Simultaneous vs Sequential: Optimal Assortment Recommendation in Multi-Store Retailing

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Problem definition: We study a multi-store assortment planning problem in which a seller centrally decides the assortment of products in each store. Each store is visited by a certain fraction of customers. Upon arrival, customers observe the products offered in the current store. The seller can either 1) show products offered in other stores to customers simultaneously (called simultaneous offering strategy) or 2) show products offered in other stores to customers if they decide not to purchase in the current store (called sequential offering strategy). Customers may incur a disutility if they choose products from other stores. We study the optimal assortment planning problem under each of the two strategies and compare their performance. **Methodology/results:** We show that under the simultaneous strategy, the optimal assortment for each store is revenue-ordered, and the seller cannot do better than if it operates each store separately. In contrast, adopting the sequential strategy can significantly increase the seller's revenue. In particular, when customers' disutilities of purchasing from other stores are universally homogeneous, the optimal assortment under the sequential offering strategy is revenue-ordered for each store, and the store with a lower (higher, resp.) demand should offer a larger (smaller, resp.) assortment to facilitate the sequential selling process. We then compare the two offering strategies in terms of the seller's revenue and customer surplus and analyze the impact of parameters on the revenue comparison. We also consider several extensions, e.g., heterogeneous valuation, capacity constraint, and partial recommendation, and find that our main results still hold qualitatively. **Managerial implications:** The sequential offering strategy may be a better strategy for sellers in practice. We provide detailed guidelines for sellers to implement the sequential strategy, such as regarding the optimal assortment decisions and when sellers can exploit the most benefits.

Key words: multi-store retailing, assortment optimization, simultaneous/sequential offering strategy

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1. Introduction

In the retail industry, a retailer often operates multiple stores under a common brand. With the rapid development of information technology, a multi-store retailer has the potential of increasing the coordination across different stores, instead of running each store independently. In particular, with state-of-the-art information technologies, the availability information of products in one store can be seamlessly shared with other stores, and such information empowers the seller to serve customers in one store with products offered only in other stores, and ultimately make a holistic product recommendation decision across different stores. An example of such information technology is Sterling Order Management, an inventory management software platform introduced by IBM, which can help sellers accurately track real-time inventory levels and coordinate different stores (IBM 2021). In luxury retail, Zegna adopted a series of Oracle Retail solutions, through which, Zegna not only gains convenient access to real-time inventory data but can also offer in-store services, including shipping products from other locations or scheduling in-store pickups (Spicer 2023).

The advancement of inventory tracking systems has provided sellers with the capability to recommend products from other stores. In practice, two main recommendation strategies have emerged. The first strategy is known as *simultaneous offering strategy*. With this approach, when a customer arrives at a store, the seller presents the assortment of products available in both the current store and other stores simultaneously. The customer is then able to choose from all the products, including those in the current store as well as those in other stores. This strategy has been widely implemented in the retail industry, where retailers utilize in-store kiosks or tablets to allow customers to browse and virtually explore products from other stores before making a purchase decision. For example, smart mirrors equipped with interactive displays can enable customers to access a wider range of products, such as makeup, hats, jewelry, eyewear, clothing, and shoes, even if these products are not physically present in the store they are visiting. Figure 1 provides an illustrative example, showcasing how full-length mirrors can allow customers to use swiping gestures to browse Superdry's new collection of winter apparel (Ferrandez 2022) in other stores.

Alternatively, the seller can adopt *sequential offering strategy*, where they initially present the assortment of products available in the current store to the customer. Only if the customer shows no interest in purchasing from the current store would the seller proceed to show products offered in other stores. This strategy allows the seller to first focus on promoting and selling the products within the current store before expanding the options to products in other locations. In practice, advanced technologies can assist retailers in detecting customers' inclination to leave without making a purchase. For instance, beacon technology utilizing Bluetooth Low Energy (BLE) can detect customers' presence and proximity to specific areas within the store (Beacon 2023). Retailers can utilize beacons to send targeted push notifications to customers' smartphones when they are near

Figure 1 Superdry's smart mirror

the exit. Such a strategy can also be implemented with the help of a salesperson. Consider the scenario of high-end retail stores that specialize in selling watches, fine jewelry, or luxury handbags. Typically, there is a salesperson who interacts with customers and makes product recommendations based on their needs. In this process, the salesperson may initially suggest products available in the current store since each store may have its own sales target. Only if the customer shows no interest in the current store's offerings would the salesperson recommend products from other stores under the same brand to retain the customer. In such a scenario, implementing a sequential offering strategy is natural and straightforward, particularly after providing the salesperson with appropriate training.

Undoubtedly, the flexibility provided by the above two strategies has the potential to bring convenience to customers and benefits to the seller. However, there are also significant challenges in the implementation process. One of the fundamental challenges is to decide what products to offer in each store. In the revenue management literature, this challenge is often known as the assortment planning problem, which aims to find the optimal set of products to offer with the goal of revenue maximization. The second challenge is to decide on a simultaneous or sequential recommendation strategy. Because the products offered in one store may affect customer purchase behavior in other stores, the presence of the second challenge further complicates the assortment planning decisions in each store. To our knowledge, no study has attempted to address these challenges. This paper fills the gap in the literature by answering the following questions:

1. Which offering strategy (simultaneous or sequential) is more beneficial to the seller?
2. Under each offering strategy, what is the optimal assortment decision in each store?
3. How does the optimal assortment decision depend on customer characteristics?

To answer these questions, we build a stylized model to analyze the two offering strategies in a multi-store retailing setting. Specifically, we consider a seller that operates m physical stores and sells

n substitutable products to a group of customers, each of whom may purchase at most one product. Each store is visited by a fraction of customers, with the total number of customers normalized to 1. In the simultaneous offering strategy, when a customer visits store i, she observes the products offered in both store i and the other $m-1$ stores simultaneously. If the customer chooses a product in the current store, she receives the full utility. However, if the customer chooses a product in another store, then her utility suffers a discount, which can be attributed to a delivery fee, waiting cost, or traveling cost. The customer chooses a product (or chooses not to purchase) according to a multinomial logit (MNL) choice model. For the sequential offering strategy, a customer visiting store i is first presented with the set of products in store i only. If she decides not to purchase any of them, then the seller subsequently presents the products available in all other stores to the customer.

We first study the optimal assortment decision under the simultaneous offering strategy. We find that the optimal assortment decision is simply to offer the optimal assortment when each store operates independently, for which it is known that a revenue-ordered assortment is optimal. Therefore, under the simultaneous offering strategy, the seller cannot do better than operating each store separately. In other words, the coordination between stores cannot generate additional benefits for the seller. The reason for this somewhat surprising result is as follows. On the one hand, the introduction of the assortments in other stores can recapture the lost demand in the current store. On the other hand, it also diverts customers from purchasing the high-priced products in the current store to purchasing the low-priced products in other stores. It turns out that the cannibalization effect always dominates under this strategy. Thus, it is optimal to offer the same revenue-ordered assortment in each store, as this can eliminate the cannibalization effect.

However, the situation differs considerably if the seller adopts the sequential offering strategy. We show that the sequential offering strategy under multi-store retailing always brings the seller higher revenue than operating each store separately. However, it is difficult to analyze the optimal assortment decisions in general, which is established to be an NP-hard problem. Therefore, we aim to explore the structural properties of the optimal assortments under certain conditions. We show that if the utility discounts (incurred when customers purchase from other stores) are homogeneous across all stores and all products, then revenue-ordered assortments are still optimal. More importantly, we find that the store with a relatively lower (higher, resp.) demand should offer an assortment larger (smaller, resp.) than the optimal assortment when each store operates separately. To understand the rationale, consider an example with two stores. If customers are not interested in products in the current store, the store (offering a smaller assortment) has a second selling opportunity by offering products in the other store, while the other store sacrifices its own revenue by offering a larger assortment. Therefore, the seller faces a trade-off between increasing the revenue of one store by implementing a two-step selling process and sacrificing the revenue of the other store by offering a larger assortment to facilitate the two-step selling process for the first store. Intuitively, the seller can exploit the largest benefit by implementing a two-step selling process in a higher-demand store and minimizing the sacrifice by offering a larger assortment in the other store.

We also study the impact of the model parameters, including the arrival fraction and utility discount, on the optimal assortments and revenue when the utility discounts are homogeneous. Given a fixed market size, we find that when the arrival rates of the stores are more differentiated, the total expected revenue is higher and, correspondingly, the optimal assortment of the store with a higher demand becomes smaller. Not surprisingly, the total expected revenue decreases in the utility discount. However, the size of the optimal assortments may not change monotonically with the utility discount.

We then compare the two strategies in terms of the seller's revenue and consumer surplus. We find that the sequential offering strategy always generates higher revenue than the simultaneous strategy. As discussed above, because of the cannibalization effect, the simultaneous strategy cannot perform better than operating each store separately (called the benchmark). However, the sequential strategy consistently outperforms the benchmark. On the negative side, the low-demand store offers a larger assortment than the benchmark case and cannot implement a two-step selling process for itself, leading to slightly lower revenue than in the benchmark case. On the positive side, the high-demand stores offer a smaller assortment to take advantage of the two-step selling process, which not only captures the lost demand in the first step, but also avoids the cannibalization effect caused by the low-priced products offered in the second step. This explains why the sequential strategy always performs better than the simultaneous strategy. However, for the same assortment set, the simultaneous offering strategy always leads to a higher consumer surplus than the sequential offering strategy. We also conduct extensive numerical experiments to investigate how model parameters affect the revenue comparison. We find that the revenue improvement of the sequential offering strategy over the simultaneous strategy becomes more significant when the arrival fractions of different stores are more differentiated or the utility discounts are smaller.

We also discuss how our results are related to the omnichannel setting where there exists an online store that carries the full set of products. We find that under certain conditions, the optimal assortments in all physical stores are still revenue-ordered for both strategies. Moreover, the introduction of the online channel enlarges the size of the assortment in physical stores under the simultaneous offering strategy. In contrast, the assortment size under the sequential offering strategy is smaller than that when each store operates separately. This is because, in the two-step selling process, the physical store would like to keep the high-priced products only since it can leverage the capability to recapture the lost demand. In terms of the seller's revenue, the sequential offering strategy still performs better than the simultaneous offering strategy, consistent with our main insight in the base model.

We consider three model extensions afterward. First, we confirm the advantages of the sequential offering strategy when customers visiting different stores have different valuations of the same product. Given the NP-hardness of the assortment problem under the sequential strategy, we provide a heuristic that yields a theoretical lower bound and good numerical performance. Second, we consider a capacitated problem in which a cardinality constraint is imposed on the size of the assortment in each store. We propose an integer programming formulation, conduct extensive numerical experiments, and observe that the sequential offering strategy still performs better than the simultaneous offering strategy in most spaces of the parameter set, which confirms that our main insight largely holds in the presence of capacity constraints. Third, we study partial recommendation where the seller is allowed to offer a subset of other stores' products to customers. We show that partial recommendation cannot improve revenue under the simultaneous offering strategy. For the sequential offering strategy, we conduct extensive numerical experiments and find that partial recommendation cannot benefit the seller in the vast majority of the cases. For the limited instances where partial recommendation did provide some benefit, the revenue improvement appears to be quite small. The overwhelming lack of incremental benefit from partial recommendation simplifies the decision-making process for sellers.

Finally, we study the joint assortment and pricing problem under both strategies. In this setting, the seller needs to jointly decide the optimal assortment and prices in each store, and customers visiting different stores have different price sensitivity parameters. We explicitly characterize the optimal assortment and pricing decisions under the simultaneous offering strategy. For the sequential offering strategy, we extend our approximation results in the base model to the new setting. Moreover, for a special case of the joint problem, we propose an efficient solution algorithm and find that the sequential offering strategy still outperforms the simultaneous offering strategy. Due to the page limit, we relegate this part to Appendix H.

2. Literature Review

Our work is closely related to three streams of research: assortment optimization, omnichannel retailing, and sequential recommendation. In the following, we review related works in each stream.

Assortment optimization. Assortment optimization is one of the most prominent research directions in revenue management. It studies the problem of offering a subset of products to customers to maximize firm profit. The seminal work of Talluri and van Ryzin (2004) studies the assortment optimization problem under the multinomial logit (MNL) model and shows the optimality of revenue-ordered assortments. Following Talluri and van Ryzin (2004), there has been a substantial body of research on this area under various customer choice models, including the mixed multinomial logit model (Bront et al. 2009, Rusmevichientong et al. 2014), nested logit models (Davis et al. 2014, Gallego and Topaloglu 2014), Markov chain choice models (Blanchet et al. 2016, Feldman and Topaloglu 2017), consider-then-choose choice models (Wang and Sahin 2018), multi-stage choice models (Gao et al. 2021). We also adopt the MNL model to characterize customers' purchasing decisions regarding substitutable products. A major difference is that the decision in the above studies is to optimize the assortment for a single store. In contrast, the seller in our model operates multiple stores and makes assortment decisions for all stores jointly. The coordination between different stores allows customers who visit one store to make a purchase in other stores, which is a key feature in our model. Such coordination substantially affects assortment decisions and presents considerable challenges to analyzing the problem.

Some recent works also study joint assortment planning problems. For example, Xu and Wang (2023) study a multi-stage assortment optimization problem where the seller sequentially decides the assortment offered in each stage with commitment. Wang (2022) proposes a hybrid model to characterize customers' sequential and simultaneous choice behavior with search cost, and analyzes the corresponding assortment and pricing problem. This strand of research usually assumes that the assortment decisions are made sequentially for a single store. However, the seller in our model needs to centrally decide the assortment in each store simultaneously by taking into account the interactions between different stores. While Chen et al. (2024) examine a personalized assortment planning problem where the retailer recommends add-ons after customers select the primary item, our analysis differs by focusing on the recommendation strategies for primary item assortments across multiple stores, rather than add-ons.

Omnichannel retailing. Our work is also closely related to the literature on omnichannel retailing. Omnichannel retailing is a new business mode that integrates both offline and online channels. It has received considerable attention from operations management scholars. Within the context of omnichannel retaling, Gao and Su (2017a) study how the "buy online and pick-up in store" option affects store operations in the presence of strategic customers. Gao and Su (2017b) investigate three information mechanisms to analyze how retailers can effectively deliver online and offline information to omnichannel customers. Harsha et al. (2019) develop omnichannel pricing solutions given that customers navigate across channels to maximize the value of their purchases. Gao et al. (2022) quantify the performance of online booking limit algorithms in the context of omnichannel fulfillment to answer the question of when to fulfill online orders using in-store inventory. Different from these works, our paper focuses on assortment decisions in a multi-store environment where customers are offered the assortments in different stores either simultaneously or sequentially, and we compare the performance of different offering strategies.

There has been an emergence of research on omnichannel assortment planning problems in recent years. For example, Dzyabura and Jagabathula (2018) analyze the problem of determining the assortment of products to showcase in the offline channel to maximize the profit from both offline and online channels. Lo and Topaloglu (2022) examine a similar problem by introducing a features tree to organize products by features. In both works, the seller's decision is only the offline assortment, and the interaction is unilateral from offline to online. In contrast, the seller in our model needs to decide the assortments for all stores, where the interaction between each pair of stores is bilateral. Chen et al. (2022) study a problem where a seller running an online channel plans to open physical stores in multiple locations. The seller needs to decide the location and product assortment jointly for each physical store by considering the impact of physical stores on the online channel. Different from our model, no customers are allowed to purchase from other physical stores, and the assortments in physical stores affect the online channel unilaterally. Moreover, our paper differs from the above works by comparing simultaneous and sequential recommendation strategies to identify the optimal selling mechanism, which is not studied in the above literature. We also discuss how our results can be related to the omnichannel setting.

Sequential recommendation. Finally, our work is also related to the study of sequential recommendation mechanisms in the revenue management literature. In the retail industry, sellers may recommend products in multiple stages or pages sequentially. Accordingly, customers evaluate the products and make purchase decisions sequentially. Gallego et al. (2020) consider a product framing model, in which the customer decides on the number of pages to view based on an exogenous distribution and selects from a consideration set consisting of products only on these pages. Gao et al. (2021) assume that an impatient customer views the product assortment starting from the first page. If the customer does not choose any product on the current page, then she continues to view the next page as long as her patience does not run out. Liu et al. (2020) and Feldman and Segev (2022) consider an assortment optimization problem where the seller offers a new assortment of products in each stage. Similar to the above literature, our work also analyzes a sequential offering strategy such that the products in other stores are offered subsequently if the customer is not interested in the products offered in the first store. However, their model dynamics unfold sequentially starting from the first page/stage. In contrast, different stores in our model are visited simultaneously by different customers as their first choice. That is, each store is the first choice for some customers but not for others.

3. Model Overview

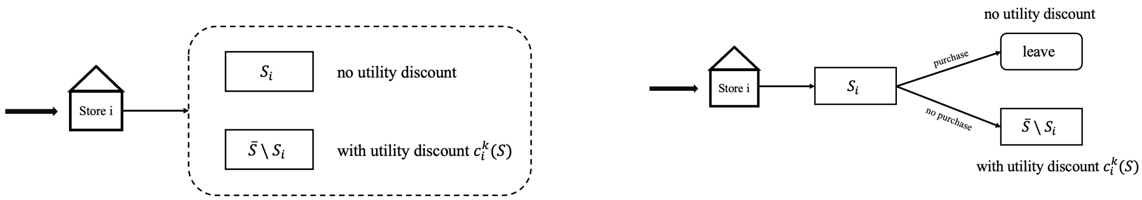
We consider a multi-store assortment planning problem where a single seller operates m stores and sells n substitute products to infinitesimal customers. The market size is normalized to 1, and we assume that a fraction λ_i of customers visit store $i \in \{1,2,\dots,m\}$, where $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. We use $\mathcal{N} = \{1,2,\dots,n\}$ to denote the full set of products. The revenue for each product $k \in \mathcal{N}$ is denoted by r_k. Without loss of generality, we assume $r_1 > r_2 > \cdots > r_n$. The intrinsic utility of product k is denoted by u_k, and we let $v_k = \exp(u_k)$ and call v_k attraction value.[1] Moreover, we assume that there is an outside option 0 with utility $u_0 \equiv 0$. We use S_i to denote the set of products offered in store i and $\boldsymbol{S} = (S_1, S_2, \dots, S_m)$ to denote the assortment planning decisions for all stores. We also let $\bar{S} = S_1 \cup S_2 \cup \cdots \cup S_m$ denote the union set of offered products. If a customer visits store i and is not interested in any product in S_i, then she may purchase a product in $\bar{S} \setminus S_i$.[2] However, in this case, the customer suffers a utility discount for this product, which is attributed to the delivery cost and/or waiting time. We use $c_{ij}^k \geq 0$ for all $i, j \in [m]$ and $k \in [n]$ to denote the utility discount when a customer visits store i but eventually purchases product k from store j. We assume $c_{ii}^k = 0$ for all $i \in [m]$ and $k \in [n]$, i.e., there is no utility discount if a customer purchases a product from the initial store she visits. Sections 3.1 and 3.2 introduce two models with different selling mechanisms within this framework, a simultaneous recommendation (SMR) model and a sequential recommendation (SQR) model, respectively.

3.1. Simultaneous Recommendation Model

In the SMR model, when a customer visits store i, the seller presents to her both the products offered in store i (i.e., S_i) and products not offered in store i but offered in other stores (i.e., $\bar{S} \setminus S_i$) *simultaneously*. Figure 2(a) depicts the customer's decision process under the SMR model, where all products (S_i and $\bar{S} \setminus S_i$) are offered simultaneously for consideration. We adopt a random utility model to characterize the customer's utility for each choice. Specifically, if the customer visiting store i decides to purchase product k where $k \in S_i$, then she receives a utility $u_k + \epsilon_{ik}$ for this product, where ϵ_{ik} is a random variable capturing customers' idiosyncrasies toward different alternatives. If the customer decides to purchase product k where $k \in \bar{S} \setminus S_i$, then she purchases from the store that leads to the least utility discount and thus receives a utility $u_k - \min_{j \in \{l | k \in S_l\}} c_{ij}^k + \epsilon_{ik}$. To simplify the notation, we let $c_i^k(\boldsymbol{S}) = \min_{j \in \{l | k \in S_l\}} c_{ij}^k$ if $\{l | k \in S_l\} \neq \emptyset$ and $c_i^k(\boldsymbol{S}) = \infty$ otherwise.

[1] Section 7.1 considers a more general case in which the same product has different attraction values in different stores.

[2] Throughout the paper, we use she/her to refer to the customer and he/his to refer to the seller.



(a) SMR model (b) SQR model

Figure 2 Customer's decision process under the two models

We assume that customer choice behavior follows the MNL choice model. That is, ϵ_{ik} follows independent and identical Gumbel distributions with scale parameter 1. Then, for each store i, given $\boldsymbol{S} = (S_1, \ldots, S_m)$, the choice probability of product k is:

$$p_{ik}^M(\boldsymbol{S}) = \begin{cases} \frac{v_k}{1+\sum_{l\in S_i} v_l + \sum_{l\in \bar{S}\setminus S_i} v_l \exp\left(-c_i^l(\boldsymbol{S})\right)} & \text{if } k \in S_i, \\ \frac{v_k \exp\left(-c_i^k(\boldsymbol{S})\right)}{1+\sum_{l\in S_i} v_l + \sum_{l\in \bar{S}\setminus S_i} v_l \exp\left(-c_i^l(\boldsymbol{S})\right)} & \text{if } k \in \bar{S} \setminus S_i, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where the superscript M in $p_{ik}^M(\boldsymbol{S})$ refers to the SMR model.

The goal of the seller is to decide the optimal assortment $\boldsymbol{S} = (S_1, \ldots, S_m)$ to maximize the total expected revenue. Therefore, the optimization problem for the seller under the SMR model is formulated as follows:

$$\max_{\boldsymbol{S}=(S_1,S_2,\ldots,S_m)} \quad \sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k p_{ik}^M(\boldsymbol{S}), \text{ where } p_{ik}^M(\boldsymbol{S}) \text{ is given in (1).} \tag{SMR}$$

3.2. Sequential Recommendation Model

In the SMR model, the customer is presented with all products that are available in at least one of the stores simultaneously. However, in practice, the seller may first present the products available in the current store only, and then show the products in other stores if the customer is not interested in any product in the current store. The sequential recommendation model considers such a two-step selling mechanism. Specifically, when a customer visits store i, her initial consideration set includes the products offered in store i (i.e., S_i) only.[3] If she purchases a product in S_i, then she leaves the market. If she decides not to purchase any of the products in S_i, then the seller shows the products

[3] We assume that the customer is not strategic, i.e., she cannot anticipate whether and what products in other stores will be offered when deciding whether or not to purchase in the current store. Such an assumption is reasonable since customers who visit physical stores directly typically do not have information about products in other stores.

not offered in store i but offered in other stores (i.e., $\bar{S}\setminus S_i$) for her consideration.[4] Figure 2(b) depicts the customer's decision process under the SQR model, where the two sets of products (S_i and $\bar{S}\setminus S_i$) are offered to the customer *sequentially*.

When the customer evaluates the products in S_i, the random part of the outside option, namely, the value of ϵ_{i0}, will be realized. We assume that if the customer decides not to purchase any product from S_i (meaning that $u_k+\epsilon_{ik}<u_0+\epsilon_{i0}$ for all $k\in S_i$), ϵ_{i0} will not be re-sampled when she evaluates the products in $\bar{S}\setminus S_i$. When she further considers the products in $\bar{S}\setminus S_i$, she chooses the product l such that $u_l-c_i^l(\boldsymbol{S})+\epsilon_{il}$ achieves the maximal utility among $\bar{S}\setminus S_i$ and $u_l-c_i^l(\boldsymbol{S})+\epsilon_{il}\geq u_0+\epsilon_{i0}$. This choice process is consistent with the impatient MNL model in Gao et al. (2021). Following the same analysis as in Gao et al. (2021), for a fixed $\boldsymbol{S}=(S_1,\ldots,S_m)$, the choice probability of product k for the customer visiting store i initially (which we denote by $p_{ik}^Q(\boldsymbol{S})$, where the superscript Q refers to the SQR model) is given in the following proposition.

Proposition 1. *(Gao et al. 2021) The choice probabilities under the SQR model are given as*

$$p_{ik}^Q(\boldsymbol{S})=\begin{cases}\frac{v_k}{1+\sum_{l\in S_i}v_l} & \text{if } k\in S_i,\\ \frac{v_k\exp(-c_i^k(\boldsymbol{S}))}{(1+\sum_{l\in S_i}v_l)(1+\sum_{l\in S_i}v_l+\sum_{l\in\bar{S}\setminus S_i}v_l\exp(-c_i^l(\boldsymbol{S})))} & \text{if } k\in\bar{S}\setminus S_i,\\ 0 & \text{otherwise.}\end{cases}\tag{2}$$

The seller's optimization problem under the SQR model is then formulated as follows:

$$\max_{\boldsymbol{S}=(S_1,S_2,\ldots,S_m)}\quad\sum_{i=1}^m\lambda_i\sum_{k=1}^n r_k p_{ik}^Q(\boldsymbol{S}),\ \text{ where } p_{ik}^Q(\boldsymbol{S}) \text{ is given in (2).}\tag{SQR}$$

Observe that the main difference between the SMR and SQR models is the seller's recommendation strategy, i.e., simultaneous offering versus sequential offering. In the following sections, we first analyze the assortment optimization problem under each model and then compare the seller's revenue to shed light on the optimal assortment recommendation strategy.

To facilitate our discussions, we define the following single-store problem:

$$R^*=\max_S\quad\frac{\sum_{k\in S}r_kv_k}{1+\sum_{k\in S}v_k}.\tag{3}$$

We denote the optimal solution to problem (3) by $\tilde{S}$. We call S a revenue-ordered assortment if $S=\{1,2,\ldots,k\}$ for some $k\in[n]$. According to Talluri and van Ryzin (2004), at least one revenue-ordered assortment is optimal for problem (3). If there are multiple optimal solutions, then we choose the smallest revenue-ordered assortment as $\tilde{S}$.

[4] In practice, if customers are not interested in products in the current store, then the products in all other stores are usually recommended simultaneously instead of sequentially, especially when customers have limited patience. This is why we focus on a two-step sequential offering strategy instead of more steps. In Section 7.3, we consider a more general case where the seller is allowed to recommend a subset of the products in other stores.

4. Analysis of the SMR Model

In this section, we analyze the assortment optimization problem under the SMR model. The optimal assortment decision for this case is given below.

THEOREM 1. *The optimal solution to the SMR problem is given by $\boldsymbol{S}^* = (\tilde{S}, \tilde{S}, \ldots, \tilde{S})$.*

The above theorem states that the optimal assortment decision is simply to offer the single-store optimal assortment in each store. Note that there is no cross-selling under this recommendation strategy. We can then conclude that the seller cannot do better than operating each store separately if he chooses to adopt the simultaneous offering strategy.[5] Hereafter, we refer to the optimal revenue that can be achieved when each store operates separately as *the benchmark*.

Below, we provide a proof sketch for Theorem 1, which can help us understand the rationale behind Theorem 1 better. There are three main steps:

Step 1: We consider a simpler case with two stores only (store 1 and store 2), which we call a two-store optimization problem. Suppose that store 2 offers S_2. We first focus on the revenue of store 1. We let $c^k = c^k_{12}$ for any $k \in [n]$ and denote $\boldsymbol{c} = (c^1, \ldots, c^n)$. Then, store 1's optimization problem becomes

$$\max_{S_1 \subseteq \mathcal{N}} \quad R(S_1, \boldsymbol{c}) \triangleq \frac{\sum_{k\in S_1} r_k v_k + \sum_{k \in S_2 \setminus S_1} r_k v_k \cdot \exp(-c^k)}{1 + \sum_{k \in S_1} v_k + \sum_{k\in S_2\setminus S_1} v_k \cdot \exp(-c^k)}. \tag{4}$$

We show that at optimality, each product in S_1 has a higher price than the expected revenue of store 1 in (4), which is a weighted average of the revenue accrued from S_1 and the revenue accrued from $S_2 \setminus S_1$. Otherwise (if there is a lower-priced product in S_1), the expected revenue of store 1 in (4) can be improved by removing that product from S_1.

Step 2: We show that the optimal objective value of problem (4) under any $\boldsymbol{c}$ is bounded above by the optimal objective value when $\boldsymbol{c} = (\infty, \ldots, \infty)$. At first glance, this result is somewhat surprising because the introduction of S_2 recaptures some lost demand in store 1 compared with when store 1 operates independently. However, because of the introduction of S_2, some customers switch from purchasing the high-priced products in S_1 to purchasing the low-priced products in $S_2 \setminus S_1$. Such "switch and buy down" behavior (which is a form of cannibalization) decreases store 1's revenue and outweighs the positive effect of recapturing the lost demand. Accordingly, when $\boldsymbol{c} = (\infty, \ldots, \infty)$, the cannibalization effect is eliminated, and store 1's revenue in (4) is maximized. The same logic applies to store 2's problem. Hence, we establish that both stores achieve the highest revenue when $\boldsymbol{c} = (\infty, \ldots, \infty)$. That is, the coordination between the two stores does not benefit the seller under the SMR strategy, and each store should offer the same assortment as when they operate independently.

[5] The result in Theorem 1 is obtained when there is no capacity constraint on stores. If there are capacity constraints, then the optimal assortments may differ across stores. More importantly, it can lead to higher revenue than when each store operates separately (we give one such example in Appendix E.1).

Step 3: We generalize the above analysis and conclusion to the problem with multiple stores. The idea is that we view stores $\{2,\ldots,m\}$ as a "proxy" store $\tilde{2}$ and reduce the general problem to a two-store problem with store 1 and store $\tilde{2}$; see Figure 3. For any assortment set $(S_1, S_2, \ldots, S_m)$, we denote $S_{\tilde{2}} = S_2 \cup S_3 \cup \cdots \cup S_m$ and $h^k = \min_{j\in\{l|k\in S_l, l=2,\ldots,m\}} c^k_{1j}$. Then, store 1's problem can be expressed as (4) by replacing S_2 with $S_{\tilde{2}}$ and c^k with h^k. Following the analysis in Steps 1 and 2, the highest revenue in store 1 is achieved when $h^k = \infty$ for any $k \in [n]$. The same logic applies to other stores. This establishes the result in Theorem 1.

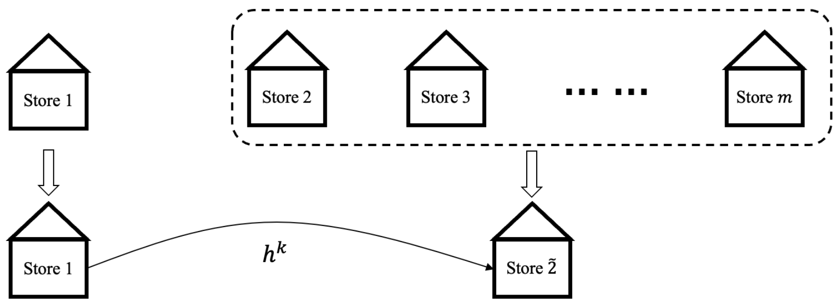


Figure 3 **Illustration of Step 3**

In the following, we discuss how our results are related to the omnichannel setting, in which there exists an online store that carries the full set of products. Without loss of generality, we assume that store m is the online store with $S_m = \mathcal{N}$ and stores $\{1, 2, \ldots, m-1\}$ are physical stores. We are interested in the optimal assortments for the physical stores. We first remark that in the presence of the online store, the optimal assortment in each physical store may not be identical and revenue-ordered, as is indicated by an example in Appendix E.2. However, under some mild conditions on the utility discounts, the following proposition holds.

Proposition 2. *For the omnichannel setting, i.e., $S_m = \mathcal{N}$, if $c^k_{im} = c^k > 0$ for each $i \in \{1, 2, \ldots, m-1\}$ and $k \in [n]$, then the optimal assortments in the first $m-1$ stores are identical and revenue-ordered. Moreover, the optimal assortment is larger than $\tilde{S}$.*

Proposition 2 shows that under certain conditions, the optimal assortments in the physical stores are still revenue-ordered. However, the assortment size is larger than that when each store operates separately, which means that the introduction of the online channel enlarges the product selection in physical stores. The underlying rationale is as follows. As argued in Step 1 above, for each physical store, all products with prices above the store's expected revenue should be included in the optimal assortment, because otherwise, the physical store can collect more revenue by adding that product to its assortment. Given that an online store offers the whole product set, the expected revenue of each physical store becomes lower due to the cannibalization effect. Therefore, more products should be included in the optimal assortment.

5. Analysis of the SQR Model

Section 4 shows that under the SMR model, the seller cannot do better than the benchmark. That is, the coordination between different stores does not benefit the seller. However, the situation is different under the SQR model. We start with the following example.

EXAMPLE 1. Consider a problem with two stores and three products with $(r_1, r_2, r_3) = (1.8, 1.1, 1)$. The arrival fractions are $(\lambda_1, \lambda_2) = (0.5, 0.5)$. The attraction values are $(v_1, v_2, v_3) = (1.2, 1.7, 2)$. For any $i, j \in [m], i \neq j$ and $k \in [n]$, we set $c_{ij}^k = 1$. One can verify that the optimal solution to the SQR model is $\{\{1\}, \{1, 2, 3\}\}$, which generates a total revenue of 1.0928. However, for the benchmark (as well as the SMR model), the optimal solution is $\{\{1, 2\}, \{1, 2\}\}$ and the corresponding total revenue is 1.0333. By adopting the SQR selling mechanism, the seller can increase the revenue by over 5% relative to the benchmark. The rationale behind this improvement is as follows. Note that product 3 brings a higher utility than products 1 and 2, so offering product 3 will shift customer demand from products 1 and 2 to product 3 whose price is the lowest. As a result, it is never optimal to include product 3 in the benchmark. One can also verify that for the benchmark, the revenues contributed by store 1 and store 2 are both 0.517. For the SQR model, the revenues contributed by store 1 and store 2 are 0.582 and 0.511, respectively. Compared with the benchmark, the revenue of store 2 decreases slightly because it offers a larger assortment than the single-store optimal assortment. However, the revenue of store 1 increases significantly, which is attributed to the benefit of implementing a two-step selling process. Specifically, customers who visit store 1 will be presented with product 1 only. Only if they decide not to purchase product 1, will store 1 recommend products 2 and 3. In this sequential process, products 2 and 3 will not shift customer demand away from product 1 (with a relatively higher price) but can recapture the lost demand from customers who do not purchase product 1. Overall, the revenue improvement in store 1 dominates the revenue loss in store 2, leading to a higher total revenue compared with the benchmark. □

Having observed the potential of the SQR strategy, this section analyzes the assortment optimization problem under the SQR model. The following proposition states that for general c_{ij}^k's, the SQR problem is NP-hard. This result sets a boundary between the solvable and intractable cases under the SQR model.

PROPOSITION 3. *For general utility discounts c_{ij}^k's, the SQR problem is NP-hard.*

Given the NP-hardness of the general SQR problem, we aim to explore the structural properties of the optimal assortments under certain conditions. We find that when the utility discounts are homogeneous across all stores and all products, revenue-ordered assortments are still optimal, based on which an efficient algorithm can be designed to find the optimal assortments. Moreover, the optimal strategy is to offer a larger assortment in the store with the lowest demand and smaller assortments in other stores. Finally, we investigate the impacts of the arrival fraction and utility discount on the optimal assortment and expected revenue.

5.1. Two-Store Problem

We start with a simpler case with two stores only, from which we derive important insights and lay the foundation for the analysis of the case with multiple stores. In the two-store case, we assume that a fraction λ of customers visit store 1 and the remaining customers visit store 2. According to the probability equation (2), the problem can be formulated as follows:

$$\begin{aligned}\max_{S_1,S_2} R(S_1,S_2) \triangleq \lambda \frac{\sum_{k\in S_1} r_k v_k}{1+\sum_{k\in S_1} v_k} + \lambda \frac{\sum_{k\in S_2\setminus S_1} r_k v_k \cdot \exp(-c_{12}^k)}{(1+\sum_{k\in S_1} v_k)(1+\sum_{k\in S_1} v_k + \sum_{k\in S_2\setminus S_1} v_k \cdot \exp(-c_{12}^k))} \\ + (1-\lambda)\frac{\sum_{k\in S_2} r_k v_k}{1+\sum_{k\in S_2} v_k} + (1-\lambda)\frac{\sum_{k\in S_1\setminus S_2} r_k v_k \cdot \exp(-c_{21}^k)}{(1+\sum_{k\in S_2} v_k)(1+\sum_{k\in S_2} v_k + \sum_{k\in S_1\setminus S_2} v_k \cdot \exp(-c_{21}^k))}.\end{aligned} \tag{5}$$

Even for the problem with two stores, the analysis of the optimal assortments is complex. This complexity arises because the assortment in one store affects the expected revenue of the other store in a complicated way. To simplify the analysis, we make the following assumption regarding the utility discounts. We say that the utility discounts are *universally homogeneous* if $c_{ij}^k = c$ with $c > 0$ holds for any $i, j \in [m], i \neq j$ and $k \in [n]$. This assumption requires the utility discounts to be homogeneous across all stores and all products, which is reasonable in practice. For example, if the utility discounts are attributed to the shipping cost, then the shipping costs between different stores in a certain region are similar. We find that when the utility discounts are universally homogeneous, revenue-ordered assortments are optimal to problem (5).

THEOREM 2. *Suppose the utility discounts are universally homogeneous. Then, there exists an optimal solution (S_1^*, S_2^*) to problem (5) such that S_1^* and S_2^* are both revenue-ordered. Moreover, when $0 \le \lambda < 0.5$, we have $S_2^* \subseteq \tilde{S} \subseteq S_1^*$; when $0.5 \le \lambda \le 1$, we have $S_1^* \subseteq \tilde{S} \subseteq S_2^*$.*

For the SQR problem with two stores, if the goal is to maximize the revenue of store 1 only, then the seller has the incentive to set $S_1 \subseteq S_2$ to facilitate a two-step selling for store 1; that is, the products in $S_2 \setminus S_1$ will be recommended to customers in store 1 if they decide not to purchase any product in S_1. However, this will sacrifice the revenue of store 2 because there is no sequential selling to customers in store 2, and the assortment S_2 performs worse than $\tilde{S}_2$ when store 2 cannot take advantage of the interaction with store 1. Therefore, the seller faces a trade-off between increasing the revenue of one store by first offering a smaller assortment then later presenting a larger assortment from the other store, and reducing the revenue of the other store by offering the larger assortment.

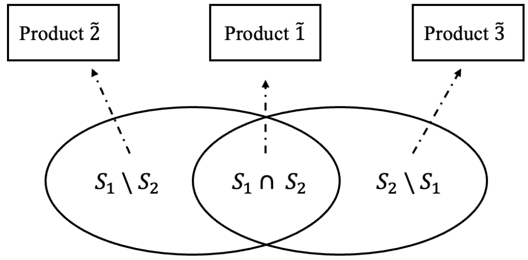


Figure 4 Illustration of the proof idea in Theorem 2

The results of Theorem 2 suggest that the optimal recommendation strategy is to facilitate a two-step selling process for the high-demand store and reduce the sacrifice of the low-demand store. As shown in Example 1, such a differentiated offering strategy could bring benefits to the seller, even when the arrival fractions of customers are the same for the two stores.

Now we present a sketch of the proof for Theorem 2, which can help us understand the rationale behind the results better. Note that when the utility discounts are universally homogeneous, we can simplify problem (5) as follows:

$$\begin{aligned}\max_{S_1,S_2}\ R(S_1,S_2) \triangleq\ & \lambda\frac{W(S_1)}{1+V(S_1)} + \lambda\frac{\beta W(S_2\setminus S_1)}{(1+V(S_1))(1+V(S_1)+\beta V(S_2\setminus S_1))} \\ & + (1-\lambda)\frac{W(S_2)}{1+V(S_2)} + (1-\lambda)\frac{\beta W(S_1\setminus S_2)}{(1+V(S_2))(1+V(S_2)+\beta V(S_1\setminus S_2))},\end{aligned} \tag{6}$$

where $V(S)=\sum_{k\in S} v_k$, $W(S)=\sum_{k\in S} r_k v_k$, and $\beta=\exp(-c)$ with $0<\beta<1$. Note that the cases when $0\le\lambda\le 0.5$ and $0.5\le\lambda\le 1$ are symmetric, so we assume $0.5\le\lambda\le 1$ hereafter. There are three main steps to establish the results in Theorem 2.[6]

Step 1: We show that there exists an optimal solution (S_1^*, S_2^*) to problem (6) such that $S_1^*\subseteq S_2^*$. That is, the store with a lower (higher) demand should offer a larger (smaller) assortment. Before proceeding, we briefly explain the idea behind the proof of the inclusion property. Given any solution (S_1,S_2), the union set of products $S_1\cup S_2$ can be partitioned into three subsets, i.e., $S_1\cap S_2$, $S_1\setminus S_2$, and $S_2\setminus S_1$. As shown in Figure 4, for each subset, we merge all products into one proxy product; specifically, we refer to $S_1\cap S_2$ as product $\tilde{1}$, $S_1\setminus S_2$ as product $\tilde{2}$, and $S_2\setminus S_1$ as product $\tilde{3}$. Now, with these proxy products, we can simply express the assortments of store 1 and store 2 as $S_1=\{\tilde{1},\tilde{2}\}$ and $S_2=\{\tilde{1},\tilde{3}\}$, respectively. Denote the revenues and attraction values of the three proxy products as R_1,R_2,R_3 and V_1,V_2,V_3, respectively. We can show that if $R_2\ge R_3$, then the assortment set $\{\{\tilde{1},\tilde{2}\},\{\tilde{1},\tilde{3}\}\}$ is dominated by either $\{\{\tilde{1}\},\{\tilde{1},\tilde{2},\tilde{3}\}\}$ or $\{\{\tilde{1},\tilde{2}\},\{\tilde{1},\tilde{2},\tilde{3}\}\}$. Otherwise (i.e., if $R_2\le R_3$), the assortment set $\{\{\tilde{1},\tilde{2}\},\{\tilde{1},\tilde{3}\}\}$ is dominated

[6] These three steps also summarize the essence of the proof for Theorem 4, which extends the results in Theorem 2 to the m-store problem where the utility discounts are not universally homogeneous.

by either $\{\{\tilde{1}\},\{\tilde{1},\tilde{2},\tilde{3}\}\}$ or $\{\{\tilde{1},\tilde{3}\},\{\tilde{1},\tilde{2},\tilde{3}\}\}$. Therefore, there must exist an optimal solution in which the inclusion property $S_1 \subseteq S_2$ holds.

Step 2: With $S_1 \setminus S_2 = \emptyset$, problem (6) can be simplified as

$$\max_{S_1,S_2:S_1\subseteq S_2} \lambda\frac{W(S_1)}{1+V(S_1)} + \lambda\frac{\beta W(S_2\setminus S_1)}{(1+V(S_1))(1+V(S_1)+\beta V(S_2\setminus S_1))} + (1-\lambda)\frac{W(S_2)}{1+V(S_2)}. \tag{7}$$

For problem (7), we first analyze the case in which S_2 is fixed, under which problem (7) reduces to the following single-store assortment optimization problem for store 1:

$$\max_{S_1:S_1\subseteq S_2} R(S_1) \triangleq \frac{W(S_1)}{1+V(S_1)} + \frac{\beta W(S_2\setminus S_1)}{(1+V(S_1))(1+V(S_1)+\beta V(S_2\setminus S_1))}. \tag{8}$$

We can show that there exists a revenue-ordered (with respect to products in S_2) assortment that is optimal to problem (8). Based on this result, we prove that there exists an optimal solution (S_1^*, S_2^*) to problem (7) such that S_1^* and S_2^* are both revenue-ordered. The proof idea is similar to the merge operation discussed in Step 1.

Step 3: We show that for the aforementioned optimal solution (S_1^*, S_2^*), the property $S_1^* \subseteq \tilde{S} \subseteq S_2^*$ holds. That is, the store with a lower (higher) demand should offer a larger (smaller) assortment than the optimal assortment in the benchmark. Combining these three steps, we establish the results in Theorem 2.

5.2. m-Store Problem

We now generalize the results in Theorem 2 to the case of m stores, where $m \geq 2$.

THEOREM 3. *Suppose that the utility discounts are universally homogeneous. Assume that* $\lambda_m = \min\{\lambda_1,\ldots,\lambda_m\}$*, then there exists an optimal solution* $(S_1^*, S_2^*, \ldots, S_m^*)$ *to the SQR problem such that* $S_1^* = S_2^* = \cdots = S_{m-1}^* \subseteq \tilde{S} \subseteq S_m^*$ *and* S_1^* *and* S_m^* *are both revenue-ordered.*

Theorem 3 states that when the utility discounts are universally homogeneous, the optimal assortment recommendation strategy with m stores is to offer a larger assortment in the store with the smallest customer arrival fraction and smaller assortments in all the stores. Additionally, the assortments in all other stores should be identical. The rationale behind this result is as follows. In the two-store problem, as shown in Theorem 2, the store with a lower demand should sacrifice its revenue by offering a larger assortment than $\tilde{S}$ to facilitate a two-step selling process for the store with a higher demand. Extending this argument to the problem with m stores, we choose to sacrifice the revenue of store m (because of its lowest demand) and boost the revenue of other stores. Note that because of the universally homogeneous disutility, customers in the first $m-1$ stores will not purchase from each other because they can always purchase from store m, which offers the largest assortment. Hence, it is not necessary to offer different assortments in the first $m-1$ stores. Because

S_1^* and S_m^* are both revenue-ordered with $S_1^* \subseteq S_m^*$, efficient algorithms can be proposed to solve the general SQR problem when the utility discounts are universally homogeneous.

Finally, it is worth noting that the condition in Theorem 3 can be generalized. We say that the utility discounts are *store-wise homogeneous* if $c_{ij}^k = c_i > 0$ holds for any $i, j \in [m], i \neq j$ and $k \in [n]$. We obtain the following results.

THEOREM 4. *Suppose that the utility discounts are store-wise homogeneous. Assume that* $\lambda_m = \min\{\lambda_1, \ldots, \lambda_m\}$ *and* $c_1 \leq c_2 \leq \cdots \leq c_m$, *then there exists an optimal solution* $(S_1^*, S_2^*, \ldots, S_m^*)$ *to the SQR problem such that* $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_{m-1}^* \subseteq \tilde{S} \subseteq S_m^*$ *and* $S_1^*, S_2^*, \ldots, S_m^*$ *are all revenue-ordered.*

Theorem 4 extends the results of Theorem 3 to the scenario where the utility discounts are store-wise homogeneous. The intuition is as follows. First, when $\lambda_m = \min\{\lambda_1, \ldots, \lambda_m\}$, as suggested by Theorem 3, it is more beneficial to offer the largest assortment in store m to facilitate the sequential selling process for other stores. Additionally, when $c_1 \leq c_2 \leq \cdots \leq c_m$, the utility discount of store m's customers is greater than that of customers in the remaining stores[7], which also implies that it is better to offer the largest assortment in store m. Therefore, these two "forces" align in the same direction, which leads to the result that S_m^* is the largest. However, unlike the situation in Theorem 3, the utility discounts in the first $m-1$ stores are not homogeneous. In fact, the selling process is the same among the first $m-1$ stores: if customers are not interested in any product in the current store, they will be offered the products in S_m^*. Note that the objective of offering the products in S_m^* in the second step is to recapture the lost demand in the first step, the effectiveness of which depends on the magnitude of customers' utility discount. The lower the utility discount, the stronger the ability to recapture the lost demand. Therefore, compared with store $m-1$, store 1 should only keep the higher priced products in S_{m-1}^* to leverage its strong ability to recapture the lost demand. This explains why store 1 offers the smallest assortment. In the following example, we show that if the two "forces" act in opposite directions, then the optimal solution may not be revenue-ordered when the utility discounts are store-wise homogeneous.

EXAMPLE 2. Consider a problem instance with two stores and three products with $(r_1, r_2, r_3) = (9, 6, 5)$. The arrival fractions are $(\lambda_1, \lambda_2) = (0.25, 0.75)$. The attraction values are $(v_1, v_2, v_3) = (1, 0.5, 0.5)$. The utility discounts are $c_1 = c_{12}^1 = c_{12}^2 = c_{12}^3 = 0.5$ and $c_2 = c_{21}^1 = c_{21}^2 = c_{21}^3 = 1$. For this problem instance, we have $\lambda_1 < \lambda_2$ and $c_1 < c_2$. One can verify that the optimal solution to problem (5) is given by $(S_1^*, S_2^*) = \{\{1, 3\}, \{1, 2\}\}$, which is not revenue-ordered. □

Before we delve into comparative statics, we discuss how the results of the SQR model can be applied to the omnichannel setting. We have the following result.

[7] This assumption is reasonable given that a faraway store may admit a low demand and lead to a high utility discount for customers in other stores.

PROPOSITION 4. *For the omnichannel setting, i.e., $S_m = \mathcal{N}$, if the utility discounts are store-wise homogeneous and $c_1 \le c_2 \le \cdots \le c_{m-1}$, then there exists an optimal solution $(S_1^*, S_2^*, \ldots, S_{m-1}^*)$ to the SQR problem such that $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_{m-1}^* \subseteq \tilde{S}$ and $S_1^*, S_2^*, \ldots, S_{m-1}^*$ are all revenue-ordered.*

Proposition 4 extends the results in Theorem 4 to the omnichannel setting.[8] It is intuitive that when the utility discounts are store-wise homogeneous, the m-store problem under the omnichannel setting can be reduced to solving a series of two-store problems consisting of a physical store and the online store, as there is no interaction between different physical stores. In addition, the size of the optimal assortment in each physical store decreases when the store-specific utility discount becomes smaller, because in the two-step selling process, the physical store would like to keep the high-priced product only since it can leverage the increasing capability to recapture the lost demand. This also explains why the assortment size in each physical store under the SQR model is smaller than that when each store operates separately, different from the SMR model where the assortment size in each physical store is larger than the benchmark case. We also note that if the utility discounts are not store-wise homogeneous, then the structure of the optimal assortment offered in each physical store could become much more complicated, as indicated by an example in Appendix E.3.

5.3. Comparative Statics

Next, we study the impact of the model parameters, including the arrival fraction and utility discount, on the optimal assortments and revenue. We focus on the scenario where the utility discounts are universally homogeneous. By Theorem 3, it suffices to study the case with two stores. We use $R_Q^*(c)$ and $R_Q^*(\lambda)$ to denote the optimal revenue of problem (6) as a function of c and λ, respectively (recall that $\beta = \exp(-c)$). We first study the effect of the utility discount c on the optimal revenue. Perhaps not surprisingly, $R_Q^*(c)$ is monotonically decreasing in c; that is, the total revenue decreases when the utility discount increases (see Appendix B.8 for a rigorous statement of this result). Recall that the store with a lower demand should sacrifice its revenue by offering a larger assortment than $\tilde{S}$ to facilitate a two-step selling process for the store with a higher demand. The revenue of the store offering a larger assortment is independent of the utility discount c. For the revenue of the other store, as discussed above, the objective of offering a larger assortment in the second step is to recapture the lost demand in the first step, the effectiveness of which depends on the magnitude of customers' utility discount. The lower the utility discount, the stronger the ability to recapture the lost demand, and thus the higher the revenue. In one extreme case where $c \to \infty$, $R_Q^*(c)$ is equivalent to the benchmark where two stores operate separately. In the other extreme case where $c = 0$, the benefit of cross-store selling is the largest.

[8] We also obtain the result in the omnichannel setting when the utility discounts are universally homogeneous. We choose not to present this result since it is very similar to Theorem 3.

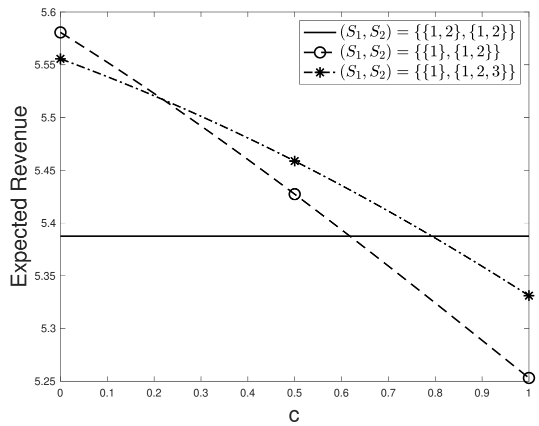
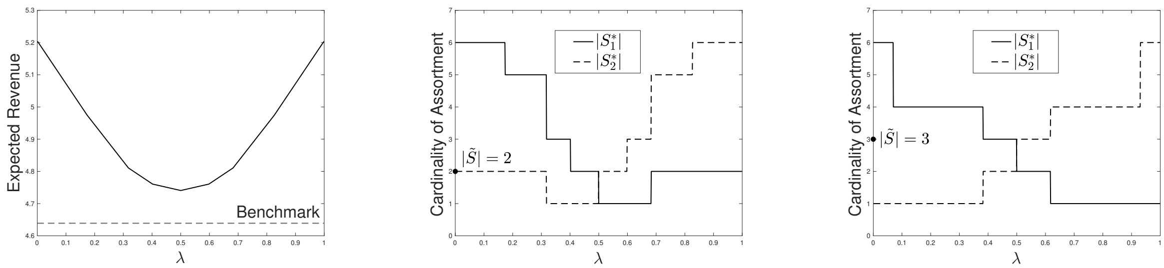


Figure 5 **Revenue of three candidate assortments in Example 3**

We then study the effect of c on the optimal assortments. Intuitively, when c is small, it is reasonable to set the assortments in the two stores differently to take advantage of cross-selling. When c is sufficiently large, the optimal assortments in both stores should be equal to $\tilde{S}$. Therefore, one may expect that as c increases, the cardinality of the smaller (larger) assortment will monotonically increase (decrease), and the optimal assortments will converge to $\tilde{S}$ when c is sufficiently large. However, the following example shows that this may not be the case.

Example 3. Consider a problem with two stores and three products with $(r_1, r_2, r_3) = (7, 6, 5)$. The arrival fractions are $(\lambda_1, \lambda_2) = (0.5, 0.5)$. The attraction values are $(v_1, v_2, v_3) = (1.1, 5.9, 6.3)$. The utility discounts are universally homogeneous and are denoted by c. Figure 5 plots the total revenue of three candidate assortment decisions when the utility discount c increases from 0 to 1.[9] When $c = 1$, the utility discount is large and the benefit of cross-selling is small, and thus the optimal assortments in both stores are $\tilde{S} = \{1, 2\}$, corresponding to the horizontal line. When c decreases to 0.5, the benefit of cross-selling increases, and the optimal assortment set becomes $\{\{1\}, \{1, 2, 3\}\}$, corresponding to the starred line. When c decreases to 0, the optimal assortment set becomes $\{\{1\}, \{1, 2\}\}$, corresponding to the dotted line. Therefore, the cardinality of the larger assortment S_2^* is not monotonically increasing as c decreases from 1 to 0. Note that compared with $\{1, 2\}$, offering $\{1, 2, 3\}$ in S_2 lowers store 2's revenue because $\{1, 2\}$ is the optimal assortment when each store operates independently. Moreover, it yields two counteracting effects on store 1's revenue. One is that offering product 3 can capture store 1 customers who are not interested in products 1 and 2. The other is the cannibalization effect, as product 3 brings a lower revenue than product 2 and attracts customers who may purchase product 2 in the absence of product 3. When c is relatively large (e.g., $c = 0.5$), the positive force dominates the cannibalization effect, making $\{1, 2, 3\}$ better than $\{1, 2\}$. As c decreases to 0, the cannibalization effect becomes more prominent, and thus it is optimal to offer $\{1, 2\}$. $\square$

[9] Other assortment combinations are never optimal for any $0 \leq c \leq 1$ and thus are not plotted.



(a) (b) (c)

Figure 6 Impact of arrival fraction λ on the optimal revenue and assortments of the SQR model.[11]

Next, we study the effect of the arrival fraction λ on the optimal assortments and revenue. We have the following proposition.

PROPOSITION 5. *Denote the optimal solution to problem* (6) *by* (S_1^*, S_2^*).[10] *We have:*

(a) $R_Q^*(\lambda)$ *is decreasing in* λ *when* $\lambda \in [0, 0.5)$ *and increasing in* λ *when* $\lambda \in (0.5, 1]$. *Moreover,* $R_Q^*(\lambda)$ *is a piecewise linear and convex function in* λ.

(b) $|S_1^*|$ *is weakly decreasing in* λ *when* $\lambda \in [0, 0.5)$ *and* $|S_2^*|$ *is weakly increasing when* $\lambda \in (0.5, 1]$.

We illustrate Proposition 5 using Figure 6. Figure 6(a) plots the optimal revenue of problem (6) for different values of λ. As shown, the optimal revenue is increasing as the two stores become more differentiated (in terms of the arrival fraction). Recall that the SQR model offers a larger (smaller) assortment to the store with a lower (higher) demand, which benefits the store with a higher demand while sacrificing the revenue of the other. As the two stores become more differentiated, i.e., λ increases from 0.5 to 1, the benefit for store 1 increases because of the growing demand, while the sacrifice for store 2 decreases because of the declining demand, resulting in an increase in the total expected revenue. In extreme cases (i.e., $\lambda = 0$ or 1), the total revenue is the largest because we only focus on customers in one store. That is, we can fully unlock the benefit of the two-step selling strategy without having to sacrifice any store.

Figures 6(b) and 6(c) plot the cardinalities of the optimal assortments for different values of λ. When $\lambda \in [0, 0.5)$, Figures 6(b) and 6(c) show that $|S_1^*|$ is weakly decreasing, which is consistent with Proposition 5(b). According to Theorem 2, when $\lambda \in [0, 0.5)$, we have $S_2^* \subseteq \tilde{S} \subseteq S_1^*$. As λ increases, the cost of offering a sub-optimal assortment in store 1 (deviating from the optimal assortment $\tilde{S}$ in its own interest) becomes more significant, which explains why $|S_1^*|$ is weakly decreasing and converges to $|\tilde{S}|$. However, $|S_2^*|$ may not be increasing when λ increases within the range $[0, 0.5]$.

[10] If there are multiple solutions at optimality, we choose the solution with the smallest $|S_1^*|$ (if $|S_1^*|$ is the same, pick the one with the smallest $|S_2^*|$) when $\lambda \in [0, 0.5)$ and the solution with the smallest $|S_2^*|$ (if $|S_2^*|$ is the same, pick the one with the smallest $|S_1^*|$) when $\lambda \in (0.5, 1]$.

[11] The problem instances for Figure 6 are described in Appendix E.5.

For example, as λ increases from 0.3 to 0.4, $|S_2^*|$ increases from 1 to 2 in Figure 6(c) but decreases from 2 to 1 in Figure 6(b), while $|S_1^*|$ decreases in both subfigures. Therefore, as the two stores become less differentiated, i.e., as λ increases from 0 to 0.5, the optimal assortment of the low-demand store (which offers a larger assortment) becomes smaller, while the optimal assortment size of the high-demand store (which offers a smaller assortment) may increase or decrease.

6. Comparison of the SMR and SQR Strategies

In Sections 4 and 5, we analyze the structure of the optimal assortment decisions under both SMR and SQR strategies. Now, we compare the two strategies in terms of firm revenue and consumer surplus.[12] We also conduct extensive numerical experiments to investigate how the model parameters affect the revenue comparison.

Theorem 5. *(a) The SQR strategy always leads to higher revenue than the SMR strategy.*

(b) For any given assortment set, the SQR strategy always leads to a lower consumer surplus than the SMR strategy.

As illustrated in Theorem 1, because of the cannibalization effect, the SMR strategy cannot outperform the benchmark where each store operates independently. However, the SQR strategy consistently outperforms the benchmark. The intuition is as follows. Following the logic of Theorem 3, the store with a lower demand should offer a larger assortment to facilitate a two-step selling process for the stores with a higher demand. In contrast to the benchmark where each store operates separately, the high-demand stores offer a smaller assortment that only includes high-priced products. Only if the customer does not make a purchase will the seller offer her the low-priced products (available in the low-demand store). In such a two-step process, the lost demand in the first step is subsequently recaptured by the low-priced product offered in the second step without incurring a cannibalization effect. This is different from the SMR strategy and can significantly increase the revenue of the high-demand stores. On the downside, the low-demand store offers a larger assortment than the benchmark case and cannot implement a two-step selling for itself, resulting in slightly lower revenue than in the benchmark case. Overall, the benefit for the high-demand stores ($m-1$ stores) outweighs the revenue loss of the low-demand store (one single store), especially when m is large. Therefore, the SQR strategy is always better than the benchmark. This explains why the SQR strategy always leads to higher revenue than the SMR strategy. We note that the same conclusion holds under the omnichannel setting.

In general, it is difficult to compare the consumer surplus of the two strategies with the optimal assortments.[13] However, we find that for the same assortment set, the SQR strategy always leads

[12] Due to the page limit, we relegate the full expressions of consumer surplus under both strategies to Appendix C.1.

[13] We give an example in Appendix E.4, where the SQR strategy with the optimal assortments yields a higher consumer surplus than the SMR strategy with the optimal assortments. Thus, a win-win outcome emerges.

α \ λ	0.1	0.2	0.3	0.4	0.5
0.5	3.58%	2.73%	2.08%	1.63%	1.26%
1	3.34%	2.50%	1.91%	1.49%	1.13%
1.5	3.05%	2.30%	1.73%	1.42%	1.12%
2	2.66%	2.08%	1.58%	1.21%	0.99%

Table 1 Effects of arrival fractions and utility discounts on the revenue improvement of SQR over SMR

to a lower consumer surplus than the SMR strategy. Under the SQR strategy, many customers are offered fewer choices in the first step, and they do not have the opportunity to purchase low-priced products in their first purchasing decisions.

We are also interested in how the model parameters affect the revenue improvement of SQR over SMR. We conduct numerical experiments to investigate the effects of the arrival fractions and utility discounts. We restrict our analysis to the case of two stores. We let $m = 2$, $n = 3$, and vary λ (the arrival fraction of store 1) from 0.1 to 0.5 with a step size of 0.1. Moreover, the utility discount c_{ij}^k is randomly sampled from a uniform distribution $U[0, \alpha]$, where α is varied from $\{0.5, 1, 1.5, 2\}$. The revenue r_i is randomly sampled from $U[0, 10]$ and the utility u_i is sampled from $U[0, 5]$. For each combination of parameters (i.e., λ and α), 10,000 problem instances are randomly generated. Table 1 summarizes the average revenue improvement of SQR over SMR. It shows that the revenue improvement increases when the arrival fractions of the two stores are more differentiated or the utility discounts decrease. As the arrival fractions are more differentiated (i.e., λ decreases from 0.5 to 0.1), the advantage of the two-step selling mechanism (first offering S_2 and then offering $S_1 \setminus S_2$ to store 2 customers) becomes more prominent. Recall that the revenue of the SMR strategy is equal to the benchmark, which is independent of λ. Therefore, the revenue improvement of SQR over SMR becomes more significant when the arrival fractions are more differentiated. Meanwhile, as the utility discount increases, the benefit of cross-selling in the SQR strategy decreases, leading to a smaller revenue improvement. This implies that practitioners can enjoy more benefits by adopting the sequential recommendation strategy when the seller operates a large number of stores or when the shipping costs (time, resp.) between different stores are relatively low (short, resp.).

7. Extensions

This section considers three model extensions to check the robustness of our main result in the base model. First, we study the case in which the same product may have different attraction values in different stores. For example, customers who visit the flagship stores downtown may have a higher valuation than those who visit outlet stores. To capture that, we use u_{ik} to denote the utility of purchasing product k in store i and let $v_{ik} = \exp(u_{ik})$. We call this case the *heterogeneous* case. Section 7.1 compares the two strategies under the heterogeneous case. Second, the base model does not impose any capacity constraint. However, in practice, because of the limited space in physical stores, the seller may face capacity constraints when making assortment decisions. Section 7.2 considers the capacitated case of the base model. Third, in Section 7.3, we consider partial recommendation where the seller is allowed to offer a subset of other stores' products, instead of the full product set.

7.1. Heterogeneous Valuation

For the SMR model under the heterogeneous case, the choice probabilities are given by:

$$p_{ik}^M(\boldsymbol{S}) = \begin{cases} \frac{v_{ik}}{1+\sum_{l\in S_i} v_{il} + \sum_{l\in \bar{S}\setminus S_i} v_{il}\exp\left(-c_i^l(\boldsymbol{S})\right)} & \text{if } k \in S_i, \\ \frac{v_{ik}\exp\left(-c_i^k(\boldsymbol{S})\right)}{1+\sum_{l\in S_i} v_{il} + \sum_{l\in \bar{S}\setminus S_i} v_{il}\exp\left(-c_i^l(\boldsymbol{S})\right)} & \text{if } k \in \bar{S}\setminus S_i, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

We define the SMR problem under the heterogeneous case as follows:

$$\max_{\boldsymbol{S}=(S_1,S_2,\ldots,S_m)} \quad \sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k p_{ik}^M(\boldsymbol{S}), \text{ where } p_{ik}^M(\boldsymbol{S}) \text{ is given in (9).} \tag{SMR-heter}$$

Similarly, for the SQR model under the heterogeneous case, we derive:

$$p_{ik}^Q(\boldsymbol{S}) = \begin{cases} \frac{v_{ik}}{1+\sum_{l\in S_i} v_{il}} & \text{if } k \in S_i, \\ \frac{v_{ik}\exp(-c_i^k(\boldsymbol{S}))}{(1+\sum_{l\in S_i} v_{il})(1+\sum_{l\in S_i} v_{il} + \sum_{l\in \bar{S}\setminus S_i} v_{il}\exp(-c_i^l(\boldsymbol{S})))} & \text{if } k \in \bar{S}\setminus S_i, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

We define the SQR problem under the heterogeneous case as follows:

$$\max_{\boldsymbol{S}=(S_1,S_2,\ldots,S_m)} \quad \sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k p_{ik}^Q(\boldsymbol{S}), \text{ where } p_{ik}^Q(\boldsymbol{S}) \text{ is given in (10).} \tag{SQR-heter}$$

A natural question is whether the SQR strategy still garners higher revenue than the SMR strategy under the heterogeneous case. We provide an affirmative answer to this question.

Proposition 6. *The optimal objective value of the SQR-heter problem is greater than that of the SMR-heter problem.*

Proposition 6 extends Theorem 5 to the heterogeneous case and confirms the robustness of our main result. Given the advantage of the SQR strategy under the heterogeneous case, we then study the corresponding assortment optimization problem. According to Proposition 3, the problem is NP-hard in general and thus we are interested in heuristic solutions with good performance. A naive idea is to offer the single-store optimal assortment in each store. We denote the optimal revenue of the SQR-heter problem as R_Q^* and the revenue of the SQR-heter problem when implementing $(\tilde{S}_1,\ldots,\tilde{S}_m)$ as R_Q^H. The following proposition shows the performance of such a heuristic.

Proposition 7. *We have $R_Q^H \ge \frac{1}{2} R_Q^*$ and also identify an instance where the equality holds.*

Proposition 7 states that by offering the single-store optimal assortment in each store, we are guaranteed to collect at least half of the optimal revenue. It also implies that by adopting the SQR strategy, the revenue improvement over the benchmark could be as large as twofold. Our numerical results in Appendix F further indicate that the performance of such a heuristic on the synthesized data can be much better than the theoretical guarantee.

7.2. Capacity Constraints

This section introduces a cardinality constraint on the offered assortment in each store, i.e., $|S_i| \leq C_i$ for each $i = 1, \ldots, m$, where C_i is a positive integer. Then, the SMR and SQR problems with cardinality constraints can be formulated as follows:

$$\begin{aligned} \max_{\boldsymbol{S}=(S_1,S_2,\ldots,S_m)} \quad & R^M(\boldsymbol{S}) = \sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k p_{ik}^M(\boldsymbol{S}) \\ \text{s.t.} \quad & |S_i| \leq C_i \quad \forall\, i = 1, \ldots, m, \end{aligned} \tag{SMR-const}$$

where $p_{ik}^M(\boldsymbol{S})$ is given in (9) and

$$\begin{aligned} \max_{\boldsymbol{S}=(S_1,S_2,\ldots,S_m)} \quad & R^Q(\boldsymbol{S}) = \sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k p_{ik}^Q(\boldsymbol{S}) \\ \text{s.t.} \quad & |S_i| \leq C_i \quad \forall\, i = 1, \ldots, m, \end{aligned} \tag{SQR-const}$$

where $p_{ik}^Q(\boldsymbol{S})$ is given in (10). For tractability, we still assume universally homogeneous utility discounts and propose an integer programming formulation for the SQR-const problem. Note that when the utility discounts are universally homogeneous, the SQR-const problem has the form

$$\begin{aligned} \max_{\boldsymbol{S}=(S_1,S_2,\ldots,S_m)} \quad & R^Q(\boldsymbol{S}) = \sum_{i=1}^m \lambda_i \left(\frac{\sum_{k\in S_i} r_k v_{ik}}{1+\sum_{k\in S_i} v_{ik}} + \frac{\sum_{k\in \bar{S}\setminus S_i} r_k v_{ik}\beta}{(1+\sum_{k\in S_i} v_{ik})(1+\sum_{k\in S_i} v_{ik} + \sum_{k\in \bar{S}\setminus S_i} v_{ik}\beta)} \right) \\ \text{s.t.} \quad & |S_i| \leq C_i \quad \forall\, i = 1, \ldots, m. \end{aligned}$$

Due to the page limit, we relegate the detailed derivation of the integer programming formulation to Appendix G. We note that a similar integer programming formulation can be proposed for the SMR-const problem.

With the integer programming formulations, we conduct numerical experiments to check the robustness of our main result. We set the number of stores $m = 2$ and vary the number of products $n \in \{3, 5, 7\}$ and arrival fraction $\lambda \in \{0.1, 0.3, 0.5\}$. For simplicity, we let $C_1 = C_2 = C$ and vary the value of $C \in \{1, 2, 3\}$. We assume $c_{ij}^k = c$ in this case, where c is randomly sampled from $U[0, 1]$. The revenue r_i is randomly sampled from $U[0, 10]$ and the utility u_{ik} is sampled from $U[0, 5]$. Then, for each combination of parameters (i.e., n, λ, and C), we randomly generate 10,000 problem instances. For each problem instance, we calculate the optimal revenues of the SMR-const problem (denoted by

			Revenue improvement of SQR over SMR			Percentage of SQR over SMR	
			Max	Mean	Min	$R_Q^* > R_M^*$	$R_Q^* \geq R_M^*$
$n=3$	$C=1$	$\lambda=0.1$	36.53%	2.82%	-2.91%	51.23%	97.48%
		$\lambda=0.3$	24.87%	1.03%	-3.22%	31.14%	94.71%
		$\lambda=0.5$	22.45%	0.53%	-2.92%	23.07%	92.40%
	$C=2$	$\lambda=0.1$	41.24%	3.49%	-0.08%	69.29%	99.98%
		$\lambda=0.3$	28.14%	2.35%	-0.01%	59.99%	99.98%
		$\lambda=0.5$	19.90%	1.81%	-0.16%	55.35%	99.98%
	$C=3$	$\lambda=0.1$	35.79%	3.49%	0.00%	69.29%	100.00%
		$\lambda=0.3$	32.23%	2.29%	0.00%	59.16%	100.00%
		$\lambda=0.5$	21.59%	1.84%	0.00%	55.90%	100.00%
$n=5$	$C=1$	$\lambda=0.1$	35.18%	3.15%	-2.14%	64.83%	96.87%
		$\lambda=0.3$	26.29%	1.33%	-3.99%	45.31%	91.70%
		$\lambda=0.5$	19.89%	0.70%	-4.35%	35.43%	88.07%
	$C=2$	$\lambda=0.1$	39.22%	3.90%	-0.12%	83.96%	99.90%
		$\lambda=0.3$	26.62%	2.66%	-0.63%	76.82%	99.96%
		$\lambda=0.5$	21.12%	2.29%	-0.11%	75.33%	99.95%
	$C=3$	$\lambda=0.1$	36.00%	3.89%	0.00%	84.60%	100.00%
		$\lambda=0.3$	31.30%	2.79%	0.00%	77.30%	100.00%
		$\lambda=0.5$	22.30%	2.33%	0.00%	75.29%	100.00%
$n=7$	$C=1$	$\lambda=0.1$	32.94%	2.99%	-2.15%	70.56%	96.29%
		$\lambda=0.3$	27.13%	1.35%	-2.91%	52.90%	90.14%
		$\lambda=0.5$	18.70%	0.74%	-3.53%	43.28%	85.78%
	$C=2$	$\lambda=0.1$	38.67%	3.74%	-0.35%	89.66%	99.91%
		$\lambda=0.3$	23.68%	2.74%	-0.10%	84.96%	99.91%
		$\lambda=0.5$	22.04%	2.24%	-0.12%	83.08%	99.91%
	$C=3$	$\lambda=0.1$	39.03%	3.94%	0.00%	90.18%	100.00%
		$\lambda=0.3$	25.33%	2.76%	0.00%	85.00%	100.00%
		$\lambda=0.5$	19.48%	2.36%	0.00%	83.99%	100.00%

Table 2 Revenue improvement of SQR over SMR with capacity constraints

R_M^*) and SQR-const problem (denoted by R_Q^*), based on which we calculate the revenue improvement of SQR over SMR $(R_Q^* - R_M^*)/R_M^*$. Table 2 reports the maximum, mean, and minimum revenue improvement over all 10,000 problem instances for each combination of parameters. It also reports the percentage of the problem instances in which the SQR strategy outperforms the SMR strategy.

Table 2 shows that the SQR strategy outperforms the SMR strategy in most spaces of the parameter set. In particular, for each combination of parameters, the percentage of the problem instances in which the SQR strategy outperforms the SMR strategy exceeds 99.9% when $C \geq 2$. Not surprisingly, when the imposed cardinality constraints are relatively loose (i.e., when the value of C is larger), the revenue improvement of SQR over SMR is more significant because the two strategies act more similarly to the unconstrained case. Even for the most extreme case where $C = 1$, the corresponding percentage is no less than 90% when the two stores are relatively differentiated (i.e., $\lambda \leq 0.3$). Therefore, our main managerial insight that the SQR strategy performs better than the SMR strategy largely holds in the presence of capacity constraints.

7.3. Partial Recommendation

In the base model, the seller always shows *all* products in other stores to customers either simultaneously or sequentially. In this section, we allow partial recommendations where the seller offers a subset of other stores' products to customers. Specifically, for the SMR strategy, for any determined assortment set $\boldsymbol{S} = (S_1, \ldots, S_m)$, the customer who visits store 1 can simultaneously evaluate the products in S_1 and a subset of other stores' products $\hat{S}_1 \subseteq \bar{S} \setminus S_1$, which is decided by the seller to maximize the expected revenue in store 1. For the SQR strategy, the customer who visits store 1 first evaluates the products in S_1 as before. If she does not purchase, then instead of recommending $\bar{S} \setminus S_1$, the seller is allowed to offer a subset $\hat{S}_1 \subseteq \bar{S} \setminus S_1$ to maximize the expected revenue.

Note that partial recommendation is a more flexible recommendation policy, which leads to revenue at least as large as that in the original full recommendation setting. Then, an important question would be when partial recommendation brings additional benefits to the seller. We have the following result for the SMR strategy.

PROPOSITION 8. *Under the SMR strategy, suppose*

(a) customers' valuations of the same product are homogeneous across different stores, then the optimal partial recommendation strategy equals $(\tilde{S}, \tilde{S}, \ldots, \tilde{S})$, and thus partial recommendation has no additional benefit in that case.

(b) customers' valuations of the same product are heterogeneous across different stores, then the optimal partial recommendation strategy equals $(\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_m)$, where $\tilde{S}_i$ is the single-store optimal assortment in store i.

To understand this result, a key observation is that the optimal revenue under partial recommendation (homogeneous or heterogeneous) is equal to that under the full recommendation by setting some of the c_{ij}^ks to infinity (so that the customer who visits store i cannot view product k in store j). For the homogeneous case, Theorem 1 has shown that for any c_{ij}^k (including $c_{ij}^k = \infty$), the optimal assortment in each store is the same as the single-store optimal assortment. Therefore, the optimal partial recommendation strategy equals $(\tilde{S}, \tilde{S}, \ldots, \tilde{S})$, which is the same as the optimal full recommendation strategy and thus admits no revenue improvement. However, for the heterogeneous case, there could be positive revenue improvement by adopting partial recommendations. In that case, the optimal partial recommendation strategy follows by offering the single-store optimal assortment in each store and not recommending products from other stores (if any). We remark that the revenue achieved by

n \ λ	0.1	0.2	0.3	0.4	0.5
3	0.13% (4.54%)	0.21% (6.93%)	0.26% (10.77%)	0.3% (11.02%)	0.31% (10.99%)
4	0.15% (4.42%)	0.25% (8.76%)	0.32% (10.64%)	0.36% (10.15%)	0.35% (11.80%)
5	0.17% (3.97%)	0.28% (6.22%)	0.35% (9.31%)	0.37% (10.87%)	0.37% (10.47%)
6	0.18% (4.54%)	0.27% (7.00%)	0.33% (9.41%)	0.38% (11.37%)	0.38% (12.67%)
7	0.17% (4.83%)	0.26% (8.46%)	0.32% (9.88%)	0.37% (10.44%)	0.38% (10.34%)

Table 3 Summarized statistics of the performance of partial recommendation under the SMR strategy.

the optimal partial recommendation strategy equals the weighted summation of each store's optimal single-store revenue, which is still dominated by the SQR strategy. That said, our main result that the SQR strategy outperforms the SMR strategy is robust against partial recommendation.

We then conduct numerical experiments to examine the magnitude of revenue improvement brought by partial recommendations under the SQR strategy. We set the number of stores $m = 2$ and vary the number of products $n \in \{3,4,5,6,7\}$ and arrival fraction $\lambda \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$. The utility discount c_{ij}^k is randomly sampled from $U[0,1]$. Similar to the settings in Section 7.2, the revenue r_i is randomly sampled from $U[0,10]$ and the utility u_{ik} is sampled from $U[0,5]$. For each combination of parameters (i.e., n, λ, and C), we randomly generate 10,000 problem instances.

Table 3 presents the performance of partial recommendation strategies relative to full recommendation under the SMR strategy. The number reported outside (inside) the bracket indicates the average (maximum) revenue improvement of the partial recommendation strategy against the full recommendation. For all combinations of parameters, the average revenue improvement is below 0.5%. One can thus conclude that when partial recommendation is allowed, the potential revenue improvement for the SMR strategy is not significant.

We then study partial recommendations under the SQR strategy. Suppose that customers' valuations are heterogeneous. The seller then aims to solve the following bi-level problem:

$$\max_{\boldsymbol{S}} \sum_{i=1}^{m} \lambda_i \left(\frac{\sum_{k\in S_i} r_k v_{ik}}{1+\sum_{l\in S_i} v_{il}} + \max_{\hat{S}_i \subseteq \bar{S}\setminus S_i} \frac{\sum_{k\in \hat{S}_i} r_k v_{ik} \exp(-c_i^k(\boldsymbol{S}))}{(1+\sum_{l\in S_i} v_{il})(1+\sum_{l\in S_i} v_{il} + \sum_{l\in \hat{S}_i} v_{il}\exp(-c_i^l(\boldsymbol{S})))} \right). \quad (11)$$

In the outer maximization, the seller decides the set of assortments $\boldsymbol{S}$ as before. Then, for a given $\boldsymbol{S}$, the seller needs to solve the following problem for each store i by deciding the optimal assortment to offer to customers who are not interested in the products in S_i:

$$\max_{\hat{S}_i \subseteq \bar{S}\setminus S_i} \frac{\sum_{k\in \hat{S}_i} r_k v_{ik} \exp(-c_i^k(\boldsymbol{S}))}{(1+\sum_{l\in S_i} v_{il})(1+\sum_{l\in S_i} v_{il} + \sum_{l\in \hat{S}_i} v_{il}\exp(-c_i^l(\boldsymbol{S})))}. \quad (12)$$

Note that the optimal solution to problem (12) must be revenue-ordered since it can be reduced to the single-store problem under a regular MNL model. However, it is generally difficult to determine the optimal solution to (11) even when we restrict our attention to the case where the utility discounts

n \ λ	0.1	0.2	0.3	0.4	0.5
3	99.92% (0.15%)	99.96% (0.43%)	99.97% (0.28%)	99.93% (0.12%)	99.95% (0.94%)
4	99.93% (0.32%)	99.88% (0.30%)	99.91% (0.21%)	99.95% (0.55%)	99.93% (0.33%)
5	99.87% (0.16%)	99.86% (0.60%)	99.88% (1.20%)	99.89% (0.22%)	99.87% (0.34%)
6	99.81% (0.36%)	99.83% (0.48%)	99.86% (0.41%)	99.90% (0.36%)	99.77% (0.67%)
7	99.87% (0.34%)	99.78% (0.60%)	99.82% (0.69%)	99.85% (0.27%)	99.82% (0.21%)

Table 4 Summarized statistics of the performance of partial recommendation under the SQR strategy.

are universally homogeneous. In this regard, we also conduct numerical experiments to examine the magnitude of revenue improvement brought by partial recommendations under the SQR strategy. The parameter settings are the same as that of the SMR strategy.

Table 4 presents the performance of partial recommendation strategies relative to full recommendation. The number reported outside the bracket indicates the percentage of problem instances where partial recommendation failed to improve revenue. Notably, this "no additional benefit" rate is 99.7% or higher, suggesting that even with heterogeneous valuation, the seller cannot improve revenues by employing a partial recommendation strategy in the vast majority of cases. For the limited instances where partial recommendation did provide some benefit, the maximum revenue improvement is recorded in the bracketed value. However, the gains appear to be quite small. Collectively, these numerical results partially support that sellers can largely rely on full recommendation strategies in practice, with no need to heavily prioritize the potential for marginal revenue improvements from partial recommendation. The overwhelming lack of incremental benefit from partial recommendation simplifies the decision-making process for sellers in this context.

8. Conclusion

Motivated by the prevalence of operating multiple stores in the retail industry, we study a multi-store assortment planning problem under two strategies, i.e., simultaneous and sequential offering strategies. We show that under the simultaneous strategy, the seller should offer the optimal assortment when each store operates separately, which implies that coordination between different stores cannot benefit the seller. However, if the seller adopts the sequential strategy, the revenue improvement relative to operating each store separately could be significant. We analyze the structural property of the optimal assortment under the sequential strategy. We find that under mild conditions, the optimal assortment is revenue-ordered for each store, while the store with a lower (higher, resp.) demand should offer a larger (smaller, resp.) assortment to facilitate the sequential selling process. We also study the impact of model parameters on the optimal assortments and revenue. Finally, we consider three extensions and study the joint assortment and pricing problem under both strategies, and we find that our main results are robust in general.

One may wonder why some retailing industries still adopt the SMR strategy since the SQR strategy performs better. We provide two possible reasons. First, as discussed in Section 7.2, when capacity constraints are imposed, the SMR strategy can lead to higher revenue than the SQR strategy in some cases. Additionally, according to Section 6, for the same assortment set, the SMR strategy always generates a higher consumer surplus than the SQR strategy. That said, under the SMR strategy, the customer is better off and thus is more likely to participate in future purchases. Second, the no-purchase decision of a customer is not easily observed in some cases, which may cause some difficulty in implementing the SQR strategy. For instance, in the example of beacon technology we mentioned earlier, the utilization of BLE may not be accurate enough to detect the customer's instant intention to leave, and it also runs the risk that the customer may disregard those notifications. In contrast, it is much more flexible to implement the SMR strategy since it does not require knowing the exact time when the no-purchase decision is made. To summarize, although the main managerial insight is that the SQR strategy outperforms the SMR strategy in terms of revenue, we do not rule out the potential benefits of the SMR strategy in other aspects, which may explain why it is still adopted in some businesses.

There are several directions for future research. First, we analyze the performance of the two offering strategies for a monopolistic seller. An immediate follow-up would be to consider the impact of competition on the performance of the two strategies. Second, it would be interesting to study the optimal assortment decision when the utility discounts follow a more general structure under the SQR strategy. Lastly, our paper assumes exogenous arrival rates. In practice, customers may update their arrival probabilities based on the assortments in different stores and the seller can adjust the assortment recommendations dynamically. It would be worthwhile to study the optimal recommendation strategy in this dynamic environment.

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Online Appendix to "Simultaneous vs Sequential: Optimal Assortment Recommendation in Multi-Store Retailing"

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Here we only provide proofs for main theorems and propositions. Results marked with "($\star\star\star$)" can be found in the full version of the paper via https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4592172.

Appendix A: Proofs for Section 4

We restate store 1's optimization problem discussed in Section 4 here for clarity.

$$\max_{S_1\subseteq\mathcal{N}} \quad R(S_1,\boldsymbol{c}) \triangleq \frac{\sum_{k\in S_1} r_k v_k + \sum_{k\in S_2\setminus S_1} r_k v_k\cdot\exp\left(-c^k\right)}{1+\sum_{k\in S_1} v_k + \sum_{k\in S_2\setminus S_1} v_k\cdot\exp\left(-c^k\right)}. \tag{A.1}$$

Given $\boldsymbol{c}$, we denote the optimal solution to (A.1) by $S_M^*(\boldsymbol{c})$ and the optimal objective value of (A.1) by $R_M^*(\boldsymbol{c})$, respectively. When there are multiple optimal assortments, to simplify the analysis, we focus on the optimal assortment that is revenue-ordered with the least elements (the existence of such an assortment will be confirmed by Proposition A.1). For instance, if $\{1,\dots,k\}$ and $\{1,\dots,k+1\}$ are both optimal to problem (A.1), then we let $S_M^*(\boldsymbol{c}) = \{1,\dots,k\}$.

A.1. Proof of Theorem 1

To prove Theorem 1, we need the following two propositions.

Proposition A.1. *For any given $\boldsymbol{c}$, there must exist a revenue-ordered assortment that is optimal to (A.1). Moreover, we have $r_i > R_M^*(\boldsymbol{c})$ if and only if $i\in S_M^*(\boldsymbol{c})$.*

The proof of Proposition A.1 is in Appendix A.2. We then study the effect of the utility discount factor. Define $\boldsymbol{c}\geq\tilde{\boldsymbol{c}}$ if $c^j\geq\tilde{c}^j$ for any $j\in[n]$ and there exists at least one i such that $c^i>\tilde{c}^i$. We have the following proposition.

Proposition A.2. *If $\boldsymbol{c}\geq\tilde{\boldsymbol{c}}$, we have $R_M^*(\boldsymbol{c})\geq R_M^*(\tilde{\boldsymbol{c}})$.*

The proof of Proposition A.2 is in Appendix A.3. We then offer a formal proof of Theorem 1.

Proof of Theorem 1: For any assortment set $\boldsymbol{S}=(S_1,S_2,\dots,S_m)$, we denote $S_{\bar{2}} = S_2\cup S_3\cup\cdots\cup S_m$ and $h^k = \min_{j\in\{l\mid k\in S_l^*, l=2,\dots,m\}} c_{1j}^k$. We then consider the following assortment optimization problem under $S_{\bar{2}}$ and $\boldsymbol{h}=(h^1,h^2,\dots,h^n)$:

$$\max_{S_1\subseteq\mathcal{N}} \quad R(S_1,\boldsymbol{h}) \triangleq \frac{\sum_{k\in S_1} r_k v_k + \sum_{k\in S_{\bar{2}}\setminus S_1} r_k v_k\cdot\exp\left(-h^k\right)}{1+\sum_{k\in S_1} v_k + \sum_{k\in S_{\bar{2}}\setminus S_1} v_k\cdot\exp\left(-h^k\right)}. \tag{A.2}$$

Notice that S_1 is only a feasible solution to problem (A.2). Thus, we must have $R(S_1,\boldsymbol{h})\leq R_M^*(\boldsymbol{h})$ where $R_M^*(\boldsymbol{h})$ denotes the optimal objective value of (A.2). Also, due to Proposition A.2, we must have $R_M^*(\boldsymbol{h})\leq R_M^*(\tilde{\boldsymbol{h}})$ where $\tilde{\boldsymbol{h}}=(\infty,\dots,\infty)$. Therefore, we have $R(S_1,\boldsymbol{h})\leq R_M^*(\tilde{\boldsymbol{h}})$. We notice that $R(S_1,\boldsymbol{h})$ is the expected revenue of store 1 under the solution $\boldsymbol{S}$. We also notice that the solution $(\tilde{S},\tilde{S},\dots,\tilde{S})$ has its expected revenue for each store equal to $R_M^*(\tilde{\boldsymbol{h}})$. Thus, the revenue obtained in store 1 under $\boldsymbol{S}$ must be less than that under $(\tilde{S},\dots,\tilde{S})$. Repeat this argument for all stores, we are able to prove Theorem 1. ∎

A.2. Proof of Proposition A.1 (⋆⋆⋆)

A.3. Proof of Proposition A.2 (⋆⋆⋆)

A.4. Proof of Proposition 2

Proof: The proof is quite similar to that of Theorem 1. We first denote the optimal solution to the following assortment optimization problem as $\hat{S}$:

$$\max_{S\subseteq\mathcal{N}} \quad \frac{\sum_{k\in S} r_k v_k + \sum_{k\in\mathcal{N}\setminus S} r_k v_k \cdot \exp\left(-c^k\right)}{1+\sum_{k\in S} v_k + \sum_{k\in\mathcal{N}\setminus S} v_k \cdot \exp\left(-c^k\right)}. \tag{A.3}$$

Due to Proposition A.1, we know that $\hat{S}$ is revenue-ordered. For any assortment set $\boldsymbol{S}=(S_1,S_2,\dots,S_{m-1},\mathcal{N})$, we denote $h^k = \min_{j\in\{l|k\in S_l, l=2,\dots,m\}} c_{1j}^k$ and consider the following problem:

$$\max_{S_1\subseteq\mathcal{N}} \quad R(S_1,\boldsymbol{h}) \triangleq \frac{\sum_{k\in S_1} r_k v_k + \sum_{k\in\mathcal{N}\setminus S_1} r_k v_k \cdot \exp\left(-h^k\right)}{1+\sum_{k\in S_1} v_k + \sum_{k\in\mathcal{N}\setminus S_1} v_k \cdot \exp\left(-h^k\right)}. \tag{A.4}$$

Clearly, we have $R(S_1,\boldsymbol{h}) \le R_M^*(\boldsymbol{h})$ where $R_M^*(\boldsymbol{h})$ denotes the optimal objective value of (A.4). Also, due to Proposition B.2, we must have $R_M^*(\boldsymbol{h}) \le R_M^*(\tilde{\boldsymbol{h}})$ where $\tilde{\boldsymbol{h}} = (c^1,c^2,\dots,c^k)$. Therefore, we have $R(S_1,\boldsymbol{h}) \le R_M^*(\tilde{\boldsymbol{h}})$. We again notice that $R(S_1,\boldsymbol{h})$ is the expected revenue of store 1 under the solution $\boldsymbol{S}$. We also notice that the solution $(\hat{S},\hat{S},\dots,\hat{S},\mathcal{N})$ has its expected revenue for each store equal to $R_M^*(\tilde{\boldsymbol{h}})$. Thus, the revenue obtained in store 1 under $\boldsymbol{S}$ must be less than that under $(\hat{S},\hat{S},\dots,\hat{S},\mathcal{N})$. Repeat this argument for all stores, we have proved that the optimal solution should be $(\hat{S},\hat{S},\dots,\hat{S},\mathcal{N})$. Moreover, we know that $R_M^*(\tilde{\boldsymbol{h}}) \le R_M^*((\infty,\dots,\infty))$, which is the optimal expected revenue of the single-store problem. Then, based on Proposition A.1, we must have that $|\hat{S}| \ge |\tilde{S}|$. ■

Appendix B: Proofs for Section 5

B.1. Proof of Theorem 2

Proof: Theorem 2 can be viewed as a special case of Theorem 4, thus the proof is omitted here. ■

B.2. Proof of Theorem 3

Proof: Theorem 3 can also be viewed as a special case of Theorem 4. The only thing we need to show is that when $c_1=c_2=\cdots=c_m=c$, we must have $S_1^*=S_2^*=\cdots=S_{m-1}^*$. To see this, we first define $\beta=\exp(-c)$ with $\beta\in(0,1)$. By Theorem 4, we know that there exists an optimal solution $(S_1^*,S_2^*,\dots,S_m^*)$ such that $S_i^*\subseteq S_m^*$ for all $i=1,2,\dots,m-1$. We can then express the total expected revenue from all m stores under this solution as:

$$\begin{aligned} R(S_1^*,\dots,S_m^*) &= \sum_{i=1}^{m-1}\lambda_i\left(\frac{W(S_i^*)}{1+V(S_i^*)}+\frac{W(S_m^*\setminus S_i^*)\cdot\beta}{(1+V(S_i^*))(1+V(S_i^*)+V(S_m^*\setminus S_i^*)\cdot\beta)}\right)+\lambda_m\frac{W(S_m^*)}{1+V(S_m^*)} \\ &= \sum_{i=1}^{m-1}\lambda_i\left(\frac{W(S_i^*)}{1+V(S_i^*)}+\frac{W(S_m^*\setminus S_i^*)\cdot\beta}{(1+V(S_i^*))(1+V(S_i^*)+V(S_m^*\setminus S_i^*)\cdot\beta)}+\frac{\lambda_m}{1-\lambda_m}\frac{W(S_m^*)}{1+V(S_m^*)}\right). \end{aligned}$$

We denote the optimal solution and the optimal objective value of the following problem as $(S^{'},S^{''})$ and R^* respectively. We know that $S^{'}$ and $S^{''}$ are both revenue-ordered and $S^{'}\subseteq S^{''}$.

$$\begin{aligned} \max_{S_1,S_2} \quad & (1-\lambda_m)\frac{W(S_1)}{1+V(S_1)}+(1-\lambda_m)\frac{\beta W(S_2\setminus S_1)}{(1+V(S_1))(1+V(S_1)+\beta V(S_2\setminus S_1))} \\ & +\lambda_m\frac{W(S_2)}{1+V(S_2)}+\lambda_m\frac{\beta W(S_1\setminus S_2)}{(1+V(S_2))(1+V(S_2)+\beta V(S_1\setminus S_2))}. \end{aligned}$$

Then, we have $R(S_1^*, \ldots, S_m^*) \leq R^*$ and the inequality holds as equality when we choose the solution as $(S^{'}, S^{'}, \ldots, S^{'}, S^{''})$. Therefore, all of the optimal solutions of the SQR problem will be dominated by a solution of $(S^{'}, S^{'}, \ldots, S^{'}, S^{''})$, which leads to the conclusion that $(S^{'}, S^{'}, \ldots, S^{'}, S^{''})$ is indeed an optimal solution. ■

B.3. Proof of Theorem 4

The proof of Theorem 4 consists of three steps. In this section, we list the results for each step and prove them in the following subsections.

Step 1:

In step 1, we first prove the following proposition.

PROPOSITION B.1. *Suppose the utility discounts are store-wise homogeneous. Assume $\lambda_m = \min\{\lambda_1, \ldots, \lambda_m\}$ and $c_m = \max\{c_1, \ldots, c_m\}$, then there exists an optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$ to the SQR problem such that $S_i^* \subseteq S_m^*$ for all $i = 1, 2, \ldots, m-1$.*

Based on Proposition B.1, we prove the following lemma.

LEMMA B.1. *Suppose the utility discounts are store-wise homogeneous. Assume $\lambda_m = \min\{\lambda_1, \ldots, \lambda_m\}$ and $c_1 \leq c_2 \leq \cdots \leq c_m$, then there exists an optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$ to the SQR problem such that $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_m^*$.*

Step 2:

In step 2, we first prove the following lemma.

LEMMA B.2. *There exists a revenue-ordered (within the products in S_2) assortment S_1^* such that it is optimal to problem* (8).

Using the result of Lemma B.2, we prove the following proposition.

PROPOSITION B.2. *Suppose the utility discounts are store-wise homogeneous. Assume $\lambda_m = \min\{\lambda_1, \ldots, \lambda_m\}$ and $c_1 \leq c_2 \leq \cdots \leq c_m$, then there exists an optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$ to the SQR problem such that $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_m^*$ and $S_1^*, S_2^*, \ldots, S_m^*$ are all revenue-ordered.*

Step 3:

Based on the results of the first two steps, we prove Theorem 4.

Restate of Theorem 4: *Suppose the utility discounts are store-wise homogeneous. Assume $\lambda_m = \min\{\lambda_1, \ldots, \lambda_m\}$ and $c_1 \leq c_2 \leq \cdots \leq c_m$, then there exists an optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$ for the SQR problem such that $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_{m-1}^* \subseteq \tilde{S} \subseteq S_m^*$ and $S_1^*, S_2^*, \ldots, S_m^*$ are all revenue-ordered.*

B.4. Proofs in Step 1

In this section, we prove Proposition B.1 and Lemma B.1 in Step 1. To prove Proposition B.1, we need the following lemma.

LEMMA B.3. ($\star\star\star$) *Consider an instance of the SQR problem, where there are three stores ($m = 3$) and four products ($n = 4$). The revenues and the attraction values of the products are denoted as (R_1, R_2, R_3, R_4) and (V_1, V_2, V_3, V_4) respectively. We set $\lambda_1 = \lambda_2 = 1/2$ and $\lambda_3 = 0$. We assume the utility discounts are store-wise homogeneous and use $c_1, c_2, c_3 > 0$ to denote them. We also assume $R_2 \geq R_3$ and $c_1 \leq c_2$. Under this problem instance, we denote three recommendation strategies as $\boldsymbol{S} = \{\{1,2\},\{1,3\},\{4\}\}$, $\boldsymbol{S_1} = \{\{1\},\{1,2,3\},\{4\}\}$ and $\boldsymbol{S_2} = \{\{1,2\},\{1,2,3\},\{4\}\}$. Then, we must have $R(\boldsymbol{S}) \leq \max(R(\boldsymbol{S_1}), R(\boldsymbol{S_2}))$.*

Utilizing the result of Lemma B.3, we prove Proposition B.1 as follows.

Proof of Proposition B.1: We prove this argument by contradiction. Without loss of generality, we suppose that $S_1^* \setminus S_m^* \neq \emptyset$. The total expected revenue generated by store 1 and store m can be written as:

$$R(S_1^*, S_m^*) = \lambda_1 \left(\frac{W(S_1^*)}{1+V(S_1^*)} + \frac{W(\bar{S} \setminus S_1^*) \cdot \beta_1}{(1+V(S_1^*))(1+V(S_1^*)+V(\bar{S} \setminus S_1^*) \cdot \beta_1)} \right) + \lambda_m \left(\frac{W(S_m^*)}{1+V(S_m^*)} + \frac{W(\bar{S} \setminus S_m^*) \cdot \beta_m}{(1+V(S_m^*))(1+V(S_m^*)+V(\bar{S} \setminus S_m^*) \cdot \beta_m)} \right),$$

where we denote $\bar{S} = S_1^* \cup S_2^* \cup \cdots \cup S_m^*$. We then define the following quantities:

$$R_1 = \frac{W(S_1^* \cap S_m^*)}{V(S_1^* \cap S_m^*)}, R_2 = \frac{W(S_1^* \setminus S_m^*)}{V(S_1^* \setminus S_m^*)}, R_3 = \frac{W(S_m^* \setminus S_1^*)}{V(S_m^* \setminus S_1^*)}, R_4 = \frac{W(\bar{S} \setminus (S_1^* \cup S_m^*))}{V(\bar{S} \setminus (S_1^* \cup S_m^*))},$$
$$V_1 = V(S_1^* \cap S_m^*), V_2 = V(S_1^* \setminus S_m^*), V_3 = V(S_m^* \setminus S_1^*), V_4 = V(\bar{S} \setminus (S_1^* \cup S_m^*)).$$

We can then reformulate the expression of $R(S_1^*, S_m^*)$ as follows:

$$R(S_1^*, S_m^*) = \lambda_1 \frac{R_1V_1 + R_2V_2}{1+V_1+V_2} + \lambda_1 \frac{\beta_1(R_3V_3 + R_4V_4)}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)} + \lambda_m \frac{R_1V_1 + R_3V_3}{1+V_1+V_3} + \lambda_m \frac{\beta_m(R_2V_2 + R_4V_4)}{(1+V_1+V_3)(1+V_1+\beta_mV_2+V_3+\beta_mV_4)} = \lambda_1\pi_1 + \lambda_m\pi_m,$$

where we denote

$$\pi_1 = \frac{R_1V_1 + R_2V_2}{1+V_1+V_2} + \frac{\beta_1(R_3V_3 + R_4V_4)}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)}.$$
$$\pi_m = \frac{R_1V_1 + R_3V_3}{1+V_1+V_3} + \frac{\beta_m(R_2V_2 + R_4V_4)}{(1+V_1+V_3)(1+V_1+\beta_mV_2+V_3+\beta_mV_4)}.$$

We then consider the following two cases:

Case 1: $R_2 \geq R_3$

We denote $\boldsymbol{S_1} = (S_1^* \cap S_m^*, S_1^* \cup S_m^*)$ and express the total expected revenue generated by store 1 and store m under $\boldsymbol{S_1}$ as follows:

$$R(\boldsymbol{S_1}) = \lambda_1 \frac{R_1V_1}{1+V_1} + \lambda_1 \frac{\beta_1(R_2V_2 + R_3V_3 + R_4V_4)}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)} + \lambda_m \frac{R_1V_1 + R_2V_2 + R_3V_3}{1+V_1+V_2+V_3} + \lambda_m \frac{\beta_mR_4V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_mV_4)} = \lambda_1\pi_1^1 + \lambda_m\pi_m^1,$$

where we denote

$$\pi_1^1 = \frac{R_1V_1}{1+V_1} + \frac{\beta_1(R_2V_2 + R_3V_3 + R_4V_4)}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)}.$$
$$\pi_m^1 = \frac{R_1V_1 + R_2V_2 + R_3V_3}{1+V_1+V_2+V_3} + \frac{\beta_mR_4V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_mV_4)}.$$

We denote $\boldsymbol{S_2} = (S_1^*, S_1^* \cup S_m^*)$ and express the total expected revenue generated by store 1 and store m under $\boldsymbol{S_2}$ as follows:

$$\begin{aligned}R(\boldsymbol{S_2}) = \lambda_1 \frac{R_1V_1 + R_2V_2}{1+V_1+V_2} + \lambda_1 \frac{\beta_1(R_3V_3 + R_4V_4)}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)} \\ + \lambda_m \frac{R_1V_1+R_2V_2+R_3V_3}{1+V_1+V_2+V_3} + \lambda_m \frac{\beta_m R_4V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_mV_4)} = \lambda_1\pi_1^2 + \lambda_m\pi_m^2,\end{aligned}$$

where we denote

$$\begin{aligned}\pi_1^2 = \frac{R_1V_1+R_2V_2}{1+V_1+V_2} + \frac{\beta_1(R_3V_3+R_4V_4)}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)}. \\ \pi_m^2 = \frac{R_1V_1+R_2V_2+R_3V_3}{1+V_1+V_2+V_3} + \frac{\beta_mR_4V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_mV_4)}.\end{aligned}$$

Based on the result of Lemma B.3, we must have:

$$\max(\pi_1^1 + \pi_m^1, \pi_1^2 + \pi_m^2) \geq \pi_1 + \pi_m.$$

We first notice that we must have $\pi_m^1 = \pi_m^2 \leq \pi_m$. Otherwise, we should have $R(S_1^*, S_m^*) < R(S_1^*, S_1^* \cup S_m^*)$ and note that by replacing the assortment in store m with $S_1^* \cup S_m^*$, the total expected revenue generated by the stores other than 1 and m will not change since the utility discounts are store-wise homogeneous. Then, the total expected revenue generated by all stores will be strictly higher if we make the replacement, which is contradictory to the optimality of $(S_1^*, \dots, S_m^*)$. Then, if $\pi_1^1 + \pi_m^1 \geq \pi_1 + \pi_m$, we must have $\pi_1^1 \geq \pi_1$. Notice that $\lambda_1(\pi_1^1 - \pi_1) \geq \lambda_m(\pi_1^1 - \pi_1) \geq \lambda_m(\pi_m - \pi_m^1)$ and this implies $R(\boldsymbol{S_1}) \geq R(\boldsymbol{S})$. By the same reasoning, we can also imply that $R(\boldsymbol{S_2}) \geq R(\boldsymbol{S})$ if $\pi_1^2 + \pi_m^2 \geq \pi_1 + \pi_m$. To conclude, we have $\max(R(\boldsymbol{S_1}), R(\boldsymbol{S_2})) \geq R(\boldsymbol{S})$ in this case. Note that under $\boldsymbol{S}$, $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$, the total expected revenue generated by the stores other than 1 and m are the same. Therefore, the optimal solution $(S_1^*, \dots, S_m^*)$ is dominated by one of the replacements (replace $\boldsymbol{S}$ with $\boldsymbol{S_1}$ or replace $\boldsymbol{S}$ with $\boldsymbol{S_2}$), which is enough to prove the desired result.

Case 2: $R_2 \leq R_3$

The proof for this case is largely the same as the proof for the first case. The only difference is that we now should let $\boldsymbol{S_2} = (S_m^*, S_1^* \cup S_m^*)$ in this case, and the result follows. ■

Based on Proposition B.1, we prove Lemma B.1 as follows.

Proof of Lemma B.1: We first define $\beta_i = \exp(-c_i)$ with $\beta_i \in (0,1)$. By Proposition B.1, we know that there exists an optimal solution $(S_1^*, S_2^*, \dots, S_m^*)$ such that $S_i^* \subseteq S_m^*$ for all $i = 1, 2, \dots, m-1$. We can then express the total expected revenue from all m stores under this solution as:

$$\begin{aligned}R(S_1^*, \dots, S_m^*) &= \sum_{i=1}^{m-1} \lambda_i \left(\frac{W(S_i^*)}{1+V(S_i^*)} + \frac{W(S_m^* \setminus S_i^*) \cdot \beta_i}{(1+V(S_i^*))(1+V(S_i^*)+V(S_m^* \setminus S_i^*) \cdot \beta_i)} \right) + \lambda_m \frac{W(S_m^*)}{1+V(S_m^*)} \\ &= \sum_{i=1}^{m-1} \lambda_i \left(\frac{W(S_i^*)}{1+V(S_i^*)} + \frac{W(S_m^* \setminus S_i^*) \cdot \beta_i}{(1+V(S_i^*))(1+V(S_i^*)+V(S_m^* \setminus S_i^*) \cdot \beta_i)} + \frac{\lambda_m}{1-\lambda_m} \frac{W(S_m^*)}{1+V(S_m^*)} \right).\end{aligned}$$

Suppose S_m^* is fixed, we know that S_i^* must be the optimal solution to the following problem:

$$\max_{S_i : S_i \subseteq S_m^*} \quad \frac{W(S_i)}{1+V(S_i)} + \frac{\beta_i W(S_m^* \setminus S_i)}{(1+V(S_i))(1+V(S_i)+\beta_i V(S_m^* \setminus S_i))}. \tag{B.1}$$

We then prove the following claim:

Claim: Suppose the optimal solution to problem (B.1) is unique, then $|S_i^*|$ must be weakly decreasing in β_i.

Proof of the claim: We prove this claim by contradiction. For $0 \le \beta'' \le \beta'$, denote the optimal solution to problem (B.1) when $\beta_i = \beta'$ and $\beta_i = \beta''$ as S' and S'', respectively. We assume $|S'| > |S''|$. Due to the optimality and uniqueness of S' and S'', we must have:

$$\frac{W(S')}{1+V(S')} + \frac{\beta' W(S_2 \setminus S')}{(1+V(S'))(1+V(S')+\beta' V(S_2 \setminus S'))} > \frac{W(S'')}{1+V(S'')} + \frac{\beta' W(S_2 \setminus S'')}{(1+V(S''))(1+V(S'')+\beta' V(S_2 \setminus S''))},$$

and

$$\frac{W(S'')}{1+V(S'')} + \frac{\beta'' W(S_2 \setminus S'')}{(1+V(S''))(1+V(S'')+\beta'' V(S_2 \setminus S''))} > \frac{W(S')}{1+V(S')} + \frac{\beta'' W(S_2 \setminus S')}{(1+V(S'))(1+V(S')+\beta'' V(S_2 \setminus S'))},$$

which then leads to the following inequality:

$$\begin{aligned} &\frac{W(S_2 \setminus S')}{1+V(S')} \left(\frac{\beta'}{1+V(S')+\beta' V(S_2 \setminus S')} - \frac{\beta''}{1+V(S')+\beta'' V(S_2 \setminus S')} \right) \\ &\quad > \frac{W(S_2 \setminus S'')}{1+V(S'')} \left(\frac{\beta'}{1+V(S'')+\beta' V(S_2 \setminus S'')} - \frac{\beta''}{1+V(S'')+\beta'' V(S_2 \setminus S'')} \right). \end{aligned} \tag{B.2}$$

We then consider the following set function in S:

$$\begin{aligned} f(S) &= \frac{W(S_2 \setminus S)}{1+V(S)} \left(\frac{\beta'}{1+V(S)+\beta' V(S_2 \setminus S)} - \frac{\beta''}{1+V(S)+\beta'' V(S_2 \setminus S)} \right) \\ &= \frac{(\beta' - \beta'') W(S_2 \setminus S)}{(1+V(S)+\beta' V(S_2 \setminus S))(1+V(S)+\beta'' V(S_2 \setminus S))}. \end{aligned}$$

It is then easy to see that when $S'' \subseteq S'$, we must have $f(S') \le f(S'')$. According to Lemma B.2, we know that S' and S'' are both revenue-ordered. Additionally, we have $|S'| > |S''|$ and thus we have $S'' \subseteq S'$ indeed. However, (B.2) implies that $f(S') > f(S'')$, which causes a contradiction. The claim has then been proved.

Then, since $c_1 \le c_2 \le \ldots \le c_m$, we must then have $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_{m-1}^* \subseteq S_m^*$ and the result in Lemma B.1 has been proved. ■

B.5. Proofs in Step 2

In this section, we prove Lemma B.2 and Proposition B.2 in Step 2. We first prove Lemma B.2 as follows.

Proof of Lemma B.2: Denote the optimal solution to problem (8) as S_1^*. Since we have $S_1^* \subseteq S_2$, S_2 is partitioned into S_1^* and $S_2 \setminus S_1^*$. If S_1^* is not revenue-ordered, then we must have product j and k with $r_j > r_k$ such that $j \in S_2 \setminus S_1^*$ and $k \in S_1^*$. Based on the optimality of S_1^*, we know that moving product j to S_1^* will at least not increase the total expected revenue, which can be written as:

$$\begin{aligned} &\frac{W(S_1^*) + r_j v_j}{1+V(S_1^*)+v_j} + \frac{\beta(W(S_2 \setminus S_1^*) - r_j v_j)}{(1+V(S_1^*)+v_j)(1+V(S_1^*)+v_j+\beta(V(S_2 \setminus S_1^*) - v_j))} \\ &\le \frac{W(S_1^*)}{1+V(S_1^*)} + \frac{\beta W(S_2 \setminus S_1^*)}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2 \setminus S_1^*))}. \end{aligned} \tag{B.3}$$

Note that (B.3) can be further transformed to

$$\begin{aligned} &\frac{r_j v_j}{1+V(S_1^*)+v_j} \left(1 - \frac{\beta}{1+V(S_1^*)+\beta V(S_2 \setminus S_1^*)+(1-\beta)v_j} \right) \le W(S_1^*) \left(\frac{1}{1+V(S_1^*)} - \frac{1}{1+V(S_1^*)+v_j} \right) \\ &\quad + \beta W(S_2 \setminus S_1^*) \left(\frac{1}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2 \setminus S_1^*))} - \frac{1}{(1+V(S_1^*)+v_j)(1+V(S_1^*)+\beta V(S_2 \setminus S_1^*)+(1-\beta)v_j)} \right). \end{aligned}$$

We also note that

$$\frac{1}{1+V(S_1^*)} - \frac{1}{1+V(S_1^*)+v_j} = \frac{v_j}{(1+V(S_1^*))(1+V(S_1^*)+v_j)},$$

and

$$\frac{1}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))}-\frac{1}{(1+V(S_1^*)+v_j)(1+V(S_1^*)+\beta V(S_2\setminus S_1^*)+(1-\beta)v_j)}$$
$$=\frac{(1+V(S_1^*))\cdot(1-\beta)v_j+(1+V(S_1^*)+\beta V(S_2\setminus S_1^*)+(1-\beta)v_j)v_j}{(1+V(S_1^*))(1+V(S_1^*)+v_j)(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*)+(1-\beta)v_j))}.$$

Therefore, we can finally transform (B.3) into the following inequality:

$$r_j\le\frac{\frac{W(S_1^*)}{1+V(S_1^*)}+\frac{\beta W(S_2\setminus S_1^*)}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))}+\frac{\beta(1-\beta)W(S_2\setminus S_1^*)}{(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*)+(1-\beta)v_j)}}{1-\frac{\beta}{1+V(S_1^*)+\beta V(S_2\setminus S_1^*)+(1-\beta)v_j}}=U_j.$$

Based on the optimality of S_1^*, we also know that moving product k out of S_1^* will at least not increase the total expected revenue, which can be written as:

$$\frac{W(S_1^*)-r_kv_k}{1+V(S_1^*)-v_k}+\frac{\beta(W(S_2\setminus S_1^*)+r_kv_k)}{(1+V(S_1^*)-v_k)(1+V(S_1^*)-v_k+\beta(V(S_2\setminus S_1^*)+v_k))}$$
$$\le\frac{W(S_1^*)}{1+V(S_1^*)}+\frac{\beta W(S_2\setminus S_1^*)}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))}.$$

With some algebraic calculations, the above inequality can be rewritten as:

$$r_k\ge\frac{\frac{W(S_1^*)}{1+V(S_1^*)}+\frac{\beta W(S_2\setminus S_1^*)}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))}+\frac{\beta(1-\beta)W(S_2\setminus S_1^*)}{(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*)-(1-\beta)v_k)}}{1-\frac{\beta}{1+V(S_1^*)+\beta V(S_2\setminus S_1^*)-(1-\beta)v_k}}=L_k.$$

Therefore, we have the following relationship:

$$L_k\le r_k<r_j\le U_j.$$

However, checking the expressions of U_j and L_k, we find that $U_j\le L_k$, which comes to a contradiction. ■

Using the result of Lemma B.2, we prove Proposition B.2. We need the following lemma.

LEMMA B.4. (⋆⋆⋆) *For $x\ge 0$, we define a function $f(x)$ as follows:*

$$f(x)=\sum_{i=1}^m\frac{a_i+b_ix}{d_i+x}.$$

If $a_i,b_i>0$ for all $i\in[m]$, $\frac{a_1}{b_1}\ge\cdots\ge\frac{a_m}{b_m}$ and $0<d_1\le\cdots\le d_m$, then $f(x)$ is quasi-convex in x for $x\ge 0$.

We then prove Proposition B.2 as follows.

Proof of Proposition B.2: Under the given conditions, denote the optimal solution to the SQR problem as $\boldsymbol{S^*}=(S_1^*,S_2^*,\ldots,S_m^*)$. By Lemma B.1, we must have $S_1^*\subseteq S_2^*\subseteq\cdots\subseteq S_m^*$. If S_m^* is revenue-ordered, then $S_1^*,S_2^*,\ldots,S_{m-1}^*$ are all revenue-ordered by Lemma B.2 and the proof is completed. Therefore, we only need to show that S_m^* is indeed revenue-ordered. We prove it by contradiction. Suppose S_m^* is not revenue-ordered, we then let k be the smallest index such that $k\notin S_m^*$. We denote $S_{m,l}^*=\{j|j<k,j\in S_m^*\}$ and $S_{m,r}^*=\{j|j>k,j\in S_m^*\}$. We consider the following two cases:

Case 1: $S_{m,l}^*\subseteq S_{m-1}^*$

Under the optimal solution $\boldsymbol{S^*}=(S_1^*,S_2^*,\ldots,S_m^*)$, the total expected revenue is written as follows:

$$R(\boldsymbol{S^*})=\sum_{i=1}^{m-1}\lambda_i\left[\frac{W(S_i^*)}{1+V(S_i^*)}+\frac{\beta_iW(S_{m-1}^*\setminus S_i^*)+\beta_iW(S_m^*\setminus S_{m-1}^*)}{(1+V(S_i^*))(1+V(S_i^*)+\beta_iV(S_{m-1}^*\setminus S_i^*)+\beta_iV(S_m^*\setminus S_{m-1}^*))}\right]$$
$$+\lambda_m\frac{W(S_{m-1}^*)+W(S_m^*\setminus S_{m-1}^*)}{1+V(S_{m-1}^*)+V(S_m^*\setminus S_{m-1}^*)}.$$

We consider a feasible solution $\boldsymbol{S_1} = (S_1^*, \ldots, S_{m-1}^*, S_m^* \cup \{k\})$ with its total expected revenue given as follows:

$$R(\boldsymbol{S_1}) = \sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S_i^*)}{1+V(S_i^*)} + \frac{\beta_i W(S_{m-1}^* \setminus S_i^*) + \beta_i W(S_m^* \setminus S_{m-1}^*) + \beta_i r_k v_k}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V(S_{m-1}^* \setminus S_i^*) + \beta_i V(S_m^* \setminus S_{m-1}^*) + \beta_i v_k)} \right]$$
$$+ \lambda_m \frac{W(S_{m-1}^*) + W(S_m^* \setminus S_{m-1}^*) + r_k v_k}{1+V(S_{m-1}^*) + V(S_m^* \setminus S_{m-1}^*) + v_k}.$$

We consider another feasible solution $\boldsymbol{S_2} = (S_1^*, \ldots, S_{m-1}^*, S_{m-1}^*)$ with its total expected revenue given as follows:

$$R(\boldsymbol{S_2}) = \sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S_i^*)}{1+V(S_i^*)} + \frac{\beta_i W(S_{m-1}^* \setminus S_i^*)}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V(S_{m-1}^* \setminus S_i^*))} \right] + \lambda_m \frac{W(S_{m-1}^*)}{1+V(S_{m-1}^*)}.$$

We aim to show the following relationship:

$$R(\boldsymbol{S^*}) \le \max\{R(\boldsymbol{S_1}), R(\boldsymbol{S_2})\}. \tag{B.4}$$

To prove (B.4), we first denote $Z^* = R(\boldsymbol{S^*}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S_i^*)}{1+V(S_i^*)}$, $Z_1 = R(\boldsymbol{S_1}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S_i^*)}{1+V(S_i^*)}$ and $Z_2 = R(\boldsymbol{S_2}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S_i^*)}{1+V(S_i^*)}$. Then, it is equivalent to prove $Z^* \le \max\{Z_1, Z_2\}$. For notation brevity, we define $R_i = \frac{W(S_{m-1}^* \setminus S_i^*)}{V(S_{m-1}^* \setminus S_i^*)}$ and $V_i = V(S_{m-1}^* \setminus S_i^*)$ for $i = 1, \ldots, m-1$ if $S_{m-1}^* \setminus S_i^* \neq \emptyset$, while define $R_i = 0$ and $V_i = 0$ otherwise. We also define $r = \frac{W(S_m^* \setminus S_{m-1}^*)}{V(S_m^* \setminus S_{m-1}^*)}$ and $v = V(S_m^* \setminus S_{m-1}^*)$ if $S_m^* \setminus S_{m-1}^* \neq \emptyset$, while define $r = 0$ and $v = 0$ otherwise. Since $S_{m,l}^* \subseteq S_{m-1}^*$, we must then have $r \le r_k$. Note that we can express Z^*, Z_1 and Z_2 as

$$Z^* = \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r v}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V_i + \beta_i v)} + \lambda_m \frac{W(S_{m-1}^*) + rv}{1+V(S_{m-1}^*)+v}.$$
$$Z_1 = \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r v + \beta_i r_k v_k}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V_i + \beta_i v + \beta_i v_k)} + \lambda_m \frac{W(S_{m-1}^*) + rv + r_k v_k}{1+V(S_{m-1}^*)+v+v_k}.$$
$$Z_2 = \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V_i)} + \lambda_m \frac{W(S_{m-1}^*)}{1+V(S_{m-1}^*)}.$$

Since $r \le r_k$, we have the following:

$$Z_1 \ge \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r(v+v_k)}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V_i + \beta_i (v+v_k))} + \lambda_m \frac{W(S_{m-1}^*) + r(v+v_k)}{1+V(S_{m-1}^*)+(v+v_k)}.$$

We then define a function $f(x)$ for $x \ge 0$ as follows:

$$f(x) = \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r x}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V_i + \beta_i x)} + \lambda_m \frac{W(S_{m-1}^*) + rx}{1+V(S_{m-1}^*)+x}.$$

We can now apply the result of Lemma B.4. Specifically, we let $a_i = \frac{\lambda_i R_i V_i}{1+V(S_i^*)}$, $b_i = \frac{\lambda_i r}{1+V(S_i^*)}$ and $d_i = \frac{1+V(S_i^*)+\beta_i V_i}{\beta_i}$ for $i = 1, \ldots, m-1$. We also let $a_m = \lambda_m W(S_{m-1}^*)$, $b_m = \lambda_m r$ and $d_m = 1 + V(S_{m-1}^*)$. Note that $\frac{a_i}{b_i} = \frac{R_i V_i}{r} = \frac{W(S_{m-1}^* \setminus S_i^*)}{r}$ for $i = 1, \ldots, m-1$ and $\frac{a_m}{b_m} = \frac{W(S_{m-1}^*)}{r}$. Thus, we have $\frac{a_m}{b_m} \ge \frac{a_1}{b_1} \ge \frac{a_2}{b_2} \ge \cdots \ge \frac{a_{m-1}}{b_{m-1}}$. Also, note that $d_i = \frac{1-\beta_i}{\beta_i}(1+V(S_i^*)) + 1 + V(S_{m-1}^*)$ for $i = 1, \ldots, m-1$. Thus, we also have $d_m \le d_1 \le d_2 \le \cdots \le d_{m-1}$. Based on these two conditions, we know that the function $f(x)$ must be quasi-convex for $x \ge 0$ and thus we have the following:

$$\max\{Z_1, Z_2\} \ge \max\{f(v+v_k), f(0)\} \ge f(v) = Z^*.$$

Thus, inequality (B.4) is proved. □

Case 2: $S^*_{m-1} \subseteq S^*_{m,l}$

Under the optimal solution $\boldsymbol{S^*} = (S^*_1, S^*_2, \ldots, S^*_m)$, the total expected revenue is written as follows:

$$R(\boldsymbol{S^*}) = \sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S^*_i)}{1+V(S^*_i)} + \frac{\beta_i W(S^*_{m,l} \setminus S^*_i) + \beta_i W(S^*_{m,r})}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V(S^*_{m,l}\setminus S^*_i)+\beta_i V(S^*_{m,r}))}\right] + \lambda_m \frac{W(S^*_{m,l})+W(S^*_{m,r})}{1+V(S^*_{m,l})+V(S^*_{m,r})}.$$

We consider a feasible solution $\boldsymbol{S_1} = (S^*_1, \ldots, S^*_{m-1}, S^*_m \cup \{k\})$ with its total expected revenue given as follows:

$$\begin{aligned} R(\boldsymbol{S_1}) = &\sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S^*_i)}{1+V(S^*_i)} + \frac{\beta_i W(S^*_{m,l} \setminus S^*_i) + \beta_i W(S^*_{m,r}) + \beta_i r_k v_k}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V(S^*_{m,l}\setminus S^*_i)+\beta_i V(S^*_{m,r})+\beta_i v_k)}\right] \\ &+ \lambda_m \frac{W(S^*_{m,l})+W(S^*_{m,r})+r_k v_k}{1+V(S^*_{m,l})+V(S^*_{m,r})+v_k}. \end{aligned}$$

We consider another feasible solution $\boldsymbol{S_2} = (S^*_1, \ldots, S^*_{m-1}, S^*_{m,l})$ with its total expected revenue given as follows:

$$R(\boldsymbol{S_2}) = \sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S^*_i)}{1+V(S^*_i)} + \frac{\beta_i W(S^*_{m,l} \setminus S^*_i)}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V(S^*_{m,l}\setminus S^*_i))}\right] + \lambda_m \frac{W(S^*_{m,l})}{1+V(S^*_{m,l})}.$$

We aim to show the following relationship:

$$R(\boldsymbol{S^*}) \le \max\{R(\boldsymbol{S_1}), R(\boldsymbol{S_2})\}. \tag{B.5}$$

To prove (B.5), we first denote $Z^* = R(\boldsymbol{S^*}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S^*_i)}{1+V(S^*_i)}$, $Z_1 = R(\boldsymbol{S_1}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S^*_i)}{1+V(S^*_i)}$ and $Z_2 = R(\boldsymbol{S_2}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S^*_i)}{1+V(S^*_i)}$. Then, it is equivalent to prove $Z^* \le \max\{Z_1, Z_2\}$. For notation brevity, we define $R_i = \frac{W(S^*_{m,l}\setminus S^*_i)}{V(S^*_{m,l}\setminus S^*_i)}$ and $V_i = V(S^*_{m,l} \setminus S^*_i)$ for $i = 1, \ldots, m-1$ if $S^*_{m,l} \setminus S^*_i \ne \emptyset$, while define $R_i = 0$ and $V_i = 0$ otherwise. We also define $r = \frac{W(S^*_{m,r})}{V(S^*_{m,r})}$ and $v = V(S^*_{m,r})$ if $S^*_{m,r} \ne \emptyset$, while define $r = 0$ and $v = 0$ otherwise. Based on the definition, we must then have $r \le r_k$. Note that we can express Z^*, Z_1 and Z_2 as

$$\begin{aligned} Z^* &= \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r v}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V_i + \beta_i v)} + \lambda_m \frac{W(S^*_{m,l}) + rv}{1+V(S^*_{m,l})+v}. \\ Z_1 &= \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r v + \beta_i r_k v_k}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V_i + \beta_i v + \beta_i v_k)} + \lambda_m \frac{W(S^*_{m,l}) + rv + r_k v_k}{1+V(S^*_{m,l})+v+v_k}. \\ Z_2 &= \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V_i)} + \lambda_m \frac{W(S^*_{m,l})}{1+V(S^*_{m,l})}. \end{aligned}$$

Since $r \le r_k$, we have the following:

$$Z_1 \ge \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r (v+v_k)}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V_i + \beta_i (v+v_k))} + \lambda_m \frac{W(S^*_{m,l}) + r(v+v_k)}{1+V(S^*_{m,l})+(v+v_k)}.$$

We then define a function $f(x)$ for $x \ge 0$ as follows:

$$f(x) = \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r x}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V_i + \beta_i x)} + \lambda_m \frac{W(S^*_{m,l}) + rx}{1+V(S^*_{m,l})+x}.$$

We can now apply the result of Lemma B.4. Specifically, we let $a_i = \frac{\lambda_i R_i V_i}{1+V(S^*_i)}$, $b_i = \frac{\lambda_i r}{1+V(S^*_i)}$ and $d_i = \frac{1+V(S^*_i)+\beta_i V_i}{\beta_i}$ for $i = 1, \ldots, m-1$. We also let $a_m = \lambda_m W(S^*_{m,l})$, $b_m = \lambda_m r$ and $d_m = 1 + V(S^*_{m,l})$. Note that $\frac{a_i}{b_i} = \frac{R_i V_i}{r} = \frac{W(S^*_{m,l}\setminus S^*_i)}{r}$ for $i = 1, \ldots, m-1$ and $\frac{a_m}{b_m} = \frac{W(S^*_{m,l})}{r}$. Thus, we have $\frac{a_m}{b_m} \ge \frac{a_1}{b_1} \ge \frac{a_2}{b_2} \ge \cdots \ge \frac{a_{m-1}}{b_{m-1}}$. Also, note that $d_i = \frac{1-\beta_i}{\beta_i}(1+V(S^*_i)) + 1 + V(S^*_{m,l})$ for $i = 1, \ldots, m-1$. Thus, we also have $d_m \le d_1 \le d_2 \le \cdots \le d_{m-1}$.

Based on these two conditions, we know that the function $f(x)$ must be quasi-convex for $x \geq 0$ and thus we have the following:

$$\max\{Z_1, Z_2\} \geq \max\{f(v+v_k), f(0)\} \geq f(v) = Z^*.$$

Thus, inequality (B.5) is proved. □

For the optimal solution $\boldsymbol{S^*} = (S_1^*, S_2^*, \ldots, S_m^*)$, suppose the second case happens, i.e., $S_{m-1}^* \subseteq S_{m,l}^*$, we then have either $R(\boldsymbol{S^*}) \leq R(S_1^*, \ldots, S_{m-1}^*, S_m^* \cup \{k\})$ or $R(\boldsymbol{S^*}) \leq R(S_1^*, \ldots, S_{m-1}^*, S_{m,l}^*)$. If the latter case happens, the proof is completed since $S_{m,l}^*$ is already revenue-ordered. If the former case happens, the proof is also completed if $S_m^* \cup \{k\}$ is revenue-ordered. Otherwise, we let $S_m^* \leftarrow S_m^* \cup \{k\}$ and follow the previous argument to analyze the current optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$, which will finally lead to a conclusion that there exists a revenue-ordered assortment S_m^* such that it is optimal.

Now, suppose the first case happens, i.e., $S_{m,l}^* \subseteq S_{m-1}^*$, we then have either $R(\boldsymbol{S^*}) \leq R(S_1^*, \ldots, S_{m-1}^*, S_m^* \cup \{k\})$ or $R(\boldsymbol{S^*}) \leq R(S_1^*, \ldots, S_{m-1}^*, S_{m-1}^*)$. If the former case happens, the proof is completed if $S_m^* \cup \{k\}$ is revenue-ordered. Otherwise, we let $S_m^* \leftarrow S_m^* \cup \{k\}$. We then fix S_m^* and update S_i^* for $i \in \{1, \ldots, m-1\}$ by solving problem (8) with $S_2 = S_m^*$ and $\beta = \beta_i$. Note that the total expected revenue will not decrease after this operation and the updated $S_1^*, \ldots, S_{m-1}^*$ should all be revenue-ordered within S_m^*. We can then follow the previous argument to analyze the updated solution. If the latter case happens, we let $S_m^* \leftarrow S_{m-1}^*$. We can also follow the previous argument to analyze the updated solution. To be specific, we now consider two cases: $S_{m,l}^* \subseteq S_{m-2}^*$ and $S_{m-2}^* \subseteq S_{m,l}^*$. If the case $S_{m,l}^* \subseteq S_{m-2}^*$ happens, for instance, we then have either $R(S_1^*, S_2^*, \ldots, S_{m-1}^*, S_m^*) \leq R(S_1^*, S_2^*, \ldots, S_{m-2}^*, S_{m-1}^* \cup \{k\}, S_m^* \cup \{k\})$ or $R(S_1^*, S_2^*, \ldots, S_{m-1}^*, S_m^*) \leq R(S_1^*, S_2^*, \ldots, S_{m-2}^*, S_{m-2}^*, S_{m-2}^*)$. If we continue the analysis, one can show that it will finally lead to a conclusion that either 1) there exists a revenue-ordered assortment S_m^* such that it is optimal or 2) the initial optimal solution $\boldsymbol{S^*}$ is dominated by a solution $(S', S', \ldots, S')$ where S' is not revenue-ordered. However, in the latter case, such a solution is further dominated by the solution $(\tilde{S}, \tilde{S}, \ldots, \tilde{S})$ where $\tilde{S}$ is the single-store optimal assortment, which is revenue-ordered. Therefore, the proof is completed. ■

B.6. Proof in Step 3

Finally, we give a proof to Theorem 4 as follows.

Proof: Based on Proposition B.2, we have shown that under the given conditions, there exists an optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$ to the SQR problem such that $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_m^*$ and $S_1^*, S_2^*, \ldots, S_m^*$ are all revenue-ordered. Therefore, to prove Theorem 4, we only need to show that $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_{m-1}^* \subseteq \tilde{S} \subseteq S_m^*$. We consider the following two cases:

Case 1: $S_1^* \subseteq \cdots \subseteq S_k^* \subseteq \tilde{S} \subset S_{k+1}^* \subseteq \cdots \subseteq S_m^*$ for $k \in \{0, \ldots, m-2\}$

Under the optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$, the total expected revenue is written as follows:

$$R(S_1^*, S_2^*, \ldots, S_m^*) = \sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S_i^*)}{1+V(S_i^*)} + \frac{\beta_i W(S_m^* \setminus S_i^*)}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V(S_m^* \setminus S_i^*))} \right] + \lambda_m \frac{W(S_m^*)}{1+V(S_m^*)}.$$

We then consider a feasible solution $(S_1^*,\dots,S_k^*,\tilde{S},\dots,\tilde{S},S_m^*)$ with its total expected revenue given as follows:

$$R(S_1^*,\dots,S_k^*,\tilde{S},\dots,\tilde{S},S_m^*)=\sum_{i=1}^{k}\lambda_i\left[\frac{W(S_i^*)}{1+V(S_i^*)}+\frac{\beta_i W(S_m^*\setminus S_i^*)}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V(S_m^*\setminus S_i^*))}\right]$$
$$+\sum_{i=k+1}^{m-1}\lambda_i\left[\frac{W(\tilde{S})}{1+V(\tilde{S})}+\frac{\beta_i W(S_m^*\setminus \tilde{S})}{(1+V(\tilde{S}))(1+V(\tilde{S})+\beta_i V(S_m^*\setminus \tilde{S}))}\right]+\lambda_m\frac{W(S_m^*)}{1+V(S_m^*)}.$$

Due to the fact that $\tilde{S}$ is the optimal assortment in the single-store problem, we must have:

$$\frac{W(S_i^*)}{1+V(S_i^*)}\le\frac{W(\tilde{S})}{1+V(\tilde{S})}. \tag{B.6}$$

Due to the fact that $|\tilde{S}|<|S_i^*|\le|S_m^*|$ for $i=k+1,\dots,m-1$, we must have:

$$V(S_i^*)\ge V(\tilde{S})\quad\text{and}\quad \beta_i W(S_m^*\setminus S_i^*)\le\beta_i W(S_m^*\setminus\tilde{S}). \tag{B.7}$$

Additionally, notice that the following relationship holds: $1+V(S_i^*)+V(S_m^*\setminus S_i^*)=1+V(\tilde{S})+V(S_m^*\setminus\tilde{S})$. Since we also have $(1-\beta_i)V(S_m^*\setminus S_i^*)\le(1-\beta_i)V(S_m^*\setminus\tilde{S})$ for $i=k+1,\dots,m-1$, we then have:

$$1+V(S_i^*)+\beta_i V(S_m^*\setminus S_i^*)\ge 1+V(\tilde{S})+\beta_i V(S_m^*\setminus\tilde{S}). \tag{B.8}$$

Therefore, combining (B.6), (B.7) and (B.8), we can show that $R(S_1^*,\dots,S_k^*,\tilde{S},\dots,\tilde{S},S_m^*)\ge R(S_1^*,S_2^*,\dots,S_m^*)$, which implies that $(S_1^*,\dots,S_k^*,\tilde{S},\dots,\tilde{S},S_m^*)$ should also be an optimal solution.

Case 2: $S_1^*\subseteq S_2^*\subseteq\cdots\subseteq S_{m-1}^*\subseteq S_m^*\subseteq\tilde{S}$

Under the optimal solution $(S_1^*,S_2^*,\dots,S_m^*)$, the total expected revenue is written as follows:

$$R(S_1^*,S_2^*,\dots,S_m^*)=\sum_{i=1}^{m-1}\lambda_i\left[\frac{W(S_i^*)}{1+V(S_i^*)}+\frac{\beta_i W(S_m^*\setminus S_i^*)}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V(S_m^*\setminus S_i^*))}\right]+\lambda_m\frac{W(S_m^*)}{1+V(S_m^*)}.$$

We denote $j=|S_m^*|+1$. Since $|\tilde{S}|>|S_m^*|$, we must have $j\in\tilde{S}$. We then consider a feasible solution $(S_1^*,\dots,S_{m-1}^*,S_m^*\cup\{j\})$ with its total expected revenue expressed as follows:

$$R(S_1^*,\dots,S_{m-1}^*,S_m^*\cup\{j\})=\sum_{i=1}^{m-1}\lambda_i\left[\frac{W(S_i^*)}{1+V(S_i^*)}+\frac{\beta_i W(S_m^*\setminus S_i^*)+\beta_i r_j v_j}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V(S_m^*\setminus S_i^*)+\beta_i v_j)}\right]+\lambda_m\frac{W(S_m^*)+r_j v_j}{1+V(S_m^*)+v_j}.$$

Then, since $\tilde{S}$ is the optimal assortment of the single-store problem and $j\in\tilde{S}$, we must have:

$$r_j\ge\frac{W(\tilde{S})}{1+V(\tilde{S})}\ge\frac{W(S_m^*)}{1+V(S_m^*)},$$

which implies the following inequality:

$$\frac{W(S_m^*)+r_j v_j}{1+V(S_m^*)+v_j}\ge\frac{W(S_m^*)}{1+V(S_m^*)}. \tag{B.9}$$

Notice that we also have the following relationship:

$$r_j\ge\frac{W(\tilde{S})}{1+V(\tilde{S})}\ge\frac{W(S_m^*\setminus S_i^*)}{1+V(S_m^*\setminus S_1^*)}\ge\frac{W(S_m^*\setminus S_i^*)}{\frac{1}{\beta_i}\cdot(1+V(S_i^*))+V(S_m^*\setminus S_i^*)}=\frac{\beta_i W(S_m^*\setminus S_i^*)}{1+V(S_i^*)+\beta_i V(S_m^*\setminus S_i^*)}. \tag{B.10}$$

Therefore, combining (B.9) and (B.10), we can show that $R(S_1^*,\dots,S_{m-1}^*,S_m^*\cup\{j\})\ge R(S_1^*,S_2^*,\dots,S_m^*)$, which implies that $(S_1^*,S_2^*,\dots,S_m^*)$ should also be an optimal solution. We can then incrementally add products to S_m^* until $S_m^*=\tilde{S}$ and conclude that $(S_1^*,\dots,S_{m-1}^*,\tilde{S})$ should also be an optimal solution. ■

B.7. Proof of Proposition 3

Proof: Our proof is mainly adapted from Rusmevichientong et al. (2014). We show that any instance of the partition problem can be reduced to an instance of the assortment feasibility problem constructed below. We first formally define the partition problem as follows.

PARTITION PROBLEM:
INPUTS: Set of items indexed by $1,2,\ldots,n$ and the size $w_i\in\mathbb{Z}_+$ associated with each item i.
QUESTION: Is there a subset $S\subseteq\{1,\ldots,n\}$ such $\sum_{i\in S} w_i=\sum_{i\in\{1,\ldots,n\}\setminus S} w_i$?

We construct an instance of the SQR problem as follows. Let $T=\frac{1}{2}\sum_{i=1}^n w_i$. There are three stores with $(\lambda_1,\lambda_2,\lambda_3)=(\frac{1+4T}{1+8T},\frac{4T}{1+8T},0)$. That is, store 3 does not contribute to the total revenue but stores 1 and 2 can view products from store 3. We consider $n+2$ products and label them as $\{1,\ldots,n,n+1,n+2\}$. The revenues of these products are set as:

$$r_i=\begin{cases}(1+8T)(3+4T) & \text{if } i=1,\ldots,n\\ 4(1+8T)(3+4T) & \text{if } i=n+1\\ 100(1+8T)(3+4T) & \text{if } i=n+2.\end{cases}$$

We first set $c_{12}^k=\infty$ and $c_{21}^k=\infty$ for $k=1,\ldots,n+2$, i.e., store 1(2) can not see any product from store 2(1). For product $n+2$, we let $v_{n+2}=1$ and $c_{13}^{n+2}=c_{23}^{n+2}=\infty$. For product $n+1$, we let $v_{n+1}=\frac{1}{2}$ and $c_{13}^{n+1}=0$ and $\exp(-c_{23}^{n+1})=\frac{2}{7}$. For product $k=1,\ldots,n$, we let $v_k=4w_i$ and $\exp(-c_{13}^k)=\frac{1}{2}$ and $\exp(-c_{23}^k)=\frac{1+4T}{7T}\in(0,1)$.

Denote the optimal solution to the above instance as (S_1^*,S_2^*,S_3^*) and $r=(1+8T)(3+4T)$ for convenience. We first argue that $\{n+2\}\in S_1^*$. If not, since $c_{13}^{n+2}=\infty$, adding product $n+2$ to S_1^* must increase the revenue of store 1 and does not affect the other stores' revenue. We then argue that $S_1^*=\{n+2\}$, i.e., all other products will not be included in S_1^*. Clearly, when $S_1^*=\{n+2\}$, the revenue of store 1 is at least $100r/(1+1)=50r$. If other products are added, the revenue of store 1 is at most $(100r+4r\cdot 1/2)/(1+1+1/2)+4r<50r$. We also note that the products chosen in S_1^* will not affect the revenue of store 2 since $c_{21}^k=\infty$ for $k=1,\ldots,n+2$. Therefore, we have shown that $S_1^*=\{n+2\}$. Due to the same reason, we have that $S_2^*=\{n+2\}$. We then argue that $\{n+1\}\in S_3^*$ since adding it will always increase the second-stage revenue of store 1 and store 2. Therefore, the problem now becomes to decide a set $S\subseteq\{1,\ldots,n\}$ to offer in S_3^*. The problem can now be equivalently represented as follows:

$$\max_{S\subseteq\{1,\ldots,n\}}\quad \frac{1+4T}{1+8T}\cdot\frac{2r+\sum_{i\in S}2w_ir}{1+\frac{1}{2}+\sum_{i\in S}2w_i}+\frac{4T}{1+8T}\cdot\frac{4r\cdot\frac{1}{7}+\sum_{i\in S}\frac{4(1+4T)w_i}{7T}r}{1+\frac{1}{7}+\sum_{i\in S}\frac{4(1+4T)w_i}{7T}}. \tag{B.11}$$

One can verify that based on our construction, (B.11) is equivalent to the formulation of the assortment feasibility problem in Rusmevichientong et al. (2014). Thus, the SQR problem under construction is NP-hard. We should comment that although the above construction requires that the revenues of products $1,\ldots,n$ are the same, it is possible to perturb those revenues a little bit but still guarantee that the optimal solution to problem (B.11) is attained by $\sum_{i\in S}w_i=T$ since we assume $w_i\in\mathbb{Z}_+$. ■

B.8. Total revenue decreases when the utility discount increases

Claim: $R_Q^*(c)$ is monotonically decreasing in c.

Proof: From the formulation of problem (6), it is easy to see that when c decreases, $\beta=\exp(-c)$ increases, and thus the optimal objective value increases. ■

B.9. Proof of Proposition 5

Proof: We first prove Proposition 5(a). It is easy to see that $R_Q^*(\lambda)$ is a symmetric function with respect to $\lambda = 0.5$. The fact that it is piecewise linear follows from the observation that for each feasible assortment (S_1, S_2), the total expected revenue $R(S_1, S_2)$ can be viewed as a linear function in λ. Then, $R_Q^*(\lambda)$ is the maximization of a finite number of linear functions, which is convex by definition. Due to symmetry, $R_Q^*(\lambda)$ must be decreasing when $\lambda \in [0, 0.5]$ and increasing when $\lambda \in [0.5, 1]$.

We then prove Proposition 5(b). Due to symmetry, it suffices to prove that $|S_1^*|$ is weakly decreasing in λ when $\lambda \in [0, 0.5)$. We prove this claim by contradiction. For $0 \le \lambda_l < \lambda_r \le 0.5$, we denote (S_1^l, S_2^l) as the optimal solution to problem (6) when $\lambda = \lambda_l$ and (S_1^r, S_2^r) as the optimal solution to problem (6) when $\lambda = \lambda_r$. Suppose we now have $|S_1^l| < |S_1^r|$. Due to Theorem 2, we know that S_1^l, S_2^l, S_1^r and S_2^r should be revenue-ordered. Also, we must have $S_2^l \subseteq \tilde{S} \subseteq S_1^l$ and $S_2^r \subseteq \tilde{S} \subseteq S_1^r$, where $\tilde{S}$ is the optimal assortment in the single-store problem. We then express the total expected revenue when $\lambda = \lambda_l$ as follows:

$$\begin{aligned} R(S_1^l, S_2^l) &= \lambda_l \frac{W(S_1^l)}{1 + V(S_1^l)} + (1-\lambda_l)\frac{W(S_2^l)}{1+V(S_2^l)} + (1-\lambda_l)\frac{\beta W(S_1^l \setminus S_2^l)}{(1+V(S_2^l))(1+V(S_2^l)+\beta V(S_1^l \setminus S_2^l))} \\ &= \lambda_l R_1(S_1^l, S_2^l) + (1-\lambda_l) R_2(S_1^l, S_2^l) = R_2(S_1^l, S_2^l) + \lambda_l (R_1(S_1^l, S_2^l) - R_2(S_1^l, S_2^l)), \end{aligned}$$

where we denote $R_1(S_1^l, S_2^l)$ ($R_2(S_1^l, S_2^l)$, resp.) as the expected revenue of store 1 (store 2, resp.) when the offered assortments are (S_1^l, S_2^l) (assuming arrival rate is 1). Similarly, we can express the total expected revenue when $\lambda = \lambda_r$ as follows:

$$R(S_1^r, S_2^r) = R_2(S_1^r, S_2^r) + \lambda_r (R_1(S_1^r, S_2^r) - R_2(S_1^r, S_2^r)).$$

Since $\tilde{S} \subseteq S_1^l$, $\tilde{S} \subseteq S_1^r$ and $|S_1^l| < |S_1^r|$, we have $R_1(S_1^l, S_2^l) \ge R_1(S_1^r, S_2^r)$. Then, we must have $R_2(S_1^l, S_2^l) \le R_2(S_1^r, S_2^r)$, since otherwise the optimal solution (S_1^r, S_2^r) when $\lambda = \lambda_r$ is dominated by the solution (S_1^l, S_2^l). We also note that $R_1(S_1^l, S_2^l) = R_1(S_1^r, S_2^r)$ and $R_2(S_1^l, S_2^l) = R_2(S_1^r, S_2^r)$ cannot hold at the same time since then the optimal solution at λ_r would be (S_1^l, S_2^l) based on our definition since $|S_1^l| < |S_1^r|$. However, this implies that the slope of the function $R_Q^*(\lambda)$ is smaller at the point $\lambda = \lambda_r$ (which is $R_1(S_1^r, S_2^r) - R_2(S_1^r, S_2^r)$) than at the point $\lambda = \lambda_l$ (which is $(R_1(S_1^l, S_2^l) - R_2(S_1^l, S_2^l))$, which is contradictory to the first argument. This completes the proof. ∎

B.10. Proof of Proposition 4

Proof: The proof is directly followed by Lemma B.2 and the proof in Lemma B.1. ∎

Appendix C: Proofs for Section 6

C.1. Proof of Theorem 5

We first prove Theorem 5(a) as follows.

Proof of Theorem 5(a): Denote the optimal objective of the SMR and the SQR problem as R_M^* and R_Q^*, respectively. By Theorem 1, we have $R_M^* \le R^*$, where R^* is the optimal revenue of the single-store problem. Note that we also have $R_Q^* \ge R^*$ since we can always offer $(\tilde{S}, \tilde{S}, \ldots, \tilde{S})$ under the SQR strategy. Therefore, we have $R_Q^* \ge R_M^*$. ∎

To prove Theorem 5(b), we need the following Proposition.

Proposition C.1. ($\star\star\star$) *When $\boldsymbol{S}$ is given, the consumer surplus under the SMR model follows:*

$$CS^M(\boldsymbol{S}) = \sum_{i=1}^{m} \lambda_i \log\left(1 + \sum_{k\in S_i} v_k + \sum_{k\in \bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)\right).$$

The consumer surplus under the SQR model follows:

$$\begin{aligned}CS^Q(\boldsymbol{S}) = &\sum_{i=1}^{m} \lambda_i \frac{\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))}{1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)} + \sum_{i=1}^{m}\lambda_i \frac{V(S_i)}{1+V(S_i)}\log(1+V(S_i))\\ &+\sum_{i=1}^{m} \lambda_i \frac{\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)}{(1+V(S_i))(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))}\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)).\end{aligned}$$

We then prove Theorem 5(b) as follows.

Proof of Theorem 5(b): The result simply follows by the inequalities below:

$$\begin{aligned}CS^Q(\boldsymbol{S}) = &\sum_{i=1}^{m} \lambda_i \frac{\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))}{1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)} + \sum_{i=1}^{m}\lambda_i \frac{V(S_i)}{1+V(S_i)}\log(1+V(S_i))\\ &+\sum_{i=1}^{m} \lambda_i \frac{\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)}{(1+V(S_i))(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))}\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))\\ \le &\sum_{i=1}^{m} \lambda_i \frac{\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))}{1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)} + \sum_{i=1}^{m}\lambda_i \frac{V(S_i)}{1+V(S_i)}\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))\\ &+\sum_{i=1}^{m} \lambda_i \frac{\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)}{(1+V(S_i))(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))}\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)) = CS^M(\boldsymbol{S}).\end{aligned}$$

Appendix D: Proofs for Section 7

D.1. Proof of Proposition 6

Proof: To facilitate our analysis, we define the seller's assortment optimization problem for the single store i under the heterogeneous case as

$$R_i^* = \max_S \quad \frac{\sum_{k\in S} r_k v_{ik}}{1+\sum_{k\in S} v_{ik}}. \tag{D.1}$$

We also denote the optimal objective value of the SQR-heter problem and the SMR-heter problem as R_Q^* and R_M^*, respectively. Suppose the optimal solution to the SMR-heter problem is $\boldsymbol{S^*} = (S_1^*, \ldots, S_m^*)$, using the same analysis in Theorem 1, we can show that the revenue for each store i with $i\in[m]$ under $\boldsymbol{S^*}$ is upper bounded by the single-store optimal revenue, namely, R_i^*. Thus, we have $R_M^* \le \sum_{i=1}^n \lambda_i R_i^*$. It is also easy to see that $R_Q^* \ge \sum_{i=1}^n \lambda_i R_i^*$ always holds, which establishes the desired result. ■

D.2. Proof of Proposition 7

Proof: To prove the approximation ratio $\frac{1}{2}$, we need the following lemma.

Lemma D.1. *Given an MNL model, suppose the revenues and the attraction values of the products are denoted as $(r_1, \ldots, r_n)$ and $(v_1, \ldots, v_n)$. Denote the optimal objective value of the following problem as R:*

$$\max_S \quad \frac{\sum_{i\in S} r_i v_i}{1+\sum_{i\in S} v_i}.$$

Given a vector $(\epsilon_1,\dots,\epsilon_n)$ *where* $0<\epsilon_i<1$ *for each* $i=1,\dots,n$*, we also denote the optimal objective value of the following problem as* R'*:*

$$\max_S \quad \frac{\sum_{i\in S} r_i\epsilon_i v_i}{1+\sum_{i\in S}\epsilon_i v_i}.$$

We then have $R\geq R'$*.*

Proof of Lemma D.1: It suffices to prove the result by showing that if we only increase the attraction value for product i from $\epsilon_i v_i$ to v_i and keep the attraction values of other products the same, the optimal expected revenue would at least not decrease. To that end, denote the optimal solution to the problem with attraction value $\epsilon_i v_i$ as S'. Then, if $i\notin S'$, the result has been proved since S' generates the same objective value for the problem with v_i. Otherwise, if $i\in S'$, we must have r_i greater than R'. Then, increasing the attraction value for product i from $\epsilon_i v_i$ to v_i will increase the optimal expected revenue. ■

Proof of the approximation ratio $\frac{1}{2}$: Denote the optimal solution to the SQR problem as $\boldsymbol{S}^*=(S_1^*,\dots,S_m^*)$. We also let $\bar{S}^*=S_1^*\cup S_2^*\cup\cdots\cup S_m^*$. We then have

$$\begin{aligned}
R_Q^H = R(\tilde{S}_1,\dots,\tilde{S}_m) &\geq \sum_{i=1}^m \lambda_i\cdot\frac{\sum_{k\in\tilde{S}_i} r_k v_{ik}}{1+\sum_{k\in\tilde{S}_i} v_{ik}}\\
&\geq \sum_{i=1}^m \frac{\lambda_i}{2}\left[\frac{\sum_{k\in S_i^*} r_k v_{ik}}{1+\sum_{k\in S_i^*} v_{ik}}+\frac{\sum_{k\in\bar{S}^*\setminus S_i^*} r_k v_{ik}\cdot\exp\left(-c_i^k(\boldsymbol{S^*})\right)}{1+\sum_{k\in\bar{S}^*\setminus S_i^*} v_{ik}\cdot\exp\left(-c_i^k(\boldsymbol{S^*})\right)}\right]\\
&\geq \sum_{i=1}^m \frac{\lambda_i}{2}\left[\frac{\sum_{k\in S_i^*} r_k v_{ik}}{1+\sum_{k\in S_i^*} v_{ik}}+\frac{\sum_{k\in\bar{S}^*\setminus S_i^*} r_k v_{ik}\cdot\exp\left(-c_i^k(\boldsymbol{S^*})\right)}{(1+\sum_{k\in S_i^*} v_{ik})(1+\sum_{k\in S_i^*} v_{ik}+\sum_{k\in\bar{S}^*\setminus S_i^*} v_{ik}\cdot\exp\left(-c_i^k(\boldsymbol{S^*})\right))}\right]\\
&= \frac{1}{2}R_Q^*.
\end{aligned}$$

Recall that $\tilde{S}_i$ is the optimal solution to the single-store problem. Therefore, the second inequality follows by applying Lemma D.1 with $\epsilon_i=\exp(-c_i^k(\boldsymbol{S^*}))$ for $i=1,\dots,n$. We construct the following instance to prove the tightness (similar to Gao et al. 2021, see Appendix G). Suppose there are two stores and two products are offered. The arrival rates are $\lambda_1=1$ and $\lambda_2=0$. The revenues of the two products are $1+\frac{1}{\epsilon}$ and 1, where $\epsilon\to 0$. The attraction values of the two products are the same in the two stores with $v_1=\epsilon$ and $v_2=\frac{1}{\epsilon}$. We also let $c_{12}^1=c_{12}^2=0$. One can verify that $\tilde{S}=\{1\}$ under this instance, and the recommendation strategy $\{\{1\},\{1\}\}$ will result in a total revenue close to 1 when $\epsilon\to 0$. However, the recommendation strategy $\{\{1\},\{2\}\}$ will result in a total revenue close to 2 when $\epsilon\to 0$. The tightness then follows from the construction. ■

D.3. Proof of Proposition 8

Proof: For any given assortment set $\boldsymbol{S}$, the expected revenue obtained by the partial recommendation (homogeneous or heterogeneous) is equal to that obtained by the full recommendation but setting some of the c_{ij}^k as infinity. Based on that, recall that in Theorem 1, we have shown that under the SMR strategy, for arbitrary c_{ij}^k (even for $c_{ij}^k=\infty$), the optimal assortment in each store is the same and should be the single-store optimal assortment $\tilde{S}$ when the valuations are homogeneous. Thus, the optimal partial recommendation strategy should be $(\tilde{S},\tilde{S},\dots,\tilde{S})$ for the homogeneous case. When the valuations are heterogeneous, we have shown in the proof of Proposition 6 that for arbitrary c_{ij}^k (even for $c_{ij}^k=\infty$), the optimal expected revenue has an upper bound of $\sum_{i=1}^m\lambda_i R_i^*$, where R_i^* is the single-store optimal revenue in store i. That said,

$\sum_{i=1}^{m} \lambda_i R_i^*$ also serves as an upper bound for any partial recommendation strategy. However, such an upper bound can be exactly achieved by considering a partial recommendation strategy with $(\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_m)$ and not recommending products from other stores (if any). Thus, the optimal partial recommendation strategy should be $(\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_m)$ for the heterogeneous case. ■

Appendix E: Some numerical examples (⋆⋆⋆)

E.1. Example where SMR improves revenue when there is capacity constraint

E.2. Example where the optimal assortments may not be revenue-ordered for the SMR model under the omnichannel setting

E.3. Example where the optimal assortments may not be revenue-ordered for the SQR model under the omnichannel setting

E.4. Example where SQR has higher consumer surplus than SMR

E.5. Problem instances for Figure 6

Appendix F: Numerical results on the heuristic performance in Proposition 7 (⋆⋆⋆)

Appendix G: The integer programming formulation for the cardinality-constrained SQR problem under universally homogeneous disutilities (⋆⋆⋆)

Appendix H: Joint Assortment and Pricing Optimization (⋆⋆⋆)

Additional Appendix

In this additional appendix, the proofs of the results marked with "$(***)$" in the online appendix are provided.

Appendix A: Additional Proofs for Section 4

Proof of Proposition A.1

Proof: It suffices to prove the latter argument of Proposition A.1. We first prove the "only if" direction. Now, suppose $r_i > R^*_M(\boldsymbol{c})$ and $i \notin S^*_M(\boldsymbol{c})$, and note that the optimal revenue can be expressed as

$$R^*_M(\boldsymbol{c}) = \frac{\sum_{j\in S^*_M(\boldsymbol{c})} r_j v_j + \sum_{j\in S_2\setminus S^*_M(\boldsymbol{c})} r_j v_j \exp(-c^j)}{1+\sum_{j\in S^*_M(\boldsymbol{c})} v_j + \sum_{j\in S_2\setminus S^*_M(\boldsymbol{c})} v_j \exp(-c^j)}.$$

Consider the following two cases. For the first case, we have $i \notin S_2$, we then consider another assortment $S^*_M(\boldsymbol{c}) \cup \{i\}$ and express its revenue as follows:

$$R(S^*_M(\boldsymbol{c})\cup\{i\},\boldsymbol{c}) = \frac{r_i v_i + \sum_{j\in S^*_M(\boldsymbol{c})} r_j v_j + \sum_{j\in S_2\setminus S^*_M(\boldsymbol{c})} r_j v_j \exp(-c^j)}{1+v_i+\sum_{j\in S^*_M(\boldsymbol{c})} v_j + \sum_{j\in S_2\setminus S^*_M(\boldsymbol{c})} v_j \exp(-c^j)}.$$

Note that the following claim holds by simple algebra:

Claim: For any $a,b,c,d>0$, if $\frac{c}{d} > \frac{a}{b}$, then $\frac{a+c}{b+d} > \frac{a}{b}$.

Since $r_i > R^*_M(\boldsymbol{c})$, it is then easy to verify that $R(S^*_M(\boldsymbol{c})\cup\{i\},\boldsymbol{c}) \geq R^*_M(\boldsymbol{c})$ in this case based on the above claim, which contradicts the optimality of $S^*_M(\boldsymbol{c})$. For the second case, we have $i \notin S_2$, we also consider the assortment $S^*_M(\boldsymbol{c}) \cup \{i\}$ and express its revenue as follows:

$$R(S^*_M(\boldsymbol{c})\cup\{i\},\boldsymbol{c}) = \frac{r_i v_i(1-\exp(-c^k)) + \sum_{j\in S^*_M(\boldsymbol{c})} r_j v_j + \sum_{j\in S_2\setminus S^*_M(\boldsymbol{c})} r_j v_j \exp(-c^j)}{1+v_i(1-\exp(-c^k))+\sum_{j\in S^*_M(\boldsymbol{c})} v_j + \sum_{j\in S_2\setminus S^*_M(\boldsymbol{c})} v_j \exp(-c^j)}.$$

Similarly, one can verify that $R(S^*_M(\boldsymbol{c})\cup\{i\},\boldsymbol{c}) \geq R^*_M(\boldsymbol{c})$ in this case, which contradicts the optimality of $S^*_M(\boldsymbol{c})$. Thus, the proof has been completed for the "only if" direction. We can use a similar argument to prove the "if" direction. ■

Proof of Proposition A.2

Proof: To prove the argument in the proposition, we only have to consider the case when $c^j > \tilde{c}^j$ and $c^i = \tilde{c}^i$ for any $i \in [n]$ and $i \neq j$. For the brevity of notation, we let $\beta^i = \exp(-c^i)$ and $\tilde{\beta}^i = \exp(-\tilde{c}^i)$ for all $i \in [n]$. Since $c^j > \tilde{c}^j$, we must have $\beta^j < \tilde{\beta}^j$. As before, we denote the optimal solution to problem (A.1) under $\boldsymbol{c}$ and $\tilde{\boldsymbol{c}}$ as $S^*_M(\boldsymbol{c})$ and $S^*_M(\tilde{\boldsymbol{c}})$ respectively. We consider the following two cases:

Case 1: $j \in S^*_M(\tilde{\boldsymbol{c}})$ or $j \notin S_2$

In this case, we have $R_M^*(\boldsymbol{c}) \geq R(S_M^*(\tilde{\boldsymbol{c}}), \boldsymbol{c}) \geq R_M^*(\tilde{\boldsymbol{c}})$.

Case 2: $j \notin S_M^*(\tilde{\boldsymbol{c}})$ and $j \in S_2$

In this case, we must have $r_j \leq R_M^*(\tilde{c})$ due to Proposition A.1. We also notice that

$$R_M^*(\boldsymbol{c}) \geq R(S_M^*(\tilde{\boldsymbol{c}}), \boldsymbol{c}) = \frac{\sum_{i \in S_M^*(\tilde{\boldsymbol{c}})} r_i v_i + \sum_{i \in S_2 \setminus S_M^*(\tilde{\boldsymbol{c}})} r_i \tilde{\beta}^i v_i - (\tilde{\beta}^j - \beta^j) v_j r_j}{1 + \sum_{i \in S_M^*(\tilde{\boldsymbol{c}})} v_i + \sum_{i \in S_2 \setminus S_M^*(\tilde{\boldsymbol{c}})} \tilde{\beta}^i v_i - (\tilde{\beta}^j - \beta^j) v_j}.$$

Notice that

$$R_M^*(\tilde{\boldsymbol{c}}) = \frac{\sum_{i \in S_M^*(\tilde{\boldsymbol{c}})} r_i v_i + \sum_{i \in S_2 \setminus S_M^*(\tilde{\boldsymbol{c}})} r_i \tilde{\beta}^i v_i}{1 + \sum_{i \in S_M^*(\tilde{\boldsymbol{c}})} v_i + \sum_{i \in S_2 \setminus S_M^*(\tilde{\boldsymbol{c}})} \tilde{\beta}^i v_i}.$$

Using the fact that $\beta^j < \tilde{\beta}^j$ and $r_j \leq R_M^*(\tilde{\boldsymbol{c}})$, we can get $R_M^*(\boldsymbol{c}) \geq R_M^*(\tilde{\boldsymbol{c}})$. Therefore, combining the above two cases, the proposition is proved.

■

Appendix B: Additional Proofs for Section 5

Proof of Lemma B.3

Proof: We let $\beta_1 = \exp(-c_1)$ and $\beta_2 = \exp(-c_2)$ with $\beta_1, \beta_2 \in (0,1)$. Then, the condition $c_1 \leq c_2$ is equivalent to $\beta_1 \geq \beta_2$. We first express $R(\boldsymbol{S})$ in terms of R_1, R_2, R_3 and R_4 as follows:

$$\begin{aligned} 2R(\boldsymbol{S}) = & \frac{R_1 V_1 + R_2 V_2}{1 + V_1 + V_2} + \frac{\beta_1 R_3 V_3 + \beta_1 R_4 V_4}{(1 + V_1 + V_2)(1 + V_1 + V_2 + \beta_1 V_3 + \beta_1 V_4)} \\ & + \frac{R_1 V_1 + R_3 V_3}{1 + V_1 + V_3} + \frac{\beta_2 R_2 V_2 + \beta_2 R_4 V_4}{(1 + V_1 + V_3)(1 + V_1 + \beta_2 V_2 + V_3 + \beta_2 V_4)} = Q_1 R_1 + Q_2 R_2 + Q_3 R_3 + Q_4 R_4, \end{aligned}$$

where we denote

$$\begin{aligned} Q_1 &= \frac{V_1}{1 + V_1 + V_2} + \frac{V_1}{1 + V_1 + V_3}. \\ Q_2 &= \frac{V_2}{1 + V_1 + V_2} + \frac{\beta_2 V_2}{(1 + V_1 + V_3)(1 + V_1 + \beta_2 V_2 + V_3 + \beta_2 V_4)}. \\ Q_3 &= \frac{\beta_1 V_3}{(1 + V_1 + V_2)(1 + V_1 + V_2 + \beta_1 V_3 + \beta_1 V_4)} + \frac{V_3}{1 + V_1 + V_3}. \\ Q_4 &= \frac{\beta_1 V_4}{(1 + V_1 + V_2)(1 + V_1 + V_2 + \beta_1 V_3 + \beta_1 V_4)} + \frac{\beta_2 V_4}{(1 + V_1 + V_3)(1 + V_1 + V_3 + \beta_2 V_2 + \beta_2 V_4)}. \end{aligned}$$

Notice that due to $\beta_1 \geq \beta_2$, we have the following relationship:

$$Q_2 \leq \frac{V_2}{1 + V_1 + V_2} + \frac{\beta_1 V_2}{(1 + V_1 + V_3)(1 + V_1 + \beta_1 V_2 + V_3 + \beta_1 V_4)} = Q_2'.$$

We express $R(\boldsymbol{S_1})$ as follows:

$$\begin{aligned} 2R(\boldsymbol{S_1}) = & \frac{R_1 V_1}{1 + V_1} + \frac{\beta_1 (R_2 V_2 + R_3 V_3 + R_4 V_4)}{(1 + V_1)(1 + V_1 + \beta_1 V_2 + \beta_1 V_3 + \beta_1 V_4)} \\ & + \frac{R_1 V_1 + R_2 V_2 + R_3 V_3}{1 + V_1 + V_2 + V_3} + \frac{\beta_2 R_4 V_4}{(1 + V_1 + V_2 + V_3)(1 + V_1 + V_2 + V_3 + \beta_2 V_4)} = Q_1^1 R_1 + Q_2^1 R_2 + Q_3^1 R_3 + Q_4^1 R_4, \end{aligned}$$

where we denote

$$Q_1^1 = \frac{V_1}{1+V_1} + \frac{V_1}{1+V_1+V_2+V_3}.$$
$$Q_2^1 = \frac{\beta_1 V_2}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)} + \frac{V_2}{1+V_1+V_2+V_3}.$$
$$Q_3^1 = \frac{\beta_1 V_3}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)} + \frac{V_3}{1+V_1+V_2+V_3}.$$
$$Q_4^1 = \frac{\beta_1 V_4}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)} + \frac{\beta_2 V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_2V_4)}.$$

We also express $R(\boldsymbol{S_2})$ as follows:

$$2R(\boldsymbol{S_2}) = \frac{R_1V_1+R_2V_2}{1+V_1+V_2} + \frac{\beta_1R_3V_3+\beta_1R_4V_4}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)} + \frac{R_1V_1+R_2V_2+R_3V_3}{1+V_1+V_2+V_3} + \frac{\beta_2R_4V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_2V_4)} = Q_1^2R_1+Q_2^2R_2+Q_3^2R_3+Q_4^2R_4,$$

where we denote

$$Q_1^2 = \frac{V_1}{1+V_1+V_2} + \frac{V_1}{1+V_1+V_2+V_3}.$$
$$Q_2^2 = \frac{V_2}{1+V_1+V_2} + \frac{V_2}{1+V_1+V_2+V_3}.$$
$$Q_3^2 = \frac{\beta_1 V_3}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)} + \frac{V_3}{1+V_1+V_2+V_3}.$$
$$Q_4^2 = \frac{\beta_1 V_4}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)} + \frac{\beta_2 V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_2V_4)}.$$

We let $\gamma = \frac{(1+V_1)(1+V_1+V_2)}{(1+V_1+V_3)(1+V_1+V_2+V_3)}$ and it is obvious that $\gamma \in (0,1)$. We make the following four claims.

Claim 1: $\gamma Q_1^1 + (1-\gamma)Q_1^2 = Q_1$.

The first claim is by simple algebra. □

Claim 2: $\gamma Q_2^1 + (1-\gamma)Q_2^2 \geq Q_2' \geq Q_2$.

To prove the second claim, it suffices to prove $\gamma Q_2^1 + (1-\gamma)Q_2^2 \geq Q_2'$. To that end, it is equivalent to prove the following inequality:

$$\frac{\gamma\beta_1}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)} + \frac{1}{1+V_1+V_2+V_3} \geq \frac{\gamma}{1+V_1+V_2} + \frac{\beta_1}{(1+V_1+V_3)(1+V_1+\beta_1V_2+V_3+\beta_1V_4)}.$$

We plug in the expression of γ and the above inequality is equivalent to:

$$\frac{\beta_1(1+V_1+V_2)}{(1+V_1+V_3)(1+V_1+V_2+V_3)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)} + \frac{1}{1+V_1+V_2+V_3} \geq \frac{1+V_1}{(1+V_1+V_3)(1+V_1+V_2+V_3)} + \frac{\beta_1}{(1+V_1+V_3)(1+V_1+\beta_1V_2+V_3+\beta_1V_4)},$$

which can be further simplified as:

$$\frac{V_3}{1+V_1+V_2+V_3} \geq \beta_1\left[\frac{1}{1+V_1+\beta_1V_2+V_3+\beta_1V_4} - \frac{1+V_1+V_2}{(1+V_1+V_2+V_3)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)}\right].$$

Notice that if the right-hand side of the above inequality is negative, then the inequality naturally holds. Otherwise, we have:

$$\begin{aligned}
&\beta_1\left[\frac{1}{1+V_1+\beta_1V_2+V_3+\beta_1V_4}-\frac{1+V_1+V_2}{(1+V_1+V_2+V_3)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)}\right]\\
&\le \frac{1}{1+V_1+\beta_1V_2+V_3+\beta_1V_4}-\frac{1+V_1+V_2}{(1+V_1+V_2+V_3)(1+V_1+\beta_1V_2+V_3+\beta_1V_4)}\\
&=\frac{1}{1+V_1+\beta_1V_2+V_3+\beta_1V_4}\cdot\frac{V_3}{1+V_1+V_2+V_3}\le\frac{V_3}{1+V_1+V_2+V_3}.
\end{aligned}$$

Therefore, we have proved the second claim. □

Claim 3: $\gamma(Q_2^1+Q_3^1)+(1-\gamma)(Q_2^2+Q_3^2)\ge Q_2'+Q_3\ge Q_2+Q_3$.

To prove the third claim, it suffices to prove $\gamma(Q_2^1+Q_3^1)+(1-\gamma)(Q_2^2+Q_3^2)\ge Q_2'+Q_3$. To that end, it is equivalent to prove the following inequality:

$$\begin{aligned}
&\frac{\gamma\beta_1(V_2+V_3)}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)}+\frac{V_2+V_3}{1+V_1+V_2+V_3}\ge\frac{\gamma V_2}{1+V_1+V_2}+\frac{V_3}{1+V_1+V_3}\\
&+\frac{\gamma\beta_1V_3}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)}+\frac{\beta_1V_2}{(1+V_1+V_3)(1+V_1+\beta_1V_2+V_3+\beta_1V_4)}.
\end{aligned}\tag{B.1}$$

Notice that the following equality holds by simple algebra:

$$\frac{V_2+V_3}{1+V_1+V_2+V_3}=\frac{\gamma V_2}{1+V_1+V_2}+\frac{V_3}{1+V_1+V_3}.$$

Therefore, to prove (B.1), it is equivalent to show the following:

$$\frac{(V_2+V_3)(1+V_1+V_2)}{1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4}\ge\frac{V_3(1+V_1)}{1+V_1+V_2+\beta_1V_3+\beta_1V_4}+\frac{V_2(1+V_1+V_2+V_3)}{1+V_1+\beta_1V_2+V_3+\beta_1V_4}.$$

To that end, we notice that:

$$\begin{aligned}
\frac{(V_2+V_3)(1+V_1+V_2)}{1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4}&=\frac{V_3(1+V_1)+V_2(1+V_1+V_2+V_3)}{1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4}\\
&\ge\frac{V_3(1+V_1)}{1+V_1+V_2+\beta_1V_3+\beta_1V_4}+\frac{V_2(1+V_1+V_2+V_3)}{1+V_1+\beta_1V_2+V_3+\beta_1V_4}.
\end{aligned}$$

Therefore, we have proved the third claim. □

Claim 4: $\gamma Q_4^1+(1-\gamma)Q_4^2\ge Q_4$.

To prove the fourth claim, it is equivalent to show the following:

$$\begin{aligned}
&\frac{\gamma\beta_1}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)}-\frac{\gamma\beta_1}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)}\\
&\ge\frac{\beta_2}{(1+V_1+V_3)(1+V_1+\beta_2V_2+V_3+\beta_2V_4)}-\frac{\beta_2}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_2V_4)}.
\end{aligned}\tag{B.2}$$

We define the following function for $x\in(0,1)$:

$$f(x)=\frac{x}{(1+a+c)(1+a+bx+c+dx)}-\frac{x}{(1+a+b+c)(1+a+b+c+dx)},$$

where $a,b,c,d\ge 0$. One can check that $f'(x)=nu/de$ where the numerator nu is defined as

$$\begin{aligned}
nu&=-\left(2bd+b^2\right)x^2-\left(2bc+(2a+2)\,b\right)x+2bdx+2bc+b^2+(2a+2)\,b\\
&=2bd(x-x^2)+b^2(1-x^2)+(2bc+(2a+2)b)(1-x)>0,
\end{aligned}$$

and the denominator de is defined as:

$$de = (d^4+2bd^3+b^2d^2)x^4+((4c+2b+4a+4)d^3+(6bc+4b^2+(6a+6)b)d^2+(2b^2c+2b^3+(2a+2)b^2)d)x^3$$
$$+((6c^2+(6b+12a+12)c+b^2+(6a+6)b+6a^2+12a+6)d^2+(6bc^2+(8b^2+(12a+12)b)c+2b^3+(8a+8)b^2+(6a^2+12a+6)b)d$$
$$+b^2c^2+(2b^3+(2a+2)b^2)c+b^4+(2a+2)b^3+(a^2+2a+1)b^2)x^2+((4c^3+(6b+12a+12)c^2+(2b^2+(12a+12)b+12a^2+24a+12)c$$
$$+(2a+2)b^2+(6a^2+12a+6)b+4a^3+12a^2+12a+4)d+2bc^3+(4b^2+(6a+6)b)c^2+(2b^3+(8a+8)b^2+(6a^2+12a+6)b)c$$
$$+(2a+2)b^3+(4a^2+8a+4)b^2+(2a^3+6a^2+6a+2)b)x+c^4+(2b+4a+4)c^3+(b^2+(6a+6)b+6a^2+12a+6)c^2$$
$$+((2a+2)b^2+(6a^2+12a+6)b+4a^3+12a^2+12a+4)c+(a^2+2a+1)b^2+(2a^3+6a^2+6a+2)b+a^4+4a^3+6a^2+4a+1>0.$$

Therefore, to prove (B.2), it suffices to prove the following by noticing that $\beta_1 \geq \beta_2$ and $f'(x)>0$:

$$\frac{\gamma}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)} - \frac{\gamma}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)}$$
$$\geq \frac{1}{(1+V_1+V_3)(1+V_1+\beta_1V_2+V_3+\beta_1V_4)} - \frac{1}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_1V_4)}.$$

We plug in the expression of γ and simplify the above inequality into the following:

$$\frac{1+V_1+V_3}{1+V_1+V_2+V_3+\beta_1V_4} + \frac{1+V_1+V_2}{1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4} \geq \frac{1+V_1+V_2+V_3}{1+V_1+\beta_1V_2+V_3+\beta_1V_4} + \frac{1+V_1}{1+V_1+V_2+\beta_1V_3+\beta_1V_4}.$$

To prove the above inequality, we first define the following amounts:

$$a_1 = 1+V_1+V_3, b_1 = 1+V_1+V_2+V_3+\beta_1V_4$$
$$a_2 = 1+V_1+V_2, b_2 = 1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4$$
$$a_3 = 1+V_1+V_2+V_3, b_3 = 1+V_1+\beta_1V_2+V_3+\beta_1V_4$$
$$a_4 = 1+V_1, b_4 = 1+V_1+V_2+\beta_1V_3+\beta_1V_4.$$

Then, we want to prove $a_1/b_1 + a_2/b_2 \geq a_3/b_3 + a_4/b_4$. Namely, $(a_1b_2+a_2b_1)/b_1b_2 \geq (a_3b_4+a_4b_3)/b_3b_4$. By simple algebra, we can find that $a_1b_2+a_2b_1 = a_3b_4+a_4b_3$. It is also easy to verify that $b_1+b_2 = b_3+b_4$ and $b_1 \geq \max(b_3,b_4)$ and $b_2 \leq \min(b_3,b_4)$. Therefore, we have $b_1b_2 \leq b_3b_4$, which establishes $(a_1b_2+a_2b_1)/b_1b_2 \geq (a_3b_4+a_4b_3)/b_3b_4$ and the fourth claim accordingly. □

Then, combining the above four claims and noticing that $R_2 \geq R_3$, we have the following:

$$2(\gamma R(\boldsymbol{S_1}) + (1-\gamma)R(\boldsymbol{S_2}) - R(\boldsymbol{S})) = \Big(\gamma Q_1^1 + (1-\gamma)Q_1^2 - Q_1\Big)R_1$$
$$+\Big(\gamma Q_2^1 + (1-\gamma)Q_2^2 - Q_2\Big)R_2 + \Big(\gamma Q_3^1 + (1-\gamma)Q_3^2 - Q_3\Big)R_3 + \Big(\gamma Q_4^1 + (1-\gamma)Q_4^2 - Q_4\Big)R_4$$
$$\geq \Big(\gamma Q_2^1 + (1-\gamma)Q_2^2 - Q_2\Big)R_3 + \Big(\gamma Q_3^1 + (1-\gamma)Q_3^2 - Q_3\Big)R_3$$
$$= \Big(\gamma(Q_2^1+Q_3^1) + (1-\gamma)(Q_2^2+Q_3^2) - Q_2 - Q_3\Big)R_3 \geq 0,$$

which establishes Lemma B.3. ■

Proof of Lemma B.4

Proof: To show that $f(x)$ is quasi-convex in x for $x \geq 0$, we use the first-order condition for quasi-convexity (see Page 99 in Boyd and Vandenberghe 2004). To be specific, we want to show that the following inequality holds for all $x, y \geq 0$:

$$f(y) \leq f(x) \Longrightarrow \nabla f(x)^T(y-x) \leq 0.$$

It is easy to see that the inequality $f(y) \le f(x)$ is equivalent to the following:

$$(y-x)\sum_{i=1}^{m} \frac{b_i d_i - a_i}{(d_i+x)(d_i+y)} \le 0,$$

and the inequality $\nabla f(x)^T(y-x) \le 0$ is equivalent to the following:

$$(y-x)\sum_{i=1}^{m} \frac{b_i d_i - a_i}{(d_i+x)^2} \le 0.$$

We first consider the case of $y > x$. For this case, we aim to show the following:

$$\sum_{i=1}^{m} \frac{b_i d_i - a_i}{(d_i+x)(d_i+y)} \le 0 \Longrightarrow \sum_{i=1}^{m} \frac{b_i d_i - a_i}{(d_i+x)^2} \le 0. \tag{B.3}$$

We define $t_i = b_i d_i - a_i$ for $i \in [m]$. We can assume that $t_i \neq 0$ for all i without loss of generality. We then derive a property of t_i as follows. Consider t_i and t_{i+1} where $i \in \{1, \ldots, m-1\}$, if $t_i > 0$, then we must have $t_{i+1} > 0$, since otherwise we have

$$d_{i+1} < \frac{a_{i+1}}{b_{i+1}} \le \frac{a_i}{b_i} < d_i,$$

which is a contradiction. Therefore, there exists a $k \in \{0, \ldots, m\}$ such that $t_1, \ldots, t_k < 0$ and $t_{k+1}, \ldots, t_m > 0$. Note that when $k = m$, inequality (B.3) holds trivially. Thus, in the following, we consider the case where $k \in \{1, \ldots, m-1\}$. To show inequality (B.3), we first prove the following claim:

Claim: Suppose $0 < d_1 \le d_2 \le \cdots \le d_m$ and for $l_1, l_2, \ldots, l_m$, there exists a $k \in \{1, \ldots, m-1\}$ such that $l_1, \ldots, l_k < 0$ and $l_{k+1}, \ldots, l_m > 0$. Then, for $0 < x < y$, we have the following:

$$\frac{l_1}{d_1+y} + \frac{l_2}{d_2+y} + \cdots + \frac{l_m}{d_m+y} \le 0 \Longrightarrow \frac{l_1}{d_1+x} + \frac{l_2}{d_2+x} + \cdots + \frac{l_m}{d_m+x} \le 0.$$

We prove the above claim by contradiction. Suppose we have $\frac{l_1}{d_1+x} + \frac{l_2}{d_2+x} + \cdots + \frac{l_m}{d_m+x} > 0$ instead, then we have

$$\left(\frac{l_1}{d_1+x} - \frac{l_1}{d_1+y}\right) + \left(\frac{l_2}{d_2+x} - \frac{l_2}{d_2+y}\right) + \cdots + \left(\frac{l_m}{d_m+x} - \frac{l_m}{d_m+y}\right) > 0,$$

which leads to the following since we have $y > x$

$$\frac{l_1}{(d_1+x)(d_1+y)} + \frac{l_2}{(d_2+x)(d_2+y)} + \cdots + \frac{l_m}{(d_m+x)(d_m+y)} > 0.$$

However, we then the following inequalities:

$$\begin{aligned}
0 &< \frac{l_1}{(d_1+x)(d_1+y)} + \frac{l_2}{(d_2+x)(d_2+y)} + \cdots + \frac{l_m}{(d_m+x)(d_m+y)} \\
&< \frac{l_1}{d_1+y} \cdot \frac{1}{d_k+x} + \cdots + \frac{l_k}{d_k+y} \cdot \frac{1}{d_k+x} + \frac{l_{k+1}}{d_{k+1}+y} \cdot \frac{1}{d_k+x} + \cdots + \frac{l_m}{d_m+y} \cdot \frac{1}{d_k+x} \\
&= \frac{1}{d_k+x}\left(\frac{l_1}{d_1+y} + \frac{l_2}{d_2+y} + \cdots + \frac{l_m}{d_m+y}\right) \le 0,
\end{aligned}$$

which leads to a contradiction. Therefore, the above claim must hold. Then, if we let $l_i = \frac{t_i}{d_i+x}$ for all $i \in [m]$, inequality (B.3) is proved for the case of $y > x$. A similar analysis can be done in the case of $y < x$. Thus, the proof is completed. ■

Appendix C: Additional Proofs for Section 6

C.1. Proof of Proposition C.1

Proof: We first calculate the consumer surplus for the SMR model. Under the SMR model, if the customer visits store i, he will simultaneously be shown the set of products in S_i (with no discount) and the set of products in $\bar{S}\setminus S_i$ (with utility discount $\exp(-c_i^k(\boldsymbol{S}))$). Due to the result in McFadden (1974), we know that the consumer surplus for this type of customer is $\log(1+\sum_{k\in S_i} v_k + \sum_{k\in\bar{S}\setminus S_i} v_k \exp(-c_i^k(\boldsymbol{S})))$. The consumer surplus for the SMR model then follows by taking a weighted sum of all types of customers.

We then calculate the consumer surplus for the SQR model. Under the SQR model, if the customer visits store i, he will first be shown the set of products in S_i. If the customer decides not to purchase any product in S_i, he will then be shown the set of products in $\bar{S}\setminus S_i$. The choice process of this customer is exactly the same as the impatient MNL model as in Gao et al. (2021). Due to the result in Gallego et al. (2023) (see Proposition 3), the consumer surplus for this type of customer is

$$\frac{\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp(-c_i^k(\boldsymbol{S})))}{1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp(-c_i^k(\boldsymbol{S}))} + \frac{V(S_i)}{1+V(S_i)}\log(1+V(S_i))$$
$$+ \frac{\sum_{k\in\bar{S}\setminus S_i} v_k \exp(-c_i^k(\boldsymbol{S}))}{(1+V(S_i))(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp(-c_i^k(\boldsymbol{S})))}\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp(-c_i^k(\boldsymbol{S}))).$$

The first term corresponds to the consumer surplus when no product is purchased. The second term corresponds to the consumer surplus when some product in S_i is purchased. The third term corresponds to the consumer surplus when some product in $\bar{S}\setminus S_i$ is purchased. The consumer surplus for the SQR model also follows by taking a weighted sum of all types of customers. ■

Appendix E: Some numerical examples

E.1. An Example where SMR improves revenue when there is capacity constraint

EXAMPLE 4. Consider a problem instance with two stores and two products with $(r_1, r_2) = (10.3, 9.9)$. The arrival fractions are $(\lambda_1, \lambda_2) = (0.5, 0.5)$. The attraction values are $(v_1, v_2) = (1.9, 1.3)$. The utility discounts are $c_{12}^1 = c_{12}^2 = c_{21}^1 = c_{21}^2 = 1$. We set the capacity constraint to 1 for each store, i.e., $|S_1| \le 1$ and $|S_2| \le 1$. Then, for this problem instance, one can verify that the optimal solution is $(S_1^*, S_2^*) = \{\{1\},\{2\}\}$ or $\{\{2\},\{1\}\}$ under the SMR model. In contrast, if the two stores operate separately (with the same capacity constraint), then the total expected revenue is 6.748. However, if we adopt the SMR strategy, then the total expected revenue is 6.943, which shows an improvement. ■

E.2. An Example where the optimal assortments may not be revenue-ordered for the SMR model under the omnichannel setting

Example 5. Consider a problem instance with three stores and four products with $(r_1, r_2, r_3, r_4) = (1, 0.5, 0.499, 0.1)$. Under the omnichannel setting, we let $S_3 = \mathcal{N}$. The arrival fractions are $(\lambda_1, \lambda_2, \lambda_3) = (0.5, 0.5, 0)$. The attraction values are $(v_1, v_2, v_3, v_4) = (1.9, 1.1, 0.7, 2.0)$. The utility discounts are:

$$C^1 = \begin{bmatrix} 0 & 1.0 & 1.0 \\ 0.3 & 0 & 0.5 \\ 0.2 & 0.6 & 0 \end{bmatrix}, C^2 = \begin{bmatrix} 0 & 0.2 & 0.5 \\ 0.7 & 0 & 0.4 \\ 0.9 & 0.4 & 0 \end{bmatrix}, C^3 = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.5 & 0 & 0.3 \\ 0.3 & 0.4 & 0 \end{bmatrix}, C^4 = \begin{bmatrix} 0 & 0.5 & 1.0 \\ 1.0 & 0 & 0.5 \\ 0.4 & 0.6 & 0 \end{bmatrix},$$

where we use C^k for $k = 1, 2, 3, 4$ to represent the matrix $[c_{ij}^k]_{i,j=1,2,3}$. One can then verify that the optimal assortments for stores 1 and 2 are $(S_1^*, S_2^*) = (\{1\}, \{1, 3\})$, which is not revenue-ordered. ■

E.3. An Example where the optimal assortments may not be revenue-ordered for the SQR model under the omnichannel setting

Example 6. Consider a problem instance with two stores and three products with $(r_1, r_2, r_3) = (1, 0.5, 0.49)$. Under the omnichannel setting, we let $S_2 = \mathcal{N}$. The arrival fractions are $(\lambda_1, \lambda_2) = (0.5, 0.5)$. The attraction values are $(v_1, v_2, v_3) = (0.4, 0.2, 1.0)$. The utility discounts are $(c_{12}^1, c_{12}^2, c_{12}^3) = (1.0, 0.3, 1.0)$. One can verify that the optimal assortment for store 1 is offering $\{1, 3\}$, which is not revenue-ordered. ■

E.4. An Example where SQR has higher consumer surplus than SMR

Example 7. Consider a problem instance with two stores and two products with $(r_1, r_2) = (4, 2)$. The arrival fractions are $(\lambda_1, \lambda_2) = (0.5, 0.5)$. The attraction values are $(v_1, v_2) = (1.3, 1.5)$. The utility discounts are $c_{12}^1 = c_{12}^2 = c_{21}^1 = c_{21}^2 = 0.1$. One can then verify that the optimal solution is $(S_1^*, S_2^*) = \{\{1\}, \{1\}\}$ under the SMR model. Meanwhile, the optimal solution is $(S_1^*, S_2^*) = \{\{1, 2\}, \{1\}\}$ under the SQR model. Based on Proposition C.1, one can also calculate that the consumer surplus under the SMR and SQR models are 0.8329 and 1.1848 respectively. ■

E.5. Problem instances for Figure 6

The problem instance for Figure 6(a) and 6(b) consists of six products with revenues $(r_1, r_2, r_3, r_4, r_5, r_6) = (8.4, 5.7, 3.9, 3.7, 3.7, 3.1)$. The attraction values are $(v_1, v_2, v_3, v_4, v_5, v_6) = (0.5, 2.6, 1.9, 2.8, 3.9, 2.5)$. The common utility discount is $c = 0.4$. The problem instance for Figure 6(c) consists of six products with revenues $(r_1, r_2, r_3, r_4, r_5, r_6) = (7.3, 6.6, 6.2, 5.4, 3.2, 3.1)$. The common attraction values are $(v_1, v_2, v_3, v_4, v_5, v_6) = (4, 1.5, 0.8, 3.2, 4.7, 3.5)$. The common utility discount is $c = 1$. ■

Appendix F: Numerical results on the heuristic performance in Proposition 7

In this section, we conduct numerical experiments to evaluate the performance of the proposed heuristic in Proposition 7 on synthesized data. We set the number of stores $m = 2$ and vary the number of products $n \in \{3, 4, 5, 6, 7\}$ and arrival fraction $\lambda \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$. The utility discount c_{ij}^k is randomly sampled from $U[0, 1]$. The revenue r_i is randomly sampled from $U[0, 10]$ and the utility u_{ik} is sampled from $U[0, 5]$. For each combination of parameters (i.e., n and λ), 10,000 problem instances are randomly generated. The performance of the heuristic is evaluated by the average ratio of R_Q^H/R_Q^*, which is summarized in Table 5. From Table 5, we observe that the proposed heuristic performs much better than the theoretical lower bound and attains near-optimal revenue over the instances.

n \ λ	0.1	0.2	0.3	0.4	0.5
3	98.65%	99.04%	99.30%	99.42%	99.47%
4	98.45%	98.85%	99.14%	99.29%	99.35%
5	98.36%	98.80%	99.08%	99.24%	99.26%
6	98.31%	98.79%	99.05%	99.22%	99.27%
7	98.31%	98.72%	99.02%	99.18%	99.26%

Table 5 Performance of the proposed heuristic

Appendix G: The integer programming formulation for the cardinality-constrained SQR problem under universally homogeneous disutilities

In this section, we propose an integer programming formulation for the following problem:

$$\max_{\boldsymbol{S}=(S_1,S_2,\dots,S_m)} \quad R^Q(\boldsymbol{S}) = \sum_{i=1}^m \lambda_i \left(\frac{\sum_{k\in S_i} r_k v_{ik}}{1+\sum_{k\in S_i} v_{ik}} + \frac{\sum_{k\in \bar{S}\setminus S_i} r_k v_{ik}\beta}{(1+\sum_{k\in S_i} v_{ik})(1+\sum_{k\in S_i} v_{ik} + \sum_{k\in \bar{S}\setminus S_i} v_{ik}\beta)} \right)$$
$$\text{s.t.} \quad |S_i| \le C_i \quad \forall\, i = 1,\dots,m,$$

where $\beta = \exp(-c)$ is the common utility discount. To formulate the problem (SQR-const) into an integer program, we follow the following steps.

Step 1: In the first step, we aim to express the objective function $R^Q(\boldsymbol{S})$ in terms of binary variables. To that end, we first define a set of binary variables $x_{ik} \in \{0,1\}$ for each $i \in [m]$ and $k \in [n]$, representing the decision of whether to include product k in S_i. We also introduce a set of auxiliary variables z_{ik} (as defined below) for each $i \in [m]$ and $k \in [n]$:

$$z_{ik} = \begin{cases} 0 & \text{if } x_{1k} = \dots = x_{i-1,k} = x_{i+1,k} = \dots = x_{mk} = 0, \\ \beta & \text{otherwise.} \end{cases} \tag{G.1}$$

Intuitively, we let $z_{ik} = \beta$ if at least one of the stores other than i offers product k. We then transform (G.1) into a set of linear constraints as follows:

$$\begin{cases} z_{ik} \ge \beta x_{jk} \quad \forall j \in [m] \text{ and } j \ne i, \\ z_{ik} \le \beta(x_{1k} + \dots + x_{i-1,k} + x_{i+1,k} + \dots + x_{mk}), \\ 0 \le z_{ik} \le \beta. \end{cases} \tag{\textbf{S1-a}}$$

Now, using x_{ik} and z_{ik}, we can reformulate the objective function $R^Q(\boldsymbol{S})$ as follows:

$$\begin{aligned} R^Q(\boldsymbol{S}) &= \sum_{i=1}^m \lambda_i \left(\frac{\sum_{k\in S_i} r_k v_{ik}}{1+\sum_{k\in S_i} v_{ik}} + \frac{\sum_{k\in \mathcal{N}\setminus S_i} r_k v_{ik} z_{ik}}{(1+\sum_{k\in S_i} v_{ik})(1+\sum_{k\in S_i} v_{ik} + \sum_{k\in \mathcal{N}\setminus S_i} v_{ik} z_{ik})} \right). \\ &= \sum_{i=1}^m \lambda_i \left(\frac{\sum_{k=1}^n r_k v_{ik} x_{ik}}{1+\sum_{k=1}^n v_{ik} x_{ik}} + \frac{\sum_{k=1}^n r_k v_{ik} z_{ik}(1-x_{ik})}{(1+\sum_{k=1}^n v_{ik} x_{ik})(1+\sum_{k=1}^n v_{ik} x_{ik} + \sum_{k=1}^n v_{ik} z_{ik}(1-x_{ik}))} \right) \end{aligned} \tag{G.2}$$

Define $\tilde{x}_{ik} = z_{ik}(1-x_{ik})$ for each $i \in [m]$ and $k \in [n]$, which can then be linearized by introducing the following set of inequalities:

$$\begin{cases} z_{ik} - \tilde{x}_{ik} \le x_{ik}, \\ \tilde{x}_{ik} \le z_{ik}, \\ \tilde{x}_{ik} \le 1 - x_{ik}, \\ \tilde{x}_{ik} \ge 0. \end{cases} \tag{\textbf{S1-b}}$$

Now, using x_{ik} and $\tilde{x}_{ik}$, we can further reformulate $R^Q(\boldsymbol{S})$ as follows:

$$\begin{aligned} R^Q(\boldsymbol{S}) = &\sum_{i=1}^m \lambda_i \Bigg(\frac{\sum_{k=1}^n r_k v_{ik} x_{ik}}{1+\sum_{k=1}^n v_{ik}x_{ik}} - \frac{\sum_{k=1}^n r_k v_{ik} x_{ik}}{(1+\sum_{k=1}^n v_{ik}x_{ik})(1+\sum_{k=1}^n v_{ik}x_{ik} + \sum_{k=1}^n v_{ik}\tilde{x}_{ik})} \\ &+ \frac{\sum_{k=1}^n r_k v_{ik} x_{ik} + \sum_{k=1}^n r_k v_{ik}\tilde{x}_{ik}}{(1+\sum_{k=1}^n v_{ik}x_{ik})(1+\sum_{k=1}^n v_{ik}x_{ik} + \sum_{k=1}^n v_{ik}\tilde{x}_{ik})} \Bigg). \\ = &\sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k \cdot \frac{v_{ik}x_{ik} + v_{ik}\tilde{x}_{ik}}{(1+\sum_{k=1}^n v_{ik}x_{ik})(1+\sum_{k=1}^n v_{ik}x_{ik} + \sum_{k=1}^n v_{ik}\tilde{x}_{ik})} \\ &+ \sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k v_{ik} x_{ik} \cdot \sum_{k=1}^n \frac{v_{ik}x_{ik} + v_{ik}\tilde{x}_{ik}}{(1+\sum_{k=1}^n v_{ik}x_{ik})(1+\sum_{k=1}^n v_{ik}x_{ik} + \sum_{k=1}^n v_{ik}\tilde{x}_{ik})}. \end{aligned} \tag{G.3}$$

Step 2: In the second step, we simplify the fractional form of (G.3) by using the change of variables technique. Specifically, we let

$$\begin{cases} \pi_{i0} = \dfrac{1}{1+\sum_{k=1}^n v_{ik}x_{ik}}, \\ \pi_{ik} = \dfrac{v_{ik}x_{ik}}{1+\sum_{k=1}^n v_{ik}x_{ik}} \quad \forall\, k \in [n], \\ \omega_{i0} = \dfrac{1}{(1+\sum_{k=1}^n v_{ik}x_{ik})(1+\sum_{k=1}^n v_{ik}x_{ik} + \sum_{k=1}^n v_{ik}\tilde{x}_{ik})}, \\ \omega_{ik} = \dfrac{v_{ik}x_{ik} + v_{ik}\tilde{x}_{ik}}{(1+\sum_{k=1}^n v_{ik}x_{ik})(1+\sum_{k=1}^n v_{ik}x_{ik} + \sum_{k=1}^n v_{ik}\tilde{x}_{ik})} \quad \forall\, k \in [n]. \end{cases} \tag{G.4}$$

Now, using ω_{ik} in (G.4), $R^Q(\boldsymbol{S})$ in (G.3) can be simplified to (G.5) by imposing a set of constraints:

$$R^Q(\boldsymbol{S}) = \sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k \omega_{ik} + \sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k v_{ik} x_{ik} \sum_{k=1}^n \omega_{ik}. \tag{G.5}$$

$$\text{s.t.} \quad \sum_{k=1}^n \pi_{ik} + \pi_{i0} = 1, \tag{\textbf{S2-a}}$$

$$\sum_{k=1}^n \omega_{ik} + \omega_{i0} = \pi_{i0}, \tag{\textbf{S2-b}}$$

$$\pi_{ik} = v_{ik} x_{ik} \pi_{i0}, \tag{G.6}$$

$$\omega_{ik} = v_{ik}(x_{ik} + \tilde{x}_{ik})\omega_{i0}, \tag{G.7}$$

for each $i \in [m]$ and $k \in [n]$.

Step 3: In the final step, we transform the nonlinear terms in the objective function and the constraints using the same technique in Step 1. To transform the nonlinear terms in (G.5), we define $y_{i,k_1,k_2} = x_{ik_1}\omega_{ik_2}$ for each $i \in [m]$ and $k_1, k_2 \in [n]$ and impose the following set of constraints:

$$\begin{cases} \omega_{ik_2} - y_{i,k_1,k_2} \le 1 - x_{ik_1}, \\ y_{i,k_1,k_2} \le \omega_{i,k_2}, \\ y_{i,k_1,k_2} \le x_{ik_1}, \\ y_{i,k_1,k_2} \ge 0. \end{cases} \tag{\textbf{S3-a}}$$

We can then rewrite (G.5) as follows:

$$R^Q(\boldsymbol{S}) = \sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k \omega_{ik} + \sum_{i=1}^m \lambda_i \sum_{k_1=1}^n r_{k_1} v_{ik_1} \sum_{k_2=1}^n y_{i,k_1,k_2}. \tag{G.8}$$

To transform the nonlinear terms in (E.6) and (E.7), we define $d_{ik} = x_{ik}\pi_{i0}$, $e_{ik} = x_{ik}\omega_{i0}$ and $f_{ik} = \tilde{x}_{ik}\omega_{i0}$ for each $i \in [m]$ and $k \in [n]$ and impose the following three sets of constraints:

$$\begin{cases} \pi_{i0} - d_{ik} \le 1 - x_{ik}, \\ d_{ik} \le \pi_{i0}, \\ d_{ik} \le x_{ik}, \\ d_{ik} \ge 0, \end{cases} \quad \begin{cases} \omega_{i0} - e_{ik} \le 1 - x_{ik}, \\ e_{ik} \le \omega_{i0}, \\ e_{ik} \le x_{ik}, \\ e_{ik} \ge 0, \end{cases} \quad \begin{cases} \beta\omega_{i0} - f_{ik} \le \beta - \tilde{x}_{ik}, \\ f_{ik} \le \beta\omega_{i0}, \\ f_{ik} \le \tilde{x}_{ik}, \\ f_{ik} \ge 0. \end{cases} \tag{\textbf{S3-b}}$$

We can then rewrite (E.6) and (E.7) as follows:

$$\pi_{ik} = v_{ik} d_{ik}. \tag{\textbf{S3-c}}$$

$$\omega_{ik} = v_{ik}(e_{ik} + f_{ik}). \tag{\textbf{S3-d}}$$

Note that the cardinality constraints can be represented by

$$\sum_{k=1}^n x_{ik} \le C_i \quad \forall i = 1, \ldots, m. \tag{G.9}$$

To summarize, for the SQR problem under cardinality constraints, we establish the following mixed integer programming formulation:

$$\begin{aligned} \max_{\substack{x_{ik}\in\{0,1\},\tilde{x}_{ik},z_{ik},\pi_{i0},\pi_{ik},\omega_{i0},\omega_{ik} \\ y_{i,k_1,k_2},d_{ik},e_{ik},f_{ik}}} \quad & \sum_{i=1}^m \lambda_i \sum_{k=1}^n r_k \omega_{ik} + \sum_{i=1}^m \lambda_i \sum_{k_1=1}^n r_{k_1} v_{ik_1} \sum_{k_2=1}^n y_{i,k_1,k_2} \\ \text{s.t.} \quad & (\textbf{S1-a}), (\textbf{S1-b}), (\textbf{S2-a}), (\textbf{S2-b}), (\textbf{S3-a}), (\textbf{S3-b}), (\textbf{S3-c}), (\textbf{S3-d}), (\text{G.9}). \end{aligned}$$

Appendix H: Joint Assortment and Pricing Optimization

In the main body of this paper, we assume that product prices are exogenously given. In some situations, the seller can also decide prices across different stores. In this section, we study the joint assortment and pricing optimization problem under both the SMR and SQR models.

H.1. SMR Model

We first formulate the joint problem under the SMR model. We use p_{ik} to denote the price of product k in store i. For simplicity, the production costs of all products are normalized to zero. We assume that customers' utilities have store-specific parameters. In particular, for a customer who visits store i and purchases product $k \in S_i$ with price p_{ik}, her utility is given by $\alpha_{ik} - \beta_{ik}p_{ik} + \epsilon_{ik}$, where $\alpha_{ik} \in \mathbb{R}^+$ and $\beta_{ik} \in \mathbb{R}^+$ represent the intrinsic utility and price sensitivity of product k in store i, respectively, and ϵ_{ik} are i.i.d Gumbel variables. Let $v_{ik} = \exp(\alpha_{ik} - \beta_{ik}p_{ik})$. If the customer purchases a product from other stores, then her utility and the corresponding price are denoted by

$$\tilde{v}_{ik}(\boldsymbol{S}) = \exp(\alpha_{ik} - \beta_{ik}p_{lk} - c_{il}^k) \ \textit{and}\ \tilde{p}_{ik}(\boldsymbol{S}) = p_{lk}, \ \textit{where}\ k \notin S_i, l = \arg\max_{\{j|k\in S_j\}} \alpha_{ik} - \beta_{ik}p_{jk} - c_{ij}^k.$$

Note that the values of $\tilde{v}_{ik}$ and $\tilde{p}_{ik}$ are dependent on the decided assortments $\boldsymbol{S}$. Let $\boldsymbol{P} = \{p_{ik}, i \in [m], k \in [n]\}$. With these notations, we formulate the joint problem under the SMR model as follows:

$$\max_{\boldsymbol{S},\boldsymbol{P}} \quad R^M(\boldsymbol{S},\boldsymbol{P}) \triangleq \sum_{i=1}^m \lambda_i \cdot \frac{\sum_{k\in S_i} p_{ik} v_{ik} + \sum_{k\in \bar{S}\setminus S_i} \tilde{p}_{ik}(\boldsymbol{S})\tilde{v}_{ik}(\boldsymbol{S})}{1+\sum_{k\in S_i} v_{ik} + \sum_{k\in\bar{S}\setminus S_i} \tilde{v}_{ik}(\boldsymbol{S})}. \tag{H.1}$$

To tackle problem (H.1), we first define the following single-store problem for analysis purposes:

$$\max_{S,\{p_k, k\in[n]\}} \quad \frac{\sum_{k\in S} p_k \exp(\alpha_k - \beta_k p_k)}{1+\sum_{k\in S}\exp(\alpha_k - \beta_k p_k)}, \tag{H.2}$$

where $\alpha_k \in \mathbb{R}^+$ and $\beta_k \in \mathbb{R}^+$. Problem (H.2) is the joint assortment and pricing problem for a single store under the MNL model, where the optimal solution is to offer the full product set. Let θ_i^* be the unique solution to the equation $\theta_i = \sum_{k\in\mathcal{N}} e^{\hat{\alpha}_{ik} - \beta_{ik}\theta_i}$ where $\hat{\alpha}_{ik} = \alpha_{ik} - \log(\beta_{ik}) - 1$ for $k \in \mathcal{N}$. We have the following theorem.

Theorem H.1. *The optimal solution to problem* (H.1) *is* $\boldsymbol{S}^* = (\mathcal{N},\ldots,\mathcal{N})$ *and* $p_{ik}^* = \theta_i^* + \frac{1}{\beta_{ik}}$ *for each* $i \in [m]$ *and* $k \in [n]$.

Proof: For any recommendation strategy $\boldsymbol{S}$ and prices p_{ik}, the expected revenue of store i is given by $(\sum_{k\in S_i} p_{ik}v_{ik} + \sum_{k\in\bar{S}\setminus S_i}\tilde{p}_{ik}(\boldsymbol{S})v_{ik}(\boldsymbol{S}))/(1+\sum_{k\in S_i} v_{ik} + \sum_{k\in\bar{S}\setminus S_i}\tilde{v}_{ik}(\boldsymbol{S}))$. Now, letting $\alpha_k = \alpha_{ik}$ and $\beta_k = \beta_{ik}$, consider a feasible solution to problem (H.2), where $S = \bar{S}$, $p_k = p_{ik}$ for $k \in S_i$ and $p_k = \frac{1}{\beta_{ik}}(\alpha_{ik} - \log(\tilde{v}_{ik}(\boldsymbol{S})))$ for $k \in \bar{S}\setminus S_i$. Note that $p_k \geq \tilde{p}_{ik}(\boldsymbol{S})$ since $c_{ij}^k \geq 0$. Therefore, the expected revenue of store i is upper bounded by the expected revenue of the single-store problem under the feasible solution, and thus is further upper bounded by the optimal revenue for the single-store problem (H.2). Note that the constructed solution in Theorem H.1 is exactly the optimal solution to problem (H.2) for each store (see, e.g., Wang 2012), and thus the expected revenue of each store matches the latter upper bound, which proves the optimality of the constructed solution. ■

Theorem H.1 shows that the optimal assortment in each store is to offer the full set of products $\mathcal{N}$, and therefore the optimal revenue under the SMR strategy is equal to that by operating different stores separately. Moreover, the optimal prices are also equal to those in the single-store price optimization problem, which is known to follow the "constant adjusted markup" property (see, e.g., Wang 2012). This extends our results under the SMR strategy in the base model to a more general setting.

H.2. SQR Model

We then study the joint problem under the SQR strategy, which can be formulated as follows:

$$\max_{\boldsymbol{S},\boldsymbol{P}} R^Q(\boldsymbol{S},\boldsymbol{P}) \triangleq \sum_{i=1}^m \lambda_i \left(\frac{\sum_{k\in S_i} p_{ik}v_{ik}}{1+\sum_{k\in S_i} v_{ik}} + \frac{\sum_{k\in\bar{S}\setminus S_i}\tilde{p}_{ik}(\boldsymbol{S})\tilde{v}_{ik}(\boldsymbol{S})}{(1+\sum_{k\in S_i} v_{ik})(1+\sum_{k\in S_i} v_{ik} + \sum_{k\in\bar{S}\setminus S_i}\tilde{v}_{ik}(\boldsymbol{S}))} \right). \tag{H.3}$$

Conceptually, the joint problem is more challenging than the pure assortment problem under the SQR model, which is NP-hard even with homogeneous customer valuation. Thus, we aim to find approximation solutions to problem (H.3) with guarantee. Recall that Proposition 7 establishes a performance guarantee of 1/2 for a heuristic that offers the single-store optimal assortment. We are interested in the performance of the single-store optimal solution in the joint problem.

PROPOSITION H.1. *The optimal solution* $(\boldsymbol{S},\boldsymbol{P})$ *to problem* (H.1) *can achieve at least 1/2 of the optimal revenue in the joint problem* (H.3).

Proof: Denote the optimal solution to problem (H.3) as $(\boldsymbol{S}^*,\boldsymbol{P}^*)$. The optimal revenue can be written as:

$$R(\boldsymbol{S}^*,\boldsymbol{P}^*)=\sum_{i=1}^m \lambda_i\left(\frac{\sum_{k\in S_i} p^*_{ik}v^*_{ik}}{1+\sum_{k\in S_i} v^*_{ik}}+\frac{\sum_{k\in\bar{S}\setminus S_i}\tilde{p}^*_{ik}(\boldsymbol{S})\tilde{v}^*_{ik}(\boldsymbol{S})}{(1+\sum_{k\in S_i} v^*_{ik})(1+\sum_{k\in S_i} v^*_{ik}+\sum_{k\in\bar{S}\setminus S_i}\tilde{v}^*_{ik}(\boldsymbol{S}))}\right).$$

It is obvious that the value of $\sum_{k\in S_i} p^*_{ik}v^*_{ik}/(1+\sum_{k\in S_i} v^*_{ik})$ must be smaller than the optimal objective value of problem (H.2). Clearly, if we can show that the second term in the big brackets is also smaller than the optimal objective value of problem (H.2), then the proof is completed. We then notice that

$$\frac{\sum_{k\in\bar{S}\setminus S_i}\tilde{p}^*_{ik}(\boldsymbol{S})\tilde{v}^*_{ik}(\boldsymbol{S})}{(1+\sum_{k\in S_i} v^*_{ik})(1+\sum_{k\in S_i} v^*_{ik}+\sum_{k\in\bar{S}\setminus S_i}\tilde{v}^*_{ik}(\boldsymbol{S}))}\le\frac{\sum_{k\in\bar{S}\setminus S_i}\tilde{p}^*_{ik}(\boldsymbol{S})\tilde{v}^*_{ik}(\boldsymbol{S})}{1+\sum_{k\in\bar{S}\setminus S_i}\tilde{v}^*_{ik}(\boldsymbol{S})}.$$

Now, letting $\alpha_k=\alpha_{ik}$ and $\beta_k=\beta_{ik}$, consider a feasible solution to problem (H.2), where $S=\bar{S}\setminus S_i$, $p_k=\frac{1}{\beta_{ik}}(\alpha_{ik}-\log(\tilde{v}^*_{ik}(\boldsymbol{S})))$ for any $k\in\bar{S}\setminus S_i$. Note that $p_k\ge\tilde{p}_{ik}(\boldsymbol{S})$ since $c^k_{ij}\ge 0$. Therefore, the value of $\sum_{k\in\bar{S}\setminus S_i}\tilde{p}^*_{ik}(\boldsymbol{S})\tilde{v}^*_{ik}(\boldsymbol{S})/(1+\sum_{k\in\bar{S}\setminus S_i}\tilde{v}^*_{ik}(\boldsymbol{S}))$ is upper bounded by the expected revenue of the single-store problem under the feasible solution, and thus is further upper bounded by the optimal objective value of problem (H.2). The proof is then completed. ■

Proposition H.1 shows that by adopting the single-store optimal solution, we can have a revenue guarantee of 1/2. That said, the approximation results in Proposition 7 can be extended to the joint problem setting under the SQR model.

Given that it is hard to analyze the general joint problem, we look at a special case of (H.3) with two stores, where store 1 holds an assortment S and store 2 holds the full set of products $\mathcal{N}$. One may consider store 2 as an online store and store 1 as a physical store. For tractability, we impose an assumption $\beta_{ik}=\beta_k$ for $i\in\{1,2\}$, which means that customers visiting different stores have the same price sensitivity toward the same product. Then, the problem can be formulated as follows:

$$\begin{aligned}\max_{S\subseteq\mathcal{N},\boldsymbol{p}_1,\boldsymbol{p}_2}\quad & R^Q(S,\boldsymbol{p}_1,\boldsymbol{p}_2)\triangleq(1-\lambda)\frac{\sum_{k\in\mathcal{N}} p_{2k}\exp(\alpha_{2k}-\beta_k p_{2k})}{1+\sum_{k\in\mathcal{N}}\exp(\alpha_{2k}-\beta_k p_{2k})}+\lambda\left(\frac{\sum_{k\in S} p_{1k}\exp(\alpha_{1k}-\beta_k p_{1k})}{1+\sum_{k\in S}\exp(\alpha_{1k}-\beta_k p_{1k})}\right. \\ &\left.+\frac{\sum_{k\in\mathcal{N}\setminus S} p_{2k}\exp(\alpha_{1k}-\beta_k p_{2k}-c^k_{12})}{(1+\sum_{k\in S}\exp(\alpha_{1k}-\beta_k p_{1k}))(1+\sum_{k\in S}\exp(\alpha_{1k}-\beta_k p_{1k})+\sum_{k\in\mathcal{N}\setminus S}\exp(\alpha_{1k}-\beta_k p_{2k}-c^k_{12}))}\right),\end{aligned} \tag{H.4}$$

where $\boldsymbol{p}_1=(p_{11},\dots,p_{1n})$ and $\boldsymbol{p}_2=(p_{21},\dots,p_{2n})$. Letting $V_1=\sum_{k\in S}\exp(\alpha_{1k}-\beta_k p_{1k})$, we simplify problem (H.4) as follows:

$$\begin{aligned}\max_{S\subseteq\mathcal{N},\boldsymbol{p}_1,\boldsymbol{p}_2}\quad & \underbrace{\lambda\frac{\sum_{k\in S} p_{1k}\exp(\alpha_{1k}-\beta_k p_{1k})}{1+V_1}}_{L_1}+\underbrace{\lambda\frac{\sum_{k\in\mathcal{N}\setminus S} p_{2k}\exp(\alpha_{1k}-\beta_k p_{2k}-c^k_{12})}{(1+V_1)(1+V_1+\sum_{k\in\mathcal{N}\setminus S}\exp(\alpha_{1k}-\beta_k p_{2k}-c^k_{12}))}}_{L_2} \\ & +\underbrace{(1-\lambda)\frac{\sum_{k\in\mathcal{N}} p_{2k}\exp(\alpha_{2k}-\beta_k p_{2k})}{1+\sum_{k\in\mathcal{N}}\exp(\alpha_{2k}-\beta_k p_{2k})}}_{L_3}.\end{aligned} \tag{H.5}$$

For any given S, note that problem (H.5) can be decomposed into two subproblems, where the first one is to maximize L_1 over $\boldsymbol{p}_1$, and the second one is to maximize L_2 and L_3 over $\boldsymbol{p}_2$. For the first subproblem, the optimal prices also follow the "constant adjusted markup" property and thus can be efficiently calculated. For the second one, we find that it shares similarities with the pricing problem under the finite-mixture logit (FML) model in van de Geer and den Boer (2022). Hence, the algorithm in van de Geer and den Boer (2022) can be used to calculate the optimal $\boldsymbol{p_2}$. Thus, problem (H.4) with a given S can be solved. We also note that problem (H.4) with given prices (optimizing over S) can be solved using an integer program similar to that in Section 7.2. We iteratively repeat this procedure until we reach the optimal solution. Such an iterative process is summarized into a coordinate ascent-type algorithm. We then conduct numerical experiments to compare the revenue in problem (H.4) with its counterpart under the SMR model. We find that the SQR strategy still outperforms the SMR strategy when we incorporate the pricing decisions. For the clarity of exposition, we relegate the algorithm and numerical results to Appendix H.3 and Appendix H.4.

H.3. Algorithm Design

The algorithm for problem (H.4) with any given S is presented as follows.

Algorithm 1 Algorithm for problem (H.4) with any given S

Input: Model parameters λ, α_{1k}, α_{2k}, β_k, c_{12}^k, S, sufficiently small $\Delta, \epsilon > 0$, sufficiently large $M > 0$

Output: The optimal solution $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ to problem (H.4) for any given S.

1: Initialize $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*) = (\mathbf{0}, \mathbf{0})$.

2: **for** $V_1 = 0 : \Delta : \sum_{k \in S} \exp(\alpha_{1k})$ **do**

3: Calculate $\tilde{\boldsymbol{p}}_{\mathbf{1}}$: Denote θ as the unique root to the equation $\sum_{k \in S} \exp(\alpha_{1k} - 1 - \beta_{1k}\theta) = V_1$ and let $\tilde{p}_{1k} = \frac{1}{\beta_{1k}} + \theta$ for $k \in S$.

4: Calculate $\tilde{\boldsymbol{p}}_{\mathbf{2}}$: Reduce the optimization problem regarding p_{2k} into an instance of the pricing problem under the FML model (van de Geer and den Boer 2022). The parameters of the instance are denoted as (ω_c, a_{ck}, b_k) for $c \in \{1, 2\}$ and $k \in \mathcal{N}$. In particular, we let

- $\omega_1 = \frac{\lambda}{1+V_1} / (\frac{\lambda}{1+V_1} + 1 - \lambda)$ and $\omega_2 = (1-\lambda)/(\frac{\lambda}{1+V_1} + 1 - \lambda)$.
- $a_{1k} = -M$ for $k \in S$, $a_{1k} = \alpha_{1k} - c_{12}^k - \log(1 + V_1)$ for $k \in \mathcal{N} \setminus S$, and $a_{2k} = \alpha_{2k}$ for $k \in \mathcal{N}$.
- $b_k = \beta_k$ for $k \in \mathcal{N}$.

Utilize Algorithm BNB(ϵ) to obtain $\tilde{\boldsymbol{p}}_{\mathbf{2}}$.

5: Calculate $R^Q(\tilde{\boldsymbol{p}}_{\mathbf{1}}, \tilde{\boldsymbol{p}}_{\mathbf{2}})$. If $R^Q(\tilde{\boldsymbol{p}}_{\mathbf{1}}, \tilde{\boldsymbol{p}}_{\mathbf{2}}) \geq R^Q(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$, then update $\boldsymbol{p}_1^* \leftarrow \tilde{\boldsymbol{p}}_{\mathbf{1}}$ and $\boldsymbol{p}_2^* \leftarrow \tilde{\boldsymbol{p}}_{\mathbf{2}}$.

In Algorithm 1, we first conduct a grid search on V_1 on the interval $[0, \sum_{k \in S} \exp(\alpha_{1k})]$ since $p_{1k} \geq 0$. The grid length is controlled by the parameter Δ. Then, for a given value of V_1, we calculate $\tilde{\boldsymbol{p}}_{\mathbf{1}}$ and $\tilde{\boldsymbol{p}}_{\mathbf{2}}$ respectively using the approaches mentioned in Section H.2. When calculating $\tilde{\boldsymbol{p}}_{\mathbf{1}}$, it is equivalent to solving the following:

$$\max_{p_{1k}} \quad \sum_{k \in S} p_{1k} \cdot \exp(\alpha_{1k} - \beta_k p_{1k}) \quad \text{s.t.} \quad \sum_{k \in S} \exp(\alpha_{1k} - \beta_k p_{1k}) = V_1.$$

Using KKT conditions, one can easily verify that the optimal solution follows $\tilde{p}_{1k} = \frac{1}{\beta_k} + \theta$, where the value of θ is the unique solution to the equation $\sum_{k \in S} \exp(\alpha_{1k} - 1 - \beta_k \theta) = V_1$. When calculating $\tilde{\boldsymbol{p}}_{\mathbf{2}}$, we utilize the algorithm BNB(ϵ) developed in van de Geer and den Boer (2022), where $\epsilon > 0$ is the tolerance of error. The assumption of $\beta_{ik} = \beta_k$ is adopted in van de Geer and den Boer (2022) to avoid introducing non-linear constraints in their reparameterization (see Remark 2 therein). Note that we set $a_{1k} = -M$ for $k \in S$ with a sufficiently large $M > 0$ to ensure that no matter what values we assign to p_{1k} for $k \in S$, the attraction value of that product stays zero. We would like to remark that the idea of the algorithm cannot be easily generalized to the case of multiple physical stores. The major impediment is that the decision process of the customer involves finding the product in other stores that provide the largest utility, which is generally difficult to be reformulated to a problem similar to the pricing problem under the FLM model.

In the following, we discuss how to solve problem (H.4) based on Algorithm 1. The key idea is to adopt a coordinate ascent-type algorithm. In particular, we first fix $S = S^0 = \mathcal{N}$ and solve problem (H.4) using Algorithm 1. Denote the optimal solution to the pure pricing problem as $(\boldsymbol{p}_1^1, \boldsymbol{p}_2^1)$. Then, fix $(\boldsymbol{p}_1, \boldsymbol{p}_2) = (\boldsymbol{p}_1^1, \boldsymbol{p}_2^1)$ and solve problem (H.4) to obtain the optimal assortment in store 1, denoted as S^1. Notice that one can adopt the integer programming formulation in Appendix G to solve the pure assortment optimization problem. Specifically, we let $v_{1k} = \exp(\alpha_{1k} - \beta_k p_{1k})$ and $\tilde{v}_{1k} = \exp(\alpha_{1k} - \beta_k p_{2k} - c_{12}^k)$ for $k \in \mathcal{N}$ for convenience. We also define $x_k \in \{0,1\}$ as the binary variable to indicate whether product k is included in S. Then, the pure assortment optimization problem is equivalent to solving the following problem:

$$\max_{\boldsymbol{x}=(x_1,\ldots,x_n)} \quad \frac{\sum_{k=1}^n v_{1k} p_{1k} x_k}{1 + \sum_{k=1}^n v_{1k} x_k} + \frac{\sum_{k=1}^n \tilde{v}_{1k} p_{2k} (1 - x_k)}{(1 + \sum_{k=1}^n v_{1k} x_k)(1 + \sum_{k=1}^n v_{1k} x_k + \sum_{k=1}^n \tilde{v}_{1k}(1 - x_k))}. \tag{H.6}$$

Note that problem (H.6) shares the same formulation as problem (G.2) if we let $m = 1$ and $z_{1k} = \tilde{v}_{1k}/v_{1k}$. Therefore, problem (H.6) can be efficiently solved by an integer program. Then, fix $S = S^1$, we solve the pure pricing problem again. We repeat the above steps until $S^{t+1} = S^t$. The algorithm for problem (H.4) is presented as follows.

Algorithm 2 Algorithm for solving problem (H.4)

Input: Model parameters λ, α_{1k}, α_{2k}, β_k, c_{12}^k, sufficiently small $\Delta, \epsilon > 0$, sufficiently large $M > 0$

Output: The optimal solution $(S^*, \boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ to problem (H.4).

1: Initialize $t = 0$, $S^t = \mathcal{N}$ and $(\boldsymbol{p}_1^t, \boldsymbol{p}_2^t) = (\mathbf{0}, \mathbf{0})$.
2: **while** True **do**
3: Fix $S = S^t$ and solve the pure pricing problem (H.4) using Algorithm 1. Obtain the optimal solution as $(\boldsymbol{p}_1^{t+1}, \boldsymbol{p}_2^{t+1})$.
4: Fix $(\boldsymbol{p}_1, \boldsymbol{p}_2) = (\boldsymbol{p}_1^{t+1}, \boldsymbol{p}_2^{t+1})$ and solve the pure assortment optimization problem (H.6) using the integer program formulation in Appendix G. Obtain the optimal solution as S^{t+1}.
5: **if** $S^{t+1} = S^t$ and $(\boldsymbol{p}_1^{t+1}, \boldsymbol{p}_2^{t+1}) = (\boldsymbol{p}_1, \boldsymbol{p}_2)$ **then** break.
6: $t = t + 1$
7: Let $S^* = S^{t+1}$ and $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*) = (\boldsymbol{p}_1^{t+1}, \boldsymbol{p}_2^{t+1})$.

H.4. Computational Experiments

In this section, we conduct some numerical experiments to compare the revenue of the two strategies for the two-store joint problem where store 2 offers the full set of products $\mathcal{N}$. For the SQR strategy, the problem (H.4) can be solved using Algorithm 2. For the SMR strategy, we aim to solve the following problem:

$$\begin{aligned} \max_{S\subseteq\mathcal{N},\boldsymbol{p}_1,\boldsymbol{p}_2} \quad R^M(\boldsymbol{p}_1,\boldsymbol{p}_2) \triangleq & \lambda \frac{\sum_{k\in S} p_{1k}\exp(\alpha_{1k}-\beta_k p_{1k}) + \sum_{k\in\mathcal{N}\setminus S} p_{2k}\exp(\alpha_{1k}-\beta_k p_{2k}-c_{12}^k)}{1+\sum_{k\in S}\exp(\alpha_{1k}-\beta_k p_{1k}) + \sum_{k\in\mathcal{N}\setminus S}\exp(\alpha_{1k}-\beta_k p_{2k}-c_{12}^k)} \\ & + (1-\lambda)\frac{\sum_{k\in\mathcal{N}} p_{2k}\exp(\alpha_{2k}-\beta_k p_{2k})}{1+\sum_{k\in\mathcal{N}}\exp(\alpha_{2k}-\beta_k p_{2k})} \end{aligned} \tag{H.7}$$

According to Theorem H.1, one can conclude that the optimal solution to problem (H.7) follows $S^* = \mathcal{N}$ and $p_{ik}^* = \theta_i^* + \frac{1}{\beta_{ik}}$ for $i = 1,2$ and $k \in \mathcal{N}$. Recall that θ_i^* is the unique solution to the equation $\theta_i = \sum_{k\in\mathcal{N}} e^{\hat{\alpha}_{ik}-\beta_{ik}\theta_i}$ where $\hat{\alpha}_{ik} = \alpha_{ik} - \log(\beta_{ik}) - 1$ for $k \in \mathcal{N}$. Therefore, the optimal expected revenue for the SMR strategy can be calculated. We then compute the revenue improvement of the SQR strategy compared to the SMR strategy over randomly generated instances. The model parameters are generated as follows:

- $\lambda \sim U[0,1]$.
- $\alpha_{ik} \sim U[0,a]$, where $a > 0$ is to be selected.
- $\beta_{ik} \sim 0.1 + U[0,b]$, where $b > 0$ is to be selected.
- $c_{12}^k \sim U[0,1]$.

Note that we add a constant term when generating β_{ik} to avoid numerical issues when the value of b is close to zero. We vary the number of products $n \in \{3,5,7\}$, the value of $a \in [2,4,6]$, and the value of $b \in [0.5,1,2]$. For each combination of model parameters, we generate 1000 instances and calculate the average revenue improvement. The computational results are summarized as follows:

a \ b	0.5	1	2
2	0.95%	1.28%	1.34%
4	1.41%	1.49%	1.60%
6	1.75%	1.71%	1.72%

(a) $n = 3$

a \ b	0.5	1	2
2	1.64%	1.88%	2.02%
4	2.29%	2.67%	2.57%
6	2.68%	2.78%	2.69%

(b) $n = 5$

a \ b	0.5	1	2
2	2.08%	2.50%	2.77%
4	2.83%	3.25%	3.13%
6	3.22%	3.55%	3.49%

(c) $n = 7$

Table 6 Summarized computational results on the revenue improvement of SQR over SMR

Table 6 shows that in the two-store joint assortment and pricing setting, the SQR strategy still garners a considerable revenue improvement over the SMR strategy. In particular, when the number of products and the intrinsic utilities are larger, the revenue improvement is more prominent. Table 6 confirms the robustness of our managerial insight that the SQR strategy outperforms the SMR strategy even when we incorporate the pricing decisions.