UNIFORM WEAK SHARP MINIMA FOR MULTIOBJECTIVE OPTIMIZATION PROBLEMS *

CHUNHAI HU†, XIAOQI YANG‡ AND XI YIN ZHENG§

Abstract. Noting that there exist abnormal phenomenons in the existing weak sharp minima for multiobjective optimization problems (MOP), this paper introduces and studies uniform weak sharp minima for (MOP). We first provide several characterizations for a piecewise linear multiobjective optimization problem with respect to the natural vector partial order induced by $\mathbb{R}^n_+$ to have uniform weak sharp minima. Under a mild assumption, a piecewise linear multiobjective optimization problem with respect to an arbitrary vector partial order is proved to be equivalent to a piecewise linear multiobjective optimization problem with respect to the natural vector partial order. Based on such an interesting equivalence, this paper mainly establishes the uniform bounded and global weak sharp minima for a general piecewise linear multiobjective optimization problem with respect to any vector partial order.

Key words. Multiobjective optimization, uniform weak sharp minima, polyhedron.

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1. Introduction and main results.

The weak sharp minima for scalar optimization problems have been recognized to be useful in convergence analysis of optimal algorithms and are closely related to error bounds (cf. [4, 12, 21, 27, 29, 36, 39] and the references therein). Recall that a proper lower semicontinuous real-valued function ψ on a Hilbert space X has weak sharp minima at $\bar{x} \in S(\psi) := \{x \in X : \psi(x) = \inf_{u\in X} \psi(u)\}$ if there exist $\kappa, \delta \in (0, +\infty)$ such that

$$\kappa d(x, S(\psi)) \leq \psi(x) - \inf_{u\in X} \quad (u) \quad \forall x \in B(\bar{x}, \delta), \tag{1.1}$$

where $B(\bar{x}, \delta)$ denotes the open ball with center $\bar{x}$ and radius δ. The weak sharp minima property in the sense of (1.1) means that every minimizing sequence $\{x_k\}$ of ψ (i.e. $\psi(x_k) \to \inf_{x\in X} \psi(x)$) can arbitrarily approach $S(\psi)$ as $k \to \infty$ and the infinitesimal order of $d(x_k, S(\psi))$ is greater than or equal to the one of $\psi(x_k) - \inf_{x\in X} \psi(x)$. This may be one of the reasons why the weak sharp minima for scalar optimization problems are useful in convergence analysis of algorithms.

In contrast that the scalar optimization has a unique optimal value $\inf_{u\in X} \psi(u)$, a multiobjective optimization problem usually has infinitely many optimal values. This leads to more complex issues for studying weak sharp minima of a multiobjective optimization problem. Let $f_1, \cdots, f_n, \varphi_1, \cdots, \varphi_\nu$ be lower semicontinuous real-valued

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†Department of Mathematics, Yunnan University, Kunming 650091, P. R. China (562357698@qq.com)

‡Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong (xiao.qi.yang@polyu.edu.hk)

§Department of Mathematics, Yunnan University, Kunming 650091, P. R. China (xyzheng@ynu.edu.cn, Corresponding author)

functions on a Hilbert space X, and consider the following multiobjective optimization problem with respect to the natural vector partial order induced by $\mathbb{R}^n_+$

$$\text{(MOP)}\quad \mathbb{R}^n_+\text{−min} f(x) := (f_1(x), \cdots, f_n(x)) \text{ subject to } \varphi_1(x) \le 0, \cdots, \varphi_k(x) \le 0.$$

Recall that a feasible point ω in $\Omega := \{x \in X : \varphi_j(x) \le 0, \ j = 1 \cdots, k\}$ is a weak Pareto solution (resp. a Pareto solution) of (MOP) if there exists no $\omega' \in \Omega$ such that

$$f_i(\omega') < f_i(\omega) \ \big(\text{resp. } f(\omega') \ne f(\omega) \text{ and } f_i(\omega') \le f_i(\omega)\big), \ i = 1, \cdots, n.$$

The sets of all weak Pareto solutions and all Pareto solutions of (MOP) are denoted by S_w and S, respectively. It is well known that many problems in practice can be cast into (MOP) (cf. [10, 14, 26, 30, 34] and the references therein).

In the case that all f_i and φ_j in (MOP) are linear, Deng and Yang [6] proved that the corresponding linear multiobjective optimization problem has the following weak sharp minima: *there exists* $\kappa > 0$ *such that*

$$\kappa d(x, S) \le d(f(x), V) \quad \forall x \in \Omega, \tag{1.2}$$

where $V := f(S)$ *is the set of all Pareto optimal values of* (MOP). Under the assumption that each f_i is piecewise linear and the feasible set Ω is a convex polyhedron, some authors further established the following result on weak sharp minima for (MOP) (cf. [37, 40, 41] and the references therein): *For any* $r > 0$ *sufficiently large there exists* $\kappa \in (0, +\infty)$ *such that*

$$\kappa d(x, S) \le d(f(x) + \mathbb{R}^n_+, V) + d(x, \Omega) \quad \forall x \in rB_X. \tag{1.3}$$

It is well known that the Pareto optimal value set V of mutiobjective optimization problem (MOP) may contain excrescent and abnormal Pareto optimal values (cf.[3, 15, 16, 43] and the references therein). However the abnormal Pareto optimal values play the same role as the normal Pareto optimal values in inequality (1.3), which may lead to the weak sharp minima for (MOP) in the sense of (1.3) being useless (because of the existence of abnormal Pareto optimal values). This is one of motivations for us to introduce the new notion of uniform weak sharp minima for (MOP).

In the scalar optimization case (i.e. $n = 1$), $V = \{\inf_{x\in\Omega} f(x)\}$ is a singleton. However, even in the case of $n = 2$, the Pareto optimal value set V of (MOP) contains uncountably infinitely many elements, and sometimes one may be interested in some specific Pareto optimal value y and the set of Pareto solutions corresponding to y. For convenience, for each $y \in V$, let $S(y)$ denote the set of all Pareto solutions corresponding to the Pareto optimal value y, that is,

$$S(y) := \{\omega \in \Omega : \ f(\omega) \le_{\mathbb{R}^n_+} y\} = \Omega \cap f^{-1}(y - \mathbb{R}^n_+) = \Omega \cap f^{-1}(y).$$

Clearly, $S = \bigcup\limits_{y\in V} S(y)$ and hence

$$d(x, S) = \inf_{y\in V} d(x, S(y)) \text{ and } d(f(x) + \mathbb{R}^n_+, V) = \inf_{y\in V} d(y, f(x) + \mathbb{R}^n_+).$$

It is worth mentioning that the weak sharp minima in sense of either (1.2) or (1.3) can yield the following ill-posed phenomenon: *there exist some specific* $\bar{y}$ *in the Pareto* *optimal value* V *and a bounded minimizing sequence* $\{x_k\}$ *of* (MOP) *with respect to* $\bar{y}$ *(i.e.* $\lim_{k\to\infty} d(\bar{y}, f(x_k)+\mathbb{R}^n_+)=0$) *such that* $\{x_k\}$ *cannot approach the set* $S(\bar{y})$ *of all Pareto solutions corresponding to* $\bar{y}$*.* In fact, take $X=l^2$, let $f(x)=\|x\|(-1,1)$ for all $x\in\mathbb{R}e_1$, $f(x)=\frac{\|x\|(-1,1)}{k+1}$ for $k\in\mathbb{N}$ and $x\in\mathbb{R}e_{k+1}$ and $f(x)=+\infty_{\mathbb{R}^2}$ otherwise, where e_k is an element in l^2 whose the k-th coordinate is 1 and all the other coordinates are 0, and $\varphi_j(x)=0$ for all j and $x\in l^2$. Then, $\Omega=X$, $V=f(X)=\mathbb{R}_+(-1,1)$, $S=\mathrm{dom}(f)$ and hence $d(x,S)=0$ for all $x\in\mathrm{dom}(f)$. It follows that

$$\kappa d(x,S)\le d(f(x)+\mathbb{R}^n_+,V)+d(x,\Omega)\quad \forall(\kappa,x)\in(0,\ +\infty)\times X \tag{1.4}$$

and hence the corresponding multiobjetive optimization problem (MOP) has the weak sharp minima in the sense of (1.2) and (1.3). On the other hand, for $\bar{y}=(0,0)\in V$, it is clear that $S(\bar{y})=\{0\}$,

$$d(\bar{y},f(e_{k+1})+\mathbb{R}^2_+)=\frac{1}{k+1}\quad\text{and}\quad d(e_{k+1},S(\bar{y}))=1\ \ \forall k\in\mathbb{N}. \tag{1.5}$$

The first equality in (1.5) implies that $\{e_k\}$ is a minimizing sequence of (MOP) with respect to the Pareto optimal value $\bar{y}=(0,0)$, but the second equality in (1.5) shows that the minimizing sequence $\{e_k\}$ with respect to $\bar{y}$ can not approach the set $S(\bar{y})$ of all Pareto solutions corresponding to $\bar{y}$ as $k\to\infty$ (perhaps the weak sharp minima in the sense of (1.2) or (1.3) leads to the even worse situation that some minimizing sequence $\{e_k\}$ with respect to a "good" Pareto optimal value y_g arbitrarily approaches the Pareto solution set $S(y_a)$ corresponding to some abnormal Pareto optimal value y_a as $k\to\infty$).

In the convergence analysis of optimal algorithms for multiobjective optimization problems, one sometimes constructs an iterative sequence converging to a Pareto solution $\bar{x}$ corresponding to *an unknown Pareto optimal value* $\bar{y}$ *in advance* (see [2, 5, 13, 25, 35]). Moreover, several authors considered algorithms to find a set of many representational Pareto solutions (or Pareto optimal values) (cf. [7, 9, 18, 28, 33] and the references therein). This is another motivation for us to introduce and study the following notions of uniform weak sharp minima.

Definition 1.1. *Multiobjective optimization problem (MOP) is said to have uniform global weak sharp minima with respect to all Pareto optimal values if there exists* $\kappa\in(0,\ +\infty)$ *such that for all* $y=(y_1,\cdots,y_n)\in V$*,*

$$\kappa d(x,S(y))\le\sum_{i=1}^n[f_i(x)-y_i]_+ +\sum_{j=1}^k[\varphi_j(x)]_+\quad\forall x\in X, \tag{1.6}$$

while (MOP) is said to have uniform bounded weak sharp minima with respect to all Pareto optimal values if for any $r>0$ *sufficiently large there exists* $\kappa\in(0,\ +\infty)$ *such that for all* $y=(y_1,\cdots,y_n)\in V\cap rB_{\mathbb{R}^n}$*,*

$$\kappa d(x,S(y))\le\sum_{i=1}^n[f_i(x)-y_i]_+ +\sum_{j=1}^k[\varphi_j(x)]_+\quad\forall x\in rB_X.$$

Let V_w denote the sets of all weak Pareto optimal values of multiobjective optimization problem (MOP), that is, $V_w=f(S_w)$. For any $y\in V_w$, one can define the weak Pareto solution set $S_w(y)$ of (MOP) corresponding to the weak Pareto optimal value y as $S_w(y) = \Omega \cap f^{-1}(y - \mathbb{R}^n_+)$. Similarly, one can define the notions of uniform weak sharp minima with respect to all weak Pareto optimal values.

This paper mainly considers uniform weak sharp minima for piecewise linear multiobjective optimization problems. In particular, in the remainder of this paper, we always assume that all f_i and φ_j in (MOP) are piecewise linear, and we use (PLMOP) to denote the corresponding piecewise linear multiobjective optimization problem. The family of all piecewise linear functions is much larger than that of all linear functions and there exists a wide class of functions that can be approximated by piecewise linear functions. Moreover, one may use piecewise linear functions generated by finitely many test data to establish mathematical models for some practical problems. Therefore, from the viewpoint of theoretical interest as well as for applications, it is important to study piecewise linear problems. In fact, piecewise linear multiobjective optimization problems have been studied by many authors (cf. [11, 22, 23, 32, 37, 38, 40, 42]).

In Section 2, noting that piecewise linear functions are closely related to nonconvex polyhedra, we first establish some elaborate partitions of a nonconvex polyhedron, which play a key role in the proofs of our main results. In Section 3, we prove that if all f_i and φ_j are convex functions then (PLMOP) has uniform global weak sharp minima with respect to both all Pareto optimal values and all weak Pareto optimal values. However, dropping the convexity assumption, (PLMOP) does not even necessarily have uniform bounded weak sharp minima with respect to either Pareto optimal values or weak Pareto optimal values. Without any convexity assumption on f_i and φ_j, we prove that piecewise linear multiobjective optimization problem (PLMOP) has uniform bounded weak sharp minima with respect to all weak Pareto optimal values if and only if $y \mapsto S_w(y)$ is inner semicontinuous on V_w, that is, $S_w(y) \subset \left\{ x \in X : \lim\limits_{y' \xrightarrow{V_w} y} d(x, S_w(y')) = 0 \right\}$ for all $y \in V_w$. Moreover, we prove that (PLMOP) always has near uniform bounded weak sharp minima with respect to weak Pareto optimal values, that is, for any $\varepsilon > 0$ sufficiently small there exists a ε-small set O_ε (in the sense of both measure and topology) in V_w such that (PLMOP) has uniform bounded weak sharp minima with respect to all weak Pareto optimal values in $V_w \setminus O_\varepsilon$.

The natural vector partial order induced by $\mathbb{R}^n_+$ seems sometimes restrictive as one also needs to choose some suitable kinds of partial orders for objective vectors in some practical applications. This motivates us to consider the following piecewise linear multiobjective optimization problem with a vector partial order induced by a general closed convex cone C in $\mathbb{R}^n$:

$$(\text{PLMOP})_{\text{C}} \quad C{-}\min f(x) = (f_1(x), \cdots, f_n(x)) \ \text{ subject to } \varphi_1(x) \le 0, \cdots, \varphi_k(x) \le 0.$$

In the case that the ordering cone C is non-polyhedral, $(\text{PLMOP})_C$ is not a genuine piecewise linear problem even when all f_i and φ_j are linear. In Section 4, under the weak C-convexity assumption of f on Ω which is satisfied if $f(\Omega)$ is complete with respect to $\le_C$, we prove that $(\text{PLMOP})_C$ with C being a non-polyhedral cone is equivalent to a piecewise linear multiobjective optimization problem with respect to the natural vector partial order. Based on this equivalence and the main results obtained in Section 3, we establish the uniform bounded and global weak sharp minima for a general piecewise linear problem $(\text{PLMOP})_C$.

2. Partitions of a nonconvex polyhedron. For convenience of the readers, we first recall some notions and results in convex analysis, which will be used in our later analysis (see [31, 39] for more details). Throughout the remainder of this paper, let X be a normed space with the dual space X^*. For a set A in X, the relative interior $\text{ri}(A)$ of A is defined by

$$\text{ri}(A) := \{x \in A : \; B(x,r) \cap \text{aff}(A) \subset A \text{ for some } r > 0\},$$

where $\text{aff}(A)$ is the affine subspace generated by A. Recall that a nonempty set P in X is a convex polyhedron if there exist $x_i^* \in X^*$ and $t_i \in \mathbb{R}$ $(i = 1, \cdots, m)$ such that

$$P := \{x \in X : \; \langle x_i^*, x\rangle \leq t_i, \; i = 1, \cdots, m\}.$$

It is known that the relative interior of P is always nonempty,

$$\text{aff}(P) = \{x \in X : \; \langle x_i^*, x\rangle = t_i, \; i \in I_P\}$$

and

$$\text{ri}(P) = \{x \in X : \; \langle x_i^*, x\rangle < t_i, \; i \in \overline{1m} \setminus I_P\} \cap \{x \in X : \; \langle x_i^*, x\rangle = t_i, \; i \in I_P\},$$

where $\overline{1m} := \{1, \cdots, m\}$ and $I_P := \{i \in \overline{1m} : \; P \subset \{x \in X : \; \langle x_i^*, x\rangle = t_i\}\}$ (cf. [42, Corollary 2.1]). From the above definition, it is easy to verify the following lemma.

LEMMA 2.1. *Let A and B be convex polyhedra in a normned space X. Suppose that $\text{aff}(A) = \text{aff}(B)$ and $\text{ri}(A) \cap B = \emptyset$. Then $A \cap \text{ri}(B) = \emptyset$.*

Recently, Zheng and Yang [42] proved that if $P_1, \cdots, P_m$ are convex polyhedra in a normed space X such that the interior $\text{int}(P_i)$ of P_i is nonempty $(i = 1, \cdots, m)$ then there exist convex polyhedra $Q_1, \cdots, Q_n$ in X with $\text{int}(Q_j) \neq \emptyset$ $(j = 1, \cdots, n)$ such that $\bigcup_{i=1}^m P_i = \bigcup_{j=1}^n Q_j$ and $\text{int}(Q_j) \cap Q_{j'} = \emptyset$ for all $j, j' \in \overline{1n}$ with $j \neq j'$. In the proof of [42, Proposition 2.5], the authors obtained the following result.

LEMMA 2.2. *Let P, $Q_1, \cdots, Q_k$ be convex polyhedra in a normed space X such that $\text{int}(P) \neq \emptyset$ and $P \setminus \bigcup\limits_{i=1}^{k} Q_i \neq \emptyset$. Then there exist convex polyhedra H_j with $\text{int}(H_j) \neq \emptyset$ $(j = 1, \cdots, n)$ such that $\bigcup_{j=1}^n \text{int}(H_j) \subset \text{int}(P) \setminus \bigcup_{i=1}^k Q_i \subset \bigcup\limits_{j=1}^{n} H_j$ and $\text{int}(H_j) \cap \bigcup\limits_{j' \in \overline{1n} \setminus \{j\}} H_{j'} = \emptyset$ for all $j \in \overline{1n}$.*

Dropping the assumption of $\text{int}(P) \neq \emptyset$, we have the following more general result.

LEMMA 2.3. *Let P, $Q_1, \cdots, Q_k$ be convex polyhedra in a normed space X such that $P \setminus \bigcup_{i=1}^k Q_i \neq \emptyset$. Then there exist convex polyhedra $H_1, \cdots, H_n$ in X such that*

$$\text{aff}(P) = \text{aff}(H_j), \;\; \text{ri}(H_j) \cap H_{j'} = \emptyset \quad \forall j, j' \in \overline{1n} \text{ with } j \neq j' \tag{2.1}$$

and

$$\bigcup_{j=1}^{n} \text{ri}(H_j) \subset \text{ri}(P) \setminus \bigcup_{i=1}^{k} Q_i \subset \bigcup_{j=1}^{n} H_j. \tag{2.2}$$

Proof. Take a point $\tilde{x}$ in P, and let

$$\tilde{X} := \text{aff}(P) - \tilde{x},\ \tilde{P} := P - \tilde{x} \ \text{ and } \ \tilde{Q}_i := (Q_i - \tilde{x}) \cap \tilde{X} \ \ \forall i \in \overline{1k}.$$

Then $\tilde{X}$ is a linear subspace of X,

$$\text{int}_{\tilde{X}}(\tilde{P}) := \left\{ x \in \tilde{P} :\ B(x,r) \cap \tilde{X} \subset \tilde{P} \text{ for some } r > 0 \right\} = \text{ri}(P) - \tilde{x} \neq \emptyset$$

and $\tilde{P} \setminus \bigcup_{i=1}^{k} \tilde{Q}_i = -\tilde{x} + P \setminus \bigcup_{i=1}^{k} Q_i \neq \emptyset$. By Lemma 2.2, there exist convex polyhedra $\tilde{H}_1, \cdots, \tilde{H}_n$ in $\tilde{X}$ such that

$$\text{int}_{\tilde{X}}(\tilde{H}_j) \neq \emptyset,\ \ \text{int}_{\tilde{X}}(\tilde{H}_j) \cap \tilde{H}_{j'} = \emptyset \quad \forall j, j' \in \overline{1n} \text{ with } j \neq j'$$

and $\bigcup_{j=1}^{n} \text{int}_{\tilde{X}}(\tilde{H}_j) \subset \text{int}_{\tilde{X}}(\tilde{P}) \setminus \bigcup_{i=1}^{k} \tilde{Q}_i \subset \bigcup_{j=1}^{n} \tilde{H}_j$. Since $\tilde{X} = \text{aff}(P) - \tilde{x}$ is a convex polyhedron, each $\tilde{H}_j$ is a convex polyhedron in X. Let $H_j := \tilde{H}_j + \tilde{x}$ ($j \in \overline{1n}$). Then, H_j is a convex polyhedron, $\text{ri}(H_j) = \text{int}_{\tilde{X}}(\tilde{H}_j) + \tilde{x} \neq \emptyset$, and (2.1) and (2.2) hold. The proof is complete. □

To prove the main result in this section, we need the following lemma.

LEMMA 2.4. *Let P, $\tilde{Q}_1, \cdots, \tilde{Q}_k$ be convex polyhedra in a normed space such that*

$$\tilde{Q}_j \cap \text{ri}(\tilde{Q}_{j'}) = \emptyset \quad \forall j \in \overline{1k} \text{ and } \forall j' \in \overline{1k} \setminus \{j\}. \tag{2.3}$$

Then $P \cup \bigcup_{j=1}^{k} \tilde{Q}_j = \text{cl}\big(\text{ri}(P) \setminus \bigcup_{j=1}^{k} \tilde{Q}_j\big) \cup \bigcup_{j \in J_1} \text{cl}(\text{ri}(\tilde{Q}_j \setminus P)) \cup \bigcup_{j \in J_2} P \cap \tilde{Q}_j$, *where*

$$J_1 := \{j \in \overline{1k} :\ \tilde{Q}_j \setminus P \neq \emptyset\} \ \text{ and } \ J_2 := \{j \in \overline{1k} : \text{aff}(\tilde{Q}_j) = \text{aff}(P \cap \tilde{Q}_j) = \text{aff}(P)\}.$$

Proof. By the definition of J_1, one has

$$P \cup \bigcup_{j=1}^{k} \tilde{Q}_j = \left(P \setminus \bigcup_{j=1}^{k} \tilde{Q}_j \right) \cup \bigcup_{j \in J_1} (\tilde{Q}_j \setminus P) \cup \bigcup_{j=1}^{k} P \cap \tilde{Q}_j.$$

Since $P \setminus \bigcup_{j=1}^{k} \tilde{Q}_j \subset \text{cl}\big(\text{ri}(P) \setminus \bigcup_{j=1}^{k} \tilde{Q}_j\big) \subset P$ and $\tilde{Q}_j \setminus P \subset \text{cl}(\text{ri}(\tilde{Q}_j) \setminus P) \subset \tilde{Q}_j$, one has $P \cup \bigcup_{j=1}^{k} \tilde{Q}_j = \text{cl}\big(\text{ri}(P) \setminus \bigcup_{j=1}^{k} \tilde{Q}_j\big) \cup \bigcup_{j \in J_1} \text{cl}(\text{ri}(\tilde{Q}_j \setminus P)) \cup \bigcup_{j=1}^{k} P \cap \tilde{Q}_j$. Thus, it suffices to prove that

$$P \cap \tilde{Q}_{j'} \subset \text{cl}\left(\text{ri}(P) \setminus \bigcup_{j=1}^{k} \tilde{Q}_j \right) \cup \bigcup_{j \in J_1} \text{cl}(\text{ri}(\tilde{Q}_j \setminus P)) \quad \forall j' \in \overline{1k} \setminus J_2. \tag{2.4}$$

To prove this, let $j' \in \overline{1k} \setminus J_2$. Then

$$\text{aff}(\tilde{Q}_{j'}) \neq \text{aff}(P \cap \tilde{Q}_{j'}) \ \text{ or } \ \text{aff}(\tilde{Q}_{j'}) = \text{aff}(P \cap \tilde{Q}_{j'}) \neq \text{aff}(P).$$

First suppose that $\text{aff}(\tilde{Q}_{j'}) \neq \text{aff}(P \cap \tilde{Q}_{j'})$. For any $x \in P \cap \tilde{Q}_{j'}$ ($\subset \text{cl}(\text{ri}(\tilde{Q}_{j'}))$) and any $\varepsilon > 0$, take $u \in \text{ri}(\tilde{Q}_{j'})$ and $\eta > 0$ such that $B(u,\eta) \cap \text{aff}(\tilde{Q}_{j'}) \subset \text{ri}(\tilde{Q}_{j'})$ and $B(u,\eta) \subset B(x,\varepsilon)$. Then $B(u,\eta) \cap \text{aff}(\tilde{Q}_{j'}) \not\subset P$ and hence

$$\emptyset \neq B(u,\eta) \cap \big(\text{ri}(\tilde{Q}_{j'}) \setminus P\big) \subset B(x,\varepsilon) \cap \big(\text{ri}(\tilde{Q}_{j'}) \setminus P\big).$$

This shows that $j' \in J_1$ and $x \in \mathrm{cl}\big(\mathrm{ri}(\tilde{Q}_{j'}) \setminus P\big)$. Hence

$$P \cap \tilde{Q}_{j'} \subset \mathrm{cl}\left(\mathrm{ri}(\tilde{Q}_{j'}) \setminus P\right), \tag{2.5}$$

which implies (2.4). Next suppose that $\mathrm{aff}(\tilde{Q}_{j'}) = \mathrm{aff}(P \cap \tilde{Q}_{j'}) \neq \mathrm{aff}(P)$. Then $\mathrm{ri}(P \cap \tilde{Q}_{j'}) \subset \mathrm{ri}(\tilde{Q}_{j'})$ and $P \cap \tilde{Q}_{j'} \subset \mathrm{cl}(\mathrm{ri}(P) \setminus \tilde{Q}_{j'})$ (similar to the proof of (2.5)). This and (2.3) show that

$$\mathrm{ri}(P \cap \tilde{Q}_{j'}) \subset \mathrm{cl}(\mathrm{ri}(P) \setminus \tilde{Q}_{j'}) \setminus \bigcup_{j \in \overline{1k} \setminus \{j'\}} \tilde{Q}_j \subset \mathrm{cl}\left(\mathrm{ri}(P) \setminus \bigcup_{j=1}^{k} \tilde{Q}_j\right).$$

Hence $P \cap \tilde{Q}_{j'} \subset \mathrm{cl}\big(\mathrm{ri}(P) \setminus \bigcup_{j=1}^{k} \tilde{Q}_j\big)$, which implies (2.4). The proof is complete. □

The following propositions provide interesting partitions of a nonconvex polyhedron and will play a key role in the proofs of our main results.

PROPOSITION 2.5. *Let $P_1, \cdots, P_m$ be convex polyhedra in a normed space X. Then there exist convex polyhedra $Q_1, \cdots, Q_n$ in X satisfying the following properties:*
(i) $\bigcup_{i=1}^{m} P_i = \bigcup_{j=1}^{n} Q_j$ *and* $Q_j \cap \mathrm{ri}(Q_{j'}) = \emptyset$ *for all* $j, j' \in \overline{1n}$ *with* $j \neq j'$.
(ii) *For any $j \in \overline{1n}$, there exists a subset I_j of $\overline{1m}$ such that*

$$\mathrm{ri}(Q_j) \cap \bigcup_{i' \in \overline{1m} \setminus I_j} P_{i'} = \emptyset, \;\; Q_j \subset P_i \text{ and } \mathrm{aff}(Q_j) = \mathrm{aff}(P_i) \quad \forall i \in I_j. \tag{2.6}$$

Proof. Let $P_1, \cdots, P_k$ be convex polyhedra, and suppose that there exist convex polyhedra $\tilde{Q}_1, \cdots, \tilde{Q}_{\tilde{n}}$ in X such that

$$\bigcup_{i=1}^{k} P_i = \bigcup_{j=1}^{\tilde{n}} \tilde{Q}_j, \;\; \tilde{Q}_j \cap \mathrm{ri}(\tilde{Q}_{j'}) = \emptyset \quad \forall j, j' \in \overline{1\tilde{n}} \text{ with } j \neq j' \tag{2.7}$$

and for any $j \in \overline{1\tilde{n}}$ there exists a subset $\tilde{I}_j$ of $\overline{1k}$ such that

$$\mathrm{ri}(\tilde{Q}_j) \cap \bigcup_{i \in \overline{1k} \setminus \tilde{I}_j} P_i = \emptyset, \tag{2.8}$$

$$\tilde{Q}_j \subset P_i \;\text{ and }\; \mathrm{aff}(\tilde{Q}_j) = \mathrm{aff}(P_i) \quad \forall i \in \tilde{I}_j. \tag{2.9}$$

Let P_{k+1} be any convex polyhedron in X. Then, by induction, it suffices to prove that there exist convex polyhedra $Q_1, \cdots, Q_n$ in X such that (i) and (ii) hold with $m = k+1$. To prove this, let $J_1 := \{j \in \overline{1\tilde{n}} : \tilde{Q}_j \setminus P_{k+1} \neq \emptyset\}$ and

$$J_2 := \left\{j \in \overline{1\tilde{n}} : \mathrm{aff}(\tilde{Q}_j) = \mathrm{aff}(P_{k+1} \cap \tilde{Q}_j) = \mathrm{aff}(P_{k+1})\right\}.$$

Then, by (2.7) and Lemma 2.4, we can repartition the union $\bigcup_{i=1}^{k+1} P_i$ as follows

$$\bigcup_{i=1}^{k+1} P_i = \mathrm{cl}\left(\mathrm{ri}(P_{k+1}) \setminus \bigcup_{j=1}^{\tilde{n}} \tilde{Q}_j\right) \cup \bigcup_{j \in J_1} \mathrm{cl}(\mathrm{ri}(\tilde{Q}_j \setminus P_{k+1})) \cup \bigcup_{j \in J_2} P_{k+1} \cap \tilde{Q}_j. \tag{2.10}$$

By Lemma 2.3, there exist a finite index set $\tilde{\mathcal{I}}$ and convex polyhedra H_ι in X ($\iota \in \tilde{\mathcal{I}}$) such that

$$\text{(2.11)} \qquad \operatorname{aff}(P_{k+1}) = \operatorname{aff}(H_\iota), \ \ \operatorname{ri}(H_\iota) \cap H_{\iota'} = \emptyset \quad \forall \iota, \iota' \in \tilde{\mathcal{I}} \text{ with } \iota \neq \iota'$$

and $\bigcup_{\iota\in\tilde{\mathcal{I}}} \operatorname{ri}(H_\iota) \subset \operatorname{ri}(P_{k+1}) \setminus \bigcup_{j=1}^{\tilde{n}} \tilde{Q}_j \subset \bigcup_{\iota\in\tilde{\mathcal{I}}} H_\iota$ (taking $\tilde{\mathcal{I}} = \emptyset$ if $P_{k+1} \setminus \bigcup_{j=1}^{\tilde{n}} \tilde{Q}_j = \emptyset$). Hence

$$\text{(2.12)} \qquad \operatorname{cl}\left(\operatorname{ri}(P_{k+1}) \setminus \bigcup_{j=1}^{\tilde{n}} \tilde{Q}_j\right) = \bigcup_{\iota\in\tilde{\mathcal{I}}} H_\iota,$$

$$\text{(2.13)} \qquad \bigcup_{\iota\in\tilde{\mathcal{I}}} H_\iota \subset P_{k+1} \ \text{ and } \ \left(\bigcup_{\iota\in\tilde{\mathcal{I}}} \operatorname{ri}(H_\iota)\right) \cap \bigcup_{j=1}^{\tilde{n}} \tilde{Q}_j = \emptyset.$$

Moreover, by Lemma 2.3, for each $j \in J_1$ there exist a finite index set $\mathcal{I}_j$ and convex polyhedra H_{jl} in X ($l \in \mathcal{I}_j$) such that

$$\text{(2.14)} \qquad \operatorname{aff}(\tilde{Q}_j) = \operatorname{aff}(H_{jl}), \ \operatorname{ri}(H_{jl}) \cap H_{jl'} = \emptyset \quad \forall l, l' \in \mathcal{I}_j \text{ with } l \neq l'$$

and

$$\text{(2.15)} \qquad \bigcup_{l\in\mathcal{I}_j} \operatorname{ri}(H_{jl}) \subset \operatorname{ri}(\tilde{Q}_j) \setminus P_{k+1} \subset \bigcup_{l\in\mathcal{I}_j} H_{jl}.$$

For any $j \in J_1$, it follows from the inclusion in (2.13) that

$$\text{(2.16)} \quad \emptyset = \left(\bigcup_{l\in\mathcal{I}_j} \operatorname{ri}(H_{jl})\right) \cap P_{k+1} \supset \left(\bigcup_{l\in\mathcal{I}_j} \operatorname{ri}(H_{jl})\right) \cap \left(\bigcup_{\iota\in\tilde{\mathcal{I}}} H_\iota \cup \bigcup_{j\in J_2} \tilde{Q}_j \cap P_{k+1}\right),$$

$$\text{(2.17)} \qquad \operatorname{cl}\left(\operatorname{ri}(\tilde{Q}_j) \setminus P_{k+1}\right) = \bigcup_{l\in\mathcal{I}_j} H_{jl} \quad \text{and} \quad \bigcup_{l\in\mathcal{I}_j} H_{jl} \subset \tilde{Q}_j.$$

Thus, by the second equality in (2.7) and the equality in (2.13), one has

$$\text{(2.18)} \qquad \left(\bigcup_{l\in\mathcal{I}_j} H_{jl}\right) \cap \operatorname{ri}(\tilde{Q}_{j'}) = \emptyset \quad \forall j \in J_1 \text{ and } \forall j' \in \overline{1\tilde{n}} \setminus \{j\}$$

and

$$\text{(2.19)} \qquad \left(\bigcup_{\iota\in\tilde{\mathcal{I}}} \operatorname{ri}(H_\iota)\right) \cap \left(\bigcup_{j\in J_1, l\in\mathcal{I}_j} H_{jl} \cup \left(\bigcup_{j\in J_2} \tilde{Q}_j \cap P_{k+1}\right)\right) = \emptyset.$$

It follows from (2.11), the definition of J_2 and Lemma 2.1 that

$$\text{(2.20)} \qquad \left(\bigcup_{\iota\in\tilde{\mathcal{I}}} H_\iota\right) \cap \bigcup_{j\in J_2} \operatorname{ri}(\tilde{Q}_j \cap P_{k+1}) = \emptyset.$$

By Lemma 2.1, (2.14) and (2.16), one also has $\left(\bigcup_{l\in\mathcal{I}_j} H_{jl}\right)\cap \mathrm{ri}(\tilde{Q}_j\cap P_{k+1})=\emptyset$ for all $j\in J_1\cap J_2$. This and (2.18) imply that

$$\left(\bigcup_{j\in J_1, l\in\mathcal{I}_j} H_{jl}\right)\cap \mathrm{ri}(\tilde{Q}_{j'}\cap P_{k+1})=\emptyset \quad \forall j'\in J_2. \tag{2.21}$$

By the definition of J_2 and (2.7), one has

$$\mathrm{ri}(\tilde{Q}_j\cap P_{k+1})\cap(\tilde{Q}_{j'}\cap P_{k+1})\subset \mathrm{ri}(\tilde{Q}_j)\cap\tilde{Q}_{j'}=\emptyset \quad \forall j,j'\in J_2 \text{ with } j\neq j'. \tag{2.22}$$

Let $Q_1,\cdots,Q_n$ be convex polyhedra in X such that

$$\{Q_1,\cdots,Q_n\}=\left\{H_\iota:\,\iota\in\tilde{\mathcal{I}}\right\}\cup\left\{H_{jl}:\,j\in J_1,\,l\in\mathcal{I}_j\right\}\cup\left\{\tilde{Q}_j\cap P_{k+1}:\,j\in J_2\right\} \tag{2.23}$$

and $Q_j\neq Q_{j'}$ for any $j,j'\in\overline{1n}$ with $j\neq j'$. Then (2.10), (2.12) and (2.17) imply clearly that $\bigcup_{i=1}^{k+1}P_i=\bigcup_{j=1}^{n}Q_j$, while (2.11), (2.14), (2.16) and (2.19)—(2.22) imply that $\mathrm{ri}(Q_j)\cap Q_{j'}=\emptyset$ for all $j,j'\in\overline{1n}$ with $j\neq j'$. This shows that $Q_1,\cdots,Q_n$ in (2.23) satisfy (i) with $m=k+1$. Finally we prove that $Q_1,\cdots,Q_n$ in (2.23) also satisfy (ii) with $m=k+1$, that is, for each $j\in\overline{1n}$ there exists a subset I_j of $\overline{1n}$ such that (2.6) holds. To prove this, let j be an arbitrary element in $\overline{1n}$. If $Q_j=H_\iota$ for some $\iota\in\tilde{\mathcal{I}}$, then (2.11), (2.13) and the first equality in (2.7) imply that (2.6) holds with $I_j=\{k+1\}$. If there exist $j'\in J_1$ and $l\in\mathcal{I}_{j'}$ such that $Q_j=H_{j'l}$, then (2.9), (2.8), (2.14) and the first inclusion in (2.15) imply that (2.6) holds with $I_j=\tilde{I}_{j'}$. If $Q_j=\tilde{Q}_{j'}\cap P_{k+1}$ for some $j'\in J_2$, then the definition of J_2 implies $\mathrm{ri}(\tilde{Q}_{j'}\cap P_{k+1})=\mathrm{ri}(\tilde{Q}_{j'})\cap\mathrm{ri}(P_{k+1})$, and it follows from (2.9) and (2.8) that (2.6) holds with $I_j:=\tilde{I}_{j'}\cup\{k+1\}$. The proof is complete. □

Corollary 2.6. *Let $P_1,\cdots,P_m$ be convex polyhedra in a normed space X and A be the union of finitely many convex polyhedra in X such that $A\subset\bigcup_{i=1}^m P_i$. Then there exist convex polyhedra $Q_1,\cdots,Q_n$ in Y such that*

$$A=\bigcup_{j=1}^{n}Q_j,\quad Q_j\cap\mathrm{ri}(Q_{j'})=\emptyset \quad \forall j\in\overline{1n} \text{ and } \forall j'\in\overline{1n}\setminus\{j\} \tag{2.24}$$

and for any $j\in\overline{1n}$ there exists a subset I_j of $\overline{1m}$ such that

$$\mathrm{ri}(Q_j)\cap\bigcup_{i'\in\overline{1m}\setminus I_j}P_{i'}=\emptyset \text{ and } Q_j\subset P_i\ \ \forall i\in I_j. \tag{2.25}$$

Proof. By Proposition 2.5, take convex polyhedra $A_1,\cdots,A_k$ in X such that

$$A=\bigcup_{\iota=1}^{k}A_\iota \text{ and } \mathrm{ri}(A_\iota)\cap A_{\iota'}=\emptyset \quad \forall\iota\in\overline{1k} \text{ and } \iota'\in\overline{1k}\setminus\{\iota\}. \tag{2.26}$$

Then $A_\iota=\bigcup_{i=1}^m P_i^\iota$ for any $\iota\in\overline{1k}$, where $P_i^\iota:=A_\iota\cap P_i$. Again by Proposition 2.5, there exist convex polyhedra $Q_1^\iota,\cdots,Q_{n_\iota}^\iota$ in X such that

$$A_\iota=\bigcup_{j=1}^{n_\iota}Q_j^\iota,\quad \mathrm{ri}(Q_j^\iota)\cap Q_{j'}^\iota=\emptyset\ \ \forall j,j'\in\overline{1n_\iota} \text{ with } j\neq j' \tag{2.27}$$

and for each $j \in \overline{1n_\iota}$ there exists $I_j^\iota \subset \overline{1m}$ such that

$$\text{(2.28)} \qquad \mathrm{ri}(Q_j^\iota) \cap \bigcup_{i' \in \overline{1m} \setminus I_j^\iota} P_{i'}^\iota = \emptyset, \;\; Q_j^\iota \subset P_i^\iota \text{ and } \mathrm{aff}(Q_j^\iota) = \mathrm{aff}(P_i^\iota) \;\; \forall i \in I_j^\iota.$$

For each $\iota \in \overline{1k}$, let $J^\iota := \left\{ j \in \overline{1n_\iota} : \; A_\iota \neq \bigcup_{j' \in \overline{1n_\iota} \setminus \{j\}} Q_{j'}^\iota \right\}$. Then $A_\iota = \bigcup_{j' \in J^\iota} Q_{j'}^\iota$. For each $j \in J^\iota$, take $x \in \mathrm{int}(Q_j^\iota)$ and $r > 0$ such that $B(x,r) \cap \bigcup_{j' \in \overline{1n_\iota} \setminus \{j\}} Q_{j'}^\iota = \emptyset$, and hence $B(x,r) \cap Q_j^\iota = B(x,r) \cap A_\iota$. Noting that A_ι is a convex polyhedron, it follows that $\mathrm{aff}(Q_j^\iota) = \mathrm{aff}(A_\iota)$ and $\mathrm{ri}(Q_j^\iota) \subset \mathrm{ri}(A^\iota)$ for all $j \in J^\iota$. Thus, rewriting the family $\bigcup_{\iota \in \overline{1k}} \{Q_j^\iota : \; j \in J^\iota\}$ as $\{Q_1, \cdots, Q_n\}$, one sees that (2.24) and (2.25) hold (thanks to (2.26), (2.27) and (2.28)). The proof is complete. □

3. Uniform weak sharp minima for piecewise linear multiobjective optimization. Given a convex polyhedron P in a normed space X, recall that a function $\varphi : P \to \mathbb{R}$ is piecewise linear if there exist convex polyhedra P_i in X and $(x_i^*, c_i) \in X^* \times \mathbb{R}$ $(i = 1, \cdots, m)$ such that

$$\text{(3.1)} \qquad P = \bigcup_{i=1}^m P_i \quad \text{and} \quad \varphi(x) = \langle x_i^*, x \rangle + c_i \;\; \forall x \in P_i \text{ and } \forall i \in \overline{1m}.$$

Zheng and Ng proved the following result on nonnegative piecewise linear functions (cf. [40, Corollary 3.1]), which will play a key role in the proof of our main result.

Lemma 3.1. *Let P be a convex polyhedron in a normed space X and let $\varphi, \psi : P \to \mathbb{R}_+$ be piecewise linear functions such that*

$$\ker(\varphi) := \{x \in P : \; \varphi(x) = 0\} \subset \ker(\psi).$$

Then there exist $\kappa, r \in (0, +\infty)$ such that $\kappa\psi(x) \leq \varphi(x)$ for all $x \in P$ with *$\varphi(x) \leq r$.*

We also need the following uniform global metric regularity result on a convex polyhedral multifunction (cf. [8, Theorem 3C.3]).

Lemma 3.2. *Let $F : X \rightrightarrows \mathbb{R}^n$ be a convex polyhedral multifunction, that is, its graph* $\mathrm{gph}(F) = \{(x,y) : \; x \in X$ and $y \in F(x)\}$ *is a convex polyhedron in $X \times \mathbb{R}^n$. Then there exists $\tau \in (0, +\infty)$ such that*

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \forall x \in X \text{ and } \forall y \in F(X).$$

Let $f_1, \cdots, f_n, \varphi_1, \cdots, \varphi_\nu : X \to \mathbb{R}$ be piecewise linear functions. Equipping $\mathbb{R}^n$ with the so-called natural vector partial order induced by the polyhedral cone $\mathbb{R}^n_+$, this section considers the following piecewise linear multiobjective optimization problem

$$\text{(PLMOP)} \;\; \mathbb{R}^n_+\text{−min} f(x) := (f_1(x), \cdots, f_n(x)) \text{ subject to } \varphi_1(x) \leq 0, \cdots, \varphi_k(x) \leq 0.$$

Then the feasible set Ω of (PLMOP) is the union of finitely many convex polyhedra in X. The Pareto (resp. weak Pareto) solution set of (PLMOP) is denoted by S (resp. S_w), while the Pareto (resp. weak Pareto) optimal value set is denoted by V (resp. V_w). Hence $S = \Omega \cap f^{-1}(V)$, $S_w = \Omega \cap f^{-1}(V_w)$,

$$V = \{y \in f(\Omega) : \; f(\Omega) \cap (y - \mathbb{R}^n_+) = \{y\}\} \text{ and } V_w = \{y \in f(\Omega) : \; f(\Omega) \cap (y - \mathrm{int}(\mathbb{R}^n_+)) = \emptyset\}.$$

Extending Arrow, Barankin and Blackewll's classical result on the solution sets and optimal value sets for linear multiobjective optimization problems, several authors established the structure of the solution sets and optimal value sets for piecewise linear multiobjective optimization. In particular, the following lemma has been established (cf. [37, 42]).

Lemma 3.3. *The following statements on piecewise linear muiltiobjective optimization problem (PLMOP) hold.*
(i) The Pareto solution set S and the Pareto optimal value set V of (PLMOP) are the union of finitely many convex generalized polyhedra in X and $\mathbb{R}^n$, respectively.
(ii) The weak Pareto solution set S_w and the weak Pareto optimal value set V_w of (PLMOP) are the union of finitely many convex polyhedra in X and $\mathbb{R}^n$, respectively. If, in addition, $f(\Omega)+\mathbb{R}^n_+$ is convex, then the Pareto solution set S and the Pareto optimal value set V of (PLMOP) are also the unions of finitely many convex polyhedra.

Since the Pareto optimal value set V is always a subset of the weak Pareto optimal value set V_w and $S(y)=S_w(y)$ for all $y\in V$, the uniform global (resp. bounded) weak sharp minima of (PLMOP) with respect to weak Pareto optimal values implies the uniform global (resp. bounded) weak sharp minima of (PLMOP) with respect to Pareto optimal values. We first consider the uniform weak sharp minima of (PLMOP) with respect to weak Pareto optimal values in any polyhedral subset of V_w.

Theorem 3.4. *Let A be a subset of the weak Pareto optimial value set V_w of (PLMOP) such that it is the union of finitely many convex polyhedra in $\mathbb{R}^n$. Then the following properties are equivalent:*
(i) $S_w(y)\subset \liminf\limits_{y'\xrightarrow{A}y} S_w(y')$ for all $y\in A$.
(ii) (PLMOP) has uniform bounded weak sharp minima with respect to all weak Pareto optimal values in A, that is, for any $r>0$ sufficiently large there exists $\kappa\in(0,\ +\infty)$ such that for any $y=(y_1,\cdots,y_n)\in A\cap rB_{\mathbb{R}^n}$,

$$\kappa d(x,S_w(y))\le\sum_{i=1}^n[f_i(x)-y_i]_+ +\sum_{j=1}^k[\varphi_j(x)]_+\quad \forall x\in rB_X. \tag{3.2}$$

(iii) For any $r\in(0,\ +\infty)$, there exists $L\in(0,\ +\infty)$ such that

$$S_w(y)\cap rB_X\subset S_w(y')+L\|y'-y\|B_X\ \ \forall y\in V_w \text{ and } \forall y'\in A\cap rB_{\mathbb{R}^n}.$$

Proof. Let $F: X\rightrightarrows \mathbb{R}^n\times\mathbb{R}^k$ be defined by

$$F(x):=\big((f_1(x),\cdots,f_n(x))+\mathbb{R}^n_+\big)\times\big((\varphi_1(x),\cdots,\varphi_k(x))+\mathbb{R}^k_+\big)\quad \forall x\in X \tag{3.3}$$

Then, by $A\subset V_w$, one has $F^{-1}(y,0)=\Omega\cap f^{-1}(y-\mathbb{R}^n_+)=S_w(y)$ for all $y\in A$. Since $f_1,\cdots,f_n,\varphi_1,\cdots,\varphi_k$ are piecewise linear, it is easy to verify that F is a polyhedral multifunction (i.e. the graph gph(F) is the union of finitely many convex polyhedra in $X\times\mathbb{R}^n\times\mathbb{R}^k$). Hence there exist multifunctions $F_1,\cdots,F_\nu$ from X to $\mathbb{R}^n\times\mathbb{R}^k$ such that gph$(F_1),\cdots,$gph(F_ν) are convex polyhedra and gph$(F)=\bigcup\limits_{\iota=1}^{\nu}gph(F_\iota)$. It follows that

$$F(x)=\bigcup_{\iota=1}^{\nu}F_\iota(x)\ \forall x\in X\ \text{ and }\ S_w(y)=F^{-1}(y,0)=\bigcup_{\iota=1}^{\nu}F_\iota^{-1}(y,0)\quad\forall y\in A. \tag{3.4}$$

Moreover, by Lemma 3.2, there exists $L \in (0, +\infty)$ such that

$$d(x, F_\iota^{-1}(z)) \leq Ld(z, F_\iota(x)) \quad \forall (x, z) \in X \times F_\iota(X) \text{ and } \forall \iota \in \overline{1\nu}. \tag{3.5}$$

Noting that each $F_\iota(X)$ is a convex polyhedron and $A \times \{0\} \subset F(X) = \bigcup_{\iota=1}^{\nu} F_\iota(X)$, by Corollary 2.6, there exist convex polyhedra $Q_1, \cdots, Q_{n'}$ in $\mathbb{R}^n$ such that

$$A = \bigcup_{\upsilon=1}^{n'} Q_\upsilon, \quad \mathrm{ri}(Q_\upsilon) \cap \bigcup_{\upsilon' \in \overline{1n'} \setminus \{\upsilon\}} Q_{\upsilon'} = \emptyset, \tag{3.6}$$

$$Q_\upsilon \times \{0\} \subset F_\iota(X) \;\text{ and }\; (\mathrm{ri}(Q_\upsilon) \times \{0\}) \cap \bigcup_{\iota' \in \overline{1\nu} \setminus I_\upsilon} F_{\iota'}(X) = \emptyset \tag{3.7}$$

for all $\upsilon \in \overline{1n'}$ and $\iota \in I_\upsilon$, where each I_υ is a subset of $\overline{1\nu}$. Noting that each F_ι is a convex polyhedral multifunction, it is easy to verify that

$$(x, z) \mapsto G(x, z) := F_\iota^{-1}(z) - x$$

is a convex polyhedral multifunction from $X \times \mathbb{R}^n \times \mathbb{R}^k$ to X with $\mathrm{dom}(G) = X \times F_\iota(X)$. Thus, by [40, Lemma 3.3], there exist $\tau_\iota, \tilde{\tau}_\iota \in (0, +\infty)$ and a piecewise linear function $\eta_\iota : X \times F_\iota(X) \to \mathbb{R}_+$ such that

$$\tau_\iota \eta_\iota(x, z) \leq d(0, G(x, z)) = d(x, F_\iota^{-1}(z)) \leq \tilde{\tau}_\iota \eta_\iota(x, z) \quad \forall (x, z) \in X \times F_\iota(X). \tag{3.8}$$

First suppose that (i) holds. Let $\upsilon \in \overline{1n'}$ and $y \in Q_\upsilon$. Then, since $\mathrm{ri}(Q_\upsilon)$ is dense in Q_υ, there exists a sequence $\{y_k\}$ in $\mathrm{ri}(Q_\upsilon)$ converging to y. Hence

$$S_w(y) \subset \big\{x \in X : \lim_{k\to\infty} d(x, S_w(y_k)) = 0\big\}.$$

Noting that $S_w(y_k) = F^{-1}(y_k, 0) = \bigcup_{\iota=1}^{\nu} F_\iota^{-1}(y_k, 0) = \bigcup_{\iota \in I_\upsilon} F_\iota^{-1}(y_k, 0)$ (thanks to (3.4) and (3.7)), it follows that

$$S_w(y) \subset \left\{x \in X : \lim_{k\to\infty} d\left(x, \bigcup_{\iota \in I_\upsilon} F_\iota^{-1}(y_k, 0)\right) = 0\right\}.$$

Let u be an arbitrary element in $S_w(y)$. Then, since I_υ is a finite set, we can assume without loss of generality that there exists $\iota_0 \in I_\upsilon$ such that $\lim_{k\to\infty} d\left(u, F_{\iota_0}^{-1}(y_k, 0)\right) = 0$ (taking a subsequence of $\{y_k\}$ if necessary). Hence there exists a sequence $\{u_k\}$ in X such that $\|u - u_k\| \to 0$ and $u_k \in F_{\iota_0}^{-1}(y_k, 0)$ for all $k \in \mathbb{N}$. This and (3.5) imply that

$$d(u_k, F_{\iota_0}^{-1}(y, 0)) \leq Ld((y, 0), F_{\iota_0}(u_k)) \leq L\|y_k - y\| \quad \forall k \in \mathbb{N}.$$

Thus, since $y_k \to y$, one has $u \in F_{\iota_0}^{-1}(y, 0) \subset \bigcup_{\iota \in I_\upsilon} F_\iota^{-1}(y, 0)$. Hence $S_w(y) \subset \bigcup_{\iota \in I_\upsilon} F_\iota^{-1}(y, 0)$. It follows from (3.4) that $S_w(y) = \bigcup_{\iota \in I_\upsilon} F_\iota^{-1}(y, 0)$, which implies

$$d(x, S_w(y)) = \min_{\iota \in I_\upsilon} d(x, F_\iota^{-1}(y, 0)) \quad \forall (x, y) \in X \times Q_\upsilon \text{ and } \forall \upsilon \in \overline{1n'}. \tag{3.9}$$

For $\upsilon \in \overline{1n'}$, let $\kappa_\upsilon := \min_{\iota\in I_\upsilon} \tau_\iota$, $\tilde{\kappa}_\upsilon := \max_{\iota\in I_\upsilon} \tilde{\tau}_\iota$, and define $\theta_\upsilon : X \times Q_\upsilon \to \mathbb{R}_+$ by

$$\theta_\upsilon(x, y) := \min_{\iota\in I_\upsilon} \eta_\iota(x, (y, 0)) \quad \forall (x, y) \in X \times Q_\upsilon.$$

Then θ_υ is a nonnegative piecewise linear function on the convex polyhedron $X \times Q_\upsilon$ and

$$\kappa_\upsilon\theta_\upsilon(x, y) \le d(x, S_w(y)) \le \tilde{\kappa}_\upsilon\theta_\upsilon(x, y) \quad \forall (x, y) \in X \times Q_\upsilon \tag{3.10}$$

(thanks to (3.8), (3.9) and the inclusion in (3.7)). Let

$$\psi(x, y) := \sum_{i=1}^{n}[f_i(x) - y_i]_+ + \sum_{j=1}^{k}[\varphi_j(x)]_+ \ \ \forall x \in X \text{ and } \forall y = (y_1, \cdots , y_n) \in \mathbb{R}^n.$$

Then ψ is a nonnegative piecewise linear function on $X \times \mathbb{R}^n$, and

$$F^{-1}(y, 0) = \{x \in X : \ \psi(x, y) = 0\} \quad \forall y \in \mathbb{R}^n.$$

For any $\upsilon \in \overline{1n'}$, by (3.4) and (3.6), one has

$$\begin{aligned} \Lambda_\upsilon : &= \{(x, y) \in X \times Q_\upsilon : \ \psi(x, y) = 0\} \\ &= \{(x, y) \in X \times Q_\upsilon : \ x \in F^{-1}(y, 0))\} \\ &= \{(x, y) \in X \times Q_\upsilon : \ x \in S_w(y)\} \\ &= \{(x, y) \in X \times Q_\upsilon : \ d(x, S_w(y)) = 0\}. \end{aligned}$$

This and (3.10) imply that $\Lambda_\upsilon \subset \{(x, y) \in X \times Q_\upsilon : \ \theta_\upsilon(x, y) = 0\}$. Thus, by Lemma 3.1, there exist $\tau_\upsilon, r_\upsilon \in (0, +\infty)$ such that

$$\theta_\upsilon(x, y) \le \tau_\upsilon\psi(x, y) \quad \forall (x, y) \in X \times Q_\upsilon \text{ with } \psi(x, y) \le r_\upsilon. \tag{3.11}$$

Let $x \in X$ and $y \in A$ be such that $\psi(x, y) \le \delta_0 := \min\{r_\upsilon : \ \upsilon \in \overline{1n'}\}$. Then, by (3.6), there exists $\upsilon_0 \in \overline{1n'}$ such that $y \in Q_{\upsilon_0}$. Thus, by (3.10) and (3.11), one has

$$d(x, S_w(y)) \le \tilde{\kappa}_{\upsilon_0}\tau_{\upsilon_0}\psi(x, y) = \tilde{\kappa}_{\upsilon_0}\tau_{\upsilon_0}\left(\sum_{i=1}^{n}[f_i(x) - y_i]_+ + \sum_{j=1}^{k}[\varphi_j(x)]_+\right).$$

Let $r \in (0, +\infty)$ sufficiently large. Then, to prove (i)⇒(ii), it suffices to show that

$$\sup\{d(x, S_w(y)) : x \in rB_X \text{ and } y \in A \cap rB_{\mathbb{R}^n}\} < +\infty. \tag{3.12}$$

To do this, let $x \in rB_X$ and $y \in A \cap rB_{\mathbb{R}^n}$. Then, by (3.4), there exists $\iota \in \overline{1\nu}$ such that $(y, 0) \in F_\iota(X)$ and

$$d(x, S_w(y)) = d(x, F^{-1}(y, 0)) \le d(x, F_\iota^{-1}(y, 0)) \le \inf_{u\in X} \|x\| + \|u\| + d(u, F_\iota^{-1}(y, 0)).$$

This and (3.5) imply that

$$\begin{aligned} d(x, S_w(y)) &\le \inf_{u\in X} \|x\| + \|u\| + Ld((y, 0), F_\iota(u)) \\ &\le \inf_{u\in X} \|x\| + \|u\| + L(\|(y, 0)\| + d((0, 0), F_\iota(u))) \\ &\le (1 + L)r + \inf_{u\in X} \|u\| + Ld((0, 0), F_\iota(u)) \end{aligned}$$

for some constant $L \in (0,\ +\infty)$, verifying (3.12).

Since (iii)⇒(i) is trivial, it remains to show (ii)⇒(iii). Let $r, \kappa \in (0,\ +\infty)$ be such that (3.2) holds. Let $y = (y_1, \cdots, y_n) \in V_w$, $y' = (y'_1, \cdots, y'_n) \in A \cap rB_{\mathbb{R}^n}$ and $x \in S_w(y) \cap rB_X$. Then $x \in \Omega$, $y \in f(x) + \mathbb{R}^n_+$,

$$\kappa d(x, S_w(y')) \leq \sum_{i=1}^{n}[f_i(x) - y'_i]_+ + \sum_{j=1}^{k}[\varphi_j(x)]_+ = \sum_{i=1}^{n}[f_i(x) - y'_i]_+,$$

and hence

$$\sum_{i=1}^{n}[f_i(x) - y'_i]_+ \leq \sum_{i=1}^{n}[y_i - y'_i]_+ \leq L'\|y - y'\|,$$

where L' is a positive constant. It follows that $d(x, S_w(y')) \leq \frac{L'}{\kappa}\|y - y'\|$, that is, $x \in S_w(y') + \frac{L'}{\kappa}\|y - y'\|B_X$. This shows that (iii) holds with $L = \frac{L'}{\kappa}$. The proof is complete. □

The following theorem is immediate from Lemma 3.3 and Theorem 3.4 (applied to $A = V_w$).

Theorem 3.5. *The following properties on piecewise linear multiobjective optimization problem (PLMOP) are equivalent:*
(i) $S_w(y) \subset \liminf\limits_{y' \xrightarrow{V_w} y} S_w(y')$ *for all* $y \in V_w$.
(ii) (PLMOP) has uniform bounded weak sharp minima with respect to all weak Pareto optimal values, that is, for any $r > 0$ *sufficiently large there exists* $\kappa \in (0,\ +\infty)$ *such that for any* $y = (y_1, \cdots, y_n) \in V_w \cap rB_{\mathbb{R}^n}$,

$$\kappa d(x, S_w(y)) \leq \sum_{i=1}^{n}[f_i(x) - y_i]_+ + \sum_{j=1}^{k}[\varphi_j(x)]_+ \quad \forall x \in rB_X.$$

(iii) For any $r \in (0,\ +\infty)$, *there exists* $L \in (0,\ +\infty)$ *such that*

$$S_w(y) \cap rB_X \subset S_w(y') + L\|y' - y\|B_X \ \ \forall y \in V_w \text{ and } \forall y' \in V_w \cap rB_{\mathbb{R}^n}.$$

In the case that all f_i and ψ_i in (PLMOP) are convex, it can be proved that (PLMOP) always has uniform global weak sharp minima with respect to all weak Pareto optimal values. In fact, a more general result on uniform global weak sharp minima will be proved in Section 4. Next, in view of (i) in Theorem 3.4 and Theorem 3.5, we establish the following theorem.

Theorem 3.6. *Let* A *be a subset of the weak Pareto optimial value set* V_w *of (PLMOP) such that* A *is the union of finitely many convex polyhedra in* $\mathbb{R}^n$. *Then there exist convex polyhedra* $Q_1, \cdots, Q_{n'}$ *in* $\mathbb{R}^n$ *such that*

$$A = \bigcup_{\upsilon=1}^{n'} Q_\upsilon, \quad \mathrm{ri}(Q_{\upsilon'}) \cap \bigcup_{\upsilon \in \overline{1n'} \setminus \{\upsilon'\}} Q_\upsilon = \emptyset \quad \forall \upsilon' \in \overline{1n'} \tag{3.13}$$

and $S(\hat{y}) \subset \liminf_{y \xrightarrow{A} \hat{y}} S_w(y)$ *for all* $\hat{y} \in \bigcup_{\upsilon=1}^{n'} \mathrm{ri}(Q_\upsilon)$.

Proof. Let $F : X \rightrightarrows \mathbb{R}^n \times \mathbb{R}^k$ be defined by (3.3). As at the beginning of the proof of Theorem 3.4, there exist convex polyhedral multifunctions $F_1, \cdots, F_\nu : X \rightrightarrows \mathbb{R}^n \times \mathbb{R}^k$ and convex polyhedra $Q_1, \cdots, Q_{n'}$ in $\mathbb{R}^n$ such that (3.4), (3.6) and (3.7) hold. Let $\hat{y} \in \bigcup_{\upsilon=1}^{n'} \mathrm{ri}(Q_\upsilon)$ and take $\upsilon \in \overline{1n'}$ such that $\hat{y} \in \mathrm{ri}(Q_\upsilon)$. Then, by (3.6) and (3.7), one has

$$(\hat{y}, 0) \notin \left(\bigcup_{\upsilon' \in \overline{1n'} \setminus \{\upsilon\}} Q_{\upsilon'} \times \{0\} \right) \bigcup \left(\bigcup_{\iota' \in \overline{1\nu} \setminus I_\upsilon} F_{\iota'}(X) \right). \tag{3.14}$$

This and (3.4) imply that

$$S_w(\hat{y}) = F^{-1}(\hat{y}, 0) = \bigcup_{\iota \in I_\upsilon} F_\iota^{-1}(\hat{y}, 0). \tag{3.15}$$

Noting that $\left(\bigcup_{\upsilon' \in \overline{1n'} \setminus \{\upsilon\}} Q_{\upsilon'} \times \{0\}\right) \bigcup \left(\bigcup_{\iota' \in \overline{1\nu} \setminus I_\upsilon} F_{\iota'}(X) \right)$ is closed (because every convex polyhedron is closed), (3.14) means

$$r_\upsilon := d\left((\hat{y}, 0), \left(\bigcup_{\upsilon' \in \overline{1n'} \setminus \{\upsilon\}} Q_{\upsilon'} \times \{0\} \right) \bigcup \left(\bigcup_{\iota' \in \overline{1\nu} \setminus I_\upsilon} F_{\iota'}(X) \right) \right) > 0.$$

Let $\{y_k\}_{k \in \mathbb{N}}$ be an arbitrary sequence in A converging to $\hat{y}$. Then, without loss of generality, we can assume that $\|(y_k, 0) - (\hat{y}, 0)\| < r_\upsilon$ for all $k \in \mathbb{N}$. Thus, by (3.6) and (3.7), one has $(y_k, 0) \in (Q_\upsilon \times \{0\}) \setminus \bigcup_{\iota' \in \overline{1\nu} \setminus I_\upsilon} F_{\iota'}(X)$ for all $k \in \mathbb{N}$. It follows from (3.4) that

$$S_w(y_k) = F^{-1}(y_k, 0) = \bigcup_{\iota \in \overline{1\nu}} F_\iota^{-1}(y_k, 0) = \bigcup_{\iota \in I_\upsilon} F_\iota^{-1}(y_k, 0) \quad \forall k \in \mathbb{N} \tag{3.16}$$

and $(y_k, 0) \in F_\iota(X)$ for all $(k, \iota) \in \mathbb{N} \times I_\upsilon$, that is, $F_\iota^{-1}(y_k, 0) \neq \emptyset$ for all $(k, \iota) \in \mathbb{N} \times I_\upsilon$. Since each F_ι is a convex polyhedral multifunction, there exists $L \in (0, +\infty)$ such that

$$d(x, F_\iota^{-1}(y_k, 0)) \leq L d((y_k, 0), F_\iota(x)) \quad \forall (x, \iota, k) \in X \times I_\upsilon \times \mathbb{N}$$

(thanks to Lemma 3.2). Let $\hat{x}$ be an arbitrary element in $S_w(\hat{y})$. Then, by (3.16),

$$d(\hat{x}, S_w(y_k)) = d\left(\hat{x}, \bigcup_{\iota \in I_\upsilon} F_\iota^{-1}(y_k, 0) \right) = \min_{\iota \in I_\upsilon} d(\hat{x}, F_\iota^{-1}(y_k, 0)) \leq L \min_{\iota \in I_\upsilon} d((y_k, 0), F_\iota(\hat{x})),$$

that is, $d(\hat{x}, S_w(y_k)) \leq L d\left((y_k, 0), \bigcup_{\iota \in I_\upsilon} F_\iota(\hat{x})\right)$. By (3.15) and $\hat{x} \in S_w(\hat{y})$, one has $(\hat{y}, 0) \in \bigcup_{\iota \in I_\upsilon} F_\iota(\hat{x})$. Hence $d(\hat{x}, S_w(y_k)) \leq L\|(y_k, 0) - (\hat{y}, 0)\| \to 0$ as $k \to \infty$. This shows that $\hat{x} \in \liminf_{y \xrightarrow{A} \hat{y}} S_w(y)$. Since $\hat{x}$ is arbitrary in $S_w(\hat{y})$, one has $S_w(\hat{y}) \subset \liminf_{y \xrightarrow{A} \hat{y}} S_w(y)$. The proof is complete. □

By Lemma 3.3, both the weak Pareto optimal value set V_w of (PLMOP) and the closure $\mathrm{cl}(V)$ of the Pareto optimal value set V of (PLMOP) are the union of finitely many convex polyhedra in $\mathbb{R}^n$. Thus, by Theorems 3.5 and 3.6 (applied to $A = V_w$ and $A = \mathrm{cl}(V)$), we have the following result.

Corollary 3.7. *The following statements on the weak Pareto optimial value set V_w and Pareto optimal value set V of (PLMOP) hold.*
(i) There exist convex polyhedra $Q_1, \cdots, Q_{\hat{n}}$ in $\mathbb{R}^n$ such that

$$V_w = \bigcup_{\upsilon=1}^{\hat{n}} Q_\upsilon, \quad \mathrm{ri}(Q_{\upsilon'}) \cap \bigcup_{\upsilon \in \overline{1\hat{n}} \setminus \{\upsilon'\}} Q_\upsilon = \emptyset \quad \forall \upsilon' \in \overline{1\hat{n}} \tag{3.17}$$

and for any $\hat{y} \in \bigcup_{\upsilon=1}^{\hat{n}} \mathrm{ri}(Q_\upsilon)$ there exists $\delta > 0$ with the following property: for any $r \in (0, +\infty)$ there exists $L \in (0, +\infty)$ such that

$$S_w(y) \cap rB_X \subset S_w(y') + L\|y' - y\| B_X \ \ \forall y, y' \in V_w \cap B(\hat{y}, \delta). \tag{3.18}$$

Consequently, $S_w(\hat{y}) \subset \liminf_{y \xrightarrow{V_w} \hat{y}} S_w(y)$ for all $\hat{y} \in \bigcup_{\upsilon=1}^{\hat{n}} \mathrm{ri}(Q_\upsilon)$.
(ii) There exist convex polyhedra $\widetilde{Q}_1, \cdots, \widetilde{Q}_{\tilde{n}}$ in $\mathbb{R}^n$ such that

$$\mathrm{cl}(V) = \bigcup_{\upsilon=1}^{\tilde{n}} \widetilde{Q}_\upsilon, \quad \mathrm{ri}(\widetilde{Q}_{\upsilon'}) \cap \bigcup_{\upsilon \in \overline{1\tilde{n}} \setminus \{\upsilon'\}} \widetilde{Q}_\upsilon = \emptyset \quad \forall \upsilon' \in \overline{1\tilde{n}}$$

and for any $\hat{y} \in \bigcup_{\upsilon=1}^{\tilde{n}} \mathrm{ri}(\tilde{Q}_\upsilon)$ there exists $\delta > 0$ with the following property: for any $r \in (0, +\infty)$ there exists $L \in (0, +\infty)$ such that

$$S(y) \cap rB_X \subset S(y') + L\|y' - y\| B_X \ \ \forall y, y' \in V \cap B(\hat{y}, \delta).$$

Proof. By Lemma 3.3 and Theorem 3.6 (applied to $A = V_w$), there exist convex polyhedra $Q_1, \cdots, Q_{\hat{n}}$ in $\mathbb{R}^n$ such that (3.17) holds and

$$S_w(\hat{y}) \subset \liminf_{y \xrightarrow{V_w} \hat{y}} S_w(y) \quad \forall \hat{y} \in \bigcup_{\upsilon=1}^{\hat{n}} \mathrm{ri}(Q_\upsilon). \tag{3.19}$$

To prove (i), let $\hat{y} \in \bigcup_{\upsilon=1}^{\hat{n}} \mathrm{ri}(Q_\upsilon)$ and take $\upsilon \in \overline{1\hat{n}}$ such that $\hat{y} \in \mathrm{ri}(Q_\upsilon)$. Then, by (3.17), $\hat{y} \notin \bigcup_{\upsilon' \in \overline{1\hat{n}} \setminus \{\upsilon\}} Q_{\upsilon'}$. Noting that $\bigcup_{\upsilon' \in \overline{1\hat{n}} \setminus \{\upsilon\}} Q_{\upsilon'}$ is a closed set, it follows that there exists $\hat{\delta} > 0$ such that

$$B_{\|\cdot\|_\infty}[\hat{y}, \hat{\delta}] \cap V_w = B_{\|\cdot\|_\infty}[\hat{y}, \hat{\delta}] \cap Q_\upsilon \subset \mathrm{ri}(Q_\upsilon),$$

where $B_{\|\cdot\|_\infty}[\hat{y}, \hat{\delta}] := \{y \in \mathbb{R}^n : \ \|y - \hat{y}\|_\infty \le \hat{\delta}\}$ is a convex polyhedron. Hence, by (3.19), one has $S_w(y) \subset \liminf_{y' \xrightarrow{V_w} y} S_w(y')$ for all $y \in A := B_{\|\cdot\|_\infty}[\hat{y}, \hat{\delta}] \cap Q_\upsilon$. By Theorem

3.4, taking $\delta > 0$ such that $B(\hat{y}, \delta) \subset B_{\|\cdot\|_\infty}[\hat{y}, \hat{\delta}]$ and noting that $A = B_{\|\cdot\|_\infty}[\hat{y}, \hat{\delta}] \cap Q_\upsilon$ is a convex polyhedron contained in V_w, for any $r \in (0,\ +\infty)$ sufficiently large there exists $L \in (0,\ +\infty)$ such that (3.18) holds. This shows (i). Since $\mathrm{cl}(V)$ is a subset of V_w and is the union of finitely many convex polyhedra, (ii) can be proved similarly. □

In Theorem 3.6 and Corollary 3.7, noting that $Q_\nu \setminus \mathrm{ri}(Q_\nu)$ is the union of finitely many proper faces of Q_ν and that the dimension of every proper face of Q_ν is strictly less than the one of Q_ν, $V_w \setminus \bigcup_{\nu=1}^{\hat{n}} \mathrm{ri}(Q_\nu) = \bigcup_{\nu=1}^{\hat{n}} Q_\nu \setminus \mathrm{ri}(Q_\nu)$ is a null set in V_w. It follows from Corollary 3.7 that the Pareto solution mapping $y \mapsto S(y)$ of $(\mathrm{PLMOP})_C$ is locally Lipschitz-stable almost everywhere on V_w. It is worth mentioning that $y \in \bigcup_{\nu=1}^{\hat{n}} \mathrm{ri}(Q_\nu)$ cannot be improved as $y \in V_w (= \bigcup_{\nu=1}^{\hat{n}} Q_\nu)$ in Theorem 3.6 and Corollary 3.7 (see the following example).

Example. Let $X = \mathbb{R}$, $n = 2$, $f_1 = \varphi_1 = \cdots = \varphi_k \equiv 0$, and let $f_2 : \mathbb{R} \to \mathbb{R}$ be such that

$$f_2(x) := \begin{cases} x, & \text{if } x \in (-\infty,\ 0] \\ 0, & \text{if } x \in [0,\ +\infty). \end{cases}$$

Then $A = f(\mathbb{R}) = \{0\} \times (-\infty,\ 0]$. Moreover, $f(1) = \{(0, 0)\}$ and $S_w(0, -\frac{1}{\nu}) = (-\infty,\ -\frac{1}{\nu}]$ for all $\nu \in \mathbb{N}$. Hence $d(1, S_w(0, -\frac{1}{\nu})) = 1 + \frac{1}{\nu}$ for all $\nu \in \mathbb{N}$. Noting that $[f_2(1) - (-\frac{1}{\nu})]_+ = \frac{1}{\nu}$ and $[f_1(1) - 0]_+ = [\varphi_1(1)]_+ = \cdots = [\varphi_k(1)]_+ = 0$, it follows that

$$\lim_{\nu \to \infty} \frac{d(1, S_w(0, -\frac{1}{\nu}))}{[f_1(1) - 0]_+ + [f_2(1) + \frac{1}{\nu}]_+ + \sum_{j=1}^{k} [\varphi_j(1)]_+} = \lim_{\nu \to \infty} \frac{1 + \frac{1}{\nu}}{\frac{1}{\nu}} = +\infty.$$

This shows that (PLMOP) does not have uniform bounded weak sharp minima with respect to all weak Pareto optimal values, which motivates us to consider near uniform weak sharp minima.

Given a convex polyhedron P in $\mathbb{R}^n$ and $\varepsilon > 0$ sufficiently small, let

$$P^\varepsilon := \{x \in P :\ d(x, \mathrm{rbd}(P)) < \varepsilon\}, \tag{3.20}$$

where $\mathrm{rbd}(P) := P \setminus \mathrm{ri}(P)$ is the relative boundary of P.

Definition 3.1. *Let A be the union of finitely many convex polyhedra in $\mathbb{R}^n$ and $\varepsilon > 0$ sufficiently small. We say that a set A_ε is an ε-small set of A if there exist convex polyhedra $P_1, \cdots, P_m$ in $\mathbb{R}^n$ such that*

$$A = \bigcup_{i=1}^{m} P_i,\ A_\varepsilon = \bigcup_{i=1}^{m} P_i^\varepsilon \ \text{ and } \ \mathrm{ri}(P_i) \cap P_{i'} = \emptyset \ \ \forall i, i' \in \overline{1m} \text{ with } i \neq i', \tag{3.21}$$

where P_i^ε is as in (3.20).

Remark. Each P_i^ε in (3.21) is a "small" subset of P_i in the sense of topology for sufficiently small ε (thanks to (3.20)) and hence A_ε is a "small" subset of A in the sense of topology for sufficiently small ε. On the other hand, $\dim(\mathrm{rbd}(P_i)) <$

$\dim(P_i)$ and so $\mu_i(\mathrm{rbd}(P_i)) = 0$, where μ_i denotes the Lebesgue measure on P_i, and so $\lim\limits_{\varepsilon\to 0^+} \mu_i(P_i^\varepsilon \cap rB_X) = 0$ for any $r \in (0,\ +\infty)$. Hence A_ε is also a small subset of A in the sense of measure for sufficiently small ε.

Definition 3.2. *Piecewise linear multiobjective optimization problem (PLMOP) is said to have near uniform bounded weak sharp minima with respect to weak Pareto optimal values if for any $\varepsilon \in (0,\ +\infty)$ sufficiently small there exists an ε-small set O^ε of V_w such that for any $r \in (0,\ +\infty)$ sufficiently large there exists $\kappa \in (0,\ +\infty)$ satisfying the following property:*

$$\kappa d(x, S_w(y)) \leq \sum_{i=1}^{n}[f_i(x) - y_i]_+ + \sum_{j=1}^{k}[\varphi_j(x)]_+ \tag{3.22}$$

for any $x \in rB_X$ and any $y = (y_1, \cdots, y_n) \in (V_w \setminus O^\varepsilon) \cap rB_{\mathbb{R}^n}$.

The following theorem shows that piecewise linear multiobjective optimization problem (PLMOP) always has near uniform bounded weak sharp minima.

Theorem 3.8. *Piecewise linear multiobjective optimization problem (PLMOP) always has near uniform bounded weak sharp minima with respect to weak Pareto optimal values.*

Proof. By Corollary 3.7, there exist convex polyhedra $Q_1, \cdots, Q_{n'}$ in $\mathbb{R}^n$ such that

$$V_w = \bigcup_{\upsilon=1}^{n'} Q_\upsilon, \quad \mathrm{ri}(Q_\upsilon) \cap \bigcup_{\upsilon' \in \overline{1n'}\setminus\{\upsilon\}} Q_{\upsilon'} = \emptyset \ \ \forall \upsilon \in \overline{1n'} \tag{3.23}$$

and

$$S(\hat{y}) \subset \liminf_{y \xrightarrow{V_w} \hat{y}} S_w(y) \quad \forall \hat{y} \in \bigcup_{\upsilon=1}^{n'} \mathrm{ri}(Q_\upsilon). \tag{3.24}$$

For any $\varepsilon \in (0,\ +\infty)$, let

$$Q_\upsilon^\varepsilon := \{y \in Q_\upsilon :\ d(y, Q_j \setminus \mathrm{ri}(Q_\upsilon)) < \varepsilon\} \quad \forall \upsilon \in \overline{1n'}.$$

Then $O_\varepsilon := \bigcup_{\upsilon=1}^{n'} Q_\upsilon^\varepsilon$ is an ε-small set of V_w (thanks to (3.23)). For each $\upsilon \in \overline{1n'}$, take a prime generator group $\{(y_1^*, t_1), \cdots, (y_\nu^*, t_\nu)\}$ of Q_j, that is,

$$Q_\upsilon = \{y \in Y :\ \langle y_k^*, y\rangle \leq t_k,\ k \in \overline{1\nu}\} \tag{3.25}$$

and

$$Q_\upsilon \neq \big\{y \in Y :\ \langle y_k^*, y\rangle \leq t_k,\ k \in \overline{1\nu} \setminus \{k'\}\big\} \quad \forall k' \in \overline{1\nu}. \tag{3.26}$$

Thus, by [42, Corollary 2.1], there exists a subset J of $\overline{1\nu}$ such that

$$\mathrm{ri}(Q_\upsilon) = \{y \in Y :\ \langle y_k^*, y\rangle < t_k,\ k \in \overline{1\nu} \setminus J\} \cap \{y \in Y :\ \langle y_k^*, y\rangle = t_k,\ k \in J\}. \tag{3.27}$$

Let $E := \{y \in Y : \langle y_k^*, y\rangle = 0,\ k \in J\}$. From (3.25)—(3.27), it is easy to verify that $y_k^*|_E \neq 0$ for all $k \in \overline{1\nu} \setminus J$ and $y + B(0,\varepsilon) \cap E \subset Q_\upsilon$ and for all $y \in Q_\upsilon \setminus Q_\upsilon^\varepsilon$. Hence

$$\langle y_k^*, y\rangle + \big\|y_k^*|_E\big\|\varepsilon = \sup_{v \in y + B(0,\varepsilon)\cap E} \langle y_k^*, v\rangle \leq t_k \quad \forall k \in \overline{1\nu} \setminus J \text{ and } \forall y \in Q_\upsilon \setminus Q_\upsilon^\varepsilon.$$

Let

$$\widehat{Q}_v := \left\{ y \in Y : \langle y_k^*, y\rangle \le t_k - \frac{\|y_k^*|_E\|\varepsilon}{2},\ k \in \overline{1\nu} \setminus J \right\} \cap \{y \in Y : \langle y_k^*, y\rangle = t_k,\ k \in J\}.$$

Then $\widehat{Q}_v$ is a convex polyhedron and $Q_v \setminus Q_v^\varepsilon \subset \widehat{Q}_v \subset \mathrm{ri}(Q_v)$ (thanks to (3.27)). It follows from (3.23) that $V_w \setminus O_\varepsilon \subset \bigcup_{v=1}^{n'} \widehat{Q}_v \subset \bigcup_{v=1}^{n'} \mathrm{ri}(Q_v)$. Thus, by (3.24), $S(\hat{y}) \subset \liminf\limits_{y \xrightarrow{V_w} \hat{y}} S_w(y)$ for all $\hat{y} \in \bigcup_{v=1}^{n'} \widehat{Q}_v$. By Theorem 3.4 (applied to $A = \bigcup_{v=1}^{n'} \widehat{Q}_v$), (PLMOP) has uniform weak sharp minima with respect to all weak Pareto optimal values in $\bigcup_{v=1}^{n'} \widehat{Q}_v$. Hence, for any $r \in (0, +\infty)$, there exists $\tau_r > 0$ such that for any

$$\tau_r d(x, S_w(y)) \le \sum_{i=1}^{n} [f_i(x) - y_i]_+ + \sum_{j=1}^{k} [\varphi_j(x)]_+$$

for all $x \in rB_X$ and $y = (y_1, \cdots, y_n) \in (V_w \setminus O_\varepsilon) \cap rB_{\mathbb{R}^n}$. This shows that (PLMOP) always has near uniform bounded weak sharp minima with respect to all weak Pareto optimal values. The proof is complete. □

4. The general vector partial order case. For a general closed convex and pointed cone C in $\mathbb{R}^n$, let $\le_C$ denote the partial order induced by C, that is, for $y_1, y_2 \in \mathbb{R}^n$,

$$y_1 \le_C y_2\, (\text{resp.}\, y_1 <_C y_2) \Longleftrightarrow y_2 - y_1 \in C\, (\text{resp.}\, y_2 - y_1 \in \mathrm{int}(C)).$$

Given a set A in $\mathbb{R}^n$, let $\mathrm{WE}(A, C)$, $\mathrm{E}(A, C)$ and $\mathrm{Pos}(A, C)$ denote respectively the sets of all weakly efficient points, all efficient points and all positive proper efficient points of A with respect to C, that is,

$$\mathrm{WE}(A, C) = \{x \in A : A \cap (x - \mathrm{int}(C)) = \emptyset\},\ \mathrm{E}(A, C) = \{x \in A : A \cap (x - C) = \{x\}\}$$

and

$$\mathrm{Pos}(A, C) := \{x \in A : \text{there exists}\, x^* \in C^{+i}\ \text{such that}\ \langle x^*, x\rangle = \inf_{a \in A} \langle x^*, a\rangle\},$$

where $C^{+i} = \{x^* \in \mathbb{R}^n : \langle x^*, y\rangle > 0\ \ \forall y \in C \setminus \{0\}\}$. It is well-known and easy to verify that $\mathrm{Pos}(A, C) \subset \mathrm{E}(A, C) \subset \mathrm{WE}(A, C)$. Arrow, Barankin and Blackwell [1] proved that if A is a closed convex set in $\mathbb{R}^n$ and if $C \subset \mathbb{R}^n$ has a compact base then

$$\text{(ABB)} \qquad \mathrm{E}(A, C) \subset \mathrm{cl}(\mathrm{Pos}(A, C)).$$

It is known that every closed convex pointed cone in $\mathbb{R}^n$ has a compact base.

In the remainder of this paper, let C be a closed convex and pointed cone in $\mathbb{R}^n$. Equipping $\mathbb{R}^n$ with the vector partial order induced by C, consider the following piecewise linear multiobjective optimization problem

$$(\text{PLMOP})_C \quad C{-}\min f(x) = (f_1(x), \cdots, f_n(x))\ \text{subject to}\ \varphi_1(x) \le 0, \cdots, \varphi_k(x) \le 0,$$

where $f_1, \cdots, f_n, \varphi_1, \cdots, \varphi_k : X \to \mathbb{R}$ are piecewise linear functions.

In the case that the ordering cone C is not polyhedral, $(\text{PLMOP})_C$ is not a genuine piecewise linear problem even when all f_i and φ_j are linear. Under a mild assumption, we will prove that for any closed convex pointed cone C in $\mathbb{R}^n$ there exists a closed convex pointed polyhedral cone $\widehat{C}$ in $\mathbb{R}^n$ such that $(\text{PLMOP})_C$ and $(\text{PLMOP})_{\widehat{C}}$ are equivalent. Such an equivalence not only plays a key role in the proofs of our main results but also is of its own value. Recall that a vector-valued function $f : X \to \mathbb{R}^n$ is C-convex if

$$f(tx_1 + (1-t)x_2) \leq_C tf(x_1) + (1-t)f(x_2) \quad \forall t \in [0,\ 1] \text{ and } \forall x_1, x_2 \in X.$$

In contrast, we adopt the following weak C-convexity of f on a set A: *f is said to be C-convex-like if for any $x_1, x_2 \in A$ and any $t \in (0,\ 1)$ there exists $x_t \in A$ such that*

$$f(x_t) \leq_C tf(x_1) + (1-t)f(x_2).$$

It is clear that if $f(A)$ is complete with the vector partial order $\leq_C$ then f is C-convex-like on A. In particular, in the case of $n = 1$ and $C = \mathbb{R}_+$, every function $f : X \to \mathbb{R}$ is weakly C-convex on any subset of X.

THEOREM 4.1. *Suppose that the objective function f is C-convex-like on the feasible set Ω. Then there exists a convex polyhedral pointed cone $\widehat{C}$ containing C such that the following properties hold:*
(i) f is $\widehat{C}$-convex-like on Ω. If, in addition, f is C-convex, then f is also $\widehat{C}$-convex.
(ii) Piecewise linear multiobjective optimization problems $(\text{PLMOP})_C$ and $(\text{PLMOP})_{\widehat{C}}$ have the same weak Pareto optimal value set and the same Pareto optimal value set.
(iii) $(\text{PLMOP})_C$ and $(\text{PLMOP})_{\widehat{C}}$ have the same weak Pareto solution set and the same Pareto solution set.

Proof. Since f and each φ_j are piecewise linear, $f(\Omega)$ is the union of finitely many convex polyhedra in $X \times \mathbb{R}^n$. Noting that a set P in $\mathbb{R}^n$ is a convex polyhedron if and only if P is finitely generated (cf. [31, Theorem 19.1]), we have that $\text{co}(f(\Omega))$ is a convex polyhedron. Since f is C-convex-like on Ω, it is easy to verify that $f(\Omega) + C$ is a convex set. This implies that $\text{co}(f(\Omega)) \subset f(\Omega) + C$, and hence $\text{co}(f(\Omega)) + C = f(\Omega) + C$. Noting that $\text{WE}(A, C) = A \cap \text{WE}(A + C, C)$ and $\text{E}(A, C) = \text{E}(A + C, C)$ for any set A in $\mathbb{R}^n$, one has

$$\text{WE}(f(\Omega), C) = f(\Omega) \cap \text{WE}(\text{co}(f(\Omega)), C) \;\text{ and }\; \text{E}(f(\Omega), C) = \text{E}(\text{co}(f(\Omega)), C). \tag{4.1}$$

For each $y^* \in C^+ \setminus \{0\}$, let

$$F(y^*) := \left\{ y \in \text{co}(f(\Omega)) : \ \langle y^*, y \rangle = \sup_{z \in \text{co}(f(\Omega))} \langle y^*, z \rangle \right\}$$

and

$$\widehat{F}(y^*) := \left\{ y \in f(\Omega) : \ \langle y^*, y \rangle = \sup_{z \in f(\Omega)} \langle y^*, z \rangle \right\}.$$

Then $\text{Pos}(\text{co}(f(\Omega)), C) = \bigcup_{y^* \in C^{+i}} F(y^*)$ and, by the separation theorem, one also has $\text{WE}(\text{co}(f(\Omega)), C) = \bigcup_{y^* \in C^+ \setminus \{0\}} F(y^*)$. Since each $F(y^*)$ is a face of the convex polyhedron $\text{co}(f(\Omega))$ and since every convex polyhedron has only finitely many faces, there exists $y_1^*, \cdots, y_\kappa^* \in C^+ \setminus \{0\}$ and $\hat{y}_1^*, \cdots, \hat{y}_\nu^* \in C^{+i}$ such that

$$\text{(4.2)} \qquad \mathrm{WE}(\mathrm{co}(f(\Omega)), C) = \bigcup_{i=1}^{\kappa} F(y_i^*) \text{ and } \mathrm{Pos}(\mathrm{co}(f(\Omega)), C) = \bigcup_{j=1}^{\nu} F(\hat{y}_j^*).$$

Since every closed convex pointed cone in $\mathbb{R}^n$ has a compact base, there exists a compact convex set Θ in $\mathbb{R}^n$ such that $0 \notin \Theta$ and $C = \mathbb{R}_+\Theta$. Thus, by the Arrow-Barankin-Blackwell density theorem, $\mathrm{E}(\mathrm{co}(f(\Omega)), C) \subset \mathrm{cl}(\mathrm{Pos}(\mathrm{co}(f(\Omega)), C))$. Since each face $F(\hat{y}_j^*)$ is closed and since $\mathrm{E}(\mathrm{co}(f(\Omega)), C)$ contains $\mathrm{Pos}(\mathrm{co}(f(\Omega)), C)$, this and the second equality of (4.2) imply that $\mathrm{E}(\mathrm{co}(f(\Omega)), C) = \bigcup_{j=1}^{\nu} F(\hat{y}_j^*)$. Therefore, by (4.1) and the first equality in (4.2), one has

$$\text{(4.3)} \qquad \mathrm{WE}(f(\Omega), C) = \bigcup_{i=1}^{\kappa} \widehat{F}(y_i^*) \text{ and } \mathrm{E}(f(\Omega), C) = \bigcup_{j=1}^{\nu} \widehat{F}(\hat{y}_j^*).$$

By the compactness of Θ, one has

$$\text{(4.4)} \qquad r_i := \min_{y \in \Theta} \langle y_i^*, y \rangle \geq 0 \text{ and } \hat{r}_j := \min_{y \in \Theta} \langle \hat{y}_j^*, y \rangle > 0 \quad \forall (i,j) \in \overline{1\kappa} \times \overline{1\nu}.$$

For each $k \in \overline{1n}$, let e_k be an element in $\mathbb{R}^n$ such that its kth coordinate is 1 and all the other ones are 0, and let $t_k := \min_{z \in \Theta} \langle e_k, z \rangle$ and $\hat{t}_k := \max_{z \in \Theta} \langle e_k, z \rangle$. Let $\widehat{B} := \{ y \in \mathbb{R}^n : t_k \leq \langle e_k, y \rangle \leq \hat{t}_k \ \ \forall k \in \overline{1n} \}$,

$$\text{(4.5)} \ \widehat{\Theta} := \{ y \in \widehat{B} : r_i \leq \langle y_i^*, y \rangle \text{ and } \hat{r}_j \leq \langle \hat{y}_j^*, y \rangle \quad \forall (i,j) \in \overline{1\kappa} \times \overline{1\nu} \} \text{ and } \widehat{C} = \mathbb{R}_+ \widehat{\Theta}.$$

Then $\widehat{\Theta}$ is a convex polyhedron such that $0 \notin \widehat{\Theta} \supset \Theta$, and hence $\widehat{C}$ is a convex polyhedral pointed cone containing C. It follows that

$$\text{(4.6)} \qquad \mathrm{WE}(f(\Omega), \widehat{C}) \subset \mathrm{WE}(f(\Omega), C) \text{ and } \mathrm{E}(f(\Omega), \widehat{C}) \subset \mathrm{E}(f(\Omega), C).$$

On the other hand, by (4.4) and the definition of $\widehat{C}$, one has

$$y_i^* \in \widehat{C}^+ \ \ \forall i \in \overline{1\kappa} \text{ and } \hat{y}_j^* \in \widehat{C}^{+i} \ \ \forall j \in \overline{1\nu}.$$

Hence $\bigcup_{i=1}^{\kappa} \widehat{F}(y_i^*) \subset \mathrm{WE}(f(\Omega), \widehat{C})$ and $\bigcup_{j=1}^{\nu} \widehat{F}(\hat{y}_j^*) \subset \mathrm{E}(f(\Omega), \widehat{C})$. It follows from (4.3) and (4.6) that $\mathrm{WE}(f(\Omega), C) = \mathrm{WE}(f(\Omega), \widehat{C})$ and $\mathrm{E}(f(\Omega), C) = \mathrm{E}(f(\Omega), \widehat{C})$. This shows that $(\mathrm{PLMOP})_C$ and $(\mathrm{PLMOP})_{\widehat{C}}$ have both the same weak Pareto optimal value set and the same Pareto optimal value set and hence they have both the same weak Pareto solution set and the same Pareto solution set. This shows that (ii) and (iii) hold.

Noting that $\widehat{C}$ contains C, it is trivial that if f is C-convex-like on Ω then f is $\widehat{C}$-convex-like on Ω. Next suppose that f is C-convex. Then

$$f(tx_1 + (1-t)x_2) \leq_C tf(x_1) + (1-t)f(x_2) \quad \forall x_1, x_2 \in X \text{ and } \forall t \in [0,\ 1]$$

and hence

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2 \in C \subset \widehat{C} \quad \forall x_1, x_2 \in X \text{ and } \forall t \in [0,\ 1].$$

It follows that $f(tx_1 + (1-t)x_2) \le_{\widehat{C}} tf(x_1) + (1-t)f(x_2)$ for all $x_1, x_2 \in X$ and $t \in [0,\ 1]$. This shows that f is $\widehat{C}$-convex. The proof is complete. □

For $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, let $[y]_+^C := d(y, -C)$. If $\mathbb{R}^n$ is equipped the l_1-norm and $C = \mathbb{R}^n_+$ then $[y]_+^C = \sum_{i=1}^n [y_i]_+$.

THEOREM 4.2. *Suppose that the objective function f is weakly C-convex on Ω. Then the following statements on piecewise linear problem (PLMOP)$_C$ hold:*
(i) There exist convex polyhedra $Q_1, \cdots, Q_\nu$ in $\mathbb{R}^n$ such that

$$V = \bigcup_{j=1}^{\nu} Q_j, \quad \mathrm{ri}(Q_j) \cap \bigcup_{j' \in \overline{1\nu}\setminus\{j\}} Q_{j'} = \emptyset \text{ for all } j \in \overline{1\nu}, \tag{4.7}$$

and for any $y \in \bigcup_{j=1}^{\nu} \mathrm{ri}(Q_j)$ and any $r \in (0,\ +\infty)$ there exist $L, \delta \in (0,\ +\infty)$ such that

$$S(y_1) \cap rB_X \subset S(y_2) + L\|y_1 - y_2\| B_X \quad \forall y_1, y_2 \in V \cap B(y, \delta). \tag{4.8}$$

(ii) If $S(y) \subset \liminf_{y' \xrightarrow{V} y} S(y')$ for all $y \in V$ then (PLMOP)$_C$ has uniform bounded weak sharp minima with respect to all Pareto optimal values, that is, for any $r > 0$ sufficiently large there exists $\kappa \in (0,\ +\infty)$ such that for any $y = (y_1, \cdots, y_n) \in V \cap rB_{\mathbb{R}^n}$,

$$\kappa d(x, S(y)) \le [f(x) - y]_+^C + \sum_{j=1}^{k} [\varphi_j(x)]_+ \quad \forall x \in rB_X.$$

(iii) (PLMOP)$_C$ always has near uniform bounded weak sharp minima with respect to Pareto optimal values.

Proof. By Theorem 4.1, there exists a convex polyhedral pointed cone $\widehat{C}$ in $\mathbb{R}^n$ containing C such that f is weakly $\widehat{C}$-convex on Ω and (PLMOP)$_C$ and (PLMOP)$_{\widehat{C}}$ have the same Pareto optimal value set V, that is,

$$V = \{y \in f(\Omega) : \ f(\Omega) \cap (y - \widehat{C}) = \{y\}\}. \tag{4.9}$$

Take $y_1^*, \cdots, y_{k'}^* \in \mathbb{R}^n$ such that

$$\widehat{C} = \{y \in \mathbb{R}^n : \ \langle y_i^*, y \rangle \ge 0,\ i = 1, \cdots, k'\},$$

and define $T : \mathbb{R}^n \to \mathbb{R}^{k'}$ as follows

$$T(y) := (\langle y_1^*, y \rangle, \cdots, \langle y_{k'}^*, y \rangle) \quad \forall y \in \mathbb{R}^n.$$

Then, by the weak $\widehat{C}$-convexity of f on Ω, it is easy to verify that $T(f(\Omega)) + \mathbb{R}^{k'}_+$ is a convex set in $\mathbb{R}^{k'}$. Moreover, since $\widehat{C}$ is a pointed cone,

$$\{0\} = \widehat{C} \cap -\widehat{C} = \{y \in \mathbb{R}^n : \ \langle y_i^*, y \rangle = 0,\ i = 1, \cdots, k'\}$$

and hence the linear operator T is injective. This shows that T is a linear isomorphism and topological homeomorphism between $\mathbb{R}^n$ and $T(\mathbb{R}^n)$. Hence there exist $L_1, L_2 \in (0,\ +\infty)$ such that

$$L_1\|y\| \le \|T(y)\| \le L_2\|y\| \quad \forall y \in \mathbb{R}^n. \tag{4.10}$$

Consider the following piecewise linear multiobjective optimization problem

$$\text{(4.11)} \qquad \mathbb{R}^{k'}_+ - \min(T \circ f)(x) \text{ subject to } x \in \Omega.$$

Let V' denote the set of all Pareto optimal values of problem (4.11), that is,

$$V' := \left\{ z \in T(f(\Omega)) : \; T(f(\Omega)) \cap (z - \mathbb{R}^{k'}_+) = \{z\} \right\}.$$

Then, by Lemma 3.3 and the convexity of $T(f(\Omega)) + \mathbb{R}^{k'}_+$, V' is the union of finitely many convex polyhedra in $\mathbb{R}^{k'}$ and hence $V' = \text{cl}(V')$. Moreover, from (4.9) and (4.10), it is easy to verify that $V' = T(V)$ and

$$S(y) = \Omega \cap f^{-1}(y) = \Omega \cap (T \circ f)^{-1}(T(y)) \quad \forall y \in V.$$

Thus, by Corollary 3.7 and (4.10), there exist convex polyhedra $Q'_1, \cdots, Q'_{\hat{n}}$ in $\mathbb{R}^{k'}$ such that

$$\text{(4.12)} \qquad V' = \bigcup_{\upsilon=1}^{\hat{n}} Q'_\upsilon, \quad \text{ri}(Q'_{\hat{\upsilon}}) \cap \bigcup_{\upsilon \in \overline{1\hat{n}} \setminus \{\hat{\upsilon}\}} Q'_\upsilon = \emptyset \quad \forall \hat{\upsilon} \in \overline{1\hat{n}}$$

and for any $y \in T^{-1}\left(\bigcup_{\upsilon=1}^{\hat{n}} \text{ri}(Q'_\upsilon)\right)$ and any $r \in (0, +\infty)$ there exist $L_y, \delta_y \in (0, +\infty)$ such that

$$S(y_1) \cap rB_X \subset S(y_2) + L_y \|T(y_1) - T(y_2)\| \quad \forall y_1, y_2 \in V \cap B(y, \delta_y)$$

and hence $S(\hat{y}) \subset \liminf\limits_{y \xrightarrow{V} \hat{y}} S(y)$ for all $\hat{y} \in V$ with $T(\hat{y}) \in \bigcup\limits_{\upsilon=1}^{\hat{n}} \text{ri}(Q'_\upsilon)$. Let $Q_\upsilon = T^{-1}(Q'_\upsilon)$ for all $\upsilon \in \overline{1\hat{n}}$. Then, by the linearity of T and (4.10), each Q_υ is a convex polyhedron in $\mathbb{R}^n$, $V = \bigcup_{\upsilon=1}^{\hat{n}} Q_\upsilon$, $\text{ri}(Q_{\hat{\upsilon}}) \cap \bigcup\limits_{\upsilon \in \overline{1\hat{n}} \setminus \{\hat{\upsilon}\}} Q_\upsilon = \emptyset$ for all $\hat{\upsilon} \in \overline{1\hat{n}}$, and (4.8) holds with $L = L_2 L_y$ and $\delta = \delta_y$. This shows that (i) holds, and hence $S(\hat{y}) \subset \liminf\limits_{y \xrightarrow{V} \hat{y}} S(y)$ for all $\hat{y} \in \bigcup\limits_{\upsilon=1}^{\hat{n}} \text{ri}(Q_\upsilon)$. On the other hand, by (4.10) and $T(\widehat{C}) \subset \mathbb{R}^{k'}_+$, one has

$$\begin{aligned} [T(f(x)) - T(y)]^{\mathbb{R}^{k'}_+}_+ &= d_{\|\cdot\|_1}(T(f(x)) - T(y), -\mathbb{R}^{k'}_+) \\ &\le \sqrt{k'} d(T(f(x)) - T(y), -\mathbb{R}^{k'}_+) \\ &\le \sqrt{k'} d(T(f(x)) - T(y), -T(\widehat{C})) \\ &\le \sqrt{k'} L_2 d(f(x) - y, -\widehat{C}) \end{aligned}$$

for all $x \in X$ and $y \in V$. Since $\widehat{C}$ contains C,

$$d(f(x) - y, -\widehat{C}) \le d(f(x) - y, -C) = [f(x) - y]^C_+ \quad \forall (x, y) \in X \times V.$$

Hence

$$[T(f(x)) - T(y)]^{\mathbb{R}^{k'}_+}_+ \le \sqrt{k'} L_2 [f(x) - y]^C_+ \quad \forall (x, y) \in X \times V.$$

It follows from Theorems 3.4 and 3.6 that (ii) and (iii) hold. The proof is complete. □

Under the convexity assumption, we have the following uniform global weak sharp minima result.

THEOREM 4.3. *Suppose that the objective function f is C-convex and that $\psi_1, \cdots, \psi_k$ is convex. Then* $(\text{PLMOP})_C$ *has uniform global weak sharp minima with respect to all Pareto optimal values, that is, there exists $\tau > 0$ such that*

$$\tau d(x, S(y)) \leq [f(x) - y]_+^C + \sum_{j=1}^{k} [\psi_j(x)]_+ \quad \forall x \in X \text{ and } \forall y \in V. \tag{4.13}$$

Proof. By Theorem 4.1, there exist a convex polyhedral pointed cone $\widehat{C}$ in $\mathbb{R}^n$ containing C such that $(\text{PLMOP})_C$ and $(\text{PLMOP})_{\widehat{C}}$ have the same Pareto optimal value set V and f is $\widehat{C}$-convex. Define $\widehat{F} : X \rightrightarrows \mathbb{R}^n \times \mathbb{R}^k$ as follows

$$\widehat{F}(x) = (f(x) + \widehat{C}) \times \left((\psi_1(x), \cdots, \psi_k(x)) + \mathbb{R}_+^k\right) \quad \forall x \in X. \tag{4.14}$$

From the convexity of piecewise linear functions f and each ψ_j, it is easy to verify that $\widehat{F}$ is a convex polyhedral multifunction from X to $\mathbb{R}^n \times \mathbb{R}^k$; and moreover

$$\widehat{F}^{-1}(y, 0) = \{x \in \Omega : \ f(x) \leq_{\widehat{C}} y\} = \Omega \cap f^{-1}(y - \widehat{C}) \quad \forall y \in \mathbb{R}^n.$$

Hence

$$\widehat{F}^{-1}(y, 0) = S(y) \quad \forall y \in V. \tag{4.15}$$

By Lemma 3.2, there exists $\tau \in (0, +\infty)$ such that

$$\tau d\left(x, \widehat{F}^{-1}(y, 0)\right) \leq d_{\|\cdot\|_e}\left((y, 0), \widehat{F}(x)\right) \quad \forall (x, (y, 0)) \in X \times \widehat{F}(X),$$

where $\|(s_1, \cdots, s_n, t_1, \cdots, t_k)\|_e := \|(s_1, \cdots, s_n)\| + \sum_{j=1}^{k} |t_j|$ for all $(s_1, \cdots, s_n, t_1, \cdots, t_k)$ in $\mathbb{R}^n \times \mathbb{R}^k$. Noting that $V \times \{0\} \subset \widehat{F}(X)$, it follows from (4.15) that (4.13) holds. The proof is complete. □

5. Conclusions. Under the piecewise linear assumption on the objective function and the constrained functions, this paper establishes several existence results on global uniform weak sharp minima and bounded uniform weak sharp minima for multiobjective optimization problem (MOP). It should be interesting further to consider how one find an exact constant κ in the inequalities appearing in the definition of uniform weak sharp minima. We guess such an exact constant can be estimated or obtained by using the coderivative of the vector-valued objective function and the subdifferential of constrained functions. Moreover, one may also consider the uniform weak sharp minima results obtained in this paper whether can be translated to the other setting from the piecewise linear setting. In particular, one can further consider the uniform weak sharp minima for (MOP) in the setting that the objective function is quadratic and the constrained functions are linear (or piecewise linear).

Imitating the definition of a minimizing sequence in the scalar optimization, we can adopt the following notion for multiobjective optimization problem (MOP): *For a optimal value y in V (or V_w), a sequence $\{x_k(y)\}$ in the feasible set Ω is said to be* *a minimizing sequence of* (MOP) *with respect to* y *if* $\lim_{k\to\infty} d(y, f(x_k(y)) + \mathbb{R}^n_+) = 0$. In the special case that $n = 1$ and $C = \mathbb{R}_+$, $V = V_w = \{\inf_{x\in\Omega} f(x)\}$ is a singleton and this minimizing sequence notion for (MOP) is just the one for a scalar optimization problem. In practical applications, one may be interested in the set A of some specific Pareto optimal values (or some representative Pareto optimal values). The uniform weak sharp minima for (MOP) implies that all minimizing sequences $\{x_k(y)\}$ with respect to Pareto optimal values y in A approach the solution sets $S(y)$ (corresponding to y) in a uniform way for all y in A as $k \to \infty$. In the future research, we will study convergence analysis of some algorithms for (MOP) based on the uniform weak sharp minima.

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REFERENCES

[1] K.J. Arrow, E.W. Barankin and D. Blackwell, Admissible points of convex sets, Contributions to the theory of games, 2(1953), pp.87-91.
[2] H. Bonnel, A.N. Iusem and B.F. Svaiter, Proximal methods in vector optimization. SIAM J. Optim. 15(2005), pp.953-970.
[3] J.M. Borwein and D. Zhuang, Super efficiency in vector optimization. Trans. Am. Math. Soc. 338 (1993), pp.105- 122.
[4] J.V. Burke and M. C. Ferris, Weak sharp minima in mathematical programming, SIAM J. Control Optim., 31 (1993), pp. 1340-1359.
[5] G.A. Carrizo, P.A. Lotito and M.C. Maciel, Trust region globalization strategy for the nonconvex unconstrained multiobjective optimization problem. Math. Program. 159(2016), pp.339-369.
[6] S. Deng and X. Yang, Weak sharp minima in multicriteria linear programming. SIAM J. Optim. 15 (2005), pp.456-460.
[7] K. Deb and H. Jain, An evolutionary many-objective optimization algorithm using reference-point-based nondominated sorting approach, part I: solving problems with box constraints. IEEE. Trans. Evol. Comput., 18(2013), pp. 577-601.
[8] A.L. Dontchev and R.T. Rockafellar, Implicit Functions and Solution Mappings, Springer, Berlin, 2009.
[9] G. Eichfelder, P. Kirst, et al. A general branch-and-bound framework for continuous global multiobjective optimization. J. Glob. Optim., 80(2021), pp.195-227.
[10] G. Eichfelder, E. Quintana and S. Rocktaschel, A vectorization scheme for nonconvex set optimization problems. SIAM J. Optim. 32(2022), pp.1184-1209.
[11] Y. Fang, K. Meng, X. Yang, Polyhedral multi-criteria programs: the continuous case and its discontinuous generalization. Oper. Res. 60(2012), pp.398-409.
[12] M. C. Ferris, Weak Sharp Minima and Penalty Functions in Mathematical Programming, Ph.D. Thesis, University of Cambridge, Cambridge, 1988.
[13] J. Fliege, L. M. G. Drummond, and B. F. Svaiter, Newton's method for multiobjective optimization, SIAM J. Optim., 20 (2009), pp. 602–626.
[14] J. Handl, D.B. Kell and J. Knowles, Multiobjective optimization in bioinformatics and computational biology. IEEE/ACM Trans. Comput. Biol. Bioinform. 4(2007), pp.279-292.
[15] R.J. Gallagher and O.A. Saleh, Two generalizations of a theorem of Arrow, Barankin, and Blackwell. SIAM J. Control Optim. 31(1993), pp.247-256.
[16] R. Hartley, On cone efficiency, cone convexity, and cone compactness. SIAM J. Appl. Math. 34(1978), pp.211C222.
[17] L. Huerga, B. Jimenez, D.T. Luc, V. Novo, A unified concept of approximate and quasi efficient solutions and associated subdifferentials in multiobjective optimization, Math. Program., 189(2021), pp.379-407.
[18] K. Klamroth, R. Lacour, D. Vanderpooten, On the representation of the search region in multi-objective optimization. Eur. J. Oper. Res., 245(2015), pp.767-778.
[19] J. Jahn, Vector Optimization: Theory, Applications and Extensions, Springer-Verlag 2004.
[20] D.S. Kim, B.S. Mordukhovich, T.S. Pham, N.V. Tuyen, Existence of efficient and properly effi-

cient solutions to problems of constrained vector optimization, Math. Program., 190(2021), pp.259-283.

[21] A. S. Lewis and J. S. Pang, Error bounds for convex inequality systems, In: Crouzeix, J. P., ed., Generalized Convexity, Proceedings of the Fifth Sysposium on Generalized Convexity, Luminy Marseille, 1997, pp. 75-100.

[22] R. Lopez, Stability results for polyhedral complementarity problems. Comput. Math. Appl. 58 (2009), pp.1475-1486.

[23] N. N. Luan, J-C. Yao and N. D. Yen, On some generalized polyhedral convex constructions, Numer. Funct. Analy. Optim. 39 (2018), pp.537-570.

[24] D. T. Luc, Multiobjective Linear Programming-An Introduction, Springer, Cham 2016.

[25] L.R. Lucambio Pérez and L.F. Prudente, Nonlinear conjugate gradient methods for vector optimization. SIAM J. Optim. 20(2018), pp. 2690-2720.

[26] R.T. Marler and J.S. Arora, Survey of multi-objective optimization methods for engineering. Struct. Multidisc. Optim. 26(2004), pp.369-395.

[27] H. V. Ngai, M. Théra, Error bounds for systems of lower semicontinuous functions in Asplund spaces, Math. Program., 116(2009), pp. 397-427.

[28] J. Niebling and G. Eichfelder, A branch-and-bound-based algorithm for nonconvex multiobjective optimization. SIAM. J. Optim., 29(2019), pp.794-821.

[29] J. S. Pang, Error bounds in mathematical programming, Math. Program., 79(1997), pp. 299-332.

[30] P.M. Reed, D. Hadka, J.D. Herman, J.R. Kasprzyk and J.B. Kollat, Evolutionary multiobjective optimization in water resources: the past, present, and future. Adv. Water Resour. 51(2013), pp.438-456.

[31] R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ (1970).

[32] S.M. Robinson, Regularity and stability for convex multivalued functions, Math. Oper. Res., 1 (1976), pp. 130-143.

[33] M.D. Santis, G. Eichfelder, J. Niebling, S. Rocktaschel, Solving Multiobjective Mixed Integer Convex Optimization Problems. SIAM J. Optim. 30(2020), pp.3122-3145.

[34] M. Tapia and C. Coello, Applications of multi-objective evolutionary algorithms in economics and finance: a survey. In: IEEE Congress on Evolutionary Computation, pp. 532-539 (2007).

[35] J. Wang, Y. Hu, C.K.W. Yu, C. Li and X. Yang, Extended Newton methods for multiobjective optimization: majorizing function technique and convergence analysis, SIAM J. Optim. 29 (2019), pp.2388-2421.

[36] Z. Wu and J.J. Ye, On error bounds for lower semicontinuous functions, Math. Program., 92(2002), pp.301-314.

[37] X. Yang and N.D. Yen, Structure and weak sharp minimum of the Pareto solution set for piecewise linear multiobjective optimization. J. Optim. Theory Appl. 147(2010), pp.113-124.

[38] J.J. Ye, X. Yuan, S. Zheng, J. Zhang, Variational analysis perspective on linear convergence of some first order methods for nonsmooth convex optimization problems. Set-Valued Var. Anal. 29 (2021), pp. 803-837.

[39] C. Zalinescu, Weak sharp minima, well-behaving functions and global error bounds for convex inequalities in Banach spaces, Proc. 12th Baikal Internat. Conf. on Optimization Methods and Their Appl. Irkutsk, Russia, 2001, pp.272-284.

[40] X.Y. Zheng and K.F. Ng, Metric subregularity of piecewise linear multifunctions and applications to piecewise linear multiobjective optimization. SIAM J. Optim. 24 (2014), pp.154-174.

[41] X.Y. Zheng, X. Yang, Weak sharp minima for piecewise linear multiobjective optimization in normed spaces. Nonlinear Anal. 68 (2008), pp.3771-3779.

[42] X.Y. Zheng and X. Yang, Fully piecewise linear vector optimization problems. J. Optim. Theory Appl. 190 (2021), pp. 461-490.

[43] D. Zhuang, Density results for proper efficiency. SIAM J. Control Optim. 32(1994), pp.51-58.