

ON THE EQUILIBRIUM STRATEGIES FOR TIME-INCONSISTENT PROBLEMS IN CONTINUOUS TIME*

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Abstract. In a continuous-time setting, the existing notion of equilibrium strategies for time-inconsistent problems in the literature, referred to as weak equilibrium, is not fully aligned with the standard definition of equilibrium in game theory in that the agent may be willing to deviate from a given weak equilibrium strategy. To address this issue, [Y.-J. Huang and Z. Zhou, *Math. Oper. Res.*, 46 (2021), pp. 428–451] propose the notion of strong equilibrium for an infinite-time stochastic control problem in which an agent can control the generator of a time-homogeneous, continuous-time, finite-state Markov chain at each time. We study weak and strong equilibria in a general diffusion framework, provide necessary conditions for a strategy to be a strong equilibrium, and prove that strong equilibrium strategies do not exist for three investment and consumption problems. Finally, we propose a new notion of equilibrium strategies, referred to as regular equilibrium, show that it implies weak equilibrium, provide a sufficient condition under which a weak equilibrium strategy becomes a regular equilibrium, and show that this condition holds for many time-inconsistent problems.

Key words. stochastic control, time-inconsistency, continuous-time setting, equilibrium strategies, portfolio selection

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1. Introduction. When making dynamic decisions, the decision criteria of an agent at different times may not align with each other, leading to time-inconsistent behavior: an action that is optimal under the decision criterion today may no longer be optimal under the decision criterion at a certain future time. The agent's behavior in the presence of time-inconsistency then depends on whether she is aware of the inconsistency and whether she can commit her future selves to a strategy that is optimal under the criterion today, leading to three different types of behavior: naive, precommitted, and equilibrium strategies; see, for instance, [37], [34], and [1]. In particular, an equilibrium strategy is a rational choice of the agent if she is fully aware of time-inconsistency, has no control of her future selves, and thus considers her selves at different times to be different players in a sequential game.

When the agent makes decisions discretely in time, equilibrium strategies are well defined and fully discussed in the literature. More precisely, following the setting in [7], consider an agent who has a planning horizon with a finite end date T and makes decisions at time $t \in \{0, 1, \dots, T-1\}$. The agent's decision drives a Markov state process, and the agent's decision criterion at time t is to maximize an objective function $J(t, x; \mathbf{u})$, where x stands for the Markovian state at that time and $\mathbf{u}$ represents the agent's strategy. The agent considers Markovian strategies, so $\mathbf{u}$ is a function of time $s \in \{0, 1, \dots, T-1\}$ and the Markovian state at that time. A strategy $\hat{\mathbf{u}}$ is an

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equilibrium policy if at any time $t \in \{0, 1, \dots, T-1\}$ and in any state x at that time, any deviation of the agent's self at time t from $\hat{\mathbf{u}}(t, x)$, given that her future selves still follow $\hat{\mathbf{u}}$, is suboptimal, i.e.,

$$(1.1) \quad J(t, x; \mathbf{u}_{t,u}) \leq J(t, x; \hat{\mathbf{u}})$$

for any possible action u the agent's self at time t can take, where $\mathbf{u}_{t,u}(s, x) := u$ for $s = t$ and $\mathbf{u}_{t,u}(s, y) := \hat{\mathbf{u}}(s, y)$ for $s \neq t$ and any state y .¹ As noted in Remark 2.6 of [7], the above game-theoretic definition of equilibrium strategies is consistent with one based on backward induction; for the latter definition, see, for instance, [32], [34], and [1].

Extending the notion of equilibrium strategies from the above discrete-time setting to a continuous-time setting is nontrivial because in the latter, the change of the agent's action at one instant of time does not affect the state process and thus usually has no effect on the agent's objective function either. Formalizing the idea in [37], the authors of [13] assume that the agent's self at each time t can implement her strategy in an infinitesimally small, but positive, time period, e.g., $[t, t + \epsilon)$; as such, her action has an impact on the state process. Later, this idea was extended to a stochastic setting in [17] and [6]. Formally, we can define $\mathbf{u}_{t,\epsilon,\mathbf{a}}(s, y) := \mathbf{a}(s, y)$ for $s \in [t, t + \epsilon)$ and $\mathbf{u}_{t,\epsilon,\mathbf{a}}(s, y) := \hat{\mathbf{u}}(s, y)$ for $s \notin [t, t + \epsilon)$, where $\mathbf{a}$ stands for the strategy that the agent's self at time t chooses to implement in the period $[t, t + \epsilon)$. The authors of [13], [17], and [6] define $\hat{\mathbf{u}}$ to be an equilibrium policy if the following holds for any time $t \in [0, T)$, Markovian state x , and action $\mathbf{a}$:

$$(1.2) \quad \Delta^{\hat{\mathbf{u}}}(t, x; \mathbf{a}) := \limsup_{\epsilon \downarrow 0} \frac{J(t, x; \mathbf{u}_{t,\epsilon,\mathbf{a}}) - J(t, x; \hat{\mathbf{u}})}{\epsilon} \leq 0.$$

As noted in Remark 3.5 of [5], condition (1.2) does not necessarily imply

$$(1.3) \quad J(t, x; \mathbf{u}_{t,\epsilon,\mathbf{a}}) \leq J(t, x; \hat{\mathbf{u}})$$

for sufficiently small ϵ ; the latter condition is a natural definition of equilibrium strategies from the game-theoretic point of view and is a natural counterpart of the condition (1.1) in the discrete-time setting. However, nearly all studies of time-inconsistent problems in the literature still use the first-order condition (1.2) to define equilibrium strategies. The only exception is [29], which considers an infinite-time stochastic control problem in which an agent can control the generator of a time-homogeneous, continuous-time, finite-state Markov chain at each time to maximize the expected running reward in an infinite horizon. The authors exemplify that equilibrium strategies defined through the first-order condition (1.2), named *weak equilibrium*, are not necessarily a *strong equilibrium* that is defined through the direct comparison of objective function values as in (1.3). The authors assume that at each time the agent

¹Here, consistent with most studies on time-inconsistency in the literature, at each time t , the agent's action is contingent on the value of a certain state process, which itself depends on the agent's action in the past. As such, although the agent's future self follows a given strategy $\hat{\mathbf{u}}$, the agent's action u today affects the agent's action in the future indirectly through the value of the state process in the future. In [37], the authors assume that the agent's action at each time is *not* contingent on any state process, and this formulation has been used in [22, 23] as well. Such a formulation may not be desirable in some settings. For instance, in the problem studied by [37], an agent chooses the amount of consumption at each time in a given period, and the sum of the consumption is fixed at a given budget level. In the definition of an equilibrium policy, if the agent's consumption amount today does not affect her consumption amount in the future, the agent today actually does not have any freedom to choose the consumption amount due to the budget constraint.

can implement a time-homogeneous strategy only, provide a characterization of the strong equilibrium strategies, and prove the existence of strong equilibrium under certain conditions.

The framework in [29] cannot be applied to most time-inconsistent problems in the literature, such as the mean-variance portfolio selection problems in [8] and [2] and the optimal consumption-investment problem in [17], for three reasons. First, [29] assumes the state process to be a finite-state Markov chain, while the state processes in the above problems are diffusions. Second, the objective function in [29], namely the expected running reward, cannot account for mean-variance criteria in the problems in [8] and [2]. Third, [29] considers the infinite-time setting and thus time-homogeneous control strategies only, but in all of the above problems, the time horizon is finite, and the agents in those problems can take time-inhomogeneous strategies. To address the above issues, in the present paper we consider a diffusion framework and study weak and strong equilibria in this framework. This framework is sufficiently general to include the problems in [8], [2], and [17] as well as an optimal consumption problem with a bequest as special examples; see section 2.3.

Therefore, the issues of the possibility of an agent deviating from a weak equilibrium and whether a strong equilibrium exists for time-inconsistent problems in diffusion settings, albeit economically fundamental and important, remain unsolved. In this work, we provide an answer. To this end, we first complement the existing literature by providing a rigorous treatment of weak equilibrium. In particular, we highlight two features in the definition of weak equilibrium that are largely overlooked in the literature. The first is the set, denoted as $\mathbf{D}$, of alternatives $\mathbf{a}$ that the agent's self at time t can implement in the period $[t, t + \epsilon)$. The existing studies on time-inconsistent problems do not agree on the choice of the set $\mathbf{D}$: some assume it to be the set of all constant strategies, and others assume it to include nonconstant strategies as well. It is, however, unclear whether the choice of $\mathbf{D}$ has any real impact on determining whether a strategy is an equilibrium. The second is the set of states in which the actions of the agent's future selves are relevant to the agent's self today. The studies in the literature assume that the agent today is concerned about the actions of her future selves in *all* states, so a strategy is an equilibrium only if she will not deviate from it at all times and in all states. We argue that the agent's self today is not concerned about whether her future selves deviate from a given strategy in the states that cannot be reached under this strategy. Thus, in defining an equilibrium strategy, we demand the agent at each time not deviate from this strategy in reachable (not all) states. See the detailed discussion in section 3.1. After formalizing the definition of weak equilibrium, we prove a sufficient and necessary condition for a strategy to be a weak equilibrium and show that whether a strategy is a weak equilibrium does not depend on the choice of $\mathbf{D}$. We further apply this condition to find certain weak equilibrium strategies for the problem of optimal consumption with a bequest and to rigorously verify that the strategies found in [2] and [17] for the problems studied therein are indeed weak equilibrium.

We then prove two necessary conditions for a strategy with certain regularity to be a strong equilibrium, assuming that the set $\mathbf{D}$ of alternative strategies that the agent at each time can switch to includes nonconstant strategies. Using these two conditions, we prove that the problems studied in [2] and [17] and the optimal consumption problem with a bequest do *not* have strong equilibrium strategies that are smooth to a certain degree (i.e., either four-times or six-times differentiable in the state variable). When $\mathbf{D}$ contains constant strategies only, we show that the particular weak equilibrium strategies derived in the literature and in the present paper for the

problem in [17] and the optimal consumption problem with a bequest are not strong equilibria. The particular weak equilibrium derived in [2] for the problem therein, however, is indeed a strong equilibrium when $\mathbf{D}$ contains constant strategies only. This shows that whether a strategy is a strong equilibrium depends on the choice of $\mathbf{D}$, the set of alternative strategies, contrasting with the finding for weak equilibrium.

The above nonexistence result is unexpected because it contrasts with the finding in [29] that strong equilibrium exists for an infinite-time stochastic control problem with a finite-state Markov chain. Note that in the aforementioned three time-inconsistent problems, the model parameters are constant, the controls are unconstrained, and the weak equilibrium strategies that are already derived in the literature are infinitely smooth. Thus, the nonexistence of strong equilibrium for these three problems, albeit in the class of strategies that are smooth to a certain degree, is a strong indication that the notion of strong equilibrium is too restrictive. This motivates us to propose a new notion of equilibrium strategies, referred to as regular equilibrium. This notion builds on exact evaluation of the change in value caused by deviating from a given strategy, namely (1.3), as in the notion of strong equilibrium, but only allows the agent to choose alternatives whose value at the current time and state is different from that of the given strategy. We prove that regular equilibrium implies weak equilibrium but the reverse implication is not true. Indeed, we show that the particular weak equilibrium strategy for the optimal consumption problem with a bequest is not a regular equilibrium. Moreover, we provide a sufficient condition under which weak equilibrium implies regular equilibrium. It turns out that this condition holds for the problems studied in [2] and [17], so the regular equilibrium exists for these two problems. Thus, the notion of regular equilibrium is a step forward from the notion of weak equilibrium towards a better definition of equilibrium and is mathematically viable because existence can be established for many time-inconsistent problems. On the other hand, the notion of weak equilibrium is still useful: as long as the above sufficient condition holds, a weak equilibrium becomes a regular equilibrium, so taking any alternatives that are different from the weak equilibrium strategy at the current time and state leads to a smaller objective function value.

To summarize, our contribution is threefold. First, we conduct a rigorous study of weak equilibrium strategies in a diffusion framework. Second, we provide necessary conditions for a strategy to be a strong equilibrium and prove the nonexistence of strong equilibrium with sufficient regularity in three time-inconsistent problems. Third, we propose a new notion of equilibrium strategies, which is stronger than and in general different from the notion of weak equilibrium, and provide a sufficient condition under which weak equilibrium strategies are also equilibria under the new notion.

While most studies of equilibrium strategies for continuous-time control problems, including ours, build on the *localization approach* as pioneered in [37] and [13], where we compare a given strategy with one that deviates from it locally in time by either (1.2) or (1.3), there are two alternative approaches to extending the notion of equilibrium strategies from the discrete-time setting to the continuous-time setting. The first one is to divide the planning horizon in the continuous-time setting into a finite number of nonoverlapping subperiods, assume that the agent's self at the beginning of each subperiod can control her self in this period only, and finally send the length of each period to zero to obtain a certain strategy in the limit; see, for instance, [36], [40], [39], [33], and [38]. Under some conditions, the limit, if it exists, is indeed a weak equilibrium. Example 5.1 in [29] shows that such a limit is not necessarily a strong

equilibrium.

The other approach, referred to as the *fixed-point approach*, is exclusively applicable to optimal stopping problems because it exploits special structures of those problems. This approach employs a recursive algorithm to define equilibrium strategies: Initially, the agent at each time decides whether to stop or not based on the pre-committed strategy at that time, so effectively the agent would implement the naive strategy at any time. At each time, being aware of her future selves implementing the naive strategy, the agent would adjust her strategy today accordingly. Anticipating such adjustment in the future, the agent's self today would further change her strategy accordingly. Such a mental process continues until the strategy converges. See for instance, [24], [25], and [11].² Noting the possible multiplicity of the equilibrium stopping strategies as defined in the fixed-point approach, [28] and [26] further consider the optimal equilibrium stopping rule that maximizes the objective function value for an infinite-horizon, continuous-time optimal stopping problem under nonexponential discounting. In a very recent paper [3], the authors note that for many problems, with the notion of equilibrium stopping strategies in the fixed-point approach, deviation from the action of stopping to the action of continuing does not change the objective value because in this notion the agent can change her action at one instant of time only. Naming the above notion mild equilibrium, the authors compare it to weak equilibrium stopping strategies that are obtained by considering optimal stopping problems to be a special class of control problems with the control taking binary values and applying the localization approach with the first-order condition (1.2) used. Such weak equilibrium stopping strategies have been formulated and employed by [9] and [12]. The authors of [3] also propose strong equilibrium stopping strategies that are defined by the localization approach with the direct comparison of objective function values as in (1.3). Assuming the objective function to be a multiplication of a discount function and a Markov process taking values in a finite or countably infinite state space, the authors prove that the optimal mild equilibrium stopping strategy is a strong equilibrium strategy. Our work differs from [3] in three respects: First, the control process in our model can take values in an arbitrary subset of the Euclidean space. Second, the controlled state process in our model is a diffusion. Third, we consider a general objective function that accounts for time-inconsistency arising not only from nonexponential discounting but also from state-dependent preferences and from nonlinear dependence on the expected terminal state, such as the mean-variance criterion.

Our results also help to address some concerns raised in [39] on the localization approach to time-inconsistent literature. In the last paragraph of page 4198 therein, the authors of [39] comment that in the localization approach, it is unclear whether the feedback strategy $\mathbf{u}_{t,\epsilon,\mathbf{a}}$, which is obtained by pasting two feasible strategies $\hat{\mathbf{u}}$ and $\mathbf{a}$, is still feasible. We define the set of feasible strategies formally and show that $\mathbf{u}_{t,\epsilon,\mathbf{a}}$ is indeed feasible; see section 2.2. The authors of [39] also comment that the extended

²The authors of [19] and [18] consider a piecewise step discount function in the optimal stopping problems studied therein. With this particular discount function, there are only countably many future selves, so the authors first assume a finite number of future selves, then apply backward induction as in the discrete-time case, and finally send the number of future selves to infinity to define equilibrium stopping times. As noted by the authors of [27] in their Remark 2.5, the backward induction approach and the fixed point approach yield the same equilibrium in a finite-horizon discrete-time setting, but they build on different types of game-theoretic thinking. Moreover, the fixed point approach can be easily applied to the infinite-horizon setting and to the continuous-time setting.

Hamilton–Jacobi–Bellman (HJB) equation in [6] is derived informally. We offer a rigorous proof in the present paper; see Theorem 3.3 and the sufficient condition in Lemma B.2 in Appendix B for the assumption we make to derive the theorem.

Recently, the authors of [21] considered non-Markovian equilibrium strategies for time-inconsistent problems. They define an equilibrium strategy to be one that the agent is not willing to deviate from for a short period of time unless the deviation leads to an incremental reward that is positively proportional to the length of the period; see Definition 2.6 therein. Thus, their definition resembles the notion of weak equilibrium in a Markovian setting. The authors then show that the equilibrium exists if and only if a system of backward stochastic differential equations (BSDEs) is well posed. The authors also consider strict equilibrium, which is analogous to the notion of strong equilibrium, and argue that the BSDE system can also characterize strict equilibrium provided that a certain condition holds; see Remarks 2.7(iv), 3.4, and 6.4 therein. The nonexistence of strong equilibrium as shown in the present paper implies that the condition in [21] does not hold in general. Indeed, as we will show in Theorem 3.3, the rate of incremental reward if the agent deviates from $\hat{\mathbf{u}}$ to an alternative $\mathbf{a}$ is zero if $\mathbf{a}$ and $\hat{\mathbf{u}}$ take the same value at the current time and state, and this fails the condition in [21]. The authors of [21] also comment in their Remark 2.7(v) that one can define the so-called f -equilibrium by assuming that the incremental reward of deviating from the equilibrium strategy is positively proportional to a function f of the length of the period. The derivation of the two necessary conditions of the strong equilibrium in the present paper is based on a certain analysis that is related to the choice of f to be a quadratic function and a cubic function, respectively. Thus, the nonexistence of strong equilibrium derived in the present paper implies that the f -equilibrium might not exist in general.

Finally, let us emphasize that the focus of the present paper is to show the nonexistence of strong equilibrium and thus to motivate the notion of regular strong equilibrium, rather than to derive a general existence result for weak equilibrium. For the latter, there is some recent advancement; see, for instance, [39], [38], and [21].

The remainder of the paper is organized as follows. In section 2 we introduce the general diffusion framework of time-inconsistent stochastic control problems and exemplify it by the three investment and consumption problems, including those studied in [2] and [17] and an optimal consumption problem with a bequest. In section 3, we define the notion of weak equilibrium and prove a sufficient and necessary condition for a strategy to be a weak equilibrium. In section 4, we define the notion of strong equilibrium and provide two necessary conditions for a strategy to be a strong equilibrium. We then prove the nonexistence of strong equilibrium for the above three investment and consumption problems. In section 5, we define the notion of regular equilibrium and prove a sufficient condition under which a weak equilibrium strategy is a regular equilibrium. Section 6 is the conclusion. A lemma on stochastic differential equations (SDEs) and some sufficient conditions for the assumptions made in this paper are presented in the appendices. All the proofs are placed in the online supplementary materials available at <http://ssrn.com/abstract=3881455>.

2. The model.

2.1. Notation. We introduce some notation first. Throughout the paper, the state space $\mathbb{X}$ is either $(0, +\infty)$ or $\mathbb{R}^n$, the n -dimensional Euclidean space. Denote the Euclidean norm of a vector x as $\|x\|$. By default, $x \in \mathbb{R}^n$ denotes a column vector. When a vector x is understood as a row vector, we write it as $x \in \mathbb{R}^{1 \times n}$. Denote by $A^\top$ the transpose of matrix A . Denote by $\text{tr}(A)$ the trace of a square matrix A . For

a differentiable function ξ that maps $x \in \mathbb{R}^m$ to $\xi(x) \in \mathbb{R}^n$, its derivative, denoted as $\xi_x(x)$, is an n -by- m matrix with the entry in the i th row and j th column denoting the derivative of the i th component of ξ with respect to the j th component of x . In particular, for a mapping ξ from $\mathbb{R}^m$ to $\mathbb{R}$, $\xi_x(x)$ is an m -dimensional row vector, and we further denote by ξ_{xx} the Hessian matrix.

Consider ξ that maps $(z, x) \in \mathbb{Z} \times \mathbb{X}$ to $\xi(z, x) \in \mathbb{R}^l$, where $\mathbb{Z}$ is a certain set. ξ is *locally Lipschitz in $x \in \mathbb{X}$, uniformly in $z \in \mathbb{Z}$* , if there exist a sequence of compact sets $\{\mathbb{X}_k\}_{k \geq 1}$ with $\cup_{k \geq 1} \mathbb{X}_k = \mathbb{X}$ and a sequence of positive numbers $\{L_k\}_{k \geq 1}$ such that for any $k \geq 1$, $\|\xi(z, x) - \xi(z, x')\| \leq L_k \|x - x'\| \forall z \in \mathbb{Z}, x, x' \in \mathbb{X}_k$. ξ is *global Lipschitz in $x \in \mathbb{X}$, uniformly in $z \in \mathbb{Z}$* , if there exists constant $L > 0$ such that $\|\xi(z, x) - \xi(z, x')\| \leq L \|x - x'\| \forall z \in \mathbb{Z}, x, x' \in \mathbb{X}$. In the case $\mathbb{X} = \mathbb{R}^n$, ξ is of *linear growth in $x \in \mathbb{X}$, uniformly in $z \in \mathbb{Z}$* , if there exists $L > 0$ such that $\|\xi(z, x)\| \leq L(1 + \|x\|) \forall z \in \mathbb{Z}, x \in \mathbb{X}$. In the case $\mathbb{X} = (0, +\infty)$, ξ has a *bounded norm in $x \in \mathbb{X}$, uniformly in $z \in \mathbb{Z}$* , if there exists $L > 0$ such that $\|\xi(z, x)\| \leq Lx \forall z \in \mathbb{Z}, x \in \mathbb{X}$. ξ is of *polynomial growth in $x \in \mathbb{X}$, uniformly in $z \in \mathbb{Z}$* , if there exist $L > 0$ and integer $\gamma \geq 1$ such that $\|\xi(z, x)\| \leq L(1 + \varphi_{2\gamma}(x)) \forall z \in \mathbb{Z}, x \in \mathbb{X}$, where $\varphi_{2\gamma}(x) = \|x\|^{2\gamma}$ when $\mathbb{X} = \mathbb{R}^n$ and $\varphi_{2\gamma}(x) = x^{2\gamma} + x^{-2\gamma}$ when $\mathbb{X} = (0, +\infty)$.

Here and hereafter, for a derivative index $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\alpha_1, \dots, \alpha_n$ are nonnegative integers, denote $|\alpha| := \alpha_1 + \dots + \alpha_n$. Fix integer $r \geq 0$. For ξ that maps $x \in \mathbb{X}$ to $\xi(x) \in \mathbb{R}^l$, we say $\xi \in \mathfrak{C}^r(\mathbb{X})$ if $\xi(x)$ has up to r th order continuous derivatives in $x \in \mathbb{X}$. We say $\xi \in \bar{\mathfrak{C}}^r(\mathbb{X})$ if $\xi \in \mathfrak{C}^r(\mathbb{X})$ and $\frac{\partial^\alpha \xi(x)}{\partial x^\alpha} := \frac{\partial^{\alpha_1 + \dots + \alpha_n} \xi(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ is of polynomial growth in $x \in \mathbb{X}$ for any derivative index α with $|\alpha| \leq r$. Also denote $\mathfrak{C}^0(\mathbb{X})$ and $\bar{\mathfrak{C}}^0(\mathbb{X})$ as $\mathfrak{C}(\mathbb{X})$ and $\bar{\mathfrak{C}}(\mathbb{X})$, respectively.

Fix integers $r \geq 0$, $q \geq 2r$, and real numbers $a < b$. Consider ξ that maps $(t, x) \in [a, b] \times \mathbb{X}$ to $\xi(t, x) \in \mathbb{R}^l$. We say $\xi \in \mathfrak{C}^{r,q}([a, b] \times \mathbb{X})$ if for any derivative index α with $|\alpha| \leq q - 2j$ and $j = 0, \dots, r$, the partial derivative $\frac{\partial^{j+\alpha} \xi(t, x)}{\partial t^j \partial x^\alpha} := \frac{\partial^{j+\alpha_1 + \dots + \alpha_n} \xi(t, x)}{\partial t^j \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ exists for any $(t, x) \in (a, b) \times \mathbb{X}$ and can be extended to and continuous on $[a, b] \times \mathbb{X}$. We say $\xi \in \bar{\mathfrak{C}}^{r,q}([a, b] \times \mathbb{X})$ if $\xi \in \mathfrak{C}^{r,q}([a, b] \times \mathbb{X})$ and $\frac{\partial^{j+\alpha} \xi(t, x)}{\partial t^j \partial x^\alpha}$ is of polynomial growth in $x \in \mathbb{X}$, uniformly in $t \in [a, b]$, for any derivative index α with $|\alpha| \leq q - 2j$ and $j = 0, \dots, r$. Also denote $\mathfrak{C}^{0,0}([a, b] \times \mathbb{X})$ and $\bar{\mathfrak{C}}^{0,0}([a, b] \times \mathbb{X})$ as $\mathfrak{C}([a, b] \times \mathbb{X})$ and $\bar{\mathfrak{C}}([a, b] \times \mathbb{X})$, respectively. We say $\xi \in \mathfrak{C}_b^\infty([a, b] \times \mathbb{X})$ if $\xi(t, x)$ is bounded and infinitely differentiable in $(t, x) \in [a, b] \times \mathbb{X}$ with bounded derivatives.

Fix certain set $D \subseteq \mathbb{R}^m$ and certain set $\mathbb{Z}$. Consider ξ that maps $(z, x, u) \in \mathbb{Z} \times \mathbb{X} \times D$ to $\xi(z, x, u) \in \mathbb{R}^l$. $\xi(z, x, u)$ is *locally Lipschitz in $(x, u) \in \mathbb{X} \times D$, uniformly in $z \in \mathbb{Z}$* , if there exist a sequence of compact sets $\{\mathbb{X}_k\}_{k \geq 1}, \{D_k\}_{k \geq 1}$ with $\cup_{k \geq 1} \mathbb{X}_k = \mathbb{X}, \cup_{k \geq 1} D_k = D$ and a sequence of positive numbers $\{L_k\}_{k \geq 1}$ such that for any $k \geq 1$, $\|\xi(z, x, u) - \xi(z, x', u')\| \leq L_k [\|x - x'\| + \|u - u'\|] \forall z \in \mathbb{Z}, x, x' \in \mathbb{X}_k, u, u' \in D_k$. It is of *polynomial growth in $(x, u) \in \mathbb{X} \times D$, uniformly in $z \in \mathbb{Z}$* , if there exist integer $\gamma \geq 1$ and constant $L > 0$ such that $\|\xi(z, x, u)\| \leq L(1 + \varphi_{2\gamma}(x))(1 + \|u\|^{2\gamma})$ for all $(z, x, u) \in \mathbb{Z} \times \mathbb{X} \times D$.

Fix integers $r \geq 0, q \geq 2r$, real numbers $a < b$, and certain open set $\tilde{D} \subseteq \mathbb{R}^m$. Consider ξ that maps $(t, x, u) \in [a, b] \times \mathbb{X} \times \tilde{D}$ to $\xi(t, x, u) \in \mathbb{R}^l$. We say $\xi \in \mathfrak{C}^{r,q,q}([a, b] \times \mathbb{X} \times \tilde{D})$ if for any derivative indices α and β with $|\alpha| + |\beta| \leq q - 2j$ and $j = 0, \dots, r$, $\frac{\partial^{j+\alpha+\beta} \xi(t, x)}{\partial t^j \partial x^\alpha \partial u^\beta} := \frac{\partial^{j+\alpha_1 + \dots + \alpha_n + \beta_1 + \dots + \beta_m} \xi(t, x)}{\partial t^j \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} \partial u_1^{\beta_1} \dots \partial u_m^{\beta_m}}$ exists for any $(t, x, u) \in (a, b) \times \mathbb{X} \times \tilde{D}$ and can be extended to and continuous on $[a, b] \times \mathbb{X} \times \tilde{D}$. For any other set $D \subseteq \mathbb{R}^m$ and ξ that maps $(t, x, u) \in [a, b] \times \mathbb{X} \times D$ to $\xi(t, x, u) \in \mathbb{R}^l$, we say $\xi \in \mathfrak{C}^{r,q,q}([a, b] \times \mathbb{X} \times D)$ if ξ can be extended to $[a, b] \times \mathbb{X} \times \tilde{D}$ and $\xi \in \bar{\mathfrak{C}}^{r,q,q}([a, b] \times \mathbb{X} \times \tilde{D})$ for some open set $\tilde{D} \supset D$. Finally, we say $\xi \in \bar{\mathfrak{C}}^{r,q,q}([a, b] \times \mathbb{X} \times D)$

if $\xi \in \mathfrak{C}^{r,q,q}([a,b] \times \mathbb{X} \times D)$ and $\frac{\partial^{j+\alpha+\beta} \xi(t,x)}{\partial t^j \partial x^\alpha \partial u^\beta}$ is of polynomial growth in $(x,u) \in \mathbb{X} \times D$, uniformly in $t \in [a,b]$, for any derivative indices α and β with $|\alpha| + |\beta| \leq q - 2j$ and $j = 0, \dots, r$.

Finally, denote by $\bar{\mathfrak{C}}_{\text{pw}}^{r,q}([0,T] \times \mathbb{X})$ the set of functions ξ with the following property: there exists a partition $0 = t_0 < t_1 < \dots < t_{N-1} < t_N =: T$ of $[0,T]$ such that for any $i = 1, \dots, N$, ξ , when restricted on $[t_{i-1}, t_i] \times \mathbb{X}$, can be extended to $[t_{i-1}, t_i] \times \mathbb{X}$ with $\xi \in \bar{\mathfrak{C}}^{r,q}([t_{i-1}, t_i] \times \mathbb{X})$. Note that $\xi(t,x)$ can be discontinuous at $t = t_i, i = 1, \dots, N$. Define $\mathfrak{C}_{\text{pw}}^{r,q}([0,T] \times \mathbb{X})$, $\mathfrak{C}_{\text{pw}}^{r,q,q}([0,T] \times \mathbb{X} \times D)$, and $\bar{\mathfrak{C}}_{\text{pw}}^{r,q,q}([0,T] \times \mathbb{X} \times D)$ similarly.

2.2. Time-inconsistent stochastic control problems. Consider an agent who makes dynamic decisions in a given period $[0,T]$, and for any $t \in [0,T]$, the agent at that time faces the following stochastic control problem:

$$(2.1) \quad \begin{cases} \max_{\mathbf{u}} & J(t,x;\mathbf{u}) \\ \text{subject to} & dX^{\mathbf{u}}(s) = \mu(s, X^{\mathbf{u}}(s), \mathbf{u}(s, X^{\mathbf{u}}(s)))ds \\ & + \sigma(s, X^{\mathbf{u}}(s), \mathbf{u}(s, X^{\mathbf{u}}(s)))dW(s), \quad s \in [t,T], \quad X^{\mathbf{u}}(t) = x. \end{cases}$$

The agent's dynamic decisions are represented by $\mathbf{u}$, which takes values in $\mathbb{U}$, a certain subset of $\mathbb{R}^m$. $W(t) := (W_1(t), \dots, W_d(t))^\top$, $t \geq 0$, is a standard d -dimensional Brownian motion, and we denote by $(\mathcal{F}_t)_{t \geq 0}$ the filtration generated by the Brownian motion augmented by all null sets. The controlled diffusion process $X^{\mathbf{u}}$ under $\mathbf{u}$ takes values in $\mathbb{X}$, which as previously mentioned is set to be either $(0, +\infty)$ or $\mathbb{R}^n$. μ and σ are measurable mappings from $[0,T] \times \mathbb{X} \times \mathbb{U}$ to $\mathbb{R}^n$ and to $\mathbb{R}^{n \times d}$, respectively, where n stands for the dimension of $\mathbb{X}$. We consider feedback strategies, so each $\mathbf{u}$ is a mapping from $[0,T] \times \mathbb{X}$ to $\mathbb{U}$. The agent's objective function at time t is given as follows:

$$(2.2) \quad J(t,x;\mathbf{u}) = \mathbb{E}_{t,x} \left[\int_t^T C(t,x,s, X^{\mathbf{u}}(s), \mathbf{u}(s, X^{\mathbf{u}}(s)))ds + F(t,x, X^{\mathbf{u}}(T)) \right] + G(t,x, \mathbb{E}_{t,x}[X^{\mathbf{u}}(T)]),$$

where C is a measurable mapping from $[0,T] \times \mathbb{X} \times [0,T] \times \mathbb{X} \times \mathbb{U}$ to $\mathbb{R}$, and F and G are measurable mappings from $[0,T] \times \mathbb{X} \times \mathbb{X}$ to $\mathbb{R}$. Here and hereafter, $\mathbb{E}_{t,x}[Z]$ denotes the expectation of Z conditional on $X^{\mathbf{u}}(t) = x$.

For any feedback strategy $\mathbf{u}$, denote

$$\begin{aligned} \mu^{\mathbf{u}}(t,x) &:= \mu(t,x, \mathbf{u}(t,x)), \quad \sigma^{\mathbf{u}}(t,x) := \sigma(t,x, \mathbf{u}(t,x)), \\ \Upsilon^{\mathbf{u}}(t,x) &:= \sigma(t,x, \mathbf{u}(t,x))\sigma(t,x, \mathbf{u}(t,x))^\top, \quad C^{\tau,y,\mathbf{u}}(t,x) := C(\tau,y,t,x, \mathbf{u}(t,x)). \end{aligned}$$

With a slight abuse of notation, $u \in \mathbb{U}$ also denotes the feedback strategy $\mathbf{u}$ such that $\mathbf{u}(t,x) = u \forall (t,x) \in [0,T] \times \mathbb{X}$, so $\mathbb{U}$ also stands for the set of all *constant* strategies.

DEFINITION 2.1. A feedback strategy $\mathbf{u}$ is feasible if the following hold:

- (i) $\mu^{\mathbf{u}}, \sigma^{\mathbf{u}}$ are locally Lipschitz in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$.
- (ii) $\mu^{\mathbf{u}}$ and $\sigma^{\mathbf{u}}$ are of linear growth in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$, when $\mathbb{X} = \mathbb{R}^n$ and have bounded norm in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$, when $\mathbb{X} = (0, +\infty)$.
- (iii) For each fixed $(\tau,y) \in [0,T] \times \mathbb{X}$, $C^{\tau,y,\mathbf{u}}(t,x)$ and $F(\tau,y,x)$ are of polynomial growth in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$.

- (iv) For each fixed $(\tau, y) \in [0, T) \times \mathbb{X}$ and $x \in \mathbb{X}$, $\mu^{\mathbf{u}}(t, x)$ and $\sigma^{\mathbf{u}}(t, x)$ are right-continuous in $t \in [0, T)$ and $\lim_{t' \geq t, (t', x') \rightarrow (t, x)} C^{\tau, y, \mathbf{u}}(t', x') = C^{\tau, y, \mathbf{u}}(t, x)$ for any $t \in [0, T)$.

Denote the set of feasible strategies as $\mathbf{U}$.

Conditions (i) and (ii) in Definition 2.1 ensure the existence and uniqueness of the solution to the SDE in (2.1); see Lemma A.2 in Appendix A. Moreover, condition (iii) in Definition 2.1, together with Lemma A.2, implies that

$$(2.3) \quad \mathbb{E}_{t,x} \left[\sup_{s \in [t, T]} |C^{\tau, y, \mathbf{u}}(s, X^{\mathbf{u}}(s))| + |F(\tau, y, X^{\mathbf{u}}(T))| + \|X^{\mathbf{u}}(T)\| \right]$$

is of polynomial growth in $x \in \mathbb{X}$, uniformly in $t \in [0, T]$. In particular, the objective function $J(t, x; \mathbf{u})$ is well defined. Condition (iv) in Definition 2.1 imposes a mild continuity requirement. This requirement is not necessary for the well-posedness of the problem (2.1) but will be used in the following study, so we include it in the definition of feasibility for convenience.

We impose the following assumption.

Assumption 2.2. Any $u \in \mathbb{U}$ is feasible.

Here and hereafter, we use $\mathbf{D}$, which is a subset of $\mathbf{U}$, to denote the set of alternative strategies that at each time t the agent can choose to implement in an infinitesimally small time period. For given $t \in [0, T)$, $\epsilon \in (0, T - t)$, $\hat{\mathbf{u}} \in \mathbf{U}$, and $\mathbf{a} \in \mathbf{D}$, define

$$(2.4) \quad \mathbf{u}_{t, \epsilon, \mathbf{a}}(s, y) := \begin{cases} \mathbf{a}(s, y), & s \in [t, t + \epsilon), y \in \mathbb{X}, \\ \hat{\mathbf{u}}(s, y), & s \notin [t, t + \epsilon), y \in \mathbb{X}. \end{cases}$$

One can see that $\mathbf{u}_{t, \epsilon, \mathbf{a}}$ is feasible because both $\hat{\mathbf{u}}$ and $\mathbf{a}$ are feasible.

2.3. Examples. Next, we provide several examples that fit into the general framework (2.1).

2.3.1. Mean-variance problem in [2]. Consider the following problem at time t discussed in [2]:

$$(2.5) \quad \begin{cases} \max_{\mathbf{u}} & J(t, x; \mathbf{u}) := \mathbb{E}_{t,x}[X^{\mathbf{u}}(T)] - \frac{\gamma}{2} \text{var}_{t,x}(X^{\mathbf{u}}(T)) \\ \text{subject to} & dX^{\mathbf{u}}(s) = (rX^{\mathbf{u}}(s) + b\mathbf{u}(s, X^{\mathbf{u}}(s)))ds \\ & + \bar{\sigma}\mathbf{u}(s, X^{\mathbf{u}}(s))dW(s), \quad s \in [t, T], \quad X^{\mathbf{u}}(t) = x, \end{cases}$$

where constant $r \in \mathbb{R}$ stands for the risk-free rate, constant $b \neq 0$ stands for the mean return rate of a stock, constant $\bar{\sigma} > 0$ stands for the stock's volatility, constant $\gamma > 0$ is the agent's risk aversion degree, $x \in \mathbb{R}$ is the agent's wealth level at time t , $\mathbf{u}(s, X^{\mathbf{u}}(s))$ is the dollar amount invested in a stock at time s , $X^{\mathbf{u}}(s)$ is the agent's wealth at time s , and $\text{var}_{t,x}$ denotes the variance operator conditional on $X^{\mathbf{u}}(t) = x$.

We can see that (2.5) is a special case of (2.1) with the dimension of Brownian motion $d = 1$, the state space $\mathbb{X} = \mathbb{R}$, the control space $\mathbb{U} = \mathbb{R}$, $\mu(t, x, u) = rx + bu$, $\sigma(t, x, u) = \bar{\sigma}u$, $F(\tau, y, x) = x - \frac{\gamma}{2}x^2$, $G(\tau, y, z) = \frac{\gamma}{2}z^2$, and $C(\tau, y, t, x, u) \equiv 0$.

The authors of [8] consider a similar mean-variance problem, which differs from (2.5) only in that $\frac{\gamma}{2}$ is replaced by $\frac{\bar{\gamma}}{2x}\mathbf{1}_{x \neq 0} + K\mathbf{1}_{x=0}$ for some constant $\bar{\gamma} > 0$ and $K \in \mathbb{R}$. All of the analysis of problem (2.5) in the following can apply to the problem in [8] as well and leads to similar conclusions; see [30, Chapter A.6].

2.3.2. Investment and consumption in [17]. Consider the following investment consumption problem studied by [17]:

$$(2.6) \quad \begin{cases} \max_{\mathbf{u}} & \mathbb{E}_{t,x} \left[\int_t^T h(s-t) \frac{(\zeta(s, X^{\mathbf{u}}(s)) X^{\mathbf{u}}(s))^{1-\gamma}}{1-\gamma} ds + h(T-t) \frac{(X^{\mathbf{u}}(T))^{1-\gamma}}{1-\gamma} \right] \\ \text{subject to} & dX^{\mathbf{u}}(s) = X^{\mathbf{u}}(s) \left[(r + b\theta(s, X^{\mathbf{u}}(s)) - \zeta(s, X^{\mathbf{u}}(s))) ds \right. \\ & \quad \left. + \bar{\sigma}\theta(s, X^{\mathbf{u}}(s)) dW(s) \right], \quad s \in [t, T], \quad X^{\mathbf{u}}(t) = x, \end{cases}$$

where $r \in \mathbb{R}$, $b \neq 0$, and $\bar{\sigma} > 0$ are constants, representing the risk-free rate, the mean return rate, and volatility of a stock, respectively. The agent's control $\mathbf{u}(s, X^{\mathbf{u}}(s)) = (\zeta(s, X^{\mathbf{u}}(s)), \theta(s, X^{\mathbf{u}}(s)))^T$ has two components: $\zeta(s, X^{\mathbf{u}}(s))$, which stands for the consumption propensity, and $\theta(s, X^{\mathbf{u}}(s))$, which stands for the percentage of the wealth invested in the stock at time s . The agent's wealth process is denoted as $X^{\mathbf{u}}$. The parameter $\gamma > 0$ measures the agent's relative risk aversion; when $\gamma = 1$, the function $z^{1-\gamma}/(1-\gamma)$ in the objective function of (2.6) and in the following is understood as $\ln z$. h is a deterministic discount function so that the discounting in the period from t to s is $h(s-t)$, and it is assumed to be nonnegative, in $\mathcal{C}^2([0, T])$, and it satisfies $h(0) = 1$.

We can see that (2.6) is a special case of (2.1) with the dimension of Brownian motion $d = 1$, the state space $\mathbb{X} = (0, +\infty)$, the control space $\mathbb{U} = (0, +\infty) \times \mathbb{R}$, $\mu(t, x, u) = (r + b\theta - \zeta)x$, $\sigma(t, x, u) = \bar{\sigma}\theta x$, $F(\tau, y, x) = h(T - \tau)x^{1-\gamma}/(1-\gamma)$, $G(\tau, y, z) \equiv 0$, and $C(\tau, y, t, x, u) = h(t - \tau)(\zeta x)^{1-\gamma}/(1-\gamma)$, where $u = (\zeta, \theta)^T$.

2.3.3. Optimal consumption with a bequest. Consider an agent who is endowed with $bdt + \bar{\sigma}dW(t)$ at each instant $t \in [0, T]$, where b and $\bar{\sigma}$ are two constants in $\mathbb{R}$ and $W(t)$, $t \in [0, T]$, is a standard Brownian motion. The agent decides the amount of consumption at each time $t \in [0, T]$ with the objective to maximize her utility for the consumption stream and a bequest for her descendants at time T . Formally, the agent faces the following problem:

$$(2.7) \quad \begin{cases} \max_{\mathbf{u}} & \mathbb{E}_{t,x} \left[\int_t^T h(s-t) \mathbf{u}(s, X^{\mathbf{u}}(s)) ds + \tilde{h}(T-t) X^{\mathbf{u}}(T) \right] \\ \text{subject to} & dX^{\mathbf{u}}(s) = (b - \mathbf{u}(s, X^{\mathbf{u}}(s))) ds + \bar{\sigma} dW(s), \quad s \in [t, T], \quad X^{\mathbf{u}}(t) = x. \end{cases}$$

Here, $\mathbf{u}(s, X^{\mathbf{u}}(s))ds$ and $X^{\mathbf{u}}(s)$ stand for the agent's consumption and wealth at time s , respectively, and we allow $\mathbf{u}(s, X^{\mathbf{u}}(s))$ to be negative. For the consumption at time s , the agent applies a discount $h(s-t)$ to evaluate it at time t , where h is a deterministic function. For the wealth at time T , which is a bequest for the agent's descendants, the agent applies a discount $\tilde{h}(T-t)$, where $\tilde{h}$ is another deterministic function. h and $\tilde{h}$ are assumed to be nonnegative, in $\mathcal{C}^2([0, T])$, and they satisfy $h(0) = \tilde{h}(0) = 1$. The agent is risk-neutral with respect to the random discounted value of the consumption and bequest, which explains the objective function in (2.7).

The above problem is a special case of (2.1) with the dimension of Brownian motion $d = 1$, the state space $\mathbb{X} = \mathbb{R}$, the control space $\mathbb{U} = \mathbb{R}$, $\mu(t, x, u) = b - u$, $\sigma(t, x, u) = \bar{\sigma}$, $F(\tau, y, x) = \tilde{h}(T - \tau)x$, $G(\tau, y, z) \equiv 0$, and $C(\tau, y, t, x, u) = h(t - \tau)u$.

The following assumption is needed in the derivation of a weak equilibrium for problem (2.7) but is not needed in the proof of nonexistence of strong equilibrium.

Assumption 2.3. Suppose that $h'(0) \neq 0$, $\tilde{h}'(0) = 0$, and that ψ , which is the

unique solution to the equation

$$(2.8) \quad \psi(t) = \frac{1}{h'(0)} \left[h'(T-t) - \tilde{h}'(T-t) - \int_t^T \psi(s) h''(s-t) ds \right], \quad t \in [0, T],$$

satisfies $\psi \in \mathfrak{C}^1[0, T]$ and $\psi(t) > 0$ for all $t \in [0, T]$.

The requirement $\tilde{h}'(0) = 0$ holds for the discount function used in [17], $\tilde{h}(t) = (1 + \tilde{\rho}t)e^{-\tilde{\rho}t}$ for some constant $\tilde{\rho}$, but does not hold for some other commonly used discount functions, such as generalized hyperbolic discount functions and pseudo-exponential discount functions. The existence of a unique solution ψ of (2.8) in $\mathfrak{C}^1[0, T]$ can be established by the classical results of Volterra equations, and a sufficient condition for the positivity of ψ is that $[h'(t) - \tilde{h}'(t)]/h'(0) > 0$ and $h''(t)/h'(0) \leq 0$ for all $t \in [0, T]$; see Lemma B.1 in Appendix B.³ Note that typically $h'(0) < 0$, so the sufficient condition stipulates that (i) $h''(t) \geq 0, t \in [0, T]$, and (ii) $h'(t) - \tilde{h}'(t) < 0, t \in [0, T]$. Condition (i) means that h is convex, which holds for a typical discount function. Condition (ii), which is the same as $h(t) - \tilde{h}(t)$ becoming smaller, i.e., more negative, when t becomes larger, simply means that the difference in how the agent discounts future consumption and future bequest becomes larger when the future moment is more distant. Condition (ii) holds in particular when the agent does not discount the bequest (i.e., $\tilde{h} \equiv 1$) but discounts the consumption (i.e., $h'(t) < 0, t \in [0, T]$).

3. Weak equilibrium. We introduce the generator of the controlled state process first. Given $\mathbf{u} \in \mathbf{U}$ and interval $[a, b] \subseteq [0, T]$, consider ξ that maps $(t, x) \in [a, b] \times \mathbb{X}$ to $\xi(t, x) \in \mathbb{R}$. Suppose $\xi \in \mathfrak{C}^{1,2}([a, b] \times \mathbb{X})$, and denote its first-order partial derivative with respect to t , first-order partial derivative with respect to x , and second-order partial derivative with respect to x as ξ_t , ξ_x , and ξ_{xx} , respectively. Define the following generator:

$$(3.1) \quad \mathcal{A}^{\mathbf{u}}\xi(t, x) = \xi_t(t, x) + \xi_x(t, x)\mu^{\mathbf{u}}(t, x) + \frac{1}{2}\text{tr}(\xi_{xx}(t, x)\Upsilon^{\mathbf{u}}(t, x)), \quad t \in [a, b], x \in \mathbb{X}.$$

For $\xi \in \mathfrak{C}^{1,2}([a, b] \times \mathbb{X})$ that maps $(t, x) \in [a, b] \times \mathbb{X}$ to $\xi(t, x) = (\xi_1(t, x), \dots, \xi_l(t, x))^{\top} \in \mathbb{R}^l$, we denote $\mathcal{A}^{\mathbf{u}}\xi(t, x) = (\mathcal{A}^{\mathbf{u}}\xi_1(t, x), \dots, \mathcal{A}^{\mathbf{u}}\xi_l(t, x))^{\top}$. Finally, $(\mathcal{A}^{\mathbf{u}})^2\xi$ and $(\mathcal{A}^{\mathbf{u}})^3\xi$ denote the functions obtained by applying the generator $\mathcal{A}^{\mathbf{u}}$ twice and thrice, respectively, to ξ .

Here and hereafter, $\hat{\mathbf{u}} \in \mathbf{U}$ denotes a given strategy, and we examine whether it is an equilibrium strategy. For each fixed $(\tau, y) \in [0, T] \times \mathbb{X}$, denote

$$(3.2) \quad f^{\tau, y}(t, x) := \mathbb{E}_{t, x}[F(\tau, y, X^{\hat{\mathbf{u}}}(T))], \quad g(t, x) := \mathbb{E}_{t, x}[X^{\hat{\mathbf{u}}}(T)], \quad t \in [0, T], x \in \mathbb{X}.$$

In addition, for fixed $(\tau, y) \in [0, T] \times \mathbb{X}$ and $s \in [0, T]$, denote

$$(3.3) \quad c^{\tau, y, s}(t, x) := \mathbb{E}_{t, x}[C^{\tau, y, \hat{\mathbf{u}}}(s, X^{\hat{\mathbf{u}}}(s))], \quad t \in [0, s], x \in \mathbb{X}.$$

In the following, $\mathcal{A}^{\mathbf{u}}f^{\tau, y}$ denotes the function that is obtained by applying the operator $\mathcal{A}^{\mathbf{u}}$ to $f^{\tau, y}(t, x)$ as a function of (t, x) while fixing (τ, y) , and $(\mathcal{A}^{\mathbf{u}})^2f^{\tau, y}$ is defined likewise. Then, $\mathcal{A}^{\mathbf{u}}f^{t, x}(t, x)$ denotes the value of $\mathcal{A}^{\mathbf{u}}f^{\tau, y}$ at (t, x) while (τ, y) is also set at (t, x) , and $(\mathcal{A}^{\mathbf{u}})^2f^{t, x}(t, x)$, $(\mathcal{A}^{\mathbf{u}})^3f^{t, x}(t, x)$ are interpreted similarly. The above notation also applies to $C^{\tau, y, \mathbf{u}}$ and $c^{\tau, y, s}$.

³We thank an anonymous referee for providing us with the idea of this lemma and its proof.

To define equilibrium strategies, the agent's selves at different times in $[0, T)$ are regarded as different players in a sequential game, and the agent's self at time 0 is the *first* player in the game. Thus, the initial state at time 0 is relevant. Throughout the paper, we assume that the initial value of the controlled diffusion process is a constant $x_0 \in \mathbb{X}$. Given feasible strategy $\hat{\mathbf{u}}$, Lemma A.2 in Appendix A implies that for any $(t, x) \in [0, T) \times \mathbb{X}$, there exists a unique strong solution to the SDE in (2.1) under the strategy $\hat{\mathbf{u}}$. Particularly, we denote by $X^{\hat{\mathbf{u}}}(\cdot; 0, x_0)$ the solution with initial time 0 and initial state x_0 . Denote by $\bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}}$ the support of the distribution of $X^{\hat{\mathbf{u}}}(t; 0, x_0)$.⁴ Moreover, denote by $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ the set of *reachable states* at time t from the initial state x_0 at time 0 and following the strategy $\hat{\mathbf{u}}$; i.e.,

$$(3.4) \quad \mathbb{X}_t^{x_0, \hat{\mathbf{u}}} := \mathring{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}} \cup \{x \in \partial \bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}} : \mathbb{P}(X^{\hat{\mathbf{u}}}(t; 0, x_0) \in \partial \bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}} \cap \mathbb{B}_\delta(x)) > 0 \forall \delta > 0\},$$

where $\mathring{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}}$ is the interior of $\bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}}$ in $\mathbb{X}$, $\partial \bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}}$ is the boundary of $\bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}}$ in $\mathbb{X}$, and $\mathbb{B}_\delta(x)$ denotes the ball centered at x with radius δ .⁵ By definition, we have $\mathbb{P}(X^{\hat{\mathbf{u}}}(t; 0, x_0) \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}) = 1$.

The set of reachable states slightly differs from the support of the distribution of $X^{\hat{\mathbf{u}}}(t; 0, x_0)$ because the former excludes some points on the boundary of the support that are actually not reachable. For example, consider $\mathbb{X} = \mathbb{R}$ and suppose $X^{\hat{\mathbf{u}}}(\cdot; 0, x_0)$ is a geometric Brownian motion with $x_0 > 0$. Then, $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}} = (0, +\infty)$ and $\bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}} = [0, +\infty)$. Note that $X^{\hat{\mathbf{u}}}(t; 0, x_0)$ never visits 0, which is in the support of the distribution of $X^{\hat{\mathbf{u}}}(t; 0, x_0)$ but not in the set of reachable states.

3.1. Definition of weak equilibrium. We first study weak equilibrium. Although such strategies have been proposed and studied in the literature, our definition is slightly different, and our analysis addresses some technical issues that are overlooked in the literature; see the discussion in the following.

DEFINITION 3.1 (Weak equilibrium). $\hat{\mathbf{u}} \in \mathbf{U}$ is a weak equilibrium strategy if for any $x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$, $t \in [0, T)$, and $\mathbf{a} \in \mathbf{D}$, we have

$$(3.5) \quad \Delta^{\hat{\mathbf{u}}}(t, x; \mathbf{a}) := \limsup_{\epsilon \downarrow 0} \frac{J(t, x; \mathbf{u}_{t, \epsilon, \mathbf{a}}) - J(t, x; \hat{\mathbf{u}})}{\epsilon} \leq 0.$$

With $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ replaced by $\mathbb{X}$, Definition 3.1 becomes the one used in the literature to study equilibrium strategies in time-inconsistent problems; see, for instance, [13, 14, 15], [6], [2], [8], and [10], where $\mathbf{D}$ is set to be $\mathbb{U}$, and [17], [16], and [5], where $\mathbf{D}$ is essentially set to be $\mathbf{U}$.

There are two reasons why in the above definition we demand the condition (3.5) hold for any $x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ rather than for any $x \in \mathbb{X}$. First, although the agent's self today cannot control the action of her future selves, her action today actually determines the state process in the future based on which her future selves make decisions. Thus, the agent's self today is not concerned about whether her future selves deviate from a given strategy in the states that cannot be attained under this strategy. This argument motivates the authors of [35] to propose a similar notion of

⁴The support of a distribution on $\mathbb{X}$ is defined to be the smallest closed set in $\mathbb{X}$ such that its complement has zero measure under this distribution.

⁵We can equivalently define $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ to be the union of $\mathring{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}}$ and the smallest relatively close subset $\mathbb{S}$ of $\partial \bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}}$ such that $\mathbb{P}(X^{\hat{\mathbf{u}}}(t; 0, x_0) \in \mathbb{S}) = \mathbb{P}(X^{\hat{\mathbf{u}}}(t; 0, x_0) \in \partial \bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}})$.

equilibrium strategies in a discrete-time, deterministic setting.⁶ Second, the strategy derived in [8] for their problem is a weak equilibrium strategy under Definition 3.1, but is not if $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ is replaced by $\mathbb{X}$. Indeed, this strategy does not satisfy the extended HJB equation for $x < 0$ in Definition 2 therein.⁷

As previously mentioned, the set of reachable states $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ slightly differs from the support of the distribution of $X^{\hat{\mathbf{u}}}(t; 0, x_0)$. The reason we use the former rather than the latter in the definition of equilibrium strategies can be seen from the median maximization problem studied by [20]. Those authors derive an equilibrium strategy by demanding that the agent's self at any time t not deviate from it in any reachable state at time t , similar to Definition 3.1. The agent therein, however, will deviate from the equilibrium strategy if she were at a state that is in the support of the state process but not reachable. The deviation in this state, however, should not invalidate an equilibrium strategy because it is not reachable.

Finally, let us comment that our analysis in the following does not use any structure of the set of reachable states $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$. Thus, all the results in the present paper still hold with $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ replaced by $\bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}}$ or $\mathbb{X}$ in the definition of equilibrium. Indeed, we would have encountered fewer technical challenges if $\bar{\mathbb{X}}_t^{x_0, \hat{\mathbf{u}}}$ or $\mathbb{X}$ were used because these two sets have nicer properties than $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$.

3.2. Comments on technical assumptions. Before we proceed, let us make three comments. First, in the following analysis of weak equilibrium and other notions of equilibrium strategies, we need to make various assumptions on the smoothness and growth conditions of the functions $f^{\tau, y}$, g , and $c^{\tau, y, s}$ as defined in (3.2) and (3.3); see Assumptions 3.2, 4.2, and 4.7 in the following. These assumptions might not be straightforward to verify in general because the functions $f^{\tau, y}$, g , and $c^{\tau, y, s}$ depend on $\hat{\mathbf{u}}$ in a nontrivial way.⁸ Thus, we provide sufficient conditions for these assumptions in Appendix B. Roughly speaking, as long as $\hat{\mathbf{u}}$ and the model parameters— μ , σ , C , F , and G —are sufficiently smooth locally in time and satisfy certain growth conditions, the above assumptions hold. Second, we do not assume any structure on $\mathbb{U}$, in which the control takes values. In particular, $\mathbb{U}$ can be a closed set or an open set, so our results are broadly applicable. Third, our framework can also apply to deterministic time-inconsistent problems by simply setting σ at 0.

3.3. Sufficient and necessary conditions for weak equilibrium. We impose the following assumption in order to study weak equilibrium strategies.

⁶The authors of [35] propose that a strategy $(s_0^*, s_1^*, \dots)$, where s_t^* stands for the agent's feedback action at time t , is an equilibrium strategy if for any t , $(s_0^*, \dots, s_{t-1}^*, s_t, s_{t+1}^*, \dots)$ is dominated by $(s_0^*, \dots, s_{t-1}^*, s_t^*, s_{t+1}^*, \dots)$ for any s_t . The authors argue that the above definition is more desirable than the following one, which is based on a model in [36]: $(s_0^*, s_1^*, \dots)$ is an equilibrium strategy if for any time t , $(s_0, \dots, s_{t-1}, s_t, s_{t+1}^*, \dots)$ is dominated by $(s_0, \dots, s_{t-1}, s_t^*, s_{t+1}^*, \dots)$ for any $(s_0, \dots, s_t)$. Note that in the former definition, one is only concerned about the agent's action at time t in the state that is attainable by $(s_0^*, \dots, s_{t-1}^*)$. In the latter definition, one needs to consider the agent's action at time t in any state because $(s_0, \dots, s_{t-1})$ is arbitrary.

⁷The extended HJB equation in Definition 2 of [8] is equivalent to equations (4.1)–(4.3) therein, and the equation (4.1) can be reformulated as the equation in the fourth last line of page 11 of [8]. Note that all the above equations need to hold for all $x \in \mathbb{R}$ because it is assumed therein in the definition of equilibrium strategies that (3.5) needs to hold for any $x \in \mathbb{X} = \mathbb{R}$. The last equation in the above, however, does not imply the equation in the second to last line of the same page when $x < 0$ because $f_{xx}(t, x, x) > 0$ in this case and, consequently, this equation is not satisfied by the equilibrium strategy derived by [8] for $x < 0$; i.e., that strategy is not an equilibrium under the definition used by [8].

⁸Other assumptions in the present paper, namely Assumptions 2.2 and 4.5, are easy to verify because they involve the model parameters, i.e., μ , σ , C , F , and G , only.

Assumption 3.2. For any fixed $(\tau, y) \in [0, T] \times \mathbb{X}$ and $t \in [0, T]$, there exists $\tilde{t} \in (t, T]$ such that (i) $f^{\tau, y}, g \in \tilde{\mathcal{C}}^{1,2}([t, \tilde{t}] \times \mathbb{X})$; (ii) $c^{\tau, y, s} \in \mathcal{C}^{1,2}([t, \tilde{t} \wedge s] \times \mathbb{X})$ for each fixed $s \in (t, T]$ and $\frac{\partial^{j+\alpha} c^{\tau, y, s}(t', x')}{\partial t^j \partial x^\alpha}$ is of polynomial growth in $x' \in \mathbb{X}$, uniformly in $t' \in [t, \tilde{t} \wedge s]$ and $s \in (t, T]$, for any α with $|\alpha| \leq 2 - 2j$ and $j = 0, 1$; and (iii) $G(\tau, y, z)$ is continuously differentiable with respect to z .

In the following, we always consider sufficiently small ϵ , and $o(1)$ denotes a generic function of ϵ such that $\lim_{\epsilon \downarrow 0} |o(1)| = 0$.

THEOREM 3.3. *Suppose Assumptions 2.2 and 3.2 hold. Then, for any $(t, x) \in [0, T] \times \mathbb{X}$ and $\mathbf{a} \in \mathbf{U}$, we have*

$$(3.6) \quad J(t, x; \mathbf{u}_{t, \epsilon, \mathbf{a}}) - J(t, x; \hat{\mathbf{u}}) = \epsilon \Delta^{\hat{\mathbf{u}}}(t, x; \mathbf{a}) + \epsilon o(1),$$

with $\Delta^{\hat{\mathbf{u}}}(t, x; \mathbf{a}) = \Gamma^{t, x, \hat{\mathbf{u}}}(t, x; \mathbf{a})$, where for any $(\tau, y) \in [0, T] \times \mathbb{X}$,

$$(3.7) \quad \begin{aligned} \Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{a}) := & C^{\tau, y, \mathbf{a}}(t, x) - C^{\tau, y, \hat{\mathbf{u}}}(t, x) + \int_t^T \mathcal{A}^{\mathbf{a}} c^{\tau, y, s}(t, x) ds + \mathcal{A}^{\mathbf{a}} f^{\tau, y}(t, x) \\ & + G_z(\tau, y, g(t, x)) \mathcal{A}^{\mathbf{a}} g(t, x). \end{aligned}$$

Moreover, $\Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{a}) = \Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; \tilde{\mathbf{a}})$ for any $\mathbf{a}, \tilde{\mathbf{a}} \in \mathbf{U}$ with $\mathbf{a}(t, x) = \tilde{\mathbf{a}}(t, x)$, and, in particular, $\Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{a}) = 0$ if $\mathbf{a}(t, x) = \hat{\mathbf{u}}(t, x)$. Consequently, suppose $\mathbf{U} \subseteq \mathbf{D} \subseteq \mathbf{U}$. Then, $\hat{\mathbf{u}}$ is a weak equilibrium strategy if and only if

$$(3.8) \quad \Delta^{\hat{\mathbf{u}}}(t, x; u) \leq 0 \quad \forall u \in \mathbf{U}, x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}, t \in [0, T].$$

Suppose the time t self of the agent deviates from $\hat{\mathbf{u}}$ by taking an alternative strategy $\mathbf{a}$ in a small time period with length ϵ , resulting in the deviating strategy $\mathbf{u}_{t, \epsilon, \mathbf{a}}$. Theorem 3.3 provides the first-order derivative of the objective value of the deviating strategy $\mathbf{u}_{t, \epsilon, \mathbf{a}}$ with respect to ϵ at $\epsilon = 0$. More importantly, it shows that this derivative depends on the value of the alternative strategy $\mathbf{a}$ at the current time t and state x only.

As a result of the first-order expansion (3.6), Theorem 3.3 provides a sufficient and necessary condition (3.8) for $\hat{\mathbf{u}}$ to be a weak equilibrium strategy. Note that this condition does not depend on the choice of $\mathbf{D}$ because, as we have mentioned, the first-order derivative of $J(t, x; \mathbf{u}_{t, \epsilon, \mathbf{a}})$ with respect to ϵ at $\epsilon = 0$ depends on the value of $\mathbf{a}$ at (t, x) only. Thus, in the literature, where weak equilibrium strategies are discussed, different choices of $\mathbf{D}$ lead to the same set of equilibrium strategies.

Let us comment that the proof of (3.6) is essentially the same as the analysis performed in [5] to derive the so-called extended HJB equation, and this analysis is also used in [17] and [2] to find the weak equilibrium strategies for certain specific stochastic control problems studied therein. To perform the analysis in the aforementioned papers, however, one needs to assume certain conditions that are not explicitly spelled out therein.⁹ In the present paper, we explicitly state the assumptions we

⁹Certain conditions are needed to derive the equation in lines 14–17 of page 343 in [5]; in particular, $\mathcal{A}^{\mathbf{u}} f^{\tau, y}(t, x)$ needs to be right-continuous in t . The authors of [5], however, did not assume $\mathbf{u}$ to be right-continuous in t ; see Definition 2.2 therein. The authors of [17] consider $\mathbf{D}$ to be the set of all progressively measurable processes; see equations (3.2) and (3.3) therein. In particular, $\mathbf{u} \in \mathbf{D}$ is not necessarily right-continuous in t . However, for the limit in lines 13–15 of page 78 to hold for any $t \in [0, T]$, one may have to impose certain continuity conditions. The authors of [2] did not define equilibrium strategies explicitly. In the derivation of equation (14) therein, they implicitly assume a certain right-continuity property of the alternative strategy that the agent's self at each time t can implement.

need, namely Assumptions 2.2 and 3.2, and prove (3.6) rigorously. Moreover, because $\Delta^{\hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) = 0$, as shown in Theorem 3.3, condition (3.8) is equivalent to following extended HJB equation:¹⁰

$$(3.9) \quad \max_{u \in \mathbb{U}} \Delta^{\hat{\mathbf{u}}}(t, x; u) = 0, x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}, t \in [0, T).$$

Finally, in the above study of weak equilibrium strategies, we do not assume any structure on $\mathbb{U}$, the set in which the control process takes values. In particular, $\mathbb{U}$ can be a discrete set. Thus, Theorem 3.3 can be applied to optimal stopping and optimal switching problems as well.

3.4. Weak equilibrium strategies in examples. Next, we derive weak equilibrium strategies for the three problems in section 2.3. The first two problems have been studied, and certain weak equilibrium strategies have been derived in the literature. With the help of Theorem 3.3, we now are able to establish these equilibrium strategies rigorously.

PROPOSITION 3.4. *Suppose $\mathbb{U} \subseteq \mathbf{D} \subseteq \mathbf{U}$.*

(i) *Consider problem (2.5). Define*

$$(3.10) \quad \hat{\mathbf{u}}(t, x) = \frac{1}{\gamma} \frac{b}{\bar{\sigma}^2} e^{-r(T-t)}, t \in [0, T], x \in \mathbb{R}.$$

Then, Assumptions 2.2 and 3.2 hold, $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}} = \mathbb{R} \forall t \in (0, T)$, and we have

$$(3.11) \quad \Delta^{\hat{\mathbf{u}}}(t, x; u) = -\frac{b^2}{2\gamma\bar{\sigma}^2} + e^{r(T-t)}bu - \frac{\gamma}{2}e^{2r(T-t)}\bar{\sigma}^2u^2, t \in [0, T], x \in \mathbb{R}, u \in \mathbb{R}.$$

Moreover, $\Delta^{\hat{\mathbf{u}}}(t, x; u) \leq 0$, and thus $\hat{\mathbf{u}}$ is a weak equilibrium strategy.

(ii) *Consider problem (2.6). Define*

$$(3.12) \quad \hat{\mathbf{u}}(t, x) = \left(\hat{\zeta}(t, x), \hat{\theta}(t, x) \right) := \left(k(t)^{-\frac{1}{\gamma}}, \frac{b}{\gamma\bar{\sigma}^2} \right), t \in [0, T], x \in (0, +\infty),$$

where $k(t)$ solves the following equation:

$$(3.13) \quad k(t) = \int_t^T h(s-t)e^{(1-\gamma)(r+b^2/(2\gamma\bar{\sigma}^2))(s-t)}k(s)^{-\frac{1-\gamma}{\gamma}}e^{-(1-\gamma)\int_t^s k(z)^{-1/\gamma}dz}ds \\ + h(T-t)e^{(1-\gamma)(r+b^2/(2\gamma\bar{\sigma}^2))(T-t)}e^{-(1-\gamma)\int_t^T k(s)^{-1/\gamma}ds}, t \in [0, T].$$

Denote $\hat{\zeta}(t, x)$ as $\hat{\zeta}(t)$ because it does not depend on x , and denote $\hat{\theta}(t, x)$ as $\hat{\theta}$ because it does not depend on t or x . Then, Assumptions 2.2 and 3.2 hold,

¹⁰The equation (3.9) takes a different form from equation (4.1) in the extended HJB equation proposed in [5] because in the latter the authors introduce the so-called continuation value function V , which is the agent's objective function value when she follows the equilibrium strategy. The introduction of the value function necessitates an additional assumption: $f^{\tau, y}(t, x)$ needs to be twice differentiable in y ; see Theorem 5.2 in [5].

and we have

$$\begin{aligned} \Delta^{\hat{\mathbf{u}}}(t, x; u) &= \frac{(\zeta x)^{1-\gamma}}{1-\gamma} - \frac{(\hat{\zeta}(t)x)^{1-\gamma}}{1-\gamma} + x^{1-\gamma} \hat{\zeta}(t)^{-\gamma} \left[\hat{\zeta}(t) - \zeta - b(\hat{\theta} - \theta) \right. \\ (3.14) \quad &\left. + \frac{1}{2} \gamma \bar{\sigma}^2 (\hat{\theta}^2 - \theta^2) \right], \quad t \in [0, T], x \in (0, +\infty), u = (\zeta, \theta) \in (0, +\infty) \times \mathbb{R}. \end{aligned}$$

Moreover, $\Delta^{\hat{\mathbf{u}}}(t, x; u) \leq 0$, and thus $\hat{\mathbf{u}}$ is a weak equilibrium strategy.

- (iii) Consider problem (2.7), and suppose Assumption 2.3 holds. Fix any constant $b_0 \in \mathbb{R}$ and define

$$(3.15) \quad \hat{\mathbf{u}}(t, x) := b_0 + k(t)x, \quad t \in [0, T], x \in \mathbb{R},$$

where $k(t) = -\psi'(t)/\psi(t)$, $t \in [0, T]$, with $\psi(t)$ solving (2.8). Then, Assumptions 2.2 and 3.2 hold, and we have

$$(3.16) \quad \Delta^{\hat{\mathbf{u}}}(t, x; u) \equiv 0, \quad t \in [0, T], x \in \mathbb{R}, u \in \mathbb{R}.$$

Consequently, $\hat{\mathbf{u}}$ is a weak equilibrium strategy.

4. Strong equilibrium.

4.1. Definition of strong equilibrium. Note that even when condition (3.5) is true, it is still possible that $J(t, x; \mathbf{u}_{t, \epsilon, \mathbf{a}}) > J(t, x; \hat{\mathbf{u}})$ for sufficiently small ϵ and certain $\mathbf{a}$, and thus the time- t self of the agent deviates from $\hat{\mathbf{u}}$. This observation motivates the authors of [29] to define strong equilibrium in their study of a continuous-time, infinite-horizon, finite-state Markov chain control problem. We mimic their definition in our diffusion framework.

DEFINITION 4.1 (Strong equilibrium). $\hat{\mathbf{u}} \in \mathbf{U}$ is a strong equilibrium strategy if for any $x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$, $t \in [0, T]$, and $\mathbf{a} \in \mathbf{D}$, there exists $\epsilon_0 \in (0, T - t)$ such that

$$(4.1) \quad J(t, x; \mathbf{u}_{t, \epsilon, \mathbf{a}}) - J(t, x; \hat{\mathbf{u}}) \leq 0 \quad \forall \epsilon \in (0, \epsilon_0].$$

Condition (4.1) stipulates that the time- t self of the agent is not willing to deviate from $\hat{\mathbf{u}}$ by implementing an alternative strategy $\mathbf{a}$ in an arbitrarily small period, and if this is true for all $\mathbf{a}$ that the agent can choose, then $\hat{\mathbf{u}}$ is a strong equilibrium strategy. By definition, any strong equilibrium strategy must be a weak one. The reverse implication, however, is not true in general, as we will see in the following analysis.

4.2. Second-order conditions for strong equilibrium strategies. The following assumption is needed in the study of strong equilibrium.

Assumption 4.2. For any given $(\tau, y) \in [0, T] \times \mathbb{X}$ and $t \in [0, T]$, there exists $\tilde{t} \in (t, T]$ such that $\hat{\mathbf{u}} \in \bar{\mathfrak{C}}^{1,2}([t, \tilde{t}] \times \mathbb{X})$ and the following hold: (i) $f^{\tau, y}, g \in \bar{\mathfrak{C}}^{2,4}([t, \tilde{t}] \times \mathbb{X})$; (ii) $c^{\tau, y, s} \in \mathfrak{C}^{2,4}([t, \tilde{t} \wedge s] \times \mathbb{X})$ for any $s \in (t, T]$ and $\frac{\partial^{j+\alpha} c^{\tau, y, s}(t', x')}{\partial t^j \partial x^\alpha}$ is of polynomial growth in $x' \in \mathbb{X}$, uniformly in $t' \in [t, \tilde{t} \wedge s]$ and $s \in (t, T]$, for any α with $|\alpha| \leq 4 - 2j$ and $j = 0, 1, 2$; (iii) $G(\tau, y, z)$ is twice continuously differentiable in z and $\mu, \sigma, C(\tau, y, \cdot, \cdot, \cdot) \in \bar{\mathfrak{C}}^{1,2,2}([t, \tilde{t}] \times \mathbb{X} \times \mathbb{U})$; and (iv) for any $x \in \mathbb{X}$, $\lim_{s \downarrow t} c_x^{\tau, y, s}(t, x) = c_x^{\tau, y, t}(t, x)$, $\lim_{s \downarrow t} c_{xx}^{\tau, y, s}(t, x) = c_{xx}^{\tau, y, t}(t, x)$, and $\lim_{s \downarrow t} c_t^{\tau, y, s}(t, x)$ exists with the limit denoted as $c_t^{\tau, y, t}(t, x)$.

PROPOSITION 4.3. Suppose Assumptions 2.2 and 4.2 hold. Then, for any $(t, x) \in [0, T] \times \mathbb{X}$ and $\mathbf{a} \in \mathbf{U} \cap \bar{\mathcal{C}}^{1,2}([0, T] \times \mathbb{X})$, we have

$$(4.2) \quad J(t, x; \mathbf{u}_{t, \epsilon, \mathbf{a}}) - J(t, x; \hat{\mathbf{u}}) = \epsilon \Delta^{\hat{\mathbf{u}}}(t, x; \mathbf{a}) + \frac{1}{2} \epsilon^2 \Lambda^{t, x, \hat{\mathbf{u}}}(t, x; \mathbf{a}) + \epsilon^2 o(1),$$

where $\Delta^{\hat{\mathbf{u}}}(t, x; \mathbf{a})$ is given in Theorem 3.3 and

$$(4.3) \quad \begin{aligned} \Lambda^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{a}) &:= \mathcal{A}^{\mathbf{a}} C^{\tau, y, \mathbf{a}}(t, x) - \mathcal{A}^{\hat{\mathbf{u}}} C^{\tau, y, \hat{\mathbf{u}}}(t, x) - 2\mathcal{A}^{\mathbf{a}} c^{\tau, y, t}(t, x) \\ &+ \int_t^T (\mathcal{A}^{\mathbf{a}})^2 c^{\tau, y, s}(t, x) ds + (\mathcal{A}^{\mathbf{a}})^2 f^{\tau, y}(t, x) + G_z(\tau, y, g(t, x)) (\mathcal{A}^{\mathbf{a}})^2 g(t, x) \\ &+ \mathcal{A}^{\mathbf{a}} g(t, x)^{\top} G_{zz}(\tau, y, g(t, x)) \mathcal{A}^{\mathbf{a}} g(t, x) \end{aligned}$$

for any given $(\tau, y) \in [0, T] \times \mathbb{X}$.

Proposition 4.3 provides the second-order expansion of $J(t, x; \mathbf{u}_{t, \epsilon, \mathbf{a}})$ in ϵ in the neighborhood of 0. When the first-order derivative is zero, the second-order derivative determines the sign of $J(t, x; \mathbf{u}_{t, \epsilon, \mathbf{a}}) - J(t, x; \hat{\mathbf{u}})$ and thus whether $\hat{\mathbf{u}}$ is a strong equilibrium. We observe from (4.3) that, in contrast to $\Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{a})$, the value of $\mathbf{a}$ at (t, x) alone is insufficient to determine $\Lambda^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{a})$. Indeed, $\Lambda^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{a})$ also depends on $\mathbf{a}_t$, $\mathbf{a}_x$, and $\mathbf{a}_{xx}$ at (t, x) . Thus, we guess that whether $\hat{\mathbf{u}}$ is a strong equilibrium strategy in general depends on the choice of the set of alternative strategies $\mathbf{D}$. Later on, we will see an example to confirm the guess.

Theorem 3.3 shows that the first-order derivative is equal to zero for $\mathbf{a}$ with $\mathbf{a}(t, x) = \hat{\mathbf{u}}(t, x)$, so it is crucial to examine the second order derivative for those $\mathbf{a}$. To this end, we will provide an alternative representation of $\Lambda^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{a})$ for such $\mathbf{a}$, and some notation is needed.

According to Theorem 3.3, $\Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{u})$ depends on the value of $\mathbf{u}(t, x)$ only, so for fixed $(\tau, y) \in [0, T] \times \mathbb{X}$, we simply consider $\Gamma^{\tau, y, \hat{\mathbf{u}}}$ to be a mapping from $(t, x, u) \in [0, T] \times \mathbb{X} \times \mathbf{U}$ to $\Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; u)$. Then, with Assumption 4.2 in place, $\Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; u)$ is twice continuously differentiable in u , and we denote the first-order and second-order derivatives as $\Gamma_u^{\tau, y, \hat{\mathbf{u}}}(t, x; u) \in \mathbb{R}^{1 \times m}$ and $\Gamma_{uu}^{\tau, y, \hat{\mathbf{u}}}(t, x; u) \in \mathbb{R}^{m \times m}$, respectively. Again, by Assumption 4.2 and assuming $\mathbf{u}(t, x)$ to be continuously differentiable in x , $\Gamma_u^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$ has a continuous first-order derivative in x , denoted as $\nabla_x \Gamma_u^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x)) \in \mathbb{R}^{m \times n}$. Note that $\nabla_x \Gamma_u^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$ is different from $\Gamma_{ux}^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$, the latter being the partial derivative of $\Gamma_u^{\tau, y, \hat{\mathbf{u}}}(t, x; u)$ with respect to x , evaluated at $u = \mathbf{u}(t, x)$. $\nabla_x \Gamma_u^{t, x, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$ then denotes $\nabla_x \Gamma_u^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$ evaluated at $(\tau, y) = (t, x)$.

Denote by $\mu_{u_i}(t, x, u)$ the partial derivative of $\mu(t, x, u)$ with respect to the i th component of $u = (u_1, \dots, u_m)^{\top}$, and for given $\mathbf{u}$, denote $\mu_{u_i}^{\mathbf{u}}(t, x) := \mu_{u_i}(t, x, \mathbf{u}(t, x))$, $i = 1, \dots, m$. Define σ_{u_i} and $\sigma_{u_i}^{\mathbf{u}}$ similarly, and denote $\Upsilon_{u_i}^{\mathbf{u}}(t, x) := \sigma_{u_i}^{\mathbf{u}}(t, x) \sigma^{\mathbf{u}}(t, x)^{\top} + \sigma^{\mathbf{u}}(t, x) \sigma_{u_i}^{\mathbf{u}}(t, x)^{\top}$, $i = 1, \dots, m$. Given $[a, b] \subseteq [0, T]$, $\mathbf{u} \in \mathbf{U}$, and $\xi \in \mathcal{C}^{1,2}([a, b] \times \mathbb{X})$ taking values in $\mathbb{R}$, denote by $\mathcal{B}^{\mathbf{u}} \xi(t, x)$ the partial derivative of $\mathcal{A}^{\mathbf{u}} \xi(t, x)$ with respect to u evaluated at $u = \mathbf{u}(t, x)$, i.e., $\mathcal{B}^{\mathbf{u}} \xi(t, x) := ((\mathcal{B}^{\mathbf{u}} \xi)_1(t, x), \dots, (\mathcal{B}^{\mathbf{u}} \xi)_m(t, x)) \in \mathbb{R}^{1 \times m}$ with

$$(\mathcal{B}^{\mathbf{u}} \xi)_i(t, x) := \xi_x(t, x) \mu_{u_i}^{\mathbf{u}}(t, x) + \frac{1}{2} \text{tr}(\xi_{xx}(t, x)^{\top} \Upsilon_{u_i}^{\mathbf{u}}(t, x)), \quad t \in [a, b], x \in \mathbb{X}.$$

For $\xi \in \mathcal{C}^{1,2}([a, b] \times \mathbb{X})$ taking values in $\mathbb{R}^l$, $\mathcal{B}^{\mathbf{u}} \xi$ is the collection of $\mathcal{B}^{\mathbf{u}}$ applied to each component of ξ and is arranged to take values in $\mathbb{R}^{l \times m}$.

PROPOSITION 4.4. *Suppose Assumptions 2.2 and 4.2 hold. Then, for fixed $x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$, $t \in [0, T]$, and any $\mathbf{a} \in \mathbf{U} \cap \bar{\mathcal{C}}^{1,2}([0, T] \times \mathbb{X})$ with $\mathbf{a}(t, x) = \hat{\mathbf{u}}(t, x)$, we have, for any $(\tau, y) \in [0, T] \times \mathbb{X}$,*

$$(4.4) \quad \begin{aligned} \Lambda^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{a}) &= \Gamma_u^{\tau, y, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) \mathcal{A}^{\hat{\mathbf{u}}}(\mathbf{a} - \hat{\mathbf{u}})(t, x) \\ &+ \frac{1}{2} \text{tr} \left\{ \Upsilon^{\hat{\mathbf{u}}}(t, x) (\mathbf{a}_x(t, x) - \hat{\mathbf{u}}_x(t, x))^{\top} \Gamma_{uu}^{\tau, y, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) (\mathbf{a}_x(t, x) - \hat{\mathbf{u}}_x(t, x)) \right\} \\ &+ \text{tr} \left\{ \Upsilon^{\hat{\mathbf{u}}}(t, x) (\nabla_x \Gamma_u^{\tau, y, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) - \mathcal{B}^{\hat{\mathbf{u}}} g(t, x)^{\top} G_{zz}(\tau, y, g(t, x)) g_x(t, x))^{\top} \right. \\ &\quad \left. (\mathbf{a}_x(t, x) - \hat{\mathbf{u}}_x(t, x)) \right\}. \end{aligned}$$

Suppose that $\hat{\mathbf{u}}(t, x)$ is a strong equilibrium; then it must be a weak equilibrium, and thus condition (3.8) holds. Then, for $\mathbf{a}$ with $\mathbf{a}(t, x) = \hat{\mathbf{u}}(t, x)$, Theorem 3.3 implies that $\Delta^{\hat{\mathbf{u}}}(t, x; \mathbf{a}) = \Delta^{\hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}) = 0$, so the second-order derivative $\Lambda^{t, x, \hat{\mathbf{u}}}(t, x; \mathbf{a})$ determines whether the agent would switch from $\hat{\mathbf{u}}$ to $\mathbf{a}$. For such $\mathbf{a}$, $\Lambda^{t, x, \hat{\mathbf{u}}}(t, x; \mathbf{a})$ is given by (4.4). The first term is zero when $\hat{\mathbf{u}}(t, x)$ is in the interior of $\mathbb{U}$ because $\hat{\mathbf{u}}(t, x)$ is the maximizer of $\Gamma^{t, x, \hat{\mathbf{u}}}(t, x; u)$ in $u \in \mathbb{U}$. The second and third terms are quadratic and linear, respectively, in $\mathbf{a}_x(t, x) - \hat{\mathbf{u}}_x(t, x)$. If we allow any alternative strategy $\mathbf{a}$, $\mathbf{a}_x(t, x) - \hat{\mathbf{u}}_x(t, x)$ can take any value. As a result, as long as the linear coefficient in the third term is not equal to zero, the sum of the second and third terms is positive for some $\mathbf{a}$, and thus $\Lambda^{t, x, \hat{\mathbf{u}}}(t, x; \mathbf{a})$ is positive for this $\mathbf{a}$, implying that the agent would switch to $\mathbf{a}$ and thus $\hat{\mathbf{u}}$ is not a strong equilibrium. Thus, a necessary condition for $\hat{\mathbf{u}}$ to be a strong equilibrium is that the linear coefficient in the third term of $\Lambda^{t, x, \hat{\mathbf{u}}}(t, x; \mathbf{a})$ in (4.4) is zero. To formalize the above discussion, we need to strengthen Assumption 2.2 as follows.

Assumption 4.5.

- (i) $\mu(t, x, u)$ and $\sigma(t, x, u)$ are locally Lipschitz in $(x, u) \in \mathbb{X} \times \mathbb{U}$, uniformly in $t \in [0, T]$.
- (ii) For any compact set $D \subseteq \mathbb{U}$, $\mu(t, x, u)$ and $\sigma(t, x, u)$ are of linear growth in $x \in \mathbb{X}$, uniformly in $(t, u) \in [0, T] \times D$ when $\mathbb{X} = \mathbb{R}^n$, and have bounded norm in $x \in \mathbb{X}$, uniformly in $(t, u) \in [0, T] \times D$ when $\mathbb{X} = (0, +\infty)$.
- (iii) For any compact set $D \subseteq \mathbb{U}$ and $(\tau, y) \in [0, T] \times \mathbb{X}$, $C(\tau, y, t, x, u)$ and $F(\tau, y, x)$ are of polynomial growth in $x \in \mathbb{X}$, uniformly in $(t, u) \in [0, T] \times D$.
- (iv) For any $(\tau, y) \in [0, T] \times \mathbb{X}$, $x \in \mathbb{X}$, and $u \in \mathbb{U}$, $\mu(t, x, u)$ and $\sigma(t, x, u)$ are right-continuous in $t \in [0, T]$ and $\lim_{t' \geq t, (t', x', u') \rightarrow (t, x, u)} C(\tau, y, t', x', u') = C(\tau, y, t, x, u)$ for any $t \in [0, T]$.

It is easy to see that Assumption 4.5 implies Assumption 2.2 and essentially stipulates that all sufficiently smooth and bounded strategies are feasible.

THEOREM 4.6. *Suppose that Assumptions 4.5 and 4.2 hold and that $\mathbf{D} \supseteq \mathbf{U} \cap \mathcal{C}_b^{\infty}([0, T] \times \mathbb{X})$. Then, a necessary condition for $\hat{\mathbf{u}}(t, x)$ to be a strong equilibrium strategy is that for any $x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ and $t \in [0, T]$ with $\hat{\mathbf{u}}(t, x)$ in the interior of $\mathbb{U}$, the following system of equations holds:*

$$(4.5) \quad \begin{cases} \Gamma_u^{t, x, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) = 0, \\ \Upsilon^{\hat{\mathbf{u}}}(t, x) (\nabla_x \Gamma_u^{t, x, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) - \mathcal{B}^{\hat{\mathbf{u}}} g(t, x)^{\top} G_{zz}(t, x, g(t, x)) g_x(t, x))^{\top} = 0. \end{cases}$$

The first equation in (4.5) is simply the differential form of the extended HJB equation (3.9), which is a sufficient and necessary condition for $\hat{\mathbf{u}}$ to be a weak equilibrium. If $\hat{\mathbf{u}}$ is a strong equilibrium, the second equation of (4.5) also needs to hold. The left-hand side of this equation consists of two terms. The first term, which involves $\nabla_x \Gamma_u^{t,x,\hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x))$, arises from the dependence of C , F , and G in the objective function (2.2) on the current state x . Indeed, suppose that x is in the interior of $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ and $\hat{\mathbf{u}}(t, x)$ is in the interior of $\mathbb{U}$. Then, there exists a neighborhood of x in the interior of $\mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ such that for any x' in this neighborhood, $\hat{\mathbf{u}}(t, x')$ is also in the interior of $\mathbb{U}$ by its continuity in x' . As a result, we can differentiate with respect to x on both sides of the first equation in (4.5) to derive from the chain rule that $\nabla_x \Gamma_u^{t,x,\hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) + \frac{\partial \Gamma_u^{\tau,y,\hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x))}{\partial y} \Big|_{(\tau,y)=(t,x)} = 0$, where we recall the definition of $\nabla_x \Gamma_u^{t,x,\hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x))$ in the discussion preceding Proposition 4.4. As a result, $\nabla_x \Gamma_u^{t,x,\hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) = 0$ if and only if $\frac{\partial \Gamma_u^{\tau,y,\hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x))}{\partial y} \Big|_{(\tau,y)=(t,x)} = 0$, and by (3.7), the latter is the case if C , F , and G do not depend on the current state x . On the other hand, the second term, which involves $G_{zz}(t, x, g(t, x))$, clearly arises from the nonlinear dependence of G on $\mathbb{E}_{t,x}[X^{\hat{\mathbf{u}}}(T)]$: This term vanishes if the nonlinearity is absent.

As one may guess, the two equations in (4.5) can hardly hold at the same time. If this is the case, then the strong equilibrium does not exist. As discussed above, the second equation arises from two sources of time-inconsistency: the dependence of C , F , and G on the current state and the nonlinear dependence of G on the mean of the terminal state. Thus, one may conjecture that the strong equilibrium does not exist with the above two sources of time-inconsistency. Indeed, in section 4.4 we will prove that for the mean-variance problem (2.5), in which the quadratic dependence on the mean of the terminal wealth in the objective function therein causes time-inconsistency, the strong equilibrium does not exist.

For problems (2.6) and (2.7), none of C , F , and G depends on the current state and $G \equiv 0$; the time inconsistency in these two problems arises from the dependence of C and F on the current time. Thus, for these two problems, the second equation in (4.5) is automatically satisfied. For deterministic time-inconsistent problems, the second equation in (4.5) also holds because $\Upsilon^{\hat{\mathbf{u}}} \equiv 0$ in this case. Therefore, to study the existence of strong equilibrium for problems (2.6) and (2.7) and for deterministic time-inconsistent problems, the information provided by the second-order expansion of $J(t, x; \mathbf{u}_{t,\epsilon,\mathbf{a}}) - J(t, x; \hat{\mathbf{u}})$ in ϵ is insufficient. To this end, we need to investigate the third-order expansion, which will be conducted in the next subsection.

4.3. Third-order conditions for strong equilibrium strategies. To examine the third-order expansion of $J(t, x; \mathbf{u}_{t,\epsilon,\mathbf{a}}) - J(t, x; \hat{\mathbf{u}})$ in ϵ , we need additional smoothness and growth conditions.

Assumption 4.7. For any given $(\tau, y) \in [0, T] \times \mathbb{X}$ and $t \in [0, T]$, there exists $\tilde{t} \in (t, T]$ such that $\hat{\mathbf{u}} \in \tilde{\mathcal{C}}^{2,4}([t, \tilde{t}] \times \mathbb{X})$ and the following hold: (i) $f^{\tau,y}, g \in \tilde{\mathcal{C}}^{3,6}([t, \tilde{t}] \times \mathbb{X})$; (ii) $c^{\tau,y,s} \in \tilde{\mathcal{C}}^{3,6}([t, \tilde{t} \wedge s] \times \mathbb{X})$ for any $s \in (t, T]$ and $\frac{\partial^{j+\alpha} c^{\tau,y,s}(t', x')}{\partial t^j \partial x^\alpha}$ is of polynomial growth in $x' \in \mathbb{X}$, uniformly in $t' \in [t, \tilde{t} \wedge s]$ and $s \in (t, T]$, for any α with $|\alpha| \leq 6 - 2j$ and $j = 0, 1, 2, 3$; (iii) $G(\tau, y, z)$ is thrice continuously differentiable in z and $\mu, \sigma, C(\tau, y, \cdot, \cdot, \cdot) \in \tilde{\mathcal{C}}^{2,4,4}([t, \tilde{t}] \times \mathbb{X} \times \mathbb{U})$; and (iv) for any $x \in \mathbb{X}$, $\lim_{s \downarrow t} \xi^{\tau,y,s}(t, x) = \xi^{\tau,y,t}(t, x)$ for $\xi^{\tau,y,s}$ to be any of $\frac{\partial^\alpha c^{\tau,y,s}}{\partial x^\alpha}$ with $|\alpha| \leq 4$ and $\lim_{s \downarrow t} \xi^{\tau,y,s}(t, x)$ exists, with the limit denoted by $\xi^{\tau,y,t}(t, x)$, for $\xi^{\tau,y,s}$ to be any of $\frac{\partial^{j+\alpha} c^{\tau,y,s}}{\partial t^j \partial x^\alpha}$ with $|\alpha| \leq 4 - 2j$ and $j = 1, 2$.

Although the examination of whether $\hat{\mathbf{u}}$ is a strong equilibrium based on the third-order expansion of $J(t, x; \mathbf{u}_{t, \epsilon, \mathbf{a}}) - J(t, x; \hat{\mathbf{u}})$ in ϵ is conceptually similar to that based on the second-order expansion, we only manage to obtain a complete result in the case in which $G \equiv 0$ and the dimension of the state space is one (i.e., $\mathbb{X}$ is $\mathbb{R}$ or $(0, +\infty)$) because the calculation involved in addressing the general case is excessively demanding. The result in this special case, however, is sufficient for our use to analyze problems (2.6) and (2.7); see section 4.4 below. Notationwise, for any $\mathbf{u} \in \mathbf{U}$ with sufficient smoothness, $\nabla_{xx}\Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$ denotes the second-order derivative of $\Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$ with respect to x , and $\nabla_{xx}\Gamma^{t, x, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$ stands for $\nabla_{xx}\Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$ with (τ, y) set to be (t, x) . $\nabla_t\Gamma^{\tau, y, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$ and $\nabla_t\Gamma^{t, x, \hat{\mathbf{u}}}(t, x; \mathbf{u}(t, x))$ are defined similarly.

THEOREM 4.8. *Suppose $\mathbb{X} = \mathbb{R}$ or $(0, +\infty)$, $G = 0$, Assumptions 4.5 and 4.7 hold, and $\mathbf{D} \supseteq \mathbf{U} \cap \mathfrak{C}_b^\infty([0, T] \times \mathbb{X})$. Then, a necessary condition for $\hat{\mathbf{u}}(t, x)$ to be a strong equilibrium is that for any $x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ and $t \in [0, T]$ with $\hat{\mathbf{u}}(t, x)$ in the interior of $\mathbf{U}$, the following system of equations holds:*

$$(4.6) \quad \begin{cases} \Gamma_u^{t, x, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) = 0, \\ \Upsilon^{\hat{\mathbf{u}}}(t, x) \nabla_x \Gamma_u^{t, x, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) = 0, \\ \Upsilon^{\hat{\mathbf{u}}}(t, x) \nabla_{xx} \Gamma_u^{t, x, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) = 0, \\ \nabla_t \Gamma_u^{t, x, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) + \mu(t, x, \hat{\mathbf{u}}(t, x))^\top \nabla_x \Gamma_u^{t, x, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x)) = 0. \end{cases}$$

Because we assume $G = 0$, (4.5) reduces to the first two equations of (4.6). If $\hat{\mathbf{u}}$ is a strong equilibrium, the third and fourth equations of (4.6) also need to hold. By the same reasoning as in the discussion following Theorem 4.6, the third equation, which involves $\nabla_{xx}\Gamma_u^{t, x, \hat{\mathbf{u}}}(t, x; \hat{\mathbf{u}}(t, x))$, arises from the dependence of C and F in the objective function (2.2) on the current state. Similarly, the fourth equation arises from the dependence of C and F on the current time t and the current state.

4.4. Nonexistence of strong equilibrium. Now, we are ready to discuss strong equilibrium strategies for the examples in section 2.3.

PROPOSITION 4.9. *Suppose $\mathbf{D} \supseteq \mathbf{U} \cap \mathfrak{C}_b^\infty([0, T] \times \mathbb{X})$.*

- (i) *Consider problem (2.5). Then, any $\hat{\mathbf{u}} \in \tilde{\mathfrak{C}}_{\text{pw}}^{1,4}([0, T] \times \mathbb{R})$ such that $\hat{\mathbf{u}}(t, x)$ is global Lipschitz in $x \in \mathbb{R}$, uniformly in $t \in [0, T]$, is not a strong equilibrium strategy.*
- (ii) *Consider problem (2.6), and suppose that $h''(0) \neq h'(0)^2$. Then, any bounded $\hat{\mathbf{u}} \in \tilde{\mathfrak{C}}_{\text{pw}}^{2,6}([0, T] \times (0, +\infty))$ such that $\mu^{\hat{\mathbf{u}}}(t, x)$ and $\sigma^{\hat{\mathbf{u}}}(t, x)$ are global Lipschitz in $x > 0$, uniformly in $t \in [0, T]$, is not a strong equilibrium strategy.*
- (iii) *Consider problem (2.7) with $\bar{\sigma} \neq 0$, and suppose that $\tilde{h}''(0) \neq \tilde{h}'(0)^2$ or $h'(0) \neq \tilde{h}'(0)$. Then, any $\hat{\mathbf{u}} \in \tilde{\mathfrak{C}}_{\text{pw}}^{2,6}([0, T] \times \mathbb{R})$ such that $\hat{\mathbf{u}}(t, x)$ is global Lipschitz in $x \in \mathbb{R}$, uniformly in $t \in [0, T]$, is not a strong equilibrium strategy.*

In Proposition 4.9(ii) and (iii), the conditions $h''(0) \neq h'(0)^2$ and $\tilde{h}''(0) \neq \tilde{h}'(0)^2$ hold for commonly used nonexponential discounting functions. For example, consider the following two forms of h that are used in [17]: $h(t) = (1 + \lambda t)e^{-\rho t}$ with $\lambda \neq 0$ and $h(t) = \lambda e^{-\rho_1 t} + (1 - \lambda)e^{-\rho_2 t}$ with $\lambda(1 - \lambda) \neq 0$ and $\rho_1 \neq \rho_2$. It is straightforward to see that for both forms, $h''(0) \neq h'(0)^2$.

Proposition 4.9 proves the nonexistence of strong equilibrium among smooth strategies for the three problems presented in section 2.3. In particular, the weak equilibrium strategies derived in Proposition 3.4 are not strong equilibria.

The idea of proving Proposition 4.9 is to show that the systems of equations (4.5) and (4.6) cannot hold for t close to the terminal time T . Because these two systems

of equations are derived under certain regularity assumptions on $\hat{\mathbf{u}}$, the nonexistence result in Proposition 4.9 is restricted to the set of strategies that are smooth up to a certain degree. On the other hand, Proposition 4.9 still holds if the regularity assumptions therein are imposed only in a subperiod towards the end time T , i.e., in $[T_1, T]$ for some $T_1 \in [0, T)$, because we only need to examine (4.5) and (4.6) for t close to T .¹¹

The above nonexistence result relies on the assumption that the set $\mathbf{D}$ of alternative strategies the agent can choose is larger than $\mathbb{U}$ so that the agent can choose $\mathbf{a}$ with arbitrary partial derivatives with respect to time and state. Given that in some of the existing studies of weak equilibrium in the literature, $\mathbf{D}$ is set to be $\mathbb{U}$, it is natural for us to examine whether the strong equilibrium exists in this case. Although we do not have a complete answer, we are able to examine whether the weak equilibrium strategies derived in Proposition 3.4 are strong equilibria for $\mathbf{D} = \mathbb{U}$.

PROPOSITION 4.10. *Suppose $\mathbf{D} = \mathbb{U}$.*

- (i) *Consider problem (2.5). Then, $\hat{\mathbf{u}}$ as defined in (3.10) is a strong equilibrium strategy.*
- (ii) *Consider problem (2.6), and recall $\hat{\mathbf{u}}$ as defined in (3.12). Then, for any $t \in [0, T)$ and $x > 0$, setting $u = \hat{\mathbf{u}}(t, x)$, we have $J(t, x; \mathbf{u}_{t, \epsilon, u}) - J(t, x; \hat{\mathbf{u}}) = \frac{x^{1-\gamma}}{6} \Theta(t) \epsilon^3 + \epsilon^3 o(1)$, where*

$$\Theta(t) = \hat{\zeta}(t)^{-\gamma} \left[-2 \left(h'(0) + (1-\gamma) \left(r + \frac{b^2}{2\gamma\bar{\sigma}^2} \right) + \gamma \hat{\zeta}(t) \right) \hat{\zeta}'(t) + \gamma \hat{\zeta}(t)^{-1} \left(\hat{\zeta}'(t) \right)^2 \right]. \quad (4.7)$$

Consequently, if there exists $t \in [0, T)$ such that $\Theta(t) > 0$, then $\hat{\mathbf{u}}$ is not a strong equilibrium strategy.

- (iii) *Consider problem (2.7), and suppose that Assumption 2.3 holds and $h, \tilde{h} \in \mathfrak{C}^3([0, T])$. Then, $\hat{\mathbf{u}}$ as defined in (3.15) is not a strong equilibrium strategy.*

In Proposition 4.10(ii), for some commonly used forms of the nonexponential discounting function h and reasonable parameter values, $\Theta(t)$ is indeed positive. For instance, following [17], we set $b = 0.07$, $\bar{\sigma} = 0.2$, $r = 0.05$, and $T = 3$, and we consider two types of discounting functions $h(t) = (1 + \lambda t)e^{-\rho t}$ and $h(t) = \lambda e^{-\rho_1 t} + (1 - \lambda)e^{-\rho_2 t}$ with $\lambda = 0.25$. By setting $\gamma = 0.5$, $\rho = 0.8$, $\rho_1 = 0.1$, and $\rho_2 = 0.8$, we plot $\Theta(t)/|\Theta(0)|$ as a function of t in the left panel of Figure 1, represented by the solid line when $h(t) = (1 + \lambda t)e^{-\rho t}$ and by the dashed line when $h(t) = \lambda e^{-\rho_1 t} + (1 - \lambda)e^{-\rho_2 t}$. The right panel plots $\Theta(t)/|\Theta(0)|$ with a different set of parameters: $\gamma = 0.9$, $\rho = 1.2$, $\rho_1 = 0.1$, and $\rho_2 = 1.4$. In all these plots, $\Theta(t)$ is positive for certain values of t .

Proposition 4.10 shows that for problems (2.6) and (2.7), the weak equilibrium strategies derived in Proposition 3.4 are not strong equilibrium even when $\mathbf{D} = \mathbb{U}$. On the other hand, for problem (2.5), the weak equilibrium derived in Proposition 3.4 is a strong equilibrium when we set $\mathbf{D} = \mathbb{U}$. Recall that Proposition 4.9 shows the nonexistence of strong equilibrium when we set $\mathbf{D} \supseteq \mathbb{U} \cap \mathfrak{C}_b^\infty([a, b] \times \mathbb{X})$. We then conclude that whether a strategy is a strong equilibrium can possibly depend on the choice of the set $\mathbf{D}$, i.e., on whether the agent is allowed to take nonconstant alternative strategies. This is in contrast to the case of weak equilibrium: as seen

¹¹Because this generalization is not significant and its proof becomes unduly complicated, we choose not to present them here. They are available from the authors upon request.

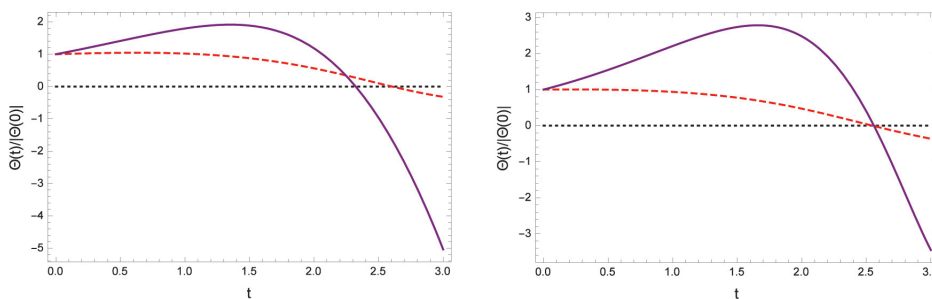


FIG. 1. $\Theta(t)/|\Theta(0)|$ as a function of $t \in [0, T]$, where Θ is as given by (4.7). The solid lines correspond to the discounting function $h(t) = (1 + \lambda t)e^{-\rho t}$, and the dashed lines correspond to the discounting function $h(t) = \lambda e^{-\rho_1 t} + (1 - \lambda)e^{-\rho_2 t}$. We set $b = 0.07$, $\bar{\sigma} = 0.2$, $r = 0.05$, and $T = 3$ in both the left and right panels. We set $\gamma = 0.5$, $\rho = 0.8$, $\rho_1 = 0.1$, and $\rho_2 = 0.8$ in the left panel and $\gamma = 0.9$, $\rho = 1.2$, $\rho_1 = 0.1$, and $\rho_2 = 1.4$ in the right panel.

from Theorem 3.3, whether a strategy is a weak equilibrium does not depend on the choice of $\mathbf{D}$.

5. Regular equilibrium. The nonexistence of strong equilibrium strategies as shown in section 4.4 leads to a dilemma: On the one hand, the notion of weak equilibrium is based on the first-order approximation of the change in value caused by deviating from a given strategy, so the agent may still be willing to deviate from a given weak equilibrium strategy. On the other hand, the notion of strong equilibrium, which does not use the above approximation, is too restrictive in that strong equilibrium strategies do not exist. This dilemma motivates us to consider the following new notion of equilibrium strategies.

DEFINITION 5.1 (Regular equilibrium). $\hat{\mathbf{u}} \in \mathbf{U}$ is a regular equilibrium strategy if for any $x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$, $t \in [0, T]$, and $\mathbf{a} \in \mathbf{D}$ with $\mathbf{a}(t, x) \neq \hat{\mathbf{u}}(t, x)$, there exists $\epsilon_0 \in (0, T - t)$ such that (4.1) holds.

When examining whether to deviate from a given strategy $\hat{\mathbf{u}}$ at a given time t with state x , the agent would naturally consider in her calculation only those alternatives $\mathbf{a}$ that are different from the given strategy in an infinitesimally small period of time. If the agent employs “approximation” in examining whether $\mathbf{a}$ and $\hat{\mathbf{u}}$ are different by comparing their values at the current time and state (t, x) only, the agent would then only consider alternatives $\mathbf{a}$ with $\mathbf{a}(t, x) \neq \hat{\mathbf{u}}(t, x)$, leading to the notion of regular equilibrium.

Note that the notion of regular equilibrium is not as economically sound as the notion of strong equilibrium due to the above “approximation;” the agent may still be willing to deviate from a regular equilibrium strategy to take another very similar, but different, strategy. We need this “approximation,” however, to expect the existence of regular equilibrium in many time-inconsistent control problems because we already showed that the strong equilibrium does not exist.

One possible economic justification for regular equilibrium is inertia, the tendency to maintain one’s current plan without changing it; see, for instance, the discussion of portfolio inertia in [4] and [31]. The agent never considers the possibility of a small deviation from $\hat{\mathbf{u}}$ to another very similar strategy $\mathbf{a}$ that yields the same action as $\hat{\mathbf{u}}$ at the current time with the current state, because such a small change in the strategy is outweighed by the intrinsic inertia of the agent.

The following theorem shows that regular equilibrium implies weak equilibrium, provides a sufficient condition under which a strategy is a regular equilibrium strategy, and proves a sufficient condition under which regular equilibrium and weak equilibrium are equivalent.

THEOREM 5.2. *Suppose Assumptions 2.2 and 3.2 hold and $\mathbb{U} \subseteq \mathbf{D} \subseteq \mathbf{U}$. Recall $\Delta^{\hat{\mathbf{u}}}(t, x; u)$ as given in Theorem 3.3. Then, the following are true:*

- (i) *If $\hat{\mathbf{u}}$ is a regular equilibrium strategy, it is also a weak equilibrium strategy.*
- (ii) *$\hat{\mathbf{u}}$ is a regular equilibrium strategy if the following hold:*

$$(5.1) \quad \Delta^{\hat{\mathbf{u}}}(t, x; u) < 0 \quad \forall u \in \mathbb{U} \text{ with } u \neq \hat{\mathbf{u}}(t, x), x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}, t \in [0, T).$$

- (iii) *Suppose that for any $x \in \mathbb{X}_t^{x_0, \hat{\mathbf{u}}}$ and $t \in [0, T)$, the maximization of $\Delta^{\hat{\mathbf{u}}}(t, x; u)$ in u admits a unique maximizer, which in particular holds when $\mathbb{U}$ is a convex set and $\Delta^{\hat{\mathbf{u}}}(t, x; u)$ is strictly quasi-concave in u . Then, $\hat{\mathbf{u}}$ is a weak equilibrium strategy if and only if it is a regular equilibrium strategy.*

PROPOSITION 5.3. *Suppose $\mathbb{U} \subseteq \mathbf{D} \subseteq \mathbf{U}$.*

- (i) *Consider problem (2.5). Then, $\hat{\mathbf{u}}$ as defined in (3.10) is a regular equilibrium strategy.*
- (ii) *Consider problem (2.6). Then, $\hat{\mathbf{u}}$ as defined in (3.12) is a regular equilibrium strategy.*
- (iii) *Consider problem (2.7), and suppose that Assumption 2.3 holds and $h, \tilde{h} \in \mathcal{C}^3([0, T])$. Then, $\hat{\mathbf{u}}$ as defined in (3.15) is not a regular equilibrium strategy.*

Suppose that we have identified a weak equilibrium strategy $\hat{\mathbf{u}}$. Then, $\hat{\mathbf{u}}(t, x)$ must be the maximizer of $\Delta^{\hat{\mathbf{u}}}(t, x; u)$ in $u \in \mathbb{U}$. Theorem 5.2 shows that if the maximizer is unique, $\hat{\mathbf{u}}$ is also a regular equilibrium. Proposition 5.3 shows that for many time-inconsistent problems, e.g., problems (2.5) and (2.6), the maximizer of $\Delta^{\hat{\mathbf{u}}}(t, x; u)$ in $u \in \mathbb{U}$ is indeed unique.

The above discussion shows that the notion of weak equilibrium is still useful: To ensure that the agent is unwilling to deviate from a weak equilibrium to take a large class of alternative strategies, where such a class is specified in the notion of regular equilibrium strategies, we only need to check whether the maximizer of the maximization problem in the extended HJB equation (3.9) is unique.

Finally, Proposition 5.3(iii) shows that a weak equilibrium can possibly not be a regular equilibrium strategy. Thus, the notion of regular equilibrium is a step forward from the notion of weak equilibrium towards a better definition of equilibrium.

6. Conclusion. The existing notion of equilibrium strategies for continuous-time time-inconsistent problems is not aligned with the standard definition of equilibrium in game theory; i.e., this notion does not imply that the self of an agent at each time is unwilling to deviate from a given equilibrium strategy. To address this issue, the authors of [29] propose the notion of strong equilibrium for an infinite-time horizon, continuous-time stochastic control problem in which an agent can control the generator of a time-homogeneous, continuous-time, finite-state Markov chain. We considered a general diffusion framework, which includes as special cases the investment and consumption problems studied in [8], [2], and [17] as well as another optimal consumption problem with a bequest, and studied weak and strong equilibria in this framework.

We proved a sufficient and necessary condition for a strategy to be a weak equilibrium and showed that it is independent of whether or not the set of alternative

strategies that the agent can switch to includes nonconstant strategies. We then examined the weak equilibrium strategies derived in [2] and [17] for the problems studied therein and found a weak equilibrium strategy for the optimal consumption problem with a bequest.

We then formulated the notion of strong equilibrium and proved two necessary conditions for a strategy to be a strong equilibrium. Each of the two conditions consists of a system of equations, and these equations give us insights that strong equilibrium can hardly exist. We confirmed these insights by showing that none of the above three investment and consumption problems admits any strong equilibrium with sufficient regularity if the set of alternative strategies includes nonconstant ones.

The above nonexistence result motivates us to propose the notion of regular equilibrium: in this notion the agent also directly evaluates the change in value caused by deviating from a given strategy, as in the notion of strong equilibrium, but she is only allowed to choose alternatives that are different from the given strategy at the current time and state. We proved that a regular equilibrium strategy must be a weak equilibrium and provided a sufficient condition under which the converse is also true. We then verified that the investment and consumption strategies in [2] and [17] satisfy this condition, showing that the notion of weak equilibrium is still useful as long as this condition holds. Finally, we found that the particular weak equilibrium strategy we derived for the optimal consumption problem with a bequest is not a regular equilibrium, showing that regular equilibrium is in general different from weak equilibrium.

Finally, while Theorem 3.3, which concerns weak equilibrium, can apply to problems with a finite control set $\mathbb{U}$, such as optimal stopping problems and optimal switching problems, the results about strong equilibrium in section 4 cannot apply to those problems. Indeed, for the latter results, we need to assume that the control strategies are smooth in the state variable; see Assumptions 4.2 and 4.7 and the assumption that $\mathbf{D} \supseteq \mathbf{U} \cap \mathcal{C}_b^\infty([0, T] \times \mathbb{X})$ as imposed in Theorems 4.6 and 4.8. Thus, it remains an open question whether strong equilibrium strategies exist for problems with a finite control set, such as for optimal stopping problems. On the other hand, for the case of a continuous control set $\mathbb{U}$, the result of nonexistence of strong equilibrium derived in Proposition 4.9 is restricted to the set of sufficiently regular strategies. The approach in the present paper to studying strong equilibrium, which is to do Taylor expansion of the increment reward of taking an alternative strategy and committing to it in a period of time with respect the length of the period, does not apply to strategies that are not sufficiently regular. Thus, it remains an open question whether there exists a strong equilibrium strategy without sufficient regularity.

Appendix A. A lemma on stochastic differential equations. For each $t \in [0, T]$ and $x \in \mathbb{X}$, consider the following SDE taking values in $\mathbb{X}$:

$$(A.1) \quad dX(s) = \tilde{\mu}(s, X(s))ds + \tilde{\sigma}(s, X(s))dW(s), \quad s \in [t, T], \quad X(t) = x,$$

where $(W(s))_{s \geq 0}$ is a d -dimensional standard Brownian motion, $\tilde{\mu}$ is a mapping from $[0, T] \times \mathbb{X}$ to $\mathbb{R}^n$, and $\tilde{\sigma}$ is a mapping from $[0, T] \times \mathbb{X}$ to $\mathbb{R}^{n \times d}$. We make the following assumption.

Assumption A.1.

- (i) $\tilde{\mu}(t, x)$ and $\tilde{\sigma}(t, x)$ are measurable in $(t, x) \in [0, T] \times \mathbb{X}$ and are locally Lipschitz in $x \in \mathbb{X}$, uniformly in $t \in [0, T]$.
- (ii) When $\mathbb{X} = \mathbb{R}^n$, $\tilde{\mu}(t, x)$ and $\tilde{\sigma}(t, x)$ are of linear growth in $x \in \mathbb{X}$, uniformly in $t \in [0, T]$, and when $\mathbb{X} = (0, +\infty)$, $\tilde{\mu}(t, x)$ and $\tilde{\sigma}(t, x)$ have bounded norm in

$x \in \mathbb{X}$, uniformly in $t \in [0, T]$.

LEMMA A.2. Suppose Assumption A.1 holds. Then, for any $t \in [0, T]$ and $x \in \mathbb{X}$, there exists a unique strong solution to the SDE (A.1). Moreover, when $\mathbb{X} = \mathbb{R}^n$, for any integer $\gamma \geq 1$, there exists a constant $L > 0$ such that

$$(A.2) \quad \mathbb{E}_{t,x} \left[\sup_{s \in [t, T]} \|X(s)\|^{2\gamma} \right] \leq L(1 + \|x\|^{2\gamma}) \quad \forall t \in [0, T], x \in \mathbb{X},$$

and when $\mathbb{X} = (0, +\infty)$, for any $\gamma \in \mathbb{R}$, there exists a constant $L > 0$ such that

$$(A.3) \quad \mathbb{E}_{t,x} \left[\sup_{s \in [t, T]} |X(s)|^\gamma \right] \leq Lx^\gamma \quad \forall t \in [0, T], x \in \mathbb{X}.$$

Appendix B. Sufficient conditions for assumptions. In this section, we provide sufficient conditions for Assumptions 2.3, 3.2, 4.2, and 4.7, respectively.

LEMMA B.1. Suppose h and $\tilde{h}$ are nonnegative, in $\mathfrak{C}^2([0, T])$, $h(0) = \tilde{h}(0) = 1$, $h'(0) \neq 0$, and $\tilde{h}'(0) = 0$. Then, (2.8) admits a unique solution in $\mathfrak{C}^1[0, T]$. Moreover, if $[h'(t) - \tilde{h}'(t)]/h'(0) > 0$ and $h''(t)/h'(0) \leq 0$ for all $t \in [0, T]$, then $\psi(t) > 0$ for any $t \in [0, T]$.

LEMMA B.2. Consider a strategy $\hat{\mathbf{u}}$, and suppose the following two conditions hold:

- (i) For each fixed $(\tau, y) \in [0, T] \times \mathbb{X}$, $\mu \in \bar{\mathfrak{C}}_{\text{pw}}^{0,2,2}([0, T] \times \mathbb{X} \times \mathbb{U})$, $\sigma \in \bar{\mathfrak{C}}_{\text{pw}}^{0,2,2}([0, T] \times \mathbb{X} \times \mathbb{U})$, $F(\tau, y, \cdot) \in \bar{\mathfrak{C}}^2(\mathbb{X})$, $C(\tau, y, s, x, u)$ is right-continuous in $s \in [0, T]$ for any $(x, u) \in \mathbb{X} \times \mathbb{U}$, $C(\tau, y, s, \cdot, \cdot) \in \mathfrak{C}^{2,2}(\mathbb{X} \times \mathbb{U})$ for each fixed $s \in [0, T]$, $\frac{\partial^{\alpha+\beta} C(\tau, y, s, x, u)}{\partial x^\alpha \partial u^\beta}$ is of polynomial growth in $(x, u) \in \mathbb{X} \times \mathbb{U}$, uniformly in $s \in [0, T]$, for any derivative indices α and β with $|\alpha| + |\beta| \leq 2$, and $G(\tau, y, z)$ is continuously differentiable with respect to z .
- (ii) $\hat{\mathbf{u}} \in \bar{\mathfrak{C}}_{\text{pw}}^{0,2}([0, T] \times \mathbb{X})$, $\mu^{\hat{\mathbf{u}}}$ and $\sigma^{\hat{\mathbf{u}}}$ are global Lipschitz in $x \in \mathbb{X}$, uniformly in $t \in [0, T]$, and in the case $\mathbb{X} = (0, +\infty)$, $\mu^{\hat{\mathbf{u}}}$ and $\sigma^{\hat{\mathbf{u}}}$ have bounded norm in $x \in \mathbb{X}$, uniformly in $t \in [0, T]$.

Then, $\hat{\mathbf{u}} \in \mathbf{U}$, and Assumption 3.2 holds.

LEMMA B.3. Consider a strategy $\hat{\mathbf{u}}$ and suppose the following two conditions hold:

- (i) For any fixed $(\tau, y) \in [0, T] \times \mathbb{X}$, $\mu \in \bar{\mathfrak{C}}_{\text{pw}}^{1,4,4}([0, T] \times \mathbb{X} \times \mathbb{U})$, $\sigma \in \bar{\mathfrak{C}}_{\text{pw}}^{1,4,4}([0, T] \times \mathbb{X} \times \mathbb{U})$, $F(\tau, y, \cdot) \in \bar{\mathfrak{C}}^4(\mathbb{X})$, $C(\tau, y, \cdot, \cdot, \cdot) \in \bar{\mathfrak{C}}_{\text{pw}}^{1,2,2}([0, T] \times \mathbb{X} \times \mathbb{U})$, $C(\tau, y, s, \cdot, \cdot) \in \mathfrak{C}^{4,4}(\mathbb{X} \times \mathbb{U})$ for each fixed $s \in [0, T]$, $\frac{\partial^{\alpha+\beta} C(\tau, y, s, x, u)}{\partial x^\alpha \partial u^\beta}$ is of polynomial growth in $(x, u) \in \mathbb{X} \times \mathbb{U}$, uniformly in $s \in [0, T]$, for any derivative indices α and β with $|\alpha| + |\beta| \leq 4$, and $G(\tau, y, z)$ is twice continuously differentiable with respect to z .
- (ii) $\hat{\mathbf{u}} \in \bar{\mathfrak{C}}_{\text{pw}}^{1,4}([0, T] \times \mathbb{X})$, $\mu^{\hat{\mathbf{u}}}$ and $\sigma^{\hat{\mathbf{u}}}$ are global Lipschitz in $x \in \mathbb{X}$, uniformly in $t \in [0, T]$, and in the case $\mathbb{X} = (0, +\infty)$, $\mu^{\hat{\mathbf{u}}}$ and $\sigma^{\hat{\mathbf{u}}}$ have bounded norm in $x \in \mathbb{X}$, uniformly in $t \in [0, T]$.

Then, $\hat{\mathbf{u}} \in \mathbf{U}$, and Assumption 4.2 holds.

LEMMA B.4. Consider a strategy $\hat{\mathbf{u}}$, and suppose the following two conditions hold:

- (i) For any fixed $(\tau, y) \in [0, T] \times \mathbb{X}$, $\mu \in \bar{\mathfrak{C}}_{\text{pw}}^{2,6,6}([0, T] \times \mathbb{X} \times \mathbb{U})$, $\sigma \in \bar{\mathfrak{C}}_{\text{pw}}^{2,6,6}([0, T] \times \mathbb{X} \times \mathbb{U})$, $F(\tau, y, \cdot) \in \bar{\mathfrak{C}}^6(\mathbb{X})$, $C(\tau, y, \cdot, \cdot, \cdot) \in \bar{\mathfrak{C}}_{\text{pw}}^{2,4,4}([0, T] \times \mathbb{X} \times \mathbb{U})$, $C(\tau, y, s, \cdot, \cdot) \in \mathfrak{C}^{6,6}(\mathbb{X} \times \mathbb{U})$ for each fixed $s \in [0, T]$, $\frac{\partial^{\alpha+\beta} C(\tau, y, s, x, u)}{\partial x^\alpha \partial u^\beta}$ is of polynomial growth

in $(x, u) \in \mathbb{X} \times \mathbb{U}$, uniformly in $s \in [0, T]$, for any derivative indices α and β with $|\alpha| + |\beta| \leq 6$, and $G(\tau, y, z)$ is thrice continuously differentiable with respect to z .

- (ii) $\hat{\mathbf{u}} \in \tilde{\mathcal{C}}_{\text{pw}}^{2,6}([0, T] \times \mathbb{X})$, $\mu^{\hat{\mathbf{u}}}$ and $\sigma^{\hat{\mathbf{u}}}$ are global Lipschitz in $x \in \mathbb{X}$, uniformly in $t \in [0, T]$, and in the case $\mathbb{X} = (0, +\infty)$, $\mu^{\hat{\mathbf{u}}}$ and $\sigma^{\hat{\mathbf{u}}}$ have bounded norm in $x \in \mathbb{X}$, uniformly in $t \in [0, T]$.

Then, $\hat{\mathbf{u}} \in \mathbf{U}$, and Assumption 4.7 holds.

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REFERENCES

- [1] N. BARBERIS, *A model of casino gambling*, Management Sci., 58 (2012), pp. 35–51.
- [2] S. BASAK AND G. CHABAKAURI, *Dynamic mean-variance asset allocation*, Rev. Financial Stud., 23 (2010), pp. 2970–3016.
- [3] E. BAYRAKTAR, J. ZHANG, AND Z. ZHOU, *Equilibrium concepts for time-inconsistent stopping problems in continuous time*, Math. Finance, 31 (2021), pp. 508–530.
- [4] Y. BILIAS, D. GEORGARAKOS, AND M. HALIASSOS, *Portfolio inertia and stock market fluctuations*, J. Money Credit Banking, 42 (2010), pp. 715–742.
- [5] T. BJÖRK, M. KHAPKO, AND A. MURGOCI, *On time-inconsistent stochastic control in continuous time*, Finance Stoch., 21 (2017), pp. 331–360.
- [6] T. BJÖRK AND A. MURGOCI, *A General Theory of Markovian Time Inconsistent Stochastic Control Problems*, 2010, <https://doi.org/10.2139/ssrn.1694759>.
- [7] T. BJÖRK AND A. MURGOCI, *A theory of Markovian time-inconsistent stochastic control in discrete time*, Finance Stoch., 18 (2014), pp. 545–592.
- [8] T. BJÖRK, A. MURGOCI, AND X. Y. ZHOU, *Mean-variance portfolio optimization with state dependent risk aversion*, Math. Finance, 24 (2014), 1.
- [9] S. CHRISTENSEN AND K. LINDENSJÖ, *On finding equilibrium stopping times for time-inconsistent Markovian problems*, SIAM J. Control Optim., 56 (2018), pp. 4228–4255, <https://doi.org/10.1137/17M1153029>.
- [10] M. DAI, H. JIN, S. KOU, AND Y. XU, *A dynamic mean-variance analysis for log returns*, Management Sci., 67 (2021), pp. 1093–1108.
- [11] S. EBERT AND P. STRACK, *Never, Ever Getting Started: On Prospect Theory without Commitment*, 2017, <https://doi.org/10.2139/ssrn.2765550>.
- [12] S. EBERT, W. WEI, AND X. Y. ZHOU, *Weighted discounting—on group diversity, time-inconsistency, and consequences for investment*, J. Econom. Theory, 189 (2020), 105089.
- [13] I. EKELAND AND A. LAZRAK, *Being Serious about Non-commitment: Subgame Perfect Equilibrium in Continuous Time*, preprint, <https://arxiv.org/abs/math/0604264>, 2006.
- [14] I. EKELAND AND A. LAZRAK, *Equilibrium Policies When Preferences Are Time Inconsistent*, preprint, <https://arxiv.org/abs/0808.3790>, 2008.
- [15] I. EKELAND AND A. LAZRAK, *The golden rule when preferences are time inconsistent*, Math. Financ. Econ., 4 (2010), pp. 29–55.
- [16] I. EKELAND, O. MBODJI, AND T. A. PIRVU, *Time-consistent portfolio management*, SIAM J. Financial Math., 3 (2012), pp. 1–32, <https://doi.org/10.1137/100810034>.
- [17] I. EKELAND AND T. PIRVU, *Investment and consumption without commitment*, Math. Financ. Econ., 2 (2008), pp. 57–86.

- [18] S. R. GRENADIER AND N. WANG, *Investment under uncertainty and time-inconsistent preferences*, J. Financ. Econ., 84 (2007), pp. 2–39.
- [19] C. HARRIS AND D. LAIBSON, *Instantaneous gratification*, Quart. J. Econom., 128 (2013), pp. 205–248.
- [20] X. D. HE, Z. JIANG, AND S. KOU, *Portfolio Selection under Median and Quantile Maximization*, preprint, <https://arxiv.org/abs/2008.10257>, 2020.
- [21] C. HERNÁNDEZ AND D. POSSAMAÏ, *Me, Myself and I: A General Theory of Non-Markovian Time-Inconsistent Stochastic Control for Sophisticated Agents*, preprint, <https://arxiv.org/abs/2002.12572>, 2020.
- [22] Y. HU, H. JIN, AND X. Y. ZHOU, *Time-inconsistent stochastic linear-quadratic control*, SIAM J. Control Optim., 50 (2012), pp. 1548–1572, <https://doi.org/10.1137/110853960>.
- [23] Y. HU, H. JIN, AND X. Y. ZHOU, *Time-inconsistent stochastic linear-quadratic control: Characterization and uniqueness of equilibrium*, SIAM J. Control Optim., 55 (2017), pp. 1261–1279, <https://doi.org/10.1137/15M1019040>.
- [24] Y.-J. HUANG AND A. NGUYEN-HUU, *Time-consistent stopping under decreasing impatience*, Finance Stoch., 22 (2018), pp. 69–95.
- [25] Y.-J. HUANG, A. NGUYEN-HUU, AND X. Y. ZHOU, *General stopping behaviors of naïve and non-committed sophisticated agents, with application to probability distortion*, Math. Finance, 30 (2020), pp. 310–340.
- [26] Y.-J. HUANG AND Z. WANG, *Optimal equilibria for multidimensional time-inconsistent stopping problems*, SIAM J. Control Optim., 59 (2021), pp. 1705–1729, <https://doi.org/10.1137/20M1343774>.
- [27] Y.-J. HUANG AND Z. ZHOU, *The optimal equilibrium for time-inconsistent stopping problems—the discrete-time case*, SIAM J. Control Optim., 57 (2019), pp. 590–609, <https://doi.org/10.1137/17M1139187>.
- [28] Y.-J. HUANG AND Z. ZHOU, *Optimal equilibria for time-inconsistent stopping problems in continuous time*, Math. Finance, 30 (2020), pp. 1103–1134.
- [29] Y.-J. HUANG AND Z. ZHOU, *Strong and weak equilibria for time-inconsistent stochastic control in continuous time*, Math. Oper. Res., 46 (2021), pp. 428–451.
- [30] Z. JIANG, *Time-Inconsistent Stochastic Control Problems in Continuous Time*, Ph.D. thesis, The Chinese University of Hong Kong, Hong Kong, 2020.
- [31] H. H. KIM, R. MAURER, AND O. S. MITCHELL, *Time is money: Rational life cycle inertia and the delegation of investment management*, J. Financ. Econ., 121 (2016), pp. 427–447.
- [32] D. LAIBSON, *Golden eggs and hyperbolic discounting*, Quart. J. Econom., 112 (1997), pp. 443–477.
- [33] H. MEI AND J. YONG, *Equilibrium strategies for time-inconsistent stochastic switching systems*, ESAIM Control Optim. Calc. Var., 25 (2019), 64.
- [34] T. O'DONOGHUE AND M. RABIN, *Doing it now or later*, Amer. Econom. Rev., 89 (1999), pp. 103–124.
- [35] B. PELEG AND M. E. YAARI, *On the existence of a consistent course of action when tastes are changing*, Rev. Econom. Stud., 40 (1973), pp. 391–401.
- [36] R. A. POLLAK, *Consistent planning*, Rev. Econom. Stud., 35 (1968), pp. 201–208.
- [37] R. H. STROTZ, *Myopia and inconsistency in dynamic utility maximization*, Rev. Econom. Stud., 23 (1955–1956), pp. 165–180.
- [38] H. WANG AND J. YONG, *Time-inconsistent stochastic optimal control problems and backward stochastic Volterra integral equations*, ESAIM Control Optim. Calc. Var., 27 (2021), 22.
- [39] Q. WEI, J. YONG, AND Z. YU, *Time-inconsistent recursive stochastic optimal control problems*, SIAM J. Control Optim., 55 (2017), pp. 4156–4201, <https://doi.org/10.1137/16M1079415>.
- [40] J. YONG, *Time-inconsistent optimal control problems and the equilibrium HJB equation*, Math. Control Relat. Fields, 2 (2012), pp. 271–329, <https://doi.org/10.3934/mcrf.2012.2.271>.