefore, it can be observed that models incorporating two reference points exhibit stronger performance in both the CW and AY return metrics. Lastly, utilizing reference points within uncertain intervals, as opposed to fixed values, not only aligns the model more closely with investors' decision-making behaviors in uncertain markets but also bolsters the model's performance under the SR and CR risk-adjusted return metrics. Consequently, the performance of the model TUP is optimal under these two metrics.

Hence, based on this case study, introducing two reference points in the portfolio selection model based on prospect theory appears to offer a potential benefit in enhancing the model's ability to pursue higher returns, which could encourage investments in stocks with higher growth potential. Simultaneously, when the reference points encompass uncertain alongside fixed values, the model exhibits heightened resilience against risk. These findings underscore the ability of the portfolio selection model incorporating two uncertain reference points to achieve more robust returns. Consequently, we can assert that the fuzzy portfolio selection model based on prospect theory proposed in this study is indeed effective.

C. Comparison study on real market data

Problem description: This section includes a comparison study based on real market data, discussing the potential impact and role of the fuzzy portfolio selection model proposed in this study in the real market through comparison with three portfolio selection models. In this case, 31 stocks were selected from NYSE and NASDAQ, and the return rates of these stocks from the period January, 4, 2016 to October, 7, 2016 (40 weeks) expressed in fuzzy variables can be obtained from [57].

Comparison models: Buy & Hold (B&H) model is often chosen as a benchmark model due to its ability to intuitively display the average return performance of the stock-pool. Expected utility-CVaR (EU-C). The objective of this model is to maximize the expected return of the portfolio selection scheme while adopts the Conditional Value-at-Risk (CVaR) as the risk constraint. Prospect theory-based mean-variance (PT-MV). Srivastava et al. [58] propose a prospect theory-based portfolio selection model within the framework of mean-variance under uncertainty with ambiguity.

Based on experts' knowledge, the uncertain interval of the reference point minimum requirement was assigned as a triangular variable $(0.992, 1.000, 1.007)$ which is a reflection of the risk-free rate and the average return rate of NYSE and NASDAQ in the past three months. And the uncertain interval of the reference point goal was assigned as a triangular variable $(1.013, 1.051, 1.072)$ which is based on the return rate prediction of the stocks in the pool. Other parameters were adopted the similar assignments in **Tab**. (XI). Similar to the ablation study, ICPSO was adopted to find the optimal solution of the aforementioned portfolio selection models, and these solutions are the stock position of each model which will be tested by real market data. The metric values of these fuzzy portfolio selection models were listed in **Tab**. (XV), the best performances are highlighted in boldface for better illustration.

TABLE XV: Performances under selected metrics in comparison study.

Metric	Return		Risk		RAROC	
	Cw	Ay	Md	Vol	Sr	Cr
TUP	**1.127**	**0.165**	0.116	0.129	**1.279**	**1.422**
B&H	1.018	0.023	**0.073**	**0.074**	0.311	0.311
EU-C	1.101	0.131	0.105	0.173	0.757	1.248
PT-MV	1.077	0.103	0.084	0.122	0.844	1.226

Under the metrics of cumulative wealth and annual yield, B&H model performs the worst, indicating its inferior ability to generate investment returns. This is due to B&H model's strategy of evenly distributing liquid cash across all invest-able stocks without any selection, which only reflects the average return of the stock pool. Therefore, this model is widely chosen as a benchmark in the field of portfolio research. However, due to its overly diversified investment strategy, it performs exceptionally well in terms of volatility. PT-MV model employs prospect theory in characterizing investor behavior paradigms within the mean-variance framework. Mean-variance has been well-observed of risk-concentration, as discussed previously.

The fuzzy portfolio selection model proposed in this study, based on prospect theory with two uncertain interval reference points, takes into account not only the loss aversion effect but also the pursuit of returns by investors, thus exhibiting optimal performance under the metrics of cumulative wealth and annual yield. This demonstrates that the use of a variant of prospect theory with two reference points can effectively enhance the model's ability to generate returns, which is the goal of many investors. In terms of volatility, this model outperforms the PT-MV model, proving that the introduction of prospect theory in portfolio research can effectively reduce investment volatility. Finally, on the Sharpe ratio metric, this model performs optimally. Therefore, we can conclude that while achieving optimal investment returns, this model's ability to control risk has not deteriorated. It effectively balances the achievement of portfolio returns and risk aversion.

To further validate the robustness of the proposed portfolio model under different market conditions, we compare the performance of the SSE 50 Index during Q4 2022 with that of comparison study market over the given time period. As presented in **Tab**. (XIV), the cumulative wealth for the SSE 50 Index is 1.0354, corresponding to an annualized yield of 15.75%, which significantly outperforms the comparison market index with a cumulative wealth of 1.018 and an annualized yield of only 2.30%.

However, this superior return from the SSE 50 Index comes at the cost of higher volatility and downside risk. Specifically, the SSE 50 Index exhibits an annualized volatility of 23.21% and a maximum drawdown of 11.20%, compared to 7.40% and 7.30% respectively for the comparison market index. Despite the increased risk exposure, the SSE 50 Index demonstrates a stronger risk-adjusted performance. The Sharpe ratio and Calmar ratio reach 0.679 and 1.406 respectively, markedly surpassing the comparison's corresponding values of 0.311.

These results suggest that while the SSE 50 Index entails higher market fluctuations, it also delivers superior excess returns per unit of risk. This indicates a more favorable risk-reward trade-off and a higher risk premium embedded in the SSE 50 Index during the observed period. In contrast, the benchmark index exhibits a more stable yet conservative return pattern, making it more suitable for risk-averse investors.

In summary, the two market indices display distinct characteristics in terms of risk and return. The SSE 50 Index, with higher volatility and stronger compensation for risk, provides a suitable testing ground for behavior-based portfolio optimization. The benchmark index, on the other hand, reflects a low-risk, low-return profile aligned with more conservative investment strategies.

VII. Conclusions

Traditional portfolio selection models, often based on expected utility theory, fail to adequately account for behavioral biases such as loss aversion, return chasing, and the psychological lower bounds that influence investor decisions under extreme market conditions. This research seeks to bridge these gaps by proposing new model that better reflect the complexities of real-world investor behavior while providing robust solutions to high-dimensional portfolio optimization problems. This research employs a theory-centric strategy through enhancements to prospect theory to mitigate the research gap. Develop a prospect theory-based multiple reference points model to capture return chasing, addressing dynamic investor behavior and mitigating loss aversion-driven conservatism. Integrate risk constraints into portfolio selection for diversification and enhanced risk control.

Specifically, this study proposes a fuzzy model with two uncertain interval reference points for representing prospect theory. Based on this model, a portfolio selection framework is recommended with the mean-absolute deviation as a constraint. In comparison to existing works, this model closely aligns with real-world investor behavior patterns and demonstrates robust risk resilience. Simultaneously, it amalgamates the advantages of expected utility theory and prospect theory, achieving a balance between risk and return. It does not err on the side of excessive conservatism, avoiding missing opportunities, nor does it become overly aggressive, leading to excessive volatility. Two case study based on real-market data validates the incorporation of two reference points empowers the model to attain higher returns while reference points with uncertain interval bolster its risk resilience to a certain extent.

Furthermore, to obtain optimal solutions for the intricate optimization problems addressed in this study, we introduce adaptive cooperative particle swarm optimization algorithm. This algorithm incorporates adaptive and cooperative strategies, whereby the optimal learning rate sequences is derived via the deep deterministic policy gradient algorithm to achieve the best adaptive strategy. The optimal cooperative strategy is attained by decomposing the entire swarm randomly into multiple sub-swarms. An computational study, based on test fitness functions, verifies the capability of this algorithm to rapidly yield solutions of sufficient precision.

Considering that capital markets are inherently random and subject to various disturbances, implementing adaptive intervals for fuzzy variables which could be derived from machine learning methods, to better address these uncertainties and disruption is an interesting and meaningful direction for prospect theory based portfolio selection research. Meanwhile, drawing on recent advances in the field of fuzzy sets, utilizing more complex and sophisticated membership functions to characterize and represent preferences regarding returns and risks is a valuable approach for portfolio selection. In this study, deep deterministic policy gradient was used to obtain the optimal learning rate sequences. However, due to the inherent randomness of particle swarm optimization algorithm, these optimal sequences may vary depending the characteristics of the input fitness functions and the dimensionality of the searching space. Therefore, designing more efficient methods to obtain the optimal sequences based on the structure of the specific problem will be a meaningful and challenging research direction.

References

[1] H. Markowitz, "Portfolio selection," *The Journal of Finance*, 1952.

[2] B. Li, R. Zhang, and Y. Sun, "Multi-period portfolio selection based on uncertainty theory with bankruptcy control and liquidity," *Automatica*, vol. 147, p. 110751, 2023.

[3] H. Markowitz, P. Todd, G. Xu, and Y. Yamane, "Computation of mean-semivariance efficient sets by the critical line algorithm," *Annals of Operations Research*, vol. 45, pp. 307–317, 1993.

[4] H. Konno and H. Yamazaki, "Mean-absolute deviation portfolio optimization model and its applications to tokyo stock market," *Management Science*, vol. 37, no. 5, pp. 519–531, 1991.

[5] L. Chen, R. Gao, Y. Bian, and H. Di, "Elliptic entropy of uncertain random variables with application to portfolio selection," *Soft Computing*, vol. 25, pp. 1925–1939, 2021.

[6] X. Guo, R. H. Chan, W.-K. Wong, and L. Zhu, "Mean–variance, mean–var, and mean–cvar models for portfolio selection with background risk," *Risk Management*, vol. 21, pp. 73–98, 2019.

[7] V. Dixit and M. K. Tiwari, "Project portfolio selection and scheduling optimization based on risk measure: a conditional value at risk approach," *Annals of Operations Research*, vol. 285, no. 1-2, pp. 9–33, 2020.

[8] L. Adam and M. Branda, "Risk-aversion in data envelopment analysis models with diversification," *Omega*, vol. 102, p. 102338, 2021.

[9] S. Wang and S. Zhu, "On fuzzy portfolio selection problems," *Fuzzy Optimization and Decision Making*, vol. 1, pp. 361–377, 2002.

[10] H. Xu, D. Yu, and Y.-J. Liu, "Observer-based fuzzy adaptive predefined time control for uncertain nonlinear systems with full-state error constraints," *IEEE Transactions on Fuzzy Systems*, vol. 32, no. 3, pp. 1370–1382, 2023.

[11] S. Sui, D. Shen, W. Bi, S. Tong, and C. P. Chen, "Observer-based event-triggered bipartite consensus for nonlinear multi-agent systems: Asymmetric full-state constraints," *IEEE Transactions on Automation Science and Engineering*, pp. 1–11, 2024.

[12] B. Wang, Y. Li, S. Wang, and J. Watada, "A multi-objective portfolio selection model with fuzzy value-at-risk ratio," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 6, pp. 3673–3687, 2018.

[13] C. Li, Y. Wu, Z. Lu, J. Wang, and Y. Hu, "A multiperiod multiobjective portfolio selection model with fuzzy random returns for large scale securities data," *IEEE Transactions on Fuzzy Systems*, vol. 29, no. 1, pp. 59–74, 2020.

[14] J.-H. Syu, J. C.-W. Lin, C.-J. Wu, and J.-M. Ho, "Stock selection system through suitability index and fuzzy-based quantitative characteristics," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 1, pp. 322–334, 2022.

[15] X. Wang, B. Wang, S. Liu, H. Li, T. Wang, and J. Watada, "Fuzzy portfolio selection based on three-way decision and cumulative prospect theory," *International Journal of Machine Learning and Cybernetics*, vol. 13, no. 1, pp. 293–308, 2022.

[16] X. Gong, L. Min, and C. Yu, "Multi-period portfolio selection under the coherent fuzzy environment with dynamic risk-tolerance and expected-return levels," *Applied Soft Computing*, vol. 114, p. 108104, 2022.

[17] H. Jalota, P. K. Mandal, M. Thakur, and G. Mittal, "A novel approach to incorporate investor's preference in fuzzy multi-objective portfolio selection problem using credibility measure," *Expert Systems with Applications*, vol. 212, p. 118583, 2023.

[18] X. Wang, B. Wang, T. Li, H. Li, and J. Watada, "Multi-criteria fuzzy portfolio selection based on three-way decisions and cumulative prospect theory," *Applied Soft Computing*, vol. 134, p. 110033, 2023.

[19] W. Guo, W.-G. Zhang, and X. Chen, "Portfolio selection models considering fuzzy preference relations of decision makers," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2024.

[20] A. Gu and H. I. Yoo, "Prospect theory and mutual fund flows," *Economics Letters*, vol. 201, p. 109776, 2021.

[21] D. Luo, "Ambiguity, risk, and sentiment," *Available at SSRN 4336463*, 2023.

[22] Z. Gao, W. Jiang, W. A. Xiong, and W. Xiong, "Daily momentum and new investors in an emerging stock market," National Bureau of Economic Research, Tech. Rep., 2023.

[23] D. Kahneman and A. Tversky, "Prospect theory: An analysis of decision under risk," in *Handbook of the fundamentals of financial decision making: Part I.* World Scientific, 2013, pp. 99–127.

[24] X. D. He and X. Y. Zhou, "Portfolio choice under cumulative prospect theory: An analytical treatment," *Management Science*, vol. 57, no. 2, pp. 315–331, 2011.

[25] J. Bi, H. Jin, and Q. Meng, "Behavioral mean-variance portfolio selection," *European Journal of Operational Research*, vol. 271, no. 2, pp. 644–663, 2018.

[26] L. Gan, Y. Hu, X. Chen, G. Li, and K. Yu, "Application and outlook of prospect theory applied to bounded rational power system economic decisions," *IEEE Transactions on Industry Applications*, vol. 58, no. 3, pp. 3227–3237, 2022.

[27] X. Zhou, L. Wang, H. Liao, S. Wang, B. Lev, and H. Fujita, "A prospect theory-based group decision approach considering consensus for portfolio selection with hesitant fuzzy information," *Knowledge-Based Systems*, vol. 168, pp. 28–38, 2019.

[28] G. M. Ferro, T. Kovalenko, and D. Sornette, "Quantum decision theory augments rank-dependent expected utility and cumulative prospect theory," *Journal of Economic Psychology*, vol. 86, p. 102417, 2021.

[29] S. Yadav, P. Gupta, M. K. Mehlawat, and A. Kumar, "A multiobjective multiperiod portfolio selection approach with different investor attitudes under an uncertain environment," *Soft Computing*, vol. 28, no. 13, pp. 8013–8050, 2024.

[30] K. Gao, L. Sun, Y. Yang, F. Meng, and X. Qu, "Cumulative prospect theory coupled with multi-attribute decision making for modeling travel behavior," *Transportation Research Part A: Policy and Practice*, vol. 148, pp. 1–21, 2021.

[31] A. Grant, D. Johnstone, and O. K. Kwon, "A cumulative prospect theory explanation of gamblers cashing-out," *Journal of Mathematical Psychology*, vol. 102, p. 102534, 2021.

[32] R. Oprea, "Simplicity equivalents," *Working Paper*, 2022.

[33] J. Chapman, E. Snowberg, S. W. Wang, and C. Camerer, "Looming large or seeming small? attitudes towards losses in a representative sample," National Bureau of Economic Research, Tech. Rep., 2022.

[34] X. T. Wang, "Decision heuristics as predictors of public choice," *Journal of Behavioral Decision Making*, vol. 21, no. 1, pp. 77–89, 2008.

[35] G. J. Koop and J. G. Johnson, "The use of multiple reference points in risky decision making," *Journal of Behavioral Decision Making*, vol. 25, no. 1, pp. 49–62, 2012.

[36] M. Pfiffelmann, T. Roger, and O. Bourachnikova, "When behavioral portfolio theory meets markowitz theory," *Economic Modelling*, vol. 53, pp. 419–435, 2016.

[37] H. Shefrin and M. Statman, "Behavioral portfolio theory," *Journal of Financial and Quantitative Analysis*, vol. 35, no. 2, pp. 127–151, 2000.

[38] D. Kahneman and A. Tversky, "Prospect theory: an analysis of decisions under risk." *Econometrica*, vol. 47, pp. 263–291, 1979.

[39] X. T. Wang and J. G. Johnson, "A tri-reference point theory of decision making under risk." *Journal of Experimental Psychology: General*, vol. 141, no. 4, p. 743, 2012.

[40] A. Tversky and D. Kahneman, "Advances in prospect theory: Cumulative representation of uncertainty," *Journal of Risk and Uncertainty*, vol. 5, pp. 297–323, 1992.

[41] B. Liu and K. Iwamura, "Chance constrained programming with fuzzy parameters," *Fuzzy sets and systems*, vol. 94, no. 2, pp. 227–237, 1998.

[42] B. Liu and Y.-K. Liu, "Expected value of fuzzy variable and fuzzy expected value models," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 4, pp. 445–450, 2002.

[43] S. Wang, J. Watada, and W. Pedrycz, "Value-at-risk-based two-stage fuzzy facility location problems," *IEEE Transactions on Industrial Informatics*, vol. 5, no. 4, pp. 465–482, 2009.

[44] J. Kennedy and R. Eberhart, "Particle swarm optimization," in *Proceedings of the IEEE International Conference on Neural Networks*, vol. 4. IEEE, 1995, pp. 1942–1948.

[45] Z. Wang, Z. Zhan, S. Kwong, H. Jin, and J. Zhang, "Adaptive granularity learning distributed particle swarm optimization for large-scale optimization," *IEEE Transactions on Cybernetics*, vol. 51, no. 3, pp. 1175–1188, 2020.

[46] D. Silver, G. Lever, N. Heess, T. Degris, D. Wierstra, and M. Riedmiller, "Deterministic policy gradient algorithms," in *International Conference on Machine Learning*. Pmlr, 2014, pp. 387–395.

[47] T. P. Lillicrap, J. J. Hunt, A. Pritzel, N. Heess, T. Erez, Y. Tassa, D. Silver, and D. Wierstra, "Continuous control with deep reinforcement learning," *arXiv preprint arXiv:1509.02971*, 2015.

[48] D. Wang, D. Tan, and L. Liu, "Particle swarm optimization algorithm: an overview," *Soft computing*, vol. 22, no. 2, pp. 387–408, 2018.

[49] K. Tang, X. Li, P. N. Suganthan, Z. Yang, and T. Weise, "Benchmark functions for the cec'2010 special session and competition on large-scale global optimization," *Nature Inspired Computation and Applications Laboratory, USTC, China, Tech. Rep*, 2009.

[50] A. G. Gad, "Particle swarm optimization algorithm and its applications: a systematic review," *Archives of computational methods in engineering*, vol. 29, no. 5, pp. 2531–2561, 2022.

[51] Q. Al-Tashi, H. M. Rais, S. J. Abdulkadir, and S. Mirjalili, "Feature selection based on grey wolf optimizer for oil & gas reservoir classification," in *2020 International Conference on Computational Intelligence (ICCI)*. IEEE, 2020, pp. 211–216.

[52] B. Xia, T. Liu, T. Ding, and Z. Wang, "An improved pso localization algorithm for uwb sensor networks," *Wireless Personal Communications*, vol. 117, pp. 2207–2223, 2021.

[53] M. G. Omran, A. P. Engelbrecht, and A. Salman, "Particle swarm optimization for pattern recognition and image processing," in *Swarm intelligence in data mining*. Springer, 2006, pp. 125–151.

[54] M. Kandakoglu, G. Walther, and S. Ben Amor, "The use of multi-criteria decision-making methods in project portfolio selection: a literature review and future research directions," *Annals of Operations Research*, vol. 332, no. 1, pp. 807–830, 2024.

[55] P. K. Mandal and M. Thakur, "Higher-order moments in portfolio selection problems: A comprehensive literature review," *Expert Systems with Applications*, vol. 238, p. 121625, 2024.

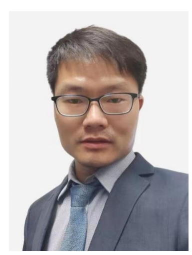
[56] A. Georgantas, M. Doumpos, and C. Zopounidis, "Robust optimization approaches for portfolio selection: a comparative analysis," *Annals of Operations Research*, vol. 339, no. 3, pp. 1205–1221, 2024.

[57] L. Dymova, K. Kaczmarek, and P. Sevastjanov, "A new approach to the bi-criteria multi-period fuzzy portfolio selection," *Knowledge-Based Systems*, vol. 234, p. 107582, 2021.

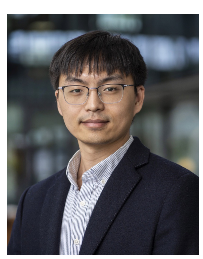
[58] S. Srivastava, A. Aggarwal, and A. Mehra, "Portfolio selection by cumulative prospect theory and its comparison with mean-variance model," *Granular Computing*, vol. 7, no. 4, pp. 903–916, 2022.

Xianhe Wang is currently working toward the Ph.D. degree in systems engineering in systems engineering at Hong Kong Polytechnic University, Hong Kong, China. His research interests include robust optimization in finance and risk management.

Bo Wang (Member, IEEE) received the B.S. degree in software engineering from Southeast University, Nanjing, China, in 2007, and the M.S. and Ph.D. degrees in data science from the Graduate School of Information, Production and Systems, Waseda University, Japan, in 2009 and 2012. He is currently an associate professor with the School of Management and Engineering, Nanjing University, Nanjing, China. He was a Research Assistant of the Global COE Program, Waseda University, Ministry of Education, Culture, Sports, Science and Technology, Japan. He was a Special Research Fellow of the Japan Society for the Promotion of Science (JSPS). He is a Committee Member of Chinese Association of Automation (CAA) Energy Internet Committee, and a Committee Member of Systems Engineering Society of Jiangsu. His research interests include power system planning, renewable generation forecasting, data-driven decision-making, portfolio selection and artificial intelligence algorithms.



Long Teng (Member, IEEE) received the Ph.D. degree in control engineering from Nanyang Technological University, Singapore, in 2018. From August 2017 to September 2018, he worked with Singapore Technology Engineering (Electronics, Satellite Systems), Singapore, and a start-up for rehabilitation robotics, Shenzhen, China. From October 2018 to October 2020, he was a Post-Doctoral Researcher with the Department of Materials and Production, Aalborg University, Aalborg, Denmark. He is currently a Research Assistant Professor with the Department of Industrial and Systems Engineering, The Hong Kong Polytechnic University, Hong Kong. His current research interests include assistive robots, bioinspired robots, control system theory and application to robotics, and machine learning with applications to robotics.



Yaoxin Wu Yaoxin Wu received the Ph.D. degree in computer science from Nanyang Technological University, Singapore, in 2023. In the same year, he joined the Information Systems group within the Department of Industrial Engineering and Innovation Sciences at Eindhoven University of Technology as an Assistant Professor. His research interests include deep learning, combinatorial optimization and integer programming.