SIAM/ASA J. UNCERTAINTY QUANTIFICATION
Vol. 12, No. 2, pp. 579–613

Quantifying the Effect of Random Dispersion for Logarithmic Schrödinger Equation*

Jianbo Cui† and Liying Sun‡

Abstract. This paper is concerned with the random effect of the noise dispersion for the stochastic logarithmic Schrödinger equation emerged from the optical fibre with dispersion management. The well-posedness of the logarithmic Schrödinger equation with white noise dispersion is established via the regularization energy approximation and a spatial scaling property. For the small noise case, the effect of the noise dispersion is quantified by the proven large deviation principle under additional regularity assumptions on the initial datum. As an application, we show that for the regularized model, the exit from a neighborhood of the attractor of deterministic equation occurs on a sufficiently large time scale. Furthermore, the exit time and exit point in the small noise case, as well as the effect of large noise dispersion, is also discussed for the stochastic logarithmic Schrödinger equation.

Key words. stochastic nonlinear Schrödinger equation, logarithmic nonlinearity, noise dispersion, large deviation principle, exit problem

MSC codes. 35Q55, 35R60, 60H15, 60F10

DOI. 10.1137/23M1578619

1. Introduction.

The stochastic nonlinear Schrödinger equation describes the propagation of varying envelops of a wave packet in media with both weakly nonlinear and dispersive responses [40], and has been applied in nonlinear optics, hydrodynamics, biology, crystals, and Fermi–Pasta–Ulam chains of atoms. In nonlinear optics, the spantaneous emission noise in the stochastic nonlinear Schrödinger equation appears since the amplifiers are placed along the fiber line to compensate for the loss caused by the weak damping in the fiber [30]. Due to the inherent uncertainty on the amplified signal and quantum considerations, amplification is intrinsically associated with small noise [37]. In the context of crystal and Fermi–Pasta–Ulam chains of atoms, the noise accounts for thermal effects.

*Received by the editors June 12, 2023; accepted for publication (in revised form) March 13, 2024; published electronically June 7, 2024.
https://doi.org/10.1137/23M1578619

Funding: The research of the first author was partially supported by the Hong Kong Research Grant Council ECS grant 25302822, GRF grant 15302823, NSFC grant 12301526, the internal grants P0039016, P0045336, and P0046811 from Hong Kong Polytechnic University, and the CAS AMSS-PolyU Joint Laboratory of Applied Mathematics. The research of the second author was supported by National Natural Science Foundation of China grant 12101596.

†Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong (jianbo.cui@polyu.edu.hk).

‡Corresponding author. Academy for Multidisciplinary Studies, Capital Normal University, Beijing 100048, China (liyingsun@lsec.cc.ac.cn).

In this paper, we are interested in studying the random effect of the noise dispersion for the following stochastic logarithmic Schrödinger equation (SLogSE):

$$\begin{aligned} du^\epsilon(t) = \mathbf{i}\sqrt{\epsilon}\Delta u^\epsilon(t)\circ dB(t) + \mathbf{i}\lambda\log(|u^\epsilon(t)|^2)u^\epsilon(t)dt \\ + \mathbf{i}V[u^\epsilon(t)]u^\epsilon(t)dt - \mu u^\epsilon(t)dt \end{aligned} \tag{1.1}$$

emerged from the optical fibre with dispersion management [1, 2]. Here $u^\epsilon(0)=u_0$, $\lambda\in\mathbb{R}\backslash\{0\}$ shows the force of nonlinear interaction of the logarithmic potential, $\epsilon>0$ characterizes the intensity of noise dispersion, and $\mu>0$ is the weak damping coefficient over long distances. The Laplacian operator is defined on $\mathbb{R}^d$, $V[\cdot]$ represents a nonlocal interaction defined by $V[u](y)=\int_{\mathbb{R}^d}V(y-x)|u(x)|^2dx$ for some function V, $B(\cdot)$ is the standard Brownian motion on a complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, and the symbol "$\circ$" means that the stochastic integral is understood in the Stratonovich sense.

In the last two decades, there have been fruitful results and studies on stochastic nonlinear Schrödinger equations with polynomial and smooth nonlinearities driven by random forces. For instance, results on the well-posedness and the effect of a noise on the blow-up phenomenon have been proved in [3, 8, 18, 22, 23, 24, 39] and references therein. Due to the loss of analytical expression of the solution, several numerical methods have been designed to simulate the behaviors of stochastic nonlinear Schrödinger equations, such as in [7, 11, 14, 15, 17, 26], just to name a few. Recently, more and more attention has been paid on stochastic nonlinear Schrödinger equations with random dispersion emerged from dispersion management (see, e.g., [25, 28, 31, 36, 38]). In [25], it has been shown that the stochastic nonlinear Schrödinger equation with white noise dispersion is the limit of a nonlinear Schrödinger equation with a scaling sequence of real-valued stationary random processes. Many researches are also devoted to its numerical study (see, e.g., [4, 6, 13, 16, 38]). In contrast, less is known on the effect of white noise dispersion on stochastic nonlinear Schrödinger equations with logarithmic nonlinearities, which is one motivation of this current study.

In the determinstic case, it is known that the logarithmic nonlinearity makes the logarithmic Schrödinger equation,

$$\partial_t u(t) = \mathbf{i}\Delta u(t) + \mathbf{i}\lambda\log(|u(t)|^2)u(t), \tag{1.2}$$

unique among many nonlinear wave equations. This equation has played an important role since the seminal article [5], followed by the first mathematical work [10]. Recently, it has been shown in [9] that when $\lambda<0$, the modulus of the solution converges to a universal Gaussian profile by scaling in space via the dispersion rate. The idea of scaling in space and time plays an important role in studying (1.2). Inspired by such idea, one may formally define $e^{-\mathbf{i}l(t)|x|^2}v(t,l(t)x)=u(t,x)$ for $x\in\mathbb{R}^d$ and some positive (or negative) and continuously differentiable function $l(\cdot)$. Then it follows that

$$\begin{aligned} &\frac{d}{dt}e^{-\mathbf{i}l(t)|x|^2}v(t,l(t)x) \\ &= \partial_t v(t,l(t)x)e^{-\mathbf{i}l(t)|x|^2} + \dot{l}(t)\nabla_x v(t,l(t)x)\cdot xe^{-\mathbf{i}l(t)|x|^2} - \mathbf{i}\dot{l}(t)|x|^2 e^{-\mathbf{i}l(t)|x|^2}v(t,l(t)x) \\ &= \mathbf{i}\Delta(e^{-\mathbf{i}l(t)|x|^2}v(t,l(t)x)) + \mathbf{i}\lambda\log(|v(t,l(t)x)|^2)v(t,l(t)x)e^{-\mathbf{i}l(t)|x|^2}. \end{aligned}$$

Direct calculations yield that v satisfies

$$\begin{aligned}\partial_t v(t,x) &= \mathbf{i}(l(t))^2\Delta v(t,x) + \mathbf{i}\lambda \log(|v(t,x)|^2)v(t,x) \\ &\quad + (4(l(t))^2 - \dot{l}(t))\nabla v \cdot x + (\mathbf{i}\dot{l}(t)|x|^2 - \mathbf{i}4(l(t))^2|x|^2 + 2dl(t))v(t,x).\end{aligned} \tag{1.3}$$

Thus, under suitable conditions on $l(t)$, the properties of (1.2) can be transformed into those of (1.3) with the dispersion $l^2(t)$ via the above scaling technique. Unfortunately, this approach fails for studying (1.1) since $\epsilon\dot{B}(t) \neq (l(t))^2$ for any $l(t)$.

Since (1.1) involves a logarithmic nonlinearity, the techniques in [25, 28] cannot be used directly to show its well-posedness. To deal with the singularity caused by the possible vacuum of logarithmic nonlinearity, we will exploit the idea and functional setting in [9, 20] where the regularized approximation is used. Moreover, in section 2 we present several subtle estimates on the uniform bound of the regularized approximations such that the global well-posedness of (1.1) could be established under certain assumptions on V. We also find an interesting scaling result in space for (1.1) and its regularization approximation. More precisely, denote the solution $u^{\epsilon,\delta}(t,\cdot)$ of the following regularized SlogSE:

$$\begin{aligned}\mathrm{d}u^{\epsilon,\delta}(t) &= \mathbf{i}\Delta u^{\epsilon,\delta}(t) \circ \sqrt{\epsilon}\mathrm{d}B(t) + \mathbf{i}\lambda f_\delta(|u^{\epsilon,\delta}(t)|^2)u^{\epsilon,\delta}(t)\mathrm{d}t \\ &\quad + \mathbf{i}V[u^{\epsilon,\delta}(t)]u^{\epsilon,\delta}(t)\mathrm{d}t - \mu u^{\epsilon,\delta}(t)\mathrm{d}t\end{aligned} \tag{1.4}$$

with $u^{\epsilon,\delta}(0) = u_0$. Here

$$f_\delta(|x|^2) := \log\left(\frac{\delta + |x|^2}{1+\delta|x|^2}\right) \tag{1.5}$$

is an approximation of the logarithmic function $\log(|x|^2), x \in \mathbb{R}^d$ with $\delta > 0$. For convenience, we also denote $f_0(\cdot) = \log(\cdot)$ and $u^{\epsilon,0} = u^\epsilon$. Then defining $v^\delta(t, \epsilon^{\frac{1}{4}}x) := u^{\epsilon,\delta}(t,x)$, v^δ should satisfy

$$\begin{aligned}\mathrm{d}v^\delta(t) &= \mathbf{i}\Delta v^\delta(t) \circ \mathrm{d}B(t) + \mathbf{i}\lambda f_\delta(|v^\delta(t)|^2)v^\delta(t)\mathrm{d}t \\ &\quad + \mathbf{i}V[v^\delta(t)]v^\delta(t)\mathrm{d}t - \mu v^\delta(t)\mathrm{d}t,\end{aligned} \tag{1.6}$$

where $v^\delta(0,x) = u_0(\epsilon^{-\frac{1}{4}}x)$. It can be seen that (1.4) is equivalent to (1.6) up to a scaling in space. As a by-product, we can have the following scaling properties, i.e., for $\alpha \geq 0$,

$$\begin{aligned}\|u^{\epsilon,\delta}(t,\cdot)\|_{L^2_\alpha} &= \|v^\delta(t,\epsilon^{\frac{1}{4}}(\cdot))\|_{L^2_\alpha} = \|v^\delta(t,\cdot)\|_{L^2_\alpha}, \\ \|\nabla u^{\epsilon,\delta}(t,\cdot)\| &= \|\nabla v^\delta(t,\epsilon^{\frac{1}{4}}(\cdot))\|\epsilon^{\frac{1}{4}} = \|\nabla v^\delta(t,\cdot)\|\epsilon^{\frac{1}{4}}\end{aligned}$$

for any $u^{\epsilon,\delta}(t) \in L^2_\alpha \cap H^1$ and $t \geq 0$. Here $\|\cdot\|$ is the standard L^2-norm, $H^{\mathbf{s}}, \mathbf{s} \geq 0$, is the standard Sobolev space and the weighted Sobolev space $L^2_\alpha, \alpha \geq 0$, is defined by

$$L^2_\alpha := \{z \in L^2 | x \mapsto (1+|x|^2)^{\frac{\alpha}{2}} z(x) \in L^2\}$$

with the norm $\|z\|_{L^2_\alpha} := \|(1+|\cdot|^2)^{\frac{\alpha}{2}} z\|$.

Another motivation of this work lies on the study of the random effect of the noise dispersion for stochastic nonlinear Schrödinger equations. In the small noise case, the large deviation principle (LDP) of stochastic nonlinear Schrödinger equations with polynomial nonlinearities driven by additive and multiplicative noises have been addressed in [33, 34]. Then the LDP are applied to studying the asymptotic of the time jitter in soliton transmission in [27] and to quantifying the exit time and exit points for the exit problem from a basin of attractor for weakly damped stochastic nonlinear Schrödinger equations in [35]. When the LDP holds, the first order of the probability of rare event is that of Boltzmann theory and the square of the amplitude of the small noise acts as the temperature. The rate functional of LDP generally characterizes the transition between two states and the exit from the basin of attractor of the deterministic system, which is also related to *minimum action paths*. However, it is still unclear what are the asymptotic behaviors and LDP of (1.1) due to the singularity on the possible vacuum.

Thanks to the established well-posedness result of (1.1) and its regularization approximation (1.4), we are able to partially answer the LDP problem of (1.1). By imposing additional regularity on the initial datum, in section 3 we derive the LDP and its rate functional for the regularization approximation of (1.1), and then pass to the limit to prove the LDP of the original system via the strong convergence property and Varadhan's contraction principle (see, e.g., [12]). As an application, in section 4, we use the LDP to study the exit problem from a neighborhood of an asymptotically stable equilibrium point for the regularized equation (1.4) (or, equivalently, (1.6)), and prove that on an exponentially large time scale, the exit from the domains of attraction for (1.4) occurs due to large fluctuations. We would like to remark that it is still hard to establish the LDP and quantify the exit problem for (1.1) directly when the considered support is $\mathcal{C}([0,T];H^1\cap L^2_\alpha)$. In section 5, we give further discussions on the related topics, including giving a rough result on the exiting points for (1.1), proving an exponential estimate on a special rare event, and presenting the effect of the large dispersion, which may help understand the dynamics of (1.1). Several numerical tests are presented in section 6.

2. Logarithmic Schrödinger equation with random dispersion. The rigorous derivation of (1.1) can be understood in the way of [25, 38], once the well-posedness of (1.1) is established. Namely, (1.1) can be viewed as the limit of the following Schrödinger equation as $\sigma \to 0$:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \mathbf{i}\frac{\epsilon}{\sigma}m\left(\frac{1}{\sigma^2}\right)\Delta z + \mathbf{i}V[z]z + \mathbf{i}\lambda\log(|z|^2)z - \mu z,$$

under suitable conditions on the centered stationary random process $m(\cdot)$. Thus, (1.1) also belongs to the category of stochastic Wasserstein Hamiltonian flows in the sense of [19]. Recall that the mild solution of (1.4) is defined by a stochastic process $u^{\epsilon,\delta}$ satisfying

$$\begin{aligned} u^{\epsilon,\delta}(t) =& S_{\sqrt{\epsilon}B}(t,0)u_0 + \int_0^t \mathbf{i}\lambda S_{\sqrt{\epsilon}B}(t,r)f_\delta(u^{\epsilon,\delta}(r))\mathrm{d}r \\ &+ \mathbf{i}\int_0^t S_{\sqrt{\epsilon}B}(t,r)(V([u^{\epsilon,\delta}(r)])u^{\epsilon,\delta}(r))\mathrm{d}r - \int_0^t \mu S_{\sqrt{\epsilon}B}(t,r)u^{\epsilon,\delta}(r)\mathrm{d}r, \text{ a.s.} \end{aligned}$$

Here $S_{\sqrt{\epsilon}B}(t,r) = e^{\mathbf{i}\Delta\sqrt{\epsilon}(B(t)-B(r))}$, where $0 \le r \le t$. Due to $B(0) = 0$, we denote $S_{\sqrt{\epsilon}B}(t) := S_{\sqrt{\epsilon}B}(t,0)$. To prove the well-posedness of (1.1), thanks to the spatial scaling technique, our idea is to study the regularization approximation (1.6) at first and then to pass to the limit on δ.

Throughout this paper, we assume that $V \in \mathcal{C}_b^m(\mathbb{R}^d)$, for some $m \in \mathbb{N}^+$, i.e., the function space whose element is bounded, and has m continuous and bounded derivatives. We use C to denote various constants which may change from line to line. Due to the Leibniz rule and integration by parts, similar to [6], one can verify that the mapping $\mathbf{i}V[(\cdot)](\cdot)$ satisfies, for $v,h \in H^k, k=0,1$,

$$\|\mathbf{i}V[v]v\|_{H^k} \le C(k,V)\|v\|^2\|v\|_{H^k}, \tag{2.1}$$

$$\|\partial_v(\mathbf{i}V[v]v)\cdot h\|_{H^k} \le C(k,V)(\|v\|^2\|h\|_{H^k} + \|v\|\|h\|\|v\|_{H^k}). \tag{2.2}$$

Thanks to the definition (1.5) of f_δ, [20, Lemma 1] yields that for $v,w \in L^2$,

$$\|\lambda \mathbf{i} f_\delta(|v|^2)v\| \le |\lambda||\log(\delta)|\|v\|, \tag{2.3}$$

$$|\langle \lambda \mathbf{i} f_\delta(|v|^2)v - \lambda \mathbf{i} f_\delta(|w|^2)w, v-w\rangle| \le 4|\lambda|\|v-w\|^2, \tag{2.4}$$

$$\|\lambda \mathbf{i} f_\delta(|v|^2)v - \lambda \mathbf{i} f_\delta(|w|^2)w\| \le |\lambda|(|\log(\delta)|+2)\|v-w\|. \tag{2.5}$$

Here the complex inner product is defined by $\langle w,z\rangle := \mathrm{Re}\int_{\mathbb{R}^d} \bar{w}(x)z(x)\mathrm{d}x$. We suppose that the deterministic initial value $u_0 \in L^2_\alpha \cap H^1$ for $\alpha \in [0,1]$. For convenience, we denote $L^p := L^p(\mathbb{R}^d;\mathbb{C})$ and $H = L^2$.

In the following, we will frequently use the weighted Sobolev embedding inequality (see [20, Lemma 6]), i.e., for $d \in \mathbb{N}^+, \eta \in (0,1)$, and $\alpha > \frac{d\eta}{2-2\eta}$, it holds that

$$\|z\|_{L^{2-2\eta}} \le C\|z\|^{1-\frac{d\eta}{2\alpha(1-\eta)}}\|z\|_{L^2_\alpha}^{\frac{d\eta}{2\alpha(1-\eta)}}, \; z \in L^2_\alpha, \tag{2.6}$$

and the Gagliardo–Nirenberg interpolation inequality, i.e., for $\eta' > 0$ and $d \in \mathbb{N}^+$ such that $\frac{\eta' d}{2+2\eta'} \in (0,1)$,

$$\|z\|_{L^{2+2\eta'}} \le C\|z\|^{1-\frac{\eta' d}{2+2\eta'}}\|\nabla z\|^{\frac{\eta' d}{2+2\eta'}}, \; z \in L^{2+2\eta'} \cap H^1. \tag{2.7}$$

Now we are in a position to present the well-posedness result of (1.4). To apply the Itô formula rigorously, one needs to use suitable approximation procedures as in [20]. Here we omit these tedious and standard arguments. Recall that $u^\epsilon = u^{\epsilon,0}$ denotes the solution of the original problem.

Theorem 1. *Let $T > 0$, $\delta \ge 0$ and $\alpha \in [0,1]$. There exists a unique mild solution $u^{\epsilon,\delta} \in \mathcal{C}([0,T];H)$, a.s., of (1.4) satisfying for any $p \ge 2$,*

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]}\|u^{\epsilon,\delta}(t)\|_{H^1}^p\right] + \mathbb{E}\left[\sup_{t\in[0,T]}\|u^{\epsilon,\delta}(t)\|_{L^2_\alpha}^p\right] \\ &\quad \le C\left(T,\lambda,\mu,p,\|u_0\|\right)\left(\|u_0\|_{H^1}^p + \|u_0\|_{L^2_\alpha}^p\right). \end{aligned} \tag{2.8}$$

Proof. Since $v^\delta(t,\epsilon^{\frac{1}{4}}x) := u^{\epsilon,\delta}(t,x)$, it suffices to prove the well-posedness of (1.6). The proof combines the following three steps.

Step 1. We first prove the well-posedness of (1.6) for any fixed $\delta > 0$. Since $V \in \mathcal{C}_b^m(\mathbb{R}^d)$ for some $m \ge 1$ and f_δ is Lipschitz (2.5) when $\delta > 0$, one can use the procedures in

[20, section 2] and [25] to give the well-posedness of (1.6). Let us briefly introduce these standard procedures. By [25, Proposition 3.1], we have the unitary property of $S_{\kappa B}(t,r)$, i.e.,

$$\|S_{\kappa B}(t,r)z\|_{H^{\mathbf{s}}} = \|z\|_{H^{\mathbf{s}}} \tag{2.9}$$

for any $\kappa \in \mathbb{R}$, $z \in H^{\mathbf{s}}$ and $\mathbf{s} \geq 0$. Besides, (2.1)–(2.2) yield the local Lipschitz property of $V[\cdot](\cdot)$, i.e., for $v, w \in H$,

$$\|V[z]z - V[w]w\| \leq C(\|z\|^2 + \|w\|^2)\|z - w\|. \tag{2.10}$$

Then by constructing a mapping $\Gamma : H \to \mathcal{C}([0,\tau]; H)$ defined as

$$\begin{aligned}\Gamma(w)(t) := S_B(t,0)w + \int_0^t \mathbf{i}\lambda S_B(t,r) f_\delta(\Gamma(w)(r))\mathrm{d}r \\ + \mathbf{i}\int_0^t S_B(t,r)V[\Gamma(w)(r)]\Gamma(w)(r)\mathrm{d}r - \int_0^t \mu S_B(t,r)\Gamma(w)(r)\mathrm{d}r,\end{aligned}$$

where $t \in [0,\tau], \tau > 0$, to be determined later. Thanks to (2.9) with $\kappa = 1$, (2.10), and (2.5), the contraction mapping principle yields that there exists a local mild solution of (1.6) satisfying

$$\begin{aligned}v^\delta(t) = S_B(t,0)v(0) + \int_0^t \mathbf{i}\lambda S_B(t,r) f_\delta(v^\delta(r))\mathrm{d}r \\ + \mathbf{i}\int_0^t S_B(t,r)V[v^\delta(r)]v^\delta(r)\mathrm{d}r - \int_0^t \mu S_B(t,r)v^\delta(r)\mathrm{d}r, \text{a.s.},\end{aligned}$$

where $t \in [0,\tau]$ for some small $\tau > 0$ only depending on $\|v(0,\cdot)\| = \|u(0, \frac{\cdot}{\epsilon^{\frac14}})\|$, λ, μ, and $|\log(\delta)|$. To extend the local solution to be the global one in $[0,T]$ for any $T > 0$, we need a priori H-estimate. From the Itô formula and the chain rule it follows that for any $t \geq 0$

$$\|v^\delta(t)\| = e^{-\mu t}\|v^\delta(0)\|, \text{ a.s.} \tag{2.11}$$

Thus, one can construct a global mild solution of (1.6) thanks to (2.11). To see the uniqueness of the solution, assume that $v^\delta(\cdot), \widetilde{v}^\delta(\cdot)$ are two different solutions of (1.6). By applying the chain rule to study the evolution of $\|v^\delta(t) - \widetilde{v}^\delta(t)\|^2$ as in [20], and using (2.10), (2.4), and (2.11), the uniqueness of the solution follows. To sum up, for a fixed $\delta > 0$, there exists a unique global solution $v^\delta \in L^p(\Omega; \mathcal{C}([0,T]; H))$ for $T > 0$ and $p \geq 1$.

Step 2. In order to study the well-posedness for the case of $\delta = 0$, we consider the moment estimates under the H^1-norm and the weighted Sobolev L^2_α-norm with $\alpha \in [0,1]$. Notice that

$$(f_\delta(|x|^2))' = \frac{2x}{|x|^2 + \delta} - \frac{2\delta x}{\delta|x|^2 + 1}$$

for $x \in \mathbb{R}$ and $\delta > 0$. By the Itô formula and integration by parts, it holds that

$$\begin{aligned}&\mathrm{d}\frac{1}{2}\|\nabla v^\delta(t)\|^2 \\ &= -\langle \mathbf{i}\Delta v^\delta(t), \Delta v^\delta(t)\rangle \circ \mathrm{d}B(t) + \langle \lambda \mathbf{i} f_\delta(|v^\delta(t)|^2)\nabla v^\delta(t), \nabla v^\delta(t)\rangle \mathrm{d}t \\ &\quad + 2\langle \lambda \mathbf{i}\left(\frac{\mathrm{Re}(\bar{v}^\delta(t)\nabla v^\delta(t))}{|v^\delta(t)|^2 + \delta}v^\delta(t) - \frac{\delta \mathrm{Re}(\bar{v}^\delta(t)\nabla v^\delta(t))}{\delta|v^\delta(t)|^2 + 1}v^\delta(t)\right), \nabla v^\delta(t)\rangle \mathrm{d}t \\ &\quad + \langle \mathbf{i}V([v^\delta(t)])\nabla v^\delta(t), \nabla v^\delta(t)\rangle \mathrm{d}t + \langle \mathbf{i}\nabla V([v^\delta(t)])v^\delta(t), \nabla v^\delta(t)\rangle \mathrm{d}t - \mu\|\nabla v^\delta(t)\|^2 \mathrm{d}t.\end{aligned} \tag{2.12}$$

Using the skew-symmetric property of the complex inner product, the first two terms in (2.12) are zeros. Then applying Hölder's inequality, the fact that $V \in \mathcal{C}_b^m$ for some $m \geq 1$ and that $\frac{2|x|^2}{|x|^2+\delta} \leq 2$ and $\frac{2\delta|x|^2}{\delta|x|^2+1} \leq 2$, as well as Young's inequality, we achieve that

$$\begin{aligned}
&\mathrm{d}\frac{1}{2}\|\nabla v^\delta(t)\|^2 \\
&\leq 2|\lambda|\left\langle \frac{|v^\delta(t)|^2}{|v^\delta(t)|^2+\delta} + \frac{|v^\delta(t)|^2}{\delta|v^\delta(t)|^2+1}, |\nabla v^\delta(t)|^2 \right\rangle \mathrm{d}t \\
&\quad + C_V\|v^\delta(t)\|^2(\|\nabla v^\delta(t)\|^2 + \|v^\delta(t)\|\|\nabla v^\delta(t)\|)\mathrm{d}t - \mu\|\nabla v^\delta(t)\|^2\mathrm{d}t \\
&\leq 2|\lambda|\|\nabla v^\delta(t)\|^2\mathrm{d}t + C_V\|v^\delta(t)\|^2(\frac{3}{2}\|\nabla v^\delta(t)\|^2 + \frac{1}{2}\|v^\delta(t)\|^2)\mathrm{d}t - \mu\|\nabla v^\delta(t)\|^2\mathrm{d}t,
\end{aligned}$$

where C_V only depends on V. By Gronwall's inequality and using (2.11), we obtain that

$$\begin{aligned}
\|\nabla v^\delta(t)\|^2 &\leq \exp\left(\int_0^t (4|\lambda| + 3C_V\|v^\delta(s)\|^2 - 2\mu)\mathrm{d}s\right)\|\nabla v^\delta(0)\|^2 \\
&\quad + \int_0^t \exp\left(\int_s^t (4|\lambda| + 3C_V\|v^\delta(r)\|^2 - 2\mu)\mathrm{d}r\right) C_V\|v^\delta(s)\|^2\mathrm{d}s \\
&\leq \exp\left((4|\lambda| - 2\mu)t + 3C_V\frac{1-e^{-2\mu t}}{2\mu}\|v^\delta(0)\|^2\right)\|\nabla v^\delta(0)\|^2 \\
&\quad + \int_0^t \exp\left((4|\lambda| - 2\mu)(t-s) + 3C_V\|v^\delta(0)\|^2\frac{e^{-2\mu s} - e^{-2\mu t}}{2\mu}\right) C_V e^{-2\mu s}\|v^\delta(0)\|^2\mathrm{d}s \\
&\leq \exp\left((4|\lambda| - 2\mu)t + 3C_V\frac{1-e^{-2\mu t}}{2\mu}\|v^\delta(0)\|^2\right)\|\nabla v^\delta(0)\|^2 \\
&\quad + \exp\left((4|\lambda| - 2\mu)t + 3C_V\|v^\delta(0)\|^2\frac{1}{\mu}\right)\frac{1-e^{-4|\lambda|t}}{4|\lambda|}C_V\|v^\delta(0)\|^2.
\end{aligned} \tag{2.13}$$

Thus, by taking supremum over $[0,T]$ and taking pth moment, we conclude that there exists $C(V,\lambda,\mu,T,\|u_0\|) > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla v^\delta(t)\|^{2p}\right] \leq \exp\Big(C(V,\lambda,\mu,T,\|u_0\|)p\Big)\left(\|\nabla v^\delta(0)\|^{2p} + \|v^\delta(0)\|^{2p}\right). \tag{2.14}$$

By applying the Itô's formula to the weighted Sobolev norm, thanks to the fact that $\langle \mathbf{i}f_\delta(v^\delta)v^\delta, (1+|x|^2)^\alpha v^\delta\rangle = 0$, we have that

$$\begin{aligned}
\mathrm{d}\frac{1}{2}\|v^\delta(t)\|_{L^2_\alpha}^2 &= \langle \mathbf{i}\Delta v^\delta(t), (1+|x|^2)^\alpha v^\delta(t)\rangle \mathrm{d}B(t) - \frac{1}{2}\langle (1+|x|^2)^\alpha v^\delta(t), \Delta^2 v^\delta(t)\rangle \mathrm{d}t \\
&\quad + \frac{1}{2}\langle (1+|x|^2)^\alpha \Delta v^\delta(t), \Delta v^\delta(t)\rangle \mathrm{d}t - \mu\|v^\delta(t)\|_{L^2_\alpha}^2\mathrm{d}t.
\end{aligned} \tag{2.15}$$

To proceed, recall that $\Delta u = \sum_{i=1}^d \partial_{x_i}^2 u$ and $\Delta^2 u = \sum_{i,j=1}^d \partial_{x_i}^2\partial_{x_j}^2 u$. By using integration by parts formula,

$$
\begin{aligned}
&-\frac{1}{2}\langle (1+|x|^2)^{\alpha}u, \sum_{i,j=1}^{d}\partial^2_{x_i}\partial^2_{x_j}u\rangle = -\frac{1}{2}\sum_{i,j=1}^{d}\langle \partial^2_{x_j}[(1+|x|^2)^{\alpha}u], \partial^2_{x_i}u\rangle\\
&= -\frac{1}{2}\sum_{i,j=1}^{d}\langle (1+|x|^2)^{\alpha-1}2\alpha u, \partial^2_{x_i}u\rangle - \sum_{i,j=1}^{d}\langle (1+|x|^2)^{\alpha-1}2\alpha x_j\partial_{x_j}u, \partial^2_{x_i}u\rangle\\
&\quad -\frac{1}{2}\sum_{i,j=1}^{d}\langle (1+|x|^2)^{\alpha}\partial^2_{x_j}u, \partial^2_{x_i}u\rangle - 2\sum_{i,j=1}^{d}\langle \alpha(\alpha-1)|x_j|^2(1+|x|^2)^{\alpha-2}u, \partial^2_{x_i}u\rangle.
\end{aligned}
$$

Note that the third term $-\frac{1}{2}\sum_{i,j=1}^{d}\langle (1+|x|^2)^{\alpha}\partial^2_{x_j}u, \partial^2_{x_i}u\rangle$ in the above equality will be eliminated due to the third term in (2.15). The other terms will be bounded by using integration by parts, (2.11) and (2.14). For the first term and last term, using integration by parts once and $(1+|x|^2)^{-1}|x|^{\zeta} \le C_{\zeta}$ for $\zeta \in [0,2]$, as well as Young's inequality, we have that

$$
\begin{aligned}
&\left|-\frac{1}{2}\sum_{i,j=1}^{d}\langle (1+|x|^2)^{\alpha-1}2\alpha u, \partial^2_{x_i}u\rangle - 2\sum_{i,j=1}^{d}\langle \alpha(\alpha-1)|x_j|^2(1+|x|^2)^{\alpha-2}u, \partial^2_{x_i}u\rangle\right|\\
&\le C(\|\nabla u\|^2 + \|u\|^2).
\end{aligned}
$$

Furthermore, by using integration by parts recurrently, for $j \ne i$, it holds that

$$
\begin{aligned}
&-2\langle (1+|x|^2)^{\alpha-1}2\alpha x_j\partial_{x_j}u, \partial^2_{x_i}u\rangle\\
&= -\langle \partial_{x_i}u, (1+|x|^2)^{\alpha-1}2\alpha\partial_{x_i}u + (1+|x|^2)^{\alpha-2}2(\alpha-1)2\alpha x_i u\rangle\\
&\quad -\langle \partial_{x_i}u, (1+|x|^2)^{\alpha-2}2(\alpha-1)2\alpha x_j^2\partial_{x_i}u\rangle\\
&\quad -\langle \partial_{x_i}u, (1+|x|^2)^{\alpha-3}2(\alpha-2)2(\alpha-1)2\alpha x_j^2 x_i u\rangle\\
&\quad +\langle \partial_{x_j}u, (1+|x|^2)^{\alpha-2}2(\alpha-1)2\alpha x_i x_j\partial_{x_i}u\rangle\\
&\quad +\langle \partial_{x_j}u, (1+|x|^2)^{\alpha-3}2(\alpha-2)2(\alpha-1)2\alpha x_i x_j x_i u\rangle\\
&\quad +\langle \partial_{x_j}u, (1+|x|^2)^{\alpha-2}2(\alpha-1)2\alpha x_j u\rangle\\
&\quad +\langle (1+|x|^2)^{\alpha-2}2(\alpha-1)x_i 2\alpha x_j\partial_{x_j}u, \partial_{x_i}u\rangle.
\end{aligned}
$$

Thanks to $(1+|x|^2)^{-1}|x|^{\zeta} \le C_{\zeta}$ for $\zeta \in [0,2]$, it follows that for $j \ne i$,

$$
|\langle (1+|x|^2)^{\alpha-1}2\alpha x_j\partial_{x_j}u, \partial^2_{x_i}u\rangle| \le C(\|\partial_{x_i}u\|^2 + \|u\|^2 + \|\partial_{x_j}u\|^2). \tag{2.16}
$$

Similarly, one can verify (2.16) for $j = i$. By the Burkholder-Davis-Gundy inequality and Gronwall's inequality, as well as (2.11), (2.14), and (2.16), we obtain that

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\|v^{\delta}(t)\|^{2p}_{L^2_{\alpha}}\right] \le \exp\Big(C(V,\lambda,\mu,T,\|u_0\|)p\Big)\Big(\|v^{\delta}(0)\|^{2p}_{H^1} + \|v^{\delta}(0)\|^{2p}_{L^2_{\alpha}}\Big). \tag{2.17}
$$

The estimates (2.14) and (2.17) lead to (2.8) for $\delta > 0$ by the spatial scaling property, i.e., $\|v^{\delta}(t,\cdot)\| = \|u^{\epsilon,\delta}(t, \frac{\cdot}{\epsilon^{\frac{1}{4}}})\|$ and $\|\nabla v^{\delta}(t,\cdot)\| = \epsilon^{-\frac{1}{4}}\|\nabla u^{\epsilon,\delta}(t, \frac{\cdot}{\epsilon^{\frac{1}{4}}})\|$.

Step 3. We show that the limit of v^δ as $\delta\to 0$ is the unique mild solution of (1.1). Notice that $v^\delta(0,x)=u_0(\epsilon^{-\frac14}x)$ is independent of δ. We take arbitrary small positive parameters $\delta_1,\delta_2>0$ such that $\delta_1\ge\delta_2$. Then considering the evolution of $\frac12\|v^{\delta_1}(t)-v^{\delta_2}(t)\|^2$, it holds that

$$\begin{aligned}
&\frac12\|v^{\delta_1}(t)-v^{\delta_2}(t)\|^2\\
&=\frac12\|v^{\delta_1}(0)-v^{\delta_2}(0)\|^2+\int_0^t\langle v^{\delta_1}(s)-v^{\delta_2}(s),\mathbf{i}\Delta(v^{\delta_1}(s)-v^{\delta_2}(s))\rangle\circ \mathrm{d}B(s)\\
&\quad+\int_0^t\langle v^{\delta_1}(s)-v^{\delta_2}(s),\lambda\mathbf{i}f_{\delta_1}(|v^{\delta_1}(s)|^2)v^{\delta_1}(s)-\lambda\mathbf{i}f_{\delta_2}(|v^{\delta_2}(s)|^2)v^{\delta_2}(s)\rangle\mathrm{d}s\\
&\quad+\int_0^t\langle v^{\delta_1}(s)-v^{\delta_2}(s),\mathbf{i}V[v^{\delta_1}(s)]v^{\delta_1}(s)-\mathbf{i}V[v^{\delta_2}(s)]v^{\delta_2}(s)\rangle\mathrm{d}s\\
&\quad-\int_0^t\mu\|v^{\delta_1}(s)-v^{\delta_2}(s)\|^2\mathrm{d}s\\
&=:\frac12\|v^{\delta_1}(0)-v^{\delta_2}(0)\|^2+I_1(t)+I_2(t)+I_3(t)+I_4(t).
\end{aligned}$$

Since $B(\cdot)$ is independent of the spatial variable, it follows that $I_1=0$. The property of the logarithmic function and (2.4) yield that

$$\begin{aligned}
|I_2(t)|&\le C(\lambda)\int_0^t\|v^{\delta_1}(s)-v^{\delta_2}(s)\|^2\mathrm{d}s\\
&\quad+\int_0^t|\langle v^{\delta_1}(s)-v^{\delta_2}(s),\mathbf{i}\lambda(f_{\delta_1}(|v^{\delta_1}(s)|^2)-f_{\delta_2}(|v^{\delta_2}(s)|^2))v^{\delta_2}(s)\rangle|\mathrm{d}s\\
&\le C(\lambda)\int_0^t\|v^{\delta_1}(s)-v^{\delta_2}(s)\|^2\mathrm{d}s\\
&\quad+|\lambda|\int_0^t\int_{\mathcal{O}}|v^{\delta_1}(s)-v^{\delta_2}(s)||\log(\delta_1+|v^{\delta_1}(s)|^2)-\log(\delta_2+|v^{\delta_2}(s)|^2)||v^{\delta_2}(s)|\mathrm{d}x\mathrm{d}s\\
&\quad+|\lambda|\int_0^t\int_{\mathcal{O}}|v^{\delta_1}(s)-v^{\delta_2}(s)||\log(1+\delta_1|v^{\delta_1}(s)|^2)-\log(1+\delta_2|v^{\delta_2}(s)|^2)||v^{\delta_2}(s)|\mathrm{d}x\mathrm{d}s\\
&\le C(\lambda)\int_0^t\|v^{\delta_1}(s)-v^{\delta_2}(s)\|^2\mathrm{d}s\\
&\quad+|\lambda|\int_0^t\int_{\mathcal{O}}|v^{\delta_1}(s)-v^{\delta_2}(s)|\log\left(1+\frac{(\delta_1-\delta_2)}{\delta_2+|v^{\delta}(s)|^2}\right)|v^{\delta_2}(s)|\mathrm{d}x\mathrm{d}s\\
&\quad+|\lambda|\int_0^t\int_{\mathcal{O}}|v^{\delta_1}(s)-v^{\delta_2}(s)||\log\left(1+\frac{(\delta_1-\delta_2)|v^{\delta_2}|^2}{1+\delta_2|v^{\delta_2}(s)|^2}\right)||v^{\delta_2}(s)|\mathrm{d}x\mathrm{d}s\\
&=:C(\lambda)\int_0^t\|v^{\delta_1}(s)-v^{\delta_2}(s)\|^2ds+I_{21}+I_{22}.
\end{aligned}$$

Applying Young's inequality, using the fact that for any $\theta>0$, $\log(1+|x|^2)\le C(\theta)|x|^{2\theta}$ for all $x\in\mathbb{R}$, (2.6) and (2.7), it follows that for $\eta\in[0,\frac{2\alpha}{2\alpha+d})$, $\frac{\eta' d}{2\eta'+2}\in[0,1)$, and $\alpha\in(0,1]$,

$$I_{21} + I_{22} \le \int_0^t C|\lambda| \|v^{\delta_1}(s) - v^{\delta_2}(s)\|^2 \mathrm{d}s + \int_0^t C|\lambda|\delta_1^{\eta} \|v^{\delta_2}(s)\|_{L^{2-2\eta}}^{2-2\eta} \mathrm{d}s$$
$$+ \int_0^t C|\lambda|\delta_1^{\eta'} \|v^{\delta_2}(s)\|_{L^{2+2\eta'}}^{2+2\eta'} \mathrm{d}s$$
$$\le \int_0^t C|\lambda| \|v^{\delta_1}(s) - v^{\delta_2}(s)\|^2 \mathrm{d}s + \int_0^t C\delta_1^{\eta} \|v^{\delta_2}(s)\|_{L^2_\alpha}^{\frac{d\eta}{\alpha}} \|v^{\delta_2}(s)\|^{2-2\eta-\frac{d\eta}{\alpha}} \mathrm{d}s$$
$$+ \int_0^t C\delta_1^{\eta'} \|v^{\delta_2}(s)\|^{d\eta'} \|\nabla v^{\delta_2}(s)\|^{2\eta'+2-d\eta'} \mathrm{d}s.$$

Using (2.2) yields that

$$|I_3(t)| = C(m,V) \int_0^t (\|v^{\delta_1}(s)\|^2 + \|v^{\delta_2}(s)\|^2) \|v^{\delta_1}(s) - v^{\delta_2}(s)\|^2 \mathrm{d}s,$$

which, together with the mass evolution law (2.11), implies that

$$|I_3(t)| \le C(m,V)(1 + \|u_0\|^2) \int_0^t \|v^{\delta_1}(s) - v^{\delta_2}(s)\|^2 \mathrm{d}s.$$

Based on the estimates of I_1-I_3 and using Gronwall's inequality, we have

$$\|v^{\delta_1} - v^{\delta_2}\|_{L^{2p}(\Omega;\mathcal{C}([0,T];H))} \le C(V,\lambda,T,p,\mu,\|u_0\|)(\delta_1^{\frac{\eta}{2}} + \delta_1^{\frac{\eta'}{2}}). \tag{2.18}$$

Thus for any $\delta_n \to 0$, $\{v^{\delta_n}\}_n$ forms a Cauchy sequence in $L^{2p}(\Omega;\mathcal{C}([0,T];H))$. Then standard arguments as in [20, section 4] show that there exists a unique limit of the Cauchy sequence v^δ as $\delta \to 0$, which is also the mild solution of (1.6) with $\delta = 0$. By the spatial scaling property between v^δ and $u^{\epsilon,\delta}$ and using Fatou's lemma, we get (2.8) for $\delta \to 0$ and complete the proof. ■

Remark 1. Let us emphasize several aspects on Theorem 1.

- From (2.13), by imposing some additional assumptions on the damping coefficient μ, one may expect the uniform estimate over the time interval. For instance, if, in addition, we assume that $2|\lambda| \le \mu$, the upper bound of $\mathbb{E}[\|\nabla v^\delta(t)\|^2]$ is uniform w.r.t. the time variable t. More precisely, using the scaling property and (2.13), one can derive that

$$\sup_{t\ge 0} \mathbb{E}[\|\nabla u^{\epsilon,\delta}(t)\|^2] \le \exp\left(3C_V \frac{1}{2\mu}\|u_0\|^2\right) \|\nabla u_0\|^2$$
$$+ \exp\left(3C_V\|u_0\|^2 \frac{1}{\mu}\right) \frac{1}{4|\lambda|} C_V \epsilon^{\frac{1}{2}} \|u_0\|^2.$$

  As a by-product, the upper bound of $\mathbb{E}[\|v^\delta(t)\|_{L^2_\alpha}^2]$ is also independent of the time variable t.
- To verify the limit $v^* := \lim_{\delta\to 0} v^\delta$ is the mild solution of the original problem, one could use the arguments in the proof (Step 2) of [20, Theorem 1]. More precisely, we need to apply the convergence rate result (2.18) to verify that as $\delta \to 0$,

$$\int_0^t \mathbf{i}\lambda S_B(t,r) f_\delta(|v^\delta(r)|^2) v^\delta(r) \mathrm{d}r - \int_0^t \mathbf{i}\lambda S_B(t,r) f_0(|v^*(r)|^2) v^*(r)\mathrm{d}r \to 0,$$
$$\mathbf{i}\int_0^t S_B(t,r) V[v^\delta(r)] v^\delta(r)\mathrm{d}r - \mathbf{i}\int_0^t S_B(t,r)V[v^*(r)]v^*(r)\mathrm{d}r \to 0,$$

as well as $\int_0^t \mu S_B(t,r)v^\delta(r)\mathrm{d}r - \int_0^t \mu S_B(t,r)v^*(r)\mathrm{d}r \to 0$. To derive the a priori bound on v^*, by using the strong convergence property of v^δ, (2.14), (2.17), as well as the arguments in the proof (Step 2) of [20, Theorem 1], one has that

$$\begin{aligned}
(2.19)\qquad & \mathbb{E}[\|v^*(t)\|^{2p}_{H^1}] + \mathbb{E}[\|v^*(t)\|^{2p}_{L_\alpha}] \\
&\le \sup_{\delta>0} \mathbb{E}[\|v^\delta(t)\|^{2p}_{H^1}] + \sup_{\delta>0}\mathbb{E}[\|v^\delta(t)\|^{2p}_{L^2_\alpha}] \\
&\le \exp\Big(C(V,\lambda,\mu,T,\|u_0\|)p\Big)\Big(\|v^\delta(0)\|^{2p}_{H^1} + \|v^\delta(0)\|^{2p}_{L^2_\alpha}\Big).
\end{aligned}$$

- By an interpolation argument as in [20], one can also prove that for any $\delta_n \to 0$, $\{v^{\delta_n}\}_n$ forms a Cauchy sequence in $L^{2p}(\Omega;\mathcal{C}([0,T];L^2_{\alpha_1}))\cap L^{2p}(\Omega;\mathcal{C}([0,T];H^{\mathbf{s}}))$ with $\mathbf{s}\in(0,1)$ and $\alpha_1 \in (0,\alpha)$.

3. Large deviation principle.

It is known that the large deviation principle (LDP) characterizes the limiting behavior as $\epsilon \to 0$ of a family of random variables $\{X^\epsilon\}_{\epsilon>0}$ and can be regarded as an extension or refinement of the law of large numbers and central limit theorem. The prototype for large deviations result deals with the "rough" logarithmic asymptotics of the probability $\mathbb{P}(X^\epsilon \in A)$ where $A \in \mathcal{B}$ is open or closed. More precisely, we say $\{X^\epsilon\}_{\epsilon>0}$ on a topological space $(\mathcal{X},\mathcal{B})$ with $\mathcal{B}$ being the completed Borel σ-field satisfies the LDP with a rate function I if for any closed set $F\subset \mathcal{X}$, the upper bound holds

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^\epsilon \in F) \le -\inf_{y\in F} I(y),$$

and if for any open set $G \subset \mathcal{X}$, the lower bound holds

$$\liminf_{\epsilon\to 0} \log \mathbb{P}(X^\epsilon \in G) \ge -\inf_{y\in G} I(y).$$

For more equivalent definitions of LDP, we refer to [29, Chapter 1]. Thanks to Schilder's theorem (see, e.g., [29, Chapter 5]), it is known that $\sqrt{\epsilon}B(\cdot)$ satisfies the LDP with a good rate functional

$$I^W(g) = \begin{cases} \frac{\|g\|^2_{\mathcal{H}_P}}{2}, & g\in \mathcal{H}_P, \\ +\infty, & g \notin \mathcal{H}_P, \end{cases}$$

where $\mathcal{H}_P$ is the Cameron–Martin space of the standard Brownian motion defined by $\{g \in W^{1,2}([0,T];\mathbb{R}) \mid g(0)=0\}$ equipped with the norm $\|g\|_{\mathcal{H}_P} := \sqrt{\int_0^T |h|^2 dt}$, where $h=\dot{g}$. Define $L^\delta_{u_0}$ is as the solution operator of the skeleton equation,

$$L_{u_0}^{\delta}(g)(t) = S_g(t)u_0 + \int_0^t S_g(t,s)\mathbf{i}\lambda f_\delta(|L_{u_0}^{\delta}(g)(s)|^2)L_{u_0}^{\delta}(g)(s)\mathrm{d}s + \int_0^t S_g(t,s)\mathbf{i}V[L_{u_0}^{\delta}(g)(s)]L_{u_0}^{\delta}(g)(s)\mathrm{d}s - \int_0^t S_g(t,s)\mu L_{u_0}^{\delta}(g)(s)\mathrm{d}s$$

with $S_g(t,s) = \exp(\mathbf{i}\Delta(g(t)-g(s)))$. For simplicity, we also denote $\mathcal{X}^{1,\alpha} = L^2_\alpha \cap H^1$ equipped with the norm $\|\cdot\|_{\mathcal{X}^{1,\alpha}} := \sqrt{\|\cdot\|^2_{L^2_\alpha} + \|\cdot\|^2_{H^1}}$. For the considered models, we have the following LDP result.

Proposition 1. *Let $T>0$, $\delta \geq 0$, and $\alpha \in [0,1]$. The family $\{u^{\epsilon,\delta}\}_{\epsilon>0}$ satisfies an LDP with a good rate function*

$$I_{u_0}^{\delta,T}(z) = \frac{1}{2}\inf_{L_{u_0}^{\delta}(g)=z} \|g\|^2_{\mathcal{H}_P},\ z \in \mathcal{C}([0,T];H).$$

Namely, for every a,ρ,κ,γ positive,

(i) *there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0,\epsilon_0)$ and $\|u_0\|_{\mathcal{X}^{1,\alpha}} \leq \rho$ and $\widetilde{a} \in (0,a]$,*

$$\mathbb{P}(d_{\mathcal{C}([0,T];\mathcal{H})}(u^{\epsilon,\delta}, \mathcal{K}_T^{u_0}(\widetilde{a})) \geq \gamma) < \exp\Big(-\frac{\widetilde{a}-\kappa}{\epsilon}\Big);$$

(ii) *there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0,\epsilon_0)$ and $\|u_0\|_{\mathcal{X}^{1,\alpha}} \leq \rho$ and $w \in \mathcal{K}_T^{u_0,\delta}(a)$,*

$$\mathbb{P}(\|u^{\epsilon,\delta} - w\|_{\mathcal{C}([0,T];H)} < \gamma) > \exp\Big(-\frac{I_{u_0}^{\delta,T}(w)-\kappa}{\epsilon}\Big).$$

Here $\mathcal{K}_T^{u_0,\delta}(a) = (I_{u_0}^{\delta,T})^{-1}([0,a])$, i.e.,

$$\mathcal{K}_T^{u_0,\delta}(a) = \Big\{y \in \mathcal{C}([0,T];H) \mid y = L_{u_0}^{\delta}(g), \frac{1}{2}\int_0^T |h|^2 dt \leq a, h = \dot{g}\Big\}.$$

Proof. For the case that $\delta > 0$, by the contraction principle, it suffices to prove the continuity of $L_{u_0}^{\delta}$, which is obtain in Lemma 1.

Notice that $u^{\epsilon,\delta}$ is proven to be an exponentially good approximation of u^δ in Lemma 2. By [29, Theorem 4.2.23], to verify that $I_{u_0}^{\delta,T}$ is the good rate function for (1.4) with $\delta = 0$, it suffices to show that

$$\lim_{\delta\to 0} \sup_{\{g:\|g\|_{\mathcal{H}_P} \leq R\}} \|L_{u_0}^{\delta}(g) - L_{u_0}^{0}(g)\| = 0 \tag{3.1}$$

for every positive $R < +\infty$. Since $g \in \mathcal{H}_P$, by similar arguments as in the proof of Theorem 1, thanks to the isometry property of $S_g(t,s)$ for any $t,s \geq 0$, it is not hard to show that there exists a unique solution for

$$\mathrm{d}L_{u_0}^{\delta}(g)(t) = \mathbf{i}\Delta L_{u_0}^{\delta}(g)(t)\mathrm{d}g(t) + \mathbf{i}\lambda f_\delta(|L_{u_0}^{\delta}(g)(t)|^2)L_{u_0}^{\delta}(g)(t)\mathrm{d}t + \mathbf{i}V[L_{u_0}^{\delta}(g)(t)]L_{u_0}^{\delta}(g)(t)\mathrm{d}t - \mu L_{u_0}^{\delta}(g)(t)\mathrm{d}t,$$

satisfying

$$\|L^\delta_{u_0}(g)(t)\| = e^{-\mu t}\|u_0\|,\ \|\nabla L^\delta_{u_0}(g)(t)\| \le C(V,\lambda,T,\mu)\|\nabla u_0\|, \tag{3.2}$$

$$\|L^\delta_{u_0}(g)(t)\|_{L^2_\alpha} \le C(V,\lambda,T,\mu)(\|u_0\|_{L^2_\alpha} + \|u_0\|_{H^1}\|g\|_{W^{1,2}([0,T];\mathbb{R})}). \tag{3.3}$$

The properties (2.2) and (2.4) yield that

$$\begin{aligned}
&\mathrm{d}\|L^\delta_{u_0}(g)(t) - L^0_{u_0}(g)(t)\|^2\\
&\quad\le C(\lambda)\|(f_\delta(|L^0_{u_0}(g)(t)|^2)L^0_{u_0}(g)(t) - \log(|L^0_{u_0}(g)(t)|^2))L^0_{u_0}(g)(t)\|^2\mathrm{d}t\\
&\qquad + C(V,\lambda,\mu)(1+\|L^\delta_{u_0}(g)(t)\|^2 + \|L^0_{u_0}(g)(t)\|^2)\|L^\delta_{u_0}(g)(t) - L^0_{u_0}(g)(t)\|^2\mathrm{d}t.
\end{aligned}$$

Using the fact that for any $\theta>0$, $\log(1+|x|^2)\le C(\theta)|x|^{2\theta}$ for all $x\in\mathbb{R}$, we get

$$\begin{aligned}
&\|(f_\delta(|u_1|^2)u_1 - \log(|u_1|^2))u_1\|^2\\
&\quad\le C\left(\|\log(1+\delta|u_1|^2)u_1\|^2 + \left\|\log\left(1+\frac{\delta}{|u_1^2|}\right)u_1\right\|^2\right)\\
&\quad\le C(\eta')\delta^{\eta'}\|u_1\|^{2\eta'+2}_{L^{2\eta'+2}} + C(\eta)\delta^{\eta}\|u_1\|^{2-2\eta}_{L^{2-2\eta}}.
\end{aligned}$$

According to (2.7) and (2.6), it follows that for $\eta\in[0,\frac{2\alpha}{2\alpha+d})$, $\frac{\eta' d}{2\eta'+2}\in[0,1)$, and $\alpha\in(0,1]$,

$$\begin{aligned}
&\|(f_\delta(|u_1|^2)u_1 - \log(|u_1|^2))u_1\|^2\\
&\quad\le C(\delta^\eta+\delta^{\eta'})(\|u_1\|^{\frac{d\eta}{\alpha}}_{L^2_\alpha}\|u_1\|^{2-2\eta-\frac{d\eta}{\alpha}} + \|u_1\|^{d\eta'}\|\nabla u_1\|^{2\eta'+2-d\eta'}),
\end{aligned} \tag{3.4}$$

where $u_1\in L^2_\alpha\cap H^1$. Combining the above estimates with (3.4) in the proof of Lemma 2, (3.1) follows. ∎

Now we present the continuity result of $L^\delta_{u_0}$ such that the contraction principle is applicable in the case that $\delta>0$.

Lemma 1. *Let $T>0$, $\delta>0$, $\alpha\in[0,1]$, and $g\in\mathcal{H}_P$. The operator $L^\delta_{u_0}$ is continuous from $\mathcal{C}_0([0,T];\mathbb{R})$ to $\mathcal{C}([0,T];H)$.*

Proof. For any $g_1,g_2\in\mathcal{H}_P$, using the mild formulations of $L^\delta_{u_0}(g_1)$ and $L^\delta_{u_0}(g_2)$, it holds that

$$\begin{aligned}
&\|L^\delta_{u_0}(g_1)(t) - L^\delta_{u_0}(g_2)(t)\| \le \|S_{g_1}(t)u_0 - S_{g_2}(t)u_0\|\\
&+\left\|\int_0^t (S_{g_1}(t,s)-S_{g_2}(t,s))\Big(\mathbf{i}\lambda f_\delta(|L^\delta_{u_0}(g_1)(s)|^2)L^\delta_{u_0}(g_1)(s) - \mu L^\delta_{u_0}(g_1)(s)\Big)\mathrm{d}s\right\|\\
&+\left\|\int_0^t (S_{g_1}(t,s)-S_{g_2}(t,s))\mathbf{i}V[L^\delta_{u_0}(g_1)(s)]L^\delta_{u_0}(g_1)(s)\mathrm{d}s\right\|\\
&+\left\|\int_0^t S_{g_2}(t,s)\mathbf{i}\lambda\Big(f_\delta(|L^\delta_{u_0}(g_1)(s)|^2)L^\delta_{u_0}(g_1)(s) - f_\delta(|L^\delta_{u_0}(g_2)(s)|^2)L^\delta_{u_0}(g_2)(s)\right.\\
&\quad\left. + \mu L^\delta_{u_0}(g_2)(s) - \mu L^\delta_{u_0}(g_1)(s)\Big)\mathrm{d}s\right\|\\
&+\left\|\int_0^t S_{g_2}(t,s)\mathbf{i}\Big(V[L^\delta_{u_0}(g_1)(s)]L^\delta_{u_0}(g_1)(s) - V[L^\delta_{u_0}(g_2)(s)]L^\delta_{u_0}(g_2)(s)\Big)\mathrm{d}s\right\|.
\end{aligned}$$

On the one hand, for any $u_0 \in H^1$, by Fourier's transformation, it holds that for $\|g_1 - g_2\|_{\mathcal{C}([0,T];\mathbb{R})} \to 0$,

$$\|(S_{g_1}(t) - S_{g_2}(t))u_0\| = \|(I - S_{g_2-g_1}(t))u_0\| \le C\|u_0\|_{H^1}|g_2(t) - g_1(t)|^{\frac{1}{2}} \to 0.$$

As a consequence, by (3.2)–(3.3), (2.1), and (2.3), we have that for $\|g_1 - g_2\|_{\mathcal{C}([0,T];\mathbb{R})} \to 0$,

$$\begin{aligned}
&\Big\| \int_0^t (S_{g_1}(t,s) - S_{g_2}(t,s))\Big(\mathbf{i}\lambda f_\delta(|L_x^\delta(g_1)(s)|^2)L_x^\delta(g_1)(s) - \mu L_x^\delta(g_1)(s)\Big)\mathrm{d}s\Big\| \\
&\quad + \Big\| \int_0^t (S_{g_1}(t,s) - S_{g_2}(t,s))\mathbf{i}V[L_{u_0}^\delta(g_1)(s)]L_{u_0}^\delta(g_1)(s)\mathrm{d}s\Big\| \\
&\le C(\lambda,\mu)\int_0^t \|(S_{g_1}(t,s) - S_{g_2}(t,s))\|_{\mathcal{L}(H;H^1)}\Big(\|V[L_{u_0}^\delta(g_1)(s)]L_{u_0}^\delta(g_1)(s)\|_{H^1} \\
&\quad + \|f_\delta(|L_{u_0}^\delta(g_1)(s)|^2)L_{u_0}^\delta(g_1)(s)\|_{H^1} + \|L_{u_0}^\delta(g_1)(s)\|_{H^1}\Big)\mathrm{d}s \\
&\le C(T,V,\lambda,|\log(\delta)|,\mu)(1+\|u_0\|^2)\int_0^t |g_1(t) - g_1(s) - g_2(t) + g_1(s)|^{\frac{1}{2}}\|u_0\|_{H^1}\mathrm{d}s \to 0.
\end{aligned}$$

Making use of properties (2.2) and (2.5), it follows that

$$\begin{aligned}
&\Big\| \int_0^t S_{g_2}(t,s)\mathbf{i}\lambda\Big(f_\delta(|L_{u_0}^\delta(g_1)(s)|^2)L_{u_0}^\delta(g_1)(s) - f_\delta(|L_{u_0}^\delta(g_2)(s)|^2)L_{u_0}^\delta(g_2)(s) \\
&\quad + \mu L_{u_0}^\delta(g_2)(s) - \mu L_x^\delta(g_1)(s)\Big)\mathrm{d}s\Big\| \\
&\quad + \Big\| \int_0^t S_{g_2}(t,s)\mathbf{i}\Big(V[L_{u_0}^\delta(g_1)(s)]L_{u_0}^\delta(g_1)(s) - V[L_{u_0}^\delta(g_2)(s)]L_{u_0}^\delta(g_2)(s)\Big)\mathrm{d}s\Big\| \\
&\le C(V,\lambda,|\log(\delta)|,T,\mu)(1+\|u_0\|^2)\int_0^t \|L_{u_0}^\delta(g_1)(s) - L_{u_0}^\delta(g_2)(s)\|\mathrm{d}s.
\end{aligned}$$

Then Gronwall's inequality yields that

$$\begin{aligned}
&\|L_{u_0}^\delta(g_1) - L_{u_0}^\delta(g_2)\|_{\mathcal{C}([0,T];H)} \\
&\quad \le C(V,\lambda,|\log(\delta)|,T,\mu)(1+\|u_0\|^2)(1+\|u_0\|_{H^1})|g_1 - g_2|_{\mathcal{C}([0,T];\mathbb{R})},
\end{aligned}$$

which completes the proof. ■

Note that if one can prove the continuity of $L_{u_0}^\delta$ when $\delta = 0$, then the contraction principle directly yields the LDP of u^ϵ (recall that $u^\epsilon = u^{\epsilon,0}$). However, there is a technical issue in controlling the some terms involving with f_0 in the proof of Lemma 1. For example, we only have $\|f_\delta(|L_{u_0}^\delta(g_1)|^2)L_{u_0}^\delta(g_1)\|_{H^1} \le C(1+|\log(\delta)|)\|L_{u_0}^\delta(g_1)\|_{H^1}$ instead of a uniform bound w.r.t. δ. Next, we show that $u^{\epsilon,\delta}$ is an exponentially good approximation of u^ϵ to prove its LDP when $\delta = 0$.

Lemma 2. *Let $T > 0$, $\delta \ge 0$, and $\alpha \in [0,1]$. The sequence $\{u^{\epsilon,\delta}\}_{\delta>0}$ is an exponentially good approximation of u^ϵ, i.e., for any $\delta_1 > 0$,*

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\epsilon\log\mathbb{P}(\|u^{\epsilon,\delta} - u^\epsilon\|_{\mathcal{C}([0,T];H)} > \delta_1) = -\infty.$$

Proof. We prove that $u^{\epsilon,\delta}$ is an exponentially good approximation of u^ϵ by the strong convergence property under the $L^p(\Omega)$-norm. By the chain rule, the properties (2.4) and (2.2), it holds that

$$\begin{aligned}
&\mathrm{d}\|u^{\epsilon,\delta}(t)-u^\epsilon(t)\|^2\\
&\quad=2\langle \mathbf{i}\Delta(u^{\epsilon,\delta}(t)-u^\epsilon(t)),u^{\epsilon,\delta}(t)-u^\epsilon(t)\rangle\circ \mathrm{d}B(t)\\
&\qquad+2\langle \mathbf{i}\lambda(f_\delta(|u^{\epsilon,\delta}(t)|^2)u^{\epsilon,\delta}(t)-\log(|u^\epsilon(t)|^2)u^\epsilon(t)),u^{\epsilon,\delta}(t)-u^\epsilon(t)\rangle \mathrm{d}t\\
&\qquad+2\langle \mathbf{i}(V[u^{\epsilon,\delta}(t)]u^{\epsilon,\delta}(t)-V[u^\epsilon(t)]u^\epsilon(t)),u^{\epsilon,\delta}(t)-u^\epsilon(t)\rangle \mathrm{d}t\\
&\qquad-2\mu\|u^{\epsilon,\delta}(t)-u^\epsilon(t)\|^2\mathrm{d}t\\
&\quad\le C(\lambda)\|(f_\delta(|u^\epsilon(t)|^2)u^\epsilon(t)-\log(|u^\epsilon(t)|^2))u^\epsilon(t)\|^2\mathrm{d}t\\
&\qquad+C(V,\lambda,\mu)(1+\|u^{\epsilon,\delta}(t)\|^2+\|u^\epsilon(t)\|^2)\|u^{\epsilon,\delta}(t)-u^\epsilon(t)\|^2\mathrm{d}t.
\end{aligned}$$

By using (3.4) and Gronwall's inequality, we get that

$$\begin{aligned}
\|u^{\epsilon,\delta}(t)-u^\epsilon(t)\|^2 &\le e^{C(V,\lambda,\mu)(1+\|u_0\|^2)T}(\delta^\eta+\delta^{\eta'})(1+\|u_0\|^{2-2\eta-\frac{d\eta}{\alpha}}+\|u_0\|^{d\eta'})\\
&\quad\times\int_0^T(\|u^\epsilon(t)\|_{L^2_\alpha}^{\frac{d\eta}{\alpha}}+\|\nabla u^\epsilon(t)\|^{2\eta'+2-d\eta'})\mathrm{d}t.
\end{aligned}$$

Then according to (2.19) and a scaling argument, applying Chebyshev's inequality, it holds that

$$\begin{aligned}
&\epsilon\log\mathbb{P}(\|u^{\epsilon,\delta}-u^\epsilon\|_{\mathcal{C}([0,T];H)}\ge\delta_1)\\
&\quad\le\epsilon\log\mathbb{P}(\|u^{\epsilon,\delta}-u^\epsilon\|_{\mathcal{C}([0,T];H)}^{\frac{1}{\epsilon}}\ge\delta_1^{\frac{1}{\epsilon}})\\
&\quad\le\epsilon\log\frac{C(V,\lambda,T,\mu,u_0)^{\frac{1}{\epsilon}}\mathbb{E}\left[\left(\int_0^T(\|u^\epsilon(t)\|_{L^2_\alpha}^{\frac{d\eta}{\alpha}}+\|\nabla u^\epsilon(t)\|^{2\eta'+2-d\eta'})dt\right)^{\frac{1}{\epsilon}}\right]\delta^{\max(\frac{\eta}{2\epsilon},\frac{\eta'}{2\epsilon})}}{\delta_1^{\frac{1}{\epsilon}}}\\
&\quad\le C_1(V,\lambda,T,\mu,u_0)+\log\left(\frac{\delta^{\frac{1}{2}\max(\eta,\eta')}}{\delta_1}\right).
\end{aligned}$$

Then letting $\delta\to0$, we have completed the proof for the exponentially good approximation property. ∎

Remark 2. From the above analysis, it can be seen that the strongly convergent approximation with explicit convergence rate and finite moment bounds under $L^p(\Omega)$-norm for sufficiently large $p\ge1$ is also an exponentially good approximation for a stochastic (partial) differential equation. By a classical approach in [29, Chapter 5], one could also prove the uniform LDP for (1.4) with $\delta\ge0$, for any compact set $K\subset L^2_\alpha\cap H^1$, for any $A\in\mathcal{B}(\mathcal{C}([0,T];H))$,

$$\begin{aligned}
-\sup_{u_0\in K}\inf_{w\in\mathrm{Int}(A)}I_{u_0}^{\delta,T}(w)&\le\liminf_{\epsilon\to0}\epsilon\log\inf_{u_0\in K}\mathbb{P}(u^\epsilon\in A)\\
&\le\limsup_{\epsilon\to0}\epsilon\log\sup_{u_0\in K}\mathbb{P}(u^\epsilon\in A)\le-\inf_{w\in\bar{A},u_0\in K}I_{u_0}^{\delta,T}(w).
\end{aligned}\tag{3.5}$$

In the following, we present a useful LDP result of (1.4) with $\delta>0$ when the domain of $I_{u_0}^\delta$ is considered on $\mathcal{C}([0,T];\mathcal{X}^{1,1})$, whose proof is in the appendix. For any $a\ge0$, now we denote $K_T^{u_0,\delta}(a)=(I_{u_0}^{\delta,T})^{-1}([0,a])$, i.e.,

$$K_T^{u_0,\delta}(a)=\Big\{y\in\mathcal{C}([0,T];\mathcal{X}^{1,1})|y=L_{u_0}^{\delta}(g),\frac{1}{2}\int_0^T|h|^2\mathrm{d}t\le a, h=\dot{g}\Big\}.$$

Theorem 2. *Let $\delta>0, T>0$, $u_0\in\mathcal{X}^{1,1}$, and $d\le 2$. For every a,ρ,κ,γ positive,*

(i) *there exists $\epsilon_0>0$ such that for every $\epsilon\in(0,\epsilon_0)$ and $\|u_0\|_{\mathcal{X}^{1,1}}\le\rho$ and $\widetilde{a}\in(0,a]$,*

$$\mathbb{P}(d_{\mathcal{C}([0,T];\mathcal{X}^{1,1})}(u^{\epsilon,\delta,u_0},K_T^{u_0}(\widetilde{a}))\ge\gamma)<\exp\Big(-\frac{\widetilde{a}-\kappa}{\epsilon}\Big);$$

(ii) *there exists $\epsilon_0>0$ such that for every $\epsilon\in(0,\epsilon_0)$ and $\|u_0\|_{\mathcal{X}^{1,1}}\le\rho$ and $w\in K_T^{u_0,\delta}(a)$,*

$$\mathbb{P}(\|u^{\epsilon,\delta,u_0}-w\|_{\mathcal{C}([0,T];\mathcal{X}^{1,1})}<\gamma)>\exp\Big(-\frac{I_{u_0}^{\delta,T}(w)-\kappa}{\epsilon}\Big).$$

The assumption $d\le 2$ is technical for deriving a priori estimates in the Sobolev space of higher order. It should be remarked that our current approach does not give the LDP result of Theorem 2 in $\mathcal{C}([0,T];\mathcal{X}_1)$ for the case $\delta=0$ due to the strong singularity of the logarithmic nonlinearity in the Sobolev space of higher order. For further discussion, we refer to section 5.

4. Application: Exit from a basin of attraction.

This section is devoted to the first exit time from a neighborhood of an asymptotically stable equilibrium point for (1.4) with $\delta>0$ and $\mu>0$. We refer to [29, 32] for more backgrounds for the exit problems and its connection to LDP. In the following, we assume that $V=0$ for the following reasons. One main reason is that several computations can be simplified according to this assumption. For instance, in (4.2), if $V\ne 0$, we may need a stronger condition, like $\mu\ge 2|\lambda|+C_V$, to derive the attractor in $H^{\mathbf{s}},\mathbf{s}>0$, where $C_V>0$ depends on V. Besides, a lot of terms related to V should be calculated as in the proof of sections 4 and 5. For example, in the proof of Proposition 4, one may need to control the exponential moments of $V[u^\epsilon]$ if $V\ne 0$. Thanks to the assumption $V\in\mathcal{C}_b^m$ and mass evolution (2.11), we believe that the current approach in this section is also available for the case that $V\ne 0$ under a stronger dissipative condition on the coefficients. Considering the following infinite-dimensional ordinary differential equation with weakly damping effect,

$$\mathrm{d}w=\mathbf{i}\lambda f_\delta(|w|^2)w\mathrm{d}t-\mu w\mathrm{d}t, \tag{4.1}$$

it holds that

$$|w(t,x)|=e^{-\mu t}|w(0,x)|,$$
$$w(t,x)=\exp\Big(-\mu t+\int_0^t\mathbf{i}f_\delta(|w(0,x)|^2e^{-2\mu s})\mathrm{d}s\Big)w(0,x).$$

Thus, 0 is the unique attractor of the above equation in H. Note that in the H-topology, the exit problems are trivial since

$$\|w^{\epsilon,\delta}(t)\|\le e^{-\mu t}\|w^{\epsilon,\delta}(0)\|.$$

Similarly, one may consider the topology in L^2_α since 0 is also the attractor of the considered equation. Taking $\alpha=1$, we obtain

$$\|w(t)\|_{L^2_1}\le e^{-\mu t}\|w(0)\|_{L^2_1}.$$

However, 0 may be not the attractor in $H^{\mathbf{s}}, \mathbf{s} > 0$. Let us briefly explain the reason for this possibility. For example, when $\mathbf{s} = 1$, if $\langle \lambda \frac{f_\delta'(|w(t)|^2)}{\|\nabla w(t)\|^2} \mathrm{Re}(\bar{w}(t)\nabla w(t))w(t), \nabla w(t)\rangle > 2\mu$,

$$\mathrm{d}\|\nabla w(t)\|^2 = \langle \mathbf{i}\lambda f_\delta'(|w(t)|^2)\mathrm{Re}(\bar{w}(t)\nabla w(t))w(t), \nabla w(t)\rangle \mathrm{d}t - 2\mu\|\nabla w(t)\|^2 \mathrm{d}t > 0.$$

One numerical example is also shown in Example 3 in section 6. This implies that $\|\nabla w(t)\|^2$ may increase. Instead, if $\mu > 2|\lambda|$, by Young's inequality,

$$\begin{aligned} \mathrm{d}\|\nabla w(t)\|^2 &= \langle \mathbf{i}\lambda f_\delta'(|w(t)|^2)\mathrm{Re}(\bar{w}(t)\nabla w(t))w(t), \nabla w(t)\rangle \mathrm{d}t - 2\mu\|\nabla w(t)\|^2 \mathrm{d}t \\ &\le (-2\mu + 4|\lambda|)\|\nabla w(t)\|^2 \mathrm{d}t, \end{aligned} \tag{4.2}$$

which implies that 0 is always an attractor in H^1.

We will consider the exit problem under $\|\cdot\|_{\mathcal{X}^{1,1}}$-norm. Consider an open bounded domain D containing 0 in the interior of $\mathcal{X}^{1,1}$ such that $D \subset \mathcal{B}_R^0$ for some $R > 0$. The above analysis indicates that D is invariant under the deterministic flow of (4.1) if $\mu > 2|\lambda|$.

Define $\tau^{\epsilon,\delta,u_0} = \inf\{t \ge 0 \mid u^{\epsilon,\delta,u_0}(t) \in D^c\}$ the first exit time of the regularized SlogS equation from D. Similar to [35], we introduce

$$\overline{M^\delta} = \inf\{I_0^{\delta,T}(y) \mid y(T) \in (\overline{D})^c, T > 0\}.$$

For any sufficient positive $\rho > 0$, set

$$M_\rho^\delta = \inf\{I_{u_0}^{\delta,T}(y) \mid \|u_0\|_{\mathcal{X}^{1,1}} \le \rho, y(T) \in (D_{-\rho})^c, T > 0\}$$

with $D_{-\rho} := D\backslash\mathcal{N}^0(\partial D, \rho)$ where $\mathcal{N}^0(\partial D, \rho)$ is the open neighborhood of ∂D with the distance ρ. Here ∂D is the boundary of D in $\mathcal{X}^{1,1}$. Define $\underline{M^\delta} = \lim_{\rho\to 0} M_\rho^\delta$. It can be seen that $\underline{M^\delta} \le \overline{M^\delta}$. Below, we shall prove that the lower bound of $\underline{M^\delta}$ is strictly larger than 0 thanks to special structure of the skeleton equation.

Lemma 3. *Let $\delta > 0$, $\mu > 2|\lambda|$, and $\alpha = 1$. Then it holds that $0 < \inf_{\delta>0} \underline{M^\delta} \le \inf_{\delta>0} \overline{M^\delta}$.*

Proof. Let $d(0, \partial D) > 0$ denote the distance between 0 and ∂D. Choose ρ small enough such that $\mathcal{B}_\rho^0 \subset D$ and that the distance between $\mathcal{B}_\rho^0$ and $(D_{-\rho})^c$ is larger than $\frac{1}{2}d(0, \partial D)$. By studying the functional $\mathcal{H}(L_{u_0}^\delta(g)) := \frac{1}{2}\|L_{u_0}^\delta(g)\|_{\mathcal{X}^{1,1}}^2$, it follows that

$$\begin{aligned} &\mathrm{d}\mathcal{H}(L_{u_0}^\delta(g)(t)) \\ &= \langle 2x L_{u_0}^\delta(g)(t), \mathbf{i}\nabla L_{u_0}^\delta(g)(t)\rangle h(t)\mathrm{d}t - \mu\mathcal{H}(L_{u_0}^\delta(g)(t))\mathrm{d}t \\ &\quad + \langle \nabla L_{u_0}^\delta(g)(t), \mathbf{i}\lambda f_\delta'(|L_{u_0}^\delta(g)(t)|^2)\mathrm{Re}(\overline{L_{u_0}^\delta(g)}(t)\nabla L_{u_0}^\delta(g)(t))L_{u_0}^\delta(g)(t)\rangle \mathrm{d}t \\ &\le 2\|L_{u_0}^\delta(g)(t)\|_{L_1^2}\|\nabla L_{u_0}^\delta(g)(t)\|h(t)\mathrm{d}t - \mu\mathcal{H}(L_{u_0}^\delta(g)(t))\mathrm{d}t + 2|\lambda|\|\nabla L_{u_0}^\delta(g)(t)\|^2 \mathrm{d}t, \end{aligned}$$

where $h = \dot{g}$. Using the Duhamel formula, we obtain that

$$\mathcal{H}(L_{u_0}^\delta(g)(T)) - e^{-(\mu-2|\lambda|)T}\mathcal{H}(L_{u_0}^\delta(g)(0)) \le \int_0^T e^{-(\mu-2|\lambda|)(T-s)} CR^2|h(s)|\mathrm{d}s.$$

As a consequence, if $L_{u_0}(g)(T) \in (D_{-\rho})^c$, it holds that

$$\frac{d^2(0,\partial D) - 4d(0,\partial D)\rho}{16} \le CR^2\sqrt{\int_0^T e^{-2(\mu-2|\lambda|)(T-s)}\mathrm{d}s}\|h\|_{L^2([0,T];\mathbb{R})}.$$

Taking ρ sufficiently small leads to

$$\frac{d^2(0,\partial D)}{32CR^2\sqrt{2\mu-4|\lambda|}} \le \|h\|_{L^2([0,T];\mathbb{R})},$$

thus the desired results follow. ∎

We are not able to prove $\underline{M^\delta} = \overline{M^\delta}$ due to loss of the approximate controllability at this current study. The argument to prove the approximate controllability of the original system (i.e., $\delta = 0$) is hard since the nonlinearity is not locally Lipschitz and the Schrödinger group relies on the control. In the ideal case, it is expected that

$$\lim_{\delta\to 0} M^\delta = \inf_{v\in\partial D} U(0,v),$$

with the quasi-potential defined by

$$U(u_0,u_1) := \inf\{I_{u_0}^{0,T}(y) | y \in \mathcal{C}([0,T],\mathcal{X}^{1,1}), y(0)=u_0, y(T)=u_1, T>0\}.$$

Define $\sigma_\rho^{\epsilon,\delta,u_0} := \inf\{t\ge 0 \mid u^{\epsilon,\delta,u_0}(t)\in \mathcal{B}_\rho^0 \cup D^c\}$. Below we give the tail estimate of $\sigma_\rho^{\epsilon,\delta,u_0}$ whose proof is similar to [35, Lemma 3.6]. The main difference of the proofs lies on the continuity estimate of the related skeleton equation. For the completeness, we provide a proof in the appendix.

Lemma 4. *Let $\delta>0$, $\mu>2|\lambda|$, and $\alpha=1$. For every ρ small enough and L positive with $\mathcal{B}_\rho^0\subset D$, there exists $T>0$ and ϵ_0 such that for every $u_0\in D$ and $\epsilon\in(0,\epsilon_0)$,*

$$\mathbb{P}(\sigma_\rho^{\epsilon,\delta,u_0} > T) \le \exp\Big(-\frac{L}{\epsilon}\Big).$$

The below lemma indicates that at the time $\sigma_\rho^{\epsilon,\delta,u_0}$, the probability that u^{ϵ,δ,u_0} escapes D is at most exponentially small. Since the procedure of the proof is same as in that of [35, Lemma 3.7] by using the LDP in Theorem 2, we put its proof in the appendix for completeness.

Lemma 5. *Let $\delta>0$, $\mu>2|\lambda|$, and $\alpha=1$. For every $\rho>0$ such that $\mathcal{B}_\rho^0\subset D$ and $u_0\in D$, there exists $L>0$ such that*

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}\Big(u^{\epsilon,\delta,u_0}\left(\sigma_\rho^{\epsilon,\delta,u_0}\right)\in\partial D\Big) \le -L.$$

The following lemma concerns the exponential tail estimate of $\mathcal{H}(u^{\epsilon,\delta,u_0})$ whose proof is slightly different from that in [35, Lemma 3.8]. More precisely, in [35], to study the exit problem, one needs the exponential tail estimates of the mass function $\|\cdot\|^2$ in the additive noise case and the energy function in the multiplicative noise case (i.e., the driving noise is $\sqrt{\epsilon}\mathbf{i}udW(t)$ with W being the H^1-valued Wiener process). In our case, we introduce the functional $\mathcal{H}(u^{\epsilon,\delta,u_0})$ since the considered equation with white noise dispersion has no a clear energy evolution law as pointed out in [25].

Lemma 6. *Let $\delta > 0$, $\mu > 2|\lambda|$, and $\alpha = 1$. For every $\rho > 0$ and $L > 0$ such that $\mathcal{B}^0_{2\rho} \subset D$, there exists $T < +\infty$ such that*

$$\limsup_{\epsilon\to 0} \epsilon \log \sup_{u_0 \in S^0_\rho} \mathbb{P}\left(\sup_{t\in[0,T]} \left(\mathcal{H}(u^{\epsilon,\delta,u_0}) - \mathcal{H}(u_0)\right) \ge \frac{3}{2}\rho^2 \right) \le -L,$$

where $\mathcal{S}^0_\rho$ is the sphere centered at 0 with the radius ρ in $\mathcal{X}^{1,1}$.

Proof. The evolution of $\mathcal{H}(u^{\epsilon,\delta,u_0})$ yields that

$$\begin{aligned}
&\mathcal{H}(u^{\epsilon,\delta,u_0}(t)) - \mathcal{H}(u_0)\\
&= \sqrt{\epsilon}\int_0^t \langle \mathbf{i}2xu^{\epsilon,\delta,u_0}, \nabla u^{\epsilon,\delta,u_0}\rangle \mathrm{d}W(s) - \frac{1}{2}\int_0^t \epsilon\langle (1+|x|^2)u^{\epsilon,\delta,u_0}, \Delta^2 u^{\epsilon,\delta,u_0}\rangle \mathrm{d}s\\
&\quad + \frac{1}{2}\int_0^t \epsilon\langle (1+|x|^2)\Delta u^{\epsilon,\delta,u_0}, \Delta u^{\epsilon,\delta,u_0}\rangle \mathrm{d}s - \mu\int_0^t \mathcal{H}(u^{\epsilon,\delta,u_0}(s))\mathrm{d}s\\
&\quad + \int_0^t \mathbf{i}2\lambda\langle \partial_x f_\delta(|u^{\epsilon,\delta,u_0}|^2)\mathrm{Re}(\overline{u^{\epsilon,\delta,u_0}}\nabla u^{\epsilon,\delta,u_0})u^{\epsilon,\delta,u_0}, \nabla u^{\epsilon,\delta,u_0}\rangle \mathrm{d}s.
\end{aligned}$$

By integration by parts, Holder's and Young's inequalities, it is enough to show for $\epsilon \in (0,\epsilon_0)$ with small $\epsilon_0 > 0$ and for small $T(\rho,L) < 1$ that

$$\limsup_{\epsilon\to 0} \epsilon \log \sup_{u_0\in S^0_\rho} \mathbb{P}\left(\sup_{t\in[0,T(\rho,L)]} |Z(t)| \ge \frac{\rho^2}{\sqrt{\epsilon}} \right) \le -L,$$

where $Z(t) = \int_0^t \langle 2xu^{\epsilon,\delta,u_0}, \mathbf{i}\nabla u^{\epsilon,\delta,u_0}\rangle \mathrm{d}W(s)$. By using [21, Proposition 4.31], it holds that for any $b > 0$,

$$\mathbb{P}\left(\sup_{[0,T(\rho,L)]} |Z(t)| \ge \frac{\rho^2}{\sqrt{\epsilon}} \right) \le \frac{b\epsilon}{\rho^4} + \mathbb{P}\left(\int_0^{T(\rho,L)} 4\|xu^{\epsilon,\delta,u_0}\|^2 \|\nabla u^{\epsilon,\delta,u_0}\|^2 \mathrm{d}s \ge b \right).$$

Note that by Chebyshev's inequality and the moment bound of $\mathcal{H}(u^{\epsilon,\delta,u_0})$, it holds that for $q = \frac{1}{\epsilon}$,

$$\begin{aligned}
\mathbb{P}\left(4\int_0^{T(\rho,L)} \|xu^{\epsilon,\delta,u_0}\|^2 \|\nabla u^{\epsilon,\delta,u_0}\|^2 \mathrm{d}s \ge b \right) &\le \frac{4^q T(\rho,L)^q \sup_{t\in[0,T(\rho,L)]} \mathbb{E}\left[\mathcal{H}(u^{\epsilon,\delta,u_0})^{2q}\right]}{b^q}\\
&\le \frac{4^q T(\rho,L)^q C(\lambda,\mu,T(\rho,L))^q q\rho^{4q}}{b^q}.
\end{aligned}$$

Taking $b = (\frac{1}{L'})^{\frac{1}{\epsilon}}$ for a sufficient large L', it follows that

$$\begin{aligned}
&\epsilon \log \sup_{u_0\in S^0_\rho} \mathbb{P}\left(4\int_0^{T(\rho,L)} \|xu^{\epsilon,\delta,u_0}\|^2 \|\nabla u^{\epsilon,\delta,u_0}\|^2 \mathrm{d}s \ge b \right)\\
&\quad \le \log\left(4T(\rho,L)\right) - \log(b) + \log\left(\rho^4\right) + \epsilon\log\left(\frac{1}{\epsilon}\right) + \log\left(C(\lambda,\mu,T(\rho,L))\right).
\end{aligned}$$

As a consequence,

$$\limsup_{\epsilon\to 0}\epsilon\log\sup_{u_0\in S_\rho^0}\mathbb{P}\left(\sup_{t\in[0,T(\rho,L)]}|Z(t)|\geq\frac{\rho^2}{\sqrt{\epsilon}}\right)$$
$$\leq\max\left(\log\left(\frac{1}{L'}\right)-\log\left(\rho^4\right),\log\left(4T(\rho,L)\right)-\log\left(\frac{1}{L'}\right)+\log\left(\rho^4\right)+\log\left(C(\lambda,\mu,T(\rho,L))\right)\right).$$

To complete the proof, one just takes $T(\rho,L)\leq\frac{1}{4C(\lambda,\mu,T(\rho,L))\rho^4L'}$ with L' sufficiently large. ■

Now we are able to present the theorem to characterize the first exit time from a given domain D for the regularized problem ($\delta>0$). Its proof is in the appendix.

Proposition 2. *Let $\delta>0$, $\mu>2|\lambda|$, and $\alpha=1$. For every $u_0\in D$ and small $\kappa>0$, there exists positive L such that*

$$\limsup_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\tau^{\epsilon,\delta,u_0}\notin\left(e^{\frac{\underline{M^\delta}-\kappa}{\epsilon}},e^{\frac{\overline{M^\delta}+\kappa}{\epsilon}}\right)\right)\leq -L, \tag{4.3}$$

and for every $u_0\in D$,

$$\underline{M^\delta}\leq\liminf_{\epsilon\to 0}\epsilon\log\mathbb{E}\left[\tau^{\epsilon,\delta,u_0}\right]\leq\limsup_{\epsilon\to 0}\epsilon\log\mathbb{E}\left[\tau^{\epsilon,\delta,u_0}\right]\leq\overline{M^\delta}. \tag{4.4}$$

Furthermore, for every small $\kappa>0$, there exists $L>0$ such that

$$\limsup_{\epsilon\to 0}\epsilon\log\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,\delta,u_0}\geq e^{\frac{\overline{M^\delta}+\kappa}{\epsilon}}\right)\leq -L, \tag{4.5}$$

$$\limsup_{\epsilon\to 0}\log\sup_{u_0\in D}\mathbb{E}[\tau^{\epsilon,\delta,u_0}]\leq\overline{M^\delta}. \tag{4.6}$$

Remark 3. Since the regularized problem with $f_\delta(|u^{\epsilon,\delta}|^2)u^{\epsilon,\delta}$ is like some classical mass-subcritical nonlinear Schrödinger equation, it should be mentioned that one may use the Strichartz estimates in [25] and modified arguments in [35] to derive similar properties in Proposition 2 for the exit problem in L^2 (or $L^p, p\geq 2$). However, different from [35], to study the exit problem in H^1 for the considered problem, the functional $\mathcal{H}$ has been introduced to deal with those terms caused by the white noise dispersion $\mathbf{i}\sqrt{\epsilon}\Delta u^{\epsilon,\delta}\circ dB(t)$ (see the proof of Lemmas 6 and Step 2 in the proof of Theorem 1 for details).

Note that the results in this section reply on the assumption $D\in\mathcal{B}_R^0$ with $R>0$ similar as in [35]. At this stage, we only have the following estimate between R and the first time of exit for the regularized problem, i.e., for any $T>0$,

$$\limsup_{\epsilon\to 0}\epsilon\log\mathbb{P}(\tau^{\epsilon,\delta,u_0}\leq T)\leq -(R^2-\||x|u_0\|^2),$$

The proof is similar to that of Proposition 3 and thus is omitted here. It would be interesting and complicated to study the delicate dependence on R of LDP result (especially for the dependence on R of $\inf_{\delta>0}\underline{M^\delta},\inf_{\delta>0}\overline{M^\delta}$). But this is beyond the scope of the current work.

5. Further discussions. For the limit model, i.e., (1.4) with $\delta=0$, there are still a lot of unclear parts on the random effect of the noise. Below we briefly discuss some potential and interesting aspects in studying the effect of stochastic dispersion on the logarithmic Schrödinger equation.

5.1. Exit time and exit points. Similar to [35], one can also formally characterize the exit points of the regularized problem. However, it is still difficult to derive a rigorous result on the exit points of (1.4) with $\delta=0$ and $V=0$ since Proposition 2 may fail if $\delta\to 0$. Thanks to the special structures (2.11) and (2.12) when $\mu>2|\lambda|$, we have the following rough result on the exit points.

Proposition 3. *Let $\delta\geq 0$, $\mu>2|\lambda|$, and $\alpha=1$. For (1.4), the exit from an open bounded domain D containing 0 in the interior of $\mathcal{X}^{1,1}$ appears in $\mathcal{X}^{1,1}$ if and only if the exit appears in L^2_1.*

The proof is straightforward by noticing that

$$\|u^{\epsilon,\delta}(t)\|\leq e^{-\mu t}\|u_0\|,\quad \|\nabla u^{\epsilon,\delta}(t)\|\leq e^{-(\mu-2|\lambda|)t}\|\nabla u_0\|.$$

Furthermore, we have the following finding on the exit time of (1.4) with $\delta=0$ based on the exponential integrability property.

Proposition 4. *Let $\delta=0$, $\mu>2|\lambda|$, and $\alpha=1$. Assume that $R>\||x|u_0\|$. Then for any $T>0$, it holds that*

$$\limsup_{\epsilon\to 0}\epsilon\log\mathbb{P}(\tau^{\epsilon,u_0}\leq T)\leq -(R^2-\||x|u_0\|^2),$$

where τ^{ϵ,u_0} is the exit time from the ball $\mathcal{B}^0_R$ in L^2_1.

Proof. According to Proposition 3, it suffices to analyze the exit problem in L^2_1. For any $T>0$, it follows that

$$\mathbb{P}\left(\tau^{\epsilon,u_0}\leq T\right)\leq \mathbb{P}\left(\sup_{t\in[0,T]}\||x|u^{\epsilon,u_0}\|\geq R\right).$$

By applying [15, Lemma 3.1], it holds that for any $p\geq 0$,

$$\begin{aligned}
&\exp\left(\frac{p\||x|u^{\epsilon,u_0}(t)\|^2}{\epsilon}\right)\leq \exp\left(\frac{p\||x|u_0\|^2}{\epsilon}\right)\\
&+\int_0^t\left(C(\lambda,p)\|u^{\epsilon,u_0}(s)\|^2_{H^1}-\frac{\mu p}{\epsilon}\||x|u^{\epsilon,u_0}(t)\|^2\right)\exp\left(\frac{p\||x|u^{\epsilon,u_0}(s)\|^2}{\epsilon}\right)\mathrm{d}s\\
&+\int_0^t p\exp\left(\frac{p\||x|u^{\epsilon,u_0}(t)\|^2}{\epsilon}\right)\langle \mathbf{i}\nabla u^{\epsilon,u_0},2xu^{\epsilon,u_0}\rangle\sqrt{\epsilon}\mathrm{d}B(t).
\end{aligned}$$

Combining the above estimate with the BDG inequality, we can obtain that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,T]}\exp\left(\frac{p\||x|u^{\epsilon,u_0}(t)\|^2}{\epsilon}\right)\right]\\
&\leq \exp\left(\frac{p\||x|u_0\|^2}{\epsilon}\right)\exp\left(\int_0^t C(\lambda,p,\mu)e^{-2(\mu-2|\lambda|)s}\mathrm{d}s\|u_0\|^2_{H^1}\right).
\end{aligned}$$

Taking $p=1$, by Chebyshev's inequality, we have that

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in[0,T]}\||x|u^{\epsilon,u_0}\|\geq R\right)\\&\leq \exp\left(-\frac{R^2}{\epsilon}\right)\mathbb{E}\left[\sup_{s\in[0,T]}\exp\left(\frac{\||x|u^{\epsilon,u_0}(t)\|^2}{\epsilon}\right)\right]\\&\leq \exp\left(-\frac{R^2-\||x|u_0\|^2}{\epsilon}\right)\exp\left(C(\lambda,\mu)\frac{1}{2(\mu-2|\lambda|)}\|u_0\|^2_{H^1}\right).\end{aligned}$$

If $R^2>\||x|u_0\|^2$, then it holds that for any $T>0$,

$$\limsup_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\tau^{\epsilon,u_0}\leq T\right)\leq -\left(R^2-\||x|u_0\|^2\right).\qquad \blacksquare$$

In order to predict more delicate results on the exit time and exit points for the considered model, we present several numerical tests to simulate its asymptotic behaviors in section 6. The rigorous numerical analysis of (1.1) will be the future work.

5.2. Effect on the large dispersion. Another interesting problem lies on the effect of large random dispersion for (1.4) with $\delta=0$, i.e., $\epsilon\to+\infty$. For example, let us assume that $V=0$ and $\mu=0$. In order to avoid confusion, we denote the solution of (1.4) with $\delta=0$ by X^ϵ for large enough $\epsilon>0$. By the analysis in section 2, it is not hard to obtain

$$\begin{aligned}\|X^\epsilon(t)\|=\|u_0\|,\quad \|X^\epsilon(t)\|_{H^1}\leq e^{C(\lambda,\|u_0\|)t}\|u_0\|_{H^1},\\ \|X^\epsilon(t)\|_{L^p(\Omega;L^2_\alpha)}\leq e^{C(\lambda,\|u_0\|,p)t}\epsilon\|u_0\|_{H^1}+C(p)\|u_0\|_{L^2_\alpha}.\end{aligned}$$

Since the decaying estimate of $S_{\sqrt{\epsilon}B}(t,s)$ holds [25], i.e., for $\frac{1}{p}+\frac{1}{p'}=1$ and $s<t$,

$$\begin{aligned}\|S_B(t,s)z\|_{L^p}&=\left\|\frac{1}{4\pi\mathbf{i}(\sqrt{\epsilon}B(t)-\sqrt{\epsilon}B(s))^{\frac{d}{2}}}\int_{\mathbb{R}^d}\exp\left(\mathbf{i}\frac{|x-y|^2}{4(\sqrt{\epsilon}B(t)-\sqrt{\epsilon}B(s))}\right)z\mathrm{d}y\right\|_{L^p}\\&\leq C_d\epsilon^{-\frac{d}{2}(\frac12-\frac1p)}(B(t)-B(s))^{-d(\frac12-\frac1p)}\|z\|_{L^{p'}},\end{aligned}$$

thus it holds that for $p'=2-2\eta$ and $p=\frac{2-2\eta}{1-2\eta}$ with $\eta\in[0,\frac12)$,

$$\begin{aligned}\|X^\epsilon(t)\|_{L^p}\leq&\|S_B(t)u_0\|_{L^p}+\left\|\int_0^t S_B(t,s)\log(|X^\epsilon|^2)X^\epsilon\mathrm{d}s\right\|_{L^p}\\\leq& C_d\epsilon^{-\frac{d}{2}(\frac12-\frac1p)}B(t)^{-d(\frac12-\frac1p)}\|u_0\|_{L^{p'}}\\&+C_d\int_0^t\epsilon^{-\frac{d}{2}(\frac12-\frac1p)}(B(t)-B(s))^{-d(\frac12-\frac1p)}\|\log(|X^\epsilon|^2)X^\epsilon\|_{L^{p'}}\mathrm{d}s.\end{aligned}$$

Note that by the weighted Sobolev inequality, Gagliardo–Nirenberg inequality, and the properties of logarithmic function, it holds that for $\eta\in(0,1),\alpha>\frac{d\eta}{2-2\eta}$,

$$\|X^\epsilon\|_{L^{2-2\eta}}\leq C\|X^\epsilon\|^{1-\frac{d\eta}{2\alpha(1-\eta)}}\|X^\epsilon\|^{\frac{d\eta}{2\alpha(1-\eta)}}_{L^2_\alpha}$$

and that for $\eta_1, \eta_1' > 0$ small enough and $\alpha_1 > \frac{d(\eta+\eta_1)}{2-2\eta-2\eta_1}$,

$$\begin{aligned}\|\log(|X^\epsilon|^2)X^\epsilon\|_{L^{2-2\eta}} &\le C\left(\|X^\epsilon\|_{L^{2-2\eta-2\eta_1}} + \|X^\epsilon\|_{L^{2-2\eta-2\eta_1'}}\right)\\ &\le C\|u_0\|^{1-\frac{d(\eta+\eta_1)}{2\alpha_1(1-\eta-\eta_1)}}\|X^\epsilon\|_{L^2_{\alpha_1}}^{\frac{d(\eta+\eta_1)}{2\alpha_1(1-\eta-\eta_1)}}\\ &\quad + C\|u_0\|^{\frac{d\eta+\eta_1'}{2+2\eta+2\eta_1'}}\|\nabla X^\epsilon\|^{1-\frac{d(\eta+\eta_1')}{2+2\eta+2\eta_1'}}.\end{aligned}$$

Thanks to the a priori estimate on X^ϵ, it holds that

$$\begin{aligned}&\|\log(|X^\epsilon|^2)X^\epsilon\|_{L^{p'}}\\ &\le e^{C(\lambda,\|u_0\|,d,\eta,\eta_1,\eta_1')t}\left[1 + \left(\epsilon\|u_0\|_{H^1} + \|u_0\|_{L^2_\alpha}\right)^{\frac{d(\eta+\eta_1)}{2\alpha_1(1-\eta-\eta_1)}} + \|\nabla u_0\|^{1-\frac{d(\eta+\eta_1')}{2+2\eta+2\eta_1'}}\right].\end{aligned}$$

Then one may expect that

$$\|X^\epsilon(t)\|_{L^p} \sim O\left(\epsilon^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})+\frac{d(\eta+\eta_1)}{2\alpha(1-\eta-\eta_1)}}\right), \quad \text{a.s.}$$

Note that the above asymptotic estimate is not trivial especially in the case where the Sobolev embedding theorem $H^1 \hookrightarrow L^p, p \ge \frac{2d}{(d-2)^+}$, does not hold.

6. Numerical tests.

In this section, we present several numerical experiments to predict some dynamics of (1.1) via (1.4) and certain modifications of numerical approximations in [6]. To be more specific, we split (1.4) into three subsystems that have explicit solutions (see, e.g., [6, 14]), and obtain the following splitting approximation:

$$\begin{aligned}v_n &= e^{\mathbf{i}\Delta\sqrt{\epsilon}(B(t_{n+1})-B(t_n))}(e^{\mathbf{i}V[u_n]\tau}u_n),\\ u_{n+1} &= e^{-\mu\tau}e^{\mathbf{i}\lambda\tau\log\left(\frac{\delta+|v_n|^2}{1+\delta|v_n|^2}\right)}v_n.\end{aligned}$$

Here τ is the time stepsize, $t_n = n\tau$, $\{u_n\}_{n\in\mathbb{Z}}$ is the numerical solution. To get an implementable full discrete scheme, we further apply the classical spatial central difference method with the space stepsize $h > 0$ to $\mathbf{i}\Delta$.

Example 1. In this test, we consider the exit problem of $\mathcal{B}_R^0$ with $R = 1$ under L_1^2-norm when $\mu > 2|\lambda|$. Let $d = 1$, $V = 0$, $\lambda = 0.005$, $\mu = 0.012$, $\epsilon = 2^{-8}$, $\delta = 10^{-6}$, $\tau = 2^{-9}$, $h = 2^{-4}$, the terminal time $T = 100$, and the sample size $N = 2000$. To approximate the original problem defined on $\mathbb{R}$, we consider the truncated domain $[-2\pi, 2\pi]$ and the initial value $u_0 = \sin(x)e^{-|x|^2}$.

In Figure 1, it can be seen that there are two trajectories of $\|u(\cdot,t)\|_{L_1^2}$ which exit the ball $\mathcal{B}_1^0$ among 2000 samples. The exiting time $\tau^{\epsilon,\delta,u_0}$ of these two trajectories is 29.4238 and 49.0684, respectively. In contrast, the L^2-norm and H^1-norm are exponentially decaying which verifies (2.11), the first bullet point in Remark 1 and Proposition 3.

Example 2. In this test, we study the influence of V on the profile of the solution of (1.1). Let $d = 1$, $\lambda = 0.05$, $\mu = 0$, $\epsilon = 2^{-8}$, $\delta = 10^{-6}$, $\tau = 2^{-10}$, $h = 2^{-6}$, the terminal time $T = 100$. We consider the truncated domain $[-4\pi, 4\pi]$ and the initial value $u_0 = e^{-|x|^2}$. We present

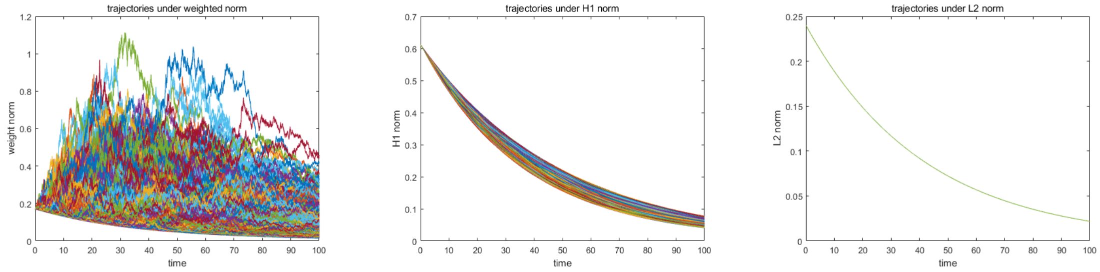

Figure 1. *Example 1: time evolution of the solution under different metrics (left: L^1_2-seminorm, middle: H^1-seminorm, right: L^2-norm).*

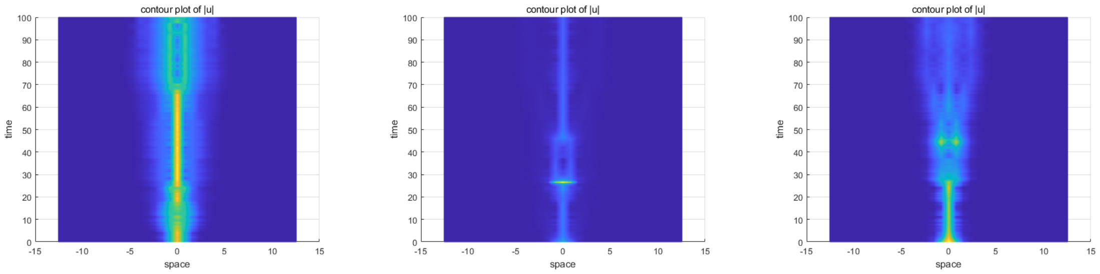

Figure 2. *Example 2: contour plot of the space-time evolution of the modulus $u(t,x)$ (left: $V=0$, middle: $V=2|x|^2e^{-|x|^2}$, right: $V=\sin^2(x)$).*

the contour plot of the modulus $|u(t,x)|$ for $V=0$, $V=2|x|^2e^{-|x|^2}$, $V=\sin^2(x)$ where the trajectories are simulated with one realization of a Brownian motion.

As we can see, in Figure 2, when $V=0$, the modulus is still confined, and the effect of the dispersive noise and the nonlinearity effect stay well-balanced resulting in alternating contractions and expansions of the profile. For the case where $V=2|x|^2e^{-|x|^2}$, it seems that the balance between dispersion and nonlinearity is observed for a while, and the contraction wins after $T=50$ even if the final amplitude is slightly smaller than the initial amplitude. With respect to $V=\sin^2(x)$, the solution behaves well-balanced in $[0,70]$ and the dispersion finally wins. Note that all different simulations that we have done give similar observations.

Example 3. In this example, we present some tests on the influence of V for the exit problem $\mathcal{B}^0_R$ with $R=10$ under L^2_1-norm and H^1-norm. Let $d=1$, $V=2|x|e^{-|x|^2}$, $\lambda=0.005$, $\mu=0.012$, $\epsilon=2^{-8}$, $\delta=10^{-6}$, $\tau=2^{-9}$, $h=2^{-4}$, the terminal time $T=100$, and the sample size $N=2000$. We consider the truncated domain $[-2\pi,2\pi]$ and the initial value $u_0=\sin(x)e^{-|x|^2}$.

From Figure 3, we observe that the exit probability from $\mathcal{B}^0_R$ is increasing due to the effect of the nonlinear interaction V compared with that in Example 1. Indeed, there are more trajectories that exit B^0_R at a certain time less than $T=100$. Besides, the phenomenon of the exit appears in both L^2_1 and H^1. In contrast, the exit event never happens in L^2.

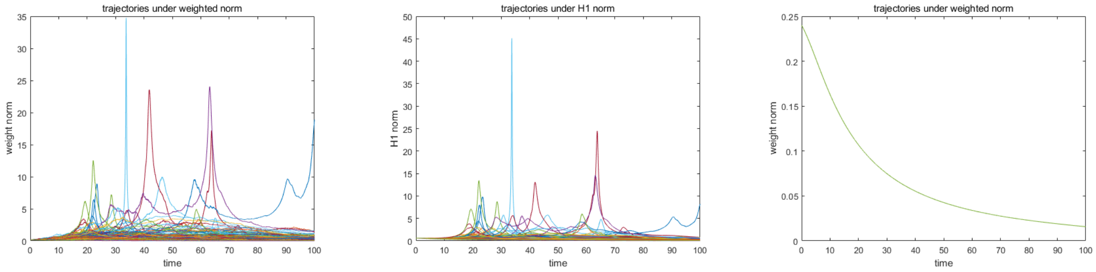

Figure 3. *Example 3: time evolution of the solution under different metrics (left: L^1_2-seminorm, middle: H^1-seminorm, right: L^2-norm).*

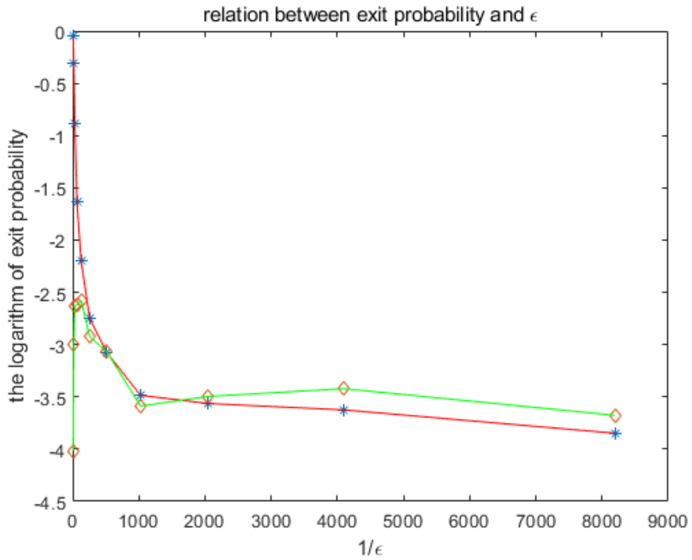

Figure 4. *Example 4: the red line is about the relationship between $\frac{1}{\epsilon}$ and $\log(\mathbb{P}(\tau^{\epsilon,\delta,u_0}_{L^2_1}))$. The green line is about the relationship of $\frac{1}{\epsilon}$ and $\log(\mathbb{P}(\tau^{\epsilon,\delta,u_0}_{H^1}))$.*

Example 4. In this example, we provide some evidences on the LDP of the exit problem via simulating the rare event based on the Monte-Carlo method. We set the truncated domain $[-2\pi, 2\pi]$, $h = 2^{-4}, \tau = 2^{-8}$, $\delta = 10^{-6}$, $\lambda = 0.01, \mu = 0.01$, the terminal time $T = 100$, and the sample size $N = 3000$. Let $V = 2|x|^2 e^{-|x|^2}$ and $u_0 = \sin(x)e^{-|x|^2}$. The exit domain is set as $D = \mathcal{B}^0_R$ with $R = 2$ under L^2_1-seminorm and H^1-seminorm. To measure the dependence of the exit problem on $\frac{1}{\epsilon}$, we choose ϵ as $2^{-j}, j = 3, 4, \ldots, 12$.

Via the Monte-Carlo method, we can approximate the exit probability by accounting the number of exit trajectories where the exit happens before the time T. Then in Figure 4, we plot the relationship between $\frac{1}{\epsilon}$ and the logarithm of the exit probability $\log(\mathbb{P}(\tau^{\epsilon,\delta,u_0}_{L^2_1} \le T))$ under L^2_1-norm and $\log(\mathbb{P}(\tau^{\epsilon,\delta,u_0}_{H^1} \le T))$ under H^1-norm, respectively. Note that if the LDP in section 3 holds, the function $\log(\mathbb{P}(\tau^{\epsilon,\delta,u_0}_{L^2_1} \le T))$ (or $\log(\mathbb{P}(\tau^{\epsilon,\delta,u_0}_{H^1} \le T))$) w.r.t. $\frac{1}{\epsilon}$ is more like a linear function as $\epsilon \to 0$. Our numerical observation coincides with this fact.

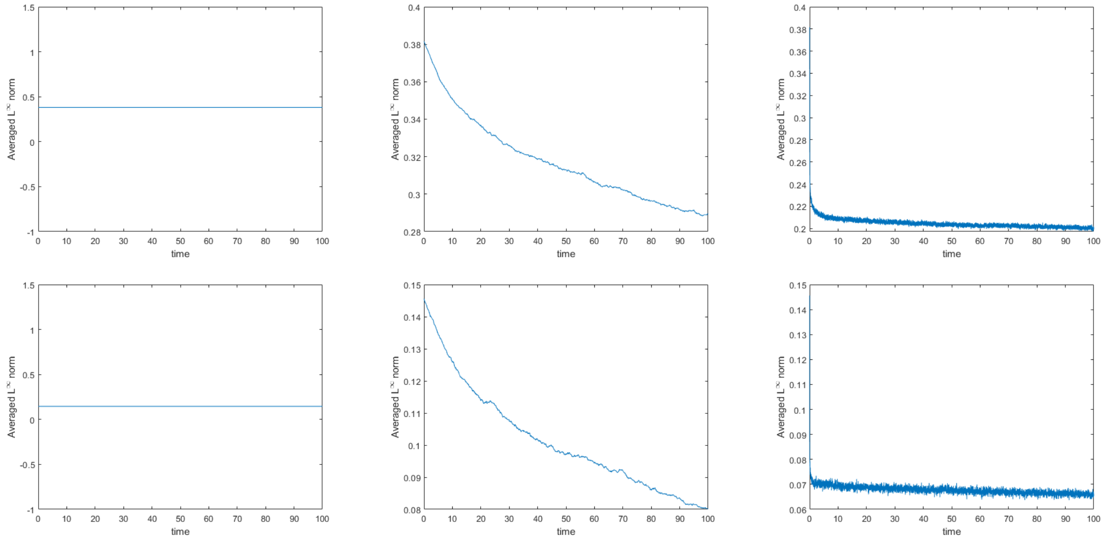

Figure 5. *Example* 5*: first row: time evolution of* $\frac{1}{N}\sum_{i=1}^{N}\|u^{(i)}(t,\cdot)\|_{L^\infty}$ *in* $d=1$ *under different scales of noise; second row: time evolution of* $\frac{1}{N}\sum_{i=1}^{N}\|u^{(i)}(t,\cdot)\|_{L^\infty}$ *in* $d=2$ *under different scales of noise (left:* $\epsilon=0$*, middle:* $\epsilon=2^{-7}$*, right:* $\epsilon=2^{7}$*).*

Example 5. This example predicts some evolutions of the profile of the solution of (1.1) when $d=1$ and $d=2$. We test the effect of noise intensity on the profile of averaged L^∞-norm of $u(t,\cdot)$, i.e., $\frac{1}{N}\sum_{i=1}^{N}\|u^{(i)}(t,\cdot)\|_{L^\infty}$ as time goes to $T=100$. To this end, we set $\epsilon=0$, $\epsilon=2^7$, $\epsilon=2^{-7}$. When $d=1$, we let the truncated domain be $[-4\pi,4\pi]$, $h=2^{-4}, \tau=2^{-9}$, $\delta=10^{-6}$, $\lambda=0.1, \mu=0$, $V=0$, the sample size $N=1000$ and $u_0=\sin(x)e^{-|x|^2}$. This case is presented in the first row of Figure 5. When $d=2$, we set the truncated domain as $[-2\pi,2\pi]\times[-2\pi,2\pi]$, $h=2^{-4}$, $\tau=2^{-6}$, $\delta=10^{-6}$, $\lambda=0.2, \mu=0$, $V=0$, the sample size $N=1000$ and $u_0=\sin(x)\sin(y)e^{-|x|^2-|y|^2}$. This case is shown in the second row of Figure 5.

From Figure 5, in the case that $d=1$ and $d=2$. it is observed that when $\epsilon=0$, the numerical solution keeps the L^∞-norm, which coincides with the original property of the solution of (4.1) with $\mu=0$. When considering (1.1), it turns out that even the small noise ($\epsilon=2^{-7}$) will lead to a rather strong diffusion of the expected amplitude as time evolves. When the noise intensity is large ($\epsilon=2^7$), an interesting finding is that the decaying rate of the solution in the large noise case is faster than that in the small noise case.

Appendix A.

Proof of Theorem 2. First, we show that under the $\mathcal{C}([0,T];\mathcal{X}^{1,1})$-norm, the considered model satisfies the LDP with the same good rate function in section 3. From the arguments in section 3, it suffices to show that the trajectories of (1.4) and its skeleton equation are continuous in $\mathcal{X}^{1,1}$. For simplicity, we only present the detailed proof for the continuity of $L^\delta_{u_0}(g)$ in $\mathcal{X}^{1,1}$ since the other argument is similar.

Define $W^2_{1,x}:=\{z\in H^1|\int_{\mathbb{R}^d}(1+|x|^2)(|z(x)|^2+|\nabla z(x)|^2)\mathrm{d}x<\infty\}$. Let us consider a sequence of approximations $u_0^{R'}\in H^2\cap W^2_{1,x}$ to u_0 such that $\|u_0^{R'}-u_0\|_{\mathcal{X}^{1,1}}\to 0$ as $R'\to+\infty$.

By using the mild formulation of $L^\delta_{u_0^{R'}}(h)(t)$ and the Gagliardo–Nirenberg inequality $\|w\|_{L^4} \le C_d\|\nabla w\|^{\frac{d}{4}}\|w\|^{1-\frac{d}{4}}$, we obtain that for $d\le 2$,

$$
\begin{aligned}
&\|L^\delta_{u_0^{R'}}(g)(t)\|_{H^2}\\
&\le \|u_0^{R'}\|_{H^2} + \int_0^t |\lambda| \Big\| S_g(t,s) f_\delta(|L^\delta_{u_0^{R'}}(g)(s)|^2) L^\delta_{u_0^{R'}}(g)(s)\Big\|_{H^2} \mathrm{d}s\\
&\quad + \int_0^t \mu \Big\| S_g(t,s) L^\delta_{u_0^{R'}}(g)(s)\Big\|_{H^2}\mathrm{d}s\\
&\le \|u_0^{R'}\|_{H^2} + \int_0^t C(\lambda,\delta,\mu)\left(\|L^\delta_{u_0^{R'}}(g)(s)\|_{H^2} + \|L^\delta_{u_0^{R'}}(g)(s)\|_{H^2}^{\frac{d}{2}}\|L^\delta_{u_0^{R'}}(g)(s)\|_{H^1}^{2-\frac{d}{2}}\right)\mathrm{d}s.
\end{aligned}
$$

As a consequence, using the isometry property of $S_g(t,s)$ for any $t,s\ge 0$, the global estimate holds,

$$
\sup_{t\in[0,T]} \|L^\delta_{u_0^{R'}}(g)(t)\|_{H^2} \le C(\lambda,\delta,\mu,T,\|u_0^{R'}\|_{H^1})\|u_0^{R'}\|_{H^2}.
$$

Next, we deal with the $W^2_{1,x}$-estimate. By the chain rule and integration by parts, it follows that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|x\nabla L^\delta_{u_0^{R'}}(g)(t)\|^2\\
&\le 4\Big\langle x\nabla L^\delta_{u_0^{R'}}(g)(t), \mathbf{i}\Delta L^\delta_{u_0^{R'}}(g)(t)\Big\rangle g(t)\\
&\quad + 4|\lambda|\Big\langle |x|^2\nabla L^\delta_{u_0^{R'}}(g)(t), f_\delta'(|L^\delta_{u_0^{R'}}(g)(t)|^2)\mathrm{Re}\Big(\overline{L^\delta_{u_0^{R'}}(g)(t)}\nabla L^\delta_{u_0^{R'}}(g)(t)\Big) L^\delta_{u_0^{R'}}(g)(t)\Big\rangle\\
&\quad - 2\mu\|x\nabla L^\delta_{u_0^{R'}}(g)(t)\|^2.
\end{aligned}
$$

Gronwall's inequality yields that

$$
\sup_{t\in[0,T]}\|x\nabla L^\delta_{u_0^{R'}}(g)(t)\| \le C(T,\delta,\mu,\lambda)\left(\|x\nabla u_0^{R'}\| + \|u_0^{R'}\|_{H^2}\right).
$$

Now we prove the convergence of $L^\delta_{u_0^{R'}}(g)$ in $\mathcal{X}^{1,1}$ as $R'\to\infty$, which implies that $L^\delta_{u_0^{R'}}(g)\in \mathcal{C}([0,T];\mathcal{X}^{1,1})$. On the one hand, by the unitary property of S_g and the properties of f_δ, we have that

$$
\begin{aligned}
&\|L^\delta_{u_0^{R'}}(g) - L^\delta_{u_0}(g)\|_{H^1}\\
&\le \|S_g(t,0)(u_0^{R'} - u_0)\|_{H^1}\\
&\quad + \int_0^t |\lambda| \|S_g(t-s)(f_\delta(|L^\delta_{u_0^{R'}}(g)|^2)L^\delta_{u_0^{R'}}(g) - f_\delta(|L^\delta_{u_0}(g)|^2)L^\delta_{u_0}(g)\|_{H^1}\mathrm{d}s\\
&\quad + \int_0^t |\mu| \|S_g(t-s)(L^\delta_{u_0^{R'}}(g) - L^\delta_{u_0}(g))\|_{H^1}\mathrm{d}s\\
&\le \|u_0^{R'} - u_0\|_{H^1} + \int_0^t |\lambda|(|\log(\delta)|+2)\|L^\delta_{u_0^{R'}}(g)(s) - L^\delta_{u_0}(g)(s)\|_{H^1}\mathrm{d}s\\
&\quad + \int_0^t \mu\|L^\delta_{u_0^{R'}}(g)(s) - L^\delta_{u_0}(g)(s)\|_{H^1}\mathrm{d}s.
\end{aligned}
$$

Gronwall's inequality yields that

$$\text{(A.1)}\qquad \sup_{t\in[0,T]}\|L^{\delta}_{u_0^{R'}}(g)(t)-L^{\delta}_{u_0}(g)(t)\|_{H^1}\le C(|\lambda|,\delta,T,\mu)\|u_0^{R'}-u_0\|_{H^1}\to 0 \text{ as } R'\to\infty.$$

On the other hand, recalling the fact that $h=\dot g$, applying the chain rule and integration by parts, as well as Young's inequality, one can derive

$$\begin{aligned}
&\|x(L^{\delta}_{u_0^{R'}}(g)(t)-L^{\delta}_{u_0}(g)(t))\|^2\\
&\le \|x(u_0^{R'}-u_0)\|^2-2\mu\int_0^t\|x(L^{\delta}_{u_0^{R'}}(g)(s)-L^{\delta}_{u_0}(g)(s))\|^2\mathrm{d}s\\
&\quad+\int_0^t 4\langle x(L^{\delta}_{u_0^{R'}}(g)(s)-L^{\delta}_{u_0}(g)(s)),\mathbf{i}\nabla(L^{\delta}_{u_0^{R'}}(g)(s)-L^{\delta}_{u_0}(g)(s))\rangle h(s)\mathrm{d}s\\
&\le \|x(u_0^{R'}-u_0)\|^2+\int_0^t 2\|\nabla(L^{\delta}_{u_0^{R'}}(g)(s)-L^{\delta}_{u_0}(g))\|h^2(s)\mathrm{d}s\\
&\quad+\int_0^t(2-2\mu)\|x(L^{\delta}_{u_0^{R'}}(g)-L^{\delta}_{u_0}(g))\|^2\mathrm{d}s.
\end{aligned}$$

According to (A.1) and Gronwall's inequality, we obtain that

$$\text{(A.2)}\qquad \begin{aligned}&\sup_{t\in[0,T]}\|L^{\delta}_{u_0^{R'}}(g)(t)-L^{\delta}_{u_0}(g)(t)\|_{L^2_1}\\
&\quad\le C(\mu,T)\|u_0^{R'}-u_0\|_{L^2_1}+C(|\lambda|,\delta,T,\mu)\|u_0^{R'}-u_0\|_{H^1}\to 0 \text{ as } R'\to\infty.\end{aligned}$$

Moreover, one can verify that $u^{\epsilon,u_0^{R'},\delta}$ is an exponentially good approximation of $u^{\epsilon,\delta}$ and thus $I^{\delta}_{u_0}$ is also a good rate function under $\mathcal{C}([0,T];\mathcal{X}^{1,1})$ like in Lemma 2.

Now we are able to prove the desired result. Since I^W is a good rate function, $K_T^{u_0}(\widetilde a)$ is compact set of $\mathcal{C}([0,T];\mathcal{X}^{1,1})$ for any $\widetilde a$. Define

$$A^{u_0}_{\widetilde a}:=\{v\in\mathcal{C}([0,T];\mathcal{X}^{1,1})\mid d_{\mathcal{C}([0,T];\mathcal{X}^{1,1})}\left(v,K_T^{u_0}(\widetilde a)\right)\ge\gamma\}.$$

Choosing g such that $I^W(g)<\widetilde a$, it follows that

$$\mathbb{P}\left(u^{\epsilon,\delta,u_0}\in A^{u_0}_{\widetilde a}\right)\le\mathbb{P}\Big(\|u^{\epsilon,\delta,u_0}-L^{\delta}_{u_0}(g)\|_{\mathcal{C}([0,T];\mathcal{X}^{1,1})}\ge\gamma\Big).$$

Then by the LDP, there exists $\epsilon_0>0$ such that for every $\epsilon\in(0,\epsilon_0)$,

$$\epsilon\log\mathbb{P}\left(u^{\epsilon,u_0,\delta}\in A^{(u_0)}_{\widetilde a}\right)\le-(\widetilde a-\kappa),$$

which implies the upper bound. Next, we consider the lower bound. Due to the continuity of $L^{\delta}_{u_0}(\cdot)$, for any $\|u_0\|_{\mathcal{X}^{1,1}}\le\rho$ and $w\in K_T^{u_0,\delta}(a)$, there exists g such that $w=L^{\delta}_{u_0}(g)$ and $I^{\delta}_{u_0}(w)=I^W(g)$. By the LDP and the fact that $B^w_\gamma=\{v\mid\|v-w\|_{\mathcal{C}([0,T];\mathcal{X}^{1,1})}<\gamma\}$ is an open set, there exists $\epsilon_0'>0$ such that for every $\epsilon\in(0,\epsilon_0')$,

$$\begin{aligned}\epsilon\log\mathbb{P}\left(\|u^{\epsilon,u_0,\delta}-w\|_{\mathcal{C}([0,T];\mathcal{X}^{1,1})}<\gamma\right)&\ge-\inf_{v\in B^w_\gamma}I^{\delta,T}_{u_0}(v)-\kappa\\
&\ge -I^{\delta,T}_{u_0}(w)-\kappa.\end{aligned}$$

We complete the proof. ∎

Proof of Lemma 4. For the case where u_0 belongs to $\mathcal{B}^0_\rho$, the result is straightforward. Below we focus on the case that $u_0 \in D\backslash\mathcal{B}^0_\rho$. Since D is uniformly attracted to zero by the deterministic flows (4.1), there exists a positive time $T_1 > 0$ such that for every u_1 in $\mathcal{N}^0(D\backslash\mathcal{B}^0_\rho, \frac{\rho}{8})$ and $t \geq T_1$, $L^\delta_{u_1}(0)(t) \in \mathcal{B}^0_{\frac{\rho}{8}}$. Notice that

$$\sup_{u_1\in\mathcal{N}^0(D\backslash\mathcal{B}^0_\rho,\frac{\rho}{8})} \|L^\delta_{u_1}(0)\|_{\mathcal{C}([0,T_1],\mathcal{X}^{1,1})} \leq C(T_1, |\lambda|, \mu, R).$$

Next, we prove that for sufficient large $T \geq T_1$,

$$\begin{aligned}\mathcal{T}_\rho := &\Big\{ y \in \mathcal{C}([0,T];\mathcal{X}^{1,1}) \mid \forall t \in [0,T], y(t) \in \mathcal{N}^0\Big(D\backslash\mathcal{B}^0_\rho, \frac{\rho}{8}\Big)\Big\} \\ &\subset (K^{u_0,\delta}_T(2L))^c.\end{aligned} \tag{A.3}$$

It suffices to consider $y = L^\delta_{u_0}(g) \in \mathcal{T}_\rho$. Indeed, we have that

$$\|L^\delta_{u_0}(g) - L^\delta_{u_0}(0)\|_{\mathcal{C}([0,T_1];\mathcal{X}^{1,1})} \geq \|L^\delta_{u_0}(g)(T_1) - L^\delta_{u_0}(0)(T_1)\|_{\mathcal{X}^{1,1}} \geq \frac{3}{4}\rho.$$

On the other hand, using the approximation $u_0^{R'} \in H^2 \cap W^2_{1,x}$ to u_0 on $\mathcal{X}^{1,1}$, there exists small $t_1 > 0$ such that

$$\begin{aligned}&\|L^\delta_{u_0}(g) - L^\delta_{u_0}(0)\|_{\mathcal{C}([0,t_1];\mathcal{X}^{1,1})} \\ &\leq C(R',\delta,R)\Big|\int_0^{t_1} |h(s)|\mathrm{d}s\Big|^{\frac{1}{2}} + \mu t_1 \|L^\delta_{u_0^{R'}}(g) - L^\delta_{u_0^{R'}}(0)\|_{\mathcal{C}([0,t_1];\mathcal{X}^{1,1})} \\ &\quad + |\lambda||\log(\delta)|t_1\|L^\delta_{u_0}(g) - L^\delta_{u_0}(0)\|_{\mathcal{C}([0,t_1];\mathcal{X}^{1,1})} \\ &\leq C(R',\delta,R)t_1^{\frac{1}{4}}\Big|\int_0^{t_1} |h(s)|^2\mathrm{d}s\Big|^{\frac{1}{4}} + \alpha t_1\|L^\delta_{u_0}(g) - L^\delta_{u_0}(0)\|_{\mathcal{C}([0,t_1];\mathcal{X}^{1,1})} \\ &\quad + |\lambda||\log(\delta)|t_1\|L^\delta_{u_0}(g) - L^\delta_{u_0}(0)\|_{\mathcal{C}([0,t_1];\mathcal{X}^{1,1})}.\end{aligned}$$

Letting t_1 sufficiently small such that $|\lambda||\log(\delta)|t_1 + \alpha t_1 \leq \frac{1}{2}$ and $C(R',\delta,R)t_1^{\frac{1}{4}} \leq \frac{1}{2}$, we obtain that

$$\|L^\delta_{u_0}(g) - L^\delta_{u_0}(0)\|_{\mathcal{C}([0,t_1];\mathcal{X}^{1,1})} \leq \Big|\int_0^{t_1} |h(s)|^2\mathrm{d}s\Big|^{\frac{1}{4}}.$$

Then by iterating the above estimate for each small interval $[kt_1,(k+1)t_1]$ for $k = 1,\dots,\lfloor T_1/t_1\rfloor$, ($\lfloor\cdot\rfloor$ is the floor function), it follows that

$$\|L^\delta_{u_0}(g) - L^\delta_{u_0}(0)\|_{\mathcal{C}([0,T_1];\mathcal{X}^{1,1})} \leq 2^{\lfloor T_1/t_1\rfloor+1}\|h\|^{\frac{1}{2}}_{L^2([0,T_1];\mathbb{R})}.$$

Thus, we conclude that

$$\frac{1}{2^{4\lfloor T_1/t_1\rfloor+5}}\left(\frac{3}{4}\rho\right)^4 \leq \frac{1}{2}\|h\|^2_{L^2([0,T_1];\mathbb{R})}.$$

Similarly, for any $[T_1, 2T_1]$, by the inverse triangle inequality and the fact that 0 is an attractor of the deterministic flow, one has that

$$\|L^{\delta}_{L^{\delta}_{u_0}(g)(T_1)}(g) - L^{\delta}_{L^{\delta}_{u_0}(g)(T_1)}(0)\|_{\mathcal{C}([0,T_1];\mathcal{X}^{1,1})} \geq \frac{1}{2}\rho.$$

This also implies that

$$\frac{1}{2}\|h\|^2_{L^2([T_1,2T_1];\mathbb{R})} \geq \frac{1}{2^{4\lfloor T_1/t_1\rfloor+5}}\left(\frac{1}{2}\rho\right)^4 =: M''.$$

Therefore, we obtain that

$$\frac{1}{2}\|h\|^2_{L^2([0,2T_1];\mathbb{R})} \geq 2M''.$$

For any $T>0$, there exists a positive number j such that $T > jT_1$. Iterating the above arguments, one has $\frac{1}{2}\|h\|^2_{L^2([0,jT_1];\mathbb{R})} \geq jM''$. To obtain (A.3), we will take $jM'' > 2L$.

Finally, we are able to show the desired result since

$$\begin{aligned}\mathbb{P}\left(\sigma_\rho^{\epsilon,\delta,u_0} > T\right) &= \mathbb{P}\left(\forall t\in[0,T], u^{\epsilon,\delta,u_0} \in D\backslash \mathcal{B}^0_\rho\right)\\ &= \mathbb{P}\left(d_{\mathcal{C}([0,T];\mathcal{X}^{1,1})}\left(u^{\epsilon,\delta,u_0}, \mathcal{T}^c_\rho\right) > \frac{\rho}{8}\right)\\ &\leq \mathbb{P}\left(d_{\mathcal{C}([0,T];\mathcal{X}^{1,1})}\left(u^{\epsilon,\delta,u_0}, K^{u_0}_T(2L)\right) \geq \frac{\rho}{8}\right).\end{aligned}$$

By Theorem 2 (i), it follows that for small $\epsilon>0$ and $\rho \leq R$,

$$\mathbb{P}\left(d_{\mathcal{C}([0,T];\mathcal{X}^{1,1})}\left(u^{\epsilon,\delta,u_0}, K^{u_0}_T(2L)\right) \geq \frac{\rho}{8}\right) \leq e^{-\frac{L}{\epsilon}}. \qquad \blacksquare$$

Proof of Lemma 5. When $u_0 \in \mathcal{B}^0_\rho$, the desired bound is trivial. Thus we only deal with the case that $u_0 \in D\backslash\mathcal{B}^0_\rho$. Since zero is the attractor of the deterministic flow, define $T = \inf\{t \geq 0 | L^\delta_{u_0}(0)(t) \in \mathcal{B}^0_{\frac{\rho}{2}}\}$ and it holds that

$$\mathbb{P}\left(u^{\epsilon,\delta,u_0}(\sigma_\rho^{\epsilon,\delta,u_0}) \in \partial D\right) \leq \mathbb{P}\left(\|u^{\epsilon,\delta,u_0} - L^\delta_{u_0}(0)\|_{\mathcal{C}([0,T];\mathcal{X}^{1,1})} \geq \min\left(\frac{\rho}{2}, \frac{d(0,\partial D)}{2}\right)\right).$$

By the LDP, we get

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\|u^{\epsilon,\delta,u_0} - L^\delta_{u_0}(0)\|_{\mathcal{C}([0,T];\mathcal{X}^{1,1})} \geq \min\left(\frac{\rho}{2}, \frac{d(0,\partial D)}{2}\right)\right) \leq -L,$$

where

$$L = \inf_{\|L^\delta_{u_0}(0) - L^\delta_{u_0}(g)\|_{\mathcal{C}([0,T];\mathcal{X}^{1,1})} \geq \min(\frac{\rho}{2}, \frac{d(0,\partial D)}{2})} \|h\|^2_{L^2([0,T];\mathbb{R})} > 0. \qquad \blacksquare$$

Proof of Proposition 2. Let us first prove the upper bound estimate (4.5)–(4.6). Fix $\kappa > 0$ small enough and choose g and T_1' such that $L_0^\delta(g)(T_1') \in (\overline{D})^c$ and

$$I_0^{\delta,T_1}(L_0^\delta(g)(T_1')) = \frac{1}{2}\|h\|^2_{L^2([0,T_1'];\mathbb{R})} \leq \overline{M^\delta} + \frac{\kappa}{6}.$$

Let d_0 denote the positive distance between $L_0^\delta(g)(T_1')$ and $\overline{D}$. Since f_δ is Lipschitz for a fixed δ, there exist a small ball $\mathcal{B}_\rho^0 \subset D$ such that if u_0 belongs to $\mathcal{B}_\rho^0$,

$$\|L_{u_0}^\delta(g) - L_0^\delta(g)\|_{C([0,T_1'];\mathcal{X}^{1,1})} < \frac{d_0}{2}.$$

By the LDP in Theorem 2 and triangle inequality, there exists ϵ_1' such that for every $\epsilon \in (0, \epsilon_1')$,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,\delta,u_0} < T_1'\right) &\geq \mathbb{P}\left(\|u^{\epsilon,\delta,u_0} - L_0^\delta(g)\|_{C([0,T_1'];\mathcal{X}^{1,1})} < d_0\right)\\ &\geq \mathbb{P}\left(\|u^{\epsilon,\delta,u_0} - L_{u_0}^\delta(g)\|_{C([0,T_1'];\mathcal{X}^{1,1})} < \frac{d_0}{2}\right)\\ &\geq \exp\left(-\frac{I_{u_0}^{\delta,T_1'} + \frac{\kappa}{6}}{\epsilon}\right).\end{aligned}$$

From Lemma 4, there exist T_2' and ϵ_2' such that for $\epsilon \in (0, \epsilon_2')$,

$$\inf_{u_0 \in D} \mathbb{P}\left(\sigma_\rho^{\epsilon,\delta,u_0} \leq T_2\right) \geq \frac{1}{2}.$$

Applying the Markov property, we get that for $\epsilon \in (0, \epsilon_1' \wedge \epsilon_2')$,

$$\begin{aligned}\inf_{u_0 \in D} \mathbb{P}\left(\tau^{\epsilon,\delta,u_0} \leq T_1' + T_2'\right) &\geq \inf_{u_0 \in D} \mathbb{P}\left(\sigma_\rho^{\epsilon,\delta,u_0} \leq T_2'\right) \inf_{u_0 \in B_\rho^0} \mathbb{P}\left(\tau^{\epsilon,\delta,u_0} \leq T_1'\right)\\ &\geq \exp\left(-\frac{I_{u_0}^{\delta,T_1'} + \frac{\kappa}{3}}{\epsilon}\right),\end{aligned}$$

where we have used the fact that ϵ is small enough such that $\frac{1}{2} \geq e^{-\frac{\kappa}{6\epsilon}}$. Then for any $k \geq 1$, using the property of conditional probability, it holds that

$$\begin{aligned}&\mathbb{P}\left(\tau^{\epsilon,\delta,u_0} > (k+1)(T_1' + T_2')\right)\\ &= \left[1 - \mathbb{P}\left(\tau^{\epsilon,\delta,u_0} \leq (k+1)(T_1' + T_2') \middle| \tau^{\epsilon,\delta,u_0} > k(T_1' + T_2')\right)\right]\\ &\quad \times \mathbb{P}\left(\tau^{\epsilon,\delta,u_0} > k(T_1' + T_2')\right)\\ &\leq \left(1 - \inf_{u_0 \in D} \mathbb{P}\left(\tau^{\epsilon,\delta,u_0} \leq T_1' + T_2'\right)\right) \mathbb{P}\left(\tau^{\epsilon,\delta,u_0} > k(T_1' + T_2')\right)\\ &\leq \left(1 - \inf_{u_0 \in D} \mathbb{P}\left(\tau^{\epsilon,\delta,u_0} \leq T_1' + T_2'\right)\right)^k.\end{aligned}$$

Notice that $I_{u_0}^{\delta,T_1'}(L_{u_0}^{\delta}(g)) = I_0^{\delta,T_1'}(L_0^{\delta}(g)) = \frac{1}{2}\|h\|^2_{L^2([0,T_1';\mathbb{R}])}$. Thus, we also have

$$\begin{aligned}\sup_{u_0\in D}\mathbb{E}\left[\tau^{\epsilon,\delta,u_0}\right] &= \sup_{u_0\in D}\int_0^{+\infty}\mathbb{P}\left(\tau^{\epsilon,\delta,u_0}>t\right)\mathrm{d}t\\ &\le (T_1'+T_2')\sum_{k=0}^{\infty}\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,\delta,u_0}>k(T_1'+T_2')\right)\\ &\le \frac{(T_1'+T_2')}{1-\inf\limits_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,\delta,u_0}\le T_1'+T_2'\right)}\\ &\le (T_1'+T_2')\exp\left(\frac{\overline{M^{\delta}}+\frac{\kappa}{2}}{\epsilon}\right).\end{aligned}$$

One can take ϵ small enough such that

$$\sup_{u_0\in D}\mathbb{E}\left[\tau^{\epsilon,\delta,u_0}\right]\le\exp\left(\frac{\overline{M^{\delta}}+\frac{2\kappa}{3}}{\epsilon}\right), \tag{A.4}$$

which implies (4.6). The Chebyshev inequality yields that

$$\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,\delta,u_0}\ge e^{\frac{\overline{M^{\delta}+\kappa}}{\epsilon}}\right)\le e^{-\frac{\overline{M^{\delta}+\kappa}}{\epsilon}}\sup_{u_0\in D}\mathbb{E}\left[\tau^{\epsilon,\delta,u_0}\right]\le e^{-\frac{\kappa}{3\epsilon}}.$$

As a consequence, (4.5) follows.

Below we are in a position to deal with lower bound estimate of $\tau^{\epsilon,\delta,u_0}$. By Lemma 3, we can choose $\rho>0$ small enough such that $\underline{M^{\delta}}-\frac{\kappa}{4}\le M_{\rho}^{\delta}$ and $\mathcal{B}_{2\rho}^0\subset D$, where $0<\frac{\kappa}{4}<\inf_{\delta>0}\underline{M^{\delta}}$. Similar to [29], one can define two sequences of stopping times,

$$\begin{aligned}\tau_k &= \inf\{t\ge\theta_k \mid u^{\epsilon,\delta,u_0}(t)\in\mathcal{B}_{\rho}^0\cup D^c\},\\ \theta_{k+1} &= \inf\{t>\tau_k \mid u^{\epsilon,\delta,u_0}(t)\in\mathcal{S}_{2\rho}^0\},\end{aligned}$$

where $\theta_0=0, k\in\mathbb{N}$, and $\theta_{k+1}=+\infty$ if $u^{\epsilon,\delta,u_0}(\tau_k)\in\partial D$.

Choosing T_3 in Lemma 6 that satisfies $L=\underline{M^{\delta}}-\frac{3\kappa}{4}$, there exists ϵ small enough such that for all $k\ge1$ and $u_0\in D$,

$$\mathbb{P}\left(\theta_k-\tau_{k-1}\le T_3\right)\le e^{-\frac{\underline{M^{\delta}}-\frac{3\kappa}{4}}{\epsilon}}.$$

Note that the escapes before mT_3 with $m\in\mathbb{N}^+$ occur in three cases, i.e., Case 1, the escape occurs without passing $\mathcal{B}_{\rho}^0$; Case 2, the trajectory of $u^{\epsilon,u_0,\delta}$ crosses $S_{2\rho}^0$ with k times and then escapes at τ_k; Case 3: the escape occurs after τ_m which implies that there exists at least a length of one interval $[\tau_{k-1},\theta_k]$ which is smaller than T_3.

As a consequence, for $u_0\in D$,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,\delta,u_0}\le mT_3\right) &\le \mathbb{P}\left(\tau^{\epsilon,\delta,u_0}=\tau_0\right)+\sum_{k=1}^{m}\mathbb{P}\left(\tau^{\epsilon,\delta,u_0}=\tau^k\right)+\sum_{k=1}^{m}\mathbb{P}\left(\theta_k-\tau_{k-1}<T_3\right)\\ &\le \mathbb{P}\left(\tau^{\epsilon,\delta,u_0}=\tau_0\right)+\sum_{k=1}^{m}\mathbb{P}\left(\tau^{\epsilon,\delta,u_0}=\tau^k\right)+me^{-\frac{\underline{M^{\delta}}-\frac{3\kappa}{4}}{\epsilon}}.\end{aligned} \tag{A.5}$$

Below we will estimate $\mathbb{P}(\tau^{\epsilon,\delta,u_0} = \tau^k), k \geq 1$, by using the fact that

$$\mathbb{P}\left(\tau^{\epsilon,\delta,u_0} = \tau_k\right) \leq \mathbb{P}\left(\tau^{\epsilon,\delta,u_0} \leq T_4, \tau^{\epsilon,\delta,u_0} = \tau_k\right) + \sup_{y \in S^0_{2\rho}} \mathbb{P}\left(\sigma^{\epsilon,\delta,y}_\rho > T_4\right)$$

for all $T_4 > 0$ with $y = u^{\epsilon,\delta,u_0}(\theta_{k-1}) \in \mathcal{S}^0_{2\rho}$. On the one hand, choosing T_4 as the time in Lemma 5 and $L = \underline{M}^\delta - \frac{3\kappa}{4}$, we obtain that

$$\mathbb{P}\left(\sigma^{\epsilon,\delta,y}_\rho > T_4\right) \leq e^{-\frac{\underline{M}^\delta - \frac{3\kappa}{4}}{\epsilon}}.$$

On the other hand, using the LDP in Theorem 2, there exists ϵ small enough such that for $u_1 \in B^0_\rho$,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,\delta,u_1} \leq T_4\right) &\leq \mathbb{P}\Big(d_{\mathcal{C}([0,T_4];\mathcal{X}^{1,1})}\Big(u^{\epsilon,\delta,u_1}, K^{u_1}_{T_4}\Big(M^\delta_\rho - \frac{\kappa}{4}\Big)\Big) \geq \rho\Big)\\ &\leq e^{-\frac{M^\delta_\rho - \frac{\kappa}{2}}{\epsilon}} \leq e^{-\frac{\underline{M}^\delta - \frac{3\kappa}{4}}{\epsilon}}.\end{aligned}$$

As a consequence, we have that

$$\mathbb{P}\left(\tau^{\epsilon,u_0,\delta} \leq T_4, \tau^{\epsilon,u_0,\delta} = \tau_k\right) \leq \mathbb{P}\left(\tau^{\epsilon,\delta,u^{\epsilon,\delta,u_0}(\tau^{k-1})} \leq T_4, \tau_k - \tau_{k-1} \leq T_4\right) \leq e^{-\frac{\underline{M}^\delta - \frac{3\kappa}{4}}{\epsilon}}.$$

Combining the above estimates and (A.5), it holds that for ϵ small enough,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,\delta,u_0} \leq mT_3\right) &\leq \mathbb{P}\left(\tau^{\epsilon,\delta,u_0} = \tau_0\right) + 3me^{-\frac{\underline{M}^\delta - \frac{3\kappa}{4}}{\epsilon}}\\ &\leq \mathbb{P}\left(u^{\epsilon,\delta,u_0}(\sigma^{\epsilon,\delta,u_0}_\rho) \in \partial D\right) + 3me^{-\frac{\underline{M}^\delta - \frac{3\kappa}{4}}{\epsilon}}.\end{aligned}$$

Using Lemma 5, taking $m = \lfloor \frac{1}{T_3} \exp(\frac{\underline{M}^\delta - \kappa}{\epsilon}) \rfloor$, we obtain (4.3). Applying the Chebyshev's inequality, the desired lower bound on $\mathbb{E}[\tau^{\epsilon,\delta,u_0}]$ follows. ■

Acknowledgments. The authors would like to thank the anonymous referees who provided useful and detailed comments to improve this paper.

REFERENCES

[1] G. P. Agrawal, *Applications of Nonlinear Fiber Optics*, Academic Press, San Diego, 2001.

[2] G. P. Agrawal, *Nonlinear Fiber Optics*, 3rd ed., Academic Press, San Diego, 2001.

[3] V. Barbu, M. Röckner, and D. Zhang, *Stochastic nonlinear Schrödinger equations with linear multiplicative noise: Rescaling approach*, J. Nonlinear Sci., 24 (2014), pp. 383–409.

[4] R. Belaouar, A. de Bouard, and A. Debussche, *Numerical analysis of the nonlinear Schrödinger equation with white noise dispersion*, Stoch. Partial Differ. Equ. Anal. Comput., 3 (2015), pp. 103–132.

[5] I. Bialynicki-Birula and J. Mycielski, *Nonlinear wave mechanics*, Ann. Phys., 100 (1976), pp. 62–93, https://doi.org/10.1016/0003-4916(76)90057-9.

[6] C.-E. Bréhier and D. Cohen, *Strong rates of convergence of a splitting scheme for Schrödinger equations with nonlocal interaction cubic nonlinearity and white noise dispersion*, SIAM/ASA J. Uncertain. Quantif., 10 (2022), pp. 453–480, https://doi.org/10.1137/20M1378168.

[7] C.-E. Bréhier and D. Cohen, *Analysis of a splitting scheme for a class of nonlinear stochastic Schrödinger equations*, Appl. Numer. Math., 186 (2023), pp. 57–83, https://doi.org/10.1016/j.apnum.2023.01.002.
[8] Z. Brzeźniak, F. Hornung, and L. Weis, *Uniqueness of martingale solutions for the stochastic nonlinear Schrödinger equation on 3d compact manifolds*, Stoch. Partial Differ. Equ. Anal. Comput., 10 (2022), pp. 828–857, https://doi.org/10.1007/s40072-022-00238-w.
[9] R. Carles and I. Gallagher, *Universal dynamics for the defocusing logarithmic Schrödinger equation*, Duke Math. J., 167 (2018), pp. 1761–1801, https://doi.org/10.1215/00127094-2018-0006.
[10] T. Cazenave and A. Haraux, *Équation de Schrödinger avec non-linéarité logarithmique*, C. R. Acad. Sci. Paris Sér. A-B, 288 (1979), pp. A253–A256.
[11] C. Chen and J. Hong, *Symplectic Runge–Kutta semidiscretization for stochastic Schrödinger equation*, SIAM J. Numer. Anal., 54 (2016), pp. 2569–2593, https://doi.org/10.1137/151005208.
[12] X. Chen, *Random Walk Intersections*, in Large Deviations and Related Topics, Math. Surveys Monogr. 157, American Mathematical Society, Providence, RI, 2010, https://doi.org/10.1090/surv/157.
[13] D. Cohen and G. Dujardin, *Exponential integrators for nonlinear Schrödinger equations with white noise dispersion*, Stoch. Partial Differ. Equ. Anal. Comput., 5 (2017), pp. 592–613, https://doi.org/10.1007/s40072-017-0098-1.
[14] J. Cui and J. Hong, *Analysis of a splitting scheme for damped stochastic nonlinear Schrödinger equation with multiplicative noise*, SIAM J. Numer. Anal., 56 (2018), pp. 2045–2069, https://doi.org/10.1137/17M1154904.
[15] J. Cui, J. Hong, and Z. Liu, *Strong convergence rate of finite difference approximations for stochastic cubic Schrödinger equations*, J. Differential Equations, 263 (2017), pp. 3687–3713.
[16] J. Cui, J. Hong, Z. Liu, and W. Zhou, *Stochastic symplectic and multi-symplectic methods for nonlinear Schrödinger equation with white noise dispersion*, J. Comput. Phys., 342 (2017), pp. 267–285, https://doi.org/10.1016/j.jcp.2017.04.029.
[17] J. Cui, J. Hong, Z. Liu, and W. Zhou, *Strong convergence rate of splitting schemes for stochastic nonlinear Schrödinger equations*, J. Differential Equations, 266 (2019), pp. 5625–5663.
[18] J. Cui, J. Hong, and L. Sun, *On global existence and blow-up for damped stochastic nonlinear Schrödinger equation*, Discrete Contin. Dyn. Syst. Ser. B, 24 (2019), pp. 6837–6854.
[19] J. Cui, S. Liu, and H. Zhou, *Stochastic Wasserstein Hamiltonian flows*, J. Dynam. Differential Equations, https://doi.org/10.1007/s10884-023-10264-4.
[20] J. Cui and L. Sun, *Stochastic logarithmic Schrödinger equations: Energy regularized approach*, SIAM J. Math. Anal., 55 (2023), pp. 3044–3080, https://doi.org/10.1137/21M1442425.
[21] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, 2nd ed., Encyclopedia Math. Appl. 152, Cambridge University Press, Cambridge, 2014.
[22] A. de Bouard and A. Debussche, *A stochastic nonlinear Schrödinger equation with multiplicative noise*, Comm. Math. Phys., 205 (1999), pp. 161–181.
[23] A. de Bouard and A. Debussche, *On the effect of a noise on the solutions of the focusing supercritical nonlinear Schrödinger equation*, Probab. Theory Related Fields, 123 (2002), pp. 76–96.
[24] A. de Bouard and A. Debussche, *Blow-up for the stochastic nonlinear Schrödinger equation with multiplicative noise*, Ann. Probab., 33 (2005), pp. 1078–1110.
[25] A. de Bouard and A. Debussche, *The nonlinear Schrödinger equation with white noise dispersion*, J. Funct. Anal., 259 (2010), pp. 1300–1321, https://doi.org/10.1016/j.jfa.2010.04.002.
[26] A. Debussche and L. Di Menza, *Numerical simulation of focusing stochastic nonlinear Schrödinger equations*, Phys. D, 162 (2002), pp. 131–154.
[27] A. Debussche and E. Gautier, *Small noise asymptotic of the timing jitter in soliton transmission*, Ann. Appl. Probab., 18 (2008), pp. 178–208, https://doi.org/10.1214/07-AAP449.
[28] A. Debussche and Y. Tsutsumi, *1D quintic nonlinear Schrödinger equation with white noise dispersion*, J. Math. Pures Appl. (9), 96 (2011), pp. 363–376, https://doi.org/10.1016/j.matpur.2011.02.002.
[29] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, 2nd ed., Appl. Math. (N. Y.) 38, Springer-Verlag, New York, 1998.
[30] E. Desurvire, *Erbium-Doped Fiber Amplifiers: Device and System Developments*, John Wiley & Sons, Wiley, 2002, https://books.google.com.hk/books?id=WfxSAAAAMAAJ.

[31] R. Duboscq and A. Réveillac, *On a stochastic Hardy-Littlewood-Sobolev inequality with application to Strichartz estimates for a noisy dispersion*, Ann. H. Lebesgue, 5 (2022), pp. 263–274, https://doi.org/10.5802/ahl.122.

[32] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Grundlehren Math. Wiss. 260, Springer-Verlag, New York, 1984, translated from the Russian by Joseph Szücs, https://doi.org/10.1007/978-1-4684-0176-9.

[33] E. Gautier, *Large deviations and support results for nonlinear Schrödinger equations with additive noise and applications*, ESAIM Probab. Stat., 9 (2005), pp. 74–97, https://doi.org/10.1051/ps:2005005.

[34] E. Gautier, *Uniform large deviations for the nonlinear Schrödinger equation with multiplicative noise*, Stochastic Process. Appl., 115 (2005), pp. 1904–1927, https://doi.org/10.1016/j.spa.2005.06.011.

[35] E. Gautier, *Exit from a basin of attraction for stochastic weakly damped nonlinear Schrödinger equations*, Ann. Probab., 36 (2008), pp. 896–930, https://doi.org/10.1214/07-AOP344.

[36] M. Hofmanová, M. Knöller, and K. Schratz, *Randomized exponential integrators for modulated nonlinear Schrödinger equations*, IMA J. Numer. Anal., 40 (2020), pp. 2143–2162, https://doi.org/10.1093/imanum/drz050.

[37] I. A. Khovanov, D. G. Luchinsky, R. Mannella, and P. V. E. Mc Clintock, *Fluctuational Escape and Related Phenomena in Nonlinear Optical Systems*, 2nd ed., Modern Nonlinear Optics, Part 3 119, Advances in Chemical Physics, Wiley, 2001.

[38] R. Marty, *Local error of a splitting scheme for a nonlinear Schrödinger-type equation with random dispersion*, Commun. Math. Sci., 19 (2021), pp. 1051–1069, https://doi.org/10.4310/CMS.2021.v19.n4.a8.

[39] T. Oh and M. Okamoto, *On the stochastic nonlinear Schrödinger equations at critical regularities*, Stoch. Partial Differ. Equ. Anal. Comput., 8 (2020), pp. 869–894, https://doi.org/10.1007/s40072-019-00163-5.

[40] C. Sulem and P. Sulem, *The Nonlinear Schrödinger Equation*, Self-Focusing and Wave Collapse, Appl. Math. Sci. 139, Springer-Verlag, New York, 1999.