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Mosco convergence of gradient forms with non-convex potentials II

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Abstract

This article provides a scaling limit for a family of skew interacting Brownian motions in the context of mesoscopic interface models. Let $M, d \in \mathbb{N}$, $y_1, \ldots, y_M \in \mathbb{R}$, $f \in C_b(\mathbb{R})$. For $N \in \mathbb{N}$ we consider a k_N-dimensional, skew reflecting distorted Brownian motion $(X_t^{N,i})_{i=1,\ldots,k_N}$, $t \geq 0$, and investigate its scaling limit for $N \to \infty$. The drift includes skew reflections at height levels $\tilde{y}_j := N^{1-\frac{d}{2}} y_j$ with intensities β_j/N^d for $j = 1, \ldots, M$. The corresponding SDE is given by

$$\mathrm{d}X_t^{N,i} = -\left(A_N X_t^N\right)_i \mathrm{d}t - \frac{1}{2} N^{-\frac{d}{2}-1} f\left(N^{\frac{d}{2}-1} X_t^{N,i}\right) \mathrm{d}t$$

$$+ \sum_{j=1}^{M} \frac{1-e^{-\beta_j/N^d}}{1+e^{-\beta_j/N^d}} \, \mathrm{d}l_t^{N,i,\tilde{y}_j} + \mathrm{d}B_t^{N,i},$$

where $(B_t^{N,i})_{t\geq0}$, $i = 1, \ldots, k_N$, are independent Brownian motions, $A_N \in \mathbb{R}^{k_N \times k_N}$ is symmetric positive definite and $l_t^{N,i,\tilde{y}_j}$ denotes the local time of $(X_t^{N,i})_{t\geq0}$ at $\tilde{y}_j$. We prove the weak convergence of the equilibrium laws of

$$u_t^N = \Lambda_N \circ X_{N^2 t}^N, \quad t \geq 0,$$

for $N \to \infty$, choosing suitable injective, linear maps $\Lambda_N : \mathbb{R}^{k_N} \to \{h \,|\, h : \mathbb{R}^d \supset D \to \mathbb{R}\}$, where D is an open domain. The scaling limit is a distorted Ornstein–Uhlenbeck process whose state space is the Hilbert space $H = L^2(D, \mathrm{d}z)$. We characterize a class of height maps, such that the scaling limit of the dynamic is not influenced by the particular choice of $(\Lambda_N)_{N\in\mathbb{N}}$ within that class.

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1 Introduction

The reaction-diffusion stochastic partial differential equation of the form

$$\frac{\partial u}{\partial t} = -Au - \frac{1}{2}g'(u) + \dot{W}, \quad t \geq 0, \tag{1.1}$$

over the spatial domain $D = (0, 1)$ with Dirichlet boundary conditions has been analysed in [4]. There, the equation is suggested as a model for the random fluctuations of an effective physical interface at a mesoscopic level. Above, $\dot{W}(t, z)$, $t \geq 0$, $z \in D$, denotes space-time white-noise and $-A$ is the second-order differential operator $\frac{1}{2}\frac{\partial^2}{\partial z^2}$. Equation 1.1 is an informal equation, since $g : \mathbb{R} \to \mathbb{R}$ is merely a function of bounded variation and not necessarily differentiable or continuous. However, $g'(u)$ can be made sense of as a continuous additive functional of the solution u.

This article covers some relevant topics for scaling limits related to Eq. 1.1. In a more general fashion, we consider a spatial domain D which is a bounded, open domain of $\mathbb{R}^d$ with $d \in \mathbb{N}$ and A denotes any self-adjoint operator on $H := L^2(D, \mathrm{d}z)$ such that $\frac{1}{2}A^{-1}$ is the covariance operator of a Gaussian measure on H. For such g, A and domain D we consider Eq. 1.1 as an informal equation modelling a mesoscopic dynamic interface. As a standard example in this text, we take $D = (0, 1)^2$ and set $A = -\Delta + \Delta^2$ with operator domain $H^{4,2}(D) \cap H_0^{3,2}(D)$, resulting in the Gaussian measure for a semiflexible polymer model, such as considered in Cipriani et al. [6]. For $N \in \mathbb{N}$ we construct a system of k_N interacting skew Brownian motions. Their equilibrium laws converge weakly to the stationary solution of the reaction-diffusion Eq. 1.1 under a scaling limit for $N \to \infty$. The identification of a diffusion process on $\mathbb{R}^{k_N}$ with a process on H is done via suitable height maps $\Lambda_N : \mathbb{R}^{k_N} \to H$, $N \in \mathbb{N}$. Different choices for the sequence $(\Lambda_N)_{N \in \mathbb{N}}$, such as piecewise constant interpolation or piecewise linear, continuous interpolation might serve for that purpose. We show the invariance of the scaling limit for a general class of height maps which includes the two mentioned types of interpolations.

The approximation of an infinite dimensional random variable or an infinite dimensional diffusion process by the corresponding discretized objects is an essential aspect of an interface model in statistical mechanics. It captures the transition from a microscopic perspective to a macro- or mesoscopic point of view. In this article, a discretization of D is considered. We denote by $\overline{D}$ the topological closure of D. For each $N \in \mathbb{N}$ a finite set of grid points G_N is selected as a subset of the lattice with $G_N \subseteq N\overline{D} \cap \mathbb{Z}^d$ and a bijection $I_N : \{1, \ldots, k_N\} \to G_N$, $k_N = |G_N|$, is fixed. By interpolating between the nodes $\frac{1}{N}G_N := \{z \in \overline{D} \mid Nz \in G_N\}$ we can identify a finite family of real random variables $X^{N,i}$, or a family of real-valued diffusion processes $(X_t^{N,i})_{t \geq 0}$, over the index set $i \in \{1, \ldots k_N\}$ with an H-valued random variable, or a diffusion on H. Like this, a stochastic model for the phenomenons of a $(d+1)$-dimensional physical interface can be defined. The interface's height at a node z such that $Nz = I_N(i)$ for some $i \in \{1, \ldots, k_N\}$ is given by $X^{N,i}$, respectively $(X_t^{N,i})_{t \geq 0}$. For each $N \in \mathbb{N}$ the interface at this scale is thoroughly determined by its height values on the set of nodes $z \in \frac{1}{N}G_N$.

In this frame, the main interests of this survey are:

- Can we approximate a diffusion, informally given by Eq. 1.1, with a sequence of process $\{X_t^{N,i} \in \mathbb{R} \mid i = 1, \ldots, k_N, t \geq 0\}$, which we identify with a sequence of diffusion processes on H via the height maps $(\Lambda_N)_N$, if we let $N \to \infty$?
- Does a change of the interpolation method affect the asymptotic behaviour for $N \to \infty$ of this stochastic interface model?

The relevant asymptotic statement, to which both questions refer, is the weak measure convergence of equilibrium fluctuations. In the article, for technical reasons, the second problem has to be covered first. In the first part of this survey, we address the imminent question, to what extent the height map influences the scaling limit. We give an answer under the assumption that

$$\Lambda_N x(z) := N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} x_i \, \Xi(Nz - I_N(i)), \quad x \in \mathbb{R}^{k_N}, z \in D, \tag{1.2}$$

where $\Xi : \mathbb{R}^d \to [0, 1]$ is independent of N and $I_N : \{1, \ldots, k_N\} \to N\overline{D} \cap \mathbb{Z}^d$ is a bijection identifying the index of a microscopic component with a lattice point within $N\overline{D}$. As it turns out, the choice of the function Ξ is irrelevant to the scaling limit, as long as it meets some basic and plausible assumptions. This result is obtained for a static random interface at first and then generalized to the dynamic case. The second part of the article is dedicated to characterize the asymptotic distribution of $(u_t^N)_{t \geq 0}$ for $N \to \infty$. The main result proves the weak convergence of equilibrium laws. It is based on the recent progress in Grothaus and Wittmann [15] on the topic of Mosco–Kuwae–Shioya convergence of gradient forms with non-convex potentials.

The two most common interpolation methods to encounter in this context are either a piecewise constant interpolation from $\mathbb{R}^{k_N}$ into H or a piecewise linear interpolation from $\mathbb{R}^{k_N}$ into the continuous functions $C(\overline{D}) \subset H$. Usually in concrete examples (e.g. in [4, 6, 8, 9, 28]), one of those two methods is fixed. However, it would be desirable to be able to transfer a convergence result, which is proven for one method, to the other without doing extra calculations. In this article, we give the abstract arguments why this is possible. The interpolation methods taken into account are of the following general type. Let $N \in \mathbb{N}$. Considering a static model at first, the starting point is a random vector with values in $\mathbb{R}^{k_N}$. So, we want to define the height map Λ_N as a continuous linear operator from $\mathbb{R}^{k_N}$ into H. To this end, each index $i = 1, \ldots, k_N$ is spatially associated with the node $\frac{1}{N} I_N(i) \in \overline{D}$. We want to assume spatial homogeneity, i.e. the image set of the unit vectors $\{\mathbf{e}_i \mid i = 1, \ldots k_N\}$ under Λ_N is a set of shifted versions of one and the same archetype function $\Xi : \mathbb{R}^d \to [0, 1]$, rescaled by a factor c_N. This motivates the definition

$$\Lambda_N x(z) := c_N \sum_{i=1}^{k_N} x_i \, \Xi(Nz - I_N(i)), \quad z \in D, \, x \in \mathbb{R}^{k_N}. \tag{1.3}$$

The factor c_N rescales the interface's amplitude, or height. In this text, the choice is always $c_N := N^{\frac{d}{2}-1}$. This is merely a convention. However, the choice seems natural. As an example, we consider the one-dimensional $\nabla\varphi$ interface model with $D = (0, 1)$ and $G_N = \{1, \ldots, N\}$. If we set $X^{N,0} := 0$ and define a family of real random variables $\{X^{N,i}\}$ over the index $i \in G_N$ such that the law of

$$\left(X^{N,i} - X^{N,i-1}\right)_{i=1,\ldots,N+1} \in \mathbb{R}^{N+1}$$

is the $(N+1)$-dimensional standard normal distribution, then the scaling limit under Λ_N is the standard Brownian motion on $[0, 1]$. More precisely, let

$$Y^N := X^{N,1}\mathbf{e}_1 + \cdots + X^{N,N}\mathbf{e}_N,$$

and

$$\Lambda_N x(z) := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i \, \mathbf{1}_{[-1,0)}(Nz - i)$$

$$= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i \, \mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N})}(z), \quad z \in (0, 1), \; x \in \mathbb{R}^N.$$

Then, the law of $(\Lambda_N \circ Y^N)_{N \in \mathbb{N}}$ converges towards the law of a Brownian motion on $[0, 1]$, in the sense of weak measure convergence on $H = L^2((0, 1), \mathrm{d}z)$. The particular choice of Ξ in Eq. 1.3 is irrelevant to the question of existence of a scaling limit in the sense of weak measure convergence on H. This invariance principle is proven in Theorem 2.8 for the general setting in which D is a bounded open domain of $\mathbb{R}^d$ and $\Xi : \mathbb{R}^d \to [0, 1]$ meets the properties specified in Condition 2.1. A similar statement holds for the dynamic case and is stated in Theorem 3.5.

To motivate the first question posed above, we recall the stochastic partial differential equation, discussed in [4]. Let $g : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation and μ^{BB} denote the law of a Brownian bridge between 0 and 0 on the interval $[0, 1]$. In [4], the reader finds a definition of the weak solution for

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial z^2} - \frac{1}{2} \int_{\mathbb{R}} \frac{\partial}{\partial z} l_{t,z}^a \, g(\mathrm{d}a) + \dot{W},$$

$$u(t, 0) = u(t, 1) = 0,$$

$$u(0, z) = u_0(z), \quad z \in [0, 1], \tag{1.4}$$

where $(l_{t,z}^a)_{t \geq 0, z \in [0,1]}$ is the family of local times at $a \in \mathbb{R}$ accumulated over $[0, z]$ by the process $(u(t, r))_{t \geq 0, r \in [0,1]}$ and $\dot{W}(t, z)$, $t \geq 0$, $z \in [0, 1]$, denotes space-time white-noise. Then, the authors construct a weak solution via Dirichlet form techniques. The invariant measure m of Eq. 1.4 is the probability on H whose Radon–Nikodym derivative w.r.t. μ^{BB} is proportional to the density $\exp(-\int_{(0,1)} g \circ h \, \mathrm{d}z)$, $h \in H$. The Dirichlet form of the Markov process from [4] solving Eq. 1.4 is given by

$$\mathcal{E}(u, v) = \frac{1}{2} \int_H \langle \nabla u, \nabla v \rangle_H \, \mathrm{d}m, \quad u, v \in \mathcal{D}(\mathcal{E}),$$

on $L^2(H, m)$. Here, ∇ denotes the weak gradient in the Gaussian Sobolev space $W^{1,2}(H, \mu^{\mathrm{BB}})$, which coincides with the domain $\mathcal{D}(\mathcal{E})$, as the density $\mathrm{d}m / \mathrm{d}\mu^{\mathrm{BB}}$ is bounded from below and above by positive constants. The stationary law of the diffusion process associated to $\mathcal{E}$ can be identified with the scaling limit of a system of interacting skew Brownian motions. As usual, we use the index N to indicate the level of scaling. For each $N \in \mathbb{N}$ we define a probability measure on $\mathbb{R}^N$ by

$$\mathrm{d}m_{A_N}(x) := \frac{1}{Z_N} \exp\left(-x^{\mathrm{T}} A_N x - \frac{1}{N} \sum_{i=1}^{N} g\left(N^{-\frac{1}{2}} x_i\right) \right) \mathrm{d}x, \tag{1.5}$$

$$Z_N := \int_{\mathbb{R}^N} \exp\left(-x^{\mathrm{T}} A_N x - \frac{1}{N} \sum_{i=1}^{N} g\left(N^{-\frac{1}{2}} x_i\right) \right) \mathrm{d}x,$$

which has the symmetric, positive operator $A_N : \mathbb{R}^N \to \mathbb{R}^N$ as a parameter. The Gaussian measure on $\mathbb{R}^N$ with density proportional to $\exp(-x^T A_N x)$ is denoted by μ_{A_N}. The stationary law of the Markov process on $\mathbb{R}^N$ associated to the Dirichlet form

$$\mathcal{E}^N(u, v) = \frac{1}{2} \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dm_{A_N}(x), \quad u, v \in \mathcal{D}(\mathcal{E}^N), \tag{1.6}$$

on $L^2(\mathbb{R}^N, m_{A_N})$ is the natural candidate to yield a finite-dimensional approximation for the solution of Eq. 1.4. The Domain $\mathcal{D}(\mathcal{E}^N)$ of Eq. 1.6 coincides with the Gaussian Sobolev space $W^{1,2}(\mathbb{R}^N, \mu_{A_N})$. We denote the components of that process by $(X_t^{N,i})_{t \geq 0}$, $i = 1, \ldots, N$. Given a decomposition

$$g(y) = g_0(y) + \sum_{j=1}^{M} \beta_j \mathbf{1}_{(-\infty, y_j]}(y), \quad y \in \mathbb{R},$$

for some $M \in \mathbb{N}$, $\beta_j, y_j \in \mathbb{R}$ for $j = 1, \ldots, M$ and $g_0 \in C_b^1(\mathbb{R})$, the processes $(X_t^{N, \cdot})_{t \geq 0}$ satisfy the system of stochastic differential equations

$$dX_t^{N,i} = -\left(A_N X_t^{N, \cdot}\right)_i dt - \frac{1}{2} N^{-\frac{3}{2}} g_0'\left(X_t^{N,i}/\sqrt{N}\right) dt$$

$$+ \sum_{j=1}^{M} \frac{1 - e^{-\beta_j/N}}{1 + e^{-\beta_j/N}} \, dl_t^{N,i,\tilde{y}_j} + dB_t^{N,i}, \quad t \geq 0, \, i = 1, \ldots, N, \tag{1.7}$$

in the weak sense, where $(B_t^{N,i})_{t \geq 0}$, denote independent Brownian motions and $l_t^{N,i,\tilde{y}_j}$ is the local time of $(X_t^{N,i})_{t \geq 0}$ at $\tilde{y}_j := \sqrt{N} y_j$.

A result related to the mesoscopic scaling limit of Eq. 1.7 is provided in [4]. It refers to the subsequence $N = 2^l$, $l \in \mathbb{N}$, and a particular choice for A_N. Let

$$T_N : H \ni h \mapsto N \sum_{i=1}^{N} \left\langle \mathbf{1}_{\left[\frac{i-1}{N}, \frac{i}{N}\right)}, h \right\rangle_H \mathbf{1}_{\left[\frac{i-1}{N}, \frac{i}{N}\right)}$$

be the orthogonal projection and

$$\Lambda_N : \mathbb{R}^N \ni x \mapsto N^{-\frac{1}{2}} \sum_{i=1}^{N} x_i \mathbf{1}_{\left[\frac{i-1}{N}, \frac{i}{N}\right)},$$

the piecewise constant interpolating height map. If $A_N : \mathbb{R}^N \to \mathbb{R}^N$ in Eq. 1.5 is the positive, symmetric operator such that

$$\mu^{BB} \circ \left(\Lambda_N^{-1} \circ T_N\right)^{-1} \propto \exp(-x^T A_N x) \, dx, \tag{1.8}$$

then the laws of

$$u_t^N := \Lambda_N X_{N^2 t}^{N, \cdot} = N^{-\frac{1}{2}} \sum_{i=1}^{N} X_{N^2 t}^{N,i} \mathbf{1}_{\left[\frac{i-1}{N}, \frac{i}{N}\right)}, \quad t \geq 0,$$

converges weakly for $N = 2^l \overset{l}{\to} \infty$, as a sequence of measures on $C([0, \infty), H^{-1,2}(0, 1))$. Its limit is a diffusion process with state space $C([0, 1])$ and a stationary solution of Eq. 1.4.

This statement corresponds to [4, Lem. 5.8]. The authors argue via Mosco convergence of Dirichlet forms ([4, Thm. 5.6]) in the framework of Kuwae–Shioya (see [18, 20]).

The portrayed example of a scaling limit is significantly extended in this article. The spatial dimension of the mesoscopic interface model is now given by $d \in \mathbb{N}$, replacing $(0, 1)$ with a general open, bounded domain $D \subset \mathbb{R}^d$. We derive the scaling limit for $N \to \infty$ of a k_N-sized family of interacting skew Brownian motions $(X_t^{N,i})_{t \geq 0}$, $i = 1, \ldots, N$. Analogous as in the example above, we consider the stationary process $(X_t^{N,\cdot})_{t \geq 0}$ associated to the gradient-type Dirichlet form on $L^2(\mathbb{R}^{k_N}, m_N)$, where now

$$dm_{A_N}(x) := \frac{1}{Z_N} \exp\left(-x^{\mathrm{T}} A_N x - \frac{1}{N^d} \sum_{i=1}^{k_N} g\left(N^{\frac{d}{2}-1} x_i\right) \right) dx,$$

$$Z_N := \int_{\mathbb{R}^{k_N}} \exp\left(-x^{\mathrm{T}} A_N x - \frac{1}{N^d} \sum_{i=1}^{k_N} g\left(N^{\frac{d}{2}-1} x_i\right) \right) dx, \qquad (1.9)$$

with a symmetric positive operator $A_N : \mathbb{R}^{k_N} \to \mathbb{R}^{k_N}$ and a function of bounded variation $g : \mathbb{R} \to \mathbb{R}$. Each index $i \in \{1, \ldots, k_N\}$ corresponds to a grid point $p_i \in N\overline{D} \cap \mathbb{Z}^d$. From now on, we write m_N instead of m_{A_N}. For a height map of the type Eq. 1.3, the weak measure convergence of the law of the H-valued process

$$u_t^N := \Lambda_N X_{N^2 t}^{N,\cdot}$$
$$= N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} X_{N^2 t}^{N,i} \, \Xi(N \cdot -p_i), \quad t \geq 0,$$

for $N \to \infty$ is shown. This holds indifferently from the choice of Ξ. Moreover, in our model, the sequence $(A_N)_{N \in \mathbb{N}}$ is not pre-defined through the limiting Gaussian, as it is in Eq. 1.8. The only assumption is the existence of a Gaussian measure μ such that $\mu_{A_N} \circ \Lambda_N^{-1} \Rightarrow \mu$ in the sense of weak measure convergence on H, where as above μ_N denotes the Gaussian on $\mathbb{R}^N$ such that $\mu_N \propto \exp(-x^{\mathrm{T}} A_N x)\, dx$. A suitable state space for the convergence of the equilibrium laws of $(u_t^N)_{t \geq 0}$ for $N \to \infty$ is the dual H_0' of a densely included separable Hilbert space H_0 in the Gelfand triple $H_0 \subset H \subset H_0'$, assuming the inclusion is a Hilbert–Schmidt operator. For the limiting process an infinite-dimensional version of the Fukushima decomposition is available due to [23].

The outline of this article is as follows. Section 2 deals with a static interface model over a bounded, open domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$. We fix Borel probability measures, m_N on $\mathbb{R}^{k_N}$ for $N \in \mathbb{N}$, and m on $H = L^2(D, dz)$. The measure m_N represents the law of a $|k_N|$-sized family of real random variables. The asymptotic element m represents the law of a static random interface over D in the $(d + 1)$-dimensional space, which takes values in H. Theorem 2.8 provides a perturbation result for the height map Eq. 1.3 regarding the asymptotic behaviour of the sequence $(m_N)_{N \in \mathbb{N}}$. The validity of $m_N \circ \Lambda_N^{-1} \Rightarrow m$ (in the sense of weak measure convergence on H) is not affected by a swap of Ξ in Eq. 1.3, as long as the function Ξ meets certain criteria, specified in Condition 2.1. Lemma 2.4, Proposition 2.7 and Theorem 2.8 are taken from [26]. Their proofs have been slightly revised for better readability. The examples in Section 2 treat the wetting model of [9] and the polymer model in [6]. In each of those two cases the asymptotic equivalence of piecewise linear, continuously interpolating height maps and piecewise constant interpolating height maps is established.

Section 3 starts in the setting of the preceding section. A dynamic analogue of Theorem 2.8 is derived, together with a tightness result in the context of Dirichlet forms. Then,

the Fukushima decomposition for the diffusion processes which are relevant to the topic of microscopic interface models is stated. For the mesoscopic scaling limit of such processes an infinite-dimensional version of the Fukushima decomposition by Röckner-Zhu-Zhu is provided. At first, we assume that a symmetric, regular, local and conservative Dirichlet form $\mathcal{E}^N$ on $L^2(\mathbb{R}^{k_N}, m_N)$ with Carré-du-Champs operator Γ_N is given. For the tightness result, we require each linear functional $l : \mathbb{R}^{k_N} \to \mathbb{R}$ to be a member of $\mathcal{D}(\mathcal{E})$ and uniform bounds for $\Gamma_N(l, l)$ in terms of the Lipschitz constant of l (see Eq. 3.1), independent from N. We consider a separable Hilbert space H_0, densely included in H, such that the inclusion $H_0 \hookrightarrow H$ has a finite Hilbert–Schmidt norm. Via the Gelfand triple

$$H_0 \subset H = H' \subset H_0'$$

we obtain a suitable state space H_0' for the tightness result. We look at the image measure $\mathbb{P}_N$ of the equilibrium law of the diffusion process $(X_t^N)_{t \geq 0}$ associated to $\mathcal{E}^N$ under the time-scaled, H-valued process

$$u_t^N := \Lambda_N \circ X_{N^2 t}^N, \quad t \geq 0.$$

Proposition 3.4 states the tightness of $(\mathbb{P}_N)_{N \in \mathbb{N}}$ as a family of probabilities on $C([0, \infty), H_0')$, provided the tightness of the invariant measures. We consider another height map $\widetilde{\Lambda}_N$, which is equivalent to Λ_N in the sense that it is also built on Condition 2.1, and define $\widetilde{\mathbb{P}}_N$ as the equilibrium law under

$$u_t^N := \widetilde{\Lambda}_N \circ X_{N^2 t}^N, \quad t \geq 0.$$

Then, the weak measure convergence of $(\widetilde{\mathbb{P}}_N)_{N \in \mathbb{N}}$ and that of $(\mathbb{P}_N)_{N \in \mathbb{N}}$ on $C([0, \infty), H_0')$ are two equivalent statements. This is shown in Corollary 3.5, a dynamic version of Theorem 2.8. The Fukushima decomposition for $(X_t^N)_{t \geq 0}$ is stated for the case, where $m_N = m_{A_N}$ is of the type Eq. 1.9 and $\mathcal{E}^N$ is the standard gradient form with that reference measure. The process is a system of skew interacting Brownian motions. The infinite-dimensional analogue is a Röckner–Zhu–Zhu decomposition for the diffusion associated to the gradient from on H with a perturbed Gaussian as a reference measure. The perturbing density is proportional to $\exp(-\int_D g \circ h \, \mathrm{d}z)$, $h \in H$. It is the potential of the non-linear drift term in the reaction-diffusion Eq. 1.1. Section 3 closes with an observation preparing the subsequent and final Section 4. The weak convergence of the equilibrium laws $(\mathbb{P}_N)_{N \in \mathbb{N}}$ is equivalent to the Mosco–Kuwae–Shioya convergence of the image forms of $N^2 \mathcal{E}^N$ under the map Λ_N. By virtue of the above mentioned Corollary 3.5, we may w.l.o.g. assume $\Xi = \mathbf{1}_{[-1,0)^d}$. In this case, the image form of $N^2 \mathcal{E}^N$ under Λ_N is again a standard gradient form. Its state space is $\mathrm{Im}(\Lambda_N) \subset H$ and the reference measure is $m_N \circ \Lambda_N^{-1}$.

Section 4 deals with Mosco–Kuwae–Shioya convergence of gradient forms on H. To provide the desired result for the perturbed Gaussian models considered above, we apply the more abstract criterion for Mosco convergence, developed in [15]. The main results are stated in Theorem 4.10 and Corollary 4.11.

The most important results of this article are:

- Let m_N on $\mathbb{R}^{k_N}$, $N \in \mathbb{N}$, and m on $H = L^2(D, \mathrm{d}z)$ be Borel probability measures. We give a family $\mathcal{A}$ of functions $\mathbb{R}^d \to [0, 1]$ from which to choose Ξ in Eq. 1.3. Then, either the asymptotic statement of

$$\lim_{N \to \infty} \int_{\mathbb{R}^{k_N}} f(\Lambda_N x) m_N(\mathrm{d}x) = \int_H f \, \mathrm{d}m \quad \text{for all } f \in C_b(H) \qquad (1.10)$$

holds true for every choice $\Xi \in \mathcal{A}$, or there is no $\Xi \in \mathcal{A}$ at all to choose in Eq. 1.3 in order to obtain Eq. 1.10. The family $\mathcal{A}$ is general enough to accommodate the most relevant

choices for Ξ. So, as a consequence of this result (see Theorem 2.8), it is irrelevant to the validity of Eq. 1.10, whether one chooses piecewise linear, continuously interpolating height maps, or piecewise constant interpolating height maps (see Examples 2.2, 2.5, 2.6 and 2.9). The family $\mathcal{A}$ comprises all functions specified in Condition 2.1.

- Let μ be a non-degenerate centered Gaussian measure on H and μ_N be a non-degenerate centered Gaussian measure on $\mathbb{R}^{k_N}$ such that $\mu_N \circ \Lambda_N^{-1}$ converges to μ for $N \to \infty$, in the sense of weak measure convergence on H. Further, given functions of bounded variation $g, g_N : \mathbb{R} \to \mathbb{R}$, $N \in \mathbb{N}$, we set

$$\rho : H \ni h \mapsto \frac{1}{Z} \exp\left(\int_D g \circ h \, dz\right), \quad Z := \int_H \exp\left(\int_D g \circ h \, dz\right) d\mu(h)$$

and

$$\rho_N : \mathbb{R}^{k_N} \ni x \mapsto \frac{1}{Z_N} \exp\left(\frac{1}{N^d} \sum_{i=1}^{k_N} g_N\left(N^{\frac{d}{2}-1} x_i\right)\right),$$

$$Z_N := \int_{\mathbb{R}^{k_N}} \exp\left(\frac{1}{N^d} \sum_{i=1}^{k_N} g_N\left(N^{\frac{d}{2}-1} x_i\right)\right) d\mu_N(x).$$

We assume that $(g_N)_{N\in\mathbb{N}}$ is an approximation for g in a generalized sense of local uniform convergence (as defined in Section 4). Now, we consider the distorted Ornstein–Uhlenbeck process $(X_t)_{t\geq 0}$ on H whose Dirichlet form is the standard gradient form with reference measure $\rho\mu$ (as defined in Remark 3.8) on the one hand. On the other hand, we look at the skew reflecting distorted Brownian motion $(X_t^N)_{t\geq 0}$ on $\mathbb{R}^{k_N}$ whose Dirichlet form is the Euclidean standard gradient form with reference measure $\rho_N\mu_N$ (as defined in 3.9). Then, the equilibrium laws of the latter process under the scaling

$$\Lambda_N \circ X_{N^2 t}^N, \quad t \geq 0,$$

weakly converge towards the equilibrium law of $(X_t)_{t\geq 0}$ as $N \to \infty$ (see Corollary 4.11).

2 The Static Model and the Height Map Λ_N

Let $d \in \mathbb{N}$ and D be a bounded, open domain of $\mathbb{R}^d$ with topological closure $\overline{D}$. $\mathcal{B}_b(D)$ denotes the bounded, measurable functions from D to $\mathbb{R}$. We assume that for $N \in \mathbb{N}$ we are given a set of grid points $G_N \subset N\overline{D} \cap \mathbb{Z}^d$ and an injective linear map Λ_N from $\mathbb{R}^{|G_N|}$ to $\mathcal{B}_b(D)$. Intuitively, Λ_N, which is called the height map, associates a microscopic state $x \in \mathbb{R}^{|G_N|}$ with a physical interface in the $(d + 1)$-dimensional space, the graph of $\Lambda_N x$. It is defined over the domain D. In this text, a microscopic state is usually denoted with the variable x, while the variable z is usually used to represent a point in the domain of the interface, i.e. a point in D. We denote the Lebesgue measures in the corresponding dimension by dx, respectively dz. Moreover, we set $H := L^2(D, dz)$ with its inner product $\langle \cdot, \cdot \rangle$. $C_b(H)$ denotes the space of bounded, continuous functions from H to $\mathbb{R}$.

In this section, we fix Borel probability measures, m_N on $\mathbb{R}^{|G_N|}$ for $N \in \mathbb{N}$, and m on H. Then we discuss an asymptotic stability property for the sequence $(m_N)_{N\in\mathbb{N}}$. We ask whether the weak measure convergence,

$$\lim_{N\to\infty} \int_{\mathbb{R}^{|G_N|}} f(\Lambda_N x) m_N(dx) = \int_H f \, dm \quad \text{for } f \in C_b(H), \tag{2.1}$$

is preserved under a slight modification of the height map Λ_N, while $(m_N)_{N\in\mathbb{N}}$ and m are fixed. This question is tackled under the additional assumption that the height map Λ_N is of a particular type. We assume that a bijection $\{1,\ldots,|G_N|\} \to G_N$, $i \mapsto p_i$, is given such that

$$\Lambda_N x(z) = N^{\frac{d}{2}-1} \sum_{i=1}^{|G_N|} x_i \, \Xi(Nz - p_i), \quad x \in \mathbb{R}^{|G_N|}, \ z \in D, \tag{2.2}$$

for some measurable function $\Xi : \mathbb{R}^d \to [0,1]$, which is independent of the parameter N. The scaling factor of $N^{\frac{d}{2}-1}$ is merely a convention, as any re-scaling of the height maps Eq. 2.2 could simply be absorbed by considering the accordingly re-defined measures $(m_N)_{N\in\mathbb{N}}$ in Eq. 2.1.

As an answer to the stability question raised above, we can explicitly give a set $\mathcal{A}$ of functions $\mathbb{R}^d \to [0,1]$ such that, either Eq. 2.1 holds for every choice $\Xi \in \mathcal{A}$ in Eq. 2.2, or there is no $\Xi \in \mathcal{A}$ at all to choose in Eq. 2.2 in order to obtain Eq. 2.1. As an consequence of this section's main result, Theorem 2.8, it is irrelevant to the validity of Eq. 2.1 whether one chooses linearly interpolating height maps, or piecewise constant interpolating height maps (see Examples 2.2, 2.5, 2.6 and 2.9). $\mathcal{A}$ is the set of all functions specified by the following condition.

Condition 2.1 $\Xi : \mathbb{R}^d \to [0,1]$ is measurable with the following properties:

(i) Ξ vanishes outside the centered d-cube with side-length 2, i.e. $\mathrm{supp}[\Xi] \subseteq [-1,1]^d$.
(ii) The integral $\int_{\mathbb{R}^d} \Xi(z)\,\mathrm{d}z$ equals 1.
(iii) The family $\{\Xi(\cdot - \mathbf{k}) \mid \mathbf{k} \in \mathbb{Z}^d\}$, containing all shifts of Ξ by lattice points, form a partition of unity, i.e.

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \Xi(z - \mathbf{k}) = 1, \quad z \in \mathbb{R}^d.$$

(iv) The strict inequality $\displaystyle\int_{\mathbb{R}^d} \Xi(z)^2\,\mathrm{d}z > \frac{1}{2}$ holds true.

One obvious choice for Ξ to meet the criteria of Condition 2.1 is the indicator function $\mathbf{1}_{[-\frac{1}{2},\frac{1}{2})^d}$ of a semi-open cube with side-length 1. Alternatively, the tent function can provide a legitimate choice for Ξ. We discuss it for the cases $d = 1$ and $d = 2$ in the proceeding examples. For an arbitrary index set I and functions $f_i : \mathbb{R}^d \to \mathbb{R}$, $i \in I$, let

$$(f_i \wedge f_j)(z) := \min\{f_i(z), f_j(z)\},$$
$$\Big(\bigwedge_{i\in I} f_i\Big)(z) := \inf\{f_i(z) \mid i \in I\}$$

for $z \in \mathbb{R}^d$. The positive part $z \mapsto \max(\{0, f(z)\})$ of a function $f : \mathbb{R}^d \to \mathbb{R}$ is denoted by f^+. Moreover, $\pi_i : \mathbb{R}^d \to \mathbb{R}$ denotes the projection onto the i-th coordinate for $i \in \{1,\ldots,d\}$.

Example 2.2 THE TENT FUNCTION

(i) Let $d = 1$ and 'id' be the identity on $\mathbb{R}$. The continuous, piecewise linear function

$$\Xi := \big((1 + \mathrm{id}) \wedge (1 - \mathrm{id})\big)^+ \tag{2.3}$$

meets the criteria of Condition 2.1, as easily verified. To check the fourth criterion, for example, we calculate

$$\int_{\mathbb{R}} \Xi(z)^2 \, dz = \int_{-1}^{0} (1+z)^2 \, dz + \int_{0}^{1} (1-z)^2 \, dz = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} > \frac{1}{2}.$$

(ii) Now we consider the case $d = 2$. The continuous function

$$\Xi := \left[\bigwedge_{i,j \in \{1,2\}} \left((1 + \pi_i - \pi_j) \wedge (1 + \pi_i) \wedge (1 - \pi_i) \right) \right]^{+} \tag{2.4}$$

meets the criteria of Condition 2.1. Ξ is the function which results from taking the minimum over a family of six affine linear functions,

$$\left\{ \begin{array}{lll} 1 + \pi_1 - \pi_2, & 1 + \pi_1, & 1 - \pi_1, \\ \\ 1 + \pi_2 - \pi_1, & 1 + \pi_2, & 1 - \pi_2 \end{array} \right\},$$

and then take its positive part. Simply comparing the values of those six affine linear functions, we get

$$\Xi(z) = \begin{cases} 1 - z_1 & \text{if } 0 \leq z_2 < z_1 < 1, \\ 1 - z_2 & \text{if } 0 \leq z_1 \leq z_2 < 1, \\ 1 + z_1 - z_2 & \text{if } -1 \leq z_2 - 1 < z_1 < 0, \\ 1 + z_1 & \text{if } -1 \leq z_1 \leq z_2 < 0, \\ 1 + z_2 & \text{if } -1 \leq z_2 < z_1 < 0, \\ 1 + z_2 - z_1 & \text{if } -1 \leq z_1 - 1 \leq z_2 < 0, \\ 0 & \text{else.} \end{cases} \tag{2.5}$$

The fourth criterion of Condition 2.1 can be checked with elementary calculations. It holds

$$\int_{\{0 \leq \pi_2 < \pi_1 < 1\}} (1 - \pi_1)^2(z) \, dz = \int_{0}^{1} \int_{0}^{z_1} (1 - z_1)^2 \, dz_2 \, dz_1 = \int_{0}^{1} z_1 (1 - z_1)^2 \, dz_1 = \frac{1}{12}$$

and

$$\int_{\{-1 \leq \pi_1 - 1 \leq \pi_2 < 0\}} (1 + \pi_2 - \pi_1)^2(z) \, dz = \int_{0}^{1} \int_{z_1 - 1}^{0} (1 + z_2 - z_1)^2 \, dz_2 \, dz_1$$

$$= \int_{0}^{1} -\frac{3}{2} z_1^3 + \frac{3}{2} z_1^2 - \frac{3}{2} z_1 + \frac{5}{6} \, dz_1 = \frac{5}{24}.$$

For reasons of symmetry it holds

$$\frac{1}{12} = \int_{\{0 \leq \pi_2 < \pi_1 < 1\}} (1 - \pi_1)^2(z) \, dz = \int_{\{0 \leq \pi_1 \leq \pi_2 < 1\}} (1 - \pi_2)^2(z) \, dz$$

$$= \int_{\{-1 \leq \pi_2 < \pi_1 < 0\}} (1 + \pi_2)^2(z) \, dz = \int_{\{-1 \leq \pi_1 \leq \pi_2 < 0\}} (1 + \pi_1)^2(z) \, dz$$

and also

$$\frac{5}{24} = \int_{\{-1 \leq \pi_1 - 1 \leq \pi_2 < 0\}} (1 + \pi_2 - \pi_1)^2(z) \, dz = \int_{\{-1 \leq \pi_2 - 1 < \pi_1 < 0\}} (1 + \pi_1 - \pi_2)^2(z) \, dz.$$

Hence, from Eq. 2.5 it follows

$$\int_{\mathbb{R}^d} \Xi(z)^2 \, dz = 4 \cdot \frac{1}{12} + 2 \cdot \frac{5}{24} = \frac{3}{4} > \frac{1}{2}$$

as desired. A more extensive analysis of the tent function can be found in [15] and [26]. In both of these references, a general dimension $d \in \mathbb{N}$ is considered, complementing the definitions of Eqs. 2.4 and 2.3 with

$$\Xi := \left[\bigwedge_{i,j \in \{1,\ldots,d\}} \left((1 + \pi_i - \pi_j) \wedge (1 + \pi_i) \wedge (1 - \pi_i) \right) \right]^+$$

for a case $d > 2$. For the d-dimensional tent function, where $d \in \mathbb{N}$, items (i) to (iii) of Condition 2.1 are proven in [15, Sect. 2.1], respectively [26, Sect. 3.1.1].

Before turning back to the general case, another remark about the interpolating property of the tent function is inserted, which is exploited later on, in Examples 2.5 & 2.6. For the abstract results of this section, Remark 2.3 does not play any role. The reader who is just interested in the main results of Section 2, Proposition 2.7 and Theorem 2.8, may skip it.

Remark 2.3   (i) We consider the one-dimensional tent function of Example 2.2 (i). For any given sequence of values $(\lambda_k)_{k \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$, the function

$$\mathbb{R} \ni z \mapsto \sum_{k \in \mathbb{Z}} \lambda_k \, \Xi(z - k) \tag{2.6}$$

is the unique element of the space of piecewise linear, continuous functions

$$\left\{ f \in C(\mathbb{R}) \,\middle|\, \text{for each } k \in \mathbb{Z} \text{ there is an affine linear function} \right.$$

$$\left. a_k : \mathbb{R} \to \mathbb{R} \text{ with } a_k(z) = f(z) \text{ for } z \in [k, k+1] \right\}$$

which interpolates the sample $\{(k, \lambda_k) \mid k \in \mathbb{Z}\}$. So, Eq. 2.6 has the alternative representation

$$\sum_{k \in \mathbb{Z}} \lambda_k \, \Xi(z - k) = \sum_{k \in \mathbb{Z}} \mathbf{1}_{[k,k+1)}(z) \left(\lambda_k + (z - k)(\lambda_{k+1} - \lambda_k) \right). \tag{2.7}$$

(ii) For the two-dimensional case, an analogous representation as in Eq. 2.7 can be derived, which displays the connection between the tent function and piecewise linear interpolation. The semi-open unit square $[0, 1)^2$ is the disjoint union of two triangles $T_{1,0}$ and $T_{2,0}$, where

$$T_{1,0} := \{z \in \mathbb{R}^2 \mid 0 \le z_2 < z_1 < 1\} \quad \text{and} \quad T_{2,0} := \{z \in \mathbb{R}^2 \mid 0 \le z_1 \le z_2 < 1\}.$$

Shifting those triangles results in a partition of $\mathbb{R}^2$ through the family of disjoint sets

$$T_{i,\mathbf{k}} := \{\mathbf{k} + z \mid z \in T_{i,0}\}, \quad i = 1, 2, \quad \mathbf{k} \in \mathbb{Z}^2.$$

The representation of the two-dimensional tent function from Eq. 2.5 shows that the function Ξ considered in Example 2.2 (ii) is a member of the linear space

$$Q := \left\{ f \in C(\mathbb{R}^2) \,\middle|\, \text{for each } i \in \{1, 2\} \text{ and } \mathbf{k} \in \mathbb{Z}^2 \text{ there is an affine linear function} \right.$$

$$\left. a_{i,\mathbf{k}} : \mathbb{R}^2 \to \mathbb{R} \text{ with } a_{i,\mathbf{k}}(z) = f(z) \text{ for } z \in T_{i,\mathbf{k}} \right\}.$$

The same applies to all shifted functions $\Xi(\cdot - \mathbf{k})$, $\mathbf{k} \in \mathbb{Z}^2$, of course. Let $(\lambda_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^2} \in \mathbb{R}^{\mathbb{Z}^2}$. Since $\Xi(\mathbf{l} - \mathbf{k}) = \delta_{\mathbf{l},\mathbf{k}}$ for $\mathbf{l}, \mathbf{k} \in \mathbb{Z}^2$, the function

$$\mathbb{R}^2 \ni z \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^2} \lambda_{\mathbf{k}} \Xi(z - \mathbf{k}) \tag{2.8}$$

is an element in Q which interpolates the sample $\{(\mathbf{k}, \lambda_{\mathbf{k}}) \mid \mathbf{k} \in \mathbb{Z}^2\}$. It is the unique element of Q with that property, as every affine linear function is uniquely determined by its values on the corner points of a triangle in the two-dimensional plane. Denoting the unit vectors of $\mathbb{R}^2$ by $\mathbf{e}_1, \mathbf{e}_2$, the corner points of $T_{1,\mathbf{k}}$ are $\mathbf{k}, \mathbf{k} + \mathbf{e}_1, \mathbf{k} + \mathbf{e}_1 + \mathbf{e}_2$, while the corner points of $T_{2,\mathbf{k}}$ are $\mathbf{k}, \mathbf{k} + \mathbf{e}_2, \mathbf{k} + \mathbf{e}_1 + \mathbf{e}_2$, for every $\mathbf{k} \in \mathbb{Z}^2$. This argumentation shows that the map of Eq. 2.8 has the following representation, which reflects its interpolating property:

$$\sum_{\mathbf{k} \in \mathbb{Z}^2} \lambda_{\mathbf{k}} \Xi(z - \mathbf{k})$$
$$= \sum_{\mathbf{k} \in \mathbb{Z}^2} \mathbf{1}_{T_{1,\mathbf{k}}}(z)\Big(\lambda_{\mathbf{k}} + \pi_1(z - \mathbf{k})(\lambda_{\mathbf{k}+\mathbf{e}_1} - \lambda_{\mathbf{k}}) + \pi_2(z - \mathbf{k})(\lambda_{\mathbf{k}+\mathbf{e}_1+\mathbf{e}_2} - \lambda_{\mathbf{k}+\mathbf{e}_1})\Big)$$
$$+ \mathbf{1}_{T_{2,\mathbf{k}}}(z)\Big(\lambda_{\mathbf{k}} + \pi_2(z - \mathbf{k})(\lambda_{\mathbf{k}+\mathbf{e}_2} - \lambda_{\mathbf{k}}) + \pi_1(z - \mathbf{k})(\lambda_{\mathbf{k}+\mathbf{e}_1+\mathbf{e}_2} - \lambda_{\mathbf{k}+\mathbf{e}_2})\Big). \tag{2.9}$$

This implies the interpolating property of the map Eq. 2.2 if $d = 2$ and Ξ is chosen as the tent function Eq. 2.4.

On the Hilbert space $l^2(\mathbb{Z}^d)$ of all square-summable, multi-indexed sequences $(\lambda_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ the infinite (multi-indexed) matrix

$$A_{\mathbf{k},\mathbf{l}} := \int_{\mathbb{R}^d} \Xi(z - \mathbf{k})\, \Xi(z - \mathbf{l})\, \mathrm{d}z, \quad \mathbf{k}, \mathbf{l} \in \mathbb{Z}^d,$$

corresponds to a symmetric linear operator A. The lower estimate in Proposition 2.7 below is based on the existence of an upper bound for the operator norm of A^{-1}. Item (iv) of Condition 2.1 is a sufficient criterion for A^{-1} to be a bounded linear operator on $l^2(\mathbb{Z}^d)$, as the next lemma shows.

Lemma 2.4 *Let* $(\lambda_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ *such that* $\{\mathbf{k} \in \mathbb{Z}^d \mid \lambda_{\mathbf{k}} \neq 0\}$ *is a finite set and* Ξ *as in Condition 2.1. It holds*

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\mathbf{l} \in \mathbb{Z}^d} \lambda_{\mathbf{k}} A_{\mathbf{k},\mathbf{l}} \lambda_{\mathbf{l}} \geq c \sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}}^2, \quad where \quad c := 2 \int_{\mathbb{R}^d} \Xi(z)^2 \, \mathrm{d}z - 1.$$

Proof Since $A_{\mathbf{k},\mathbf{k}} = \frac{c+1}{2}$ for $\mathbf{k} \in \mathbb{Z}^d$,

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\mathbf{l} \in \mathbb{Z}^d} \lambda_{\mathbf{k}} A_{\mathbf{k},\mathbf{l}} \lambda_{\mathbf{l}} = \sum_{\mathbf{k} \in \mathbb{Z}^d} A_{\mathbf{k},\mathbf{k}} \lambda_{\mathbf{k}}^2 + \sum_{\substack{\mathbf{k},\mathbf{l} \in \mathbb{Z}^d \\ \mathbf{l} \neq \mathbf{k}}} A_{\mathbf{k},\mathbf{l}} \lambda_{\mathbf{k}} \lambda_{\mathbf{l}}$$
$$= \frac{1}{2} \sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}}^2 + \sum_{\substack{\mathbf{k},\mathbf{l} \in \mathbb{Z}^d \\ \mathbf{l} \neq \mathbf{k}}} A_{\mathbf{k},\mathbf{l}} \lambda_{\mathbf{k}} \lambda_{\mathbf{l}} + \frac{c}{2} \sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}}^2. \tag{2.10}$$

Moreover, since $\int_{\mathbb{R}^d} \Xi(z)\,dz = 1$ and $\sum_{\mathbf{l} \in \mathbb{Z}^d} \Xi(z - \mathbf{l}) = 1$ for $z \in \mathbb{R}^d$,

$$1 = 2 \int_{\mathbb{R}^d} \Xi(z - \mathbf{k})\,dz - 1 = 2 \sum_{\mathbf{l} \in \mathbb{Z}^d} A_{\mathbf{k},\mathbf{l}} - 1$$

$$= 2A_{\mathbf{k},\mathbf{k}} - 1 + 2 \sum_{\substack{\mathbf{l} \in \mathbb{Z}^d \\ \mathbf{l} \neq \mathbf{k}}} A_{\mathbf{k},\mathbf{l}} = c + 2 \sum_{\substack{\mathbf{l} \in \mathbb{Z}^d \\ \mathbf{l} \neq \mathbf{k}}} A_{\mathbf{k},\mathbf{l}}, \quad \mathbf{k} \in \mathbb{Z}^d. \tag{2.11}$$

Multiplying Eq. 2.11 by $\lambda_{\mathbf{k}}^2/2$ for each $\mathbf{k} \in \mathbb{Z}^2$ and then summing up over $\mathbf{k} \in \mathbb{Z}^d$ leads to

$$\frac{1}{2} \sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}}^2 = \frac{c}{2} \sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}}^2 + \sum_{\substack{\mathbf{k},\mathbf{l} \in \mathbb{Z}^d \\ \mathbf{l} \neq \mathbf{k}}} A_{\mathbf{k},\mathbf{l}} \lambda_{\mathbf{k}}^2. \tag{2.12}$$

Rewriting the right-hand side of Eq. 2.10 under use of Eq. 2.12, one obtains

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\mathbf{l} \in \mathbb{Z}^d} \lambda_{\mathbf{k}} A_{\mathbf{k},\mathbf{l}} \lambda_{\mathbf{l}} = \sum_{\substack{\mathbf{k},\mathbf{l} \in \mathbb{Z}^d \\ \mathbf{l} \neq \mathbf{k}}} A_{\mathbf{k},\mathbf{l}} \big(\lambda_{\mathbf{k}}^2 + \lambda_{\mathbf{k}} \lambda_{\mathbf{l}}\big) + c \sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}}^2$$

$$= \frac{1}{2} \sum_{\mathbf{k},\mathbf{l} \in \mathbb{Z}^d} A_{\mathbf{k},\mathbf{l}} \big(\lambda_{\mathbf{k}} + \lambda_{\mathbf{l}}\big)^2 + c \sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}}^2 \geq c \sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}}^2$$

as desired. $\qquad\qquad\square$

We make a further assumption on the height maps Λ_N, $N \in \mathbb{N}$, concerning the sets of Grid points G_N. We assume there is a sequence of increasing sets $D_1 \subseteq D_2 \subseteq \ldots$ contained in $\overline{D}$ such that

$$dz\left(\overline{D} \setminus \bigcup_{N \in \mathbb{N}} D_N\right) = 0$$

and for $N \in \mathbb{N}$ we have

$$D_N \text{ is open w.r.t. the trace topology of } \overline{D},$$

$$G_N = N D_N \cap \mathbb{Z}^d. \tag{2.13}$$

To shorten notation, let $k_N := |G_N|$ and

$$\xi_N^p(z) := \Xi(Nz - p), \quad z \in \mathbb{R}^d, \ p \in G_N.$$

Then, according to Eq. 2.2 there is a bijection $\{1, \ldots, k_N\} \to G_N, i \mapsto p_i$, such that, setting $\xi_{N,i} := \xi_N^{p_i}$ for $i \in 1, \ldots, k_N$,

$$\Lambda_N x(z) = N^{\frac{d}{2} - 1} \sum_{i=1}^{k_N} x_i \, \xi_{N,i}(z), \quad x \in \mathbb{R}^{k_N}, \ z \in D. \tag{2.14}$$

Example 2.5 PINING MODEL

In the one-dimensional case with $D = (0, 1)$, we can select $D_N = (0, 1]$ for $N \in \mathbb{N}$, and Ξ as in Example 2.2 (i). So, $G_N = \{1, \ldots, N\}$ and the natural ordering leads to the definition $p_i := i, i = 1, \ldots, N$. Choosing this particular set-up in Eq. 2.14 yields

$$\Lambda_N x(z) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i \max\left(\left\{0, \min\left(\{1 - i + Nz, 1 + i - Nz\}\right)\right\}\right),$$

$z \in (0, 1)$, $x \in \mathbb{R}^N$. As a consequence of Remark 2.3 (i), $\Lambda_N x(\cdot)$ restricts to an affine linear function on each interval $(\frac{i-1}{N}, \frac{i}{N})$, $i = 1, \ldots, N$, and interpolates the values $\frac{x_1}{\sqrt{N}}, \ldots, \frac{x_N}{\sqrt{N}}$ over the grid points $\frac{1}{N}, \frac{2}{N}, \ldots, 1$. Setting $x_0 := 0$, a different way to represent this particular height map is

$$\Lambda_N x(z) = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \left(x_i + (Nz - i)(x_{i+1} - x_i) \right) \mathbf{1}_{[\frac{i}{N}, \frac{i+1}{N})}(z), \quad z \in (0, 1), \ x \in \mathbb{R}^N,$$

as follows from Eq. 2.7 after re-scaling. We cite a result from [9]. Let $V : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that $\exp(-V(\cdot))$ is continuous, $V(0) < \infty$ and

$$\int_{\mathbb{R}} e^{-V(s)} \, \mathrm{d}s =: \kappa \in \mathbb{R} \quad \text{as well as} \quad \frac{1}{\kappa} \int_{\mathbb{R}} s^2 e^{-V(s)} \, \mathrm{d}s := \sigma^2 \in \mathbb{R}.$$

For $x \in \mathbb{R}^N$, $N \in \mathbb{N}$, setting $x_0 := 0$, $x_{N+1} := 0$, let

$$H_N(x) := \sum_{i=0}^{N} V(\sigma x_{i+1} - \sigma x_i).$$

Now, for $\beta \in [0, \infty)$ we define a probability measure on $\mathbb{R}^N$ (with support on $[0, \infty)^N$) as

$$m_N^\beta(\mathrm{d}x) := \frac{1}{Z_{\beta,N}} \exp(-H_N(x)) \prod_{i=1}^{N} \left(\mathbf{1}_{[0,\infty)}(x_i) \, \mathrm{d}x_i + \beta \delta_0(\mathrm{d}x_i) \right)$$

$$\text{with} \quad Z_{\beta,N} := \int_{[0,\infty)^N} \exp(-H_N(x)) \prod_{i=1}^{N} \left(\mathbf{1}_{[0,\infty)}(x_i) \, \mathrm{d}x_i + \beta \delta_0(\mathrm{d}x_i) \right)$$

and let

$$\beta_c := \frac{\kappa}{1 + \sum_{N=1}^{\infty} Z_{0,N}}.$$

Due to [9, Thm. 1] the sequence $(m_N^\beta \circ \Lambda_N)_{N \in \mathbb{N}}$ has a limit m^β regarding weak measure convergence on $C([0, 1])$ (equipped with the supremum-norm) for every $\beta \in [0, \infty)$. Hence, in particular, this is true regarding weak measure convergence on $H = L^2((0, 1), \mathrm{d}z)$. The model undergoes a phase transition as the limit m^β depends on the pinning strength β as follows:

- If $0 \leq \beta < \beta_c$, then m^β is the law of the three-dimensional Bessel bridge between 0 and 0 on the time interval $[0, 1]$.
- In the critical case m^{β_c} is the law of the one-dimensional Bessel bridge between 0 and 0 on the time interval $[0, 1]$.
- If $\beta_c < \beta$, then m^β assigns the value 1 to the singleton of the constant zero-function. on $[0, 1]$.

Example 2.6 SEMIFLEXIBLE POLYMERS AND $\Delta\varphi$- MODEL

In the two-dimensional case with $D = (0, 1)^2$, we can choose

$$D_N = \left(\frac{2}{N}, \frac{N-2}{N} \right) \times \left(\frac{2}{N}, \frac{N-2}{N} \right),$$

w.l.o.g. assuming $N \geq 6$. As usual, let $G_N := ND_N \cap \mathbb{Z}^2$. We cite a result on weak measure convergence from Cipriani et al. [6]. The method, used there, interpolates a state $(\lambda_{\mathbf{k}})_{\mathbf{k} \in G_N} \in \mathbb{R}^{G_N}$ by identifying it with the interface $S((\lambda_{\mathbf{k}})_{\mathbf{k} \in G_N}, \cdot) : D \to \mathbb{R}$,

$$
S\big((\lambda_{\mathbf{k}})_{\mathbf{k} \in G_N}, z\big)
$$

$$
:= \begin{cases} \lambda_{[Nz]} + \pi_1(Nz - [Nz])(\lambda_{[Nz]+\mathbf{e}_1} - \lambda_{[Nz]}) + \pi_2(Nz - [Nz])(\lambda_{[Nz]+\mathbf{e}_1+\mathbf{e}_2} - \lambda_{[Nz]+\mathbf{e}_1}) \\ \lambda_{[Nz]} + \pi_2(Nz - [Nz])(\lambda_{[Nz]+\mathbf{e}_2} - \lambda_{[Nz]}) + \pi_1(Nz - [Nz])(\lambda_{[Nz]+\mathbf{e}_1+\mathbf{e}_2} - \lambda_{[Nz]+\mathbf{e}_2}) \end{cases}
$$

$$
\text{if} \quad \pi_2(Nz - [Nz]) < \pi_1(Nz - [Nz]) \text{ in the first case,}
$$

$$
\pi_1(Nz - [Nz]) \leq \pi_2(Nz - [Nz]) \text{ in the second case}
$$

(2.15)

(cf. [6, Eq. (2.6)]), where $[Nz]$ denotes the element in $\mathbb{Z}^2$ with $Nz - [Nz] \in [0, 1)^2$ for $z \in \mathbb{Z}^2$. This is just a differently scaled version of Eq. 2.9, as we show now. For $T_{1,\mathbf{k}}, T_{2,\mathbf{k}}$ as in Remark 2.3 (ii), $\mathbf{k} \in \mathbb{Z}^2$ and $z \in \mathbb{R}^2$ it holds

$$
\mathbf{1}_{T_{1,\mathbf{k}}}(Nz) = \begin{cases} 1 & \text{if} \quad \mathbf{k} = [Nz] \quad \wedge \quad \pi_2(Nz - \mathbf{k}) < \pi_1(Nz - \mathbf{k}) \\ 0 & \text{else,} \end{cases}
$$

while

$$
\mathbf{1}_{T_{2,\mathbf{k}}}(Nz) = \begin{cases} 1 & \text{if} \quad \mathbf{k} = [Nz] \quad \wedge \quad \pi_1(Nz - \mathbf{k}) \leq \pi_2(Nz - \mathbf{k}) \\ 0 & \text{else.} \end{cases}
$$

By Remark 2.3 (ii), with Ξ as in Example 2.2 (ii), it holds

$$
S\big((\lambda_{\mathbf{k}})_{\mathbf{k} \in G_N}, z\big)
$$

$$
= \sum_{\mathbf{k} \in L_N} \mathbf{1}_{T_{1,\mathbf{k}}}(Nz)\Big(\lambda_{\mathbf{k}} + \pi_1(Nz - \mathbf{k})(\lambda_{\mathbf{k}+\mathbf{e}_1} - \lambda_{\mathbf{k}}) + \pi_2(Nz - \mathbf{k})(\lambda_{\mathbf{k}+\mathbf{e}_1+\mathbf{e}_2} - \lambda_{\mathbf{k}+\mathbf{e}_1})\Big)
$$

$$
+ \sum_{\mathbf{k} \in G_N} \mathbf{1}_{T_{2,\mathbf{k}}}(Nz)\Big(\lambda_{\mathbf{k}} + \pi_2(Nz - \mathbf{k})(\lambda_{\mathbf{k}+\mathbf{e}_2} - \lambda_{\mathbf{k}}) + \pi_1(Nz - \mathbf{k})(\lambda_{\mathbf{k}+\mathbf{e}_1+\mathbf{e}_2} - \lambda_{\mathbf{k}+\mathbf{e}_2})\Big)
$$

$$
= \sum_{\mathbf{k} \in G_N} \lambda_{\mathbf{k}} \, \Xi(Nz - \mathbf{k}).
$$

Consequently, if we fix an arbitrary bijection $I_N : \{1, \ldots, k_N\} \to G_N$, where $k_N := |G_N|$, and initialize the height map from Eq. 2.14 by choosing Ξ from Example 2.2 (ii), then Λ_N and the interpolation Eq. 2.15 from [6] are related as follows:

$$
\Lambda_N x(z) = S\big((\widetilde{x}_{\mathbf{k}})_{\mathbf{k} \in L_N}, z\big), \quad z \in D,
$$

$$
\text{where} \quad \widetilde{x}_{\mathbf{k}} := x_{I_N^{-1}(\mathbf{k})}, \quad \mathbf{k} \in G_N, x \in \mathbb{R}^{k_N}.
$$

For $x \in \mathbb{R}^{k_N}$ we extend $(\widetilde{x}_{\mathbf{k}})_{\mathbf{k}}$ to an element of $\mathbb{R}^{\mathbb{Z}^2}$ by setting $\widetilde{x}_{\mathbf{k}} = 0$ for $\mathbf{k} \in \mathbb{Z}^2 \setminus G_N$. For a scaling $\alpha : \mathbb{N} \to (0, \infty)$ we define

$$
H_N^\alpha(x) := \sum_{\mathbf{k} \in \mathbb{Z}^2} \Big[\frac{1}{16} \sum_{\substack{\mathbf{l} \in \mathbb{Z}^2: \\ |\mathbf{l}-\mathbf{k}|_{\text{euc}}=1}} (\widetilde{x}_{\mathbf{k}} - \widetilde{x}_{\mathbf{l}})^2 + \alpha(N)\Big(\frac{1}{2d} \sum_{\substack{\mathbf{l} \in \mathbb{Z}^2: \\ |\mathbf{l}-\mathbf{k}|_{\text{euc}}=1}} (\widetilde{x}_{\mathbf{k}} - \widetilde{x}_{\mathbf{l}}) \Big)^2 \Big]
$$

for $N \in \mathbb{N}$ and $x \in \mathbb{R}^{k_N}$. Let m_N^α be the Gaussian measure on $\mathbb{R}^{k_N}$ with density

$$
\mathbb{R}^{k_N} \ni x \mapsto \frac{1}{Z_N} \exp(-4H_N^\alpha(x)), \quad \text{where} \quad Z_N := \int_{\mathbb{R}^{k_N}} \exp(-4H_N^\alpha(x)) \, dx.
$$

By virtue of [6, Thm. 2.1 (1) & (2)], the sequence $(m_N^\alpha \circ \Lambda_N)_{N \in \mathbb{N}}$ has a limit m^α regarding weak measure convergence on $C(\overline{D})$ (equipped with the supremum-norm) in the following cases, depending on the asymptotic behaviour of $(\alpha(N))_{N \in \mathbb{N}}$.

- If $\alpha(N) \gg N^2$, then the limit m^α is the law of the continuous Gaussian process on $\overline{D}$ whose covariance operator on H is the inverse of the operator

$$\Delta^2 \quad \text{with domain } H^{4,2}(D) \cap H_0^{3,2}(D).$$

- If $\alpha(N) \sim 2N^2$, then the limit m^α is the law of the continuous Gaussian process on $\overline{D}$ whose covariance operator on H is the inverse of the operator

$$-\Delta + \Delta^2 \quad \text{with domain } H^{4,2}(D) \cap H_0^{3,2}(D).$$

In particular, this is true regarding weak measure convergence on $H = L^2(D, \mathrm{d}z)$.

Proposition 2.7 *It holds*

$$cN^{-2}|x|_{\text{euc}}^2 \le |\Lambda_N x|_H^2 \le 3^d N^{-2}|x|_{\text{euc}}^2$$

for $x \in \mathbb{R}^{k_N}$, where $0 < c \le 1$ is a constant, independent of N.

Proof The claimed inequality holds for all $x \in \mathbb{R}^{k_N}$ and some constant c, if and only if

$$\frac{c}{N^d} \sum_{p \in G_N} y_p^2 \le \Big| \sum_{p \in G_N} y_p \, \xi_N^p \Big|_H^2 \le \frac{3^d}{N^d} \sum_{p \in G_N} y_p^2 \quad \text{for all } y \in \mathbb{R}^{G_n}. \tag{2.16}$$

We choose $c := 2 \int_{\mathbb{R}^d} \Xi(z)^2 \, \mathrm{d}z - 1$ and prove Eq. 2.16. Note that $0 < c \le 1$, since Ξ meets Condition 2.1. Let $y \in \mathbb{R}^{G_N}$ and $\lambda \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\lambda_{\mathbf{k}} = y_p$, if $\mathbf{k} = p$ for some $p \in G_N$, and $\lambda_{\mathbf{k}} = 0$ else. Due to Lemma 2.4, it holds

$$c \sum_{p \in G_N} y_p^2 = c \sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}}^2 \le \int_{\mathbb{R}^d} \Big(\sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}} \Xi(z - \mathbf{k}) \Big)^2 \mathrm{d}z$$

$$= N^d \int_{\mathbb{R}^d} \Big(\sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}} \Xi(Nz - \mathbf{k}) \Big)^2 \mathrm{d}z = N^d \int_{\mathbb{R}^d} \Big(\sum_{p \in G_N} y_p \, \xi_N^p(Nz - p) \Big)^2 \mathrm{d}z$$

$$= N^d \Big| \sum_{p \in G_N} y_p \, \xi_N^p \Big|_H^2.$$

this proves the first inequality of Eq. 2.16. On the other hand,

$$\big| \langle y_p \, \xi_N^p, \, y_q \, \xi_N^q \rangle_H \big| \le \frac{y_p^2 + y_q^2}{2} \int_{\mathbb{R}^d} \Xi(Nz - p) \Xi(Nz - q) \, \mathrm{d}z = \frac{y_p^2 + y_q^2}{2N^d} \tag{2.17}$$

for $y \in \mathbb{R}^{G_N}$ and $p, q \in G_N$. On top of that, $\max_{i=1,\dots,d} |p_i - q_i| \le 1$ is a necessary condition for $\langle \xi_N^p, \xi_N^q \rangle_H \ne 0$, due to Condition 2.1 (i). Now, the missing upper bound of Eq. 2.16 is evident from

$$\Big| \sum_{p \in G_N} y_p \, \xi_N^p \Big|_H^2 \le \sum_{\substack{p,q \in G_N: \\ \max_i |p_i - q_i| \le 1}} \frac{y_p^2}{2N^d} + \sum_{\substack{p,q \in G_N: \\ \max_i |p_i - q_i| \le 1}} \frac{y_q^2}{2N^d} = \frac{3^d}{N^d} \sum_{p \in G_N} y_p^2,$$

where we first used Eq. 2.17 and then $\#\{q \in G_N \mid \max_{i=1,\dots,d} |p_i - q_i| \le 1\} = 3^d$ for $p \in G_N$. $\qquad\square$

We now compare the validity of Eq. 2.1 for two different choices, Ξ and $\widetilde{\Xi}$, assuming that each of them meets Condition 2.1. So, for $p \in G_N$ (as in Eq. 2.13) we have

$$\xi_N^p(z) := \Xi(Nz - p), \quad z \in \mathbb{R}^d,$$

on the one hand, and

$$\widetilde{\xi}_N^p(z) := \widetilde{\Xi}(Nz - p), \quad z \in \mathbb{R}^d,$$

on the other. This leads to two different height maps,

$$\Lambda_N x(z) = N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} x_i \xi_{N,i}(z), \quad x \in \mathbb{R}^{k_N}, \; z \in D,$$

with $\xi_{N,i} := \xi_N^{p_i}$, as well as

$$\widetilde{\Lambda}_N x(z) = N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} x_i \widetilde{\xi}_{N,i}(z), \quad x \in \mathbb{R}^{k_N}, \; z \in D,$$

with $\widetilde{\xi}_{N,i} := \widetilde{\xi}_N^{p_i}$. It turns out, these two resulting height maps are asymptotically equivalent in the sense specified by the subsequent theorem.

Theorem 2.8 *Let m_N be a Borel probability measure on $\mathbb{R}^{k_N}$ for $N \in \mathbb{N}$ and m be a Borel probability measure on H. Regarding the weak convergence of measures on H it holds*

$$m_N \circ \Lambda_N^{-1} \Rightarrow m \quad \text{is equivalent to} \quad m_N \circ \widetilde{\Lambda}_N^{-1} \Rightarrow m.$$

Proof At first, we prove the convergence

$$\lim_{N\to\infty} \int_D \Big| \sum_{p \in G_N} \varphi\Big(\frac{p}{N}\Big) \xi_N^p(z) - \varphi(z) \Big|^2 \, dz = 0 \tag{2.18}$$

for $\varphi \in C(\overline{D})$. To verify Eq. 2.18, consider that

$$\text{supp}[\Xi(N \cdot -\mathbf{k})] \subseteq \frac{1}{N}\Big([\mathbf{k}_1 - 1, \mathbf{k}_1 + 1] \times \cdots \times [\mathbf{k}_d - 1, \mathbf{k}_d + 1]\Big)$$

$$\subseteq \Big\{ z \in \mathbb{R}^d \, \Big| \, |z - \tfrac{\mathbf{k}}{N}|_{\text{euc}} \leq \tfrac{\sqrt{d}}{N} \Big\}$$

for $\mathbf{k} \in \mathbb{Z}^d$. So, with Eq. 2.13 and Condition 2.1 (iii), we have

$$\limsup_{N\to\infty} \int_D \Big| \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus G_N} \Xi(Nz - \mathbf{k}) \Big|^2 \, dz \leq \limsup_{N\to\infty} dz\big(D \cap B_{\sqrt{d}/N}^{\text{euc}}(\overline{D} \setminus D_N)\big) = 0,$$

where $B_{\sqrt{d}/N}^{\text{euc}}(\overline{D} \setminus D_N) := \{z' \in \mathbb{R}^d \mid |z' - z|_{\text{euc}} < \sqrt{d}/N \text{ for some } z \in \overline{D} \setminus D_N\}$. Then, the estimate

$$\Big(\int_D \Big| \sum_{p \in G_N} \varphi\Big(\frac{p}{N}\Big) \xi_N^p(z) - \varphi(z) \Big|^2 \, dz \Big)^{\frac{1}{2}}$$

$$= \Big(\int_D \Big| \sum_{p \in G_N} \varphi\Big(\frac{p}{N}\Big) \xi_N^p(z) - \varphi(z) \sum_{\mathbf{k} \in \mathbb{Z}^d} \Xi(Nz - \mathbf{k}) \Big|^2 \, dz \Big)^{\frac{1}{2}}$$

$$\leq \Big(\int_D \Big| \sum_{p \in G_N} \Big(\varphi\Big(\frac{p}{N}\Big) - \varphi(z)\Big) \xi_N^p(z) \Big|^2 \, dz \Big)^{\frac{1}{2}} + \sup_{z \in \overline{D}} |\varphi(z)| \Big(\int_D \Big| \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus G_N} \Xi(Nz - \mathbf{k}) \Big|^2 \, dz \Big)^{\frac{1}{2}}$$

leads to the desired equality in Eq. 2.18, because

$$\left(\int_D \Big| \sum_{p \in G_N} \Big(\varphi(\frac{p}{N}) - \varphi(z) \Big) \xi_N^p(z) \Big|^2 \, dz \right)^{\frac{1}{2}}$$

$$\leq \sup_{\substack{z,z' \in \overline{D} \\ |z-z'|_{\text{euc}} \leq \sqrt{d}/N}} |\varphi(z) - \varphi(z')| \left(\int_D \sum_{p \in G_N} \xi_N^p(z)^2 \, dz \right)^{\frac{1}{2}}$$

$$\leq \sqrt{dz(D)} \sup_{\substack{z,z' \in \overline{D} \\ |z-z'|_{\text{euc}} \leq \sqrt{d}/N}} |\varphi(z) - \varphi(z')|.$$

In the last inequality, $\xi_N^p(z)^2 \leq \xi_N^p(z)$ and $\sum_{p \in G_N} \xi_N^P(z) \leq 1$ is used. Now, since Eq. 2.18 is proven,

$$\lim_{N \to \infty} \int_D \Big| \sum_{p \in G_N} \varphi(\frac{p}{N}) \widetilde{\xi}_N^p(z) - \varphi(z) \Big|^2 \, dz = 0 \tag{2.19}$$

holds likewise, of course.

The rest of this proof is dedicated to show

$$m_N \circ \Lambda_N^{-1} \Rightarrow m \quad \text{implies} \quad m_N \circ \widetilde{\Lambda}_N^{-1} \Rightarrow m$$

regarding weak measure convergence on H. If so, then the other direction holds automatically. So, we assume $m_N \circ \Lambda_N^{-1} \Rightarrow m$ from here on. First, the tightness of $(m_N \circ \widetilde{\Lambda}_N^{-1})_{N \in \mathbb{N}}$ is shown. For this purpose, our first claim is that for each compact set $K_1 \subset H$ there exists a compact set $K_2 \subset H$ such that

$$\bigcup_{N \in \mathbb{N}} \widetilde{\Lambda}_N \circ \Lambda_N^{-1}(K_1) \subseteq K_2. \tag{2.20}$$

To prove Eq. 2.20, we show that the set on the left-hand side is totally bounded. It follows from Proposition 2.7 that $\widetilde{\Lambda}_N \circ \Lambda_N^{-1}$ is a continuous linear operator from $\text{Im}(\Lambda_N)$ to $\text{Im}(\widetilde{\Lambda}_N)$ for $N \in \mathbb{N}$ with

$$\left| \widetilde{\Lambda}_N \circ \Lambda_N^{-1} h \right|_H \leq 3^{\frac{d}{2}} c^{-\frac{1}{2}} |h|_H, \quad h \in \text{Im}(\Lambda_N), \tag{2.21}$$

where $c := 2 \int_{\mathbb{R}^d} \Xi(z)^2 \, dz - 1 \in (0, \infty)$ is the constant from that proposition. In the following, the open δ-ball $\{h' \in H \mid |h - h'|_H < \delta\}$ with centre $h \in H$ is denoted by $B(h, \delta)$ for $\delta \in (0, \infty)$.

Let $\varepsilon > 0$ be fixed. Since K_1 is totally bounded and $C(\overline{D})$ is dense in H, there exist $M \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_M \in C(\overline{D})$ such that

$$K_1 \subseteq \bigcup_{l=1}^M B(\varphi_l, 3^{-\frac{d}{2}-1} c^{\frac{1}{2}} \varepsilon). \tag{2.22}$$

Due to Eqs. 2.18 and 2.19, there exists $N_\varepsilon \in \mathbb{N}$ and $x_{N,l} \in \mathbb{R}^{k_N}$ for $l = 1, \ldots, M, N > N_\varepsilon$, such that

$$\widetilde{\Lambda}_N x_{N,l} \in B(\varphi_l, \varepsilon/3) \quad \text{and} \quad \Lambda_N x_{N,l} \in B(\varphi_l, 3^{-\frac{d}{2}-1} c^{\frac{1}{2}} \varepsilon), \quad l = 1, \ldots, M, N > N_\varepsilon.$$

For each $N \in \mathbb{N}$ the set $\widetilde{\Lambda}_N \circ \Lambda_N^{-1}(K_1)$ is a compact and therefore there exists $k \in \mathbb{N}$ and $\eta_1, \ldots, \eta_k \in C(\overline{D})$ such that

$$\bigcup_{N=1}^{N_\varepsilon} \widetilde{\Lambda}_N \circ \Lambda_N^{-1}(K_1) \subseteq \bigcup_{l=1}^k B(\eta_l, \varepsilon).$$

Next, we show

$$\bigcup_{N=N_\varepsilon+1}^{\infty} \widetilde{\Lambda}_N \circ \Lambda_N^{-1}(K_1) \subseteq \bigcup_{l=1}^{M} B(\varphi_l, \varepsilon),\tag{2.23}$$

which would prove the total boundedness of $\bigcup_{N\in\mathbb{N}} \widetilde{\Lambda}_N \circ \Lambda_N^{-1}(K_1)$, as the choice of ε is arbitrary. To prove Eq. 2.23, we assume $h \in \widetilde{\Lambda}_N \circ \Lambda_N^{-1}(K_1)$ for some $N > N_\varepsilon$. Using Eq. 2.22, there exists $l \in \{1, \ldots, M\}$ and $y \in \Lambda_N^{-1}(B(\varphi_l, 3^{-\frac{d}{2}-1}c^{\frac{1}{2}}\varepsilon))$ with $\widetilde{\Lambda}_N y = h$. Now, Eq. 2.21 and the choices of $x_{N,l}$ yield

$$\begin{aligned}
|\widetilde{\Lambda}_N y - \varphi_l|_H &= \left|\widetilde{\Lambda}_N \circ \Lambda_N^{-1}(\Lambda_N y - \Lambda_N x_{N,l}) + \widetilde{\Lambda}_N x_{N,l} - \varphi_l\right|_H \\
&\leq 3^{\frac{d}{2}} c^{-\frac{1}{2}} \left|\Lambda_N y - \Lambda_N x_{N,l}\right|_H + \left|\widetilde{\Lambda}_N x_{N,l} - \varphi_l\right|_H \\
&\leq 3^{\frac{d}{2}} c^{-\frac{1}{2}} \left(\left|\Lambda_N y - \varphi_l\right|_H + \left|\varphi_l - \Lambda_N x_{N,l}\right|_H\right) + \left|\widetilde{\Lambda}_N x_{N,l} - \varphi_l\right|_H \\
&\leq 3^{\frac{d}{2}} c^{-\frac{1}{2}} \left(3^{-\frac{d}{2}-1}c^{\frac{1}{2}}\varepsilon + 3^{-\frac{d}{2}-1}c^{\frac{1}{2}}\varepsilon\right) + \varepsilon/3 = \varepsilon.
\end{aligned}$$

This proves Eq. 2.23. The proof for the existence of a compact set $K_2 \subset H$ satisfying Eq. 2.20 for a fixed compact set K_1 is complete.

Now, the tightness of $(m_N \circ \widetilde{\Lambda}_N^{-1})_N$ is an immediate consequence of the tightness of $(m_N \circ \Lambda_N^{-1})_N$. Indeed, for $0 < \delta < 1$ and a compact set $K_\delta \subset H$ such that

$$\inf_{N\in\mathbb{N}} m_N(\Lambda_N^{-1}(K_\delta)) \geq 1 - \delta,$$

we can choose a compact set $K_\delta' \subset H$ with $\Lambda_N^{-1}(K_\delta) \subseteq \widetilde{\Lambda}_N^{-1}(K_\delta')$ for all $N \in \mathbb{N}$ due to Eq. 2.20. In particular,

$$\inf_{N\in\mathbb{N}} m_N \circ \widetilde{\Lambda}_N^{-1}(K_\delta') \geq \inf_{N\in\mathbb{N}} m_N \circ \Lambda_N^{-1}(K_\delta) \geq 1 - \delta.$$

Identifying the accumulation points of $(m_N \circ \widetilde{\Lambda}_N^{-1})_N$ is the only thing left to do in this proof. We make use of [11, Theorem 4.5 of Chapter 3]. It is sufficient to show

$$\lim_{N\to\infty} \int_{\mathbb{R}^{k_N}} F(\widetilde{\Lambda}_N x)\, dm_N(x) = \int_H F(h)\, dm(h)\tag{2.24}$$

for any $F : H \to \mathbb{R}$ of the form $F(h) = g(\langle f_1, h\rangle_H, \ldots, \langle f_q, h\rangle_H)$, where $q \in \mathbb{N}$, $g : \mathbb{R}^m \to \mathbb{R}$ is a bounded, Lipschitz continuous function and $f_1, \ldots, f_q : D \to \mathbb{R}$ are continuous with compact support contained in D. The reason is that the family of all functions $F : H \to \mathbb{R}$ of this form separates the points of H. Let F, g and $f_1, \ldots, f_q$, $q \in \mathbb{N}$, be chosen in this manner. We have

$$\begin{aligned}
\limsup_{N\to\infty} &\left|\int_{\mathbb{R}^{k_N}} F(\widetilde{\Lambda}_N x)\, dm_N(x) - \int_H F(h)\, dm(h)\right| \\
&\leq \limsup_{N\to\infty} \left|\int_{\mathbb{R}^{k_N}} F(\widetilde{\Lambda}_N x)\, dm_N(x) - \int_{\mathbb{R}^{k_N}} F(\Lambda_N x)\, dm_N(x)\right| \\
&\qquad + \limsup_{N\to\infty} \left|\int_{\mathbb{R}^{k_N}} F(\Lambda_N x)\, dm_N(x) - \int_H F(h)\, dm(h)\right|.
\end{aligned}$$

Due to the weak measure convergence $m_N \circ \Lambda_N^{-1} \Rightarrow m$, the second term of the right-hand

side vanishes. Now, we address the first summand. Let $\varepsilon > 0$. There exists a compact set $K_\varepsilon \subset H$ with

$$m_N \circ \Lambda_N^{-1}(K_\varepsilon) \geq 1 - \frac{\varepsilon}{4\sup_{h \in h} |F(h)|}$$

for $N \in \mathbb{N}$. Let L denote the Lipschitz constant of g. We have

$$\left| \int_{\mathbb{R}^{k_N}} F(\widetilde{\Lambda}_N x) \, dm_N(x) - \int_{\mathbb{R}^{k_N}} F(\Lambda_N x) \, dm_N(x) \right|$$
$$\leq \int_{\Lambda_N^{-1}(K_\varepsilon)} \left| F(\widetilde{\Lambda}_N x) - F(\Lambda_N x) \right| dm_N(x) + 2\|F\|_\infty \int_{\mathbb{R}^{k_N} \setminus \Lambda_N^{-1}(K_\varepsilon)} dm_N$$
$$\leq L \int_{\Lambda_N^{-1}(K_\varepsilon)} \left(\sum_{k=1}^{q} \left| \langle f_k, \widetilde{\Lambda}_N x - \Lambda_N x \rangle_H \right|^2 \right)^{1/2} dm_N(x) + \frac{\varepsilon}{2}.$$

So, Eq. 2.24 follows, if there exists $M_\varepsilon \in \mathbb{N}$ such that

$$\left| \langle f_k, \widetilde{\Lambda}_N x - \Lambda_N x \rangle_H \right| \leq \frac{\varepsilon}{2L\sqrt{q}} \quad \text{for } x \in \Lambda_N^{-1}(K_\varepsilon), \ k = 1, \ldots, q \text{ and } N \geq M_\varepsilon. \quad (2.25)$$

We define

$$\omega_N(z) := \max_{k=1,\ldots,q} \sup_{\substack{z' \in \overline{D} \\ |z-z'|_{\text{euc}} \leq \sqrt{d}/N}} |f_k(z') - f_k(z)|, \quad z \in \overline{D}.$$

By virtue of Eq. 2.20 there exists a compact set K'_ε such that

$$\bigcup_{N \in \mathbb{N}} \widetilde{\Lambda}_N \circ \Lambda_N^{-1}(K_\varepsilon) \subseteq K'_\varepsilon. \quad (2.26)$$

Then, we choose $M_\varepsilon^{(1)}, M_\varepsilon^{(2)} \in \mathbb{N}$ such that

$$\omega_{M_\varepsilon^{(1)}}(z) \leq \varepsilon \left(4L \max_{h \in K_\varepsilon \cup K'_\varepsilon} |h|_H \right)^{-1} \left(q \int_D dz \right)^{-\frac{1}{2}}, \quad z \in D \quad (2.27)$$

and

$$\bigcup_{k=1}^{q} \left\{ z \in \mathbb{R}^d \ \middle| \ |z - z'|_{\text{euc}} \leq \sqrt{d}/M_\varepsilon^{(2)} \text{ for some } z' \in \text{supp}[f_k] \right\} \subset D.$$

We set $M_\varepsilon := \max(\{M_\varepsilon^{(1)}, M_\varepsilon^{(2)}\})$. Let $k \in \{1, \ldots, q\}$, $x \in \Lambda_N^{-1}(K_\varepsilon)$ and $N \geq M_\varepsilon$.

$$\left| \langle f_k, \widetilde{\Lambda}_N x - \Lambda_N x \rangle_H \right|$$
$$\leq \left| \langle f_k, \widetilde{\Lambda}_N x \rangle_H - N^{-\frac{d}{2}-1} \sum_{i=1}^{k_N} f_k\left(\frac{p_i}{N}\right) x_i \right| + \left| N^{-\frac{d}{2}-1} \sum_{i=1}^{k_N} f_k\left(\frac{p_i}{N}\right) x_i - \langle f_k, \Lambda_N x \rangle_H \right|. \quad (2.28)$$

Next, since $\int_{\mathbb{R}^d} \widetilde{\xi}_{N,i} \, dz = N^{-d}$ and

$$\text{supp}[\widetilde{\xi}_{N,i}] \subseteq \left\{ z \in \mathbb{R}^d \ \middle| \ \left|z - \frac{p_i}{N}\right|_{\text{euc}} \leq \sqrt{d}/M_\varepsilon \right\} \subset D$$

for $i \in \{1, \ldots, k_N\}$ with $p_i/N \in \mathrm{supp}[f_k]$, it holds

$$\left| \langle f_k, \widetilde{\Lambda}_N x \rangle_H - N^{-\frac{d}{2}-1} \sum_{i=1}^{k_N} f_k\left(\frac{p_i}{N}\right) x_i \right| = \left| \langle f_k, \widetilde{\Lambda}_N x \rangle_H - N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} f_k\left(\frac{p_i}{N}\right) x_i \int_D \widetilde{\xi}_{N,i}(z)\, \mathrm{d}z \right|$$

$$= \left| N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} \int_D \left(f_k(z) - f_k\left(\frac{p_i}{N}\right) \right) x_i \widetilde{\xi}_{N,i}(z)\, \mathrm{d}z \right| \le \int_D \omega_N(z) \left| \widetilde{\Lambda}_N x(z) \right| \mathrm{d}z$$

$$\le \varepsilon \left(4L \max_{h \in K_\varepsilon \cup K'_\varepsilon} |h|_H \right)^{-1} \left(q \int_D \mathrm{d}z \right)^{-\frac{1}{2}} \int_D \left| \widetilde{\Lambda}_N x(z) \right| \mathrm{d}z \le \frac{\varepsilon}{4L\sqrt{q}}.$$

In the second to last inequality, we used Eq. 2.27, while the last inequality follows from Eq. 2.26, $x \in \Lambda_N^{-1}(K_\varepsilon)$ and Hölder's inequality $\int_D \left| \widetilde{\Lambda}_N x(z) \right| \mathrm{d}z \le |\widetilde{\Lambda}_N x(z)|_H |\mathbf{1}(\cdot)|_H$. With the analogous argumentation

$$\left| \langle f_k, \Lambda_N x \rangle_H - N^{-\frac{d}{2}-1} \sum_{i=1}^{k_N} f_k\left(\frac{p_i}{N}\right) x_i \right| \le \left(\int_D w_N(z)^2\, \mathrm{d}z \right)^{\frac{1}{2}} \max_{h \in K_\varepsilon} |h|_H \le \frac{\varepsilon}{4L\sqrt{q}}.$$

Hence Eq. 2.25 holds and thereby Eq. 2.24. This concludes the proof. $\qquad \square$

Example 2.9 (i) We consider the case of Example 2.5. and choose $\Xi = \mathbf{1}_{[-1,0)}$ instead of the one-dimensional tent function and re-define Λ_N according to Eq. 2.14 for $N \in \mathbb{N}$, i.e.

$$\Lambda_N x(z) := \frac{1}{\sqrt{N}} \sum_{i=1}^N x_i \mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N})}(z), \quad x \in \mathbb{R}^N, \ z \in (0,1).$$

Still, regarding weak measure convergence on H, it holds

$$m_N^\beta \circ \Lambda_N^{-1} \Rightarrow m^\beta,$$

with m_N^β, m^β exactly as in Example 2.5, because of Theorem 2.8.

(ii) In Example 2.6, we can pick the alternative choice $\Xi = \mathbf{1}_{[-\frac{1}{2}, \frac{1}{2})^d}$ instead of the two-dimensional tent function and re-define Λ_N according to Eq. 2.14 for $N \in \mathbb{N}$, $N \ge 6$, i.e.

$$\Lambda_N x(z) := \sum_{i=1}^{k_N} x_i \mathbf{1}_{[-\frac{1}{2}, \frac{1}{2})^d}(Nz - I_N(i)), \quad x \in \mathbb{R}^{k_N}, \ z \in (0,1)^2,$$

$$\text{where} \quad G_N =: (2, N-2)^2 \cap \mathbb{Z}^2, \quad k_N := (N-5)^2,$$

for some bijection $I_N : \{1, \ldots, k_N\} \to G_N$. By Theorem 2.8, it holds

$$m_N^\alpha \circ \Lambda_N^{-1} \Rightarrow m^\alpha$$

regarding weak measure convergence on H, where m_N^α, m^α are exactly as in Example 2.6.

3 The Dynamic Model

3.1 Tightness of Equilibrium Laws

We continue to work in the setting of Section 2, with $(H, \langle \cdot, \cdot \rangle) = L^2(D, \mathrm{d}z)$, Λ_N, Ξ, D_N, G_N, k_N and $p_1, \ldots, p_{k_N} \in G_N$ as in the previous section. In this section, a dynamic version of Theorem 2.8 is derived. By default, when referring to $C([0, \infty), Y)$ as a topological space for a metric space Y, we address the topology corresponding to the locally uniform convergence of continuous functions $[0, \infty) \to Y$.

We make use of the theory of Dirichlet forms for symmetric Markov processes. The basic terminology and notions of the standard textbooks [3, 12, 19] are used. From here on, we have the following additional assumptions.

Condition 3.1 (i) Let H_0 be a separable Hilbert space, densely included in H, such that the inclusion $i : H_0 \hookrightarrow H$ has a finite Hilbert Schmidt norm $\|i\|_{\mathrm{HS}}$. Identifying H with its dual H' we can interpret H as a subset of the dual H'_0 via

$$H_0 \subset H = H' \subset H'_0.$$

(ii) For each $N \in \mathbb{N}$ let m_N be a Borel probability measure on $\mathbb{R}^{k_N}$ and $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$ be a symmetric, regular, local and conservative Dirichlet form on $L^2(\mathbb{R}^{k_N}, m_N)$, which admits a Carré-du-Champs operator

$$\Gamma_N : \mathcal{D}(\mathcal{E}^N) \times \mathcal{D}(\mathcal{E}^N) \to L^1(\mathbb{R}^{k_N}, m_N).$$

Moreover, each linear functional $\mathbb{R}^{k_N} \ni x \mapsto a^{\mathrm{T}}x \in \mathbb{R}$, $a \in \mathbb{R}^{k_N}$, is a representative of an element $l_a \in \mathcal{D}(\mathcal{E}^N)$ with

$$\Gamma_N(l_a, l_a)(\cdot) \leq \gamma\, a^{\mathrm{T}}a \quad m_N\text{-a.e.,} \tag{3.1}$$

where the constant $\gamma \in (0, \infty)$ does not depend on N.

For each $N \in \mathbb{N}$ let

$$\mathbf{M}_N = \left(\Omega_N, \mathcal{F}^N, (\mathcal{F}^N_t)_{t \geq 0}, (X^N_t)_{t \geq 0}, (P^N_x)_{x \in \mathbb{R}^{k_N} \cup \{\Delta\}} \right)$$

denote a special standard process with state space $\mathbb{R}^{k_N}$ and life time ζ_N, which is properly associated with $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$. Such a process exists, because $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$ is regular. Then, $\mathbf{M}_N$ is unique up to equivalence. It holds

$$P^N_x(\{\omega \in \Omega_N \mid [0, \zeta_N(\omega)) \ni t \mapsto X^N_t(\omega) \text{ is continuous}\}) = 1 \quad \text{for } x \in \mathbb{R}^{k_N},$$

due to the local property of $\mathcal{E}^N$. The strongly continuous symmetric contraction semigroup on $L^2(\mathbb{R}^{k_N}, m_N)$ corresponding to $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$ is denoted by $(T^N_t)_{t \geq 0}$.

Remark 3.2 Let $N \in \mathbb{N}$. By conservativeness, it holds $T^N_t \mathbf{1}_{\mathbb{R}^{k_N}} = \mathbf{1}_{\mathbb{R}^{k_N}}$ for $t \geq 0$. This means there exists a set $\mathcal{N} \subset \mathbb{R}^{k_N}$ of zero capacity (referring to the 1-capacity associated with $\mathcal{E}^N$) such that $P^N_x(\{\zeta_N = \infty\}) = 1$ for $x \in \mathbb{R}^{k_N} \setminus \mathcal{N}$. W.l.o.g. $\mathbb{R}^{k_N} \setminus \mathcal{N}$ may be assumed to be $\mathbf{M}_N$-invariant, as argued in [19, Chap. IV, Cor. 6.5]. Considering the restriction $\mathbf{M}_N|_{\mathbb{R}^{k_N} \setminus \mathcal{N}}$ (as defined in [19, Chap. IV, Rem. 6.2(i)]) and then applying the procedure described in [19, Chapt.IV, Sect.3, pp. 117f.], re-defining $\mathbf{M}_N$ in such way that each element from $\mathcal{N}$ is a trap, we may assume $P^N_x(\{\zeta_N = \infty\}) = 1$ for all $x \in \mathbb{R}^{k_N}$. Furthermore, after the procedure of weeding (restricting the sample space to a subset of Ω_N as explained in [10, Chap. III, Paragraph 2, pp. 86f.]), we may w.l.o.g. assume that $\mathbf{M}_N$ is non-terminating and continuous, i.e. $\zeta_N(\omega) = \infty$ and $[0, \infty) \ni t \mapsto X^N_t(\omega) \in \mathbb{R}^{k_N}$ is continuous for every $\omega \in \Omega_N$.

As a conclusion of Remark 3.2, the following corollary holds.

Corollary 3.3 *There exists a diffusion process, i.e. a non-terminating, continuous special standard process*

$$\mathbf{M}_N = \left(\Omega_N, \mathcal{F}^N, (\mathcal{F}_t^N)_{t \geq 0}, (X_t^N)_{t \geq 0}, (P_x^N)_{x \in \mathbb{R}^{k_N}} \right)$$

on $\mathbb{R}^{k_N}$, *which is properly associated with* $\mathcal{E}^N$. *So, for every Borel measurable* $u \in L^\infty(\mathbb{R}^{k_N}, m_N)$, *the function*

$$\mathbb{R}^{k_N} \ni x \mapsto \int_{\Omega_N} u \circ X_t^N \, \mathrm{d}P_x^N$$

is a (quasi-continuous) version of $T_t^N u$ *for* $t \geq 0$. *In particular, the measure* m_N *is an invariant measure and* $\mathbf{M}_N$ *is* m_N-*symmetric.*

We analyse the distribution of $(u_t^N)_{t \geq 0} : \Omega_N \to C([0, \infty), H)$ defined by

$$u_t^N := \Lambda_N X_{N^2 t}^N, \quad t \geq 0, \tag{3.2}$$

for $N \in \mathbb{N}$. We are interested in the weak measure convergence of the equilibrium law under $(u_t^N)_{t \geq 0}$ as $N \to \infty$. A suitable path space to tackle this problem is $C([0, \infty), H_0')$. Denoting the i-th coordinate process of $(X_t^N)_{t \geq 0}$ with $(X_t^{N,i})_{t \geq 0}$ for $i = 1, \ldots, k_N$, we have

$$u_t^N = N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} X_{N^2 t}^{N,i} \, \Xi(N \cdot -p_i), \quad t \geq 0. \tag{3.3}$$

Let

$$P_N(A) := \int_{\mathbb{R}^{k_N}} P_x^N(A) \, \mathrm{d}m_N(x), \quad A \in \mathcal{F}^N.$$

The equilibrium law of $(u_t^N)_{t \geq 0}$ on $C([0, \infty), H_0')$ is denoted by $\mathbb{P}_N$, i.e.

$$\mathbb{P}_N(B) := P_N\left(\left\{ \omega \in \Omega_N \mid (u_t^N(\omega))_{t \geq 0} \in B \right\} \right) \tag{3.4}$$

for a Borel measurable set $B \subseteq C([0, \infty), H_0')$.

Proposition 3.4 *If* $(m_N \circ \Lambda_N^{-1})_{N \in \mathbb{N}}$ *is a tight family of probability measures on* H, *then* $(\mathbb{P}_N)_{N \in \mathbb{N}}$ *is tight on* $C([0, \infty), H_0')$.

Proof This proof follows a well-known idea for the derivation of a tightness result in the context of symmetric Markov processes, which has been used in [13, Theorem 6.1], [14, Theorem 5.1] or [7, Lemma 5.2] among others. As argued in each of the three cited references, the desired tightness property can be concluded from the tightness of the invariant measures $(m_N \circ \Lambda_N^{-1})_{N \in \mathbb{N}}$ together with the estimate

$$\sup_{N \in \mathbb{N}} \left(\int_{C([0,\infty), H_0')} |u_t - u_s|_{H_0'}^p \, \mathrm{d}\mathbb{P}_N(u) \right)^{\frac{1}{p}} \leq c(t-s)^{\frac{1}{2}}, \quad 0 \leq s < t < \infty, \tag{3.5}$$

for some constants $c \in (0, \infty)$ and $p \in (1, \infty)$.

The rest of the proof is dedicated to the verification of Eq. 3.5. The strategy is analogous as in [13, Theorem 6.1], [14, Theorem 5.1] or [7, Lemma 5.2]. At first we fix $N \in \mathbb{N}$ and $\varphi \in H_0$.

Let r_T be the time-reversal operator $\Omega_N \rightarrow \Omega_N$ w.r.t. time $0 < T < \infty$. There is a P_N-martingale $(M_t)_{t \geq 0}$ with quadratic variation given by

$$<M>_t = 2 \int_0^t \Gamma_N\big(\langle\varphi, \Lambda_N \cdot\rangle, \langle\varphi, \Lambda_N \cdot\rangle\big)(X_s)\,\mathrm{d}s \quad P_N\text{-a.s.}, \quad t \geq 0, \qquad (3.6)$$

such that

$$\langle\varphi, \Lambda_N X_t\rangle - \langle\varphi, \Lambda_N X_0\rangle = \frac{1}{2}M_t + \frac{1}{2}(M_{T-t} \circ r_T - M_T \circ r_T) \quad P_N\text{-a.s.}, \quad 0 \leq t \leq T. \ (3.7)$$

The formula of 3.7 is called Lyons–Zheng decomposition and stated in [12, Thm. 5.7.1]. The identification of the quadratic variation in Eq. 3.6 follows from [12, Thm. 5.1.3. & 5.2.3]. From Eqs. 3.6, 2.14 and Condition 3.1 (ii) we obtain the estimate

$$<M>_t - <M>_s \ \leq (t-s)2N^{d-2}\gamma \sum_{i=1}^{k_N} \langle\varphi, \xi_{N,i}\rangle^2 \quad P_N\text{-a.s.}, \quad 0 \leq s \leq t. \qquad (3.8)$$

Now, we derive an estimate, first using Eq. 3.7 and the Minkowski inequality, then the Burkholder–Davis–Gundy inequality and Eq. 3.6. It holds

$$\left(\int_{\Omega_N} \langle\varphi, u_t^N(\omega) - u_s^N(\omega)\rangle^4 \,\mathrm{d}P_N(\omega)\right)^{\frac{1}{4}}$$

$$\leq \frac{1}{2}\left(\int_{\Omega_N} (M_{N^2 t} - M_{N^2 s})^4 \,\mathrm{d}P_N\right)^{\frac{1}{4}} + \frac{1}{2}\left(\int_{\Omega_N} (M_{T-N^2 s} - M_{T-N^2 t})^4 \,\mathrm{d}P_N\right)^{\frac{1}{4}}$$

$$\leq \frac{C}{2}\left(\int_{\Omega_N} \big(<M>_{N^2 t} - <M>_{N^2 s}\big)^2 \,\mathrm{d}P_N\right)^{\frac{1}{4}}$$

$$+ \frac{C}{2}\left(\int_{\Omega_N} \big(<M>_{T-N^2 s} - <M>_{T-N^2 t}\big)^2 \,\mathrm{d}P_N\right)^{\frac{1}{4}}, \quad 0 \leq s \leq t \leq T/N^2. \quad (3.9)$$

with a constant $C \in (0, \infty)$ independent of N. We continue the estimate of Eq. 3.9 applying Eq. 3.8, then the Cauchy–Schwarz inequality together the estimates $\langle\xi_{N,i}, \xi_{N,i}\rangle \leq N^{-d}$ for $i = 1, \ldots, k_N$ and $\sum_{i=1}^{k_N} 1_{\mathrm{supp}[\xi_{N,i}]}(\cdot) \leq 3^d$. It holds

$$\left(\int_{\Omega_N} \langle\varphi, u_t^N(\omega) - u_s^N(\omega)\rangle^4 \,\mathrm{d}P_N(\omega)\right)^{\frac{1}{2}}$$

$$\leq C^2(t-s)N^d\gamma \sum_{i=1}^{k_N} \langle\varphi, \xi_{N,i}\rangle^2 \leq (t-s)C^2\gamma \sum_{i=1}^{k_N} \langle 1_{\mathrm{supp}[\xi_{N,i}]}\varphi, \varphi\rangle$$

$$\leq (t-s)(2 \cdot 3^d)C^2\gamma \langle\varphi, \varphi\rangle. \qquad (3.10)$$

Now, we fix an orthonormal basis $\{\varphi_i \mid i \in \mathbb{N}\}$ of H_0. Using the Minkowski inequality and then Eq. 3.10 we obtain

$$\left(\int_{\Omega_N} |u_t^N - u_s^N|_{H_0'}^4 \,\mathrm{d}P_N\right)^{\frac{1}{2}} = \left(\int_{\Omega_N} \left(\sum_{i \in \mathbb{N}} \langle\varphi_i, u_t^N - u_s^N\rangle^2\right)^2 \,\mathrm{d}P_N\right)^{\frac{1}{2}}$$

$$\leq \sum_{i \in \mathbb{N}}\left(\int_{\Omega_N} \langle\varphi_i, u_t^N - u_s^N\rangle^4 \,\mathrm{d}P_N\right)^{\frac{1}{2}} \leq (t-s)(2 \cdot 3^d)C^2\gamma \|i\|_{\mathrm{HS}}^2$$

or analogously

$$\left(\int_{C([0,\infty),H_0')} |u_t - u_s|^4_{H_0'} \, \mathrm{d}\mathbb{P}_N(u) \right)^{\frac{1}{4}} \leq C (2 \cdot 3^d \gamma)^{\frac{1}{2}} \|i\|_{\mathrm{HS}} (t-s)^{\frac{1}{2}}.$$

This concludes the proof. $\qquad\square$

As done in Theorem 2.8 we now compare two different choices, Ξ and $\widetilde{\Xi}$, assuming that each of them meets Condition 2.1. So, for $p \in G_N$ we have

$$\xi_N^p(z) := \Xi(Nz - p), \quad z \in \mathbb{R}^d,$$

on the one hand, and

$$\widetilde{\xi}_N^p(z) := \widetilde{\Xi}(Nz - p), \quad z \in \mathbb{R}^d,$$

on the other. This leads to two different height maps,

$$\Lambda_N x(z) = N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} x_i \xi_{N,i}(z), \quad x \in \mathbb{R}^{k_N}, z \in D,$$

with $\xi_{N,i} := \xi_N^{p_i}$, as well as

$$\widetilde{\Lambda}_N x(z) = N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} x_i \widetilde{\xi}_{N,i}(z), \quad x \in \mathbb{R}^{k_N}, z \in D,$$

with $\widetilde{\xi}_{N,i} := \widetilde{\xi}_N^{p_i}$. We define $\mathbb{P}_N$, respectively $\widetilde{\mathbb{P}}_N$, for these two height maps according to in Eqs. 3.2, 3.3 and 3.4.

Corollary 3.5 *Assume that $(m_N \circ \Lambda_N^{-1})_{N \in \mathbb{N}}$ is a tight family of probability measures on H. Let P be a Borel probability measure on $C([0,\infty), H_0')$. Regarding weak measure convergence on $C([0,\infty), H_0')$ it holds*

$$(\mathbb{P}_N)_{N \in \mathbb{N}} \Rightarrow P \quad \text{if and only if} \quad (\widetilde{\mathbb{P}}_N)_{N \in \mathbb{N}} \Rightarrow P.$$

Proof This corollary is a consequence of Theorem 2.8, Proposition 3.4 and the Markov property. The assumption that $(m_N \circ \Lambda_N^{-1})_{N \in \mathbb{N}}$ is tight on H (and so is $(m_N \circ \widetilde{\Lambda}_N^{-1})_{N \in \mathbb{N}}$ by Theorem 2.8) yields the tightness of $(\mathbb{P}_N)_{N \in \mathbb{N}}$ and of $(\widetilde{\mathbb{P}}_N)_{N \in \mathbb{N}}$ on H_0' due to Proposition 3.4. Therefore, it suffices to show the equivalence of weak convergence w.r.t. the finite-dimensional distributions. The statement is easily obtained as a consequence of the Markov property and multiple applications of Theorem 2.8. We demonstrate the strategy in the case of a two-dimensional distribution. We want to show that for any Borel probability measure μ on $H \times H$,

$$\lim_{N \to \infty} \int_{\mathbb{R}^{k_N}} g(\Lambda_N x) T_t^N (f \circ \Lambda_N)(x) \, \mathrm{d}m_N(x) = \int_{H \times H} g(h_1) f(h_1) \, \mathrm{d}\mu(h_1, h_2)$$

$$\text{for all } f, g \in C_b(H), \quad (3.11)$$

is equivalent to

$$\lim_{N \to \infty} \int_{\mathbb{R}^{k_N}} g(\widetilde{\Lambda}_N x) T_t^N (f \circ \widetilde{\Lambda}_N)(x) \, \mathrm{d}m_N(x) = \int_{H \times H} g(h_1) f(h_1) \, \mathrm{d}\mu(h_1, h_2)$$

$$\text{for all } f, g \in C_b(H). \quad (3.12)$$

We remark that naturally Theorem 2.8 applies regarding the weak measure convergence of finite measures (instead of probabilities), as well. Now, applying Theorem 2.8 onto Eq. 3.11 for a fixed but arbitrary choice of $f \in C_b(H)$, we see that Eq. 3.11 is equivalent to

$$\lim_{N \to \infty} \int_{\mathbb{R}^{k_N}} g(\widetilde{\Lambda}_N x) T_t^N (f \circ \Lambda_N)(x) \, dm_N(x) = \int_{H \times H} g(h_1) f(h_1) \, d\mu(h_1, h_2)$$
$$\text{for all } g, f \in C_b(H). \quad (3.13)$$

Next, using the symmetry of T_t^N for $N \in \mathbb{N}$ and applying Theorem 2.8 onto Eq. 3.13 for a fixed but arbitrary choice of $g \in C_b(H)$, it follows that Eq. 3.13 is equivalent to

$$\lim_{N \to \infty} \int_{\mathbb{R}^{k_N}} T_t^N g(\widetilde{\Lambda}_N x)(f \circ \widetilde{\Lambda}_N)(x) \, dm_N(x) = \int_{H \times H} g(h_1) f(h_1) \, d\mu(h_1, h_2)$$
$$\text{for all } g, f \in C_b(H).$$

Hence, the equivalence of Eqs. 3.11 and 3.12 is proven. This strategy can be generalized to show the equivalence of weak convergence for any finite-dimensional distributions of $(\mathbb{P}_N)_{N \in \mathbb{N}}$ and $(\widetilde{\mathbb{P}}_N)_{N \in \mathbb{N}}$ in a straight forward way by induction. $\qquad \square$

3.2 Gradient Forms and Dynamics

From here on, we make an assumption on the height map Λ_N regarding the property of the set $G_N = N D_N \cap \mathbb{Z}^d$, complementing the condition of Eq. 2.13. We assume that there exists $M \in \mathbb{N}$ such that for $N \geq M$: D_N is a bounded open set, whose closure $\overline{D}_N$ is a strict subset of D with distance from $\mathbb{R}^d \setminus D$ being larger or equal $\sqrt{d}/N$, i.e.

$$|z - z_0|_{\text{euc}} \geq \sqrt{d}/N \quad \text{if} \quad z_0 \in D_N, \ z \in \mathbb{R}^d \setminus D.$$

We note that this amendment on the definition of Λ_N is inline with the height map considered in Example 2.6. Then, for $N \in \mathbb{N}$ large enough, the support of the function $\xi_{N,i} = \Xi(N \cdot - p_i)$ is contained in D for $i = 1, \ldots, k_N$. In particular, for all asymptotic statements, we may assume that

$$\int_D \xi_{N,i}(z) \, dz = N^{-d}, \quad i = 1, \ldots, k_N.$$

Remark 3.6 (i) Let's assume the weak measure convergence of $(m_N \circ \Lambda_N^{-1})_{N \in \mathbb{N}}$ towards a Borel probability measure m on H is given. Then, Mosco convergence of Dirichlet forms in the framework of Kuwae–Shioya provides a method to identify the accumulation points granted by Proposition 3.4 and thus prove the weak convergence of equilibrium laws. This concept is well-known and exploited in many surveys, e.g. [14, 16, 17, 25, 28]. The Mosco convergence of those gradient-type Dirichlet forms, which are relevant to the scaling limit of skew interacting Brownian motions in the context of mesoscopic interface models, is the topic of the subsequent Section 4.

(ii) While the question of Mosco convergence is more involved, the pointwise convergence of gradient forms on the space

$$\mathcal{C} := \left\{ H \ni h \mapsto g(\langle \cdot, f_1 \rangle, \ldots, \langle \cdot, f_m \rangle) \,\middle|\, m \in \mathbb{N}, \ g \in C_b^1(\mathbb{R}^m), \ f_1, \ldots, f_m \in C(\overline{D}) \right\}$$

is easily shown, as the next Lemma 3.7 demonstrates. It is included at this point, because the appearing scaling factor of N^2 reflects the choice of time scale in Eq. 3.2, and also to further motivate the discussion of Section 4.

The elements in $\mathcal{C}$ are Fréchet differentiable with gradient

$$
\nabla g(\langle\,\cdot\,, f_1\rangle, \ldots, \langle\,\cdot\,, f_m\rangle)(h) = \sum_{i=1}^{m} \partial_i\, g(\langle h, f_1\rangle, \ldots, \langle h, f_m\rangle)\, f_i
$$

at a point $h \in H$, where $m \in \mathbb{N}$, $g \in C_b^1(\mathbb{R}^m)$, $f_1, \ldots, f_m \in C(\overline{D})$.

Lemma 3.7 *We assume the weak measure convergence of $(m_N \circ \Lambda_N^{-1})_{N\in\mathbb{N}}$ towards a Borel probability measure m on H.*

For $F, G \in \mathcal{C}$ it holds

$$
\lim_{N\to\infty} N^2 \sum_{i=1}^{k_N} \int_{\mathbb{R}^{k_N}} \frac{\partial(F \circ \Lambda_N)}{\partial x_i} \frac{\partial(G \circ \Lambda_N)}{\partial x_i}\, \mathrm{d}m_N(x) = \int_H \langle \nabla F, \nabla G\rangle\, \mathrm{d}m.
$$

Proof We first show that

$$
\lim_{N\to\infty} \sum_{p\in G_N} N^d \langle\xi_N^p, f\rangle \langle\xi_N^p, g\rangle = \langle f, g\rangle \tag{3.14}
$$

for $f, g \in C(\overline{D})$. The statement is equivalent to $\lim_{N\to\infty} \int_D f_N(z)g(z)\, \mathrm{d}z = \int_D f(z)g(z)\, \mathrm{d}z$ with

$$
f_N(z) := \sum_{p\in G_N} N^d \langle\xi_N^p, f\rangle \xi_N^p(z), \quad z \in D.
$$

It holds $|f_N(z)| \leq \sup_{z'\in\overline{D}} |f(z')|$ for $z \in D$. So, Lebesgue's dominated convergence can be applied. Since the increasing sets $(D_N)_{N\in\mathbb{N}}$ exhaust D up to a set of Lebesgue measure zero (see the assumptions in Eq. 2.13), it suffices to fix an arbitrary choice for $N_0 \in \mathbb{N}$ and prove that f_N converges to f pointwisely on D_{N_0} as $N \to \infty$. Let $N_0 \in \mathbb{N}$ and K be a compact set contained in D_{N_0}. Since $D_{N_0} \cap \mathbb{Z}^d \subset D_N \cap \mathbb{Z}^d = G_N$ for $N \geq N_0$, the support of a function $\Xi(Nz - \mathbf{k})$, $\mathbf{k} \in \mathbb{Z}^d \setminus G_N$, does not intersect with K, if N is chosen large enough. So, for $z_0 \in K$ and N large enough, we have

$$
\sum_{p\in G_N} \xi_N^p(z_0) = \sum_{p\in G_N} \Xi(Nz_0 - p) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \Xi(Nz_0 - \mathbf{k}) = 1.
$$

In particular, for N large enough and $z_0 \in K$, using $\int_D \xi_N^p(z)\, \mathrm{d}z = N^{-d}$ for $p \in G_N$, it holds

$$
\left| \sum_{p\in G_N} N^d \int_D \xi_N^p(z) f(z)\, \mathrm{d}z\, \xi_N^P(z_0) - f(z_0) \right|
$$

$$
= \left| \sum_{p\in G_N} N^d \int_D \xi_N^p(z) f(z)\, \mathrm{d}z\, \xi_N^P(z_0) - f(z_0) N^d \sum_{p\in G_N} \xi_N^P(z_0) \int_D \xi_N^p(z)\, \mathrm{d}z \right|
$$

$$
\leq N^d \sum_{p\in G_N} \int_{\mathbb{R}^d} \xi_N^p(z) |f(z) - f(z_0)|\, \mathrm{d}z\, \xi_N^P(z_0) \leq \max_{\substack{z\in\overline{D} \\ |z_0-z|_{\mathrm{euc}} \leq \frac{2\sqrt{d}}{N}}} |f(z) - f(z_0)|.
$$

This proves Eq. 3.14.

From Eq. 3.14, in turn, it follows that

$$
\lim_{N\to\infty} N^d \sum_{p\in G_N} \int_{\mathbb{R}^{k_N}} \langle\xi_N^p, V(\Lambda_N x)\rangle \langle\xi_N^p, g\rangle\, \mathrm{d}m_N(x) = \int_H \langle V(h), g\rangle\, \mathrm{d}m(h) \tag{3.15}
$$

if

$$V : H \ni h \; \mapsto \; G_1(h) f_1 + \cdots + G_m(h) f_m \in H, \tag{3.16}$$

for some $G_1, \ldots, G_m \in C_b(H)$ and $f_1, \ldots, f_m \in C_b(\overline{D})$. By linearity, Eq. 3.15 generalizes to

$$\lim_{N \to \infty} N^d \sum_{p \in G_N} \int_{\mathbb{R}^{k_N}} \langle \xi_N^p, V(\Lambda_N x) \rangle \langle \xi_N^p, W(\Lambda_N x) \rangle \, \mathrm{d}m_N(x) = \int_H \langle V, W \rangle \, \mathrm{d}m \tag{3.17}$$

if $W(h)$ is of the same type as $V(h)$, specified in Eq. 3.16. This proves the claim of the lemma, because

$$\frac{\partial (F \circ \Lambda_N)}{\partial x_i}(x) = N^{\frac{d}{2}-1} \langle \xi_{N,i}, \nabla F(\Lambda_N x) \rangle_H, \quad x \in \mathbb{R}^N, \quad i = 1, \ldots, k_N,$$

and ∇F is a vector field of the type specified in Eq. 3.16 for $F \in \mathcal{C}$. $\qquad\square$

Remark 3.8 (i) Let μ be a non-degenerate, centered Gaussian measure on H with covariance operator Q and let $(A, \mathcal{D}(A)) := (2Q^{-1}, \mathrm{Im}(Q))$ be defined as a positive, self-adjoint, operator on H. We consider a reference measure $m = \rho\mu$ with a density $\rho : H \to [c_1, c_2]$, where $0 < c_1 < c_2 < \infty$ and $\int_H \rho \, \mathrm{d}\mu = 1$. In this case,

$$\mathcal{E}(F, G) := \frac{1}{2} \int_H \langle \nabla F, \nabla G \rangle \rho \, \mathrm{d}\mu, \quad F, G \in \mathcal{C},$$

can be extended (in the sense of a minimal closed extension) to a local, quasi-regular and conservative Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(H, \rho\mu)$, for which $\mathcal{C}$ is a form core (see [1, 21–23]). The domain $\mathcal{D}(\mathcal{E})$ coincides with the Gaussian $(1, 2)$-Sobolev space on H w.r.t. μ. We assume that

$$\rho(h) = \exp\left(-\int_D g(h(z)) \, \mathrm{d}z \right), \quad h \in H, \tag{3.18}$$

for some bounded function $g : \mathbb{R} \to \mathbb{R}$ with bounded total variation. There exists a densely and continuously included Hilbert space $H_1 \subset H$, such that ρ is a function of bounded variation in the Gelfand triple

$$H_1 \subset H = H' \subset H_1'.$$

Indeed, this is true if H_1 can be continuously embedded into the space of continuous functions on $\overline{D}$. So, we may for example choose H_1 as the Sobolev space of square-integrable functions on D with an order which is strictly greater than $d/2$.

Let $(\nabla, \mathcal{D}(\nabla))$ be the gradient operator $L^2(H, \mu) \to L^2(H, \mu; H)$, whose domain is the Gaussian $(1, 2)$-Sobolev space w.r.t. μ. We denote the adjoint operator, a closed operator $L^2(H, \mu; H) \to L^2(H, \mu)$, by $(\nabla^*, \mathcal{D}(\nabla^*))$. Then,

$$\left\{ H \ni h \mapsto \sum_{l=1}^n f_l(h) \varphi_l \in H \;\middle|\; n \in \mathbb{N}, \; f_l \in C_b^1(H), \; \varphi_l \in \mathcal{D}(A) \cap H_1 \right\}$$

$$=: (C_b^1)_{\mathcal{D}(A) \cap H_1} \subset \mathcal{D}(\nabla^*).$$

Due to [23, Thm. 3.1], there exists a finite positive measure $\| \, \mathrm{d}\rho \|$ on H and a Borel measurable map $\sigma_\rho : H \to H_1'$ such that $|\sigma_\rho(h)|_{H_1'} = 1$ for $\| \, \mathrm{d}\rho \|$-a.e. $h \in H$ and

$$\int_H \nabla^* G(h) \rho(h) \, \mathrm{d}\mu(h) = \int_H {}_{H_1} \langle G(h), \sigma_\rho(h) \rangle_{H_1'} \| \, \mathrm{d}\rho \|(h), \quad G \in (C_b^1)_{\mathcal{D}(A) \cap H_1}.$$

(ii) Let
$$\mathbf{M} = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P_h)_{h \in H} \right)$$

be the diffusion process on H, properly associated with $\mathcal{E}$. By virtue of [23, Thm. 3.2] for quasi almost every $h \in H$ we have

$$\langle \varphi_l, X_t - X_0 \rangle = \int_0^t \langle \varphi, dW_s^h \rangle + \frac{1}{2} \int_0^t {}_{H_1}\langle \varphi, \sigma_\rho(X_s) \rangle_{H_1'} \, dL_s^{\| d\rho \|}$$
$$- \int_0^t \langle A\varphi, X_s \rangle \, ds, \quad t \geq 0, \quad P_h\text{-a.s.} \quad (3.19)$$

for $\varphi \in \mathcal{D}(A) \cap H_1$, where $(W_t^h)_{t \geq 0}$ is an $(\mathcal{F}_t)_t$-cylindrical Wiener process and $(L_t^{\| d\rho \|})_{t \geq 0}$ is the positive continuous additive functional of the process $(X_t)_{t \geq 0}$ in Revuz correspondence with $\| d\rho \|$.

Example 3.9 In this example, we fix $N \in \mathbb{N}$, a symmetric, positive, linear operator $A_N : \mathbb{R}^{k_N} \to \mathbb{R}^{k_N}$, a function $g_{N,0} \in C_b^1(\mathbb{R})$ and real numbers $y_j, \beta_j \in \mathbb{R}$ for $j = 1, \ldots, M$.

(i) With
$$g_N(y) := g_{N,0}(y) + \sum_{j=1}^M \beta_j \mathbf{1}_{(-\infty, y_j]}(y), \quad y \in \mathbb{R},$$

we define the Borel probability measure

$$dm_{A_N}(x) := \frac{1}{Z_N} \exp\left(- x^{\mathsf{T}} A_N x - \frac{1}{N^d} \sum_{i=1}^{k_N} g_N\left(N^{\frac{d}{2}-1} x_i \right) \right) dx,$$

$$Z_N := \int_{\mathbb{R}^{k_N}} \exp\left(- x^{\mathsf{T}} A_N x - \frac{1}{N^d} \sum_{i=1}^{k_N} g_N\left(N^{\frac{d}{2}-1} x_i \right) \right) dx,$$

on $\mathbb{R}^{k_N}$. The factor

$$\exp\left(- \frac{1}{N^d} \sum_{i=1}^{k_N} g_N\left(N^{\frac{d}{2}-1} x_i \right) \right), \quad x \in \mathbb{R}^{k_N},$$

should be interpreted as a discrete version of the perturbing density considered in Eq. 3.18. In the situation of Example 2.6, i.e. $d = 2$ with $N \geq 6$, $D = (0, 1)^2$, $G_N = (2, N-2)^2$ and $k_N := |G_N|$, we might choose A_N such that

$$x^{\mathsf{T}} A_N x = 4 H_N^\alpha(x), \quad x \in \mathbb{R}^{k_N}, \quad (3.20)$$

with the Hamiltonian function H_N^α defined in that example. The scaling sequence $\alpha : \mathbb{N} \to (0, \infty)$ determines the limiting distribution of the height map Λ_N for $N \to \infty$, if there is no perturbing density ($g_N \equiv 0$). Now, we state the Fukushima decomposition for the dynamic height variables from the microscopic perspective, if a perturbing density of the type above is present.

(ii) As g_N is a bounded function, we define the domain of the gradient-type Dirichlet form $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$ on $L^2(\mathbb{R}^{k_N}, m_{A_N})$ as the set of all elements in the $(1, 2)$-Sobolev space

of the Gaussian measure with density proportional to $x \mapsto \exp(-x^{\mathrm{T}} A_N x)$ on $\mathbb{R}^{k_N}$. So, let

$$\mathcal{E}^N(u, v) := \frac{1}{2} \sum_{i=1}^{k_N} \int_{\mathbb{R}^{k_N}} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dm_{A_N}, \quad u, v \in \mathcal{D}(\mathcal{E}^N).$$

The domain $\mathcal{D}(\mathcal{E}^N)$ comprises all weakly differentiable elements of $L^2(\mathbb{R}^{k_N}, m_{A_N})$ whose derivatives in any direction are again square-integrable w.r.t. m_{A_N}. The form $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$ obviously meets Condition 3.1 (ii). Let

$$\mathbf{M}_N = \left(\Omega_N, \mathcal{F}^N, (\mathcal{F}_t^N)_{t \geq 0}, (X_t^N)_{t \geq 0}, (P_x^N)_{x \in \mathbb{R}^{k_N}} \right)$$

be the associated Markov process as in the beginning of this section. Again, $(X_t^{N,i})_{t \geq 0}$ for $i = 1, \ldots, k_N$ denotes the i-th coordinate process. By Lemma A1 of the Appendix it holds

$$dX_t^{N,i} = -\left(A_N X_t^N\right)_i dt - \frac{1}{2} N^{-\frac{d}{2}-1} g'_{N,0}\left(N^{\frac{d}{2}-1} X_t^{N,i}\right) dt$$
$$+ \sum_{j=1}^{M} \frac{1 - e^{-\beta_j/N^d}}{1 + e^{-\beta_j/N^d}} \, dl_t^{N,i,\tilde{y}_j} + dB_t^{N,i}, \quad t \geq 0, \quad (3.21)$$

P_x^N-a.s. for each $x \in \mathbb{R}^{k_N}$, where $(B_t^{N,i})_{t \geq 0}$, $i = 1, \ldots, k_N$, are independent Brownian motions and $l_t^{N,i,\tilde{y}_j}$ is the local time of $(X_t^{N,i})_{t \geq 0}$ at $\tilde{y}_j := N^{1-\frac{d}{2}} y_j$. The linear term of the drift in Eq. 3.21 for the situation of Example 2.6 with Eq. 3.20 equals

$$-\left(A_N X_t^N\right)_i = -2 \, \partial_i H_N^\alpha(X_t^N).$$

We continue in the setting of Example 3.9 and explain the main motivation for the upcoming Section 4 on Mosco convergence. The following Remark 3.10 shows how the results of Section 4 (in particular, see Theorem 4.10 and Corollary 4.11) apply to this case. Let $(T_t^N)_{t \geq 0}$ denote the semigroup on $L^2(\mathbb{R}^{k_N}, m_{A_N})$ corresponding to $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$. We are interested in the weak measure convergence of the equilibrium laws $(\mathbb{P}_N)_{N \in \mathbb{N}}$ for the scaled, interpolated process

$$u_t^N := \Lambda_N X_{N^2 t}^N, \quad t \geq 0,$$

(as in Eqs. 3.4, 3.3 and 3.2). The finite-dimensional distributions of $\mathbb{P}_N$ are given in terms of the semigroup $(T_t^N)_{t \geq 0}$ for each $N \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n$ be bounded measurable functions $H \to \mathbb{R}$. Setting $\tilde{f}_l = f_l \circ \Lambda_N$ for $l = 1, \ldots, n$ the Markov property of the process $\mathbf{M}_N$ and the definition of $\mathbb{P}_N$ imply

$$\int_{C([0,\infty),H)} f_1(u_{t_1}) \cdot f_2(u_{t_1+t_2}) \cdot \cdots \cdot f_n(u_{t_1+\cdots+t_N}) \, d\mathbb{P}_N(u)$$
$$= \int_{\mathbb{R}^{k_N}} T_{N^2 t_1}^N \left(\tilde{f}_1 \cdot T_{N^2 t_2}^N \left(\cdots T_{N^2 t_{k-1}}^N \left(\tilde{f}_{k-1} \cdot T_{N^2 t_k}^N \tilde{f}_k \right) \cdots \right) \right) dm_{A_N}. \quad (3.22)$$

Further, let ρ, μ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be as in Remark 3.8. The expected limit of $(\mathbb{P}_N)_{N \in \mathbb{N}}$ is the equilibrium law associated to $\mathcal{E}$, if $(g_N)_N$ is chosen in a suitable way. Arguing as in Remark 3.6 (i) we observe the following.

Remark 3.10 By virtue of Proposition 3.4 and Eq. 3.22, it suffices prove:

- $m_{A_N} \circ \Lambda_N^{-1} \Rightarrow \rho\mu$ weakly on H.
- The image form of $(N^2 \mathcal{E}^N, \mathcal{D}(\mathcal{E}))$ under the height map Λ_N converges to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in the sense of Mosco–Kuwae–Shioya.

Thanks to Theorem 2.8 and Corollary 3.5, if these two statements are true for a particular choice for $\Xi : \mathbb{R}^d \to [0, 1]$ in the definition of Eq. 2.2 for Λ_N, then they automatically hold for any choice according to Condition 2.1. This is very useful, because for $\Xi := \mathbf{1}_{[-1,0)^d}$ the functions $\xi_{N,i} = \Xi(N \cdot -p_i)$, $i = 1, \ldots, k_N$, which are the images of the unit-vectors under $N^{1-\frac{d}{2}}\Lambda_N$, are again orthogonal to each other. Moreover,

$$\widehat{\xi}_{N,i} := N^{\frac{d}{2}} \xi_{N,i}, \quad i = 1, \ldots, k_N,$$

is normed in H, which means that $N\Lambda_N$ is an isometry from $\mathbb{R}^{k_N}$ into H. Hence, in case $\Xi := \mathbf{1}_{[-1,0)^d}$, the image form of $(N^2 \mathcal{E}^N, \mathcal{D}(\mathcal{E}))$ under the height map Λ_N simply equals

$$N^2 \mathcal{E}^N (u \circ \Lambda_N, v \circ \Lambda_N) = \mathcal{E}^N (u \circ N\Lambda_N, v \circ N\Lambda_N)$$

$$= \frac{1}{2} \sum_{i=1}^{k_N} \int_H \frac{\partial u}{\partial \widehat{\xi}_{N,i}} \frac{\partial v}{\partial \widehat{\xi}_{N,i}} \, \mathrm{d}(m_{A_N} \circ \Lambda_N)^{-1}, \quad u, v \in \mathcal{C},$$

where $\widehat{\xi}_{N,1}, \ldots, \widehat{\xi}_{N,k_N}$ is an orthonormal basis for the k_N-dimensional subspace $(\Lambda_N(\mathbb{R}^{k_N}), \langle \cdot, \cdot \rangle)$ of H. This reduces the task of the final Section 4 to the analysis of Mosco–Kuwae–Shioya convergence of gradient forms in the framework of varying L^2-spaces over the state space H. Moreover, we hint at the fact, that for $\Xi := \mathbf{1}_{[-1,0)^d}$ we have $\int_D \xi_{N,i} \, \mathrm{d}z = N^{-d}$ for $i = 1, \ldots, k_N$ and therefore

$$\exp \Big(\int_D g_N \circ (\Lambda_N x) \, \mathrm{d}z \Big) = \exp \Big(\frac{1}{N^d} \sum_{i=1}^{k_N} g_N (N^{\frac{d}{2}-1} x_i) \Big), \quad x \in \mathbb{R}^{k_N}.$$

This means,

$$\mathrm{d}(m_{A_N} \circ \Lambda_N)^{-1}(h) = \exp \Big(\int_D g_N \circ h \, \mathrm{d}z \Big) (\mathrm{d}\mu_{A_N} \circ \Lambda_N^{-1})(h),$$

where μ_{A_N} is the Gaussian measure on $\mathbb{R}^{k_N}$ with covariance operator $\frac{1}{2} A_N^{-1}$.

4 An Application of Mosco Convergence

4.1 Preliminaries and Notation

As in previous sections, $(H, \langle \cdot, \cdot \rangle) := L^2(D, \mathrm{d}z)$. Let $(\mu_N)_{N \in \mathbb{N}}$ be a sequence of Borel probabilities on H and μ be their limit in the sense of weak measure convergence on H. We assume that all these elements are centered Gaussian and on top of that, the limit μ has full topological support on H. Their covariances are denoted by

$$\mathrm{Cov}_\mu(h_1, h_2) := \int_H \langle h_1, k \rangle \langle h_2, k \rangle \, \mathrm{d}\mu(k), \quad h_1, h_2 \in H,$$

and

$$\mathrm{Cov}_{\mu_N}(h_1, h_2) := \int_H \langle h_1, k \rangle \langle h_2, k \rangle \, \mathrm{d}\mu_N(k), \quad h_1, h_2 \in H,$$

for $N \in \mathbb{N}$.

Throughout Section 4.1 we fix a sequence $(\varphi_N)_{N \in \mathbb{N}}$ and an element φ in H, such that:

- $\langle \varphi, \varphi \rangle = 1$ and there exists $\lambda \in (0, \infty)$ with $\lambda \operatorname{Cov}_\mu(\varphi, h) = \langle \varphi, h \rangle$ for $h \in H$.
- $\langle \varphi_N, \varphi_N \rangle = 1$ and there exists $\lambda_N \in (0, \infty)$ with $\lambda_N \operatorname{Cov}_{\mu_N}(\varphi_N, h) = \langle \varphi_N, h \rangle$ for $h \in H$.
- $|\varphi - \varphi_N|_H \to 0$ as well as $\lambda_N \to \lambda$ for $N \to \infty$.

Now we cite some preliminaries on the disintegration of these Gaussian measures. Under the above conditions μ is φ-quasi-invariant, i.e. the image measure of μ under the shift $H \ni h \mapsto h + r\varphi \in H$ is absolutely continuous w.r.t. μ for every $r \in \mathbb{R}$ (see [1, Prop. 5.5]). By [1, Prop. 4.2], μ and the measure

$$
\sigma_\varphi(B) := \int_{\mathbb{R}} \mu(B - r\varphi) \, dr, \quad B \in \mathcal{B}(H),
$$

are absolutely continuous w.r.t. each other and setting $\pi_\varphi h := h - \langle h, \varphi \rangle \varphi$ for $h \in H$ we have

$$
\int_H u \, d\mu = \int_{\pi_\varphi(H)} \int_{\mathbb{R}} u(h + r\varphi) \frac{d\mu}{d\sigma_\varphi}(h + r\varphi) \, dr \, d(\mu \circ \pi_\varphi^{-1})(h) \tag{4.1}
$$

for any measurable function $u : H \to [0, \infty)$. [1, Prop. 5.5] provides $\frac{d\mu}{d\sigma_\varphi}$ in an explicit form, which in our case yields

$$
\frac{d\mu}{d\sigma_\varphi}(h) = \sqrt{\frac{\lambda}{2\pi}} \exp\left(-\frac{\lambda \langle \varphi, h \rangle^2}{2} \right), \quad h \in H. \tag{4.2}
$$

From Eqs. 4.1 and 4.2 we obtain

$$
\int_H u \, d\mu = \sqrt{\frac{\lambda}{2\pi}} \int_{\pi_\varphi(H)} \int_{\mathbb{R}} u(h + r\varphi) \exp(-\tfrac{\lambda r^2}{2}) \, dr \, d(\mu \circ \pi_\varphi^{-1})(h) \tag{4.3}
$$

for any measurable function $u : H \to [0, \infty)$.

Let $N \in \mathbb{N}$. We now define a rotation $J_N : H \to H$ which maps φ_N onto φ and then disintegrate the image of μ_N under J_N in the same way as we did with μ. If $\varphi_N = \varphi$, we set $J_N := \mathrm{id}$. Otherwise, let $J_N : H \to H$ be the isometric isomorphism which leaves the orthogonal complement of $\operatorname{span}(\{\varphi, \varphi_N\})$ invariant, while on $\operatorname{span}(\{\varphi, \varphi_N\})$ its action is defined via the basis transformation

$$
\begin{array}{ccc}
\varphi_N & \longmapsto & \varphi \\[2mm]
\dfrac{\varphi - \langle \varphi_N, \varphi \rangle \varphi_N}{\sqrt{1 - \langle \varphi, \varphi_N \rangle^2}} & \longmapsto & \dfrac{\langle \varphi, \varphi_N \rangle \varphi - \varphi_N}{\sqrt{1 - \langle \varphi, \varphi_N \rangle^2}}.
\end{array}
$$

With $\tilde{\mu}_N := \mu_N \circ J_N^{-1}$ it holds

$$
\lambda_N \int_H \langle \varphi, k \rangle \langle h, k \rangle \, d\tilde{\mu}_N(k) = \lambda_N \operatorname{Cov}_{\mu_N}(J_N^{-1}\varphi, J_N^{-1}h) = \langle \varphi_N, J_N^{-1}h \rangle = \langle \varphi, h \rangle
$$

for $h \in H$. So, in analogy to the disintegration for μ, an application of [1, Prop. 4.2 & Prop. 5.5] yields that $\tilde{\mu}_N$ is φ-quasi-invariant and

$$
\int_H u \, d\tilde{\mu}_N = \sqrt{\frac{\lambda_N}{2\pi}} \int_{\pi_\varphi(H)} \int_{\mathbb{R}} u(h + r\varphi) \exp(-\tfrac{\lambda_N r^2}{2}) \, dr \, d(\tilde{\mu} \circ \pi_\varphi^{-1})(h) \tag{4.4}
$$

for any measurable function $u : H \to [0, \infty)$.

Remark 4.1 For $N \in \mathbb{N}$ we set

$$v_N := \frac{\varphi - \langle \varphi_N, \varphi \rangle \varphi_N}{\sqrt{1 - \langle \varphi, \varphi_N \rangle^2}} \quad \text{and} \quad w_N := \frac{\langle \varphi, \varphi_N \rangle \varphi - \varphi_N}{\sqrt{1 - \langle \varphi, \varphi_N \rangle^2}}.$$

For $u \in H$ with $|u|_H = 1$ it holds

$$
\begin{aligned}
|J_N u - u|_H^2 &= \left(\langle \varphi, J_N u \rangle - \langle \varphi, u \rangle \right)^2 + \left(\langle w_N, J_N u \rangle - \langle w_N, u \rangle \right)^2 \leq |\varphi_N - \varphi|_H^2 + |v_N - w_N|_H^2 \\
&= |\varphi - \varphi_N|_H^2 + \frac{|\varphi(1 - \langle \varphi_N, \varphi \rangle) + \varphi_N(1 - \langle \varphi_N, \varphi \rangle)|_H^2}{1 - \langle \varphi, \varphi_N \rangle^2} \\
&= |\varphi - \varphi_N|_H^2 + \frac{2(1 + \langle \varphi, \varphi_N \rangle)(1 - \langle \varphi, \varphi_N \rangle)^2}{1 - \langle \varphi, \varphi_N \rangle^2} \\
&= |\varphi - \varphi_N|_H^2 + 2(1 - \langle \varphi_N, \varphi \rangle).
\end{aligned}
$$

The right-hand side of this inequality converges to zero for $N \to \infty$ and is independent from u. Hence, the identity is the limit of $(J_N)_{N \in \mathbb{N}}$ w.r.t. the operator norm on H.

We want to tackle the question of Mosco convergence under the existence of perturbing densities for the Gaussian reference measure μ_N, $N \in \mathbb{N}$, and μ. For a bounded function $f : \mathbb{R} \to \mathbb{R}$ the total variation is defined as

$$\mathrm{TV}(f) := \sup_{\substack{m \in \mathbb{N} \\ x_1 \leq \cdots \leq x_m}} \sum_{i=1}^{m-1} |f(x_{i+1}) - f(x_i)| \quad \in [0, \infty].$$

For a function $f : \mathbb{R} \to \mathbb{R}$ with $\mathrm{TV}(f) < \infty$ the Jordan decomposition, as derived in [24, Chapter 5.2], states the existence of $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ such that

$$
\begin{aligned}
f = f_1 - f_2, \quad \|f_1\|_\infty + \|f_2\|_\infty \leq \|f\|_\infty + \|f\|_{\mathrm{TV}}, \\
f_1, f_2 \text{ are monotone increasing.} \quad (4.5)
\end{aligned}
$$

Until the end of Section 4 we fix bounded functions g_N for $N \in \mathbb{N}$ and g from $\mathbb{R}$ to $\mathbb{R}$ with finite total variation. Then we define

$$Z := \int_H \exp\left(\int_D g(h(z)) \, \mathrm{d}z \right) \mathrm{d}\mu, \quad \varrho(h) := \exp\left(\int_D g(h(z)) \, \mathrm{d}z \right)/Z, \quad h \in H,$$

and

$$Z_N := \int_H \exp\left(\int_D g_N(h(z)) \, \mathrm{d}z \right) \mathrm{d}\mu_N, \quad \varrho_N(h) := \exp\left(\int_D g_N(h(z)) \, \mathrm{d}z \right)/Z_N, \quad h \in H.$$

We need the following condition to be satisfied.

Condition 4.2 We assume that

$$g = g^{(1)} - g^{(2)}, \quad g_N = g_N^{(1)} - g_N^{(2)},$$

are the Jordan decompositions for g, respectively for g_N for each $N \in \mathbb{N}$, in the sense of Eq. 4.5, and that the following conditions hold:

(i) $\sup_N \|g_N\|_\infty + \|g_N\|_{\mathrm{TV}} < \infty$.

(ii) $\sup_{x \in \mathbb{R}} \inf_{|r| \leq \delta} |g^{(i)}(x) - g_N^{(i)}(x+r)| \xrightarrow{N \to \infty} 0$ for every $\delta \in (0, \infty)$ and $i = 1, 2$.

Let

$$\mathcal{F}C_b^\infty := \left\{ H \ni h \mapsto g(\langle \cdot, f_1 \rangle, \ldots, \langle \cdot, f_m \rangle) \,\Big|\, m \in \mathbb{N}, \, g \in C_b^\infty(\mathbb{R}^m), \, f_1, \ldots, f_m \in H \right\}.$$

As stated in [1, Prop. 5.5], φ_N is admissible for μ_N in the sense of [22, Def. 1.1]. Hence, the partial derivative $u \mapsto \partial u / \partial \varphi_N$ with domain $\{u \in L^2(H, \varrho_N \mu_N) \mid u$ has a representative from $\mathcal{F}C_b^\infty(H)\}$ is a well-defined, closable linear operator on $L^2(H, \varrho_N \mu_N)$. We are interested in the asymptotic behaviour for $N \to \infty$ of the symmetric Dirichlet form $(\mathcal{E}_{\varphi_N}^N, \mathcal{D}(\mathcal{E}_{\varphi_N}^N))$ on $L^2(H, \varrho_N \mu_N)$ which is the closure of the pre-Dirichlet form

$$\mathcal{E}_{\varphi_N}^N(u, v) = \int_H \frac{\partial u}{\partial \varphi_N} \frac{\partial v}{\partial \varphi_N} \varrho_N \, d\mu_N$$

with domain $\{u \in L^2(H, \varrho_N \mu_N) \mid u$ has a representative from $\mathcal{F}C_b^\infty(H)\}$.

It is asymptotically irrelevant if we consider the image form of $(\mathcal{E}_{\varphi_N}^N, \mathcal{D}(\mathcal{E}_{\varphi_N}^N))$ under J_N instead (see Lemma 4.5 (ii) and Proposition 4.6 below). This trick allows us to disintegrate the involved reference measures, which vary with N, along the same direction φ over the base $\pi_\varphi(H)$. Using a disintegration of the measures analogue to Eq. 4.4, where instead of $\tilde{\mu}_N = \mu_N \circ J_N^{-1}$ we consider the disturbed image measures

$$(\varrho_N \mu_N) \circ J_N^{-1} = \underbrace{(\varrho_N \circ J_N^{-1})}_{=: \tilde{\varrho}_N} \tilde{\mu}_N,$$

we can apply a result form [15]. The necessary preliminaries are summarized in the next remark. Let

$$\mathcal{N}_N(dr) := \sqrt{\frac{\lambda_N}{2\pi}} \exp(-\frac{\lambda_N r^2}{2}) \, dr$$

be the normal distribution on $\mathbb{R}$ with mean zero and variance $1/\lambda_N$. Further, let

$$\nu_N := (\tilde{\varrho} \tilde{\mu}_N) \circ \pi_\varphi^{-1}.$$

The Radon–Nikodym density $d\nu_N / (\tilde{\mu}_N \circ \pi_\varphi^{-1})$ coincides for $(\tilde{\mu}_N \circ \pi_\varphi^{-1})$ -a.e. $h \in \pi_\varphi(H)$ with $\int_\mathbb{R} \tilde{\varrho}_N(h + t\varphi) \, d\mathcal{N}_N(t)$, as follows from Eq. 4.4. We set

$$\rho_N(h, r) := \frac{\tilde{\varrho}_N(h + r\varphi)}{\int_\mathbb{R} \tilde{\varrho}_N(h + t\varphi) \, d\mathcal{N}_N(t)}$$

for $h \in \pi_\varphi(H)$ and $r \in \mathbb{R}$.

Remark 4.3 (i) Using Eq. 4.4 we obtain

$$\int_H u \tilde{\varrho}_N \, d\tilde{\mu}_N = \int_{\pi_\varphi(H)} \int_\mathbb{R} u(h + r\varphi) \tilde{\varrho}_N(h + r\varphi) \, d\mathcal{N}_N(r) \, d(\tilde{\mu} \circ \pi_\varphi^{-1})(h)$$

$$= \int_{\pi_\varphi(H)} \int_\mathbb{R} u(h + r\varphi) \rho_N(h, r) \, d\mathcal{N}_N(r) \, d\nu_N(h).$$

Hence,

$$I_N : L^2(H, \tilde{\varrho}_N \tilde{\mu}_N) \xrightarrow{\simeq} L^2\big(\pi_\varphi(H) \times \mathbb{R}, \rho_N(\nu_N \times \mathcal{N}_N)\big)$$

$$I_N u(h, r) := u(h + r\varphi) \quad (\nu_N \times \mathcal{N}_N) \text{ -a.e.}$$

is an isometric isomorphism. In analogy to [22, Propo. 1.2] we define

$$\mathcal{D}(\tilde{\mathcal{E}}_\varphi^N) := \left\{ u \in L^2(H, \tilde{\varrho}_N \tilde{\mu}_N) \,\middle|\, \text{for } \nu_N\text{-a.e. } h \in \pi_\varphi(H) \right.$$

$$I_N u(h, \cdot) \text{ has an absolutely continuous version } f_h$$

$$\left. \text{and } \int_{\pi_\varphi(H)} \int_{\mathbb{R}} |f_h'(r)|^2 \rho_N(h, r) \, \mathrm{d}\mathcal{N}_N(r) \, \mathrm{d}\nu_N(h) < \infty \right\}$$

and in this sense

$$\tilde{\mathcal{E}}_\varphi^N(u, v) := \int_{\pi_\varphi(H)} \int_{\mathbb{R}} u(h + \cdot \varphi)'(r) \, v'(h + \cdot \varphi)'(r) \rho_N(h, r) \, \mathrm{d}\mathcal{N}_N(r) \, \mathrm{d}\nu_N(h)$$

for $u, v \in \mathcal{D}(\tilde{\mathcal{E}}_\varphi^N)$. By [22, Thm. 5.1], the space $\{u \in L^2(H, \tilde{\varrho}_N \tilde{\mu}_N) \mid u \text{ has a representative from } \mathcal{F}C_b^\infty(H)\}$ is a form core for $(\tilde{\mathcal{E}}_\varphi^N, \mathcal{D}(\tilde{\mathcal{E}}_\varphi^N))$.

(ii) Analogously, we obtain via Eq. 4.3 the according disintegration formula for the disturbed measure $\varrho\mu$. Let

$$\mathcal{N}(\mathrm{d}r) := \sqrt{\frac{\lambda}{2\pi}} \exp(-\frac{\lambda r^2}{2}) \, \mathrm{d}r$$

be the normal distribution on $\mathbb{R}$ with mean zero and variance $1/\lambda$. We set $\nu := (\varrho\mu) \circ \pi_\varphi^{-1}$ and

$$\rho(h, r) := \frac{\varrho(h + r\varphi)}{\int_{\mathbb{R}} \varrho(h + t\varphi) \, \mathrm{d}\mathcal{N}(t)}$$

for $h \in \pi_\varphi(H), r \in \mathbb{R}$. It holds

$$\int_H u\varrho \, \mathrm{d}\mu = \int_{\pi_\varphi(H)} \int_{\mathbb{R}} u(h + r\varphi) \rho(h, r) \, \mathrm{d}\mathcal{N}(r) \, \mathrm{d}\nu(h)$$

and the map

$$I : L^2(H, \varrho\mu) \xrightarrow{\simeq} L^2\big(\pi_\varphi(H) \times \mathbb{R}, \rho(\nu \times \mathcal{N})\big)$$

$$Iu(h, r) := u(h + r\varphi) \quad (\nu \times \mathcal{N}) \text{-a.e.}$$

is an isometric isomorphism.

(iii) In analogy to [22, Propo. 1.2] let

$$\mathcal{D}(\mathcal{E}_\varphi) := \left\{ u \in L^2(H, \varrho\mu) \,\middle|\, \text{for } \nu_N\text{-a.e. } h \in \pi_\varphi(H) \right.$$

$$Iu(h, \cdot) \text{ has an absolutely continuous version } f_h$$

$$\left. \text{and } \int_{\pi_\varphi(H)} \int_{\mathbb{R}} |f_h'(r)|^2 \rho(h, r) \, \mathrm{d}\mathcal{N}(r) \, \mathrm{d}\nu(h) < \infty \right\}$$

and in this sense

$$\mathcal{E}_\varphi(u, v) := \int_{\pi_\varphi(H)} \int_{\mathbb{R}} u(h + \cdot \varphi)'(r) \, v'(h + \cdot \varphi)'(r) \rho(h, r) \, \mathrm{d}\mathcal{N}(r) \, \mathrm{d}\nu(h)$$

for $u, v \in \mathcal{D}(\mathcal{E}_\varphi)$. By [22, Thm. 5.1], the space $\{u \in L^2(H, \varrho\mu) \mid u \text{ has a representative from } \mathcal{F}C_b^\infty(H)\}$ is a form core for $(\mathcal{E}_\varphi, \mathcal{D}(\mathcal{E}_\varphi))$.

4.2 Convergence of Component Forms

Here we show the Mosco–Kuwae–Shioya convergence of the component forms defined above, with the help of [15]. The following Lemma 4.4 refers to the case without perturbing density, i.e. $g_N(\cdot) = g(\cdot) = 0$. Together with the subsequent Lemma 4.5, these form the preliminaries for the main result of Section 4.2, Proposition 4.6 below.

Lemma 4.4 *On the Borel σ-algebra of $\pi_\varphi(H) \times \mathbb{R}$ we consider the product measures*

$$\left(\tilde{\mu}_N \circ \pi_\varphi^{-1}\right) \times \mathcal{N}_N$$

for $N \in \mathbb{N}$ and their weak limit $\left(\mu \circ \pi_\varphi^{-1}\right) \times \mathcal{N}$. The criteria of [15, Cond. 3.8] are satisfied.

Proof Let $f \in C_b(\pi_\varphi(H) \times \mathbb{R})$. To verify item (i) of [15, Cond. 3.8] it suffices to argue that

$$\left| \int_{\mathbb{R}} f(h, r)\, \mathrm{d}\mathcal{N}_N(r) \right| \overset{N \to \infty}{\longrightarrow} \left| \int_{\mathbb{R}} f(h, r)\, \mathrm{d}\mathcal{N}(r) \right|$$

uniformly w.r.t. $h \in \pi_\varphi(H)$. This simply follows from the estimate

$$\left| \sqrt{\tfrac{\lambda_N}{2\pi}} \left| \int_{\mathbb{R}} f(h, r) \exp(-\tfrac{\lambda_N r^2}{2})\, \mathrm{d}r \right| - \sqrt{\tfrac{\lambda}{2\pi}} \left| \int_{\mathbb{R}} f(h, r) \exp(-\tfrac{\lambda r^2}{2})\, \mathrm{d}r \right| \right|$$

$$\leq \|f\|_\infty \int_{\mathbb{R}} \left| \sqrt{\tfrac{\lambda_N}{2\pi}} \exp(-\tfrac{\lambda_N r^2}{2}) - \sqrt{\tfrac{\lambda}{2\pi}} \exp(-\tfrac{\lambda r^2}{2}) \right| \mathrm{d}r$$

and an application of Lebesgue's dominated convergence, since $\lambda_N \to \lambda$ for $N \to \infty$. A dominating integrable function for the integrand of the right-hand side is given by $r \mapsto 2\sqrt{\tfrac{\lambda}{\pi}} \exp(-\tfrac{\lambda r^2}{4})$ if $N \in \mathbb{N}$ is large enough such that $\lambda/2 < \lambda_N < 2\lambda$.

We now address (ii) of [15, Cond. 3.8]. Let $\kappa : \mathbb{R} \to [0, 1]$ be continuous with compact support. Further, let $R_{1/l}^{\varphi,\eta}(f)$ and $I_{1/l}^{\varphi,\eta}(f)$, as defined in [15, Sect. 2.2], be the φ, η-residual and the φ, η-perturbation of a measurable, non-negative function $f : \mathbb{R} \to \mathbb{R}$ for the parameters $l \in \mathbb{N}$ and $\varphi, \eta : \mathbb{R} \to [0, 1]$, which are primal functions as defined in the beginning of [15, Sect. 2.1]. In analogy to [15] we set $\varphi_{1/l}^\alpha(r) = \varphi(lr - l\alpha)$ for $\alpha \in (1/l)\mathbb{Z}$, $r \in \mathbb{R}$. The Radon–Nikodym density of $\mathcal{N}_N$ and $\mathcal{N}$ w.r.t. the Lebesgue measure on $\mathbb{R}$ is denoted by $\tfrac{\mathrm{d}\mathcal{N}_N}{\mathrm{d}r}$, respectively $\tfrac{\mathrm{d}\mathcal{N}}{\mathrm{d}r}$. For $\mathrm{d}r$-a.e. $r \in \mathbb{R}$ it holds

$$R_{1/l}^{\varphi,\eta}\left(\kappa \tfrac{\mathrm{d}\mathcal{N}_N}{\mathrm{d}r}\right)(r) = \sqrt{\tfrac{\lambda_N}{2\pi}} \sum_{\alpha \in \frac{1}{l}\mathbb{Z}} l \int_{\mathbb{R}} \left| \kappa(t) \exp(-\tfrac{\lambda_N t^2}{2}) - \kappa(r) \exp(-\tfrac{\lambda_N r^2}{2}) \right| \eta_{1/l}^\alpha(t)\, \mathrm{d}t\, \varphi_{1/l}^\alpha(r)$$

$$\leq \sqrt{\tfrac{\lambda_N}{2\pi}} \sup_{\substack{t \in \mathbb{R} \\ |t-r| \leq 4/l}} \left| \kappa(r) \exp(-\tfrac{\lambda_N r^2}{2}) - \kappa(t) \exp(-\tfrac{\lambda_N t^2}{2}) \right|$$

$$\leq \sqrt{\tfrac{\lambda_N}{2\pi}} \sup_{\substack{t \in \mathbb{R} \\ |t-r| \leq 4/l}} (\kappa(r) + \kappa(t)) \left| \exp(-\tfrac{\lambda_N r^2}{2}) - \exp(-\tfrac{\lambda_N t^2}{2}) \right|$$

$$+ \sqrt{\tfrac{\lambda_N}{2\pi}} \sup_{\substack{t \in \mathbb{R} \\ |t-r| \leq 4/l}} |\kappa(r) - \kappa(t)| \left(\exp(-\tfrac{\lambda_N r^2}{2}) + \exp(-\tfrac{\lambda_N t^2}{2}) \right) \qquad (4.6)$$

and also

$$I_{1/l}^{\varphi,\eta}\big(\kappa\,\tfrac{d\mathcal{N}_N}{dr}\big)(r) = \sqrt{\tfrac{\lambda_N}{2\pi}}\sum_{\alpha\in\frac{1}{l}\mathbb{Z}} l\int_{\mathbb{R}}\kappa(t)\exp(-\tfrac{\lambda_N t^2}{2})\eta_{1/l}^{\alpha}(t)\,dt\,\varphi_{1/l}^{\alpha}(r)$$

$$\leq \sqrt{\tfrac{\lambda_N}{2\pi}}\exp(-\tfrac{\lambda_N r^2}{2})\sup_{\substack{t\in\mathbb{R}\\|t-r|\leq 4/l}}\Big(\kappa(t)\exp\big(-\tfrac{\lambda_N(t^2-r^2)}{2}\big)\Big). \qquad (4.7)$$

Due to Eq. 4.6, for any bounded, measurable function $f:\mathbb{R}\to\mathbb{R}$ it holds

$$\frac{\int_{\mathbb{R}} f(r)R_{1/l}^{\varphi,\eta}\big(\kappa\,\tfrac{d\mathcal{N}_N}{dr}\big)(r)\,dr}{\int_{\mathbb{R}} f^2\,d\mathcal{N}_N} \leq \int_{\mathbb{R}}\Big(R_{1/l}^{\varphi,\eta}\big(\kappa\,\tfrac{d\mathcal{N}_N}{dr}\big)(r)\Big)^2\big(\tfrac{d\mathcal{N}_N}{dr}(r)\big)^{-1}\,dr$$

$$\leq \sup_{\substack{t,r\in\mathbb{R}\\|t-r|\leq 4/l}}(\kappa(r)+\kappa(t))\Big|1-\exp(-\tfrac{\lambda_N(t^2-r^2)}{2})\Big|$$

$$+ \sup_{\substack{t,r\in\mathbb{R}\\|t-s|\leq 4/l}}(\kappa(r)-\kappa(t))\Big|1+\exp(-\tfrac{\lambda_N(t^2-r^2)}{2})\Big|.$$

The right-hand side is independent of the choices for η, φ and f and converges to zero for $l\to\infty$ uniformly w.r.t. $N\in\mathbb{N}$, because $(\lambda_N)_{N\in\mathbb{N}}$ is bounded and κ continuous with compact support. This means, for $\delta_{1/l}^{\kappa}(\mathcal{N}_N)$ defined as in [15, Sec. 2.2], it holds $\lim_{l\to\infty}\sup_{N\in\mathbb{N}}\delta_{1/l}^{\kappa}(\mathcal{N}_N)=0$.

Due to Eq. 4.7, for any bounded, measurable function $f:\mathbb{R}\to\mathbb{R}$ it holds

$$\frac{\int_{\mathbb{R}} f(r)I_{1/l}^{\varphi,\eta}\big(\kappa\,\tfrac{d\mathcal{N}_N}{dr}\big)(r)\,dr}{\int_{\mathbb{R}}|f|\,d\mathcal{N}_N} \leq \Big\|I_{1/l}^{\varphi,\eta}\big(\kappa\,\tfrac{d\mathcal{N}_N}{dr}\big)\big(\tfrac{d\mathcal{N}_N}{dr}\big)^{-1}\Big\|_{L^{\infty}(\mathbb{R},m_N)}$$

$$\leq \sup_{\substack{t,r\in\mathbb{R}\\|t-r|\leq 4}}\Big(\kappa(t)\exp\big(-\tfrac{\lambda_N(t^2-r^2)}{2}\big)\Big).$$

The value of the right-hand side is independent of η,φ,f and $l\in\mathbb{N}$ and bounded over all $N\in\mathbb{N}$. This means, for $C_{1/l}^{\kappa}(\mathcal{N}_N)$ defined as in [15, Sec. 2.2], it holds $\sup_{l\in\mathbb{N}}\sup_{N\in\mathbb{N}}C_{1/l}^{\kappa}(\mathcal{N}_N)<\infty$. This concludes the proof. $\qquad\square$

Lemma 4.5 *Under Condition 4.2, it holds:*

(i) *$\varrho_N\mu_N\Rightarrow\varrho\mu$ for $N\to\infty$ in the sense of weak measure convergence on H.*
(ii) *$\tilde{\varrho}_N\tilde{\mu}_N\Rightarrow\varrho\mu$ for $N\to\infty$ in the sense of weak measure convergence on H.*
(iii) *Let $f\in C_b(\pi_\varphi(H)\times\mathbb{R})$.*

$$\int_{\mathbb{R}} f(\cdot,r)\rho_N(\cdot,r)\,d\mathcal{N}_N(r) \xrightarrow{N\to\infty} \int_{\mathbb{R}} f(\cdot,r)\rho(\cdot,r)\,d\mathcal{N}(r) \qquad (4.8)$$

strongly in the sense of Kuwae–Shioya ([18]), within the framework of the converging Hilbert spaces $L^2(\pi_\varphi(H),\nu_N)\to L^2(\pi_\varphi(H),\nu)$.

Proof (i) We first show the claim for the case $g_N^{(2)}(\cdot)=g^{(2)}(\cdot)=0$, i.e. $g_N=g_N^{(1)}$ and $g=g^{(1)}$. The idea is to apply [15, Lemma 3.5] proving that ϱ is the strong limit of $(\varrho_N)_{N\in\mathbb{N}}$ in the sense of Kuwae–Shioya within the framework of the converging Hilbert spaces $L^2(H,\mu_N)\to L^2(H,\mu)$. In the following we use Lemma A2 for the sequence $(g_N)_N$ and g adopting

its notation. To check the assumptions of [15, Lemma 3.5] in this case, for $N, m \in \mathbb{N}$ we choose

$$G_{N,m}^{\min}(h) := \exp\left(\int_D g_{N,m}^{\min} \circ h \, dz\right), \quad h \in H,$$

as a minorante and

$$G_{N,m}^{\text{maj}}(h) := \exp\left(\int_D g_{N,m}^{\text{maj}} \circ h \, dz\right), \quad h \in H,$$

as a majorante for the function $G_N(h) := Z_N \varrho_N = \exp\left(\int_D g_N \circ h \, dz\right)$, $h \in H$.

The set of discontinuities of g is a countable subset of $\mathbb{R}$, which we denote by U_g. It holds

$$dz(\{z \in D \mid h(z) \in U_g\}) \leq \sum_{a \in U_g} dz(\{z \in D \mid h(z) = a\}) = 0 \quad d\mu(h)\text{-a.e. on } H, \quad (4.9)$$

since μ is a non-degenerate, centered Gaussian measure (see Lemma A3). In view of Eq. 4.9, we have

$$\lim_{m \to \infty} \exp\left(\int_D g_m^{\min} \circ h \, dz\right) = \exp\left(\int_D g \circ h \, dz\right) \quad d\mu(h)\text{-a.e. on } H,$$

and likewise

$$\lim_{m \to \infty} \exp\left(\int_D g_m^{\text{maj}} \circ h \, dz\right) = \exp\left(\int_D g \circ h \, dz\right) \quad d\mu(h)\text{-a.e. on } H,$$

by Lebesgue's dominated convergence. Another use of Lebesgue's dominated convergence yields

$$\lim_{m \to \infty} \int_H \left| \exp\left(\int_D g_m^{\min} \circ h \, dz\right) - \exp\left(\int_D g \circ h \, dz\right) \right|^2 d\mu(h) = 0, \quad (4.10)$$

respectively

$$\lim_{m \to \infty} \int_H \left| \exp\left(\int_D g_m^{\text{maj}} \circ h \, dz\right) - \exp\left(\int_D g \circ h \, dz\right) \right|^2 d\mu(h) = 0. \quad (4.11)$$

Moreover, Lemma A2 provides

$$\sup_{h \in H} \left| G_{N,m}^{\min}(h) - \exp\left(\int_D g_m^{\min} \circ h \, dz\right) \right| \overset{N \to \infty}{\longrightarrow} 0, \quad (4.12)$$

as well as

$$\sup_{h \in H} \left| G_{N,m}^{\text{maj}}(h) - \exp\left(\int_D g_m^{\text{maj}} \circ h \, dz\right) \right| \overset{N \to \infty}{\longrightarrow} 0 \quad (4.13)$$

for fixed $m \in \mathbb{N}$. The convergence of Eqs. 4.12 and 4.13 is stronger than the (strong) convergence w.r.t. the Kuwae–Shioya topology within the framework of converging Hilbert spaces $L^2(H, \mu_N) \to L^2(H, \mu)$. Therefore, Eqs. 4.12, 4.13, 4.10 and 4.11 together with [15, Lemma 3.5] imply that $\exp\left(\int_D g \circ h \, dz\right)$, $h \in H$, is the strong limit of $(G_N)_{N \in \mathbb{N}}$ in sense of Kuwae–Shioya. This in particular implies $Z_N \to Z$ and therefore also $\varrho_N \to \varrho$ in sense of Kuwae–Shioya.

The general case, in which

$$g = g^{(1)} - g^{(2)}, \quad g_N = g_N^{(1)} - g_N^{(2)},$$

can be handled analogously. The assumptions of [15, Lemma 3.5] are verified via Lemma A2 through the same arguments as above. In this case, for $N, m \in \mathbb{N}$ we choose

$$G_{N,m}^{\min}(h) := \exp\left(\int_D g_{N,m}^{(1)\min} \circ h \, dz\right) \exp\left(-\int_D g_{N,m}^{(2)\max} \circ h \, dz\right), \quad h \in H,$$

as a minorante and

$$G_{N,m}^{\mathrm{maj}}(h) := \exp\left(\int_D g_{N,m}^{(1)\max} \circ h \, dz\right) \exp\left(-\int_D g_{N,m}^{(2)\min} \circ h \, dz\right), \quad h \in H,$$

as a majorante for the function $G_N(h) := Z_N \varrho_N = \exp\left(\int_D g_N \, dz\right)$, $h \in H$. The arguments are analogous. This concludes the proof of (i).

(ii) As $\tilde{\varrho}_N \tilde{\mu}_N = (\varrho_N \mu_N) \circ J_N^{-1}$ for $N \in \mathbb{N}$, the statement of (ii) is an immediate consequence of

$$\sup_{\substack{h \in H \\ |h|_H = 1}} |J_N h - h|_H \overset{N \to \infty}{\longrightarrow} 0,$$

which is shown in Remark 4.1.

(iii) The proof of (iii) is very similar to that in part (i). Again, we apply [15, Lemma 3.5]. Here, we additionally assume that f is non-negative in the first step. We adopt the terminology from part (i) of this proof. As a suitable minorante to prove the convergence of Eq. 4.8 we choose the functions

$$\pi_\varphi(H) \ni h \longmapsto F_{N,m}^{\min}(h) := \frac{\int_{\mathbb{R}} f(h + r\varphi)\left(G_{N,m}^{\min} \circ J_N^{-1}\right)(h + r\varphi) \, d\mathcal{N}_N(r)}{\int_{\mathbb{R}} \left(G_{N,m}^{\mathrm{maj}} \circ J_N^{-1}\right)(h + t\varphi) \, d\mathcal{N}_N(t)}$$

for $N, m \in \mathbb{N}$. A majorante is provided by

$$\pi_\varphi(H) \ni h \longmapsto F_{N,m}^{\mathrm{maj}}(h) := \frac{\int_{\mathbb{R}} f(h + r\varphi)\left(G_{N,m}^{\mathrm{maj}} \circ J_N^{-1}\right)(h + r\varphi) \, d\mathcal{N}_N(r)}{\int_{\mathbb{R}} \left(G_{N,m}^{\min} \circ J_N^{-1}\right)(h + t\varphi) \, d\mathcal{N}_N(t)}$$

for $N, m \in \mathbb{N}$. Indeed, we have

$$F_{N,m}^{\min}(h) \leq \frac{\int_{\mathbb{R}} f(h + r\varphi)\left(G_N \circ J_N^{-1}\right)(h + r\varphi) \, d\mathcal{N}_N(r)}{\int_{\mathbb{R}} \left(G_N \circ J_N^{-1}\right)(h + t\varphi) \, d\mathcal{N}_N(t)} = \int_{\mathbb{R}} f(h, r)\rho_N(h, r) \, d\mathcal{N}_N(r)$$

and also

$$F_{N,m}^{\mathrm{maj}}(h) \geq \frac{\int_{\mathbb{R}} f(h + r\varphi)\left(G_N \circ J_N^{-1}\right)(h + r\varphi) \, d\mathcal{N}_N(r)}{\int_{\mathbb{R}} \left(G_N \circ J_N^{-1}\right)(h + t\varphi) \, d\mathcal{N}_N(t)} = \int_{\mathbb{R}} f(h, r)\rho_N(h, r) \, d\mathcal{N}_N(r)$$

for $h \in \pi_\varphi(H)$, $N.m \in \mathbb{N}$. The other assumptions of [15, Lemma 3.5] are also easily checked with similar arguments as in the proof for (i). For $m \in \mathbb{N}$ we have the uniform convergence of $(F_{N,m}^{\min})_{N \in \mathbb{N}}$ towards the function

$$\pi_\varphi(H) \ni h \longmapsto F_m^{\min}(h) := \frac{\int_{\mathbb{R}} f(h + r\varphi) \exp\left(\int_D g_m^{\min}(h + r\varphi)\right) d\mathcal{N}(r)}{\int_{\mathbb{R}} \exp\left(\int_D g_m^{\mathrm{maj}}(h + t\varphi)\right) d\mathcal{N}(t)}$$

as well as the uniform convergence of $(F_{N,m}^{\mathrm{maj}})_{N\in\mathbb{N}}$ towards the function

$$\pi_\varphi(H) \ni h \;\longmapsto\; F_m^{\min}(h) := \frac{\int_{\mathbb{R}} f(h+r\varphi)\exp\left(\int_D g_m^{\mathrm{maj}}(h+r\varphi)\right)\mathrm{d}\mathcal{N}(r)}{\int_{\mathbb{R}} \exp\left(\int_D g_m^{\min}(h+t\varphi)\right)\mathrm{d}\mathcal{N}(t)}.$$

With Lebesgue's dominated converges we conclude that

$$\pi_\varphi(H) \ni h \;\longmapsto\; \int_{\mathbb{R}} f(h,r)\rho(h,r)\,\mathrm{d}\mathcal{N}(r) = \frac{\int_{\mathbb{R}} f(h+r\varphi)\exp\left(\int_D g(h+r\varphi)\right)\mathrm{d}\mathcal{N}(r)}{\int_{\mathbb{R}} \exp\left(\int_D g(h+t\varphi)\right)\mathrm{d}\mathcal{N}(t)}$$

is the strong limit of $(F_m^{\min})_{m\in\mathbb{N}}$ and also the strong limit of $(F_m^{\mathrm{maj}})_{m\in\mathbb{N}}$ in $L^2(\pi_\varphi(H),\mathrm{d}\nu)$ as $m\to\infty$. By linearity of the claimed statement in (iii), the assumption $f(\cdot)\geq 0$ can de dropped. This concludes the proof. □

Proposition 4.6 $(\mathcal{E}_\varphi, \mathcal{D}(\mathcal{E}_\varphi))$ *is the limit of* $(\mathcal{E}_{\varphi_N}^N, \mathcal{D}(\mathcal{E}_{\varphi_N}^N))_{N\in\mathbb{N}}$ *in the sense of Mosco–Kuwae–Shioya within the framework of the converging Hilbert spaces* $L^2(H,\varrho_N\mu_N)\to L^2(H,\varrho\mu)$.

Proof The claim of the proposition is equivalent to the statement that $(\mathcal{E}_\varphi, \mathcal{D}(\mathcal{E}_\varphi))$ is the limit of $(\tilde{\mathcal{E}}_\varphi^N, \mathcal{D}(\tilde{\mathcal{E}}_\varphi^N))_{N\in\mathbb{N}}$ in the sense of Mosco–Kuwae–Shioya within the framework of the converging Hilbert spaces $L^2(H,\tilde{\varrho}_N\tilde{\mu}_N)\to L^2(H,\varrho\mu)$.

The argument for this equivalence is the following. The map $f\mapsto f\circ J_N^{-1}$ is a bijection from $C_b^1(H)$ onto itself with

$$\frac{\partial(f\circ J_N^{-1})}{\partial\varphi}(J_N h) = \frac{\partial f}{\partial\varphi_N}(h), \quad h\in H.$$

So, on the one hand, under the isometric isomorphism

$$L^2(H,\varrho\mu_N)\to L^2(H,\tilde{\varrho}_N\tilde{\mu}_N), \quad u\mapsto u\circ J_N^{-1},$$

we have

$$u\in\mathcal{D}(\mathcal{E}_{\varphi_N}^N) \text{ if and only } u\circ J_N^{-1}\in\mathcal{D}(\tilde{\mathcal{E}}_\varphi^N), \quad \mathcal{E}_{\varphi_N}^N(u,u) = \tilde{\mathcal{E}}_\varphi^N(u\circ J_N^{-1}, u\circ J_N^{-1}).$$

On the other hand, $(J_N)_{N\in\mathbb{N}}$ converges to the identity w.r.t. the operator norm on H. For $u_N\in L^2(H,\varrho_N\mu_N)$ for $N\in\mathbb{N}$ and $u\in L^2(H,\varrho\mu)$, the strong convergence of $(u_N)_N$ towards u in the framework of $L^2(H,\varrho_N\mu_N)\to L^2(H,\varrho\mu)$ is equivalent to the strong convergence of $(u_N\circ J_N^{-1})_N$ towards u in the framework of $L^2(H,\tilde{\varrho}_N\tilde{\mu}_N)\to L^2(H,\varrho\mu)$. The same holds regarding weak convergence.

To prove Mosco convergence of $(\tilde{\mathcal{E}}_\varphi^N, \mathcal{D}(\tilde{\mathcal{E}}_\varphi^N))_{N\in\mathbb{N}}$ towards $(\mathcal{E}_\varphi, \mathcal{D}(\mathcal{E}_\varphi))$, we apply [15, Thm. 3.11]. The only thing left to do is the verification of [15, Cond. 3.8]. The case without disturbing density, i.e. $\varrho_N(\cdot) = \varrho(\cdot) = 1$, is settled in Lemma 4.4. In the general case, [15, Cond. 3.8 (i)] is a direct consequence of Lemma 4.5 (iii), because the strong convergence in the sense of Kuwae–Shioya implies the convergence of respective L^2-norms. [15, Cond. 3.8 (ii)] follows from the perturbation result stated in [15, Lemma 3.12]. This concludes the proof. □

4.3 Convergence of Gradient Forms

In the final part of this article, we define analyse the Mosco–Kuwae–Shoiya convergence of gradient forms on H. We use the results of Sections 4.1 & 4.2.

The covariance $\mathrm{Cov}_\mu(h_1, h_1)$, $h_1, h_2 \in H$, defines an inner product on H which makes $(H, \mathrm{Cov}_\mu(\cdot, \cdot))$ a pre-Hilbert space. Taking the abstract completion of H w.r.t. the norm induced by Cov_μ we construct a new Hilbert space $(H_\mu, \langle \cdot, \cdot \rangle_\mu)$, in which $(H, \langle \cdot, \cdot \rangle)$ is densely and continuously included. The self-adjoint generator $(A, \mathcal{D}(A))$ of the symmetric closed form $\langle \cdot, \cdot \rangle$ with domain H on $(H_\mu, \langle \cdot, \cdot \rangle_\mu)$ is characterized by

$$H = \mathcal{D}(A^{\frac{1}{2}}), \quad \langle Au, v \rangle_\mu = \langle u, v \rangle \quad \text{for } u \in \mathcal{D}(A), \ v \in H.$$

The spectrum of A is a pure point spectrum which consists of real, positive eigenvalues. We refer to the inverse A^{-1}, which in fact can be shown to be a trace class operator from H into H_μ, as the covariance operator of μ.

Analogous structures exist regarding the Gaussian measure μ_N for $N \in \mathbb{N}$, for which we do not assume the full topological support on H. The topological support $V_N := \mathrm{supp}[\mu_N]$ is a linear subspace of H. It shall be noted at this point, that all of the three cases are possible in this setting:

- V_N may be finite-dimensional subspace.
- V_N may be a infinite-dimensional, strict subspace of H
- $V_N = H$.

In any of these cases the weak convergence of $(\mu_N)_{N \in \mathbb{N}}$ towards μ implies

$$\lim_{N \to \infty} |p_N h - h|_H = 0, \quad h \in H, \tag{4.14}$$

where p_N denotes the orthogonal projection onto V_N. The covariance of μ_N defines an inner product on V_N which makes $(V_N, \mathrm{Cov}_{\mu_N}(\cdot, \cdot))$ a pre-Hilbert space. With the same completion procedure as above we obtain a Hilbert space $(V_{\mu_N}, \langle \cdot, \cdot \rangle_{\mu_N})$, in which $(V_N, \langle \cdot, \cdot \rangle)$ is densely and continuously included. The self-adjoint generator $(A_N, \mathcal{D}(A_N))$ of the symmetric closed form $\langle \cdot, \cdot \rangle$ with domain V_N on $(H_{\mu_N}, \langle \cdot, \cdot \rangle_{\mu_N})$ is characterized by

$$V_N = \mathcal{D}(A_N^{\frac{1}{2}}), \quad \langle A_N u, v \rangle_{\mu_N} = \langle u, v \rangle \quad \text{for } u \in \mathcal{D}(A_N), \ v \in V_N.$$

Of course, if V_N is finite-dimensional, then $V_N = H_{\mu_N}$ and A_N is a continuous symmetric operator on H. The spectrum of A_N is a pure point spectrum which consists of real, positive eigenvalues. Until the end of this section, we use the following notation:

$0 < \lambda^{(1)} \leq \lambda^{(2)} \leq \ldots$ denote the eigenvalues of A. Moreover, let $\{\lambda_N^{(i)} \mid i \in I_N\}$ (ordered analogously, i.e. $\lambda_N^{(i)} \leq \lambda_N^{(j)}$ if $i \leq j$), where either $I_N = \{1, \ldots, k_N\}$, for some $k_N \in \mathbb{N}$, or $I_N = \mathbb{N}$, denote the family of eigenvalues of A_N for $N \in \mathbb{N}$. We fix an orthonormal basis $\{\eta_N^{(i)} \mid i \in I_N\}$ of $(H_{\mu_N}, \langle \cdot, \cdot \rangle_{\mu_N})$ such that $\eta_N^{(i)} \in \mathcal{D}(A_N)$, $A_N \eta_N^{(i)} = \lambda_N^{(i)} \eta_N^{(i)}$ for $i \in I_N$.

In the next lemma, we reformulate a well-known consequence of the weak measure convergence of $(\mu_N)_N$ towards μ, the convergence of the spectral structures of their covariance operators. The reader should be aware though, that the exact condition to characterize the weak convergence of Gaussian measures in terms of their covariance operators in a frame as ours, is a stronger notion of convergence: the one induced by the nuclear norm. For further reading on that topic, we refer to [5] and [2]. However, the statement of the next lemma is enough for our purpose and essential to the proof of the subsequent theorem. We set $\lambda_N^{(i)} := \infty$

and $\varphi_N^{(i)} := 0$ for $i \in \mathbb{N} \setminus I_N$ and remark that Eq. 4.14 implies $\sup(I_N) \to \infty$ for $N \to \infty$. Hence, for each $i \in \mathbb{N}$ the set $\{N \in \mathbb{N} \mid i \notin I_N\}$ is finite.

Lemma 4.7   *(i)* $\lim_{N \to \infty} \lambda_N^{(i)} = \lambda^{(i)}$ *for* $i \in \mathbb{N}$.
   (ii) There exists a subsequence $(\mu_{N_l})_{l \in \mathbb{N}}$ *of* $(\mu_N)_{N \in \mathbb{N}}$ *and an orthonormal basis* $\{\eta^{(1)}, \eta^{(2)}, \dots\}$ *of* $(H_\mu, \langle \cdot, \cdot \rangle_\mu)$ *such that* $\eta^{(i)} \in \mathcal{D}(A)$, $A\eta^{(i)} = \lambda^{(i)}\eta^{(i)}$, *and*

$$\lim_{l \to \infty} \eta_i^{N_l} = \eta_i \text{ strongly in } H \text{ for } i \in \mathbb{N}.$$

Proof The statement is exactly the content of [18, Corollary 2.5], if we understand $(H_{\mu_N}, \langle \cdot, \cdot \rangle_{\mu_N})$, $N \in \mathbb{N}$ as a sequence of converging Hilbert spaces with asymptotic space $(H_\mu, \langle \cdot, \cdot \rangle_\mu)$. There is a canonical way to do this, considering the convergence of the second moments

$$\lim_{N \to \infty} \langle p_N h, p_n h \rangle_{\mu_N} = \lim_{N \to \infty} \int_H \langle h, k \rangle^2 \, d\mu_N(k) = \int_H \langle h, k \rangle^2 \, d\mu(k) = \langle h, h \rangle_\mu$$

for $h \in H$. Moreover, by Eq. 4.14 states

$$\langle h, h \rangle = \lim_{N \to \infty} \langle p_N h, p_N h \rangle \tag{4.15}$$

for $h \in H$. So, the second criterion of [18, Def. 2.8] is fulfilled considering the quadratic form $|\cdot|_H^2$ with domain V_N on the Hilbert space $(H_{\mu_N}, \langle \cdot, \cdot \rangle_{\mu_N})$ for $N \in \mathbb{N}$ and the asymptotic form $|\cdot|_H^2$ with domain H on the limiting Hilbert space $(H_\mu, \langle \cdot, \cdot \rangle_\mu)$. The assumptions of [18, Corollary 2.5] require the compact convergence of the spectral structure of A_N towards the one of A as $N \to \infty$. We claim the following: Given $u_N \in V_N$ for $N \in \mathbb{N}$ and $u \in H$, then the weak convergence of $(u_N)_N$ towards u as a sequence in $(H, \langle \cdot, \cdot \rangle)$ implies the strong convergence of $(u_N)_N$ towards u in the topology of Kuwae–Shioya, [18, Def. 2.4]. This claim would imply the first condition for Mosco convergence, as specified in [18, Def. 2.11], and also verify asymptotic compactness as defined in [18, Def. 2.12]. So, let $u_N \in V_N$ for $N \in \mathbb{N}$ and $u \in H$ with $u_N \rightharpoonup u$ weakly in H. We want to prove the strong convergence of $(u_N)_N$ towards u in the strong topology of Kuwae–Shioya within the framework of converging Hilbert spaces $(H_{\mu_N}, \langle \cdot, \cdot \rangle_{\mu_N}) \to (H_\mu, \langle \cdot, \cdot \rangle_\mu)$. In view of [18, Lem. 2.3] it is enough to show

$$\lim_{N \to \infty} \int_H \langle u_N, k \rangle^2 \, d\mu_N(k) = \int_H \langle u, k \rangle^2 \, d\mu(k) \tag{4.16}$$

(convergence of norms) and

$$\lim_{N \to \infty} \int_H \langle u_N, k \rangle \langle h, k \rangle \, d\mu_N(k) = \int_H \langle u, k \rangle \langle h, k \rangle \, d\mu(k) \tag{4.17}$$

for $h \in H$, as Eq. 4.17 implies convergence w.r.t. the weak topology of Kuwae–Shioya (see [18, Def. 2.5] in combination with [18, Lem. 2.3]). By [5, Theorem 1] there exists $a \in (0, \infty)$ such that

$$\sup_{N \in \mathbb{N}} \int_H \exp(a|k|_H^2) \, d\mu_N(k) < \infty \tag{4.18}$$

Let $\varepsilon > 0$ and $h \in H$ be fixed. Because of Eq. 4.18 and the boundedness of $(u_N)_N$ in H, there exists $R \in (0, \infty)$ with

$$\sup_{N \in \mathbb{N}} \int_{\{k \in H \mid |k|_H > R\}} |k|_H^2 \, d\mu_N(k) < \frac{\varepsilon}{\sup_{N \in \mathbb{N}}(|u_N|_H^2 + |u_N|_H |h|_H)}.$$

In particular,

$$\sup_{N \in \mathbb{N}} \int_{\{k \in H \,|\, |k|_H > R\}} \langle u_N, k \rangle^2 \, \mathrm{d}\mu_N(k) < \varepsilon$$

as well as

$$\sup_{N \in \mathbb{N}} \int_{\{k \in H \,|\, |k|_H > R\}} \left| \langle u_N, k \rangle \langle h, k \rangle \right| \mathrm{d}\mu_N(k) < \varepsilon.$$

Now, due to the weak measure convergence $\mu_N \to \mu$ and a $\varepsilon/3$-argument (see [27, Theorem 2.2]), the integrals in Eqs. 4.16 and 4.17 converge as desired if the integrands converge uniformly on compact sets. This, however, is immediate from the weak convergence of $(u_N)_N$ towards u in H. Indeed, for an arbitrary compact set $K \subset H$, the family $\{K \ni k \mapsto \langle u_N, k \rangle \mid N \in \mathbb{N}\}$ is equicontinuous and converges pointwisely to $K \ni k \mapsto \langle u, k \rangle$. Hence, the Theorem of Arzelà–Ascoli shows

$$\sup_{k \in K} \left| \langle u_N, k \rangle - \langle u, k \rangle \right| \overset{N \to \infty}{\longrightarrow} 0.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.8 (i) The only assumption of Lemma 4.7 is the weak measure convergence of Gaussian measures on H. So, Item (ii) of that Lemma can also be read in the sense that for every subsequence of $(\mu_N)_{N \in \mathbb{N}}$ there exists a sub-subsequence with the stated properties.

  (ii) Let $i \in \mathbb{N}$. We define

$$\varphi^{(i)} := \frac{\eta^{(i)}}{\sqrt{\lambda^{(i)}}} \quad \text{and} \quad \varphi_N^{(i)} := \frac{\eta_N^{(i)}}{\sqrt{\lambda_N^{(i)}}}$$

for all $N \in \mathbb{N}$ large enough such that $i \in I_N$. Those elements are normalized in $(H, \langle \cdot, \cdot \rangle)$, $|\varphi_N^{(i)} - \varphi^{(i)}|_H \to$ as $N \to \infty$ by Lemma 4.7 and moreover

$$\lambda^{(i)} \operatorname{Cov}_\mu(\varphi^{(i)}, h) = \langle A\varphi^{(i)}, h \rangle_\mu = \langle \varphi^{(i)}, h \rangle$$

for $h \in H$. Similarly, for $N \in \mathbb{N}$ large enough such that $i \in I_N$, we have

$$\lambda_N^{(i)} \operatorname{Cov}_{\mu_N}(\varphi_N^{(i)}, h) = \lambda_N^{(i)} \operatorname{Cov}_{\mu_N}(\varphi_N^{(i)}, p_N h) = \langle A_N \varphi_N^{(i)}, p_N h \rangle_{\mu_N}$$
$$= \langle \varphi_N, v \rangle = \langle \varphi_N, v + w \rangle = \langle \varphi_N^{(i)}, p_n h \rangle = \langle \varphi_N^{(i)}, h \rangle$$

for $h \in H$. Hence, Proposition 4.6 can be applied and

$$\left(\mathcal{E}_{\varphi_N^{(i)}}^N, \mathcal{D}\big(\mathcal{E}_{\varphi_N^{(i)}}^N\big) \right) \longrightarrow (\mathcal{E}_{\varphi^{(i)}}, \mathcal{D}(\mathcal{E}_{\varphi^{(i)}}))$$

in the sense of Mosco–Kuwae–Shioya.

The infinite sum

$$\mathcal{E}(u, v) := \sum_{i=1}^{\infty} \mathcal{E}_{\varphi^{(i)}}(u, v) \quad \text{for } u, v \text{ from the domain,}$$

$$\mathcal{D}(\mathcal{E}) := \left\{ w \in \bigcap_{i \in \mathbb{N}} \mathcal{D}(\mathcal{E}_{\varphi^{(i)}}) \,\Big|\, \sum_{i=1}^{\infty} \mathcal{E}_{\varphi^{(i)}}(w, w) < \infty \right\}$$

defines a symmetric Dirichlet form on $L^2(H, \varrho\mu)$.

Lemma 4.9 *For $u \in \mathcal{D}(\mathcal{E})$ there exists a weak gradient $\nabla u \in L^2(H, H, \mu)$ which is characterized by*

$$\langle \eta, \nabla u(h) \rangle = \lim_{t \downarrow 0} t^{-1}(u(h + t\eta) - u(h))$$

for μ-a.e. $h \in H$ and $\eta \in \mathcal{D}(A)$. The limit on the right-hand side exists in μ-a.e. sense. It holds

$$\mathcal{E}(u, v) = \int_H \langle \nabla u, \nabla v \rangle \varrho \, d\mu$$

Moreover, $\mathcal{F}C_b^\infty(H)$ is dense in $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$.

Proof The statement is a consequence of [22, Thm. 1.9] together with [3, Chap. II, Thm. 4.3.2.] an the fact that ϱ is bounded from below and above by positive constants. $\square$

For $N \in \mathbb{N}$ let $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$ denote the symmetric Dirichlet form on $L^2(H, \varrho_N \mu_N)$ which is the closure of

$$\mathcal{E}^N(u, v) := \int_H \langle p_N \nabla u, p_N \nabla v \rangle \, d\mu_N \quad \text{for } u, v \text{ from the pre-domain } \mathcal{F}C_b^\infty(H).$$

Theorem 4.10 *If Condition 4.2 holds, then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the limit of $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))_{N \in \mathbb{N}}$ in the sense of Mosco–Kuwae–Shioya within the framework of the converging Hilbert spaces $L^2(H, \varrho_N \mu_N) \to L^2(H, \varrho \mu)$.*

Proof Let $u \in L^2(H, \varrho\mu)$ and $u_N \in L^2(H, \varrho_N \mu_N)$ for $N \in \mathbb{N}$ such that u is the weak limit of u in the sense of Kuwae–Shioya. We have to show

$$\mathcal{E}(u, u) \leq \liminf_{N \to \infty} \mathcal{E}^N(u_N, u_N),$$

where $\mathcal{E}$ and $\mathcal{E}^N$ are extended as quadratic forms, assigning the value ∞ outside the domain of the respective Dirichlet form. It holds

$$\mathcal{E}^N(u_N, u_N) = \sum_{i \in I_N} \mathcal{E}^N_{\varphi_N^{(i)}}(u_N, u_N)$$

extending the quadratic form associated with $\left(\mathcal{E}^N_{\varphi_N^{(i)}}, \mathcal{D}(\mathcal{E}^N_{\varphi_N^{(i)}})\right)$ in the same way, since $\varphi_N^{(i)}$, $i \in I_N$, is an orthonormal basis of V_N. We formally set $\mathcal{E}^N_{\varphi_N^{(i)}}(\cdot, \cdot) := \infty$ on $L^2(H, \varrho_N \mu_N)$ if $i \in \mathbb{N} \setminus I_N$ and remark that for every $i \in \mathbb{N}$ the set $\{N \in \mathbb{N} \mid i \in \mathbb{N} \setminus I_N\}$ is finite. As a consequence of Proposition 4.6, respectively Remark 4.8 (ii), we have

$$\mathcal{E}(u, u) = \lim_{N' \to \infty} \sum_{i \in I_{N'}} \mathcal{E}_{\varphi^{(i)}}(u, u) \leq \liminf_{N' \to \infty} \left(\sum_{i \in I_{N'}} \liminf_{N \to \infty} \mathcal{E}^N_{\varphi_N^{(i)}}(u_N, u_N) \right)$$

$$\leq \liminf_{N' \to \infty} \liminf_{N \to \infty} \left(\sum_{i \in I_{N'}} \mathcal{E}^N_{\varphi_N^{(i)}}(u_N, u_N) \right)$$

$$\leq \liminf_{N \to \infty} \left(\sum_{i \in I_N} \mathcal{E}^N_{\varphi_N^{(i)}}(u_N, u_N) \right) = \liminf_{N \to \infty} \mathcal{E}^N(u_N, u_N)$$

as desired.

In view of [18, Lem. 2.1(7)], the claim of this theorem follows if $(\mathcal{D}(\mathcal{E}^N), \mathcal{E}_1^N)$, $N \in \mathbb{N}$, form a sequence of converging Hilbert spaces with asymptotic element $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$. Since

$$\mathcal{E}(u, u) = \lim_{N \to \infty} \int_H \langle \nabla u, \nabla v \rangle \, d\mu_N \geq \limsup_{N \to \infty} \mathcal{E}^N(u, u)$$

for $u \in \mathcal{F}C_b^\infty(H)$, we must have

$$\mathcal{E}(u, u) = \lim_{N \to \infty} \mathcal{E}^N(u, u).$$

This concludes the proof. □

Corollary 4.11 *Let g_N, g be functions of bounded variation, as in Condition 4.2. For $N \in \mathbb{N}$ let $(X_t^N)_{t \geq 0}$ be the finite-dimensional system of skew interacting Brownian motions Eq. 3.21, defined in Example 3.9, and $(X_t)_{t \geq 0}$ be the diffusion process of Remark 3.8 with Röckner–Zhu–Zhu decomposition Eq. 3.19. Through the argument of Remark 3.10, we have shown the weak measure convergence of the equilibrium law of*

$$u_t^N := \Lambda_N X_{N^2 t}^N, \quad t \geq 0,$$

towards the equilibrium law of $(X_t)_{t \geq 0}$ as $N \to \infty$. The topological space to which the term of weak measure convergence refers in this context is $C([0, \infty), H_0')$ (see Condition 3.1). The statement holds for any height map Λ_N as in Eq. 2.2, if Ξ meets Condition 2.1.

Appendix

Lemma A1 *Let $N \in \mathbb{N}$. We fix a symmetric, positive, linear operator $A_N : \mathbb{R}^{k_N} \to \mathbb{R}^{k_N}$, a function $g_{N,0} \in C_b^1(\mathbb{R})$ and real numbers $y_j, \beta_j \in \mathbb{R}$ for $j = 1, \ldots, m$. With*

$$g_N(y) := g_{N,0}(y) + \sum_{j=1}^m \beta_j \mathbf{1}_{(-\infty, y_j]}(y), \quad y \in \mathbb{R},$$

we define the Borel probability measure

$$dm_{A_N}(x) := \frac{1}{Z_N} \exp\left(-x^{\mathrm{T}} A_N x - \frac{1}{N^d} \sum_{i=1}^{k_N} g_N\left(N^{\frac{d}{2}-1} x_i\right)\right) dx,$$

$$Z_N := \int_{\mathbb{R}^{k_N}} \exp\left(-x^{\mathrm{T}} A_N x - \frac{1}{N^d} \sum_{i=1}^{k_N} g_N\left(N^{\frac{d}{2}-1} x_i\right)\right) dx,$$

on $\mathbb{R}^{k_N}$. Then, the Process $(X_t^N)_{t \geq 0}$ defined in Example 3.9 satisfies

$$dX_t^{N,i} = -\left(A_N X_t^N\right)_i dt - \frac{1}{2} N^{-\frac{d}{2}-1} g_{N,0}'\left(N^{\frac{d}{2}-1} X_t^{N,i}\right) dt + \sum_{j=1}^m \frac{1 - e^{-\beta_j/N^d}}{1 + e^{-\beta_j/N^d}} dl_t^{N,i,\tilde{y}_j} + dB_t^{N,i}$$

P_x^N-a.s. for each $x \in \mathbb{R}^{k_N}$, where $(B_t^{N,i})_{t \geq 0}$, $i = 1, \ldots, k_N$, are independent Brownian motions and $l_t^{N,i,\tilde{y}_j}$ is the local time of $(X_t^{N,i})_{t \geq 0}$ at $\tilde{y}_j := N^{1-\frac{d}{2}} y_j$.

Proof The proof is analogous to [4, Thm. 5.2]. Our situation only is different from the cited theorem concerning scaling constants, the dimensions k_N, $d \in \mathbb{N}$, and the choice of A_N, which in both situations represents a positive, symmetric operator. These amendments do not touch the validity of the arguments from the proof of [4, Thm. 5.2]. Therefore, the statement of the Lemma follows directly. For the reader's convenience, we sketch the major steps of the proof.

Step (1): For $y \in \mathbb{R}$ and $i \in \{1, \ldots, k_N\}$ we define a finite Borel measure

$$\tilde{\delta}_y^i(B) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} m_{A_N}\big(B \cap \{x \in \mathbb{R}^{k_N} \,|\, |x_i - y| \leq \varepsilon\}\big), \quad B \in \mathcal{B}(\mathbb{R}^{k_N}).$$

Then, analogously to [4, Equation (5.6)], we obtain the following integration by parts formula for m_{A_N}. For $i = 1, \ldots, k_N$ and $f \in C^1(\mathbb{R}^{k_N})$ with finite Lipschitz constant, it holds

$$\int_{\mathbb{R}^{k_N}} \frac{\partial f}{\partial x_i} \, dm_{A_N} = -2 \int_{\mathbb{R}^{k_N}} f(x)(A_N x)_i \, dm_{A_N}(x)$$
$$+ N^{-\frac{d}{2}-1} \int_{\mathbb{R}^{k_N}} f(x) g_{N,0}'\big(N^{\frac{d}{2}-1} x_i\big) \, dm_{A_N}(x)$$
$$+ 2 \sum_{j=1}^{m} \frac{1 - e^{-\beta_j/N^d}}{1 + e^{-\beta_j/N^d}} \int_{\mathbb{R}^{k_N}} f(x) \, d\tilde{\delta}_{N^{1-\frac{d}{2}} y_j}^i (x). \tag{4.19}$$

Step (2): Analogously as in the proof of [4, Thm. 5.2], the positive, continuous additive functional of $(X_t^N)_{t \geq 0}$, which satisfies the Revuz correspondence w.r.t. $\tilde{\delta}_y^i$ for given $i = 1, \ldots, k_N$ and $y \in \mathbb{R}$ coincides with the local time $(l_t^{N,i,y})_{t \geq 0}$ of the component process $(X_t^{N,i})_{t \geq 0}$ at y.

Step (3): Let $(\mathcal{E}^N \mathcal{D}(\mathcal{E}^N))$ be the Dirichlet form of Example 3.9 and $u_i : \mathbb{R}^{k_N} \to \mathbb{R}$ denote the i-th coordinate projection for $i \in \{1, \ldots, k_N\}$. Then,

$$\mathcal{E}^N(u_i, f) = \frac{1}{2} \int_{\mathbb{R}^{k_N}} \partial_i f \, dm_{A_N}$$

for f as in step (1). Using Eq. 4.19 and (2), the Fukushima decomposition yields

$$X_t^{N,i} - X_0^{N,i} = M_t^{u_i} + N_t^{u_i}, \quad t \geq 0, \quad i = 1, \ldots, k_N, \tag{4.20}$$

P_x^N-a.s. for quasi-every $x \in \mathbb{R}^{k_N}$, where

$$N_t^{u_i} = -\big(A_N X_t^N\big)_i \, dt - \frac{1}{2} N^{-\frac{d}{2}-1} g_{N,0}'\big(N^{\frac{d}{2}-1} X_t^{N,i}\big) \, dt + \sum_{j=1}^{m} \frac{1-e^{-\beta_j/N^d}}{1+e^{-\beta_j/N^d}} \, dl_t^{N,i,\tilde{y}_j},$$

$\tilde{y}_j := N^{1-\frac{d}{2}} y_j$, and $(M_t^{u_i})_{t \geq 0}$ is the martingale additive functional with

$$<M^{u_i}, M^{u_j}>_t = 2 \int_0^t \Gamma_N(u_i, u_j)\big(X_s\big) \, ds = t\delta_{ij}, \quad t \geq 0, \quad i, j \in \{1, \ldots, k_N\}. \tag{4.21}$$

Step (4): Via compactness of the embedding $\mathcal{D}(\mathcal{E}) \hookrightarrow L^2(\mathbb{R}^{k_N}, m_{A_N})$ the absolute continuity criterion for the corresponding semigroup can be shown. Therefore, we have the stronger statement that Eq. 4.20 indeed holds P_x^N-a.s. for every $x \in \mathbb{R}^{k_N}$ $\qquad \square$

Lemma A2 *Let* $(g_N)_{N\in\mathbb{N}}$ *either be a sequence of bounded, monotone decreasing functions, or a sequence of bounded, monotone increasing functions on* $\mathbb{R}$ *such that there exists a bounded function* $g : \mathbb{R} \to \mathbb{R}$ *with*

$$\sup_{x\in\mathbb{R}} \inf_{|r|\leq\delta} |g(x) - g_N(x + r)| \overset{N\to\infty}{\longrightarrow} 0 \quad \text{for every } \delta \in (0, \infty).$$

Let $m \in \mathbb{N}$. *There exists a minorante* $g_{N,m}^{min} \in C_b(\mathbb{R})$ *and a majorante* $g_{N,m}^{maj} \in C_b(\mathbb{R})$ *for* $N \in \mathbb{N}$ *with*

$$-\|g_N\|_\infty \leq g_{N,m}^{min}(x) \leq g_N(x) \leq g_{N,m}^{maj} \leq \|g_N\|_\infty, \quad x \in \mathbb{R},$$

which meet the following properties:

There are asymptotic functions $g_m^{min}, g_m^{maj} \in C_b(\mathbb{R})$ *such that*

$$\lim_{N\to\infty} \|g_{N,m}^{min} - g_m^{min}\|_\infty = 0, \quad \lim_{N\to\infty} \|g_{N,m}^{maj} - g_m^{maj}\|_\infty = 0.$$

Moreover,

$$\lim_{m\to\infty} g_m^{min}(x) = g(x), \quad \lim_{m\to\infty} g_m^{maj}(x) = g(x), \quad \text{if } g \text{ is continuous at the point } x \in \mathbb{R}.$$

Proof We will construct the desired minorante and majorante for g_N via a mollifying technique with uses a partition of unity. W.l.o.g. let's assume we are in the monotone increasing case. Let $m \in \mathbb{N}$. Let $\tau_{m,i}$, $i \in \mathbb{Z}$, be a partition of union, subordinate to $(\frac{i-1}{m}, \frac{i+1}{m})$, $i \in \mathbb{Z}$. Choose $-\frac{1}{m} < r(N, m, i) < \frac{1}{m}$ such that

$$\left|g\left(\frac{i}{m}\right) - g_N\left(\frac{i}{m} + r(N, m, i)\right)\right| \leq \frac{1}{2N} + \inf_{|r|\leq 1/m} \left|g\left(\frac{i}{m}\right) - g_N\left(\frac{i}{m} + r\right)\right|, \quad i, N \in \mathbb{N}. \quad (4.22)$$

We define

$$g_{N,m}^{min} := \sum_{i\in\mathbb{Z}} g_N\left(\frac{i}{m} + r(N, m, i)\right)\tau_{m,i+2}, \quad g_{N,m}^{maj} := \sum_{i\in\mathbb{Z}} g_N\left(\frac{i}{m} + r(N, m, i)\right)\tau_{m,i-2}$$

for $N \in \mathbb{N}$ and

$$g_m^{min} := \sum_{i\in\mathbb{Z}} g\left(\frac{i}{m}\right)\tau_{m,i+2}, \quad g_m^{maj} := \sum_{i\in\mathbb{Z}} g\left(\frac{i}{m}\right)\tau_{m,i-2}.$$

We remark that $\tau_{m,i}(x) \neq 0$ necessitates $i \in \{\lfloor mx \rfloor, \lfloor mx \rfloor + 1\}$ for $i \in \mathbb{Z}$, $x \in \mathbb{R}$, and in particular, $\tau_{m,\lfloor mx \rfloor}(x) + \tau_{m,\lfloor mx \rfloor+1}(x) = 1$. The functions defined above have the desired properties indeed. It holds

$$g_{N,m}^{min}(x) = g_N\left(\frac{\lfloor mx \rfloor - 2}{m} + r(N, m, \lfloor mx \rfloor - 2)\right)\tau_{m,\lfloor mx \rfloor}(x)$$

$$+ g_N\left(\frac{\lfloor mx \rfloor - 1}{m} + r(N, m, \lfloor mx \rfloor - 1)\right)\tau_{m,\lfloor mx \rfloor+1}(x)$$

$$\leq g_N\left(\frac{\lfloor mx \rfloor - 1}{m}\right)\tau_{m,\lfloor mx \rfloor}(x) + g_N\left(\frac{\lfloor mx \rfloor}{m}\right)\tau_{m,\lfloor mx \rfloor+1}(x)$$

$$\leq g_N(x)(\tau_{m,\lfloor mx \rfloor}(x) + \tau_{m,\lfloor mx \rfloor+1}(x)) = g_N(x), \quad x \in \mathbb{R}, N \in \mathbb{N},$$

and also

$$g_{N,m}^{\mathrm{maj}}(x) = g_N\Big(\frac{\lfloor mx\rfloor + 2}{m} + r(N, m, \lfloor mx\rfloor + 2)\Big)\tau_{m,\lfloor mx\rfloor}(x)$$

$$+ g_N\Big(\frac{\lfloor mx\rfloor + 3}{m} + r(N, m, \lfloor mx\rfloor + 3)\Big)\tau_{m,\lfloor mx\rfloor+1}(x)$$

$$\geq g_N\Big(\frac{\lfloor mx\rfloor + 1}{m}\Big)\tau_{m,\lfloor mx\rfloor}(x) + g_N\Big(\frac{\lfloor mx\rfloor + 2}{m}\Big)\tau_{m,\lfloor mx\rfloor+1}(x)$$

$$\geq g_N(x)(\tau_{m,\lfloor mx\rfloor}(x) + \tau_{m,\lfloor mx\rfloor+1}(x)) = g_N(x), \quad x \in \mathbb{R}, \, N \in \mathbb{N}.$$

Let $\varepsilon > 0$. We choose N_0 is large enough such that $\inf_{r:|r|\leq 1/m} |g(x) - g_N(x+r)| < \frac{\varepsilon}{2}$ for every $x \in \mathbb{R}$ and $N \geq N_0$. For $N \geq \max\{N_0, \frac{1}{\varepsilon}\}$, using Eq. 4.22, it holds

$$|g_{N,m}^{\mathrm{min}}(x) - g_m^{\mathrm{min}}(x)| \leq \Big|g_N\Big(\frac{\lfloor mx\rfloor - 2}{m} + r(N, m, \lfloor mx\rfloor - 2)\Big) - g\Big(\frac{\lfloor mx\rfloor - 2}{m}\Big)\Big|\tau_{m,\lfloor mx\rfloor}(x)$$

$$+ \Big|g_N\Big(\frac{\lfloor mx\rfloor - 1}{m} + r(N, m, \lfloor mx\rfloor - 1)\Big) - g\Big(\frac{\lfloor mx\rfloor - 1}{m}\Big)\Big|\tau_{m,\lfloor mx\rfloor+1}(x)$$

$$\leq \Big(\frac{1}{2N} + \frac{\varepsilon}{2}\Big)\tau_{m,\lfloor mx\rfloor}(x) + \Big(\frac{1}{2N} + \frac{\varepsilon}{2}\Big)\tau_{m,\lfloor mx\rfloor+1}(x)$$

$$\leq \varepsilon\tau_{m,\lfloor mx\rfloor}(x) + \varepsilon\tau_{m,\lfloor mx\rfloor+1}(x) = \varepsilon.$$

Analogously, we get $|g_{N,m}^{\mathrm{maj}}(x) - g_m^{\mathrm{maj}}(x)| \leq \varepsilon$ for $x \in \mathbb{R}$ and $N \geq \max\{N_0, \frac{1}{\varepsilon}\}$.

Moreover, let g be continuous at $x \in \mathbb{R}$ and $m \in \mathbb{N}$ be large enough such that $|g(x) - g(y)| \leq \varepsilon$ if $y \in \mathbb{R}$, $|y - x| < \frac{3}{m}$, then

$$|g_N^{\mathrm{min}}(x) - g(x)| \leq \Big|g\Big(\frac{\lfloor mx\rfloor - 2}{m}\Big) - g(x)\Big|\tau_{m,\lfloor mx\rfloor}(x) + \Big|g\Big(\frac{\lfloor mx\rfloor - 1}{m}\Big) - g(x)\Big|\tau_{m,\lfloor mx\rfloor+1}(x)$$

$$\leq \varepsilon\tau_{m,\lfloor mx\rfloor}(x) + \varepsilon\tau_{m,\lfloor mx\rfloor+1}(x) = \varepsilon$$

and also

$$|g_N^{\mathrm{maj}}(x) - g(x)| \leq \Big|g\Big(\frac{\lfloor mx\rfloor + 2}{m}\Big) - g(x)\Big|\tau_{m,\lfloor mx\rfloor}(x) + \Big|g\Big(\frac{\lfloor mx\rfloor + 3}{m}\Big) - g(x)\Big|\tau_{m,\lfloor mx\rfloor+1}(x)$$

$$\leq \varepsilon\tau_{m,\lfloor mx\rfloor}(x) + \varepsilon\tau_{m,\lfloor mx\rfloor+1}(x) = \varepsilon.$$

In the case, where g_N, $N \in \mathbb{N}$, are monotone decreasing, we define

$$g_{N,m}^{\mathrm{min}} := \sum_{i\in\mathbb{Z}} g_N\Big(\frac{i}{m} + r(N, m, i)\Big)\tau_{m,i-2}, \quad g_{N,m}^{\mathrm{maj}} := \sum_{i\in\mathbb{Z}} g_N\Big(\frac{i}{m} + r(N, m, i)\Big)\tau_{m,i+2}$$

for $N \in \mathbb{N}$ and

$$g_m^{\mathrm{min}} := \sum_{i\in\mathbb{Z}} g\Big(\frac{i}{m}\Big)\tau_{m,i-2}, \quad g_m^{\mathrm{maj}} := \sum_{i\in\mathbb{Z}} g\Big(\frac{i}{m}\Big)\tau_{m,i+2},$$

instead. The values of $r(N, m, i)$ for $N, m, i \in \mathbb{N}$ are chosen exactly as in the first case. From here, the argumentation is analogous. This concludes the proof. $\qquad\square$

Lemma A3 *Let μ be a non-degenerate centered Gaussian measure on $H := L^2(D, \mathrm{d}z)$ for a domain $D \subseteq \mathbb{R}^d$. For each value $a \in \mathbb{R}$ the level set $\{z \in D \mid h(z) = a\}$ has Lebesgue measure zero for μ-a.e. $h \in H$.*

Proof The claim is that the dz-class defined by the composition $\mathbf{1}_{\{a\}} \circ h$ vanishes in dz-a.e. sense for μ-a.e. $h \in H$. This is shown as follows.

Let $\varphi \in H$ such that the image measure of μ under any shift $\tau_{s\varphi}h := h + s\varphi$, $h \in H$, $s \in \mathbb{R}$, is absolutely continuous w.r.t. μ itself, i.e. $\mu \circ \tau_{s\varphi}^{-1} \ll \mu$. In that case, we immediately have $\mu \ll \mu \circ \tau_{s\varphi}^{-1}$, hence the equivalence (mutual absolute continuity) $\mu \sim \mu \circ \tau_{s\varphi}^{-1}$, for $s \in \mathbb{R}$. Moreover, defining the measure

$$\sigma_\varphi(B) := \int_{\mathbb{R}} \mu \circ \tau_{s\varphi}^{-1}(B) \, \mathrm{d}s, \quad B \subseteq H \text{ Borel measurable,}$$

it holds $\sigma_\varphi \sim \mu$. Moreover,

$$\int_H u \, \mathrm{d}\mu = \int_H \int_{\mathbb{R}} u(h + s\varphi) \frac{\mathrm{d}\mu}{\mathrm{d}\sigma_\varphi}(h + s\varphi) \, \mathrm{d}s \, \mathrm{d}\mu(h)$$

for any Borel measurable function $u : H \to [0, \infty)$. Now, let $\tilde{\varphi} : D \to \mathbb{R}$ be a representative for φ and $A \subseteq D$ be a Borel measurable set such that $\tilde{\varphi}(z) \neq 0$ for $z \in A$. Since any singleton is negligible w.r.t. the Lebesgue measure, Fubini's theorem yields

$$\int_H \int_A \mathbf{1}_a(h(z)) \, \mathrm{d}z \, \mathrm{d}\mu(h) = \int_H \int_{\mathbb{R}} \int_A \mathbf{1}_a\big(h(z) + s\tilde{\varphi}(z)\big) \, \mathrm{d}z \frac{\mathrm{d}\mu}{\mathrm{d}\sigma_\varphi}(h + s\varphi) \, \mathrm{d}s \, \mathrm{d}\mu(h)$$

$$= \int_H \int_A \int_{\mathbb{R}} \mathbf{1}_{\tilde{\varphi}(z)^{-1}(a-h(z))}(s) \frac{\mathrm{d}\mu}{\mathrm{d}\sigma_\varphi}(h + s\varphi) \, \mathrm{d}s \, \mathrm{d}z \, \mathrm{d}\mu(h) = 0. \quad (4.23)$$

Due to the Cameron–Martin formula, the space $V := \{\varphi \in H \mid \mu \circ \tau_{s\varphi}^{-1} \ll \mu \text{ for } s \in \mathbb{R}\}$ is dense in H, as is shown in [3, Theorems 3.1.2 & 3.1.3 of Chapter II]. Hence, we can find an orthonormal basis $\varphi_1, \varphi_2, \ldots$ of elements from V and define Borel measurable subsets $A_1, A_2, \ldots$ of D as $A_i := \{z \in D \mid \tilde{\varphi}_i(z) \neq 0\}$ for some dz-version $\tilde{\varphi}_i$ of φ_i. The right-hand-side of Eq. 4.23 is independent of φ and in this case yields

$$\int_H \int_{A_i} \mathbf{1}_a(h(z)) \, \mathrm{d}z \, \mathrm{d}\mu(h) = 0, \quad i \in \mathbb{N}.$$

Let $Q := D \setminus \big(\bigcup_{i \in \mathbb{N}} A_i\big)$. Since $\langle \mathbf{1}_Q, \varphi_i \rangle = 0$ for every $i \in \mathbb{N}$, the set Q has Lebesgue measure zero. Therefore,

$$\int_H \int_D \mathbf{1}_a(h(z)) \, \mathrm{d}z \, \mathrm{d}\mu(h) = \int_H \int_{\bigcup_{i \in \mathbb{N}} A_i} \mathbf{1}_a(h(z)) \, \mathrm{d}z \, \mathrm{d}\mu(h)$$

$$\leq \sum_{i=1}^{\infty} \int_H \int_{A_i} \mathbf{1}_a(h(z)) \, \mathrm{d}z \, \mathrm{d}\mu(h) = 0.$$

This concludes the proof. □

Declarations

Competing Interests The authors declare no competing interests.

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