

ELECTROMAGNETIC WAVES GENERATED BY A HYBRID DIELECTRIC-PLASMONIC DIMER*

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Abstract. We know that the electric field generated by a plasmonic nano-particle (with negative permittivity) is given as a polarization of the incident electric field. Similarly, the electric field produced by a dielectric nano-particle (with positive but high permittivity) is given as a polarization of the incident magnetic field. In this work, we demonstrate that a hybrid dimer—composed of two closely coupled nano-particles, one plasmonic and the other dielectric—can polarize both the incident electric and the magnetic fields. Consequently, such hybrid dimers have the potential to modify both the electric permittivity and magnetic permeability of the surrounding medium. However, this dual modification occurs only when the two nano-particles share common resonant frequencies. We derive the asymptotic expansion of the fields generated by these hybrid dimers in the subwavelength regime for incident frequencies near their shared resonant frequencies.

Key words. Maxwell system, hybrid dimer, dielectric nano-particles, plasmonic nano-particle, plasmonic resonance, dielectric resonance

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1. Introduction and the main result.

1.1. Motivation. The interaction of light with nanoscale materials has revolutionized the understanding and manipulation of electromagnetic fields at subwavelength scales, paving the way for transformative applications in photonics, sensing, and metamaterials; see [6, 15, 23, 29, 34, 35]. Among the various nanostructures, hybrid dimers—composed of plasmonic and dielectric nano-particles represent a significant advancement due to their ability to interact with both the electric and the magnetic components of incident electromagnetic fields. Indeed, plasmonic nano-particles, characterized by their negative permittivity at optical frequencies, generate intense localized electric fields through surface plasmon resonances, effectively polarizing the incident electric field; see [5, 7, 10, 15, 20, 21, 22, 27]. In contrast, dielectric nano-particles with high positive permittivity are known for their low-loss interaction with the magnetic component of light, driven by displacement currents that produce magnetic dipole resonances; see [3, 9, 25, 30, 32]. When combined in a hybrid dimer, these nano-particles exhibit coupled electromagnetic responses, allowing simultaneous polarization of both the electric and the magnetic fields. This unique dual interaction makes hybrid dimers a promising platform for engineering media with tailored electric permittivity and magnetic permeability [24, 33].

A critical feature of hybrid dimers is their ability to achieve resonant frequency alignment between the plasmonic and dielectric components. This resonance matching

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enhances their electromagnetic coupling, enabling the formation of hybrid modes with strong field localization and enhancement. Recent studies have demonstrated that these hybrid modes can lead to significant field amplification in the gap between nano-particles, often referred to as “hot spots,” which are central to applications such as surface-enhanced Raman spectroscopy (SERS), nanoscale biosensing, and nonlinear optics [2, 26, 33].

In this work, we explore the theoretical and computational aspects of hybrid dimer interactions in the subwavelength regime. By deriving the asymptotic expansion of the fields generated near the common resonant frequencies of plasmonic and dielectric nano-particles, we aim to elucidate the underlying mechanisms driving the dual polarization effects. This approach provides a robust framework for understanding how hybrid dimers can modify the electromagnetic properties of the surrounding medium, offering new insights into their role in advanced photonic devices. Below, we list a few advantages of using hybrid dimers over homogeneous dimers of the form plasmonic-plasmonic or dielectric-dielectric dimers.

1. *Field polarization mechanisms by hybrid dimers.* These hybrid dimers combine the complementary nature of plasmonic (electric field-driven) and dielectric (magnetic field-driven) responses, creating dual polarization effects. The combination allows for both the electric permittivity ϵ and the magnetic permeability μ modulation, which is unique to hybrid structures. In a next work, we will analyze with more details the case when we have a cluster of such dimers, regularly arranged in a given bounded domain, and show how the generated effective medium is a modulation, by averaging, of both the permittivity and the permeability offering a way how to design single or double negative electromagnetic media.
2. *Field localization and enhancement.* Hybrid dimers exhibit enhanced field localization and polarization in the gap region, benefiting from the synergistic effects of plasmonic and dielectric components. Indeed, in the case of homogeneous dimers, as plasmonic-plasmonic, strong electric field enhancements are localized in the gap, but losses due to ohmic heating can dampen efficiency. Regarding dielectric-dielectric dimers, we have relatively moderate enhancement but with less ohmic loss in comparison to plasmonic counterparts. Therefore, using heterogeneous dimers might improve both the enhancement, as for plasmonics, and reduce the ohmic loss, as for dielectrics.
3. *Potential applications.* The ability of hybrid dimers to modulate both ϵ and μ enables applications in designing metamaterials with tunable refractive indices, broadband absorbers, and devices requiring simultaneous electric and magnetic field control. Homogeneous plasmonic-plasmonic dimers are widely used in SERS, photothermal therapy, and plasmonic sensing due to their strong electric field enhancements, while dielectric-dielectric dimers are ideal for applications requiring low loss, such as photonic waveguides and resonators with high-Q factors. Therefore, hybrid dimers can offer possibilities to be used in both the mentioned applications, as they share the qualities and avoid their disadvantages, to some extent.

1.2. Main results. Let D_1 and D_2 be two bounded and C^2 -regular domains in $\mathbb{R}^3$, and model $D := D_1 \cup D_2$ to stand for a dimer composed of two nano-particles D_1 and D_2 . We assume that D_1 is a dielectric nano-particle; namely, its permittivity and permeability constants enjoy the following properties:

$$\epsilon_r^{(1)} := \frac{\epsilon^{(1)}}{\epsilon_0} \quad \text{with} \quad \operatorname{Re}(\epsilon^{(1)}) \gg 1 \quad \text{and} \quad \epsilon_r^{(1)} = 1 \quad \text{outside} \quad D_1,$$

while $\mu_r^{(1)} := \frac{\mu^{(1)}}{\mu_0} = 1$ in the whole space $\mathbb{R}^3$. The nano-particle D_2 is taken to be a plasmonic one, i.e., with a moderately contrasting relative permittivity $\epsilon_r^{(2)} := \frac{\epsilon^{(2)}}{\epsilon_0}$, enjoying negative real values,

$$\Re(\epsilon_r^{(2)}) < 0 \quad \text{and} \quad \epsilon_r^{(2)} \sim 1 \quad \text{with} \quad \epsilon_r^{(2)} = 1 \quad \text{outside} \quad D_2,$$

and a permeability satisfying $\mu_r^{(2)} := \frac{\mu^{(2)}}{\mu_0} = 1$ in the whole space $\mathbb{R}^3$. More details will be given later on the related quantities. The electromagnetic wave propagation, in the time-harmonic regime, with the presence of the dimer D satisfies

$$(1.1) \quad \begin{cases} \operatorname{Curl}(E) - i k \mu_r H = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{Curl}(H) + i k \epsilon_r E = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where the total field (E, H) is of the form $(E := E^{Inc} + E^s, H := H^{Inc} + H^s)$ and the incident plane wave (E^{Inc}, H^{Inc}) is of the form

$$(1.2) \quad E^{Inc}(x, \theta, \mathbf{p}) = \mathbf{p} e^{i k \theta \cdot x} \quad \text{and} \quad H^{Inc}(x, \theta, \mathbf{p}) = (\theta \times \mathbf{p}) e^{i k \theta \cdot x},$$

with $\theta, \mathbf{p} \in \mathbb{S}^2$, $\mathbb{S}^2$ being the unit sphere, such that $\theta \cdot \mathbf{p} = 0$, as the direction of incidence and polarization, respectively, and the scattered field (E^s, H^s) satisfies the Silver–Müller radiation condition (SMRC) at infinity:

$$(1.3) \quad \sqrt{\mu_0 \epsilon_0^{-1}} H^s(x) \times \frac{x}{|x|} - E^s(x) = \mathcal{O}\left(\frac{1}{|x|^2}\right).$$

This problem is well-posed in appropriate Sobolev spaces (see [12] and [28]), and we have the following behaviors:

$$(1.4) \quad E^s(x) = \frac{e^{i k |x|}}{|x|} \left(E^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right) \right) \quad \text{as} \quad |x| \rightarrow \infty$$

and

$$H^s(x) = \frac{e^{i k |x|}}{|x|} \left(H^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right) \right) \quad \text{as} \quad |x| \rightarrow \infty,$$

where $(E^\infty(\hat{x}), H^\infty(\hat{x}))$ is the corresponding electromagnetic far-field pattern of (1.1) in the propagation direction $\hat{x} := \frac{x}{|x|}$.

Next, we present the necessary assumptions on the model (1.1) to derive the main results.

Assumption 1.1.

1. *Assumption on the dimer.* Suppose that each component D_m of D can be represented by $D_m = aB_m + z_m$ with the parameter $a > 0$ and the location z_m for $m = 1, 2$. Denote

$$a := \max\{\operatorname{diam}(D_1), \operatorname{diam}(D_2)\} \quad \text{and} \quad d := \operatorname{dist}(D_1, D_2).$$

We take

$$(1.5) \quad d = \alpha_0 a^t,$$

with t such that $0 < t < 1$, and α_0 is a positive constant independent of a .

2. *Assumptions on the permittivity and permeability of each particle.* Regarding the permittivity, we assume that

$$(1.6) \quad \eta(x) := \begin{cases} \eta_1 := \epsilon_r^{(1)} - 1 = \eta_0 a^{-2} & \text{if } x \in D_1, \\ \eta_2 := \epsilon_r^{(2)} - 1 \sim 1 \quad \text{with} \quad \Re(\epsilon_r^{(2)}) < 0 & \text{if } x \in D_2, \end{cases}$$

where η_0 is a constant in the complex plane independent of the parameter a , such that $\Re(\eta_0) \in \mathbb{R}^+$. Moreover, regarding the permeability $\mu_r^{(m)}$, we assume that $\mu_r^{(m)} = 1$ for $m = 1, 2$.

3. *Assumption on the shape of B_m .*

(a) Regarding the shape B_1 . Since

$$(1.7) \quad \mathbb{H}_0(\operatorname{div} = 0)(B_1) \equiv \operatorname{Curl}(\mathbb{H}_0(\operatorname{Curl}) \cap \mathbb{H}(\operatorname{div} = 0))(B_1),$$

where

$$\begin{aligned} & \mathbb{H}_0(\operatorname{Curl}) \cap \mathbb{H}(\operatorname{div} = 0)(B_1) \\ &:= \left\{ E \in \mathbb{L}^2(B_1), \text{ such that } \operatorname{Curl}(E) \in \mathbb{L}^2(B_1), \right. \\ & \quad \left. \operatorname{div}(E) = 0 \text{ in } B_1, \text{ and } \nu \times E = 0 \text{ on } \partial B_1 \right\} \end{aligned}$$

(see, for instance, [4, Theorem 3.17]), then there exists

$$\phi_{n,m,B_1} \in \mathbb{H}_0(\operatorname{Curl}) \cap \mathbb{H}(\operatorname{div} = 0)(B_1),$$

such that

$$(1.8) \quad e_{n,m,B_1}^{(1)} = \nabla \times \phi_{n,m,B_1} \quad \text{with} \quad \nu \times \phi_{n,m,B_1} = 0 \text{ and } \nabla \cdot \phi_{n,m,B_1} = 0,$$

where $e_{n,m,B_1}^{(1)} \in \mathbb{H}_0(\operatorname{div} = 0)(B_1)$ is an eigenfunction, corresponding to the eigenvalue $\lambda_n^{(1)}(B_1)$, related to the Newtonian operator $N_{B_1}(\cdot)$ defined, from $\mathbb{L}^2(B_1)$ to $\mathbb{L}^2(B_1)$, by¹

$$N_{B_1}(E)(x) := \int_{B_1} \frac{1}{4\pi} \frac{1}{|x-y|} E(y) dy,$$

i.e.,

$$N_{B_1}(e_{n,m,B_1}^{(1)}) = \lambda_n^{(1)}(B_1) e_{n,m,B_1}^{(1)} \quad \text{in } B_1.$$

We assume² that for B_1 ,

$$(1.9) \quad \int_{B_1} \phi_{n_0,B_1}(y) dy \otimes \int_{B_1} \phi_{n_0,B_1}(y) dy \neq 0 \quad \text{for certain } n_0 \in \mathbb{N},$$

¹The Newtonian operator $N_{B_1}(\cdot)$ is bounded from $\mathbb{L}^2(B_1)$ to $\mathbb{H}^2(B_1)$.

²To reduce the length of the notation in what follows, we eliminate the need to depend on the vector ϕ_{n,m,B_1} with respect to multiplicity index m .

and

$$\sum_n \frac{1}{\lambda_n^{(3)}(B_1)} \int_{B_1} e_{n,B_1}^{(3)}(x) dx \otimes \int_{B_1} e_{n,B_1}^{(3)}(x) dx \neq 0,$$

where $e_{n,B_1}^{(3)} \in \nabla \mathcal{H}armonic(B_1)$ is an eigenfunction, corresponding to the eigenvalue $\lambda_n^{(3)}(B_1)$, related to the Magnetization operator $\nabla M_{B_1}(\cdot)$ defined, from $\mathbb{L}^2(B_1)$ to $\mathbb{L}^2(B_1)$, by

$$\nabla M_{B_1}(E)(x) := \nabla_x \int_{B_1} \nabla_y \left(\frac{1}{4\pi |x-y|} \right) \cdot E(y) dy,$$

i.e.,

$$(1.10) \quad \nabla M_{B_1}(e_n^{(3)}) = \lambda_n^{(3)}(B_1) e_n^{(3)} \quad \text{in } B_1.$$

(b) Regarding the shape B_2 . For B_2 , we assume that

$$\sum_n \int_{B_2} \phi_{n,B_2}(y) dy \otimes \int_{B_2} \phi_{n,B_2}(y) dy \neq 0,$$

where $\phi_{n,B_2}(\cdot)$ satisfy (1.7) in B_2 , and

$$\int_{B_2} e_{n_*,B_2}^{(3)}(x) dx \otimes \int_{B_2} e_{n_*,B_2}^{(3)}(x) dx \neq 0 \quad \text{for certain } n_* \in \mathbb{N},$$

where $e_{n,B_2}^{(3)}(\cdot) \in \nabla \mathcal{H}armonic(B_2)$ is an eigenfunction, corresponding to the eigenvalue $\lambda_n^{(3)}(B_2)$, related to the Magnetization operator $\nabla M_{B_2}(\cdot)$.

To gain more information about the used spaces $\mathbb{H}_0(\operatorname{div}=0)(B_m)$, $\nabla \mathcal{H}armonic(B_m)$, and the eigensystems that relate to the Newtonian operator $N_{B_m}(\cdot)$ and the Magnetization operator $\nabla M_{B_m}(\cdot)$, with $m=1,2$, it is recommended that the readers refer to Remark 2.3.

4. *Assumption on the used incident frequency k .* Define the vector Magnetization operator $\nabla M_{B_2}^0(\cdot)$ as (2.2). Under the Helmholtz decomposition of $\mathbb{L}^2$ -space given by (2.1), denote $(\lambda_n^{(3)}(B_2), e_n^{(3)})$ as the corresponding eigensystem of $\nabla M_{B_2}^0(\cdot)$ over the subspace $\nabla \mathcal{H}armonic$. There exist complex constants c_0 and d_0 , with $\operatorname{Re}(c_0), \operatorname{Re}(d_0) \in \mathbb{R}^+$, such that

(1.11)

$$1 - k^2 \eta_1 a^2 \lambda_{n_0}^{(1)}(B_1) = \pm c_0 a^h \quad \text{and} \quad 1 + \eta_2 \lambda_{n_*}^{(3)}(B_2) = \pm d_0 a^h, \quad a \ll 1,$$

where $\lambda_{n_0}^{(1)}(B_1)$ is the eigenvalue corresponding to $e_{n_0}^{(1)}$ in B_1 , related to the Newtonian operator $N_{B_1}(\cdot)$, and $\lambda_{n_*}^{(3)}(B_2)$ is the eigenvalue to $e_{n_*}^{(3)}$ in B_2 , related to the Magnetization operator $\nabla M_{B_2}(\cdot)$.

The conditions (1.6) and (1.11) can be derived from the Lorentz model by choosing appropriate incident frequency k . Indeed, recall the Lorentz model for the relative permittivity that

$$(1.12) \quad \epsilon_r = 1 + \frac{k_p^2}{k_0^2 - k^2 - ik\xi},$$

where k_p is the plasmonic frequency, k_0 is the undamped frequency resonance of the background, and ξ is the damping frequency with $\xi \ll 1$. The details are given in Appendix A.

NOTATION. The prefix *SM* indicates elements in the supplementary materials document. For example, subsection SM1.1 refers to that subsection in M171968.SM.pdf [local/web 483KB].

Based on the above conditions, we are now in a position to state our main result.

THEOREM 1.2. Let Assumption 1.1, on the problem (1.1)–(1.2)–(1.3), which is generated by the dimer D , be satisfied. Let x be away from D ; then, for $t, h \in (0, 1)$ such that

$$4 - h - 4t > 0,$$

the scattered wave admits the following expression:

$$(1.13) \quad E^s(x) = k^2 \sum_{m=1}^2 \left[\Upsilon_k(x, z_m) \cdot \tilde{R}_m - \nabla_y \Phi_k(x, z_m) \times \tilde{Q}_m \right] + \mathcal{O}\left(a^{\min(3; 7-2h-3t; 10-2h-7t)}\right),$$

and its far-field admits the following expansion:

$$(1.14) \quad E^\infty(\hat{x}) = \frac{k^2}{4\pi} \sum_{m=1}^2 e^{-ik\hat{x} \cdot z_m} \left[(I - \hat{x} \otimes \hat{x}) \cdot \tilde{R}_m + ik\hat{x} \times \tilde{Q}_m \right] + \mathcal{O}\left(a^{\min(3; 7-2h-3t; 10-2h-7t)}\right).$$

Here, $(\tilde{Q}_1, \tilde{R}_1, \tilde{Q}_2, \tilde{R}_2)$ is the vector solution to the following algebraic system:

$$(1.15) \quad \begin{pmatrix} I_3 & 0 & -\mathcal{B}_{13} & -\mathcal{B}_{14} \\ 0 & I_3 & -\mathcal{B}_{23} & -\mathcal{B}_{24} \\ -\mathcal{B}_{31} & -\mathcal{B}_{32} & I_3 & 0 \\ -\mathcal{B}_{41} & -\mathcal{B}_{42} & 0 & I_3 \end{pmatrix} \cdot \begin{pmatrix} \tilde{Q}_1 \\ \tilde{R}_1 \\ \tilde{Q}_2 \\ \tilde{R}_2 \end{pmatrix} = \begin{pmatrix} \frac{ik\eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) \\ a^3 \mathbf{P}_{0,1}^{(2)} \cdot E^{Inc}(z_1) \\ ik\eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot H^{Inc}(z_2) \\ \frac{\eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) \end{pmatrix},$$

with

$$\begin{aligned} \mathcal{B}_{13} &:= \frac{k^4 \eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot \Upsilon_k(z_1, z_2), \\ \mathcal{B}_{23} &:= k^2 a^3 \mathbf{P}_{0,1}^{(2)} \cdot \nabla \Phi_k(z_1, z_2), \\ \mathcal{B}_{31} &:= k^4 \eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot \Upsilon_k(z_2, z_1), \\ \mathcal{B}_{41} &:= \frac{k^2 \eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot \nabla \Phi_k(z_2, z_1), \\ \mathcal{B}_{14} &:= \frac{k^2 \eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot \nabla \Phi_k(z_1, z_2), \\ \mathcal{B}_{24} &:= k^2 a^3 \mathbf{P}_{0,1}^{(2)} \cdot \Upsilon_k(z_1, z_2), \\ \mathcal{B}_{32} &:= k^2 \eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot \nabla \Phi_k(z_2, z_1), \\ \mathcal{B}_{42} &:= \frac{k^2 \eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot \Upsilon_k(z_2, z_1), \end{aligned}$$

where $\Upsilon_k(\cdot, \cdot)$ is the dyadic Green's kernel given by

$$(1.16) \quad \Upsilon_k(\cdot, \cdot) := \frac{1}{k^2} \nabla \nabla \Phi_k(\cdot, \cdot) + \Phi_k(\cdot, \cdot) I_3,$$

where $\Phi_k(\cdot, \cdot)$ is the fundamental solution of the Helmholtz equation given by

$$(1.17) \quad \Phi_k(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \neq y,$$

I_3 is the identity matrix, and $\mathbf{P}_{0,i}^{(j)}$, for $i, j = 1, 2$, are the polarization tensors defined by

$$\begin{aligned} \mathbf{P}_{0,1}^{(1)} &= \int_{B_1} \phi_{n_0, B_1}(x) dx \otimes \int_{B_1} \phi_{n_0, B_1}(x) dx, \\ \mathbf{P}_{0,1}^{(2)} &= \sum_n \frac{1}{\lambda_n^{(3)}(B_1)} \int_{B_1} e_{n, B_1}^{(3)}(x) dx \otimes \int_{B_1} e_{n, B_1}^{(3)}(x) dx, \\ \mathbf{P}_{0,2}^{(1)} &= \sum_n \int_{B_2} \phi_{n, B_2}(x) dx \otimes \int_{B_2} \phi_{n, B_2}(x) dx, \\ \mathbf{P}_{0,2}^{(2)} &= \int_{B_2} e_{n_*, B_2}^{(3)}(x) dx \otimes \int_{B_2} e_{n_*, B_2}^{(3)}(x) dx, \end{aligned}$$

with $\phi_n(\cdot)$ satisfying (1.8) and $e_n^{(3)}$ fulfilling $\nabla M_{B_2}(e_n^{(3)}) = \lambda_n^{(3)}(B_2) e_n^{(3)}$.

Moreover, if we further extract the very dominant term (Q_1, R_2) , in D , from the solution to the algebraic system (1.15), it leads to the following corollary.

COROLLARY 1.3. *Let x be away from D ; then for $t, h \in (0, 1)$ fulfilling the condition*

$$4 - h - 4t > 0,$$

the following expansion for the scattered field holds:

$$(1.18) \quad \begin{aligned} E^s(x) &= \pm k^2 a^{3-h} \left[\frac{\eta_2}{d_0} \Upsilon_k(x, z_0) \cdot \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_0) - \frac{ik}{c_0} \eta_0 \nabla_y \Phi_k(x, z_0) \times \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_0) \right] \\ &+ \mathcal{O}\left(a^{\min(3-h+t; 3; 10-2h-7t; 9-3h-5t)}\right), \end{aligned}$$

and the following expansion for the far-field holds:

$$(1.19) \quad \begin{aligned} E^\infty(\hat{x}) &= \frac{k^2}{\pm 4\pi} e^{-ik\hat{x} \cdot z_0} a^{3-h} \left[\frac{\eta_2}{d_0} (I - \hat{x} \otimes \hat{x}) \cdot \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_0) - \frac{\eta_0 k^2}{c_0} \hat{x} \times \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_0) \right] \\ &+ \mathcal{O}\left(a^{\min(3-h+t; 3; 10-2h-7t; 9-3h-5t)}\right), \end{aligned}$$

where z_0 denotes the intermediate point between z_1 and z_2 , and $\mathbf{P}_{0,1}^{(1)}$ and $\mathbf{P}_{0,2}^{(2)}$ are given in Theorem 1.2.

The analysis performed in this work is related to small-scaled nano-particles (relative to the wavelength). The results related to extended particles at low frequencies regimes can be similarly derived. We made the choice of small-scaled nano-particles because of the different applications we have in mind in material sciences and imaging, for instance. Also, we are aware that in the scattering coefficients, via the different

polarization tensors and the coefficients c_0 and d_0 , we have taken into account only the very dominant terms, neglecting the small perturbations (that are related to damping and radiative losses).

We conclude this subsection by mentioning the following compact formulas for rewriting the tensors $\mathbf{P}_{0,i}^{(j)}$, with $1 \leq i, j \leq 2$, and their values for the particular case where the nano-particles are balls.

1. The tensor

$$\mathbf{P}_{0,1}^{(1)} := \int_{B_1} \phi_{n_0,B_1}(x) dx \otimes \int_{B_1} \phi_{n_0,B_1}(x) dx$$

reduces, under the particular case of B_1 being the unit ball, i.e., $B_1 \equiv B(0, 1)$, and $n_0 = 1$, to

$$(1.20) \quad \mathbf{P}_{0,1}^{(1)} = \frac{12}{\pi^3} I_3.$$

2. We have the compact form of the tensor

$$\mathbf{P}_{0,2}^{(1)} := \sum_n \int_{B_2} \phi_{n,B_2}(x) dx \otimes \int_{B_2} \phi_{n,B_2}(x) dx = \int_{B_2} Q(x) \cdot \frac{1}{\mathbb{P}}(Q)(x) dx,$$

where Q is the matrix given by equation (SM2.58).

3. We rewrite the tensor $\mathbf{P}_{0,1}^{(2)}$ as

$$\begin{aligned} \mathbf{P}_{0,1}^{(2)} &:= \sum_n \frac{1}{\lambda_n^{(3)}(B_1)} \int_{B_1} e_{n,B_1}^{(3)}(x) dx \otimes \int_{B_1} e_{n,B_1}^{(3)}(x) dx \\ &= \int_{B_1} \nabla M_{B_1}^{-1} (I_3 \chi_{B_1})(x) dx. \end{aligned}$$

In addition, under the particular case of B_1 being the unit ball, i.e., $B_1 \equiv B(0, 1)$, we obtain

$$\mathbf{P}_{0,1}^{(2)} = 4\pi I_3.$$

4. Finally, the tensor

$$\mathbf{P}_{0,2}^{(2)} := \int_{B_2} e_{n_\star,B_2}^{(3)}(x) dx \otimes \int_{B_2} e_{n_\star,B_2}^{(3)}(x) dx$$

reduces, under the particular case of B_2 being the unit ball, i.e., $B_2 \equiv B(0, 1)$, and $n_\star = 1$, to

$$(1.21) \quad \mathbf{P}_{0,2}^{(2)} = \frac{4\pi}{27} I_3.$$

The computation details can be found in subsection A.2. We observe that for the case of balls, the main tensors $\mathbf{P}_{0,1}^{(1)}$ and $\mathbf{P}_{0,2}^{(2)}$ are not vanishing and are proportional to the identity matrix.

1.3. Discussion about the results. The estimation of the electromagnetic fields generated by a single type of nano-particles is already known in the literature; see [3, 9] for dielectric nano-particles and [10, 20] for plasmonic nano-particles. The related results correspond to those derived here by keeping only the block matrix

given by $(\mathcal{B}_{13}, \mathcal{B}_{23}, \mathcal{B}_{31}, \mathcal{B}_{41})$ or $(\mathcal{B}_{14}, \mathcal{B}_{24}, \mathcal{B}_{32}, \mathcal{B}_{42})$, respectively, in (1.15). The originality here is to have derived the fields generated by such hybrid dimers having not only different shape but also different contrasting materials. We call such hybrid dimers heterogeneous dimers, while those dimers with similar, or the same, scales are called homogeneous dimers. An analysis for a homogeneous dimer composed of two spherically shaped plasmonic nano-particles can be found in [34, 35].

The approximations of the electric fields in (1.13) and (1.14) are modeled by the vectors Q_1, Q_2, R_1 , and R_2 , which are solutions of the algebraic system (1.15). Precisely, Q_j and R_j model the magnetic and electric poles of the nano-particles D_j , $j = 1, 2$, respectively. Inverting this algebraic system, using the Born series expansions, provides one with a cascade of field approximations where the most dominant field is described in (1.19). In this corollary, we see that the generated electromagnetic field by the dimer is a combination of the electric pole generated by the plasmonic nano-particle and magnetic pole generated by the dielectric nano-particle. This shows how the dimer plays a role of dipole to generate the electromagnetic field. The higher order terms in the Born series describe the two types of contributions that are worth mentioning.

1. The first class of contributions consists in the higher order terms modeling multipoles for each nano-particle, taken in isolation. Such higher order terms are also seen when deriving the expansion for single nano-particle.
2. The second class of contributions consists in the mutual interaction between the two nano-particles. These terms model the multiple scattering between the two nano-particles.

Based on this classification of the contributions, the mutual interaction between the two nano-particles, forming the heterogeneous dimer, is richer, as compared to single nano-particles or homogeneous dimers, as these contributions enter into the game as combinations of higher order modes, for each nano-particle, with mutual interactions between the two nano-particles.

Such a mutual interaction between the two nano-particles is possible only because they are tuned to resonate at common incident frequencies. Otherwise, we can also excite the dimer with frequencies away from the common resonances but eventually near to resonances of one of the nano-particles; then the dimer will predominantly behave as a single nano-particle.

As a plasmonic-dielectric dimer has the potential of generating both electric and magnetic polarizations, we expect to use a cluster of such dimers to generate both effective electric permittivity and magnetic permeability. As we excite such systems with nearly resonating incident frequencies, we expect to be able to generate both single negative (permittivity or permeability) or eventually double negative (permittivity and permeability). Such an investigation will be reported in a forthcoming work. Let us mention that the use of a cluster of nano-particles (single nano-particle) to generate single negative permeability of single negative permittivity is already confirmed in [8] and [11], respectively.

The rest of the paper is organized as follows. In section 2, we introduce some preliminaries, including the $\mathbb{L}^2$ -Helmholtz decomposition and the Lippmann-Schwinger system of equations for the solution to (1.1). Based on the Lippmann-Schwinger system, we present the a priori estimates first for a single nano-particle and then for the dimer of nano-particles. The estimations for the related scattering coefficients, i.e., corresponding to the induced polarization tensors, are analyzed as well. In section 3, the precise form of the linear algebraic system is investigated. Section 4 is

devoted to proving Theorem 1.2 and Corollary 1.3, on the basis of the outcomes in sections 2 and 3. Appendix A will be devoted to the justification of some mentioned results in section 1.

2. Some preliminaries and a priori estimates. In this section, we present some necessary preliminaries and significant a priori estimates. For the preliminaries, we cite some key points for the completeness of the paper; see [9] for more details.

2.1. $\mathbb{L}^2(B)$ -Helmholtz decomposition. The following direct sum provides a useful decomposition of $\mathbb{L}^2(B)$ -space; see [13, Chapter IX, Table I, Page 314]:

$$(2.1) \quad \mathbb{L}^2(B) = \mathbb{H}_0(\operatorname{div} = 0)(B) \overset{\perp}{\oplus} \mathbb{H}_0(\operatorname{Curl} = 0)(B) \overset{\perp}{\oplus} \nabla \mathcal{H}armonic(B),$$

where

$$\begin{aligned} \mathbb{H}_0(\operatorname{div} = 0)(B) &:= \{E \in \mathbb{L}^2(B), \operatorname{div} E = 0 \text{ in } B, \nu \cdot E = 0 \text{ on } \partial B\}, \\ \mathbb{H}_0(\operatorname{Curl} = 0)(B) &:= \{E \in \mathbb{L}^2(B), \operatorname{Curl} E = 0 \text{ in } B, \nu \times E = 0 \text{ on } \partial B\}, \end{aligned}$$

and

$$\nabla \mathcal{H}armonic(B) := \{E : E = \nabla \psi, \psi \in \mathbb{H}^1(B), \Delta \psi = 0 \text{ in } B\}.$$

From the decomposition (2.1), we define $\overset{1}{\mathbb{P}}, \overset{2}{\mathbb{P}}$, and $\overset{3}{\mathbb{P}}$ to be the natural projectors as follows:

$$\overset{1}{\mathbb{P}} := \mathbb{L}^2(B) \longrightarrow \mathbb{H}_0(\operatorname{div} = 0)(B), \quad \overset{2}{\mathbb{P}} := \mathbb{L}^2(B) \longrightarrow \mathbb{H}_0(\operatorname{Curl} = 0)(B)$$

$$\text{and } \overset{3}{\mathbb{P}} := \mathbb{L}^2(B) \longrightarrow \nabla \mathcal{H}armonic(B).$$

2.2. Lippmann–Schwinger integral formulation of the solution. For any vector field F , we define the Newtonian operator $N_D^k(\cdot)$ and the Magnetization operator $\nabla M_D^k(\cdot)$ as follows:

$$(2.2) \quad N_D^k(F)(x) := \int_D \Phi_k(x, y) F(y) dy \quad \text{and} \quad \nabla M_D^k(F)(x) := \nabla_x \int_D \nabla_y \Phi_k(x, y) \cdot F(y) dy,$$

where $\Phi_k(\cdot, \cdot)$ is the fundamental solution of the Helmholtz equation given by (1.17). The solution to (1.1) of the integro-differential form can be formulated as the following proposition.

PROPOSITION 2.1. *The solution to the problem (1.1) satisfies*

$$(2.3) \quad E(x) + \nabla M_D^k(\eta E)(x) - k^2 N_D^k(\eta E)(x) = E^{Inc}(x), \quad x \in \mathbb{R}^3,$$

where $\eta(\cdot)$ is defined by (1.6).

Proof. The proposition can be proved by utilizing the Stratton–Chu formula directly; see [12, Theorem 6.1] for more detailed discussions. $\square$

Motivated by the study of the L.S.E given by (2.3), on the subspaces involved in the $\mathbb{L}^2$ -space decomposition (see, for instance, (2.1)), using spectral theory techniques,

it is crucial for us to discuss the Magnetization operator with vanishing frequency, i.e., $\nabla M(\cdot) := \nabla M^0(\cdot)$, and the Newtonian operator with vanishing frequency, i.e., $N(\cdot) := N^0(\cdot)$, on the L.H.S of (2.3). This idea is clarified by the following remark.

Remark 2.2. The case for a domain D being small leads to the expansions of the Magnetization operator $\nabla M_D^k(\cdot)$ and the Newtonian operator $N_D^k(\cdot)$, defined in (2.2) as

$$\begin{aligned} \nabla M_D^k(F)(x) = & \nabla M_D(F)(x) + \frac{k^2}{2} N_D(F)(x) + \frac{ik^3}{12\pi} \int_D F(y) dy \\ & - \frac{k^2}{2} \int_D \Phi_0(x, y) \frac{A(x, y) \cdot F(y)}{|x - y|^2} dy \\ & - \frac{1}{4\pi} \sum_{n \geq 3} \frac{(ik)^{n+1}}{(n+1)!} \int_D \nabla_y \nabla_y |x - y|^n \cdot F(y) dy, \quad x \in D, \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} N_D^k(F)(x) = & N_D(F)(x) + \frac{ik}{4\pi} \int_D F(y) dy \\ & + \frac{1}{4\pi} \sum_{n \geq 1} \frac{(ik)^{n+1}}{(n+1)!} \int_D |x - y|^n F(y) dy, \quad x \in D, \end{aligned} \quad (2.5)$$

where $A(\cdot, \cdot)$ is the matrix given by $A(x, y) := (x - y) \otimes (x - y)$. For more details on the derivation of (2.4) and (2.5), we refer the readers to [9, section 2.2].

In addition to the above remark, the following behaviors of the Magnetization operator and the Newtonian operator on the subspaces involved in the $\mathbb{L}^2$ -space decomposition, given by (2.1), hold.

Remark 2.3. Two points are in order.

1. The Newtonian operator $N_B(\cdot)$ projected onto the subspace $\mathbb{H}_0(\text{div} = 0)(B)$ (respectively, $\mathbb{H}_0(\text{Curl} = 0)(B)$) admits an eigensystem that we denote by $(\lambda_n^{(1)}(B); e_{n,B}^{(1)})$ (respectively, $(\lambda_n^{(2)}(B); e_{n,B}^{(2)})$). Besides, we have

$$(2.6) \quad \int_B e_{n,B}^{(j)}(y) dy = 0 \quad \forall n \in \mathbb{N} \text{ and } j = 1, 2.$$

2. For the Magnetization operator, the following relations hold.

- * $\nabla M_B^k(\cdot)$ projected onto the subspace $\mathbb{H}_0(\text{div} = 0)(B)$ is a vanishing operator, i.e.,

$$(2.7) \quad \forall E \in \mathbb{H}_0(\text{div} = 0)(B) \text{ we have } \nabla M_B^k(E) = 0.$$

- * $\nabla M_B^k(\cdot)$ projected onto the subspace $\mathbb{H}_0(\text{Curl} = 0)(B)$ satisfies

$$(2.8) \quad \forall E \in \mathbb{H}_0(\text{Curl} = 0)(B) \text{ we have } \nabla M_B^k(E) = k^2 N_B^k(E) + E \chi_B,$$

where $\chi_B(\cdot)$ is the characteristic function set.

- * $\nabla M_B(\cdot)$ projected onto the subspace $\nabla \mathcal{H}armonic$ admits an eigensystem that we denote by $(\lambda_n^{(3)}(B); e_{n,B}^{(3)})$.

For the existence and the construction of $(\lambda_n^{(j)}(B); e_{n,B}^{(j)})_{n \in \mathbb{N}, j = 1, 2, 3}$, we refer the reader to [20, section 5]. More properties for the Magnetization operator, such as the self-adjointness, positivity, spectrum, boundedness, etc., can be found in [1, 14, 16, 17, 18] and [31]. Besides, (2.6) can be proved by using (2.1) and knowing that $I_3 \in \nabla \mathcal{H}armonic$.

2.3. A priori estimates. Based on the decomposition (2.1), we present here some necessary a priori estimates related to the electric total field E , derived from the Lippmann–Schwinger equation (2.3), and some scattering coefficients, which play an important role in the proof of our main results. In order to achieve this, we will require an intermediate result that will clarify the total field estimates that can be derived from a single nano-particle, whether it's dielectric or plasmonic. This is the subject of the following lemma.

LEMMA 2.4 (estimate for just one nano-particle). *Under Assumption 1.1, we consider the problem (1.1) with only one distributed nano-particle. Let k fulfill*

$$k^2 := \frac{1 \mp c_0 a^h}{\eta_1 a^2 \lambda_{n_0}^{(1)}(B_1)} \sim 1.$$

Then, for $h < 2$, the following hold:

1. *the electric field generated by a single dielectric nano-particle*

$$\|\tilde{E}\|_{\mathbb{L}^2(B_1)} = \mathcal{O}(a^{1-h}),$$

2. *the electric field generated by a single plasmonic nano-particle*

$$\|\tilde{E}\|_{\mathbb{L}^2(B_2)} = \mathcal{O}(a^{-h}),$$

and, regardless of the nano-particle used, we have

$$(2.9) \quad \mathbb{P}^2(\tilde{E}) = 0.$$

Proof. See subsection SM1.1. □

Furthermore, in what follows, we will frequently use certain notations that require clarification.

NOTATION. *In the presence of a dimer, we denote by $E_m(\cdot)$ the restriction of $E(\cdot)$ onto the nano-particle D_m for $m = 1, 2$, i.e., $E_m(\cdot) := E|_{D_m}(\cdot)$. Besides, we denote by $\tilde{F}(\cdot)$ the vector field that we obtain by scaling $F(\cdot)$ from D to B , i.e.,*

$$\tilde{F}(\eta) = F(z + a\eta), \quad \eta \in B.$$

Based on the estimates of the total electric field generated by the presence of a single nano-particle, whether it's dielectric or plasmonic (see Lemma 2.4), we develop the following proposition to estimate the total electric field generated by the presence of a dimer.

PROPOSITION 2.5 (estimation for the dimer). *Under Assumption 1.1, we consider the problem (1.1) for the dimer D . Let k fulfill*

$$(2.10) \quad k^2 := \frac{1 \mp c_0 a^h}{\eta_1 a^2 \lambda_{n_0}^{(1)}(B_1)} \sim 1.$$

Then, for $t, h \in (0, 1)$, the following estimation holds:

$$(2.11) \quad \begin{cases} \left\| \mathbb{P}^1(\tilde{E}_1) \right\|_{\mathbb{L}^2(B_1)} = \mathcal{O}(a^{1-h}), & \left\| \mathbb{P}^1(\tilde{E}_2) \right\|_{\mathbb{L}^2(B_2)} = \mathcal{O}(a^{\min(1; 4-h-3t)}), \\ \left\| \mathbb{P}^3(\tilde{E}_1) \right\|_{\mathbb{L}^2(B_1)} = \mathcal{O}(a^2), & \left\| \mathbb{P}^3(\tilde{E}_2) \right\|_{\mathbb{L}^2(B_2)} = \mathcal{O}(a^{-h}). \end{cases}$$

In addition, we have

$$(2.12) \quad \mathbb{P}^2(\tilde{E}_1) = \mathbb{P}^2(\tilde{E}_2) = 0.$$

Proof. See subsection SM1.2. $\square$

In addition to estimating the electric total field generated by a dimer, which can be found in Proposition 2.5, it's necessary to estimate the scattering coefficients related to the problem (1.1). The purpose of the following definition is to define the scattering coefficients related to the problem (1.1), associated with the used dimer, which will be utilized to justify our derived results.

DEFINITION 2.6. We define W_1, W_2, V_1 , and V_2 to be solutions of

$$(2.13) \quad (I + \eta_1 \nabla M_{D_1}^{-k} - k^2 \eta_1 N_{D_1}^{-k})(W_1)(x) = \mathcal{P}(x, z_1), \quad x \in D_1,$$

$$(2.14) \quad (I + \eta_2 \nabla M_{D_2}^{-k} - k^2 \eta_2 N_{D_2}^{-k})(W_2)(x) = \mathbb{P}^1(\mathcal{P}(x, z_2)), \quad x \in D_2,$$

$$(2.15) \quad (I + \eta_m \nabla M_{D_m}^{-k} - k^2 \eta_m N_{D_m}^{-k})(V_m)(x) = I_3, \quad x \in D_m, \quad m = 1, 2,$$

where, for $m = 1, 2$, the operators $\nabla M_{D_m}^{-k}(\cdot)$ and $N_{D_m}^{-k}(\cdot)$ are the adjoint operators to $\nabla M_{D_m}^k(\cdot)$ and $N_{D_m}^k(\cdot)$, introduced in (2.2), and $\mathcal{P}(x, z)$ is the matrix expressed by

$$\mathcal{P}(x, z) := \begin{pmatrix} (x - z)_1 I_3 \\ (x - z)_2 I_3 \\ (x - z)_3 I_3 \end{pmatrix}.$$

Based on the estimates given in Proposition 2.5, and using the notations introduced by Definition 2.6, we provide in the following proposition the estimates related to problem (1.1) having scattering coefficients.

PROPOSITION 2.7 (estimation of the scattering coefficients). For $h \in (0, 2)$, under assumption (1.11), the following estimations hold:

1. Regarding the scattering parameter W_1 , defined by (2.13), we have

$$(2.16) \quad \left\| \mathbb{P}^1(\tilde{W}_1) \right\|_{\mathbb{L}^2(B_1)} = \mathcal{O}(a^{1-h}) \quad \text{and} \quad \left\| \mathbb{P}^j(\tilde{W}_1) \right\|_{\mathbb{L}^2(B_1)} = \mathcal{O}(a^3) \text{ for } j = 2, 3.$$

2. Regarding the scattering parameter W_2 , defined by (2.14), we have

$$(2.17) \quad \left\| \mathbb{P}^1(\tilde{W}_2) \right\|_{\mathbb{L}^2(B_2)} = \mathcal{O}(a), \quad \mathbb{P}^2(\tilde{W}_2) = 0 \quad \text{and} \quad \left\| \mathbb{P}^3(\tilde{W}_2) \right\|_{\mathbb{L}^2(B_2)} = \mathcal{O}(a^{5-h}).$$

3. Regarding the scattering parameter V_m , defined by (2.15), for $m = 1, 2$, we have

$$(2.18) \quad \left\| \tilde{V}_1 \right\|_{\mathbb{L}^2(B_1)} = \mathcal{O}(a^2) \quad \text{and} \quad \left\| \tilde{V}_2 \right\|_{\mathbb{L}^2(B_2)} = \mathcal{O}(a^{-h}).$$

Proof. See subsection SM1.3. $\square$

3. Linear algebraic system of the dimer. In this section, we shall present the linear algebraic system derived from the L.S.E (2.3) by projecting the solution E onto the two subspaces $\mathbb{H}_0(\text{div} = 0)$ and $\nabla \mathcal{H}armonic$. We start by using (1.7) to derive, for $m = 1, 2$, the following expression:

$$(3.1) \quad \begin{aligned} \mathbb{P}^1(E_m) &= \text{Curl}(F_m) \text{ in } D_m, \quad \text{with } \nu \times F_m = 0 \text{ on } \partial D_m \\ &\text{and } \text{div}(F_m) = 0 \text{ in } D_m. \end{aligned}$$

Set

$$(3.2) \quad Q_m := \eta_m \int_{D_m} F_m(y) dy \quad \text{and} \quad R_m := \eta_m \int_{D_m} \mathbb{P}^3(E_m)(y) dy,$$

where η_m is defined in (1.6).

PROPOSITION 3.1. *Under Assumption 1.1, for $t, h \in (0, 1)$, such that*

$$4 - h - 4t > 0,$$

the coefficients (Q_1, R_1, Q_2, R_2) , defined in (3.2), built up from the solution to the problem (1.1), satisfy the following algebraic system:

$$(3.3) \quad \begin{pmatrix} I_3 & 0 & -\mathcal{B}_{13} & -\mathcal{B}_{14} \\ 0 & I_3 & -\mathcal{B}_{23} & -\mathcal{B}_{24} \\ -\mathcal{B}_{31} & -\mathcal{B}_{32} & I_3 & 0 \\ -\mathcal{B}_{41} & -\mathcal{B}_{42} & 0 & I_3 \end{pmatrix} \cdot \begin{pmatrix} Q_1 \\ R_1 \\ Q_2 \\ R_2 \end{pmatrix} = \begin{pmatrix} \frac{ik\eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) \\ a^3 \mathbf{P}_{0,1}^{(2)} \cdot E^{Inc}(z_1) \\ ik\eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot H^{Inc}(z_2) \\ \frac{\eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) \end{pmatrix} + \begin{pmatrix} Error_1^{(1)} \\ Error_1^{(2)} \\ Error_2^{(1)} \\ Error_2^{(2)} \end{pmatrix}$$

with

$$\begin{aligned} \mathcal{B}_{13} &:= \frac{k^4 \eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot \Upsilon_k(z_1, z_2), \\ \mathcal{B}_{23} &:= k^2 a^3 \mathbf{P}_{0,1}^{(2)} \cdot \nabla \Phi_k(z_1, z_2), \\ \mathcal{B}_{31} &:= k^4 \eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot \Upsilon_k(z_2, z_1), \\ \mathcal{B}_{41} &:= \frac{k^2 \eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot \nabla \Phi_k(z_2, z_1), \\ \mathcal{B}_{14} &:= \frac{k^2 \eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot \nabla \Phi_k(z_1, z_2), \\ \mathcal{B}_{24} &:= k^2 a^3 \mathbf{P}_{0,1}^{(2)} \cdot \Upsilon_k(z_1, z_2), \\ \mathcal{B}_{32} &:= k^2 \eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot \nabla \Phi_k(z_2, z_1), \\ \mathcal{B}_{42} &:= \frac{k^2 \eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot \Upsilon_k(z_2, z_1), \end{aligned}$$

where

$$\begin{aligned} \mathbf{P}_{0,1}^{(1)} &:= \int_{B_1} \phi_{n_0, B_1}(x) dx \otimes \int_{B_1} \phi_{n_0, B_1}(x) dx, \\ \mathbf{P}_{0,1}^{(2)} &:= \sum_n \frac{1}{\lambda_n^{(3)}(B_1)} \int_{B_1} e_{n, B_1}^{(3)}(x) dx \otimes \int_{B_1} e_{n, B_1}^{(3)}(x) dx, \\ \mathbf{P}_{0,2}^{(1)} &= \sum_n \int_{B_2} \phi_{n, B_2}(x) dx \otimes \int_{B_2} \phi_{n, B_2}(x) dx, \\ \mathbf{P}_{0,2}^{(2)} &= \int_{B_2} e_{n_*, B_2}^{(3)}(x) dx \otimes \int_{B_2} e_{n_*, B_2}^{(3)}(x) dx, \end{aligned}$$

with $\phi_{n_0, B_1}(\cdot)$ satisfying (1.9) and $\phi_{n, B_2}(\cdot)$ being given by (1.8). Besides,

$$\begin{aligned} \text{Error}_1^{(1)} &= \mathcal{O}\left(a^{\min(3; 7-2h-3t)}\right), & \text{Error}_1^{(2)} &= \mathcal{O}\left(a^{\min(4; 7-h-4t)}\right), \\ \text{Error}_2^{(1)} &= \mathcal{O}\left(a^{\min(6; 9-h-4t)}\right), & \text{Error}_2^{(2)} &= \mathcal{O}\left(a^{\min(4-h; 7-h-4t; 6-3t; 7-2h-3t)}\right). \end{aligned}$$

Proof. See section SM2. $\square$

4. Proof of the main result. In this section, the proof of Theorem 1.2 is presented as the following four steps with all the necessary propositions given in the previous sections.

(I) Derivation of the scattered wave $E^s(x)$.

Thanks to the L.S.E given by (2.3) and the fact that $E = E^{Inc} + E^s$, we deduce that

$$E^s(x) = -\nabla M_D^k(\eta E)(x) + k^2 N_D^k(\eta E)(x),$$

which, by letting x outside D and using (1.16), can be rewritten as

$$E^s(x) = k^2 \int_D \Upsilon_k(x, y) \cdot \eta(y) E(y) dy = k^2 \sum_{m=1}^2 \eta_m \int_{D_m} \Upsilon_k(x, y) \cdot E_m(y) dy.$$

Besides, by splitting E_m as $E_m = \mathbb{P}^1(E_m) + \mathbb{P}^3(E_m)$ (see (2.12)) and using the fact that $\nabla M^k(\cdot)|_{\mathbb{H}_0(\text{div}=0)}$ is a vanishing operator we get

$$\begin{aligned} E^s(x) &= k^2 \sum_{m=1}^2 \eta_m \int_{D_m} \Phi_k(x, y) \mathbb{P}^1(E_m)(y) dy \\ &\quad + k^2 \sum_{m=1}^2 \eta_m \int_{D_m} \Upsilon_k(x, y) \cdot \mathbb{P}^3(E_m)(y) dy \\ &\stackrel{(3.1)}{=} -k^2 \sum_{m=1}^2 \eta_m \int_{D_m} \nabla_y \Phi_k(x, y) \times F_m(y) dy \\ &\quad + k^2 \sum_{m=1}^2 \eta_m \int_{D_m} \Upsilon_k(x, y) \cdot \mathbb{P}^3(E_m)(y) dy. \end{aligned}$$

On the R.H.S of the above expression, by expanding $\nabla \Phi_k(x, \cdot)$ and $\Upsilon_k(x, \cdot)$, near the center z_m , with $m = 1, 2$, and using (3.2) we obtain

$$(4.1) \quad E^s(x) = -k^2 \sum_{m=1}^2 \left[\nabla_y \Phi_k(x, z_m) \times Q_m - \Upsilon_k(x, z_m) \cdot R_m \right] + \text{Remainder},$$

where *Remainder* is given by

$$\begin{aligned} \text{Remainder} &:= -k^2 \sum_{m=1}^2 \eta_m \int_{D_m} \int_0^1 \nabla_y \nabla_y \Phi_k(x, z_m + t(y - z_m)) \cdot (y - z_m) dt \\ &\quad \times F_m(y) dy + k^2 \sum_{m=1}^2 \eta_m \int_{D_m} \int_0^1 \nabla_y \Upsilon_k(x, z_m + t(y - z_m)) \\ &\quad \times \mathcal{P}(y, z_m) dt \cdot \mathbb{P}^3(E_m)(y) dy. \end{aligned}$$

Next, we estimate the term *Remainder*.

$$\begin{aligned}
|Remainder| &\lesssim \sum_{m=1}^2 |\eta_m| \left\| \int_0^1 \nabla_y \nabla_y \Phi_k(x, z_m + t(\cdot - z_m)) \cdot (\cdot - z_m) dt \right\|_{\mathbb{L}^2(D_m)} \|F_m\|_{\mathbb{L}^2(D_m)} \\
&\quad + \sum_{m=1}^2 |\eta_m| \left\| \int_0^1 \nabla_y \Upsilon_k(x, z_m + t(\cdot - z_m)) \cdot \mathcal{P}(\cdot, z_m) dt \right\|_{\mathbb{L}^2(D_m)} \left\| \mathbb{P}^3(E_m) \right\|_{\mathbb{L}^2(D_m)} \\
&\lesssim a^4 \sum_{m=1}^2 |\eta_m| \left[\left\| \tilde{F}_m \right\|_{\mathbb{L}^2(B_m)} + \left\| \mathbb{P}^3(\tilde{E}_m) \right\|_{\mathbb{L}^2(B_m)} \right].
\end{aligned}$$

Thanks to (2.11), (1.6), and the formula (SM2.23), we deduce that

$$Remainder = \mathcal{O}(a^{4-h}).$$

Hence, (4.1) becomes

$$(4.2) \quad E^s(x) = -k^2 \sum_{m=1}^2 \left[\nabla_y \Phi_k(x, z_m) \times Q_m - \Upsilon_k(x, z_m) \cdot R_m \right] + \mathcal{O}(a^{4-h}),$$

where (Q_1, R_1, Q_2, R_2) is solution of (3.3).

(II) Derivation of the far-field $E^\infty(\hat{x})$.

To estimate the far-field, we use the fact that $E = E^{Inc} + E^s$, the formula (2.2), and formula (1.4) to obtain

$$\begin{aligned}
E^\infty(\hat{x}) &= \frac{k^2}{4\pi} (I - \hat{x} \otimes \hat{x}) \cdot \int_D e^{-ik\hat{x} \cdot y} \eta(y) E(y) dy, \quad D = D_1 \cup D_2 \\
&\stackrel{(1.6)}{=} \frac{k^2}{4\pi} \eta_1 (I - \hat{x} \otimes \hat{x}) \cdot \int_{D_1} e^{-ik\hat{x} \cdot y} E_1(y) dy \\
&\quad + \frac{k^2}{4\pi} \eta_2 (I - \hat{x} \otimes \hat{x}) \cdot \int_{D_2} e^{-ik\hat{x} \cdot y} E_2(y) dy.
\end{aligned}$$

By expanding the function $y \rightarrow e^{-ik\hat{x} \cdot y}$ at $y = z_m$, with $m = 1, 2$, we obtain

$$\begin{aligned}
E^\infty(\hat{x}) &= \frac{k^2}{4\pi} \sum_{m=1}^2 \eta_m (I - \hat{x} \otimes \hat{x}) \cdot \int_{D_m} [e^{-ik\hat{x} \cdot z_m} - ike^{-ik\hat{x} \cdot z_m} (y - z_m) \cdot \hat{x}] E_m(y) dy \\
(4.3) \quad &+ \sum_{m=1}^2 R_m^{(1)},
\end{aligned}$$

where, for $m = 1, 2$,

$$R_m^{(1)} := \frac{k^4}{8\pi} \eta_m (I - \hat{x} \otimes \hat{x}) \cdot \int_{D_m} ((y - z_m) \cdot \hat{x})^2 \int_0^1 (1-t) e^{-ik\hat{x} \cdot (z_1 + t(y - z_m))} dt E_m(y) dy.$$

It is direct to get from (1.6) that the following holds:

$$(4.4) \quad R_1^{(1)} = \mathcal{O}\left(a^{\frac{3}{2}} \|E_1\|_{\mathbb{L}^2(D_1)}\right) \quad \text{and} \quad R_2^{(1)} = \mathcal{O}\left(a^{\frac{7}{2}} \|E_2\|_{\mathbb{L}^2(D_2)}\right).$$

Based on Lemma 2.4, formula (2.9), by splitting E_m as $E_m = \mathbb{P}^1(E_m) + \mathbb{P}^3(E_m)$, we use (2.6) and the estimations derived in (4.4) to rewrite (4.3) as

$$(4.5) \quad \begin{aligned} E^\infty(\hat{x}) &= \frac{k^2}{4\pi} \sum_{m=1}^2 \eta_m (I - \hat{x} \otimes \hat{x}) \cdot e^{-ik\hat{x} \cdot z_m} \int_{D_m} \mathbb{P}^3(E_m)(y) dy \\ &\quad - i \frac{k^3}{4\pi} \sum_{m=1}^2 \eta_m (I - \hat{x} \otimes \hat{x}) \cdot e^{-ik\hat{x} \cdot z_m} \int_{D_m} \hat{x} \cdot (y - z_m) \mathbb{P}^1(E_m)(y) dy \\ &\quad + R_1^{(2)} + R_2^{(2)} + \mathcal{O}\left(a^{\frac{3}{2}} \|E_1\|_{\mathbb{L}^2(D_1)}\right) + \mathcal{O}\left(a^{\frac{7}{2}} \|E_2\|_{\mathbb{L}^2(D_2)}\right), \end{aligned}$$

where

$$R_m^{(2)} := -i \frac{k^3}{4\pi} \eta_m (I - \hat{x} \otimes \hat{x}) \cdot e^{-ik\hat{x} \cdot z_m} \int_{D_m} \hat{x} \cdot (y - z_m) \mathbb{P}^3(E_m)(y) dy, \quad m = 1, 2.$$

Then, using (1.6), the following estimation holds:

$$R_1^{(2)} = \mathcal{O}\left(a^{\frac{1}{2}} \left\| \mathbb{P}^3(E_1) \right\|_{\mathbb{L}^2(D_1)}\right) \quad \text{and} \quad R_2^{(2)} = \mathcal{O}\left(a^{\frac{5}{2}} \left\| \mathbb{P}^3(E_2) \right\|_{\mathbb{L}^2(D_2)}\right).$$

Thus, we can deduce from (4.5) that

$$\begin{aligned} E^\infty(\hat{x}) &= \frac{k^2}{4\pi} \sum_{m=1}^2 \eta_m (I - \hat{x} \otimes \hat{x}) \cdot e^{-ik\hat{x} \cdot z_m} \int_{D_m} \mathbb{P}^3(E_m)(y) dy \\ &\quad - i \frac{k^3}{4\pi} \sum_{m=1}^2 \eta_m (I - \hat{x} \otimes \hat{x}) \cdot e^{-ik\hat{x} \cdot z_m} \int_{D_m} \hat{x} \cdot (y - z_m) \mathbb{P}^1(E_m)(y) dy \\ &\quad + \mathcal{O}\left(a^{\frac{1}{2}} \left\| \mathbb{P}^3(E_1) \right\|_{\mathbb{L}^2(D_1)}\right) + \mathcal{O}\left(a^{\frac{5}{2}} \left\| \mathbb{P}^3(E_2) \right\|_{\mathbb{L}^2(D_2)}\right) \\ &\quad + \mathcal{O}\left(a^{\frac{3}{2}} \|E_1\|_{\mathbb{L}^2(D_1)}\right) + \mathcal{O}\left(a^{\frac{7}{2}} \|E_2\|_{\mathbb{L}^2(D_2)}\right), \end{aligned}$$

which, by using (3.1) and the notations given by (3.2), indicates

$$(4.6) \quad E^\infty(\hat{x}) = \frac{k^2}{4\pi} \sum_{m=1}^2 e^{-ik\hat{x} \cdot z_m} [(I - \hat{x} \otimes \hat{x}) \cdot R_m + i k \hat{x} \times Q_m] + Error^{(1)},$$

where the vectors $(Q_m, R_m)_{m=1,2}$ are the solutions of the linear algebraic system given by (1.15), and

$$\begin{aligned} Error^{(1)} &:= \mathcal{O}\left(a^{\frac{1}{2}} \left\| \mathbb{P}^3(E_1) \right\|_{\mathbb{L}^2(D_1)}\right) + \mathcal{O}\left(a^{\frac{5}{2}} \left\| \mathbb{P}^3(E_2) \right\|_{\mathbb{L}^2(D_2)}\right) + \mathcal{O}\left(a^{\frac{3}{2}} \|E_1\|_{\mathbb{L}^2(D_1)}\right) \\ &\quad + \mathcal{O}\left(a^{\frac{7}{2}} \|E_2\|_{\mathbb{L}^2(D_2)}\right), \end{aligned}$$

which can be estimated, by using Lemma 2.4 and Proposition 2.5, as $\mathcal{O}(a^{4-h})$. Hence, (4.6) becomes

$$(4.7) \quad E^\infty(\hat{x}) = \frac{k^2}{4\pi} \sum_{m=1}^2 e^{-ik\hat{x} \cdot z_m} [(I - \hat{x} \otimes \hat{x}) \cdot R_m + i k \hat{x} \times Q_m] + \mathcal{O}(a^{4-h}),$$

where (Q_1, R_1, Q_2, R_2) is solution of (3.3).

We have seen from the estimation of the scattered field $E^s(\cdot)$, given by (4.2), and the estimation of the far-field $E^\infty(\cdot)$, given by (4.7), that the determination of (Q_1, R_1, Q_2, R_2) is of great importance to achieve the estimation of both $E^s(\cdot)$ and $E^\infty(\cdot)$. The goal of the next step is to determine the dominant part related to (Q_1, R_1, Q_2, R_2) .

(III) Invertibility of the algebraic system (3.3).

We recall that (Q_1, R_1, Q_2, R_2) is the solution to

$$(4.8) \quad \begin{pmatrix} I_3 & 0 & -\mathcal{B}_{13} & -\mathcal{B}_{14} \\ 0 & I_3 & -\mathcal{B}_{23} & -\mathcal{B}_{24} \\ -\mathcal{B}_{31} & -\mathcal{B}_{32} & I_3 & 0 \\ -\mathcal{B}_{41} & -\mathcal{B}_{42} & 0 & I_3 \end{pmatrix} \cdot \begin{pmatrix} Q_1 \\ R_1 \\ Q_2 \\ R_2 \end{pmatrix} = \begin{pmatrix} \frac{ik\eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) \\ a^3 \mathbf{P}_{0,1}^{(2)} \cdot E^{Inc}(z_1) \\ ik\eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot H^{Inc}(z_2) \\ \frac{\eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) \end{pmatrix} + \begin{pmatrix} Error_1^{(1)} \\ Error_1^{(2)} \\ Error_2^{(1)} \\ Error_2^{(2)} \end{pmatrix}.$$

Notice that the R.H.S of (4.8) is perturbed by the presence of an additive vector error term. Next, we investigate the associated unperturbed algebraic system given by

$$(4.9) \quad \begin{pmatrix} I_3 & 0 & -\mathcal{B}_{13} & -\mathcal{B}_{14} \\ 0 & I_3 & -\mathcal{B}_{23} & -\mathcal{B}_{24} \\ -\mathcal{B}_{31} & -\mathcal{B}_{32} & I_3 & 0 \\ -\mathcal{B}_{41} & -\mathcal{B}_{42} & 0 & I_3 \end{pmatrix} \cdot \begin{pmatrix} \tilde{Q}_1 \\ \tilde{R}_1 \\ \tilde{Q}_2 \\ \tilde{R}_2 \end{pmatrix} = \begin{pmatrix} \frac{ik\eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) \\ a^3 \mathbf{P}_{0,1}^{(2)} \cdot E^{Inc}(z_1) \\ ik\eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot H^{Inc}(z_2) \\ \frac{\eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) \end{pmatrix}.$$

It is clear that the difference between the solution to (4.9) and the solution to (4.8) satisfies the following algebraic system:

$$\begin{pmatrix} I_3 & 0 & -\mathcal{B}_{13} & -\mathcal{B}_{14} \\ 0 & I_3 & -\mathcal{B}_{23} & -\mathcal{B}_{24} \\ -\mathcal{B}_{31} & -\mathcal{B}_{32} & I_3 & 0 \\ -\mathcal{B}_{41} & -\mathcal{B}_{42} & 0 & I_3 \end{pmatrix} \cdot \begin{pmatrix} Q_1 - \tilde{Q}_1 \\ R_1 - \tilde{R}_1 \\ Q_2 - \tilde{Q}_2 \\ R_2 - \tilde{R}_2 \end{pmatrix} = \begin{pmatrix} Error_1^{(1)} \\ Error_1^{(2)} \\ Error_2^{(1)} \\ Error_2^{(2)} \end{pmatrix},$$

which is invertible under the condition $4 - h - 4t > 0$. By inverting the above system using Born series, we obtain

$$(4.10) \quad \begin{pmatrix} Q_1 - \tilde{Q}_1 \\ R_1 - \tilde{R}_1 \\ Q_2 - \tilde{Q}_2 \\ R_2 - \tilde{R}_2 \end{pmatrix} = \sum_{n \geq 0} \mathbb{K}_n \cdot \begin{pmatrix} Error_1^{(1)} \\ Error_1^{(2)} \\ Error_2^{(1)} \\ Error_2^{(2)} \end{pmatrix},$$

where the matrix $\mathbb{K}_n$ is given by

$$(4.11) \quad \mathbb{K}_n := \begin{pmatrix} 0 & 0 & \mathcal{B}_{13} & \mathcal{B}_{14} \\ 0 & 0 & \mathcal{B}_{23} & \mathcal{B}_{24} \\ \mathcal{B}_{31} & \mathcal{B}_{32} & 0 & 0 \\ \mathcal{B}_{41} & \mathcal{B}_{42} & 0 & 0 \end{pmatrix}^n.$$

In order to evaluate the L.H.S of (4.10), we keep only the zeroth order term and the first order term on the R.H.S of the Born series to derive the following estimations:

$$\begin{aligned}
 (4.12) \quad & |Q_1 - \tilde{Q}_1| \lesssim \text{Error}_1^{(1)} + a^{3-h} d^{-3} \text{Error}_2^{(1)} + a^{3-h} d^{-2} \text{Error}_2^{(2)} = \mathcal{O}\left(a^{\min(3; 7-2h-3t)}\right), \\
 & |R_1 - \tilde{R}_1| \lesssim \text{Error}_1^{(2)} + (a^3 d^{-2} + a^{9-2h} d^{-8}) \text{Error}_2^{(1)} + a^3 d^{-3} \text{Error}_2^{(2)} \\
 & \quad = \mathcal{O}\left(a^{\min(4; 7-h-4t)}\right), \\
 & |Q_2 - \tilde{Q}_2| \lesssim \text{Error}_2^{(1)} + a^5 d^{-3} \text{Error}_1^{(1)} + (a^5 d^{-2} + a^{11-2h} d^{-8}) \text{Error}_1^{(2)} \\
 & \quad = \mathcal{O}\left(a^{\min(6; 9-h-4t; 8-3t)}\right), \\
 & |R_2 - \tilde{R}_2| \lesssim \text{Error}_2^{(2)} + a^{3-h} d^{-2} \text{Error}_1^{(1)} + a^{3-h} d^{-3} \text{Error}_1^{(2)} \\
 & \quad = \mathcal{O}\left(a^{\min(4-h; 7-h-4t; 6-3t; 7-2h-3t; 10-2h-7t)}\right).
 \end{aligned}$$

(IV) The corresponding revised Foldy–Lax approximation.

We recall first the scattered field expansion given by (4.2),

$$E^s(x) = -k^2 \sum_{m=1}^2 \left[\nabla_y \Phi_k(x, z_m) \times Q_m - \Upsilon_k(x, z_m) \cdot R_m \right] + \mathcal{O}(a^{4-h}),$$

where (Q_1, R_1, Q_2, R_2) is the solution to (3.3). Now, by using the estimates derived in (4.12), under the condition $4 - h - 4t > 0$, and the fact that x is away from D , we obtain

$$\begin{aligned}
 E^s(x) = & -k^2 \sum_{m=1}^2 \left[\nabla_y \Phi_k(x, z_m) \times \tilde{Q}_m - \Upsilon_k(x, z_m) \cdot \tilde{R}_m \right] \\
 & + \mathcal{O}\left(a^{\min(3; 7-2h-3t; 10-2h-7t)}\right),
 \end{aligned}$$

where $(\tilde{Q}_1, \tilde{R}_1, \tilde{Q}_2, \tilde{R}_2)$ is the solution to the unperturbed algebraic system (4.9). This justifies (1.13). Similarly, by recalling the far-field expression given by (4.7) and using the estimates derived in (4.12), with the condition $4 - h - 4t > 0$, and the fact that x is away from D , we can obtain

$$\begin{aligned}
 E^\infty(\hat{x}) = & \frac{k^2}{4\pi} \sum_{m=1}^2 e^{-ik\hat{x} \cdot z_m} \left[(I - \hat{x} \otimes \hat{x}) \tilde{R}_m + i k \hat{x} \otimes \tilde{Q}_m \right] \\
 & + \mathcal{O}(a^{\min(3; 7-2h-3t; 10-2h-7t)}),
 \end{aligned}$$

where $(\tilde{Q}_1, \tilde{R}_1, \tilde{Q}_2, \tilde{R}_2)$ is the solution to (4.9). This justifies (1.14) and ends the proof of Theorem 1.2.

4.1. Proof of Corollary 1.3. We start by recalling, from (4.9), that $(\tilde{Q}_1, \tilde{R}_1, \tilde{Q}_2, \tilde{R}_2)$ is a vector solution of the following algebraic system:

$$\begin{pmatrix} I_3 & 0 & -\mathcal{B}_{13} & -\mathcal{B}_{14} \\ 0 & I_3 & -\mathcal{B}_{23} & -\mathcal{B}_{24} \\ -\mathcal{B}_{31} & -\mathcal{B}_{32} & I_3 & 0 \\ -\mathcal{B}_{41} & -\mathcal{B}_{42} & 0 & I_3 \end{pmatrix} \cdot \begin{pmatrix} \tilde{Q}_1 \\ \tilde{R}_1 \\ \tilde{Q}_2 \\ \tilde{R}_2 \end{pmatrix} = \begin{pmatrix} \frac{ik\eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) \\ a^3 \mathbf{P}_{0,1}^{(2)} \cdot E^{Inc}(z_1) \\ ik\eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot H^{Inc}(z_2) \\ \frac{\eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) \end{pmatrix},$$

which is invertible under the condition $4 - h - 4t > 0$. By inverting the above system using Born series, we obtain

$$(4.13) \quad \begin{pmatrix} \tilde{Q}_1 \\ \tilde{R}_1 \\ \tilde{Q}_2 \\ \tilde{R}_2 \end{pmatrix} = \begin{pmatrix} \frac{ik\eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) \\ a^3 \mathbf{P}_{0,1}^{(2)} \cdot E^{Inc}(z_1) \\ ik\eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot H^{Inc}(z_2) \\ \frac{\eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) \end{pmatrix} + \sum_{n \geq 1} \mathbb{K}_n \cdot \begin{pmatrix} \frac{ik\eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) \\ a^3 \mathbf{P}_{0,1}^{(2)} \cdot E^{Inc}(z_1) \\ ik\eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot H^{Inc}(z_2) \\ \frac{\eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) \end{pmatrix},$$

where the matrix $\mathbb{K}_n$ is given by (4.11). By keeping the dominant part of $\sum_{n \geq 1} \mathbb{K}_n$, which is $\mathbb{K}_1$, the second term on the R.H.S of the above equation will be reduced to

$$\begin{aligned} \dots &= \mathbb{K}_1 \cdot \begin{pmatrix} \frac{ik\eta_0}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) \\ a^3 \mathbf{P}_{0,1}^{(2)} \cdot E^{Inc}(z_1) \\ ik\eta_2 a^5 \mathbf{P}_{0,2}^{(1)} \cdot H^{Inc}(z_2) \\ \frac{\eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) \end{pmatrix} \\ &\simeq \begin{pmatrix} ik\eta_2 a^5 \mathcal{B}_{13} \cdot \mathbf{P}_{0,2}^{(1)} \cdot H^{Inc}(z_2) + \frac{\eta_2}{\pm d_0} a^{3-h} \mathcal{B}_{23} \cdot \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) \\ ik\eta_2 a^5 \mathcal{B}_{23} \cdot \mathbf{P}_{0,2}^{(1)} \cdot H^{Inc}(z_2) + \frac{\eta_2}{\pm d_0} a^{3-h} \mathcal{B}_{24} \cdot \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) \\ \frac{ik\eta_0}{\pm c_0} a^{3-h} \mathcal{B}_{31} \cdot \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) + a^3 \mathcal{B}_{32} \cdot \mathbf{P}_{0,1}^{(2)} \cdot E^{Inc}(z_1) \\ \frac{ik\eta_0}{\pm c_0} a^{3-h} \mathcal{B}_{41} \cdot \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) + a^3 \mathcal{B}_{42} \cdot \mathbf{P}_{0,1}^{(2)} \cdot E^{Inc}(z_1) \end{pmatrix}. \end{aligned}$$

By estimating component by component the above vector, using the expression of the matrix $\mathcal{B}_{ij}$, given in Proposition 3.1, and plugging the obtained result into (4.13), we can obtain the following estimations:

$$(4.14) \quad \begin{aligned} \tilde{Q}_1 &= \frac{i\eta_0 k}{\pm c_0} a^{3-h} \mathbf{P}_{0,1}^{(1)} \cdot H_1^{Inc}(z_1) + \mathcal{O}(a^{9-3h-5t}), \\ \tilde{R}_1 &= a^3 \mathbf{P}_{0,1}^{(2)} \cdot E_1^{Inc}(z_1) + \frac{k^2 \eta_0}{\pm d_0} a^{6-h} \mathbf{P}_{0,1}^{(2)} \cdot \Upsilon_k(z_1, z_2) \cdot \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_2) + \mathcal{O}(a^{9-2h-5t}), \\ \tilde{Q}_2 &= i\eta_2 k a^5 \mathbf{P}_{0,2}^{(1)} \cdot H_2^{Inc}(z_2) + \frac{ik^5 \eta_0 \eta_2}{\pm c_0} a^{8-h} \mathbf{P}_{0,2}^{(1)} \cdot \Upsilon_k(z_2, z_1) \cdot \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_1) \\ &\quad + \mathcal{O}(a^{11-2h} d^{-5}), \\ \tilde{R}_2 &= \frac{\eta_2}{\pm d_0} a^{3-h} \mathbf{P}_{0,2}^{(2)} \cdot E_2^{Inc}(z_2) + \mathcal{O}(a^{\min(11-2h-5t; 14-3h-8t)}). \end{aligned}$$

It is direct to observe from the above estimations that $(\tilde{Q}_1, \tilde{R}_2)$ dominates $(\tilde{R}_1, \tilde{Q}_2)$. Consequently, by returning back to (1.13) and using the above observation we can get that

$$\begin{aligned} E^s(x) &= k^2 \left[\Upsilon_k(x, z_2) \cdot \tilde{R}_2 - \nabla_y \Phi_k(x, z_1) \times \tilde{Q}_1 \right] \\ &\quad + k^2 \left[\Upsilon_k(x, z_1) \cdot \tilde{R}_1 - \nabla_y \Phi_k(x, z_2) \times \tilde{Q}_2 \right] + \mathcal{O}\left(a^{\min(3; 7-2h-3t; 10-2h-7t)}\right). \end{aligned}$$

Since x is away from D for the scattered wave, by utilizing the estimation of $(\tilde{R}_1, \tilde{Q}_2)$, we can know that the second term on the R.H.S is estimated of order $\mathcal{O}(a^3)$. Hence, with (4.14),

$$\begin{aligned}
E^s(x) &= k^2 \left[\Upsilon_k(x, z_2) \cdot \tilde{R}_2 - \nabla_y \Phi_k(x, z_1) \times \tilde{Q}_1 \right] + \mathcal{O} \left(a^{\min(3; 7-2h-3t; 10-2h-7t)} \right) \\
&= \pm k^2 a^{3-h} \left[\frac{\eta_2}{d_0} \Upsilon_k(x, z_2) \cdot \mathbf{P}_{0,2}^{(2)} \cdot E_2^{Inc}(z_2) - \frac{ik\eta_0}{c_0} \nabla_y \Phi_k(x, z_1) \times \mathbf{P}_{0,1}^{(1)} \cdot H_1^{Inc}(z_1) \right] \\
&\quad + \mathcal{O} \left(a^{\min(3; 7-2h-3t; 10-2h-7t; 9-3h-5t)} \right),
\end{aligned}$$

which, by taking Taylor expansion of $H^{Inc}(\cdot)$ and $E^{Inc}(\cdot)$ at the intermediate point, between z_1 and z_2 with $|z_1 - z_2| \lesssim a^t$, i.e.,

$$\begin{aligned}
H_1^{Inc}(z_1) &= H^{Inc}(z_0) + ik \int_0^1 e^{ik\theta \cdot (z_0 + t(z_1 - z_0))} (\theta^\perp \times \theta) \langle \theta, (z_1 - z_0) \rangle dt \\
&= H^{Inc}(z_0) + \mathcal{O}(a^t), \\
E_2^{Inc}(z_2) &= E^{Inc}(z_0) + ik \int_0^1 e^{ik\theta \cdot (z_0 + t(z_2 - z_0))} (\theta^\perp \times \theta) \langle \theta, (z_2 - z_0) \rangle dt \\
&= E^{Inc}(z_0) + \mathcal{O}(a^t),
\end{aligned}$$

gives us under the condition $4 - h - 4t > 0$ that

$$\begin{aligned}
(4.15) \quad E^s(x) &= \pm k^2 a^{3-h} \left[\frac{\eta_2}{d_0} \Upsilon_k(x, z_2) \cdot \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_0) \right. \\
&\quad \left. - \frac{ik\eta_0}{c_0} \nabla_y \Phi_k(x, z_1) \times \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_0) \right] \\
&\quad + \mathcal{O}(a^{\min(3-h+t; 3; 10-2h-7t; 9-3h-5t)}).
\end{aligned}$$

Based on (4.15), the last step consists in taking the Taylor expansion near the intermediate point z_0 for the functions $\Upsilon_k(x, \cdot)$ and $\nabla \Phi_k(x, \cdot)$, i.e.,

$$\Upsilon_k(x, z_2) = \Upsilon_k(x, z_0) + \mathcal{O}(a^t) \quad \text{and} \quad \nabla \Phi_k(x, z_1) = \nabla \Phi_k(x, z_0) + \mathcal{O}(a^t),$$

where we can use the fact that x is away from D to derive

$$\begin{aligned}
E^s(x) &= \pm k^2 a^{3-h} \left[\frac{\eta_2}{d_0} \Upsilon_k(x, z_0) \cdot \mathbf{P}_{0,2}^{(2)} \cdot E^{Inc}(z_0) \right. \\
&\quad \left. - \frac{ik\eta_0}{c_0} \nabla_y \Phi_k(x, z_0) \times \mathbf{P}_{0,1}^{(1)} \cdot H^{Inc}(z_0) \right] \\
&\quad + \mathcal{O}(a^{\min(3-h+t; 3; 10-2h-7t; 9-3h-5t)}),
\end{aligned}$$

which justifies (1.18). Furthermore, by returning back to (1.14) we obtain

$$\begin{aligned}
E^\infty(\hat{x}) &= \frac{k^2}{4\pi} \left[e^{-ik\hat{x} \cdot z_2} (I - \hat{x} \otimes \hat{x}) \cdot \tilde{R}_2 + ik e^{-ik\hat{x} \cdot z_1} \hat{x} \times \tilde{Q}_1 \right] \\
&\quad + \frac{k^2}{4\pi} \left[e^{-ik\hat{x} \cdot z_1} (I - \hat{x} \otimes \hat{x}) \cdot \tilde{R}_1 + ik e^{-ik\hat{x} \cdot z_2} \hat{x} \times \tilde{Q}_2 \right] \\
&\quad + \mathcal{O}(a^{\min(3; 7-2h-3t; 10-2h-7t)}),
\end{aligned}$$

which, by using the expression of $(\tilde{R}_1, \tilde{Q}_2)$ in (4.14), and the fact that the second term on the R.H.S of the above equation is of order $\mathcal{O}(a^3)$, can be further simplified as

$$\begin{aligned}
E^\infty(\hat{x}) &= \frac{k^2}{4\pi} \left[e^{-ik\hat{x} \cdot z_2} (I - \hat{x} \otimes \hat{x}) \cdot \tilde{R}_2 + ik e^{-ik\hat{x} \cdot z_1} \hat{x} \times \tilde{Q}_1 \right] \\
&\quad + \mathcal{O}(a^{\min(3; 7-2h-3t; 10-2h-7t)}).
\end{aligned}$$

Now, taking Taylor expansion of $z \rightarrow e^{-ik\hat{x}\cdot z}$ at the intermediate point z_0 , between z_1 and z_2 with $|z_1 - z_2| \lesssim a^t$, we obtain

$$E^\infty(\hat{x}) = \frac{k^2}{4\pi} e^{-ik\hat{x}\cdot z_0} \left[(I - \hat{x} \otimes \hat{x}) \cdot \tilde{R}_2 + i k \hat{x} \times \tilde{Q}_1 \right] + \mathcal{O}(a^t (|\tilde{R}_2| + |\tilde{Q}_1|)) + \mathcal{O}(a^{\min(3; 7-2h-3t; 10-2h-7t)}).$$

Now, using the expression of $(\tilde{R}_2, \tilde{Q}_1)$ in (4.14), we can derive the estimation

$$|a^t (|R_2| + |Q_1|)| \lesssim a^{3-h+t}$$

and the following expression for $E^\infty(\cdot)$ under the condition $4 - h - 4t > 0$ that

$$\begin{aligned} E^\infty(\hat{x}) = & \pm \frac{k^2}{4\pi} a^{3-h} e^{-ik\hat{x}\cdot z_0} \left[\frac{\eta_2}{d_0} (I - \hat{x} \otimes \hat{x}) \cdot \mathbf{P}_{0,2}^{(2)} \cdot E_2^{Inc}(z_0) \right. \\ & \left. - k^2 \frac{\eta_0}{c_0} \hat{x} \times \mathbf{P}_{0,1}^{(1)} \cdot H_1^{Inc}(z_0) \right] \\ & + \mathcal{O}(a^{\min(3-h+t; 3; 10-2h-7t; 9-3h-5t)}), \end{aligned}$$

where the last step consists in taking Taylor expansion for the incident fields, similar to (4.15), to justify (1.19). The proof of Corollary 1.3 is now complete.

Appendix A.

A.1. Lorentz model and the common resonant frequencies. Here we show that the conditions (1.6) and (1.11) can be derived from the Lorentz model by choosing appropriate incident frequency k . Indeed, recall the Lorentz model for the relative permittivity that

$$(A.1) \quad \epsilon_r = 1 + \frac{k_p^2}{k_0^2 - k^2 - ik\xi},$$

where k_p is the plasmonic frequency, k_0 is the undamped frequency resonance of the background, and ξ is the damping frequency with $\xi \ll 1$. In D_m , with $m = 1, 2$, we respectively denote ξ_m as the damping frequencies, $k_{0,m}$ as the undamped frequencies, and $k_{p,m}$ as the according plasmonic frequencies. Let

$$k_{0,1} := k_0, \quad k_{p,1} = k_p \quad \text{in } D_1$$

and $k_{0,2}, k_{p,2}$ such that

$$(A.2) \quad k_{0,2}^2 < k_{0,1}^2 < k_{0,2}^2 + k_{p,2}^2 \quad \text{in } D_2.$$

1. Regarding the dielectric resonance. If the frequency k is chosen to be real and k^2 close to the undamped resonance frequency $k_{0,1}^2$ in the format that

$$\begin{aligned} (A.3) \quad k^2 - k_{0,1}^2 = & - \frac{k_{p,1}^2 a^2 \lambda_{n_0}^{(1)}(B_1) k_{0,1}^2}{(1 \mp \Re(c_0) a^h) \pm \frac{\Im^2(c_0) a^h}{(1 \mp \Re(c_0) a^h)} - k_{p,1}^2 a^2 \lambda_{n_0}^{(1)}(B_1)} \\ = & -k_{p,1}^2 a^2 \lambda_{n_0}^{(1)}(B_1) k_{0,1}^2 [1 + \mathcal{O}(a^h)], \end{aligned}$$

and

$$k \xi_1 = \pm \frac{\Im(c_0) a^2 (k^2 - k_{0,1}^2)}{(1 \mp \Re(c_0) a^h)} = \mp \Im(c_0) a^4 \lambda_{n_0}^{(1)}(B_1) k_{0,1}^2 k_{p,1}^2 [1 + \mathcal{O}(a^h)],$$

where $\lambda_{n_0}^{(1)}(B_1)$ is an eigenvalue of the Newtonian potential operator $N_{B_1}(\cdot)$, then there holds that

$$\Re(\eta_1) = a^{-2} \left(\lambda_{n_0}^{(1)}(B_1) k_{0,1}^2 \right)^{-1} (1 + \mathcal{O}(a^h))$$

and

$$\Im(\eta_1) = \pm \Im(c_0) \left(\lambda_{n_0}^{(1)}(B_1) k_{0,1}^2 \right)^{-1} + \mathcal{O}(a^4),$$

which implies that D_1 behaves as a dielectric nano-particle. Furthermore, by choosing k satisfying (A.3), we derive that

$$1 - k^2 \eta_1 a^2 \lambda_{n_0}^{(1)}(B_1) = \pm c_0 a^h;$$

see (1.11). For more details, we refer the readers to [9, Remarks 1.1 and 1.2].

2. Regarding the plasmonic resonance. Using the introduced notations, we start by recalling the Lorentz model for the permittivity related to the nano-particle D_2 , given by (A.1),

$$\begin{aligned} \epsilon_r^{(2)} &= 1 + \frac{k_{p,2}^2}{k_{0,2}^2 - k^2 - i k \xi_2} \\ &\stackrel{(A.3)}{=} 1 + \frac{k_{p,2}^2}{k_{0,2}^2 - \left(k_{0,1}^2 - k_{p,1}^2 a^2 \lambda_{n_0}^{(1)}(B_1) k_{0,1}^2 [1 + \mathcal{O}(a^h)] \right) - i k \xi_2}. \end{aligned}$$

Hence,

$$\operatorname{Re}(\epsilon_r^{(2)}) = 1 + \frac{k_{p,2}^2 \left[k_{0,2}^2 - k_{0,1}^2 + k_{p,1}^2 a^2 \lambda_{n_0}^{(1)}(B_1) k_{0,1}^2 [1 + \mathcal{O}(a^h)] \right]}{\left[k_{0,2}^2 - k_{0,1}^2 + k_{p,1}^2 a^2 \lambda_{n_0}^{(1)}(B_1) k_{0,1}^2 [1 + \mathcal{O}(a^h)] \right]^2 + (k \xi_2)^2}.$$

Besides, by using (A.2) and the fact that $\xi_2 \ll 1$, we can deduce that

$$\operatorname{Re}(\epsilon_r^{(2)}) < 0.$$

Then D_2 behaves like a plasmonic nano-particle. Furthermore, by choosing k such that

$$(A.4) \quad k^2 = k_{0,2}^2 + \frac{\lambda_{n_*}^{(3)}(B_2) k_{p,2}^2 (1 \mp a^h \operatorname{Re}(d_0))}{|1 \mp a^h d_0|^2},$$

and letting ξ_2 be given by

$$\xi_2 = \pm \frac{a^h \operatorname{Im}(d_0) \lambda_{n_*}^{(3)}(B_2) k_{p,2}^2}{|1 \mp a^h d_0| \sqrt{k_{0,2}^2 |1 \mp a^h d_0|^2 + \lambda_{n_*}^{(3)}(B_2) k_{p,2}^2 |1 \mp a^h d_0|}},$$

we deduce that

$$1 + \eta_2 \lambda_{n_*}^{(3)}(B_2) = \pm d_0 a^h;$$

see (1.11).

According to the analysis above, there is a frequency k that can be considered as a dielectric resonance and a plasmonic resonance under certain conditions. Indeed, by equating (A.3) and (A.4), we obtain

$$\begin{aligned} & k_{0,1}^2 \left[1 - \frac{k_{p,1}^2 a^2 \lambda_{n_0}^{(1)}(B_1)}{(1 \mp \Re(c_0) a^h) \pm \frac{\Im^2(c_0) a^h}{(1 \mp \Re(c_0) a^h)} - k_{p,1}^2 a^2 \lambda_{n_0}^{(1)}(B_1)} \right] \\ &= k_{0,2}^2 + \frac{\lambda_{n_*}^{(3)}(B_2) k_{p,2}^2 (1 \mp a^h \operatorname{Re}(d_0))}{|1 \mp a^h d_0|^2}, \end{aligned}$$

which, by keeping only the dominant terms, gives us condition

$$(A.5) \quad k_{0,1}^2 = k_{0,2}^2 + \lambda_{n_*}^{(3)}(B_2) k_{p,2}^2,$$

establishing a correlation between the Lorentz model parameters associated with the dielectric nano-particle D_1 , i.e., $k_{0,1}$, and the parameters associated with the plasmonic nano-particle D_2 , i.e., $k_{0,2}$ and $k_{p,2}$. Hence, under the condition (A.5), and the smallness assumption on both ξ_1 and ξ_2 , we deduce that the dielectric resonance equals the plasmonic resonance up to an additive small error term. To put it simply, a hybrid dielectric-plasmonic dimer can have a common resonance.

A.2. Justification of (1.20)–(1.21). The computations have been divided into four parts.

1. Computation of $\mathbf{P}_{0,1}^{(1)}$.

$$\mathbf{P}_{0,1}^{(1)} := \int_{B_1} \phi_{n_0, B_1}(x) dx \otimes \int_{B_1} \phi_{n_0, B_1}(x) dx = \frac{12}{\pi^3} I_3,$$

where we have assumed that B_1 is being the unit ball, i.e., $B_1 \equiv B(0, 1)$, and $n_0 = 1$; see [11, section 3.1, formula (3.1)].

2. Computation of $\mathbf{P}_{0,2}^{(1)}$.

$$\mathbf{P}_{0,2}^{(1)} := \sum_n \int_{B_2} \phi_{n, B_2}(x) dx \otimes \int_{B_2} \phi_{n, B_2}(x) dx,$$

which by using the fact that

$$\int_{B_2} \phi_{n, B_2}(x) dx = - \int_{B_2} Q(x) \cdot e_{n, B_2}^{(1)}(x) dx$$

gives us

$$\begin{aligned} \mathbf{P}_{0,2}^{(1)} &= \sum_n \int_{B_2} Q(x) \cdot e_{n, B_2}^{(1)}(x) dx \otimes \int_{B_2} Q(x) \cdot e_{n, B_2}^{(1)}(x) dx \\ &= \sum_n \langle Q, e_{n, B_2}^{(1)} \rangle_{\mathbb{L}^2(B_2)} \otimes \int_{B_2} Q(x) \cdot e_{n, B_2}^{(1)}(x) dx \\ &= \int_{B_2} Q(x) \cdot \sum_n \langle Q, e_{n, B_2}^{(1)} \rangle_{\mathbb{L}^2(B_2)} \otimes e_{n, B_2}^{(1)}(x) dx \\ &= \int_{B_2} Q(x) \cdot \mathbb{P}(Q)(x) dx. \end{aligned}$$

3. Computation of $\mathbf{P}_{0,1}^{(2)}$.

$$\begin{aligned}
\mathbf{P}_{0,1}^{(2)} &:= \sum_n \frac{1}{\lambda_n^{(3)}(B_1)} \int_{B_1} e_{n,B_1}^{(3)}(x) dx \otimes \int_{B_1} e_{n,B_1}^{(3)}(x) dx \\
&= \sum_n \left\langle I_3, e_{n,B_1}^{(3)} \right\rangle_{\mathbb{L}^2(B_1)} \otimes \int_{B_1} \frac{1}{\lambda_n^{(3)}(B_1)} e_{n,B_1}^{(3)}(x) dx \\
&\stackrel{(1,10)}{=} \sum_n \left\langle I_3, e_{n,B_1}^{(3)} \right\rangle_{\mathbb{L}^2(B_1)} \otimes \int_{B_1} \nabla M_{B_1}^{-1} \left(e_{n,B_1}^{(3)} \right) (x) dx \\
&= \int_{B_1} \sum_n \left\langle I_3, e_{n,B_1}^{(3)} \right\rangle_{\mathbb{L}^2(B_1)} \otimes \nabla M_{B_1}^{-1} \left(e_{n,B_1}^{(3)} \right) (x) dx \\
&= \int_{B_1} \nabla M_{B_1}^{-1} \left(\sum_n \left\langle I_3, e_{n,B_1}^{(3)} \right\rangle_{\mathbb{L}^2(B_1)} \otimes e_{n,B_1}^{(3)} \right) (x) dx \\
&= \int_{B_1} \nabla M_{B_1}^{-1} (I_3 \chi_{B_1}) (x) dx.
\end{aligned}$$

Besides, in the case where B_1 is the unit ball, i.e., $B_1 \equiv B(0, 1)$, we know that

$$\nabla M_{B_1}(I_3) = \frac{1}{3} I_3 \quad \text{in } B_1;$$

see [20, formula (1.17)]. Hence,

$$\mathbf{P}_{0,1}^{(2)} = \int_{B_1} 3 I_3 dx = 3 |B_1| I_3 = 4\pi I_3.$$

4. Computation of $\mathbf{P}_{0,2}^{(2)}$.

$$\mathbf{P}_{0,2}^{(2)} := \int_{B_2} e_{n_*,B_2}^{(3)}(x) dx \otimes \int_{B_2} e_{n_*,B_2}^{(3)}(x) dx.$$

By letting B_2 be the unit ball, i.e., $B_2 = B(0, 1)$, and the index $n_* = 1$, we obtain

$$\mathbf{P}_{0,2}^{(2)} = \frac{4\pi}{27} I_3;$$

see, for instance, [19, subsection 4.5.3].

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