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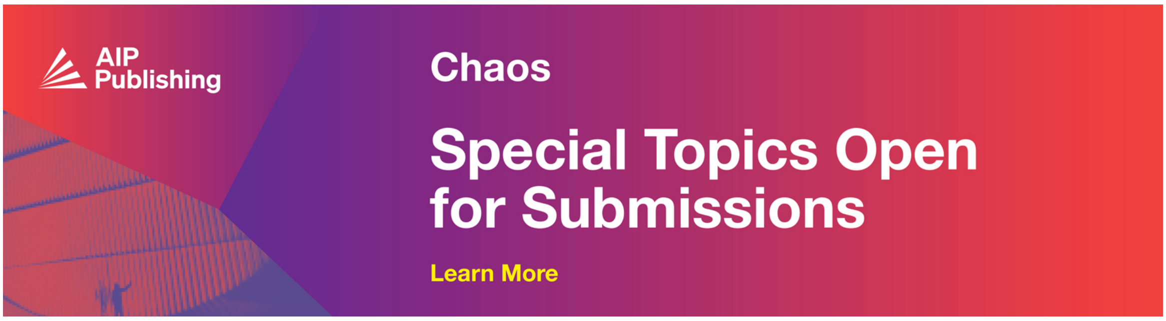

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ABSTRACT

This paper addresses the asynchronous leader-following consensus problem for networked double-integrator systems. In practical engineering contexts, there are three key factors that must be considered significantly: (1) asynchronous hybrid event- and time-triggered control, where asynchrony affects event detection, event-triggered processes, and controller updates; (2) heterogeneous networks, wherein position and velocity information are governed by distinct, independent graphs; and (3) communication time delays arising from limited bandwidth and long-distance transmission. Due to the independence of these heterogeneous networks, edge events related to position and velocity information are defined separately. When an event occurs on an edge, the connected agents sample the corresponding relative state information (position or velocity) and update their controllers accordingly. The paper proposes a control protocol based on these event rules and employs Lyapunov methods to address the leader-following consensus problem. Numerical simulations are provided to validate and illustrate the theoretical findings.

In multi-agent system control, the interaction between agents and data sampling is fundamental and central to ongoing research. Unlike traditional approaches with continuous communications and periodic sampling, event-triggered control offers a more efficient alternative that has attracted significant attention from researchers and engineers due to its potential to reduce energy consumption. To design an effective event-triggered control mechanism, it is crucial to accurately characterize operational demands posed by real-world systems. Specifically, this paper focuses on the complexities of asynchronous hybrid event- and time-triggered control, where asynchrony impacts event detection, triggering processes, and controller updates; the use of heterogeneous networks, where position and velocity information are governed by distinct, independent communication graphs; and the challenges of communication delays resulting from bandwidth limitations and long-distance transmission. Within this framework, we propose an innovative edge-based event-triggered approach that introduces two new event-detecting rules, leading to a novel control scheme that ensures leader-following consensus despite these practical constraints.

I. INTRODUCTION

The study of multi-agent systems (MAS) has emerged as a cornerstone of modern control theory, driven by its applications across various fields such as autonomous vehicles,[1,2] distributed robotics,[3,4] and smart grids.[5,6] At the heart of this research lies the consensus problem, which seeks to align the states or actions of individual agents in a decentralized system.[7–15] Among the numerous paradigms within this domain, the leader-following consensus problem has gained prominence.[16–25] In this framework, a designated leader dictates a reference trajectory, while the followers, through carefully designed control laws, synchronize their behavior to converge toward the leader's state.

Recent advances in control strategies have highlighted the importance of communication-efficient methods, as traditional continuous-time control schemes often suffer from excessive resource demands, particularly in bandwidth-constrained environments.[26–29] Event-triggered control has established itself as a powerful alternative, allowing communications and controller updates to occur only when certain predefined conditions are met, significantly reducing redundant communications, and offering a balance between system performance and resource optimization, compared to standard periodic sampling methods.[30–34] Pioneering works, such as those in Ref. 37, introduced both centralized and distributed event-triggered control frameworks, addressing consensus problems of multi-agent systems. Further extensions in Ref. 38 explored the challenges of quantized data transmission and asynchronous information updates, reinforcing the versatility of event-triggered schemes in complex MAS environments. Differing from Refs. 37–44, a novel event-triggered framework based on edge events was proposed in Ref. 45. Instead of each agent independently monitoring its state, events are triggered when a specific condition involving the state difference between two adjacent agents is satisfied. Once an event is activated, the connected agents sample their relative state information and update their control actions simultaneously based on this newly acquired data.

Unlike homogeneous networks, where agents share the same communication structure, heterogeneous networks involve multiple types of interactions through different sensors or communication channels, leading to a more complex consensus behavior. While homogeneous MAS have been extensively studied, particularly in systems governed by double-integrator dynamics, the behavior of heterogeneous systems largely remains underexplored. Previous works such as Refs. 35 and 36 have addressed consensus in double-integrator networks under various assumptions regarding velocity measurement and network connectivity, in which continuous-time sampling played a critical role in achieving consensus; but such approaches can be resource-intensive in practice. To address these limitations, we propose a novel edge-event-triggered control mechanism tailored to leader-following consensus in heterogeneous multi-agent systems. Moreover, it is crucial to accurately characterize the entire event-triggering process to ensure effective system performance and control. Traditional approaches often assume that all agents detect event conditions simultaneously,[45–47] but this assumption is both strong and unrealistic. In practice, due to factors like time jitter[48] and network latencies,[49] event detections across agents are inherently asynchronous, leading to asynchronous event triggering and controller updates. Furthermore, limitations in communication network bandwidth and the effects of long-distance data transmission exacerbate the challenge, as agents receive and use the latest state information from their neighbors with delays.[14,16,21] These issues significantly impact system performance and stability. Consequently, there is a critical need to investigate control strategies that can handle these asynchronous dynamics and communication delays.

In this paper, we address the asynchronous leader-following consensus problem for double-integrator networks with communication delays, using hybrid periodic edge-event-triggered control. The proposed approach involves each pair of adjacent agents detecting event conditions asynchronously and independently. In the context of double-integrator networks, we define event conditions separately for position and velocity information. Even if there are connections between two adjacent agents in both the position and velocity information networks, events related to different types of information remain independent. When an event is triggered, the agents involved update only the relevant information (position or velocity) in their controllers. We introduce a protocol with specifically designed event-detecting rules to achieve leader-following consensus in heterogeneous double-integrator networked systems with time delays. Additionally, we derive sufficient conditions for leader-following consensus by employing Lyapunov methods.

The rest of this paper is organized as follows. In Sec. II, we outline the key concepts of algebraic graph theory and introduce the specific model along with the edge-event rules. In Sec. III, we provide the main results in detail. In Sec. IV, we provide a simulation to demonstrate the effectiveness of our approach. In Sec. V, we conclude with a summary of our findings and discuss potential future research directions.

Notations: $\underline{n}$ denotes the set {1, 2, …, n} and $\underline{n}/1$ denotes {2, 3, …, n}. $\mathbf{0}$ is a zero vector with the appropriate dimensions. I_n denotes an $n \times n$ identity matrix. $\lambda_2(A)$ denotes the second smallest eigenvalue of matrix A. For matrices A, B, $A > B$ means $A - B$ is positive definite.

II. PRELIMINARIES

In this paper, we study the multi-agent system with $n+1$ double-integrator individuals, labeled from 1 to $n+1$, where the agent 1 is assigned as the leader, and the others as followers. The communication networks based on position information and velocity information are usually treated as the same. Here, we assume that the position information network and the velocity information network are modeled by different graphs G_p and G_v, respectively. In G_p and G_v, the information flow among the followers is undirected, but the information flow from the leader to its followers is unidirectional, i.e., the leader cannot receive information from the followers. Denote the common vertex set of G_p and G_v as $\mathscr{V} = \{\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_{n+1}\}$. Assume that there are m_p and m_v edges in G_p and G_v, and the corresponding edge sets are denoted by $E_p = \left\{e_1^p, e_2^p, \ldots, e_{m_p}^p\right\}$ and $E_v = \left\{e_1^v, e_2^v, \ldots, e_{m_v}^v\right\}$, respectively, where $e_q^p = (\mathscr{V}_{i_1}, \mathscr{V}_{j_1})$ [resp. $e_q^v = (\mathscr{V}_{i_2}, \mathscr{V}_{j_2})$] means agent i_1 (resp. i_2) and agent j_1 (resp. j_2) are connected by edge e_q^p (resp. e_q^v) in G_p (resp. G_v). Agent j is a neighbor of agent i, if agent i can receive the information of agent j, that is, $(\mathscr{V}_j, \mathscr{V}_i) \in E_p$ or E_v. The sets of neighbors of agent i in G_p and G_v are denoted by N_i^p and N_i^v, respectively. Edge weight matrices of G_p and G_v are denoted by $W_p = \mathrm{diag}\left(w_1^p, w_2^p, \ldots, w_{m_p}^p\right)$ and $W_v = \mathrm{diag}(w_1^v, w_2^v, \ldots, w_{m_v}^v)$, where $w_{q_1}^p$ and $w_{q_2}^v$ are the weights of edges $e_{q_1}^p$ and $e_{q_2}^v$, respectively. Based on graphs G_p and G_v, we construct virtual undirected graphs $\tilde{G}_p = (\mathscr{V}, \tilde{E}_p, W_p)$ and $\tilde{G}_v = (\mathscr{V}, \tilde{E}_v, W_v)$, where $\tilde{E}_p = E_p \cup \{(\mathscr{V}_i, \mathscr{V}_1) | (\mathscr{V}_1, \mathscr{V}_i) \in E_p, i \in \underline{n+1}/1\}$ and $\tilde{E}_v = E_v \cup \{(\mathscr{V}_i, \mathscr{V}_1) | (\mathscr{V}_1, \mathscr{V}_i) \in E_v, i \in \underline{n+1}/1\}$. In this paper, we assume that there exists a path from the leader through each follower. That is to say, all followers are reachable from the leader, i.e., the network is leader–follower reachable. Therefore, $\tilde{G}_p$ and $\tilde{G}_v$ are both connected. Afterward, define Laplacian matrix $L_p = [l_{ij}^p]$ of G_p as follows:

$$l_{ij}^p = \begin{cases} -a_{ij}^p, & \text{if } i \neq j, \\ \sum_{k \in N_i^p} a_{ik}^p, & \text{if } i = j, \end{cases}$$

where $a_{ij}^p > 0$, if $(\mathscr{V}_j, \mathscr{V}_i) \in E_p$, otherwise, $a_{ij}^p = 0$. For the edges between followers, assign an arbitrary orientation and define the incidence matrix $D_p = [d_{ij}^p]$ of G_p as follows:

$$d_{ij}^p = \begin{cases} -1, & \text{if } \mathscr{V}_i \text{ is the tail of the } j\text{th oriented edge in } G_p, \\ 1, & \text{if } \mathscr{V}_i \text{ is the head of the } j\text{th oriented edge in } G_p, \\ 0, & \text{otherwise.} \end{cases}$$

Analogously, by introducing a_{ij}^v, we can define the Laplacian matrix L_v and the incidence matrix D_v of G_v, as well as the Laplacian matrix $\tilde{L}_p$ (resp. $\tilde{L}_v$) of $\tilde{G}_p$ (resp. $\tilde{G}_v$). Obviously, $\tilde{L}_p = D_p^T W_p D_p$ and $\tilde{L}_v = D_v^T W_v D_v$ hold.

This paper considers the asynchronous event detection. Let $t_k^{p(q)}$ or $t_k^{p(ij)}, k = 0, 1, 2, \ldots$ denote the event-detecting times of the edge $e_q^p = (\mathscr{V}_i, \mathscr{V}_j)$ with $t_{k+1}^{p(q)} = t_k^{p(q)} + h$ or $t_{k+1}^{p(ij)} = t_k^{p(ij)} + h$, where h is the event-detecting period. Similarly, introduce the event-detecting times of the edge $e_q^v = (\mathscr{V}_i, \mathscr{V}_j)$ in G_v as $t_k^{v(q)}$ or $t_k^{v(ij)}, k = 0, 1, 2, \ldots$ with $t_{k+1}^{v(q)} = t_k^{v(q)} + h$ or $t_{k+1}^{v(ij)} = t_k^{v(ij)} + h$.

At event-detecting times, followers connected by the edge detect the corresponding event conditions. If the conditions are satisfied, the followers sample the corresponding relative state information on the edge and update their controllers. Note that the leader cannot receive information from its followers, and its controller remains unchanged. Therefore, if the edge connects the leader and a follower, only the follower performs event detection and updates its controller. Due to the independence of the position information network and the velocity information network, event detection and controller updates based on position information and velocity information are mutually independent. For example, at time $t_k^{p(ij)}$, agent j checks the event conditions of the edge $(\mathscr{V}_i, \mathscr{V}_j)$ in the network G_p. If the event occurs, agent j samples the relative position data and updates position information of its controller. In the process, agent j whether or not to update velocity information of its controller depends not only on whether there exists an edge connecting agent j in network $G_v$whose event-detecting time also occurs at $t_k^{p(ij)}$, but also on whether the event occurs.

For the sake of simplicity, let $t_{k^{p(q)}(t)}^{p(q)}$ or $t_{k^{p(ij)}(t)}^{p(ij)}$ and $t_{k_{p(q)}(t)}^{p(q)}$ or $t_{k_{p(ij)}(t)}^{p(ij)}$ denote the recent edge-event time and the recent event-detecting time for edge $e_q^p = (\mathscr{V}_i, \mathscr{V}_j)$ in G_p before or at time t, respectively. For example, in Fig. 1, we assume there are three agents and three edges in G_p. From the vertical, if there exist a line between two agents at some time, it is the event-detecting time of the edge. Furthermore, if the line is red, it indicates event of the edge occurs; otherwise, the event is not triggered. For the edge connecting agents 1 and 3, the corresponding event occurs at the recent detecting time before t. For the edge connecting agents 1 and 2, there are three event-detecting times, in which only at the second one, the event occurs. Therefore, $t_{k^{p(12)}(t)}^{p(12)}$ corresponds to the second event-detecting time. For the meanings of $t_{k_{p(12)}(t)}^{p(12)}$, $t_{k_{p(13)}(t)}^{p(13)}$, and $t_{k_{p(23)}(t)}^{p(23)}$, they are clear in Fig. 1.

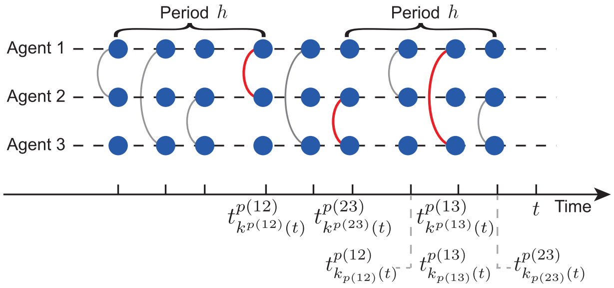

FIG. 1. Asynchronous edge-event detections.

Likewise, let $t_{k^{v(q)}(t)}^{v(q)}$ or $t_{k^{v(ij)}(t)}^{v(ij)}$ and $t_{k_{v(q)}(t)}^{v(q)}$ or $t_{k_{v(ij)}(t)}^{v(ij)}$ denote the recent edge-event time and the recent event-detecting time for edge $e_q^v = (\mathscr{V}_i, \mathscr{V}_j)$ in G_v before or at time t, respectively. Mathematically, $k^{p(q)}(t) = \max\{k | t_k^{p(q)} \leq t, \text{an edge event of } e_q^p \text{ occurs at } t_k^{p(q)}\}$; $k_{p(q)}(t) = \max\{k | t_k^{p(q)} \leq t\}$; $k^{v(q)}(t) = \max\{k | t_k^{v(q)} \leq t, \text{ an edge event of } e_q^v \text{ occurs at } t_k^{p(q)}\}$; $k_{v(q)}(t) = \max\{k | t_k^{v(q)} \leq t\}$.

A. Model

Consider leader–follower multi-agent systems as follows:

- Leader:

$$\begin{cases} \dot{x}_1(t) = v_1(t), \\ \dot{v}_1(t) = 0, \end{cases} \tag{1}$$

- Followers:

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), \end{cases} \qquad i \in \underline{n+1}/1 \tag{2}$$

where $x_i(t)$ and $v_i(t)$ are the position and velocity of agent i, respectively. The protocol $u_i(t)$ takes the following form:

$$\begin{aligned} u_i(t) = & \sum_{j \in N_i^p} a_{ij}^p \left(x_j \left(t_{k^{p(ij)}(t-\tau)}^{p(ij)} \right) - x_i \left(t_{k^{p(ij)}(t-\tau)}^{p(ij)} \right) \right) \\ & + k \sum_{j \in N_i^v} a_{ij}^v \left(v_j \left(t_{k^{v(ij)}(t-\tau)}^{v(ij)} \right) - v_i \left(t_{k^{v(ij)}(t-\tau)}^{v(ij)} \right) \right), \end{aligned} \tag{3}$$

where $k > 0$ is a parameter to be designed later and τ is the communication time delay with $\tau < h$. In many physical systems, delays (such as sensor response time and signal transmission delay) are typically much smaller than the sampling period, as the sampling period is usually set as part of the system's response time to ensure adequate feedback and control accuracy. Therefore, assuming that the time delay is smaller than the sampling period aligns with the response characteristics and control requirements of practical physical systems. Therefore, from (3), relative state information used by agent i is sampled at the most recent corresponding edge-event times before or at $t - \tau$. In additions, based on the definition of the recent edge-event time $t^{p(ij)}_{k^{p(ij)}(t-\tau)}$ (or $t^{v(ij)}_{k^{v(ij)}(t-\tau)}$), it can be observed that the controller of agent i relies on the position (or velocity) information of neighbor j, which is derived from data sampled at the recent event-triggered time before or at time $t-\tau$, rather than the latest sampled data before or at the current time t. Within the interval $(t-\tau, t]$, events may occur on the corresponding edge, according to the event rules. Consequently, communication delays inevitably affect the state update rate, further influencing the corresponding event-triggering times and frequencies. Systems (1) and (2) are said to achieve leader-following consensus, if $\lim_{t\to\infty} x_i(t) - x_1(t) = 0, \lim_{t\to\infty} v_i(t) - v_1(t) = 0, i \in \underline{n+1}/1$.

Introduce variables $y(t) = D_p^T x(t)$ and $z(t) = D_v^T v(t)$, where $y(t) = [y_1(t)y_2(t)\dots y_{m_p}(t)]^T, z(t) = [z_1(t)z_2(t)\dots z_{m_v}(t)]^T$, $x(t) = [x_1(t)x_2(t)\dots x_{n+1}(t)]^T$, and $v(t) = [v_1(t)v_2(t)\dots v_{n+1}(t)]^T$. According to the definition of incidence matrix D_p, for any $y_{q_1}(t), q_1 \in \underline{m_p}$, there exist a pair of $x_{i_1}(t), x_{j_1}(t)$, such that $y_{q_1}(t) = x_{i_1}(t) - x_{j_1}(t)$ with $d^p_{i_1q_1} = 1$ and $d^p_{j_1q_1} = -1$. Apparently, the variable $y_{q_1}(t)$ denotes the relative position information of the adjacent agents i_1, j_1 connected by $e^p_{q_1}$ at time t. Similarly, the variable $z_{q_2}(t), q_2 \in \underline{m_v}$ denotes the relative velocity information of a pair of adjacent agents connected by $e^v_{q_2}$ in G_v at time t. Furthermore, let $\hat{y}_{q_1}(t) = x_{i_1}\left(t^{p(i_1j_1)}_{k^{p(i_1j_1)}(t)}\right) - x_{j_1}\left(t^{p(i_1j_1)}_{k^{p(i_1j_1)}(t)}\right)$ with $e^p_{q_1} = (\mathscr{V}_{i_1}, \mathscr{V}_{j_1})$ and $\hat{z}_{q_2}(t) = v_{i_2}\left(t^{v(i_2j_2)}_{k^{v(i_2j_2)}(t)}\right) - v_{j_2}\left(t^{v(i_2j_2)}_{k^{v(i_2j_2)}(t)}\right)$ with $e^v_{q_2} = (\mathscr{V}_{i_2}, \mathscr{V}_{j_2})$. From $t^{p(i_1j_1)}_{k^{p(i_1j_1)}(t)}$ and $t^{v(i_2j_2)}_{k^{v(i_2j_2)}(t)}$, variables $\hat{y}_{q_1}(t)$ and $\hat{z}_{q_2}(t)$ denote the recent sampled position information on $e^p_{q_1}$ and the recent sampled velocity information on $e^v_{q_2}$, respectively.

Then, systems (1) and (2) with protocol (3) can be compactly expressed as follows:

$$\begin{cases} \dot{x}(t) = v(t), \\ \dot{v}(t) = -I_0 D_p W_p \hat{y}(t-\tau) - kI_0 D_v W_v \hat{z}(t-\tau), \end{cases} \tag{4}$$

where $I_0 = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix}$, $\hat{y}(t) = [\hat{y}_1(t)\hat{y}_2(t)\dots\hat{y}_{m_p}(t)]^T$ and $\hat{z}(t) = [\hat{z}_1(t)\hat{z}_2(t)\dots\hat{z}_{m_v}(t)]^T$.

B. Edge-event-detecting rules

Here, we present the following edge-event-detecting rules:

Rule A:

(A-I) At edge-event-detecting time $t^{p(i_1j_1)}_k$ of edge $e^p_{q_1} = (\mathscr{V}_{i_1}, \mathscr{V}_{j_1})$ in G_p, agent j_1 checks the following inequalities. If any of them is not satisfied, agent j_1 samples their relative position data with agent i_1 and update its controller accordingly. If agent i_1 does not play the role of the leader, it performs the role of agent j_1:

(I-1) if $\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right) \geq 0$,

$$a\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right) < y_{q_1}\left(t^{p(i_1j_1)}_{k}\right) < b\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right),$$

(I-2) if $\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right) < 0$,

$$b\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right) < y_{q_1}\left(t^{p(i_1j_1)}_{k}\right) < a\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right),$$

(A-II) at edge-event-detecting time $t^{v(i_2j_2)}_k$ of edge $e^v_{q_2} = (\mathscr{V}_{i_2}, \mathscr{V}_{j_2})$ in G_v, agent j_2 checks the following inequalities. If any of them is not satisfied, agent j_2 samples their relative velocity data with agent i_2 and update its controller accordingly. If agent i_2 does not play the role of the leader, it performs the role of agent j_2:

(II-1) if $\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right) \geq 0$,

$$c\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right) < z_{q_2}\left(t^{v(i_2j_2)}_{k}\right) < d\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right),$$

(II-2) if $\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right) < 0$,

$$d\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right) < z_{q_2}\left(t^{v(i_2j_2)}_{k}\right) < c\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right),$$

where $0 < a, c \leq 1$ and $b, d \geq 1$.

Rule B:

(B-I) At edge-event-detecting time $t^{p(i_1j_1)}_k$ of edge $e^p_{q_1} = (\mathscr{V}_{i_1}, \mathcal{V}_{j_1})$ in G_p with $\hat{y}_{q_1}(t) = x_{i_1}\left(t^{p(i_1j_1)}_{k^{p(i_1j_1)}(t)}\right) - x_{j_1}\left(t^{p(i_1j_1)}_{k^{p(i_1j_1)}(t)}\right)$, agent j_1 checks the following inequalities. If any of them is not satisfied, agent j_1 samples their relative position data with agent i_1 and update its controller accordingly. If agent i_1 does not play the role of the leader, it performs the role of agent j_1:

(I-1) if $\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right) \geq 0$,

$$-\frac{1-a}{2}\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right) < x_{i_1}\left(t^{p(i_1j_1)}_{k}\right) - x_{i_1}\left(t^{p(i_1j_1)}_{k^{p(i_1j_1)}\left(t^{p(i_1j_1)}_{k-1}\right)}\right) < \frac{b-1}{2}\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right),$$

(I-2) if $\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right) < 0$,

$$\frac{b-1}{2}\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right) < x_{i_1}\left(t^{p(i_1j_1)}_{k}\right) - x_{i_1}\left(t^{p(i_1j_1)}_{k^{p(i_1j_1)}\left(t^{p(i_1j_1)}_{k-1}\right)}\right) < -\frac{1-a}{2}\hat{y}_{q_1}\left(t^{p(i_1j_1)}_{k-1}\right),$$

(B-II) At edge-event-detecting time $t^{v(i_2j_2)}_k$ of edge $e^v_{q_2} = (\mathscr{V}_{i_2}, \mathscr{V}_{j_2})$ in G_v with $\hat{z}_{q_2}(t) = v_{i_2}\left(t^{v(i_2j_2)}_{k^{v(i_2j_2)}(t)}\right) - v_{j_2}\left(t^{v(i_2j_2)}_{k^{v(i_2j_2)}(t)}\right)$, agent j_2 checks the following inequalities. If any of them is not satisfied, agent j_2 samples their relative velocity data with agent i_2 and update its controller accordingly. If agent i_2 does not play the role of the leader, it performs the role of agent j_2:

(II-1) if $\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right) \geq 0$,

$$-\frac{1-c}{2}\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right) < v_{i_2}\left(t^{v(i_2j_2)}_{k}\right) - v_{i_2}\left(t^{v(i_2j_2)}_{k^{v(i_2j_2)}\left(t^{v(i_2j_2)}_{k-1}\right)}\right) < \frac{d-1}{2}\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right),$$

(II-2) if $\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right) < 0$,

$$\frac{d-1}{2}\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right) < v_{i_2}\left(t^{v(i_2j_2)}_{k}\right) - v_{i_2}\left(t^{v(i_2j_2)}_{k^{v(i_2j_2)}\left(t^{v(i_2j_2)}_{k-1}\right)}\right) < -\frac{1-c}{2}\hat{z}_{q_2}\left(t^{v(i_2j_2)}_{k-1}\right).$$

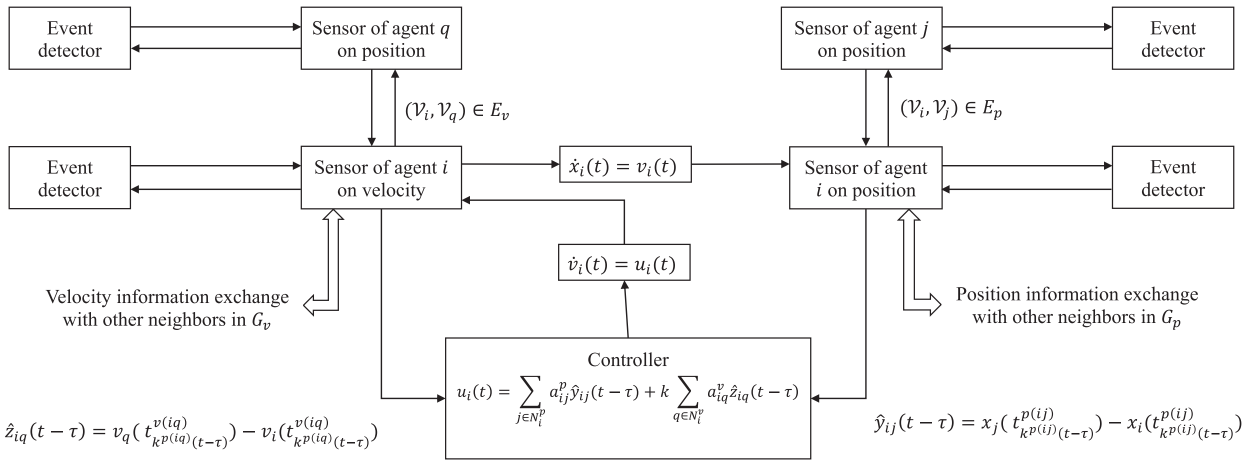

FIG. 2. Edge-event-triggered control under heterogeneous networks.

From rule A and rule B, it is clear that the event conditions based on position data and velocity data are unrelated and independent of each other. Therefore, events based on position information and velocity occur independently of each other, even if e_{q1}^p and e_{q2}^v connect the same pair of agents. Furthermore, the position information and velocity information of the controller for the same agent are updated asynchronously. More details are presented in Fig. 2. Moreover, we reveal the differences between rule A and rule B using the position information network G_p as an example. At edge-event-detecting time $t_k^{p(i_1 j_1)}$ of edge $e_{q1}^p = (\mathscr{V}_{i_1}, \mathscr{V}_{j_1})$, agents $\mathscr{V}_{i_1}$ and $\mathscr{V}_{j_1}$ should know the current relative states $y_{q_1}(t_k^{p(i_1 j_1)})$ and the latest sampled states $\hat{y}_{q_1}(t_{k-1}^{p(i_1 j_1)})$, to evaluate the event condition in rule A. That is, rule A relies on the current relative state, whereas rule B does not. Instead, in rule B, the agents only require the own latest states $x_{i_1}(t_k^{p(i_1 j_1)})$ and the latest sampled data $x_{i_1}\left(t^{p(i_1 j_1)}_{k^{p(i_1 j_1)}(t_{k-1}^{p(i_1 j_1)})}\right)$. However, this comes a higher probability of event triggering and more memory costs. Mathematically, the inequality (I-1) or (I-2) in rule A is more likely to be satisfied compared to the corresponding inequality in rule B, thereby reducing the frequency of event triggers.

III. MAIN RESULTS

In this paper, we investigate the leader-following consensus of the multi-agent system described by Eqs. (1) and (2). We introduce variables $\delta_i(t) = x_i(t) - x_1(t)$ and $\sigma_i(t) = v_i(t) - v_1(t)$ for $i \in \underline{n+1}$. Note that $\delta_1(t) = \sigma_1(t) = 0$ are introduced solely for the convenience of subsequent analysis. Systems (1) and (2) achieve leader-following consensus if $\delta_i(t) \to 0$ and $\sigma_i(t) \to 0$, as $t \to \infty$.

Construct a Lyapunov function candidate as follows:

$$V(t) = [\delta^T(t)\, \sigma^T(t)]^T \begin{bmatrix} \beta \tilde{L}_p + \dfrac{k\alpha\lambda_2(\tilde{L}_v)}{2} I_{n+1} & \dfrac{\alpha}{2} I_{n+1} \\ \dfrac{\alpha}{2} I_{n+1} & \beta I_{n+1} \end{bmatrix} \begin{bmatrix} \delta(t) \\ \sigma(t) \end{bmatrix}, \tag{5}$$

where $\alpha > 0, \beta > 0$ are parameters, to be designed later, $\delta(t) = [\delta_1(t)\ \delta_2(t) \ldots \delta_{n+1}]^T$ and $\sigma(t) = [\sigma_1(t)\ \sigma_2(t) \ldots \sigma_{n+1}(t)]^T$.

Lemma 1 *The Lyapunov function candidate (5) is positive definite, if α and β satisfy*

$$2k\beta\lambda_2(\tilde{L}_v) > \alpha. \tag{6}$$

In addition, $V(t) = 0$ if and only if $\delta(t) = \sigma(t) = \mathbf{0}$.

Proof. Let $\gamma = \frac{k\alpha\lambda_2(\tilde{L}_v)}{2}$ and

$$P = \begin{bmatrix} \beta \tilde{L}_p + \gamma I_{n+1} & \dfrac{\alpha}{2} I_{n+1} \\ \dfrac{\alpha}{2} I_{n+1} & \beta I_{n+1} \end{bmatrix}.$$

Since the Laplacian matrix $\tilde{L}_p$ is positive semi-definite, $\beta\tilde{L}_p + \gamma I_{n+1}$ is invertible with $\alpha, \beta > 0$.

Let

$$Q = \begin{bmatrix} I_{n+1} & -\dfrac{\alpha}{2}(\beta\tilde{L}_p + \gamma I_{n+1})^{-1} \\ \mathbf{0} & I_{n+1} \end{bmatrix}.$$

Then

$$Q^T P Q = \begin{bmatrix} \beta\tilde{L}_p + \gamma I_{n+1} & \mathbf{0} \\ \mathbf{0} & \beta I_{n+1} - \dfrac{\alpha^2}{4}(\beta\tilde{L}_p + \gamma I_{n+1})^{-1} \end{bmatrix}.$$

Obviously, $V(t)$ is positive definite, if $\beta I_{n+1} - \frac{\alpha^2}{4}(\beta\tilde{L}_p + \gamma I_{n+1})^{-1}$ is positive definite. Furthermore, $\beta I_{n+1} > \frac{\alpha^2}{4}(\beta\tilde{L}_p + \gamma I_{n+1})^{-1}$ holds, if $\beta > \frac{\alpha^2}{4\gamma}$. Therefore, the condition (6) guarantees the positive definiteness of $V(t)$. In addition, $\delta(t) = \sigma(t) = \mathbf{0}$ means systems (1) and (2) achieve the leader-following consensus. ☐

To simplify the following theorem, we introduce two constants,

$$\begin{aligned} M = &-\lambda_{min}(W_p) a\kappa\varphi_3' + \rho_1\lambda_n^{W_p} + \rho_2 h\lambda_n^{d_1} + \rho_3\lambda_n^{d_3} \\ &+ \rho_4\lambda_n^{F_1} + \rho_5\lambda_n^{F_3} + \rho_6||W_p|| + \rho_7||W_v||\,||F_1||, \end{aligned} \tag{7}$$

with

$$\begin{cases} \kappa = \alpha - \dfrac{2\gamma + k\alpha\lambda_n(\tilde{L}_v)}{2\lambda_2(\tilde{L}_p)}, \\ \iota = 2k\beta - \dfrac{k\alpha}{2} - \dfrac{\alpha+\gamma}{\lambda_2(\tilde{L}_v)} \\ \rho_1 = h(1/2 + h/2(k+1))((a-\kappa)b^2 + \kappa\varphi_3^2) + \varphi_1^2(kh^2/2 + 3(k+1)/4 + h(h+1)), \\ \rho_2 = (2\beta+\kappa)h/4(1+\theta_1^2) + (\alpha-\kappa)((1+\theta_1^2)(h/2 + h^2/2) + (k+1)\theta_1^2(h^3/2 + 3h/2) + h^3/2 + kh^3 + 3h/2(k+1)), \\ \rho_3 = \dfrac{\alpha+\gamma}{\lambda_2(\tilde{L}_v)}(1+\theta_1^2)(5h + 2h^2(k+1) + 4kh) + \left(\dfrac{h\iota}{2} + \dfrac{kh\alpha}{4}\right)(1+\theta_1^2), \\ \rho_4 = \dfrac{\alpha+\gamma}{\lambda_2(\tilde{L}_v)}\left(2\xi_1^2 + 2\xi_2^2\right)((k+1)(h+2) + h + kh + 2k + 5/2), \\ \rho_5 = (\alpha-\kappa)(\xi_3^2 + 2\xi_4^2)(h/2 + kh^2 + h^2 + 3k/4 + 1), \\ \rho_6 = \beta||D_p||\varphi_1 + \left(\dfrac{\beta||D_p|| + \kappa\varphi_3 + b(\alpha-\kappa)}{2}\right)||F_3||(\xi_3 + 2\xi_4) + \dfrac{k\kappa\varphi_3 + kb(\alpha-\kappa)}{4}||F_4||(\mu_3 + 2\mu_4) + b\varphi_1(\alpha-\kappa), \\ \rho_7 = (2\xi_1 + 2\xi_2)\left(\dfrac{d(\alpha+\gamma)}{\lambda_2(2\tilde{L}_v)} + \dfrac{k\alpha\varphi_4}{4} + \dfrac{\iota\varphi_4}{2}\right) \end{cases}$$

and

$$N = -\lambda_{min}(W_v)c\iota\varphi_4' + \phi_1\lambda_n^{W_v} + \phi_2\lambda_n^{d_2} + \phi_3\lambda_n^{d_4} + \phi_4\lambda_n^{F_2} + \phi_5\lambda_n^{F_4} + \phi_6||W_v|| + \phi_7||D_p||\,||F_4||, \tag{8}$$

with

$$\begin{cases} \phi_1 = \left(\iota + \dfrac{k\alpha}{2}\right)\varphi_4^2(k+1)h + \dfrac{(\alpha+\gamma)(k+1)}{\lambda_2(\tilde{L}_v)}\left(hd^2 + (h+2)\varphi_2^2\right), \\ \phi_2 = \dfrac{kh^2}{4}(1+\theta_2^2)(2\beta + \kappa + (\alpha-\kappa)(3 + 2h + 2h^2)) + (\alpha-\kappa)\left(\theta_2^2\left(\dfrac{k^2h^2}{4} + \dfrac{3kh^2(k+1)}{2}\right) + \dfrac{2k^2h^4 + 3kh^2(k+1)}{4}\right), \\ \phi_3 = \dfrac{(\alpha+\gamma)(1+\theta_2^2)}{\lambda_2(\tilde{L}_v)}kh\left((k+1)(h+2) + \dfrac{5}{2} + h + kh + 2k\right) + \left(\iota + \dfrac{k\alpha}{2}\right)\dfrac{kh(1+\theta_2^2)}{2}, \\ \phi_4 = \dfrac{\alpha+\gamma}{\lambda_2(\tilde{L}_v)}(2\mu_1^2 + 2\mu_2^2)\left((k+1)(kh+2k) + k\left(\dfrac{5}{2} + h + kh + 2k\right)\right), \\ \phi_5 = (\alpha-\kappa)(\mu_3^2 + 2\mu_4)\left(\dfrac{kh}{2} + kh^2 + \dfrac{k^2h^2}{2} + \dfrac{3k^2}{4} + k\right) \\ \phi_6 = \left(\dfrac{\alpha+\gamma}{\lambda_2(\tilde{L}_v)}\dfrac{d}{2} + \dfrac{(k\alpha + 2\iota)\varphi_4}{4}\right)||F_1||(2\xi_1 + 2\xi_2) + \left(\dfrac{(\alpha+\gamma)d}{\lambda_2(\tilde{L}_v)} + \dfrac{(k\alpha + 2\iota)\varphi_4}{2}\right)k||F_2||(2\mu_1 + 2\mu_2) + \dfrac{k\alpha}{2}\varphi_2\varphi_4 + \dfrac{\alpha+\gamma}{\lambda_2(\tilde{L}_v)}d\varphi_2, \\ \phi_7 = (\mu_3 + 2\mu_4)\left(\dfrac{k\beta}{2}||D_p|| + \dfrac{k\kappa\varphi_3}{4} + \dfrac{kb(\alpha-\kappa)}{4}\right), \end{cases}$$

where $r_1 = \max\{1 - a, b - 1\}$; $r_2 = \max\{1 - c, d - 1\}$; $\theta_1 = \max\{1 - a/b, b/a - 1\}$; $\theta_2 = \max\{1 - c/d, d/c - 1\}$; $\varphi_1 = r_1 + \theta_1$; $\varphi_2 = r_2 + \theta_2$; $\varphi_3 = 1 + \theta_1$, $\varphi_3' = \max\{1, |\theta_1 - 1|\}$; $\varphi_4 = 1 + \theta_2$, $\varphi_4' = \max\{1, |\theta_2 - 1|\}$, $\xi_1 = \sqrt{m_v}h\frac{b}{a}\theta_1$; $\xi_2 = \sqrt{m_v}h\theta_1$, $\mu_1 = \sqrt{m_v}h\frac{d}{c}\theta_2$; $\mu_2 = \sqrt{m_v}h\theta_2$, $\xi_3 = \sqrt{m_p}h^3\theta_1$; $\xi_4 = \sqrt{m_p}h^3\frac{b}{a}\theta_1$, $\mu_3 = \sqrt{m_p}h^3\theta_2$; $\mu_4 = \sqrt{m_p}h^3\frac{d}{c}\theta_2$, $\lambda_n^{W_p}$, $\lambda_n^{W_v}$; $\lambda_n^{d_1}$, $\lambda_n^{d_2}$, $\lambda_n^{d_3}$, $\lambda_n^{d_4}$; $\lambda_n^{F_1}$, $\lambda_n^{F_2}$, $\lambda_n^{F_3}$, and $\lambda_n^{F_4}$ are the maximum eigenvalues of W_p, W_v, $W_pD_p^TI_0D_pW_pD_P^TI_0D_pW_p$, $W_vD_v^TI_0D_pW_pD_P^TI_0D_vW_v$, $W_pD_p^TI_0D_vW_vD_v^TI_0D_pW_p$, $W_vD_v^TI_0D_vW_vD_v^TI_0D_vW_v$, $F_1^TW_vF_1$, $F_2^TW_vF_2$, $F_3^TW_pF_3$, and $F_4^TW_pF_4$, respectively.

Theorem 1 *Assume that the position information network G_p and the velocity information network G_v are both leader–follower reachable. For a given event-detecting period h, communication time delay τ, and parameter k, if there exist constants $\alpha, \beta, a, b, c, d$ with $0 < a, c \leq 1$ and $b, d \geq 1$ in rules A or B, such that*

$$\begin{cases} \kappa > 0, \\ \iota > 0, \\ M, N < 0, \end{cases} \tag{9}$$

then systems (1) and (2) achieve asynchronous leader-following consensus under protocol (3) under event-detecting rule A or rule B.

Proof. With $\delta(t), \sigma(t)$, we can derive the equivalent form of system (4) as follows:

$$\begin{bmatrix} \dot{\delta}(t) \\ \dot{\sigma}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_{n+1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta(t) \\ \sigma(t) \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_0 D_p W_p & k I_0 D_v W_v \end{bmatrix} \begin{bmatrix} \hat{y}(t-\tau) \\ \hat{z}(t-\tau) \end{bmatrix}. \tag{10}$$

Then, we consider the derivative of $V(t)$ along the trajectory of (10),

$$\begin{aligned} \dot{V}(t) &= 2[\delta^T(t)\,\sigma^T(t)]^T \begin{bmatrix} \beta \tilde{L}_p + \gamma I_{n+1} & \frac{\alpha}{2} I_{n+1} \\ \frac{\alpha}{2} I_{n+1} & \beta I_{n+1} \end{bmatrix} \begin{bmatrix} \dot{\delta}(t) \\ \dot{\sigma}(t) \end{bmatrix} \\ &= 2\beta \delta^T(t) \tilde{L}_p \sigma(t) + 2\gamma \delta^T \sigma + \alpha \sigma^T \sigma \\ &\quad - \alpha \delta^T I_0 D_p W_p \hat{y}(t-\tau) - 2\beta \sigma^T I_0 D_p W_p \hat{y}(t-\tau) \\ &\quad - k\alpha \delta^T I_0 D_v W_v \hat{z}(t-\tau) - 2k\beta \sigma^T I_0 D_v W_v \hat{z}(t-\tau). \end{aligned}$$

Due to $\delta_1(t) = \sigma_1(t) = 0$, $\delta^T(t) I_0 = \delta^T(t)$, and $\sigma^T(t) I_0 = \sigma^T(t)$ hold. Furthermore, by $2x^T y \le x^T x + y^T y$ and $\tilde{L}_p = D_p^T W_p D_p$, $\tilde{L}_v = D_v^T W_v D_v$ with $\delta^T \delta \le \frac{1}{\lambda_2(\tilde{L}_p)} \delta^T \tilde{L}_p \delta$ and $\sigma^T \sigma \le \frac{1}{\lambda_2(\tilde{L}_v)} \sigma^T \tilde{L}_v \sigma$,[46] we have

$$\begin{aligned} \dot{V}(t) &= 2\beta \delta^T(t) D_p W_p D_p^T \sigma(t) + 2\gamma \delta^T \sigma + \alpha \sigma^T \sigma \\ &\quad - \alpha \delta^T D_p W_p \hat{y}(t-\tau) - 2\beta \sigma^T D_p W_p \hat{y}(t-\tau) \\ &\quad - k\alpha \delta^T D_v W_v \hat{z}(t-\tau) - 2k\beta \sigma^T D_v W_v \hat{z}(t-\tau) \\ &\le 2\beta \sigma^T D_p W_p (y(t) - \hat{y}(t-\tau)) \\ &\quad + \frac{2\gamma + k\alpha \lambda_n(\tilde{L}_v)}{2\lambda_2(\tilde{L}_p)} y^T W_p (y - \hat{y}(t-\tau)) \\ &\quad - \left(\alpha - \frac{2\gamma + k\alpha \lambda_n(\tilde{L}_v)}{2\lambda_2(\tilde{L}_p)} \right) y^T W_p \hat{y}(t-\tau) \\ &\quad - \frac{k\alpha}{2} (z^T - \hat{z}^T(t-\tau)) W_v \hat{z}(t-\tau) \\ &\quad + \frac{\gamma + \alpha}{\lambda_2(\tilde{L}_v)} z^T W_v (z - \hat{z}(t-\tau)) \\ &\quad - \left(2k\beta - \frac{k\alpha}{2} - \frac{\gamma + \alpha}{\lambda_2(\tilde{L}_v)} \right) z^T W_v \hat{z}(t-\tau). \end{aligned}$$

Next, we will specify the forms of $y(t)$ and $z(t)$. Given that $z(t) = D_v^T v(t)$, we have $\dot{z}(t) = D_v^T \dot{v}(t) = -D_v^T I_0 D_p W_p \hat{y}(t-\tau) - k D_v^T I_0 D_v W_v \hat{z}(t-\tau)$. Denote $S_1 = D_v^T I_0 D_p = [s_{1ij}] \in R^{m_v \times m_p}$ and $S_2 = D_v^T I_0 D_v = [s_{2ij}] \in R^{m_v \times m_v}$. Then, we can derive that

$$\dot{z}_i(t) = -\sum_{j=1}^{m_p} s_{1ij} w_j^p \hat{y}_j(t-\tau) - k \sum_{j=1}^{m_v} s_{2ij} w_j^v \hat{z}_j(t-\tau), \; i \in \underline{m_v}. \tag{11}$$

For any $t \in [t_{k_{v(i)}(t)}^{v(i)}, t_{k_{v(i)}(t)+1}^{v(i)})$, the solution of (11) is

$$\begin{aligned} z_i(t) &= z_i\left(t_{k_{v(i)}(t)}^{v(i)}\right) - \sum_{j=1}^{m_p} s_{1ij} w_j^p \int_{t_{k_{v(i)}(t)}^{v(i)}}^{t} \hat{y}_j(s-\tau)\, ds \\ &\quad - k \sum_{j=1}^{m_v} s_{2ij} w_j^v \int_{t_{k_{v(i)}(t)}^{v(i)}}^{t} \hat{z}_j(s-\tau)\, ds, \; i \in \underline{m_v}. \end{aligned} \tag{12}$$

For $\int_{t_{k_{v(i)}(t)}^{v(i)}}^{t} \hat{y}_j(s-\tau)\, ds$, we proceed with the analysis as follows:

Case 1A: for some j, $s \in \left[t_{k_{p(j)}(t)}^{p(j)}, t_{k_{p(j)}(t)}^{p(j)} + \tau\right)$, we have

(1A-i) if $t_{k_{v(i)}(t)}^{v(i)} \in \left[t_{k_{p(j)}(t)}^{p(j)}, t\right)$,

$$\begin{aligned} \int_{t_{k_{v(i)}(t)}^{v(i)}}^{t} \hat{y}_j(s-\tau)\, ds &= \left(t - t_{k_{v(i)}(t)}^{v(i)}\right) \hat{y}_j\left(t_{k_{p(j)}(t)-1}^{p(j)}\right) \\ &= \left(t - t_{k_{v(i)}(t)}^{v(i)}\right) \hat{y}_j\left(t_{k_{v(i)}(t)-1}^{v(i)}\right), \end{aligned}$$

(1A-ii) if $t_{k_{v(i)}(t)}^{v(i)} \in \left[t_{k_{p(j)}(t)-1}^{p(j)} + \tau, t_{k_{p(j)}(t)}^{p(j)}\right)$,

$$\begin{aligned} \int_{t_{k_{v(i)}(t)}^{v(i)}}^{t} \hat{y}_j(s-\tau)\, ds &= \left(t - t_{k_{v(i)}(t)}^{v(i)}\right) \hat{y}_j\left(t_{k_{p(j)}(t)-1}^{p(j)}\right) \\ &= \left(t - t_{k_{v(i)}(t)}^{v(i)}\right) \hat{y}_j\left(t_{k_{v(i)}(t)}^{v(i)}\right), \end{aligned}$$

(1A-iii) if $t_{k_{v(i)}(t)}^{v(i)} \in \left[t_{k_{p(j)}(t)-1}^{p(j)}, t_{k_{p(j)}(t)-1}^{p(j)} + \tau\right)$,

$$\begin{aligned} \int_{t_{k_{v(i)}(t)}^{v(i)}}^{t} \hat{y}_j(s-\tau)\, ds &= \left(t - t_{k_{p(j)}(t)-1}^{p(j)} - \tau\right) \hat{y}_j\left(t_{k_{p(j)}(t)-1}^{p(j)}\right) \\ &\quad + \left(t_{k_{p(j)}(t)-1}^{p(j)} + \tau - t_{k_{v(i)}(t)}^{v(i)}\right) \hat{y}_j\left(t_{k_{v(i)}(t)-1}^{v(i)}\right). \end{aligned}$$

Therefore, the comprehensive expression for the above three situations is

$$\begin{aligned} &\int_{t_{k_{v(i)}(t)}^{v(i)}}^{t} \hat{y}_j(s-\tau)\, ds \\ &= \left(t - t_{k_{p(j)}(t)-1}^{p(j)} - \tau\right) \hat{y}_j\left(t_{k_{p(j)}(t)-1}^{p(j)}\right) \\ &\quad + \left(t_{k_{p(j)}(t)-1}^{p(j)} + \tau - t_{k_{v(i)}(t)}^{v(i)}\right) \hat{y}_j\left(t_{k_{v(i)}(t)-1}^{v(i)}\right) \\ &\quad + \eta\left(t_{k_{p(j)}(t)}^{p(j)} - t_{k_{v(i)}(t)}^{v(i)}\right) \eta\left(t_{k_{v(i)}(t)}^{v(i)} - t_{k_{p(j)}(t)-1}^{p(j)} - \tau\right) \\ &\quad \times \left(t_{k_{p(j)}(t)-1}^{p(j)} + \tau - t_{k_{v(i)}(t)}^{v(i)}\right) \left(\hat{y}_j\left(t_{k_{v(i)}(t)}^{v(i)}\right) - \hat{y}_j\left(t_{k_{v(i)}(t)-1}^{v(i)}\right)\right), \end{aligned} \tag{13}$$

where $\eta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0. \end{cases}$

Case 2A: for other j, $s \in [t^{p(j)}_{k_{p(j)}(t)} + \tau, t^{p(j)}_{k_{p(j)}(t)+1})$, we have
(2A-i) if $t^{v(i)}_{k_{v(i)}(t)} \in [t^{p(j)}_{k_{p(j)}(t)} + \tau, t)$,

$$\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{y}_j(s-\tau)\,ds = \left(t - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{y}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right),$$

(2A-ii) if $t^{v(i)}_{k_{v(i)}(t)} \in \left[t^{p(j)}_{k_{p(j)}(t)}, t^{p(j)}_{k_{p(j)}(t)} + \tau\right)$,

$$\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{y}_j(s-\tau)\,ds = \left(t - t^{p(j)}_{k_{p(j)}(t)} - \tau\right)\hat{y}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right) + \left(t^{p(j)}_{k_{p(j)}(t)} + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{y}_j\left(t^{v(i)}_{k_{v(i)}(t)-1}\right),$$

(2A-iii) if $t^{v(i)}_{k_{v(i)}(t)} \in \left[t^{p(j)}_{k_{p(j)}(t)-1} + \tau, t^{p(j)}_{k_{p(j)}(t)}\right)$,

$$\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{y}_j(s-\tau)\,ds = \left(t - t^{p(j)}_{k_{p(j)}(t)} - \tau\right)\hat{y}_j\left(t^{p(j)}_{k_{p(j)}(t)}\right) + \left(t^{p(j)}_{k_{p(j)}(t)} + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{y}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right).$$

Therefore, the comprehensive expression of the above three situations is

$$\begin{aligned}\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{y}_j(s-\tau)\,ds &= \left(t - t^{p(j)}_{k_{p(j)}(t)} - \tau\right)\hat{y}_j\left(t^{p(j)}_{k_{p(j)}(t)}\right) \\ &\quad + \left(t^{p(j)}_{k_{p(j)}(t)} + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{y}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right) \\ &\quad + \eta\left(t^{v(i)}_{k_{v(i)}(t)} - t^{p(j)}_{k_{p(j)}(t)}\right)\eta\left(t^{p(j)}_{k_{p(j)}(t)} + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\left(t^{p(j)}_{k_{p(j)}(t)} \right. \\ &\quad \left. + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\left(\hat{y}_j\left(t^{v(i)}_{k_{v(i)}(t)-1}\right) - \hat{y}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right)\right). \end{aligned} \tag{14}$$

For $\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{z}(s-\tau)ds$, we proceed with the analysis as follows:

Case 3A: for some j, $s \in [t^{v(j)}_{k_{v(j)}(t)}, t^{v(j)}_{k_{v(j)}(t)} + \tau)$, we have
(3A-i) if $t^{v(i)}_{k_{v(i)}(t)} \in \left[t^{v(j)}_{k_{v(j)}(t)}, t\right)$,

$$\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{z}_j(s-\tau)\,ds = \left(t - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)-1}\right),$$

(3A-ii) if $t^{v(i)}_{k_{v(i)}(t)} \in \left[t^{v(j)}_{k_{v(j)}(t)-1} + \tau, t^{v(j)}_{k_{v(j)}(t)}\right)$,

$$\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{z}_j(s-\tau)\,ds = \left(t - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right),$$

(3A-iii) if $t^{v(i)}_{k_{v(i)}(t)} \in \left[t^{v(j)}_{k_{v(j)}(t)-1}, t^{v(j)}_{k_{v(j)}(t)-1} + \tau\right)$,

$$\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{z}_j(s-\tau)\,ds = \left(t - t^{v(j)}_{k_{v(j)}(t)-1} - \tau\right)\hat{z}_j\left(t^{v(j)}_{k_{v(j)}(t)-1}\right) + \left(t^{v(j)}_{k_{v(j)}(t)-1} + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)-1}\right).$$

Therefore, the comprehensive expression of the above three situations is

$$\begin{aligned}\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{z}_j(s-\tau)\,ds &= \left(t - t^{v(j)}_{k_{v(j)}(t)-1} - \tau\right)\hat{z}_j\left(t^{v(j)}_{k_{v(j)}(t)-1}\right) \\ &\quad + \left(t^{v(j)}_{k_{v(j)}(t)-1} + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)-1}\right) \\ &\quad + \eta\left(t^{v(j)}_{k_{v(j)}(t)} - t^{v(i)}_{k_{v(i)}(t)}\right)\eta\left(t^{v(i)}_{k_{v(i)}(t)} - t^{v(j)}_{k_{v(j)}(t)-1} - \tau\right)\left(t^{v(j)}_{k_{v(j)}(t)-1} \right. \\ &\quad \left. + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\left(\hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right) - \hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)-1}\right)\right). \end{aligned} \tag{15}$$

Case 4A: for other j, $s \in \left[t^{v(j)}_{k_{v(j)}(t)} + \tau, t^{v(j)}_{k_{v(j)}(t)+1}\right)$, we have
(4A-i) if $t^{v(i)}_{k_{v(i)}(t)} \in \left[t^{v(j)}_{k_{v(j)}(t)} + \tau, t\right)$,

$$\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{z}_j(s-\tau)\,ds = \left(t - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right),$$

(4A-ii) if $t^{v(i)}_{k_{v(i)}(t)} \in \left[t^{v(j)}_{k_{v(j)}(t)}, t^{v(j)}_{k_{v(j)}(t)} + \tau\right)$,

$$\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{z}_j(s-\tau)\,ds = \left(t - t^{v(j)}_{k_{v(j)}(t)} - \tau\right)\hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right) + \left(t^{v(j)}_{k_{v(j)}(t)} + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)-1}\right),$$

(4A-iii) if $t^{v(i)}_{k_{v(i)}(t)} \in \left[t^{v(j)}_{k_{v(j)}(t)-1} + \tau, t^{v(j)}_{k_{v(j)}(t)}\right)$,

$$\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{z}_j(s-\tau)\,ds = \left(t - t^{v(j)}_{k_{v(j)}(t)} - \tau\right)\hat{z}_j\left(t^{v(j)}_{k_{v(j)}(t)}\right) + \left(t^{v(j)}_{k_{v(j)}(t)} + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right).$$

Therefore, the comprehensive expression of the above three situations is

$$\begin{aligned}\int_{t^{v(i)}_{k_{v(i)}(t)}}^{t} \hat{z}_j(s-\tau)\,ds &= \left(t - t^{v(j)}_{k_{v(j)}(t)} - \tau\right)\hat{z}_j\left(t^{v(j)}_{k_{v(j)}(t)}\right) \\ &\quad + \left(t^{v(j)}_{k_{v(j)}(t)} + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right) \\ &\quad + \eta\left(t^{v(i)}_{k_{v(i)}(t)} - t^{v(j)}_{k_{v(j)}(t)}\right)\eta\left(t^{v(j)}_{k_{v(j)}(t)} + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\left(t^{v(j)}_{k_{v(j)}(t)} \right. \\ &\quad \left. + \tau - t^{v(i)}_{k_{v(i)}(t)}\right)\left(\hat{z}_j(t^{v(i)}_{k_{v(i)}(t)-1}) - \hat{z}_j\left(t^{v(i)}_{k_{v(i)}(t)}\right)\right). \end{aligned} \tag{16}$$

Due to page limitations, the specific form of $z_i(t)$ is provided in Appendix A. Furthermore, we can derive the compact form of $z(t)$,

$$\begin{aligned} z(t) = \tilde{z}(t) &- \Lambda_v D_v^T I_0 D_p W_p \Theta_1(t) \hat{y}_0(t) - \Lambda_v D_v^T I_0 D_p W_p \Theta_2(t) \hat{y}(t) \\ &- k\Lambda_v D_v^T I_0 D_v W_v \Theta_3(t) \hat{z}_0(t) - k\Lambda_v D_v^T I_0 D_v W_v \Theta_4(t) \hat{z}(t) \\ &+ F_1(\Theta_1(t) \otimes I_{m_v}) Y_1 + F_1(\Theta_2(t) \otimes I_{m_v}) Y_2 \\ &+ F_1(\Theta_1(t) \otimes I_{m_v}) Y_3 + F_1(\Theta_2(t) \otimes I_{m_v}) Y_4 \\ &+ kF_2(\Theta_3(t) \otimes I_{m_v}) Z_1 + kF_2(\Theta_4(t) \otimes I_{m_v}) Z_2 \\ &+ kF_2(\Theta_3(t) \otimes I_{m_v}) Z_3 + kF_2(\Theta_4(t) \otimes I_{m_v}) Z_4, \end{aligned} \tag{17}$$

where $\Lambda_v = \mathrm{diag}\Big(t - t^{v(1)}_{k_{v(1)}(t)}, t - t^{v(2)}_{k_{v(2)}(t)}, \ldots, t - t^{v(m_v)}_{k_{v(m_v)}(t)}\Big)$, $\hat{y}_0(t) = \Big[\hat{y}_1\left(t^{p(1)}_{k_{p(1)}(t)-1}\right) \hat{y}_2\left(t^{p(2)}_{k_{p(2)}(t)-1}\right) \ldots \hat{y}_{m_p}\left(t^{p(m_p)}_{k_{p(m_p)}(t)-1}\right)\Big]^T$, $\Theta_1(t) = (1_{11}(t), 1_{12}(t), \ldots, 1_{1m_p}(t))$, where $1_{1j}(t) = 1$, if jth edge in G_p belongs to case 1 at time t; otherwise, $1_{1j}(t) = 0$. Similarly, matrices $\Theta_2(t), \Theta_3(t)$, and $\Theta_4(t)$ are diagonal matrices under cases 2, 3, and 4, respectively, with $\Theta_1(t) + \Theta_2(t) = I$ and $\Theta_3(t) + \Theta_4(t) = I$. In matrix $F_1 = \begin{bmatrix} F_{11} & & & \\ & F_{12} & & \\ & & \ddots & \\ & & & F_{1m_v} \end{bmatrix}$, F_{1i} is the ith row vector of $D_v^T I_0 D_p W_p$. $Y_1 = [Y_{11}^T Y_{12}^T \ldots Y_{1m_v}]^T$, $Y_2 = [Y_{21}^T Y_{22}^T \ldots Y_{2m_v}]^T$, $Y_3 = [Y_{31}^T Y_{32}^T \ldots Y_{3m_v}]^T$, $Y_4 = [Y_{41}^T Y_{42}^T \ldots Y_{4m_v}]^T$ (the specific forms of Y_{1i}, Y_{2i}, Y_{3i}, and Y_{4i} are provided in Appendix B). In matrix $F_2 = \begin{bmatrix} F_{21} & & & \\ & F_{22} & & \\ & & \ddots & \\ & & & F_{2m_v} \end{bmatrix}$, F_{2i} is the ith row vector of $D_v^T I_0 D_v W_v$. $Z_1 = [Z_{11}^T Z_{12}^T \ldots Z_{1m_v}]^T$, $Z_2 = [Z_{21}^T Z_{22}^T \ldots Z_{2m_v}]^T$, $Z_3 = [Z_{31}^T Z_{32}^T \ldots Z_{3m_v}]^T$, $Z_4 = [Z_{41}^T Z_{42}^T \ldots Z_{4m_v}]^T$ (the specific forms of Z_{1i}, Z_{2i}, Z_{3i}, and Z_{4i} are provided in Appendix C).

Subsequently, we will proceed with an analysis of $y(t)$.

Due to $\dot{y}(t) = D_p^T \dot{x}(t)$, we have that

$$\dot{y}_i(t) = \sum_{j=1}^{n} d_{ji}^p v_j(t). \tag{18}$$

Then for $t \in \left[t^{p(i)}_{k_{p(i)}(t)}, t^{p(i)}_{k_{p(i)}(t)+1}\right)$, the solution of (18) is

$$y_i(t) = y_i\left(t^{p(i)}_{k_{p(i)}(t)}\right) + \sum_{j=1}^{n} d_{ji}^p \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} v_j(s)\, ds. \tag{19}$$

From $\dot{v}(t) = -I_0 D_p W_p \hat{y}(t-\tau) - kI_0 D_v W_v \hat{z}(t-\tau)$, we have

$$\dot{v}_j(t) = -\sum_{s=1}^{m_p} (I_0 D_p W_p)_{js} \hat{y}_s(t-\tau) - k\sum_{s=1}^{m_v} (I_0 D_v W_v)_{js} \hat{z}_s(t-\tau). \tag{20}$$

Then for $t \in [t^{p(i)}_{k_{p(i)}(t)}, t^{p(i)}_{k_{p(i)}(t)+1})$, the solution of (20) is

$$\begin{aligned} v_j(t) = v_j\left(t^{p(i)}_{k_{p(i)}(t)}\right) &- \sum_{s=1}^{m_p} \left(I_0 D_p W_p\right)_{js} \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \hat{y}_s(\tau_1 - \tau)\, d\tau_1 \\ &- k\sum_{s=1}^{m_v} (I_0 D_v W_v)_{js} \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \hat{z}_s(\tau_1 - \tau)\, d\tau_1. \end{aligned} \tag{21}$$

Substituting (21) into (19), we derive that

$$\begin{aligned} y_i(t) = y_i\left(t^{p(i)}_{k_{p(i)}(t)}\right) &+ \sum_{j=1}^{n} d_{ji}^p \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} v_j\left(t^{p(i)}_{k_{p(i)}(s)}\right) ds \\ &- \sum_{j=1}^{n} d_{ji}^p \sum_{s=1}^{m_p} (I_0 D_p W_p)_{js} \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{y}_s(\tau_1 - \tau)\, d\tau_1\, d\tau_2 \\ &- k\sum_{j=1}^{n} d_{ji}^p \sum_{s=1}^{m_v} (I_0 D_v W_v)_{js} \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{z}_s(\tau_1 - \tau)\, d\tau_1\, d\tau_2. \end{aligned} \tag{22}$$

For $\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} v_j\left(t^{p(i)}_{k_{p(i)}(s)}\right) ds$, the term $v_j\left(t^{p(i)}_{k_{p(i)}(s)}\right)$ is a constant, if $s \in \left[t^{p(i)}_{k_{p(i)}(t)}, t\right)$. Then, we have

$$\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} v_j\left(t^{p(i)}_{k_{p(i)}(s)}\right) ds = \left(t - t^{p(i)}_{k_{p(i)}(t)}\right) v_j\left(t^{p(i)}_{k_{p(i)}(t)}\right). \tag{23}$$

For $\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{y}_s(\tau_1 - \tau)\, d\tau_1\, d\tau_2$, we proceed with the analysis as follows:

Case 1B: for some s, $t \in [t^{p(s)}_{k_{p(s)}(t)}, t^{p(s)}_{k_{p(s)}(t)} + \tau)$, we have

(1B-i) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{p(s)}_{k_{p(s)}(t)}, t\right)$,

$$\begin{aligned} &\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{y}_s(\tau_1 - \tau)\, d\tau_1\, d\tau_2 \\ &\quad = \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{p(i)}_{k_{p(i)}(\tau_2)}\right) \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(\tau_2)-1}\right) d\tau_2 \\ &\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right), \end{aligned}$$

(1B-ii) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{p(s)}_{k_{p(s)}(t)-1} + \tau, t^{p(s)}_{k_{p(s)}(t)}\right)$,

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{y}_s(\tau_1 - \tau)\, d\tau_1\, d\tau_2 \\
&\quad = \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{p(i)}_{k_{p(i)}(\tau_2)}\right) \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(\tau_2)}\right) d\tau_2 \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right),
\end{aligned}$$

(1B-iii) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{p(s)}_{k_{p(s)}(t)-1}, t^{p(s)}_{k_{p(s)}(t)-1} + \tau\right)$,

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{y}_s(\tau_1 - \tau)\, d\tau_1\, d\tau_2 \\
&= \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t^{p(s)}_{k_{p(s)}(t)-1}+\tau} \left(\tau_2 - t^{p(i)}_{k_{p(i)}(\tau_2)}\right) \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(\tau_2)-1}\right) d\tau_2 \\
&\quad + \int_{t^{p(s)}_{k_{p(s)}(t)-1}+\tau}^{t} \left(\tau_2 - t^{p(s)}_{k_{p(s)}(\tau_2)-1} - \tau\right) \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(\tau_2)-1}\right) \\
&\quad + \left(t^{p(s)}_{k_{p(s)}(\tau_2)-1} + \tau - t^{p(i)}_{k_{p(i)}(\tau_2)}\right) \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(\tau_2)-1}\right) d\tau_2 \\
&= \frac{1}{2}\left(t^{p(s)}_{k_{p(s)}(t)-1} + \tau - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right) \\
&\quad + \left(t - t^{p(s)}_{k_{p(s)}(t)-1} - \tau\right)\left(t^{p(s)}_{k_{p(s)}(t)-1} + \tau - t^{p(i)}_{k_{p(i)}(t)}\right) \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right) \\
&\quad + \frac{1}{2}\left(t - t^{p(s)}_{k_{p(s)}(t)-1} - \tau\right)^2 \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)-1}\right) \\
&= \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)-1}\right) + \frac{1}{2}\left(\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\right. \\
&\quad \left. - \left(t - t^{p(s)}_{k_{p(s)}(t)-1} - \tau\right)^2\right)\left(\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)-1}\right) - \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right).
\end{aligned}$$

Then, we derive the unified form for (1B-i), (1B-ii), and (1B-iii),

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{y}_s(\tau_1 - \tau)\, d\tau_1\, d\tau_2 \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)-1}\right) \\
&\qquad + \eta\left(t^{p(s)}_{k_{p(s)}(t)-1} + \tau - t^{p(i)}_{k_{p(i)}(t)}\right) \eta\left(t^{p(i)}_{k_{p(i)}(t)} - t^{p(s)}_{k_{p(s)}(t)-1}\right) \\
&\qquad \times \frac{1}{2}\left(\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 - \left(t - t^{p(s)}_{k_{p(s)}(t)-1} - \tau\right)^2\right) \\
&\qquad \times \left(\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)-1}\right) - \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right).
\end{aligned}$$

Case 2B: at time t, for other s, $t \in [t^{p(s)}_{k_{p(s)}(t)} + \tau, t^{p(s)}_{k_{p(s)}(t)+1})$, we have (2B-i) if $t^{p(i)}_{k_{p(i)}(t)} \in [t^{p(s)}_{k_{p(s)}(t)} + \tau, t)$,

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{y}_s(\tau_1 - \tau)\, d\tau_1\, d\tau_2 \\
&\quad = \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{p(i)}_{k_{p(i)}(\tau_2)}\right) \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(\tau_2)}\right) d\tau_2 \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right),
\end{aligned}$$

(2B-ii) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{p(s)}_{k_{p(s)}(t)}, t^{p(s)}_{k_{p(s)}(t)} + \tau\right)$,

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{y}_s(\tau_1 - \tau)\, d\tau_1\, d\tau_2 \\
&\quad = \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{p(s)}_{k_{p(s)}(\tau_2)} - \tau\right) \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(\tau_2)}\right) \\
&\qquad + \left(t^{p(s)}_{k_{p(s)}(\tau_2)} + \tau - t^{p(i)}_{k_{p(i)}(\tau_2)}\right) \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(\tau_2)-1}\right) d\tau_2 \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)-1}\right) \\
&\qquad + \frac{1}{2}\left(t - t^{p(s)}_{k_{p(s)}(t)} - \tau\right)^2 \left(\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)}\right) - \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)-1}\right)\right) \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)}\right) + \frac{1}{2}\left(\left(t - t^{p(s)}_{k_{p(s)}(t)} - \tau\right)^2\right. \\
&\qquad \left. - \left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)}\right) - \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right),
\end{aligned}$$

(2B-iii) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{p(s)}_{k_{p(s)}(t)-1} + \tau, t^{p(s)}_{k_{p(s)}(t)}\right)$,

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{y}_s(\tau_1 - \tau)\, d\tau_1\, d\tau_2 \\
&\quad = \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{p(s)}_{k_{p(s)}(\tau_2)} - \tau\right) \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(\tau_2)}\right) \\
&\qquad + \left(t^{p(s)}_{k_{p(s)}(\tau_2)} + \tau - t^{p(i)}_{k_{p(i)}(\tau_2)}\right) \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(\tau_2)-1}\right) d\tau_2 \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right) \\
&\qquad + \frac{1}{2}\left(t - t^{p(s)}_{k_{p(s)}(t)} - \tau\right)^2 \left(\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)}\right) - \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right) \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)}\right) + \frac{1}{2}\left(\left(t - t^{p(s)}_{k_{p(s)}(t)} - \tau\right)^2\right. \\
&\qquad \left. - \left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)}\right) - \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right).
\end{aligned}$$

Under case 2B, we can derive

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{y}_s(\tau_1-\tau)\,d\tau_1\,d\tau_2 \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)}\right) + \frac{1}{2}\left(\left(t - t^{p(s)}_{k_{p(s)}(t)} - \tau\right)^2\right. \\
&\qquad \left. - \left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)}\right) - \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right) \\
&\qquad + \eta\left(t^{p(s)}_{k_{p(s)}(t)} + \tau - t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)} - t^{p(s)}_{k_{p(s)}(t)}\right) \\
&\qquad \times \frac{1}{2}\left(\left(t - t^{p(s)}_{k_{p(s)}(t)} - \tau\right)^2 - \left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\right) \\
&\qquad \times \left(\hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right) - \hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right).
\end{aligned}$$

For $\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{z}_s(\tau_1-\tau)\,d\tau_1\,d\tau_2$, we proceed with the analysis as follows:

Case 3B: for some s, $t \in \left[t^{v(s)}_{k_{v(s)}(t)}, t^{v(s)}_{k_{v(s)}(t)} + \tau\right)$, we have (3B-i) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{v(s)}_{k_{v(s)}(t)}, t\right)$,

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{z}_s(\tau_1-\tau)\,d\tau_1\,d\tau_2 \\
&\quad = \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{p(i)}_{k_{p(i)}(\tau_2)}\right)\hat{z}_s\left(t^{p(i)}_{k_{p(i)}(\tau_2)-1}\right)\,d\tau_2 \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right),
\end{aligned}$$

(3B-ii) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{v(s)}_{k_{v(s)}(t)-1} + \tau, t^{v(s)}_{k_{v(s)}(t)}\right)$,

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{z}_s(\tau_1-\tau)\,d\tau_1\,d\tau_2 \\
&\quad = \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{p(i)}_{k_{p(i)}(\tau_2)}\right)\hat{z}_s\left(t^{p(i)}_{k_{p(i)}(\tau_2)}\right)\,d\tau_2 \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right),
\end{aligned}$$

(3B-iii) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{v(s)}_{k_{v(s)}(t)-1}, t^{v(s)}_{k_{v(s)}(t)-1} + \tau\right)$,

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{z}_s(\tau_1-\tau)\,d\tau_1\,d\tau_2 \\
&\quad = \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{v(s)}_{k_{v(s)}(\tau_2)-1} - \tau\right)\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(\tau_2)-1}\right) \\
&\qquad + \left(t^{v(s)}_{k_{v(s)}(\tau_2)-1} + \tau - t^{p(i)}_{k_{p(i)}(\tau_2)}\right)\hat{z}_s\left(t^{p(i)}_{k_{p(i)}(\tau_2)-1}\right)\,d\tau_2 \\
&\quad = \frac{1}{2}\left(t^{v(s)}_{k_{v(s)}(t)-1} + \tau - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right) \\
&\qquad + \left(t - t^{v(s)}_{k_{v(s)}(t)-1} - \tau\right)\left(t^{v(s)}_{k_{v(s)}(t)-1} + \tau - t^{p(i)}_{k_{p(i)}(t)}\right)\hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right) \\
&\qquad + \frac{1}{2}\left(t - t^{v(s)}_{k_{v(s)}(t)-1} - \tau\right)^2 \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right) \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right) + \frac{1}{2}\left(\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\right. \\
&\qquad \left. - \left(t - t^{v(s)}_{k_{v(s)}(t)-1} - \tau\right)^2\right)\left(\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right) - \hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right).
\end{aligned}$$

Then, we derive the unified form for (3B-i), (3B-ii), and (3B-iii),

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{z}_s(\tau_1-\tau)\,d\tau_1\,d\tau_2 \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right) \\
&\qquad + \eta\left(t^{v(s)}_{k_{v(s)}(t)-1} + \tau - t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)} - t^{v(s)}_{k_{v(s)}(t)-1}\right) \\
&\qquad \times \frac{1}{2}\left(\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 - \left(t - t^{v(s)}_{k_{v(s)}(t)-1} - \tau\right)^2\right) \\
&\qquad \times \left(\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right) - \hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right).
\end{aligned}$$

Case 4B: at time t, for other s, $t \in \left[t^{v(s)}_{k_{v(s)}(t)} + \tau, t^{v(s)}_{k_{v(s)}(t)+1}\right)$, we have (4B-i) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{v(s)}_{k_{v(s)}(t)} + \tau, t\right)$,

$$\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{z}_s(\tau_1-\tau)\,d\tau_1\,d\tau_2 \\
&\quad = \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{p(i)}_{k_{p(i)}(\tau_2)}\right)\hat{z}_s\left(t^{p(i)}_{k_{p(i)}(\tau_2)}\right)\,d\tau_2 \\
&\quad = \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right),
\end{aligned}$$

(4B-ii) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{v(s)}_{k_{v(s)}(t)}, t^{v(s)}_{k_{v(s)}(t)} + \tau\right)$,

$$
\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{z}_s(\tau_1-\tau)\, d\tau_1\, d\tau_2 \\
&= \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{v(s)}_{k_{v(s)}(\tau_2)} - \tau\right) \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(\tau_2)}\right) \\
&\quad + \left(t^{v(s)}_{k_{v(s)}(\tau_2)} + \tau - t^{p(i)}_{k_{p(i)}(\tau_2)}\right) \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(\tau_2)-1}\right) d\tau_2 \\
&= \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right) \\
&\quad + \frac{1}{2}\left(t - t^{v(s)}_{k_{v(s)}(t)} - \tau\right)^2 \left(\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)}\right) - \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right)\right) \\
&= \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)}\right) + \frac{1}{2}\left(\left(t - t^{v(s)}_{k_{v(s)}(t)} - \tau\right)^2 \right. \\
&\quad \left. - \left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\right) \left(\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)}\right) - \hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right),
\end{aligned}
$$

(4B-iii) if $t^{p(i)}_{k_{p(i)}(t)} \in \left[t^{v(s)}_{k_{v(s)}(t)-1} + \tau, t^{v(s)}_{k_{v(s)}(t)}\right)$,

$$
\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{z}_s(\tau_1-\tau)\, d\tau_1\, d\tau_2 \\
&= \int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \left(\tau_2 - t^{v(s)}_{k_{v(s)}(\tau_2)} - \tau\right) \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(\tau_2)}\right) \\
&\quad + \left(t^{v(s)}_{k_{v(s)}(\tau_2)} + \tau - t^{p(i)}_{k_{p(i)}(\tau_2)}\right) \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(\tau_2)-1}\right) d\tau_2 \\
&= \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right) \\
&\quad + \frac{1}{2}\left(t - t^{v(s)}_{k_{v(s)}(t)} - \tau\right)^2 \left(\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)}\right) - \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right)\right) \\
&= \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)}\right) + \frac{1}{2}\left(\left(t - t^{v(s)}_{k_{v(s)}(t)} - \tau\right)^2 \right. \\
&\quad \left. - \left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\right) \left(\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)}\right) - \hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right).
\end{aligned}
$$

Under case 4B, we can derive

$$
\begin{aligned}
&\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t} \int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2} \hat{z}_s(\tau_1-\tau)\, d\tau_1\, d\tau_2 \\
&= \frac{1}{2}\left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2 \hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)}\right) + \frac{1}{2}\left(\left(t - t^{v(s)}_{k_{v(s)}(t)} - \tau\right)^2 \right. \\
&\quad \left. - \left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\right) \left(\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)}\right) - \hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right) \\
&\quad + \eta\left(t^{v(s)}_{k_{v(s)}(t)} + \tau - t^{p(i)}_{k_{p(i)}(t)}\right) \eta\left(t^{p(i)}_{k_{p(i)}(t)} - t^{v(s)}_{k_{v(s)}(t)}\right) \\
&\quad \times \frac{1}{2}\left(\left(t - t^{v(s)}_{k_{v(s)}(t)} - \tau\right)^2 - \left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\right) \\
&\quad \times \left(\hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right) - \hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right).
\end{aligned}
$$

Therefore, we can derive the specific forms of $y_i(t)$ (referred to Appendix D) and $y(t)$ as follows:

$$
\begin{aligned}
y(t) &= \tilde{y}(t) + \Lambda_p D_1 \tilde{v} - \frac{1}{2}\Lambda_p^2 D_p^T I_0 D_p W_p \Theta_1(t) \hat{y}_0(t) \\
&\quad - \frac{1}{2}\Lambda_p^2 D_p^T I_0 D_p W_p \Theta_2(t) \hat{y}(t) - \frac{k}{2}\Lambda_p^2 D_p^T I_0 D_v W_v \Theta_3(t) \hat{z}_0(t) \\
&\quad - \frac{k}{2}\Lambda_p^2 D_p^T I_0 D_v W_v \Theta_4(t) \hat{z}(t) \\
&\quad + \frac{1}{2}F_3(\Theta_2(t) \otimes I_{m_p}) Y_5 + \frac{k}{2}F_4(\Theta_4(t) \otimes I_{m_p}) Z_5 \\
&\quad + \frac{1}{2}F_3(\Theta_1(t) \otimes I_{m_p}) Y_6 + \frac{1}{2}F_3(\Theta_2(t) \otimes I_{m_p}) Y_7 \\
&\quad + \frac{k}{2}F_4(\Theta_3(t) \otimes I_{m_p}) Z_6 + \frac{k}{2}F_4(\Theta_4(t) \otimes I_{m_p}) Z_7, \qquad (24)
\end{aligned}
$$

where $\tilde{y}(t) = \left[y_1\left(t^{p(1)}_{k_{p(1)}(t)}\right)\ y_2\left(t^{p(2)}_{k_{p(2)}(t)}\right) \dots y_{m_p}\left(t^{p(m_p)}_{k_{p(m_p)}(t)}\right)\right]^T$, $\Lambda_p = diag\left(t - t^{p(1)}_{k_{p(1)}(t)}, t - t^{p(2)}_{k_{p(2)}(t)}, \dots, t - t^{p(m_p)}_{k_{p(m_p)}(t)}\right)$, $\tilde{v}^T = \left[v^T\left(t^{p(1)}_{k_{p(1)}(t)}\right)\ v^T\left(t^{p(2)}_{k_{p(2)}(t)}\right) \dots v^T\left(t^{p(m_p)}_{k_{p(m_p)}(t)}\right)\right]^T$, $D_1 = \begin{bmatrix} D_{11} & & & \\ & D_{12} & & \\ & & \ddots & \\ & & & D_{1m_p} \end{bmatrix}$ with D_{1i} being the ith row vector of D_p^T, $F_3 = \begin{bmatrix} F_{31} & & & \\ & F_{32} & & \\ & & \ddots & \\ & & & F_{3m_p} \end{bmatrix}$ with F_{3i} being the ith row vector of $D_p^T I_0 D_p W_p$, $Y_5 = [Y_{51}^T Y_{52}^T \dots Y_{5m_p}^T]^T$, $Z_5 = [Z_{51}^T Z_{52}^T \dots Z_{5m_p}^T]^T$, $Y_6 = [Y_{61}^T Y_{62}^T \dots Y_{6m_p}^T]^T$, $Z_6 = [Z_{61}^T Z_{62}^T \dots Z_{6m_p}^T]^T$, $Y_7 = [Y_{71}^T Y_{72}^T \dots Y_{7m_p}^T]^T$, $Z_7 = [Z_{71}^T Z_{72}^T \dots Z_{7m_p}^T]^T$. Specific forms of $Y_{5i}, Z_{5i}, Y_{6i}, Z_{6i}, Y_{7i}$, and Z_{7i} are provided in Appendix E.

Furthermore, by introducing variables $e_y(t) = \hat{y}_0(t) - \hat{y}(t)$ and $e_z(t) = \hat{z}_0(t) - \hat{z}(t)$, we can derive

$$
\begin{aligned}
z(t) &= \tilde{z}(t) - \Lambda_v D_v^T I_0 D_p W_p \Theta_1(t) e_y(t) - \Lambda_v D_v^T I_0 D_p W_p \hat{y}(t) \\
&\quad - k\Lambda_v D_v^T I_0 D_v W_v \Theta_3(t) e_z(t) - k\Lambda_v D_v^T I_0 D_v W_v \hat{z}(t) \\
&\quad + F_1(\Theta_1(t) \otimes I_{m_v}) Y_1 + F_1(\Theta_2(t) \otimes I_{m_v}) Y_2 \\
&\quad + F_1(\Theta_1(t) \otimes I_{m_v}) Y_3 + F_1(\Theta_2(t) \otimes I_{m_v}) Y_4 \\
&\quad + kF_2(\Theta_3(t) \otimes I_{m_v}) Z_1 + kF_2(\Theta_4(t) \otimes I_{m_v}) Z_2 \\
&\quad + kF_2(\Theta_3(t) \otimes I_{m_v}) Z_3 + kF_2(\Theta_4(t) \otimes I_{m_v}) Z_4, \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
y(t) = \tilde{y}(t) &+ \Lambda_p D_1 \tilde{v} - \frac{1}{2}\Lambda_p^2 D_p^T I_0 D_p W_p \Theta_1(t) e_y(t) \\
&- \frac{1}{2}\Lambda_p^2 D_p^T I_0 D_p W_p \hat{y}(t) \\
&- \frac{k}{2}\Lambda_p^2 D_p^T I_0 D_v W_v \Theta_3(t) e_z(t) - \frac{k}{2}\Lambda_p^2 D_p^T I_0 D_v W_v \hat{z}(t) \\
&+ \frac{1}{2} F_3(\Theta_2(t) \otimes I_{m_p}) Y_5 + \frac{k}{2} F_4(\Theta_4(t) \otimes I_{m_p}) Z_5 \\
&+ \frac{1}{2} F_3(\Theta_1(t) \otimes I_{m_p}) Y_6 + \frac{1}{2} F_3(\Theta_2(t) \otimes I_{m_p}) Y_7 \\
&+ \frac{k}{2} F_4(\Theta_3(t) \otimes I_{m_p}) Z_6 + \frac{k}{2} F_4(\Theta_4(t) \otimes I_{m_p}) Z_7, \qquad (26)
\end{aligned}
$$

where $\tilde{\sigma}^T = \left[\sigma^T\left(t^{p(1)}_{k_{p(1)}(t)}\right)\sigma^T\left(t^{p(2)}_{k_{p(2)}(t)}\right)\ldots\sigma^T\left(t^{p(m_p)}_{k_{p(m_p)}(t)}\right)\right]^T$.

In Subsection II B, we have the following inequalities from rule A or rule B:

$$
\begin{cases}
a\left|\hat{y}_q\left(t_k^{p(q)}\right)\right| \le \left|y_q\left(t_k^{p(q)}\right)\right| \le b\left|\hat{y}_q\left(t_k^{p(q)}\right)\right| \\
\quad \text{and} \;\; y_q\left(t_k^{p(q)}\right)\hat{y}_q\left(t_k^{p(q)}\right) > 0, \;\; q \in \underline{m_p}, \\
c\left|\hat{z}_q\left(t_k^{v(q)}\right)\right| \le \left|z_q\left(t_k^{v(q)}\right)\right| \le d\left|\hat{z}_q\left(t_k^{v(q)}\right)\right| \\
\quad \text{and} \;\; z_q\left(t_k^{v(q)}\right)\hat{z}_q\left(t_k^{v(q)}\right) > 0, \;\; q \in \underline{m_v}.
\end{cases} \qquad (27)
$$

Furthermore, we have that

$$
\begin{cases}
0 \le \left| ||\tilde{y}(t) - \hat{y}(t)\right| | \le r_1 ||\hat{y}(t)||, \\
0 \le \left| ||\tilde{z}(t) - \hat{z}(t)\right| | \le r_2 ||\hat{z}(t)||,
\end{cases} \qquad (28)
$$

where $r_1 = \max\{1-a, b-1\}$ and $r_2 = \max\{1-c, d-1\}$. In addition, we have that

$$
\begin{cases}
|\hat{y}_q\left(t^q_{k_q(t)-1}\right) - \hat{y}_q\left(t^q_{k_q(t)}\right)| \le \theta_1 |\hat{y}_q\left(t^q_{k_q(t)}\right)|, \\
|\hat{z}_q\left(t^q_{k_q(t)-1}\right) - \hat{z}_q\left(t^q_{k_q(t)}\right)| \le \theta_2 |\hat{z}_q\left(t^q_{k_q(t)}\right)|,
\end{cases} \qquad (29)
$$

$$
\begin{cases}
|\hat{y}_q\left(t^{p(q)}_{k-1}\right)| \le \frac{b}{a} |\hat{y}_q\left(t^{p(q)}_{k}\right)|, \\
|\hat{z}_q\left(t^{v(q)}_{k-1}\right)| \le \frac{d}{c} |\hat{z}_q\left(t^{v(q)}_{k}\right)|,
\end{cases} \qquad (30)
$$

where $\theta_1 = \max\{1 - a/b, b/a - 1\}$ and $\theta_2 = \max\{1 - c/d, d/c - 1\}$. Based on the previous analysis and the definitions of $\hat{y}_0(t), \hat{y}(t), \hat{z}_0(t), \hat{z}(t)$, we have that

$$
\begin{cases}
\hat{y}(t-\tau) = \Theta_1(t)\hat{y}_0(t) + \Theta_2(t)\hat{y}(t), \\
\hat{z}(t-\tau) = \Theta_3(t)\hat{z}_0(t) + \Theta_4(t)\hat{z}(t).
\end{cases} \qquad (31)
$$

By (28), (29), and (31), we can derive that

$$
\begin{aligned}
||\tilde{y}(t) - \hat{y}(t-\tau)|| &\le ||\tilde{y}(t) - \hat{y}(t)|| + ||\Theta_1(t) e_y(t)|| \\
&\le (r_1 + \theta_1 ||\Theta_1(t)||) ||\hat{y}(t)|| \\
&= \varphi_1 ||\hat{y}(t)||, \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
||\tilde{z}(t) - \hat{z}(t-\tau)|| &\le ||\tilde{z}(t) - \hat{z}(t)|| + ||\Theta_3(t) e_z(t)|| \\
&\le (r_2 + \theta_2 ||\Theta_3(t)||) ||\hat{z}(t)|| \\
&= \varphi_2 ||\hat{z}(t)||. \qquad (33)
\end{aligned}
$$

For $||\hat{y}(t-\tau)||$ with (29) and (31), we have that

$$
\begin{aligned}
||\hat{y}(t-\tau)|| &= ||\Theta_1(t)\hat{y}_0(t) + \Theta_2(t)\hat{y}(t)|| \\
&\le (1 + \theta_1) ||\hat{y}(t)||. \qquad (34)
\end{aligned}
$$

On the other hand, we have that

$$
||\hat{y}(t-\tau)|| \ge |\theta_1 \|\Theta_1(t)\| - 1| \; ||\hat{y}(t)||. \qquad (35)
$$

Furthermore, from (35) we can derive

$$
||\hat{y}(t-\tau)|| \ge \max\{1, |\theta_1 - 1|\} ||\hat{y}(t)||. \qquad (36)
$$

Combining (34) with (36), we have

$$
\varphi_3' \hat{y}(t) \le ||\hat{y}(t-\tau)|| \le \varphi_3 \hat{y}(t), \qquad (37)
$$

where $\varphi_3 = 1 + \theta_1$ and $\varphi_3' = \max\{1, |\theta_1 - 1|\}$. Analogously, for $\|\hat{z}(t-\tau)\|$, we have

$$
\varphi_4' \hat{z}(t) \le ||\hat{z}(t-\tau)|| \le \varphi_4 \hat{z}(t), \qquad (38)
$$

where $\varphi_4 = 1 + \theta_2$ and $\varphi_4' = \max\{1, |\theta_2 - 1|\}$.

For $\tilde{\sigma}^T(t)\tilde{\sigma}(t)$, there exists some j, such that

$$
\tilde{\sigma}^T(t)\tilde{\sigma}(t) \le m_p \sigma^T\left(t^{p(j)}_{k_{p(j)}(t)}\right)\sigma\left(t^{p(j)}_{k_{p(j)}(t)}\right). \qquad (39)
$$

With the limitation of integrator interval,
for cases 1A and 1B:

$$
\begin{cases}
t^{p(j)}_{k_{p(j)}(t)-1} + \tau - t^{v(i)}_{k_{v(i)}(t)} < h, \\
t^{p(j)}_{k_{p(j)}(t)-1} + \tau - t^{p(i)}_{k_{p(i)}(t)} < h,
\end{cases} \qquad (40)
$$

for cases 2A and 2B:

$$
\begin{cases}
t^{p(j)}_{k_{p(j)}(t)} + \tau - t^{v(i)}_{k_{v(i)}(t)} < h, \\
t^{p(j)}_{k_{p(j)}(t)} + \tau - t^{p(i)}_{k_{p(i)}(t)} < h,
\end{cases} \qquad (41)
$$

for cases 3A and 3B:

$$
\begin{cases}
t^{v(j)}_{k_{v(j)}(t)-1} + \tau - t^{v(i)}_{k_{v(i)}(t)} < h, \\
t^{v(j)}_{k_{v(j)}(t)-1} + \tau - t^{p(i)}_{k_{p(i)}(t)} < h,
\end{cases} \qquad (42)
$$

for cases 4A and 4B:

$$
\begin{cases}
t^{v(j)}_{k_{v(j)}(t)} + \tau - t^{v(i)}_{k_{v(i)}(t)} < h, \\
t^{v(j)}_{k_{v(j)}(t)} + \tau - t^{p(i)}_{k_{p(i)}(t)} < h.
\end{cases} \qquad (43)
$$

Furthermore, we have that

$$
\begin{cases}
Y_1^T Y_1, Y_3^T Y_3 \le m_v h^2 \frac{b^2}{a^2} \theta_1^2 \hat{y}^T(t)\hat{y}(t) = \xi_1^2 \hat{y}^T(t)\hat{y}(t), \\
Y_2^T Y_2, Y_4^T Y_4 \le m_v h^2 \theta_1^2 \hat{y}^T(t)\hat{y}(t) = \xi_2^2 \hat{y}^T(t)\hat{y}(t), \\
Z_1^T Z_1, Z_3^T Z_3 \le m_v h^2 \frac{d^2}{c^2} \theta_2^2 \hat{z}^T(t)\hat{z}(t) = mu_1^2 \hat{z}^T(t)\hat{z}(t), \\
Z_2^T Z_2, Z_4^T Z_4 \le m_v h^2 \theta_2^2 \hat{z}^T(t)\hat{z}(t) = mu_2^2 \hat{z}^T(t)\hat{z}(t).
\end{cases} \qquad (44)
$$

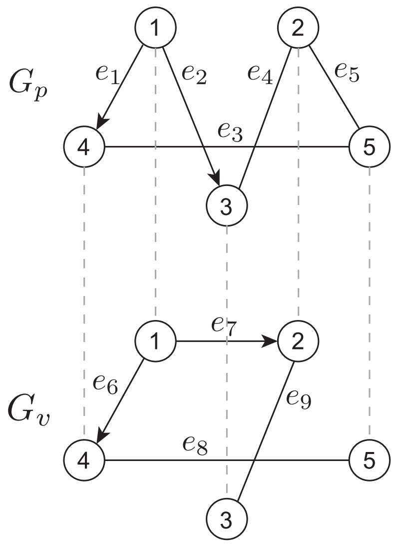

FIG. 3. Communication topology.

In addition, in $Y_5^T Y_5$, we have

$$\begin{aligned}&\left(t - t^{p(j)}_{k_{p(j)}(t)} - \tau\right)^2 - \left(t - t^{p(i)}_{k_{p(i)}(t)}\right)^2\\&= \left(t - t^{p(j)}_{k_{p(j)}(t)} - \tau + t - t^{p(i)}_{k_{p(i)}(t)}\right)\\&\quad\times \left(t - t^{p(j)}_{k_{p(j)}(t)} - \tau - t + t^{p(i)}_{k_{p(i)}(t)}\right)\\&\le h^3.\end{aligned} \tag{45}$$

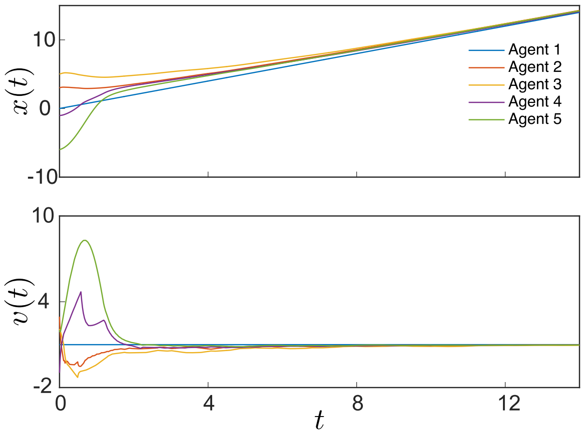

FIG. 4. State trajectory under rule *A*.

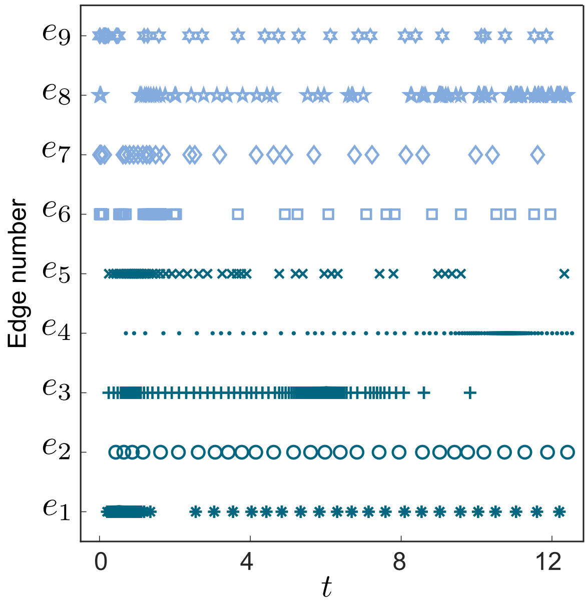

FIG. 5. Edge-event times under rule *A*.

Analogously, we can further derive

$$\begin{cases}Y_5^T Y_5 \le m_p h^6 \theta_1^2 \hat{y}^T(t)\hat{y}(t) = \xi_3^2 \hat{y}^T(t)\hat{y}(t),\\ Y_6^T Y_6, Y_7^T Y_7 \le m_p h^6 \theta_1^2 \dfrac{b^2}{a^2} \hat{y}^T(t)\hat{y}(t) = \xi_4^2 \hat{y}^T(t)\hat{y}(t),\\ Z_5^T Z_5 \le m_p h^6 \theta_2^2 \hat{z}^T(t)\hat{z}(t) = \mu_3^2 \hat{z}^T(t)\hat{z}(t),\\ Z_6^T Z_6, Z_7^T Z_7 \le m_p h^6 \theta_2^2 \dfrac{d^2}{c^2} \hat{z}^T(t)\hat{z}(t) = \mu_4^2 \hat{z}^T(t)\hat{z}(t).\end{cases} \tag{46}$$

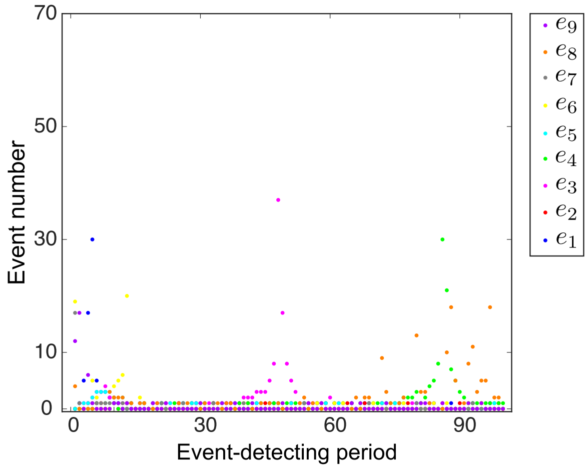

FIG. 6. The numbers of events every 0.2 s under rule *A*.

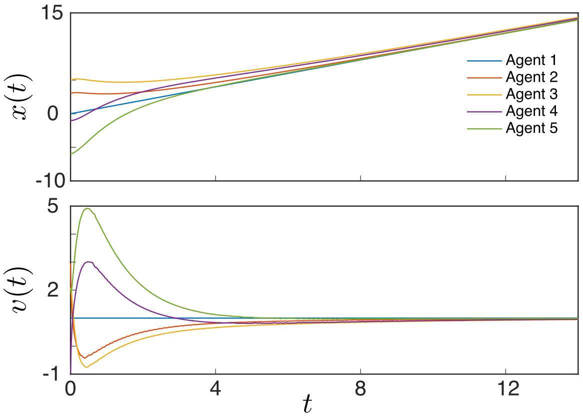

FIG. 7. State trajectory under rule *B*.

Finally, by applying the Hölder inequality, the inequality $2x^Ty \leq x^Tx + y^Ty$, and various other techniques with inequalities (27)–(46), we obtain

$$\dot{V}(t) \leq M\hat{y}^T(t)\hat{y}(t) + N\hat{z}^T(t)\hat{z}(t),$$

where M and N are given in (7) and (8), respectively. With $\gamma = \frac{k\alpha\lambda_2(\tilde{L}_v)}{2}$, $\iota > 0$ can guarantee the condition (6), which is the sufficient condition of positive definiteness of $V(t)$. Furthermore, in (9), we know $M, N < 0$. Then, $\dot{V}(t) = 0$ holds if and only if $\hat{y}(t) = \mathbf{0}$ and $\hat{z}(t) = \mathbf{0}$, which imply $\delta(t) = \sigma(t) = \mathbf{0}$. Therefore, the leader-following consensus of system (1) and (2) is achieved. □

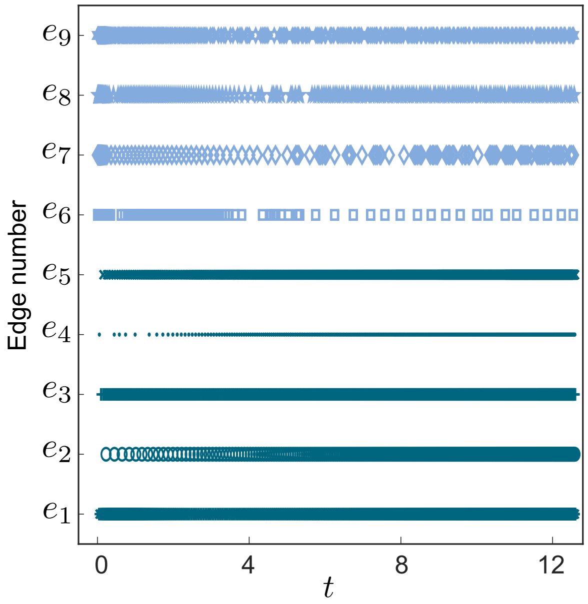

FIG. 8. Edge-event times under rule *B*.

Remark 1 Due to the complexity of symbols (7) and (8), we need to explain the existence of M and N as (7) and (8). Assume parameters $a = b = c = d = 1$, which result in $r_1 = r_2 = \theta_1 = \theta_2 = 0$ as well as $\varphi_1 = \varphi_2 = \xi_1 = \xi_2 = \xi_3 = \xi_4 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$ and $\varphi_3 = \varphi_3' = \varphi_4 = \varphi_4' = 1$. If $h = 0$, then it is clear that $M = \lambda_{\min}(W_p)\kappa < 0$ and $N = -\lambda_{\min}(W_v)\iota < 0$. Furthermore, M and N are monotone increase functions of h. Therefore, there exist $h_{\max}$ and the proper parameters such that $M, N < 0$ for $h \in (0, h_{\max})$.

IV. SIMULATION

In this section, to verify the validity of the proposed theoretical results, we present a numerical example. For systems (1) and (2) with protocol (3), heterogeneous networks are illustrated in Fig. 3, which are clearly leader–follower reachable. Assume the weights of edges in G_p and G_v are $3, 1, 2, 1, 3$ and $3, 5, 4, 5$, respectively. The system parameters are set as $h = 0.002$, $\tau = 0.001$, and $k = 4$. For the event-triggered mechanism, initial edge-event times in G_p and G_v are set to 0, 0.0005, 0.0009, 0.001, 0.0012, 0.0013, 0.0015, 0.0017, and 0.0019, respectively. The parameters are $a = 0.9$, $b = 1.1$, $c = 0.85$, and $d = 1.15$. The initial states of the system are $x(0) = [035 - 1 - 6]^T$ and $v(0) = [132 - 12]^T$.

Through numerical calculations, the corresponding results are presented in Figs. 4–9. Figures 4 and 7 show the state trajectories of systems (1) and (2) under rule *A* and rule *B*, respectively, demonstrating that leader-following consensus is ultimately achieved. Each edge of G_p and G_v performs approximately 7000 event detections in total under both rule *A* and rule *B*. The corresponding event times are depicted in Figs. 5 and 8. Additionally, the number of events for all edges in G_p and G_v under rule *A* are 89, 28, 152, 139, 47, 78, 42, 162, and 56, while under rule *B*, they are 682, 316, 479, 336, 2173, 220, 164, 216, and 379, respectively. Furthermore, the number of events per 0.2 s for each edge is presented in Figs. 6 and 9. Clearly, the number of events is significantly lower than the number of event detections, indicating that the event-triggered control mechanisms we proposed greatly reduce communication and controller-update costs for multi-agent systems.

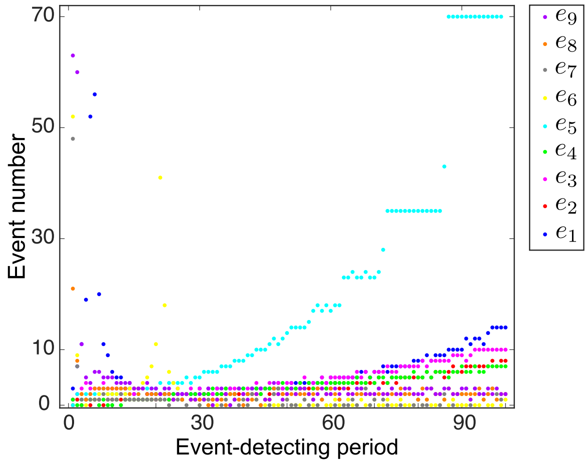

FIG. 9. The numbers of events every 0.2 s under rule *B*.

V. CONCLUSION

In conclusion, this paper presented a comprehensive approach to addressing the asynchronous leader-following consensus problem in networked double-integrator systems. By considering three critical factors, namely, asynchronous hybrid event- and time-triggered control, heterogeneous networks with distinct position and velocity graphs, and communication time delays, our proposed control protocol effectively tackled the challenges of multi-agent coordination in practical engineering scenarios. Through the use of separately defined edge events for position and velocity information, agents were able to update their controllers asynchronously based on sampled relative state information. The theoretical results, supported by Lyapunov-based analysis, were validated through numerical simulations, demonstrating the efficacy of the proposed approach in achieving consensus. Future work will continue to relax the assumption $\tau < h$ to widen more generalized frameworks and refine the event-triggering mechanisms and explore broader applications in more complex networked systems.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Gaopeng Duan: Methodology (equal); Writing – original draft (equal). **Xuecheng Yu:** Funding acquisition (equal); Writing – review & editing (equal). **Heung Wing Joseph Lee:** Project administration (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: SPECIFIC FORM OF $z_i(t)$

$$
\begin{aligned}
z_i(t) &= z_i\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right) - \sum_{j=1}^{m_p} s_{1ij} w_j^p \int_{t^{\nu(i)}_{k_{\nu(i)}(t)}}^{t} \hat{y}_j(s-\tau)ds - k\sum_{j=1}^{m_\nu} s_{2ij} w_j^\nu \int_{t^{\nu(i)}_{k_{\nu(i)}(t)}}^{t} \hat{z}_j(s-\tau)\,ds \\
&= z_i\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right) - \sum_{j=1}^{m_p} s_{1ij} w_j^p 1_{1j}(t) \Big(\left(t - t^{p(j)}_{k_{p(j)}(t)-1} - \tau\right) \hat{y}_j\left(t^{p(j)}_{k_{p(j)}(t)-1}\right) + \left(t^{p(j)}_{k_{p(j)}(t)-1} + \tau - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \hat{y}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right) \\
&\quad + \eta\left(t^{p(j)}_{k_{p(j)}(t)} - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)} - t^{p(j)}_{k_{p(j)}(t)-1} - \tau\right) \left(t^{p(j)}_{k_{p(j)}(t)-1} + \tau - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \left(\hat{y}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right) - \hat{y}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\Big) \\
&\quad - \sum_{j=1}^{m_p} s_{1ij} w_j^p 1_{2j}(t) \Big(\left(t - t^{p(j)}_{k_{p(j)}(t)} - \tau\right) \hat{y}_j\left(t^{p(j)}_{k_{p(j)}(t)}\right) + \left(t^{p(j)}_{k_{p(j)}(t)} + \tau - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \hat{y}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \\
&\quad + \eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)} - t^{p(j)}_{k_{p(j)}(t)}\right) \eta\left(t^{p(j)}_{k_{p(j)}(t)} + \tau - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \left(t^{p(j)}_{k_{p(j)}(t)} + \tau - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \left(\hat{y}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right) - \hat{y}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\right)\Big) \\
&\quad - k\sum_{j=1}^{m_\nu} s_{2ij} w_j^\nu 1_{3j}(t) \Big(\left(t - t^{\nu(j)}_{k_{\nu(j)}(t)-1} - \tau\right) \hat{z}_j\left(t^{\nu(j)}_{k_{\nu(j)}(t)-1}\right) + \left(t^{\nu(j)}_{k_{\nu(j)}(t)-1} + \tau - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \hat{z}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right) \\
&\quad + \eta\left(t^{\nu(j)}_{k_{\nu(j)}(t)} - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)} - t^{\nu(j)}_{k_{\nu(j)}(t)-1} - \tau\right) \left(t^{\nu(j)}_{k_{\nu(j)}(t)-1} + \tau - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \left(\hat{z}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right) - \hat{z}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\Big) \\
&\quad - k\sum_{j=1}^{m_\nu} s_{2ij} w_j^\nu 1_{4j}(t) \Big(\left(t - t^{\nu(j)}_{k_{\nu(j)}(t)} - \tau\right) \hat{z}_j\left(t^{\nu(j)}_{k_{\nu(j)}(t)}\right) + \left(t^{\nu(j)}_{k_{\nu(j)}(t)} + \tau - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \hat{z}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \\
&\quad + \eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)} - t^{\nu(j)}_{k_{\nu(j)}(t)}\right) \eta\left(t^{\nu(j)}_{k_{\nu(j)}(t)} + \tau - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \left(t^{\nu(j)}_{k_{\nu(j)}(t)} + \tau - t^{\nu(i)}_{k_{\nu(i)}(t)}\right) \left(\hat{z}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right) - \hat{z}_j\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\right)\Big).
\end{aligned}
$$

APPENDIX B: SPECIFIC FORMS OF Y_{1i}, Y_{2i}, Y_{3i}, AND Y_{4i}

$$Y_{1i}=\begin{bmatrix}\left(t^{p(1)}_{k_{p(1)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_1\left(t^{p(j)}_{k_{p(1)}(t)-1}\right)-\hat{y}_1\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\\ \left(t^{p(2)}_{k_{p(2)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_2\left(t^{p(j)}_{k_{p(2)}(t)-1}\right)-\hat{y}_2\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\\ \vdots\\ \left(t^{p(m_p)}_{k_{p(m_p)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_{m_p}\left(t^{p(m_p)}_{k_{p(m_p)}(t)-1}\right)-\hat{y}_{m_p}\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\end{bmatrix},$$

$$Y_{2i}=\begin{bmatrix}\left(t^{p(1)}_{k_{p(1)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_1\left(t^{p(j)}_{k_{p(1)}(t)}\right)-\hat{y}_1\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\right)\\ \left(t^{p(2)}_{k_{p(2)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_2\left(t^{p(j)}_{k_{p(2)}(t)}\right)-\hat{y}_2\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\right)\\ \vdots\\ \left(t^{p(m_p)}_{k_{p(m_p)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_{m_p}\left(t^{p(m_p)}_{k_{p(m_p)}(t)}\right)-\hat{y}_{m_p}\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\right)\end{bmatrix},$$

$$Y_{3i}=\begin{bmatrix}\eta\left(t^{p(1)}_{k_{p(1)}(t)}-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)}-t^{p(1)}_{k_{p(1)}(t)-1}-\tau\right)\left(t^{p(1)}_{k_{p(1)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_1\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)-\hat{y}_1\left(t^{p(1)}_{k_{p(1)}(t)-1}\right)\right)\\ \eta\left(t^{p(2)}_{k_{p(2)}(t)}-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)}-t^{p(2)}_{k_{p(2)}(t)-1}-\tau\right)\left(t^{p(2)}_{k_{p(2)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_2\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)-\hat{y}_2\left(t^{p(j)}_{k_{p(2)}(t)-1}\right)\right)\\ \vdots\\ \eta\left(t^{p(m_p)}_{k_{p(m_p)}(t)}-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)}-t^{p(m_p)}_{k_{p(m_p)}(t)-1}-\tau\right)\left(t^{p(m_p)}_{k_{p(m_p)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_{m_p}\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)-\hat{y}_{m_p}\left(t^{p(m_p)}_{k_{p(m_p)}(t)-1}\right)\right)\end{bmatrix},$$

$$Y_{4i}=\begin{bmatrix}\eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)}-t^{p(1)}_{k_{p(1)}(t)}\right)\eta\left(t^{p(1)}_{k_{p(1)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(t^{p(1)}_{k_{p(1)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_1\left(t^{p(1)}_{k_{p(1)}(t)}\right)-\hat{y}_1\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\\ \eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)}-t^{p(2)}_{k_{p(2)}(t)}\right)\eta\left(t^{p(2)}_{k_{p(2)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(t^{p(2)}_{k_{p(2)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_2\left(t^{p(2)}_{k_{p(2)}(t)}\right)-\hat{y}_2\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\\ \vdots\\ \eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)}-t^{p(m_p)}_{k_{p(m_p)}(t)}\right)\eta\left(t^{p(m_p)}_{k_{p(m_p)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(t^{p(m_p)}_{k_{p(m_p)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{y}_{m_p}\left(t^{p(m_p)}_{k_{p(m_p)}(t)}\right)-\hat{y}_{m_p}\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\end{bmatrix}.$$

APPENDIX C: SPECIFIC FORMS OF Z_{1i}, Z_{2i}, Z_{3i}, AND Z_{4i}

$$Z_{1i}=\begin{bmatrix}\left(t^{\nu(1)}_{k_{\nu(1)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{z}_1\left(t^{\nu(1)}_{k_{\nu(1)}(t)-1}\right)-\hat{z}_1\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\\ \left(t^{\nu(2)}_{k_{\nu(2)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{z}_2\left(t^{\nu(2)}_{k_{\nu(2)}(t)-1}\right)-\hat{z}_2\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\\ \vdots\\ \left(t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{z}_{m_\nu}\left(t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)-1}\right)-\hat{z}_{m_\nu}\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)\right)\end{bmatrix},\quad Z_{2i}=\begin{bmatrix}\left(t^{\nu(1)}_{k_{\nu(1)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{z}_1\left(t^{\nu(1)}_{k_{\nu(1)}(t)}\right)-\hat{z}_1\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\right)\\ \left(t^{\nu(2)}_{k_{\nu(2)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{z}_2\left(t^{\nu(2)}_{k_{\nu(2)}(t)}\right)-\hat{z}_2\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\right)\\ \vdots\\ \left(t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{z}_{m_\nu}\left(t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)}\right)-\hat{z}_{m_\nu}\left(t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\right)\end{bmatrix},$$

$$Z_{3i}=\begin{bmatrix}\eta\left(t^{\nu(1)}_{k_{\nu(1)}(t)}-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)}-t^{\nu(1)}_{k_{\nu(1)}(t)-1}-\tau\right)\left(t^{\nu(1)}_{k_{\nu(1)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{z}_1\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)-\hat{z}_1\left(t^{\nu(1)}_{k_{\nu(1)}(t)-1}\right)\right)\\ \eta\left(t^{\nu(2)}_{k_{\nu(2)}(t)}-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)}-t^{\nu(2)}_{k_{\nu(2)}(t)-1}-\tau\right)\left(t^{\nu(2)}_{k_{\nu(2)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{z}_2\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)-\hat{z}_2\left(t^{\nu(2)}_{k_{\nu(2)}(t)-1}\right)\right)\\ \vdots\\ \eta\left(t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)}-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\eta\left(t^{\nu(i)}_{k_{\nu(i)}(t)}-t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)-1}-\tau\right)\left(t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)-1}+\tau-t^{\nu(i)}_{k_{\nu(i)}(t)}\right)\left(\hat{z}_{m_\nu}\left(t^{\nu(i)}_{k_{\nu(i)}(t)-1}\right)-\hat{z}_{m_\nu}\left(t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)-1}\right)\right)\end{bmatrix},$$

$$Z_{4i}=\begin{bmatrix}\eta\left(t^{v(i)}_{k_{v(i)}(t)}-t^{v(1)}_{k_{v(1)}(t)}\right)\eta\left(t^{v(1)}_{k_{v(1)}(t)}+\tau-t^{v(i)}_{k_{v(i)}(t)}\right)\left(t^{v(1)}_{k_{v(1)}(t)}+\tau-t^{v(i)}_{k_{v(i)}(t)}\right)\left(\hat{z}_1\left(t^{v(1)}_{k_{v(1)}(t)}\right)-\hat{z}_1\left(t^{v(i)}_{k_{v(i)}(t)-1}\right)\right)\\ \eta\left(t^{v(i)}_{k_{v(i)}(t)}-t^{v(2)}_{k_{v(2)}(t)}\right)\eta\left(t^{v(2)}_{k_{v(2)}(t)}+\tau-t^{v(i)}_{k_{v(i)}(t)}\right)\left(t^{v(2)}_{k_{v(2)}(t)}+\tau-t^{v(i)}_{k_{v(i)}(t)}\right)\left(\hat{z}_2\left(t^{v(2)}_{k_{v(2)}(t)}\right)-\hat{z}_2\left(t^{v(i)}_{k_{v(i)}(t)-1}\right)\right)\\ \vdots\\ \eta\left(t^{v(i)}_{k_{v(i)}(t)}-t^{v(m_v)}_{k_{v(m_v)}(t)}\right)\eta\left(t^{v(m_v)}_{k_{v(m_v)}(t)}+\tau-t^{v(i)}_{k_{v(i)}(t)}\right)\left(t^{v(m_v)}_{k_{v(m_v)}(t)}+\tau-t^{v(i)}_{k_{v(i)}(t)}\right)\left(\hat{z}_{m_v}\left(t^{v(m_v)}_{k_{v(m_v)}(t)}\right)-\hat{z}_{m_v}\left(t^{v(i)}_{k_{v(i)}(t)-1}\right)\right)\end{bmatrix}.$$

APPENDIX D: SPECIFIC FORMS OF $y_i(t)$

$$\begin{aligned}y_i(t)&=y_i\left(t^{p(i)}_{k_{p(i)}(t)}\right)+\sum_{j=1}^{n}d^p_{ji}\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t}v_j\left(t^{p(i)}_{k_{p(i)}(s)}\right)ds-\sum_{j=1}^{n}d^p_{ji}\sum_{s=1}^{m_p}\left(I_0D_pW_p\right)_{js}\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t}\int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2}\hat{y}_s(\tau_1-\tau)\,d\tau_1\,d\tau_2\\ &\quad-k\sum_{j=1}^{n}d^p_{ji}\sum_{s=1}^{m_v}(I_0D_vW_v)_{js}\int_{t^{p(i)}_{k_{p(i)}(t)}}^{t}\int_{t^{p(i)}_{k_{p(i)}(\tau_2)}}^{\tau_2}\hat{z}_s(\tau_1-\tau)d\tau_1d\tau_2\\ &=y_i\left(t^{p(i)}_{k_{p(i)}(t)}\right)+\sum_{j=1}^{n}d^p_{ji}\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)v_j\left(t^{p(i)}_{k_{p(i)}(t)}\right)-\sum_{j=1}^{n}d^p_{ji}\sum_{s=1}^{m_p}\left(I_0D_pW_p\right)_{js}1_{1j}(t)\Bigg(\frac{1}{2}\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)-1}\right)\\ &\quad+\eta\left(t^{p(s)}_{k_{p(s)}(t)-1}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{p(s)}_{k_{p(s)}(t)-1}\right)\frac{1}{2}\left(\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2-\left(t-t^{p(s)}_{k_{p(s)}(t)-1}-\tau\right)^2\right)\\ &\quad\times\left(\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)-1}\right)-\hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right)\Bigg)-\sum_{j=1}^{n}d^p_{ji}\sum_{s=1}^{m_p}\left(I_0D_pW_p\right)_{js}1_{2j}(t)\\ &\quad\times\Bigg(\frac{1}{2}\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)}\right)+\frac{1}{2}\left(\left(t-t^{p(s)}_{k_{p(s)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{y}_s\left(t^{p(s)}_{k_{p(s)}(t)}\right)-\hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right)\\ &\quad+\eta\left(t^{p(s)}_{k_{p(s)}(t)}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{p(s)}_{k_{p(s)}(t)}\right)\frac{1}{2}\left(\left(t-t^{p(s)}_{k_{p(s)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right)-\hat{y}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right)\Bigg)\\ &\quad-k\sum_{j=1}^{n}d^p_{ji}\sum_{s=1}^{m_v}(I_0D_vW_v)_{js}1_{3j}(t)\Bigg(\frac{1}{2}\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right)+\eta\left(t^{v(s)}_{k_{v(s)}(t)-1}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{v(s)}_{k_{v(s)}(t)-1}\right)\\ &\quad\times\frac{1}{2}\left(\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2-\left(t-t^{v(s)}_{k_{v(s)}(t)-1}-\tau\right)^2\right)\left(\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)-1}\right)-\hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right)\Bigg)-k\sum_{j=1}^{n}d^p_{ji}\sum_{s=1}^{m_v}(I_0D_vW_v)_{js}1_{4j}(t)\\ &\quad\times\Bigg(\frac{1}{2}\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)}\right)+\frac{1}{2}\left(\left(t-t^{v(s)}_{k_{v(s)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{z}_s\left(t^{v(s)}_{k_{v(s)}(t)}\right)-\hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right)\\ &\quad+\eta\left(t^{v(s)}_{k_{v(s)}(t)}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{v(s)}_{k_{v(s)}(t)}\right)\frac{1}{2}\left(\left(t-t^{v(s)}_{k_{v(s)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)}\right)-\hat{z}_s\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)\right)\Bigg).\end{aligned}$$

APPENDIX E: SPECIFIC FORMS OF $Y_{5i}, Z_{5i}, Y_{6i}, Z_{6i}, Y_{7i}$, AND Z_{7i}

$$Y_{5i}=\begin{bmatrix}\left(\left(t-t^{p(1)}_{k_{p(1)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{y}_1\left(t^{p(i)}_{k_{p(i)}(t)}\right)-\hat{y}_1\left(t^{p(1)}_{k_{p(1)}(t)}\right)\right)\\ \left(\left(t-t^{p(2)}_{k_{p(2)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{y}_2\left(t^{p(i)}_{k_{p(i)}(t)}\right)-\hat{y}_2\left(t^{p(2)}_{k_{p(2)}(t)}\right)\right)\\ \vdots\\ \left(\left(t-t^{p(m_p)}_{k_{p(m_p)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{y}_{m_p}\left(t^{p(i)}_{k_{p(i)}(t)}\right)-\hat{y}_{m_p}\left(t^{p(m_p)}_{k_{p(m_p)}(t)}\right)\right)\end{bmatrix},$$

$$Z_{5i}=\begin{bmatrix}\left(\left(t-t^{\nu(1)}_{k_{\nu(1)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{z}_1\left(t^{p(i)}_{k_{p(i)}(t)}\right)-\hat{z}_1\left(t^{\nu(1)}_{k_{\nu(1)}(t)}\right)\right)\\ \left(\left(t-t^{\nu(2)}_{k_{\nu(2)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{z}_2\left(t^{p(i)}_{k_{p(i)}(t)}\right)-\hat{z}_2\left(t^{\nu(2)}_{k_{\nu(2)}(t)}\right)\right)\\ \vdots\\ \left(\left(t-t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{z}_{m_\nu}\left(t^{p(i)}_{k_{p(i)}(t)}\right)-\hat{z}_{m_\nu}\left(t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)}\right)\right)\end{bmatrix},$$

$$Y_{6i}=\begin{bmatrix}\eta\left(t^{p(1)}_{k_{p(1)}(t)-1}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{p(1)}_{k_{p(1)}(t)-1}\right)\left(\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2-\left(t-t^{p(1)}_{k_{p(1)}(t)-1}-\tau\right)^2\right)\left(\hat{y}_1\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)-\hat{y}_1\left(t^{p(1)}_{k_{p(1)}(t)-1}\right)\right)\\ \vdots\\ \eta\left(t^{p(m_p)}_{k_{p(m_p)}(t)-1}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{p(m_p)}_{k_{p(m_p)}(t)-1}\right)\left(\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2-\left(t-t^{p(m_p)}_{k_{p(m_p)}(t)-1}-\tau\right)^2\right)\left(\hat{y}_{m_p}\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)-\hat{y}_{m_p}\left(t^{p(m_p)}_{k_{p(m_p)}(t)-1}\right)\right)\end{bmatrix}$$

$$Y_{7i}=\begin{bmatrix}\eta\left(t^{p(1)}_{k_{p(1)}(t)}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{p(1)}_{k_{p(1)}(t)}\right)\left(\left(t-t^{p(1)}_{k_{p(1)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{y}_1\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)-\hat{y}_1\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right)\\ \vdots\\ \eta\left(t^{p(m_p)}_{k_{p(m_p)}(t)}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{p(m_p)}_{k_{p(m_p)}(t)}\right)\left(\left(t-t^{p(m_p)}_{k_{p(m_p)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{y}_{m_p}\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)-\hat{y}_{m_p}\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right)\end{bmatrix},$$

$$Z_{6i}=\begin{bmatrix}\eta\left(t^{\nu(1)}_{k_{\nu(1)}(t)-1}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{\nu(1)}_{k_{\nu(1)}(t)-1}\right)\left(\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2-\left(t-t^{\nu(1)}_{k_{\nu(1)}(t)-1}-\tau\right)^2\right)\left(\hat{z}_1\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)-\hat{z}_1\left(t^{\nu(1)}_{k_{\nu(1)}(t)-1}\right)\right)\\ \vdots\\ \eta\left(t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)-1}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)-1}\right)\left(\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2-\left(t-t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)-1}-\tau\right)^2\right)\left(\hat{z}_{m_\nu}\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)-\hat{z}_{m_\nu}\left(t^{\nu(s)}_{k_{\nu(s)}(t)-1}\right)\right)\end{bmatrix},$$

$$Z_{7i}=\begin{bmatrix}\eta\left(t^{\nu(1)}_{k_{\nu(1)}(t)}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{\nu(1)}_{k_{\nu(1)}(t)}\right)\left(\left(t-t^{\nu(1)}_{k_{\nu(1)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{z}_1\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)-\hat{z}_1\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right)\\ \vdots\\ \eta\left(t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)}+\tau-t^{p(i)}_{k_{p(i)}(t)}\right)\eta\left(t^{p(i)}_{k_{p(i)}(t)}-t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)}\right)\left(\left(t-t^{\nu(m_\nu)}_{k_{\nu(m_\nu)}(t)}-\tau\right)^2-\left(t-t^{p(i)}_{k_{p(i)}(t)}\right)^2\right)\left(\hat{z}_{m_\nu}\left(t^{p(i)}_{k_{p(i)}(t)-1}\right)-\hat{z}_{m_\nu}\left(t^{p(i)}_{k_{p(i)}(t)}\right)\right)\end{bmatrix}.$$

REFERENCES

[1] B. Piccoli, "Control of multi-agent systems: Results, open problems, and applications," Open Math. **21**, 20220585 (2023).

[2] B. Kada, M. Khalid, and M. S. Shaikh, "Distributed cooperative control of autonomous multi-agent UAV systems using smooth control," J. Syst. Eng. Electron. **31**(6), 1297–1307 (2020).

[3] J. Ota, "Multi-agent robot systems as distributed autonomous systems," Adv. Eng. Inf. **20**, 59–70 (2006).

[4] Y. Liu, J. Liu, Z. He, Z. Li, Q. Zhang, and Z. Ding, "A survey of multi-agent systems on distributed formation control," Unmanned Syst. **12**(05), 913–926 (2024).

[5] B. Chreim, M. Esseghir, and L. Merghem-Boulahia, "Energy management in residential communities with shared storage based on multi-agent systems: Application to smart grids," Eng. Appl. Artif. Intell. **126**, 106886 (2023).

[6] D. Vallejo, J. Albusac, C. Glez-Morcillo, J. J. Castro-Schez, and L. Jiménez, "A multi-agent approach to intelligent monitoring in smart grids," Int. J. Syst. Sci. **45**, 756–777 (2014).

[7] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," IEEE Trans. Autom. Control **49**, 1520–1533 (2004).

[8] W. Ren and R. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," IEEE Trans. Autom. Control **50**, 655–661 (2005).

[9] W. Sun, C. Huang, J. Lü, X. Li, and S. Chen, "Velocity synchronization of multi-agent systems with mismatched parameters via sampled position data," Chaos **26**(2), 023106 (2016).

[10] W. Ren and E. Atkins, "Second-order consensus protocols in multiple vehicle systems with local interactions," in *Proceedings of AIAA Guidance, Navigation, Control Conference, San Francisco, CA* (AIAA, 2005), AIAA-2005-6238.

[11] A. Tahbaz-Salehi and A. Jadbabaie, "A necessary and sufficient condition for consensus over random networks," IEEE Trans. Autom. Control **53**, 791–795 (2008).

[12] J. Cortés, "Finite-time convergent gradient flows with applications to network consensus," Automatica **42**, 1993–2000 (2006).

[13] Y. Dong, N. Gupta, and N. Chopra, "Content modification attacks on consensus seeking multi-agent system with double-integrator dynamics," Chaos **26**(11), 116305 (2016).

[14] M. H. Koh and R. Sipahi, "A consensus dynamics with delay-induced instability can self-regulate for stability via agent regrouping," Chaos **26**(11), 116313 (2016).

[15] G. K. Sar and D. Ghosh, "Flocking and swarming in a multi-agent dynamical system," Chaos **33**, 123126 (2023).

[16] W. Diao and W. He, "Event-triggered protocol-based adaptive impulsive control for delayed chaotic neural networks," Chaos **34**, 063132 (2024).

[17] M. J. Park, O. M. Kwon, and A. Seuret, "Weighted consensus protocols design based on network centrality for multi-agent systems with sampled-data," IEEE Trans. Autom. Control **62**, 2916–2922 (2017).

[18] R. Martinez-Guerra, C. D. Cruz-Ancona, and C. A. Perez-Pinacho, "Generalized multi-synchronization viewed as a multi-agent leader-following consensus problem," Appl. Math. Comput. **282**, 226–236 (2016).

[19] C. D. Cruz-Ancona, R. Martinez-Guerra, and C. A. Perez-Pinacho, "Generalized multi-synchronization: A leader-following consensus problem of multi-agent systems," Neurocomputing **233**, 52–60 (2017).

[20] W. Ni and D. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," Syst. Control Lett. **59**, 209–217 (2010).

[21] W. Zhu and D. Cheng, "Leader-following consensus of second-order agents with multiple time-varying delays," Automatica **46**, 1994–1999 (2010).

[22] Y. Zheng and L. Wang, "Finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements," Syst. Control Lett. **61**, 871–878 (2012).

[23] J. M. Kim, J. B. Park, and Y. H. Choi, "Leaderless and leader-following consensus for heterogeneous multi-agent systems with random link failures," IET Control Theory Appl. **8**, 51–60 (2014).

[24] J. Ma, Y. Zheng, and L. Wang, "LQR-based optimal topology of leader-following consensus," Int. J. Robust Nonlinear Control **25**, 3404–3421 (2015).

[25] N. Abaid and M. Porfiri, "Leader-follower consensus over numerosity-constrained random networks," Automatica **48**, 1845–1851 (2012).

[26] G. Duan, A. Li, T. Meng, G. Zhang, and L. Wang, "Energy cost for controlling complex networks with linear dynamics," Phys. Rev. E **99**, 052305 (2019).

[27] T. Meng, G. Duan, A. Li, and L. Wang, "Control energy scaling for target control of complex networks," Chaos, Solitons Fractals **167**, 112986 (2023).

[28] T. Qin, G. Duan, and A. Li, "Detecting the driver nodes of temporal networks," New J. Phys. **25**, 083031 (2023).

[29] A. Li, S. P. Cornelius, Y. Y. Liu, L. Wang, and A. L. Barabási, "The fundamental advantages of temporal networks," Science **358**, 1042–1046 (2017).

[30] S. M. Dibaji and H. Ishii, "Resilient consensus of double-integrator multi-agent systems," in *Proceedings of American Control Conference, Portland, OR, USA* (IEEE, 2014), pp. 5139–5144.

[31] Y. Cao and W. Ren, "Sampled-data discrete-time coordination algorithms for double-integrator dynamics under dynamic directed interaction," Int. J. Control **83**, 506–515 (2010).

[32] Y. Gao, L. Wang, G. Xie, and B. Wu, "Consensus of multi-agent systems based on sampled-data control," Int. J. Control **82**, 2193–2205 (2009).

[33] Y. Gao and L. Wang, "Sampled-data based consensus of continuous-time multi-agent systems with time-varying topology," IEEE Trans. Autom. Control **56**, 1226–1231 (2011).

[34] G. Zhang, T. Chen, and X. Chen, "Performance recovery in digital implementation of analogue systems," SIAM J. Control Optim. **45**(6), 2207–2223 (2007).

[35] W. Ren, "On consensus algorithms for double-integrator dynamics," IEEE Trans. Autom. Control **53**, 1503–1509 (2008).

[36] J. Qin, W. X. Zheng, and H. Gao, "Coordination of multiple agents with double-integrator dynamics under generalized interaction topologies," IEEE Trans. Syst. Man Cybern. B Cybern. **42**, 44–57 (2011).

[37] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," IEEE Trans. Autom. Control **57**, 1291–1297 (2012).

[38] E. Garcia, Y. Cao, H. Yu, P. Antsaklis, and D. Casbeer, "Decentralised event-triggered cooperative control with limited communication," Int. J. Control **86**, 1479–1488 (2013).

[39] Z. Zhang, F. Hao, L. Zhang, and L. Wang, "Consensus of linear multi-agent systems via event-triggered control," Int. J. Control **87**, 1243–1251 (2014).

[40] G. Zhang, X. Chen, and T. Chen, "A model predictive control approach to networked systems," in *Proceedings of 2007 46th IEEE Conference on Decision and Control, New Orleans, LA* (IEEE, 2007), pp. 3339–3344.

[41] J. Sun and Z. Zeng, "Periodic event-triggered control for networked control systems with external disturbance and input and output delays," IEEE Trans. Cybern. **53**, 6386–6394 (2022).

[42] Z. Gu, D. Yue, C. K. Ahn *et al.*, "Segment-weighted information-based event-triggered mechanism for networked control systems," IEEE Trans. Cybern. **53**, 5336–5345 (2022).

[43] Y. Wang and J. Zhao, "Periodic event-triggered sliding mode control for switched uncertain TS fuzzy systems with a logistic adaptive event-triggering scheme," IEEE Trans. Fuzzy Syst. **30**, 4115–4126 (2022).

[44] H. Yu and T. Chen, "A mean-rate event-triggered mechanism for nonlinear plants with weak time regularization," IEEE Trans. Control Network Syst. **10**, 1831–1843 (2023).

[45] F. Xiao, X. Meng, and T. Chen, "Average sampled-data consensus driven by edge events," in *Proceedings of 31st Chinese Control Conference, Hefei, China* (IEEE, 2012), pp. 6239–6244.

[46] M. Cao, F. Xiao, and L. Wang, "Second-order leader-following consensus based on time and event hybrid-driven control," Syst. Control Lett. **74**, 90–97 (2014).

[47] M. Cao, F. Xiao, and L. Wang, "Event-based second-order consensus control for multi-agent system via synchronous periodic event detection," IEEE Trans. Autom. Control **60**, 2452–2457 (2015).

[48] A. V. Balakrishnan, "On the problem of time jitter in sampling," IRE Trans. Inf. Theory **8**(3), 226–236 (1962).

[49] X. Jiang, H. Shokri-Ghadikolaei, G. Fodor, E. Modiano, Z. Pang, M. Zorzi, and C. Fischione, "Low-latency networking: Where latency lurks and how to tame it," Proc. IEEE **107**(2), 280–306 (2018).