Journal of Physics:
Conference Series

PAPER • OPEN ACCESS

A proximal bundle technique for solving equilibrium problems with inexact information

To cite this article: Yongxiu Feng *et al* 2024 *J. Phys.: Conf. Ser.* **2905** 012006

Journal of Physics: Conference Series **2905** (2024) 012006 doi:10.1088/1742-6596/2905/1/012006

A proximal bundle technique for solving equilibrium problems with inexact information

Yongxiu Feng[1], Ming Huang[1, 2], Siqi Zhang[1], Xiaodan Chao[1], and Sida Lin[1, *]

[1] School of Science, Dalian Maritime University, Dalian 116026, China
[2] Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong, China

[*]Corresponding author's e-mail: lsdsl@163.com

Abstract. In this study, we describe a bundle approach used to solve the nonsmooth equilibrium problems was described. Firstly, a general algorithm was considered, and the algorithm's validity was demonstrated by showing how it can provide a sequence that converges to an approximate solution for the equilibrium problem. Moreover, in this strategy, using the inaccurate value of the objective function and the approximate gradient of the adjacent points, a proximal bundle algorithm is created to solve equilibrium problems, and the inaccurate value makes the subproblem easy to solve and maintains convergence. So, we find a simpler way to solve the equilibrium problem.

1. Introduction

In this work, the equilibrium problem (see [8]) will be the main problem to be investigated, and it is abbreviated to (EP):

$$\text{Find } z^* \in S, \text{ satisfying } \varphi(z^*, t) \geq 0, \text{ for every } t \in S,$$

where $S \in \mathbb{R}^n$ is bounded and closed, $\mathbb{R}^n$ is nonempty and convex; and continuous differentiable function $\varphi: S \times S \to \mathbb{R}$ such that $\varphi(z, z) = 0$, where $\varphi(z, \cdot)$ is convex for every $z \in S$.

It is general, and this problem has some special cases including variational inequality problems (in [9]), complementarity problems, and fixed-point problems. One of the merits of equilibrium problems is more convenient to connect these specific issues. Furthermore, with appropriate modifications, many methods can be extended to the general equilibrium problem which is devoted to solving one of these problems.

Suppose that the set of (EP) solutions exist in the following paper. We derive the methods of the solution of (EP) based on fixed-point ideas of (EP). From $\varphi(z, z) = 0$ for each $z \in S$, it can be discovered distinctly that $z^* \in S$ will solve problem (EP) only when the problem $\min \varphi(z^*, z)$ is solved by z^*.

Then, $z^0 \in S$ is given, the sequence $\{z^j\}$ is produced by the corresponding method, for each $j \in N$,

$$z^{j+1} = \arg\min_{z \in S} \varphi(z^j, z). \tag{1}$$

While the auxiliary problem principle is more convenient for us to solve variational inequality problems, the following fixed-point property serves as the foundation for the auxiliary problem principle: $z^* \in S$ solves problem (EP) only if z^* solves the following problem

$$\min_{y\in S}\{\epsilon\varphi(z^*,t)+\omega(t)-\omega(z^*)-\langle\nabla\omega(z^*),t\rangle\},\tag{2}$$

with a smooth and strongly convex function $\omega: S\to\mathbb{R}$ and $\epsilon>0$. Then this matching fixed-point loop as follows: Given $z^j\in S$, have $z^{j+1}\in S$ which solves

$$(\mathrm{P_j})\qquad \min_{y\in S}\{\epsilon\varphi(z^j,t)+\omega(t)-\omega(z^j)-\langle\nabla\omega(z^j),t\rangle\}.$$

Since ω is strongly convex, there is only one solution to this problem. Provided certain conditions are met, this algorithm converges, there is φ strongly monotone on $S\times S$, and γ>0, which makes

$$\varphi(z,t)+\varphi(t,z)\le-\gamma\parallel t-z\parallel^2,\forall z,y\in S,\tag{3}$$

and that φ fulfils the following inequation: because $c,d>0$, this means that

$$\forall z,t,q\in S\qquad \varphi(z,t)+\varphi(t,q)\ge\varphi(z,q)-c\|t-z\|^2-d\|q-t\|^2.\tag{4}$$

When

$$\varphi(z,t)=\langle\Phi(z),t-z\rangle+\theta(t)-\theta(z),\ \forall z,t\in S\tag{5}$$

with a continuous convex function $\theta: S\to\mathbb{R}$ and mapping $\Phi: S\to\mathbb{R}^n$, where problem (EP) is simplified to $(GVIP)$ on behalf of the generalized variational inequality problem:

$$\exists z^*\in S,\text{ for each } t\in S,\text{ satisfying } \langle\Phi(z^*),t-z^*\rangle+\theta(t)-\theta(z^*)\ge 0.$$

Next, the auxiliary equilibrium problem method (for short: (AEP)) is: Haven $z^j\in S$ seek $z^{j+1}\in S$, which can solve the formulation

$$\min_{t\in S}\epsilon\left[\varphi(t)+\left\langle\Phi\left(z^j\right),t-z^j\right\rangle\right]+\omega(t)-\omega\left(z^j\right)-\left\langle\nabla\omega\left(z^j\right),t-z^j\right\rangle.\tag{6}$$

When Φ is Lipschitz continuous and strongly monotone, we can observe that (3) and (4) are satisfied respectively.

But these assumptions are overly strong in that instance. The sequence $\{z^j\}$ is produced by the (AEP) method, and when Φ is co-coercive on S, $\{z^j\}$ approaches to a solution of (GVIP) in this sense that

$$\exists\gamma>0\qquad\forall z,t\in S\qquad \langle\Phi(t)-\Phi(z),t-z\rangle\ge\gamma\|\Phi(t)-\Phi(z)\|^2.\tag{7}$$

Co-coercive property (see [10]) of Φ on S in general does not suggest that on $S\times S$, the function φ is strongly monotone. So, we try to find the algorithm's convergence under assumptions less strict than (3) and (4) and prove the solution of (EP). Moreover, we shall introduce some information about approximate function results and sub-gradients (see [2]).

It is expensive to get the exact solution and sub-gradient. Then, we propose the inexact oracle for approximating function values and sub-gradients The method is implied by inexact oracle (in [6]), which is provided as the subsequent type: at each given point $t_j\in S$ and there exists $\varepsilon\ge 0$, such that:

$$\widetilde{\varphi}_j(t_j):=\varphi_j\left(t_j\right)-\varepsilon,$$

where $\varphi_j(\cdot)$ denotes $\varphi(z^j,\cdot)$. In addition, φ is non-smooth and convex, using a calculated sub-gradient $g(\tilde{z})$ of φ at a point $\tilde{z}$ close to z can be described as $\partial\varphi(z)=\{g\in R^n|g^T(t-z)\le\varphi(t)-\varphi(z)\}$, we assume the subdifferential of φ can be described as:

$$\begin{aligned}\varphi(z)+g(\tilde{z})^T(q-z)&=\varphi(\tilde{z})+g(\tilde{z})^T(q-\tilde{z})+\alpha_0(z,\tilde{z})\\&\le\varphi(q)+\alpha_0(z,z)\qquad\text{for all } q\in\mathbb{R}^n,\end{aligned}$$

with $0\le\alpha_0(z,\tilde{z}):=\varphi(z)-\varphi(\tilde{z})-g(\tilde{z})^T(z-\tilde{z})$. Thus, $g(\tilde{z})\in\partial_{\alpha_0(z,\tilde{z})}\varphi(z)$.

Then, we hope to solve the approximate solution problem $(\tilde{P}_j)$ instead of (P_j)

$$(\widetilde{\mathrm{P}}_{\mathrm{j}})\qquad \min_{t\in S}\{\epsilon\widetilde{\varphi}(z^j,t)+\omega(t)-\omega(z^j)-\langle\nabla\omega(z^j),t\rangle\},$$

at the same time, the aforementioned (AEP) gets: Given $z^k\in S$, seek $z^{k+1}\in S$ to be the solution of the following foundation:

$$\min_{t\in S}\epsilon\left[\widetilde{\varphi}(t)+\left\langle\Phi\left(z^j\right),t-z^j\right\rangle\right]+\omega(t)-\omega\left(z^j\right)-\left\langle\nabla\omega\left(z^j\right),t-z^j\right\rangle.$$

In the next section, to solve problem (EP), a general algorithm was provided along with a convergence study of the general approach.

2. Convergence analysis of the general algorithm

This section will address the general algorithm's implementation(see, for instance [4]). If function $\tilde{\varphi}(z^j,\cdot)$ is nonlinear and convex, some ways can be challenging to solve the subproblems $(\tilde{P}_j)$. In this situation, we plan to use $\bar{\varphi}_j$ to approximate $\tilde{\varphi}(z^j,\cdot)$, where $\bar{\varphi}_j$ is piecewise linear approximate. In this way, the solution t^j of the subproblem $(\tilde{P}_j)$ with $\tilde{\varphi}(z^j,\cdot)$ substituted by $\bar{\varphi}_j$ fulfills the following inequation:

$$\tilde{\varphi}(z^j, t^j) \le \tau\bar{\varphi}_j(t^j)(0 < \mu < 1). \tag{8}$$

Let's s introduce the general algorithm, which can lay a good foundation for the next algorithm, and start with some preparation conditions that we might use.

The General Algorithm

Step 0. Given $z^0 \in S$ and $\tau \in (0,1)$. Put $j = 0$.

Step 1. Look for $\bar{\varphi}_j^i$, a τ−approximation of φ_j at the point z^j, and z^{j+1} indicate the sole solution of problem

$$(\bar{P}_j) \qquad \min_{t\in S}\{\epsilon\bar{\varphi}(z^j,t) + \omega(t) - \omega(z^j) - \langle\nabla\omega(z^j), t\rangle\},$$

Step 2. Increased i by 1 and back to Step 1.

The general algorithm can be shown to converge in two steps. Initially, we investigate the algorithm's convergence in the case where $\{z^j\}$ is limited and $\| z^{j+1} - z^j \| \to 0$. Next, necessary theorems are provided to ensure that the mentioned two attributes are content (see, for instance [1]).

Theorem 1 Suppose $\epsilon_j \ge \underline{\epsilon} > 0$ for each $j \in \mathbb{N}$. When produced sequence $\{z^j\}$ is limited satisfying $\| z^{j+1} - z^j \| \to 0, j \in \mathbb{N}$, accordingly each limit point of $\{z^j\}_{j\in\mathbb{N}}$ can be a solution of EP.

Theorem 2 Suppose that there occur $\gamma, c, d > 0$ and $h: S \times S \to \mathbb{R}$ is a nonnegative function satisfying for every $z, t, q \in S$,

$$\text{(a)}\ \varphi(z,t) \ge 0 \Rightarrow \varphi(t,z) \le -\gamma h(z,t);$$

$$\text{(b)}\ \varphi(z,q) - \varphi(t,q) - \varphi(z,t) \le ch(z,t) + d \| q - t \|^2.$$

If $\{\epsilon_j\}_{j\in\mathbb{N}}$ is not increasing, $l\epsilon_j < \frac{\beta\mu}{2d}$ for every j, and when $\frac{c}{\gamma} \le \mu \le 1$, the general algorithm make a bounded sequence $\{z_j\}_{j\in\mathbb{N}}$ and $\lim_{j\to+\infty} \| z^{j+1} - z^j \| = 0$.

Theorem 3 Suppose $\epsilon_j \ge \underline{\epsilon} > 0$ for every $j \in \mathbb{N}$ and if whole presumptions of Theorem 2 are met, there exists the sequence $\{z^j\}_{j\in\mathbb{N}}$ which is produced by the algorithm will converge to EP solution.

It is worth noting that (b), which appears in Theorem 2, can be substituted by another one. Once the series $\sum_{j=0}^{+\infty} \epsilon_j^2$ is approached, there is (b') $\varphi(z,q) - \varphi(t,q) - \varphi(z,t) \le ch(z,t) + d \| q - t \|$, so we can get more conditions to suffice convergence. To acquire the convergence, we require (a) and (b) or (a) and (b'). (a) is a monotonicity property, it indicates that φ is pseudomonotone when $h = 0$, and when $h(z,t) = \| z - t \|^2$ there φ is strongly pseudomonotone with modulus γ. The relation among them is built by the function h based on how the problem is structured. As an illustration, when $\varphi(z,t) = \theta(z) - \theta(t)$ with $\theta: S \to \mathbb{R}$ a continuous convex function. So, if (EP) is a restricted convex problem, it is sufficient to look for $h(z,t) = 0$ for any $z, t \in S$ to determine whether (a), (b) and (b') are met.

In the following part, we will consider a proximal bundle method (see, for instance [3]) to address problem (EP), in which we approximate the convex function $\varphi(z^j,\cdot)$ and then show that it approaches a result of problem (EP).

3. A proximal bundle algorithm

In this part, our goal is to obtain an algorithm for implementing the proximal bundle method by using the approximated function and subgradient (see [5]). Now, we need to state how to build a τ−approximation $\bar{\varphi}_j$ so that problem $(\bar{P}_j)$ is simpler.

When the convex function $\bar{\varphi}_j$ is piecewise linear, it is wise to build $\bar{\varphi}_j$ by creating a series of models

$$\bar{\varphi}_j^i, i = 1,2, ...$$

until $\bar{\varphi}_j^i$ is a τ−approximation of $\tilde{\varphi}_j^i$ at the point x^j for some $i_j \geq 1$. We put $z^{j+1} = t_j^{i_j}$, $\bar{\varphi}_j = \bar{\varphi}_j^{i_j}$ for $i = 1, 2, ...$ and indicate the sole result of the $(\tilde{P}_j^i)$ by t_j^i

$$(\tilde{P}_j^i) \qquad \min_{t\in S}\{\epsilon_j\bar{\varphi}_j^i(t) + \omega(t) - \omega(z^j) - \langle\nabla\omega(z^j), t\rangle\}.$$

We now assume that, for $i = 1,2, ...$, the models $\bar{\varphi}_j^i$ satisfy the following inspired conditions:

(C1) $\bar{\varphi}_j^i \leq \tilde{\varphi}$ on S;

(C2) $\bar{\varphi}_j^{i+1} \geq \tilde{\varphi}_j(t_j^i) + \langle g(t_j^i), \cdot - t_j^i\rangle$ on S;

(C3) $\bar{\varphi}_j^{i+1} \geq l_j^i$ on S;

in which $g(t_j^i)$ denotes the subgradient of $\tilde{\varphi}$ attainable at t_j^i. And $l_j^i(t) = \bar{\varphi}_j^i(t_j^i) + \langle\gamma_j^i, t - t_j^i\rangle$ $\forall t \in S$ is the affine functions, where $\gamma_j^i = \frac{1}{\epsilon_j}[\nabla\omega(z^j) - \nabla\omega(t_j^i)]$.

Serious step algorithm

Step 0. Let $z^j \in S$ and $\tau \in (0,1)$. Put $i = 1$.
Step 1. A convex function $\bar{\varphi}_j^i$ is selected to fulfill (C1)–(C3), then worked out $(\tilde{P}_j^i)$ to obtain ${t_j}^i$.
Step 2. If

$$\tilde{\varphi}_j({t_j}^i) \leq \tau\bar{\varphi}_j^i({t_j}^i),$$

put $z^{j+1} = {t_j}^i$, $i_j = i$ then STOP; z^{j+1} is a serious step.
Step 3. Increased i by 1 and back to Step 1.

Lemma 1 Given t_j^i represent the sole solution to problem $(\tilde{P}_j^i)$ for each i, and suppose that requirements (C1)–(C3) are met by the models $\bar{\varphi}_j^i$, $i \in \mathbb{N}_0$, Then

(i) $\tilde{\varphi}_j(t_j^i) - \bar{\varphi}_j^i(t_j^i) \to 0$,

(ii) $t_j^i \to \bar{t}_j \equiv \arg\min_{t\in S}\{\epsilon_j\tilde{\varphi}_j(t) + \omega(t) - \omega(z^j) - \langle\nabla\omega(z^j), t - z^j\rangle\}$,

where $i \to +\infty$.

Theorem 4 Suppose problem (EP) is not solved by z^j. After a finite number of loops i_j with $\bar{\varphi}_j^{i_j}$ (a τ-approximation of φ_j at z^j), $z^{j+1} = t_j^{i_j}$, the mentioned serious step algorithm terminates.

To improve the speed of the algorithm, we update the proximal bundle and adopt the idea of aggregate compression to manage the bundle, hoping to update the key information $(g^{n_{j+1}}, {\alpha_0}^{n_{j+1}})$ to compress the size of the bundle. In the previous part, we have mentioned the idea of the objective function and sub-gradient approximation, then we will further introduce the compression step. Given the concrete piecewise linearized function

$$\bar{\varphi}_j(z) = \varphi(z^j) + \max_{0\leq k\leq i}\{\langle g(t^k), z - z^j\rangle - \alpha_0(z^j, t^k)\}.$$

When j is too large, we want to compress it to keep only n key elements ($n \leq j$). Every pair is $(g^i, {\alpha_0}^i)$, with $g^i \in \partial_{\alpha_0((z^j,t^k)}\varphi(z^j)$.

After briefly introducing the idea of the compressed bundle and some basic concepts, combined with the general algorithm, the updated proximal bundle algorithm will be acquired (in [7]), which is effective in solving the problem (EP).

Proximal Bundle Algorithm for Solving Equilibrium Problem

Step 0. Given an original point z^0, a tolerance $\tau \in (0,1)$ and a positive sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$. Put $t_0^0 = z^0,\ j = 0,\ i = 1$.

Step 1. To get a unique optimal solution $t^i \in S$, select a convex and segmented linear function $\bar{\varphi}_j^i$ that satisfies (C1)–(C3). Then settle

$$(\bar{P}_j^i) \qquad \min_{t\in S}\{\epsilon_j \bar{\varphi}_j{}^i + \omega(t) - \omega(z_j) + \langle \nabla\omega(z^j), z - z^j\rangle,$$

where $\omega: S \to \mathbb{R}$ is differentiable and intensively convex.

Step 2. If

$$\tilde{\varphi}_j(t_j{}^i) \le \tau \bar{\varphi}_j^i(t_j{}^i),$$

set $z^{j+1} = t^i,\ t_{j+1}{}^0 = z^{j+1}$ and rise j from 1 and put $i = 0$, otherwise, z^j is remained fixed for the subsequent inner loop and set $z^{j+1} = z^j$.

Step 3. Set $g^{n_{j+1}} := g(t^{j+1})$, append $(g^{n_{j+1}},\ \alpha_0{}^{n_{j+1}})$ to the bundle, with

$$\alpha_0{}^{n_{j+1}} = \begin{cases} 0, & \text{if serious step,} \\ \varphi(z^j) - \left(\varphi(t^{j+1}) + \langle g^{j+1}, z^j - t^{j+1}\rangle\right) & \text{, if null-step.} \end{cases}$$

If serious step, improve the linearization errors:

$$\alpha_0{}^i := \alpha_0{}^i + \varphi(t^{j+1}) - \varphi(z^j) - \langle g^i, t^{j+1} - z^j\rangle, i = 1, \dots, n_{j+1} - 1.$$

Step 4. Increased i from 1 and back to Step 1.

Theorem 5 Once some j has been achieved, z^j is a solution of (EP) when the criterion $\tilde{\varphi}_j(t_j{}^i) \le \tau\bar{\varphi}_j^i(t_j{}^i)$ is never satisfied.

Theorem 6 Assume that all conditions of Theorem 2 are met, $\epsilon_j \ge \underline{\epsilon} > 0$ for every $j \in \mathbb{N}$, and the method produces the order $\{z^j\}$ which is boundless. Then $\{z^j\}$ approaches to some solution of (EP).

4. Conclusions and future work

For the equilibrium problem, we get the general form and conclusion according to the general algorithm with some properties of the problem, and then we propose a new executable bundle algorithm with inexact information, which can converge to an approximate result of the (EP) and solve it by the approximation of the imprecise data of the subdifferential and objective function for the non-smooth problem. Some specific examples of equilibrium problems, such as variational inequalities can also be solved with an inexact oracle, which is widely reliable and attractive in solving equilibrium problems. In the proximal bundle algorithm, each iteration needs to calculate the approximate point of the model to get closer to the target value. At the same time, the parameters and errors are constantly updated. Under proper conditions, the effectiveness and convergence of the algorithm are also certified.

The convergence analysis shows that this algorithm effectively solves non-smooth minimization problems. However, there are still some problems that need further study:

1. Can the proximal bundle algorithm be applied to other more extensive problems?
2. How to select more appropriate parameters to enhance our method's efficiency?
3. Whether the proximate result of the question and convergence be obtained if the approximate value of the function is an uncertain range?

Acknowledgments
This work was financially supported by the National Key Research and Development Program of China under No.2022YFB3304600; the International Postdoctoral Exchange Fellowship Program 2020 by the Office of China Postdoctoral Council under No.2020003; the National Natural Science Foundation of China under No.11701063, 11901075, 12071342; the Project by China Postdoctoral Science Foundation under No.2019M651091 and 2019M661073; the Fundamental Research Funds for the Central Universities of DMU under No.3132024194, 3132024196 and 3132024197; Dalian Youth Science and Technology Star under No.2020RQ047; the Project by Natural Science Foundation of Hebei Province under No. A2020202030.

References

[1] Diem, H. T. H., and Khanh, P. Q. (2024). Approximations of Quasi-Equilibria and Nash Quasi-Equilibria in Terms of Variational Convergence. Set-Valued and Variational Analysis, 32(1), 1.
[2] Huang, M., Yuan, J., Lin, S., Liang, X., and Liu, C. (2021). On Solving the Convex Semi-Infinite Minimax Problems via Super-linear $\mathcal{V}\,\mathcal{U}$ Incremental Bundle Technique with Partial Inexact Oracle. Asia-Pacific Journal of Operational Research, 38(05), 2140015.
[3] Huang, M., Niu, H. M., Lin, S. D., Yin, Z. R., and Yuan, J. L. (2023). A redistributed proximal bundle method for non-smooth nonconvex functions with inexact information. Journal of Industrial and Management Optimization, 19(12), 8691-8708.
[4] Huang, M., He, Y., Qiao, P. P., Lin, S. D., Li, D., and Yang, Y. (2024). Proximal decomposition of convex optimization via an alternating linearization algorithm with inexact oracles. Journal of Industrial and Management Optimization, 0-0.
[5] Huang, M., He, Y., Qiao, P., Zhang, S., and Feng, Y. (2024). Solving a Class of Non-smooth Nonconvex Optimization Problems Via Proximal Alternating Linearization Scheme with Inexact Information. Advances in Engineering Technology Research, 9(1), 497-497.
[6] Hintermüller, M. (2001). A proximal bundle method based on approximate sub-gradients. Computational Optimization and Applications, 20, 245-266.
[7] Meng, F. Y., Pang, L. P., Lv, J., and Wang, J. H. (2017). An approximate bundle method for solving non-smooth equilibrium problems. Journal of Global Optimization, 68, 537-562.
[8] Nguyen, T. T. V., Strodiot, J. J., and Nguyen, V. H. (2009). A bundle method for solving equilibrium problems. Mathematical programming, 116, 529-552.
[9] Salmon, G., Strodiot, J. J., and Nguyen, V. H. (2004). A bundle method for solving variational inequalities. SIAM Journal on Optimization, 14(3), 869-893.
[10] Zhu, D. L., and Marcotte, P. (1996). Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities. SIAM Journal on Optimization, 6(3), 714-726.