Intra-Consumer Price Discrimination with Credit Refund Policies

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Consumers often receive a full or partial refund for product returns or service cancellations. Much of the existing literature studies cash refunds, where consumers get their money back minus a fee upon a product return or service cancellation. However, not all refunds are issued in cash. Sometimes consumers receive credit that can be used for future purchases, oftentimes with an expiration term after which the credit is forfeited. We study the optimal design of credit refund policies. Different from models that consider cash refunds, we explicitly model repeated interactions between the seller and consumers over time. We assume that consumers' valuation for the product/service varies over time, and that there is an exogenous probability for product returns. Several interesting results emerge. First, a credit refund policy facilitates *intra-consumer price discrimination* for a single type of consumers with stochastic valuation. Second, an optimal policy often involves an intermediate credit expiration term, under which a consumer with a high product valuation always makes a purchase, while a consumer with a low product valuation may be induced to make a purchase as the credit approaches expiration, leading to a *demand induction* effect. Finally, a credit refund policy can be more profitable than a cash refund policy and can lead to a win-win outcome for both the firm and consumers under certain conditions. We also consider several extensions to check the robustness of our findings.

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1. Introduction

In retail and service industries, consumers often make product returns or cancel services after a purchase. For example, a traveler may cancel a flight ticket before departure due to a family or business emergency. According to a survey by the National Retail Federation (2014), the value of returned products was estimated to be $286 billion in 2014, accounting for 8.89% of total sales in the US. Toktay (2003) estimates that, for most retailers, consumer return rates range between 5% and 9% of sales. American Airlines recorded $878 million from airline change and cancellation fees in 2017, which represented about 2% of the carrier's total revenue (Sider 2018).

Upon product returns or service cancellations, consumers often receive a full or partial refund.

Hudson's Bay Company and Walmart offer a full refund for most of the products they sell, while some online retailers charge handling or restocking fees. Much of the existing literature on product returns or service cancellations studies cash refunds where consumers get their money back minus a fee upon a product return or service cancellation. However, not all refunds are issued in cash. Sometimes consumers receive credit that can be used for future purchases, oftentimes with an expiration term after which the credit is forfeited. A prominent example is the airline industry, where consumers who cancel nonrefundable fares are often issued a credit that is valid within a fixed time window (typically a year). For example, United Airlines has the following refund policy for economy class tickets:

> *"Most fares are nonrefundable, and are not eligible for voluntary refunds. However, the value of your ticket may be eligible to be applied toward the price of a new ticket for a fee. Tickets are valid one year from the date of ticket issuance."*

HelloWellness, a popular fitness center in the United States, specifies the following:

> *"Any cancellations for standard events made within 24 hours will be issued a credit only, and cannot receive a refund. The credit can be used on a future event within 6 months."*

Other examples include golfnow.com (Golfnow 2023), shopify (Abbamonte 2018), and tickets for sports and entertainment events (Ticketmaster 2021).

Abdulla et al. (2019) conduct an extensive review of the literature on consumer returns and make the following observation:

> *"The literature shows that the degree of monetary leniency, typically expressed as a refund amount or a restocking fee, is the most extensively studied return policy decision, while only a few works investigate each of the remaining leniency levers (including time, effort, scope, exchange leniency), which leave significant opportunities for future research."*

In their nomenclature, "exchange leniency" refers to the ease and convenience associated with returns, including whether a cash refund or store credit is offered. Our work studies this aspect of exchange leniency.

In this paper, we study the optimal design of credit refund policies and investigate whether and why credit refund policies can be more profitable than cash refund policies. Different from models that consider a cash refund, it is necessary to model repeated interactions between the seller and consumers over time, given that the credits consumers receive upon product returns can only be used for a later purchase. We answer the following research questions:

1. What is consumers' optimal behavior under a given credit refund policy?
2. What are the optimal price, credit refund, and expiration term for a credit refund policy?

3. Will a credit refund policy benefit both the firm and consumers, compared with a cash refund policy?

To answer these questions, we assume that a monopolist sells a nondurable product to a population of infinitesimal consumers over an infinite time horizon. In each period, there is a fixed probability that each consumer comes to the market. A consumer can be either a high-valuation consumer or a low-valuation consumer each time she arrives in the market, depending on the realization of the valuation. There is an exogenous probability that a consumer returns the product after purchase. The firm chooses its pricing and credit refund policy to maximize the long-run average revenue, where the credit refund policy includes a credit refund and an expiration term. Expired credit is forfeited.

We first analyze consumers' purchase decisions under a given price and credit refund policy. We find that a low-valuation consumer may be induced to make a purchase as the credit approaches expiration, which we call the *demand induction* effect. Specifically, there may exist a threshold τ on the expiration term such that a low-valuation consumer would not make a purchase when her credit is more than τ periods from expiration but would make a purchase otherwise. To understand this behavior, note that when the credit is far from expiration, the consumer faces a low risk of losing the credit from expiration. Therefore, a consumer would prefer to redeem the credit when her valuation is high; she would only make a purchase with a low valuation when her credit is close to expiration. This result is akin to the purchase acceleration behavior in the empirical literature that studies consumers' intertemporal purchase behavior (Inman and McAlister 1994, Kivetz et al. 2006), even though these studies do not involve credit refunds.

Building on the consumer model, we investigate the firm's decision problem and characterize the optimal price and credit refund policy. We find that there exists an intermediate optimal expiration term under which a high-valuation consumer always makes a purchase, while a low-valuation consumer may only make a purchase when she has a credit on hand. The intuition for an intermediate optimal expiration term is as follows. A longer expiration term would incentivize low-valuation consumers to delay a purchase, while a shorter expiration term would make it more likely for the credit to expire unused, and thus make them less valuable to consumers. Hence, an intermediate expiration term that balances the two effects is optimal. Importantly, consumers' optimal purchase behavior under an intermediate expiration term facilitates *intra-consumer* price discrimination. The credit refund sustains consumers in the market when their valuation is low, given that a consumer with a low valuation never makes a purchase without the credit, whereas a consumer with a high valuation always makes a purchase without the credit. Consequently, in the long run, a consumer pays a higher price when her valuation is high.

Credit refund policies are more profitable than cash refund policies when consumers do not discount their future surplus. A properly designed credit refund policy can collect the maximum revenue from high-valuation consumers. In the meantime, it can also lead low-valuation consumers to make a purchase when they have credits on hand. In contrast, under a cash refund policy, the firm either serves all consumers with a low effective price or only high-valuation consumers with a high effective price. However, when consumers discount their future surplus, credit refund policies can be less appealing than cash refund policies. This is because consumer discounting makes credits less valuable to consumers, leading to a lower consumer surplus that the firm can collect.

Adopting a credit refund policy instead of a cash refund policy can either help or hurt consumers. When the firm serves only high-valuation consumers under a cash refund policy, the aggregate consumer surplus is 0. Switching to a credit refund policy cannot hurt consumers and can lead to a win-win outcome for the firm and consumers. However, when the firm charges a low price and serves all consumers, switching to a credit refund policy can decrease both the consumer surplus and social welfare.

The main contributions of this paper are threefold. First, to the best of our knowledge, our work is the first to study credit refund policies with forward-looking consumers in the consumer returns literature. We build a stylized model to study the optimal design of credit refund policies and demonstrate that they can be more profitable than cash refund policies. Our work explains the rationale of credit refund policies and provides guidance for the optimal design of such policies in practice. Second, we show that a credit refund policy facilitates intra-consumer price discrimination, even when there is no consumer heterogeneity. Finally, we study consumers' optimal behavior under a credit refund policy and analytically show that a consumer may be induced to make a purchase as the credit approaches expiration. This result is obtained in a fully rational consumer model and does not require behavioral or psychological considerations.

The rest of the paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the model setup and analyzes consumers' decision problem under a given credit refund policy. Section 4 derives the firm's optimal choice of price and credit refund under an exogenous expiration term, while Section 5 analyzes the firm's decision problem by endogenizing the expiration term. Section 6 compares credit refund policies with cash refund policies. Section 7 considers several model extensions, and Section 8 concludes. Technical proofs and supplemental materials are relegated to the appendix.

2. Literature Review

Our work is closely related to three streams of research: consumer return policies, strategic customer behavior, and price discrimination in revenue management.

There is a significant body of research on consumer returns in the marketing and operations literature. Many papers in the consumer returns literature consider the optimal design of cash refund policies, for which the main decision is the amount of the refund or restocking fee. Shulman et al. (2009) study retailers' optimal return policies and incentives to provide product information to customers. Shulman et al. (2011) investigate the same issues in a competitive setting and find that the equilibrium restocking fees can be higher than those in a monopolistic setting. Hsiao and Chen (2012) examine the impact of quality risk (defined as the possibility of a product misfit, defect, or unconformity with the consumers' perception) on the seller's return policy. Shang et al. (2017) study how consumers' opportunistic behavior affects the seller's optimal return policy. We also study a seller's optimal design of refund policies. However, different from the aforementioned papers, we assume that the firm adopts a credit refund policy, and that consumers make repeated purchases over time.

Many authors further study how consumer refund policies affect firms' operational decisions such as pricing, inventory, and supply chain coordination. Su (2009) examines the impact of full and partial refund policies on supply chain performance and finds that the return policy may create incentive distortion under common supply contracts. Swinney (2011) considers how consumer return policies affect a firm's incentive to adopt a quick response strategy. Huang and Zhang (2019) shed light on how consumers' valuation uncertainty drives the interaction between a product line design and a refund policy design. Chen and Bell (2009) examine how consumer returns affect a firm's pricing, ordering decisions, and profit. Xie and Gerstner (2007) show that refund policies could benefit a monopolistic service provider, given that cancellations made by advance buyers enable the firm not only to collect cancellation fees, but also to resell the product to other consumers. Guo (2009) investigates the profitability of partial refund policies in a competitive environment. Ofek et al. (2011) examine the impact of product returns on competing retailers' channel choice. These papers all consider cash refund policies and their interactions with other operational decisions. It would be interesting to investigate these interactions if a firm adopts a credit refund policy instead of a cash refund policy. We consider pricing and credit refund policies jointly in our work, but do not incorporate other important elements such as inventory, supply chain coordination, and competition. Needless to say, there is ample room for further research.

Consumer returns is also a significant issue in revenue management settings. In airline revenue management, ticket cancellation has received considerable attention. Subramanian et al. (1999) analyze a classical airline seat allocation problem on a single-leg flight allowing cancellations; they find that the optimal booking limits may not always be nested according to the fare classes due to different cancellation refunds. Aydin et al. (2013) propose new models for dynamic single-leg revenue management problems that involve no-shows, cancellations, and overbooking. Bertsimas and Popescu (2003) take into account cancellations and no-shows by incorporating overbooking decisions into a seat inventory allocation problem in a network environment. Iliescu et al. (2008) use ticketing data from the Airline Reporting Corporation to estimate airline passengers' cancellation behavior. Based on Talluri and van Ryzin (2004), Sierag et al. (2015) propose a revenue management model that takes cancellations into account in addition to consumer choice behavior. This stream of research analyzes how the optimal allocation policy changes when cancellations are incorporated into the context of airline revenue management and takes the refund policies as given.

A strand of the literature considers non-cash refunds such as gift cards and store credit, which is closely related to the credit refund studied in our work. Khouja et al. (2019) compare cash refunds with non-cash refunds. However, their model setup and underlying rationale of the results are quite different from ours. To be specific, they assume a fixed consumer valuation, a positive salvage value, myopic and hedonic purchase behavior, an exogenously given redemption rate of the gift card, a full refund policy, a periodic discount, and an exogenous return duration in their main analysis. In contrast, we consider stochastic consumer valuation, a zero salvage value, forward-looking and fully rational consumers, optimal consumer redemption timing, a partial refund, and an optimal expiration term of the credit refund. There are also a few empirical papers that consider non-cash refunds. Heim and Field (2007) show that offering a cash refund or store credit is not a significant predictor of a customer's rating for a retailer's return service convenience. Heiman et al. (2015) find evidence that customers value cash refunds more than store credit. These papers analyze data collected on existing refund policies. In contrast, we analytically study the optimal design of credit refund policies. An empirical validation of our theoretical predictions is a research topic with significant potential.

Our paper considers the optimal design of a credit refund policy, together with pricing decisions in the presence of forward-looking consumers who rationally anticipate the expected value of the credit in their purchases. Therefore, it is related to the literature in pricing and strategic consumer behavior. The research in this stream abounds, and we refer readers to Shen and Su (2007), Aviv and Pazgal (2008), and Elmaghraby et al. (2008) for a comprehensive review. Most of the papers in this stream consider consumers' purchase timing decisions, and consumers make at most one purchase. In contrast, our work captures the fact that consumers interact with the seller over a long time horizon and make repeated purchases.

The broad literature on revenue management entails many tools and techniques for price discrimination, which usually exploits consumer heterogeneity. Examples include heterogeneity in valuation (Stokey 1979, Besanko and Winston 1990, Su 2007, Aviv and Pazgal 2008), waiting cost (Su 2007), arrival time (Aviv and Pazgal 2008, Dana 1998), demand quantity (Elmaghraby et al. 2008), and patience level (Besbes and Lobel 2015, Liu and Cooper 2015), etc. In contrast, we demonstrate that a credit refund policy enables *intra-consumer* price discrimination for homogeneous consumers with stochastic valuation. Such intra-consumer price discrimination was also studied by Deb (2014), who considers an intertemporal pricing model of a seller facing a single buyer with private, varying valuation over time. However, the product is nonperishable, and the consumer makes a purchase only once in Deb (2014), while our model captures the fact that the consumer makes repeated purchases for a non-durable product.

3. The Modeling Framework

We consider a monopolistic firm that offers a product to a population of infinitesimal consumers over an infinite time horizon. The market size is normalized to one. Consumers in the market can be characterized by three parameters $(\lambda, \alpha, \gamma)$. Here, λ is the probability that a consumer shows up in the market in each period. For example, it can be the probability that a consumer desires to travel in a particular month. For each purchase occasion, a consumer's valuation for the product is either v_H or v_L, where $v_H \geq v_L \geq 0$. The parameter α is the probability that a consumer's valuation is v_H. Note that our valuation assumption here explicitly captures the fact that a consumer's valuation for the product can vary over time. This assumption is reasonable for travel products. A traveler might find it more convenient or desirable to travel in certain months, in which case she has a higher valuation for a flight ticket. For ease of reference, we call a consumer with valuation v_H a high-valuation consumer and a consumer with valuation v_L a low-valuation consumer. Note that a consumer's valuation changes over time; a consumer is either a high- or a low-valuation consumer in a given period, depending on the realization of her valuation.[1] The parameter γ is the probability that a consumer returns the product after a purchase. We assume that γ is exogenous. For example, a traveler may have to cancel a flight ticket due to a change in her work schedule or a last-minute family emergency, which is exogenous.[2]

The firm chooses a credit refund policy (p, c, T) to maximize its long-run average revenue. We normalize the marginal cost of the product to 0, so we use revenue and profit interchangeably. In the triplet (p, c, T), p is the price, c is the amount of the credit refund, and T is the expiration term of the credit refund. The credit c is only valid within T periods of receiving the credit and is

[1] We consider a general distribution of consumer valuation in Appendix C.

[2] We discuss endogenous return rate in Section 7.1.

forfeited afterwards. A natural restriction is $c \leq p$; that is, the credit is less than the price paid by consumers.

In the rest of this section, we analyze the decision problem of a consumer with parameters $(\lambda, \alpha, \gamma)$ under a given credit refund policy (p, c, T) offered by the firm.

3.1 The Consumer's Decision Problem under a Given Credit Refund Policy

Before establishing the consumer's decision problem, we first introduce several assumptions. First, we assume that purchase and return occur in the same period, which is a reasonable assumption for some industries. For example, purchasing a flight ticket two weeks in advance and canceling the ticket one day before departure are within one period if the period corresponds to a month. For retail products, there is almost always a time window (say two weeks) for product returns. Our model is a good approximation if consumers do not make multiple shopping trips within the return time window. Of course, one can also consider a model in which purchase and return occur in different periods, where each period is "small." However, this will make both the consumer's and the firm's decision problems much more complicated, as the consumer needs to decide when to return the product and the firm needs to choose a product return window. We leave the analysis for such a model to future work.

Second, our formulation here assumes that the consumer does not withhold the on-hand credit when making a purchase. This assumption is valid because a credit that is not used as soon as possible when the consumer makes a purchase runs the risk of expiration while applying the credit brings an immediate benefit.

Finally, we assume that the consumer maximizes her long-run average surplus rather than total discounted surplus.[3] We make this assumption for two reasons. One is for tractability. Introducing a discount factor makes the problem more complicated and defies feasible analysis. The other is that the literature tends to attribute demand induction effect to psychological and/or behavioral factors, including consumer discounting. Instead, the demand induction effect in our work is purely driven by economic benefit.

Now, we are ready to establish the consumer's decision problem, which can be formulated as an infinite-horizon average reward dynamic program (Ross 1983). When there is an outstanding credit c, the state t is the number of periods until expiration of the credit. Also, let $t = 0$ denote the state without a credit. Let ρ^* denote the optimal average reward and $J(\cdot)$ denote the bias function.

[3] Section 7.2 considers an extension in which consumers discount future surplus and maximize total discounted surplus.

The optimality equations for the consumer's decision problem can be written as follows. We first consider the case with a credit; i.e., $t=1,\dots,T$. The optimality equations are given by

$$\begin{aligned}\rho^*+J(t)=&\alpha\lambda\max\left\{\gamma J(T)+(1-\gamma)(J(0)+v_H)-p+c,J(t-1)\right\}\\&+(1-\alpha)\lambda\max\left\{\gamma J(T)+(1-\gamma)(J(0)+v_L)-p+c,J(t-1)\right\}\\&+(1-\lambda)J(t-1),\qquad\forall t=1,\dots,T.\end{aligned}\tag{1}$$

The first two terms on the right-hand side represent situations in which the consumer shows up in the market with valuation v_H and v_L, respectively. The last term represents the situation in which the consumer does not show up in the market. When the consumer shows up in the market, she needs to decide whether to make a purchase. If a purchase is made, she pays price p net the credit c. If this purchase is returned, the consumer is offered a credit c with an expiration term T. If this purchase is not returned, the consumer enjoys the service and gains the utility $v\in\{v_L,v_H\}$ in that period. If the consumer decides not to make a purchase or does not show up in the market, then the credit c will be one period closer to expiration.

The optimality equation in the case of no credit $(t=0)$ is similar. The consumer pays the full price p if she shows up in the market and decides to make a purchase. If she makes a purchase and then returns, she receives a credit c with an expiration term T. Also, the state continues to be 0 if the consumer does not make a purchase or if she makes a purchase but does not return. Therefore, the optimality equation can be written as

$$\begin{aligned}\rho^*+J(0)=&\alpha\lambda\max\left\{\gamma J(T)+(1-\gamma)(J(0)+v_H)-p,J(0)\right\}\\&+(1-\alpha)\lambda\max\left\{\gamma J(T)+(1-\gamma)(J(0)+v_L)-p,J(0)\right\}+(1-\lambda)J(0).\end{aligned}\tag{2}$$

In the remainder of this section, we analyze the consumer's optimal decision when the credit refund policy is given by (p,c,T). We restrict ourselves to the situation in which the consumer always makes a purchase without a credit when her valuation is v_H. If the consumer does not make a purchase without a credit when her valuation is v_H, she will not stay in the market in the long run, and therefore does not contribute to the firm's long-run average revenue. This implies that the firm must choose its policy (p,c,T) to guarantee that a consumer with valuation v_H purchases without a credit. Unsurprisingly, one can verify that a consumer with valuation v_H always purchases with a credit if she purchases without a credit.

The consumer's purchase decision when her valuation is v_L is more involved. The low-valuation consumer may or may not purchase without a credit. Even if she has a credit on hand, her purchase decision may depend on the expiration term of the credit. It can be shown that the consumer is more likely to purchase with a credit when her credit is closer to expiration. That is, for a given credit refund policy (p, c, T), there may exist a threshold τ such that a low-valuation consumer makes a purchase in states $\{1, 2, \ldots, \tau\}$ and does not purchase in states $\{0, \tau+1, \ldots, T\}$.

Proposition 1 below characterizes the consumer's optimal purchase decision for any given credit refund policy (p, c, T).[4] The technical conditions in Proposition 1 involve all three policy parameters (p, c, T) and are derived by considering consumers' incentive compatibility constraints. Importantly, a lower price by itself does not guarantee consumer purchases if it is offered together with a small credit amount or a short expiration term. On the flip side, a higher price does not preclude consumers from purchasing if it is offered together with generous credit terms. A quantity of interest is the term $1-(1-\lambda)^T$, which can be interpreted as the probability that a consumer comes to the market before a newly earned credit expires.

To facilitate the presentation, we first introduce some notations. For $1 \le k \le T$, we define the following boundary prices:

$$\begin{aligned}
p^C(k) =& \gamma[1-(1-\alpha\lambda)^{T-k}(1-\lambda)^k]c + (1-\gamma)\Big\{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-k}[1-(1-\lambda)^k]\Big\}v_H \\
& + (1-\gamma)\Big\{\gamma(1-\alpha)(1-\alpha\lambda)^{T-k}[1-(1-\lambda)^k]\Big\}v_L, \\
p^D(k) =& (1-\gamma)v_L + \gamma[1-(1-\alpha\lambda)^{T-k}(1-\lambda)^k]c + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-k}[1-(1-\lambda)^k]}{\alpha+(1-\alpha)(1-\lambda)^{k-1}}(1-\lambda)^{k-1}c.
\end{aligned}$$

In the above, $p^C(k)$ is the highest price at which a high-valuation consumer is willing to purchase without a credit when the low-valuation consumer is willing to purchase in states $\{1, 2, \ldots, k\}$ only. Similarly, $p^D(k)$ is the highest price at which a low-valuation consumer is willing to purchase in state k. We can show that $p^D(k)$ decreases in k. Recall that we impose the constraint $p \ge c$, i.e., the credit is no more than the price. Therefore, we need to compare the boundaries prices $p^C(k)$ and $p^D(k)$ with c in our analysis.

Proposition 1 analyzes the consumer's decision problem.

Proposition 1 **(Consumers' Optimal Purchase Decisions)**. *Suppose the credit refund policy is given by (p, c, T). Consumers' optimal purchase decisions are as follows:*

(a) Suppose $c \le p \le \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L$, then consumers purchase in all states;

(b) Suppose $\max\{c, \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L\} < p \le \min\{p^C(T), p^D(T)\}$, then a high-valuation consumer purchases in all states and a low-valuation consumer purchases only when she has a credit on hand ($t \ge 1$);

[4] An extended version of Proposition 1, Proposition A.1 in Appendix A, contains a detailed solution to the optimality equations (1)–(2).

(c) *Suppose* $\max\{c, p^D(\tau+1)\} < p \leq \min\{p^C(\tau), p^D(\tau)\}$ *for a fixed* τ *between 1 and* $T-1$*, then a high-valuation consumer purchases in all states and a low-valuation consumer purchases only when she has a credit within* τ *periods of expiration* $(1 \leq t \leq \tau)$*.*

(d) *Suppose* $\max\{c, p^D(1)\} < p \leq \gamma[1-(1-\alpha\lambda)^T]c + (1-\gamma)v_H$*, then a high-valuation consumer purchases in all states and a low-valuation consumer does not purchase.*

(e) *If the credit refund policy* (p, c, T) *does not satisfy the conditions in Parts (a)–(d), consumers do not purchase.*

We offer a detailed explanation for the results in Proposition 1 given its importance for our subsequent analysis. Part (a) identifies the condition under which all consumers purchase. The price is set to ensure that a low-valuation consumer would purchase even without a credit; that is, the price should be lower than a low-valuation consumer's expected utility of making a purchase without a credit. Observe that $[1-(1-\lambda)^T]c$ denotes the expected value of an earned credit upon a product return, the probability of which is γ. The term v_L is the utility received by a low-valuation consumer for a product that is not returned, the probability of which is $1-\gamma$. Hence, the upper bound on price is exactly the expected value of a purchase without a credit for a low-valuation consumer. Therefore, Part (a) shows that if the price is lower than the expected value of a purchase without a credit for a low-valuation consumer, then all consumers purchase in each state.

Parts (b)–(d) consider situations in which a low-valuation consumer does not purchase without a credit, but a high-valuation consumer purchases both with and without a credit. Part (e) simply states that no consumer purchases when the conditions in Parts (a)–(d) are not satisfied. Part (b) characterizes the situation in which $\tau = T$, meaning that a low-valuation consumer always purchases whenever she has a credit on hand. Note that $p^C(T)$ and $p^D(T)$ are the boundary prices when $\tau = T$ (which implies that a high-valuation consumer purchases without a credit and a low-valuation consumer always purchases with a credit). The lower bound on the price, $\gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L$, stipulates that a low-valuation consumer does not purchase without a credit. Also, the amount of the credit cannot exceed the price.

Part (c) characterizes the situation in which $\tau < T$, meaning that a low-valuation consumer only makes a purchase when the credit will expire within τ periods. That is, a low-valuation consumer does not purchase even with a credit for $t > \tau$, but purchases with a credit in state $t \leq \tau$ due to the risk of credit expiration. The boundary price $p^C(\tau)$ is derived from the constraint that a high-valuation consumer purchases without a credit. Similarly, the boundary prices $p^D(\tau)$ and $p^D(\tau+1)$ ensure that a low-valuation consumer purchases in state τ but does not purchase in state $\tau+1$. Because $p^D(\tau)$ decreases in τ, the threshold τ decreases when the price increases. That is, a low-valuation consumer is less likely to make a purchase at a higher price.

Part (d) describes the situation in which a low-valuation consumer does not make a purchase in any state. The boundary price $p^D(1)$ is obtained from the constraint that a low-valuation consumer does not make a purchase in state 1 (which implies that the low-valuation consumer never makes a purchase in other states, given that she has the highest willingness to purchase in state 1). The upper bound on the price, $\gamma[1-(1-\alpha\lambda)^T]c+(1-\gamma)v_H$, stipulates that the high-valuation consumer makes a purchase without a credit.

4. The Firm's Decision Problem under an Exogenous Expiration Term

Due to competitive pressure or industry norm, the expiration term is often exogenous in practice. This section analyzes the firm's decision problem under an exogenous expiration term T. Section 4.1 derives the optimal price and credit refund, taking into account consumers' optimal response. Section 4.2 discusses the demand induction effect under the optimal credit refund policy. We also analyze how the firm's optimal decisions change with respect to the exogenous expiration term, which is relegated to Appendix D due to the space limit.

4.1 The Firm's Optimal Decisions

To derive the firm's optimal decisions, we formulate and solve the firm's revenue maximization problem, taking into account consumers' optimal responses. As characterized in Proposition 1, consumers' optimal responses can be divided into four cases. In our analysis, we first solve the firm's problem for each of the four cases. The overall optimal solution to the firm's problem is obtained by comparing the optimal profits across the four cases. We relegate most of the analysis to Appendix B. Section 4.1.1 provides an overview and introduces necessary notations for each of the four case. Section 4.1.2 provides some details of the analysis of Case III, because it lays the foundation for our discussion in the case of endogenous T in Section 5. Section 4.1.3 compares the four cases and gives the firm's optimal decisions.

4.1.1 An Overview of the Four Cases This section briefly summarizes the four cases corresponding to Parts (a)–(d) of Proposition 1. Additional details of the cases can be found in Appendix B.

In Case I, consumers purchase in each state, independent of their valuation realization (Proposition 1(a)). The optimal profit, price, and credit amount are denoted by π_1^*, p_1^*, and c_1^*, respectively.

In Case II, a low-valuation consumer purchases whenever she has a credit on hand (Proposition 1(b)). To facilitate our analysis, we introduce several thresholds on the credit refund c. For each fixed T and $k \in \{1, \dots, T\}$, let

$$A(k) = (1-\gamma)(v_H - v_L)\Big\{\frac{\alpha}{(1-\lambda)^{k-1}} + 1 - \alpha\Big\}, \tag{3}$$

$$B(T,k)=\frac{(1-\gamma)v_H-(1-\gamma)(v_H-v_L)\gamma(1-\alpha)(1-\alpha\lambda)^{T-k}[1-(1-\lambda)^k]}{1-\gamma\Big(1-(1-\alpha\lambda)^{T-k}(1-\lambda)^k\Big)},$$

$$C(T,k)=\frac{(1-\gamma)v_L\Big(\alpha+(1-\alpha)(1-\lambda)^{k-1}\Big)}{(1-\gamma)\Big(\alpha+(1-\alpha)(1-\lambda)^{k-1}\Big)-(1-\lambda)^{k-1}\Big(1-\gamma(1-\alpha\lambda)^{T-k+1}\Big)}. \tag{4}$$

Here, $A(k)$ is a threshold on the credit refund c such that $p^C(k)\le p^D(k)$ if and only if $c\ge A(k)$. The terms $B(T,k)$ and $C(T,k)$ are thresholds on c such that $c\le p^C(k)$ if $c\le B(T,k)$ and $c\le p^D(k)$ if $c\le C(T,k)$. We also define a threshold on the expiration term that is critical to the firm's optimal decisions:

$$T_0=\max\big\{T\in\mathbb{N}:A(T)\le B(T,T)\big\}. \tag{5}$$

Lemma A.3 in Appendix A shows that T_0 exists and is finite. If $c\in[A(T),B(T,T)]$ (by the definition of T_0, this is possible only when $T\le T_0$), then $c\le p^C(T)\le p^D(T)$. By Proposition 1(b), the optimal price is $p^C(T)$. Similarly, for any $T\ge T_0+1$, the optimal price is $p^D(T)$. The optimal profit, price, and credit amount are denoted by $(\pi^*_{2,a},p^*_{2,a},c^*_{2,a})$ when $T\le T_0$ and $(\pi^*_{2,b},p^*_{2,b},c^*_{2,b})$ when $T\ge T_0+1$.

In Case III, a low-valuation consumer purchases when she has a credit within τ periods of expiration for some $\tau<T$ (Proposition 1(c)). However, the threshold τ does not admit an explicit characterization. To find the optimal τ and profit, we solve for the maximum profit for each fixed τ and compare them. We show some details for the analysis of Case III in Section 4.1.2 below, because it lays the foundation for our discussion in the case of endogenous T in Section 5.

In Case IV, a low-valuation consumer does not purchase in any state (Proposition 1(d)). The optimal profit, price, and credit amount are denoted by π^*_4, p^*_4, and c^*_4, respectively.

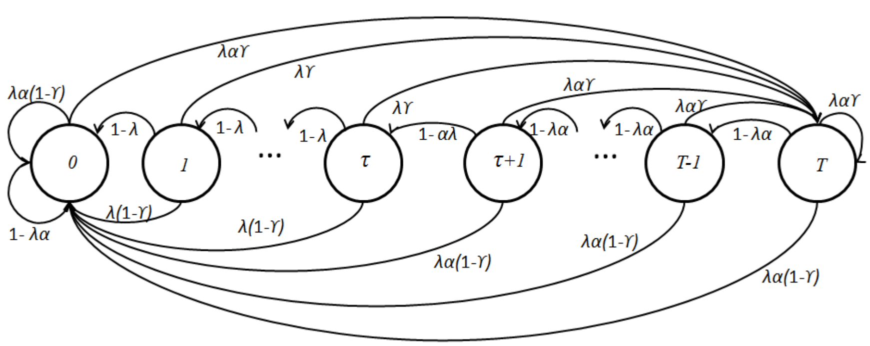


Figure 1 State transition diagram in Case III.

4.1.2 The Firm's Optimal Decision in Case III. The state transitions in Case III are illustrated in Figure 1. For each state $j\in\{0,\tau+1,\dots,T\}$, only high-valuation consumers purchase. Therefore, there is a probability $\lambda\alpha\gamma$ that a consumer purchases and returns the product (the transition from j to T), a probability $\lambda\alpha(1-\gamma)$ that a consumer purchases but does not return the product (the transition from j to 0), and a probability $1-\lambda\alpha$ that a consumer does not purchase (the transition from j to $j-1$ for $j>0$ and the transition from 0 back to 0). For each state $j\in\{1,\ldots,\tau\}$, both high- and low-valuation consumers purchase. Hence, there is a probability $\lambda\gamma$ that a consumer purchases and returns the product (the transition from j to T), a probability $\lambda(1-\gamma)$ that a consumer purchases and does not return the product (the transition from j to 0), and a probability $1-\lambda$ that a consumer does not purchase (the transition from j to $j-1$). Solving the balance equations yields the stationary probabilities; see Lemma B.3 in Appendix B.1.3.

Note that both high- and low-valuation consumers purchase in state $j\in\{1,2,\ldots,\tau\}$ and pay price $p-c$. Only high-valuation consumers purchase in state $\{0,\tau+1,\ldots,T\}$. They pay price p in state 0 and price $p-c$ in state $j\in\{\tau+1,\ldots,T\}$. Hence, the average profit contribution of each consumer is

$$\begin{aligned}&\lambda\Big\{q_0\alpha p+\sum_{j=1}^{\tau}(p-c)q_j+\sum_{j=\tau+1}^{T}\alpha(p-c)q_j\Big\}\\=&\frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^{\tau}]}\Big\{p-\gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^{\tau}]c\Big\}.\end{aligned}\tag{6}$$

The firm's revenue maximization problem is given by

$$\max_{p,c}\quad (6)\tag{7}$$

$$\text{s.t.}\quad p\le\gamma J(T)+(1-\gamma)v_H,\tag{8}$$

$$p\le\gamma J(T)+(1-\gamma)v_L+c-J(\tau-1),\tag{9}$$

$$p\ge\gamma J(T)+(1-\gamma)v_L+c-J(\tau),\tag{10}$$

$$p\ge c.\tag{11}$$

In the above, the expressions for $J(\cdot)$ are given in Proposition A.1 in Appendix A. Constraint (8) stipulates that a high-valuation consumer makes a purchase without a credit on hand. Constraint (9) ensures that a low-valuation consumer purchases when the credit is within τ periods of expiration, and constraint (10) ensures that a low-valuation consumer does not purchase when the credit is $\tau+1$ periods from expiration. We use π_3^τ to denote the profit for a fixed τ. Eventually, we compare the profits associated with different values of τ to obtain the solution to the problem (7)–(11). The optimal solution is relegated to Proposition B.3 in Appendix B.1.3.

4.1.3 Comparison. The firm's optimal decisions and profit under an exogenously fixed expiration term T can be obtained by comparing the profits across Cases I-IV. The results are summarized in Theorem 1.

THEOREM 1. *Suppose the expiration term T is exogenously fixed. Let π^*, p^*, and c^* denote the firm's optimal profit, price, and credit, respectively. We have the following results:*

(a) Suppose $T \leq T_0$*, then*[5]

$$\pi^* = \max\{\pi_1^*, \pi_{2,a}^*\},\ (p^*, c^*) = \begin{cases} (p_1^*, c_1^*), & \text{if } \pi_1^* \geq \pi_{2,a}^*, \\ (p_{2,a}^*, c_{2,a}^*), & \text{otherwise;} \end{cases}$$

(b) Suppose $T \geq T_0 + 1$*, then*[6]

$$\pi^* = \max\{\pi_1^*, \pi_3^{\hat{\tau}}\},\ (p^*, c^*) = \begin{cases} (p_1^*, c_1^*), & \text{if } \pi_1^* \geq \pi_3^{\hat{\tau}}, \\ (p^D(\hat{\tau})|_{c=c^*}, \min\{A(\hat{\tau}), C(T, \hat{\tau})\}), & \text{otherwise.} \end{cases}$$

In the above, $\hat{\tau} = \arg\max_{\tau \in \{1,\ldots,T\}} \pi_3^{\tau}$.

Recall that π_1^* is the expected profit in Case I when consumers purchase in each state, regardless of their valuation realization. Case I is unlikely to be optimal, except when v_L is close to v_H or α is small. We can also show that Case IV is always dominated by either Case II or Case III. When $T \leq T_0$, the profit in Case IV is lower than that of Case II because the firm collects the maximum revenue from all high-valuation consumers as well as some revenue from low-valuation consumers in Case II. When $T \geq T_0 + 1$, the firm can set the price and credit to achieve price discrimination between high- and low-valuation consumers, leading to a higher profit in Case III than Case IV; this result is presented in Lemma 2 later in the paper.

When v_L is not close to v_H and α is not that small (i.e., Case I is not optimal), we only need to compare the profits under Cases II and III. Theorem 1 indicates that, for a small value of T (i.e., $T \leq T_0$), Case II is optimal. Intuitively, when the expiration term is relatively short, consumers prefer to use the credit immediately, even if the realized valuation is low, given that the risk of losing the credit is high. Therefore, under the optimal price and credit refund, a low-valuation consumer always makes a purchase when she has a credit. Hence, for a relatively small T, at optimality, a low-valuation consumer always makes a purchase when she has a credit on hand (Case II).

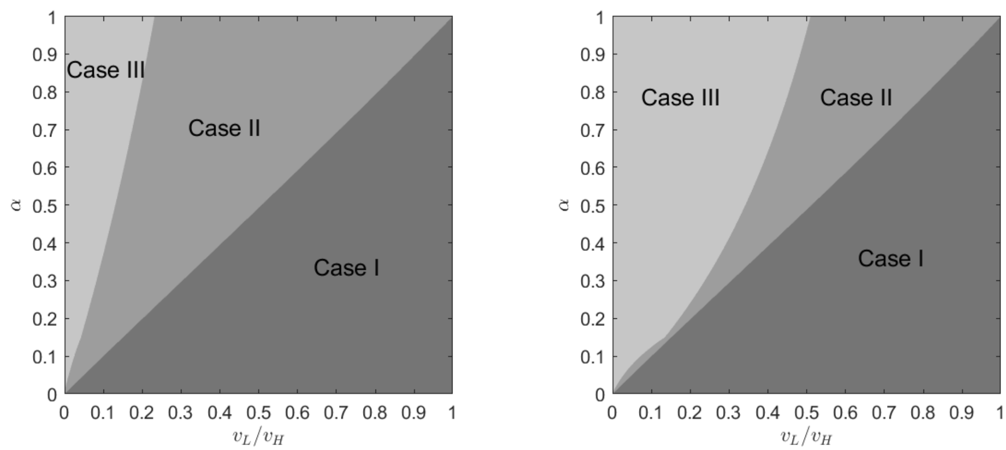
For a relatively large T (i.e., $T \geq T_0 + 1$), Theorem 1 indicates that Case III dominates Case II.[7] Intuitively, when the expiration term is relatively long, consumers face a low risk of losing the newly earned credit. Therefore, a low-valuation consumer would rather wait when her credit is far from expiration and only make a purchase when her credit is close to expiration. In particular, there is a threshold $\hat{\tau}$ such that a low-valuation consumer would only make a purchase when her credit is within $\hat{\tau}$ periods of expiration. In this case, although inducing low-valuation consumers to make a purchase in each state (except state 0) can increase the purchase probability, it also lowers

[5] Note that if $\alpha v_H \geq v_L$, then $\pi_{2,a}^* > \pi_1^*$. And, if α is sufficiently small or v_L is sufficiently close to v_H, then $\pi_1^* \geq \pi_{2,a}^*$.

[6] We point out that $C(T, \tau)$ is positive in most regions of the parameter space. If $C(T, \tau)$ is negative for any fixed T and τ, then price $p^D(\tau)$ is infeasible and cannot be optimal. We can also verify that $C(T, \hat{\tau})$ is positive.

[7] One can verify that $\pi_{2,b}^* = \pi_3^T$. That is, $\pi_3^{\hat{\tau}}$ is the maximum between the optimal profit in Case II (when $T \geq T_0 + 1$) and the optimal profit in Case III. However, our numerical investigation shows that $\pi_{2,b}^*$ is smaller than Case III's profit in most regions of the parameter space. Hence, loosely speaking, Case III is optimal for a relatively large T.

the price that the firm can charge; the price must be low enough to ensure that low-valuation consumers are willing to make a purchase in each state.[8] The firm must strike a balance between the increase in purchase probability and the decrease in price. Thus, for a relatively large T, the firm's optimal strategy is to induce the low-valuation consumers to make a purchase only when the risk of credit expiration is large enough (i.e., $t \le \hat{\tau}$). The optimal price is $p^D(\hat{\tau})$, which is derived from the constraint that low-valuation consumers purchase in state $\hat{\tau}$. We comment here that the risk of credit expiration plays a key role in determining the value of $\hat{\tau}$.



(a) $T = 3$ (b) $T = 6$

Figure 2 Regions where different cases are optimal. Note that Case IV is never optimal.

Figure 2 illustrates Theorem 1 by showing the regions of the four cases under the exogenous expiration terms $T = 3$ and $T = 6$. The horizontal axis is v_L/v_H and the vertical axis is α. We take $(\lambda, \gamma) = (0.15, 0.15)$. Figure 2 shows that Case I is optimal when v_L is close to v_H or α is small (the lower right region). When v_L/v_H is relatively small, the optimal case first switches from Case I to Case II, and then from Case II to Case III, as α increases. When v_L/v_H is relatively large, the optimal case switches from Case I to Case II as α increases. To understand the switch between Case II and Case III, note that as α increases, the proportion of low-valuation consumers decreases. Thus, Case II, which earns more profits from low-valuation consumers (by inducing them to make a purchase in each state other than 0), is less likely to be optimal. As v_L/v_H increases (the low-valuation consumers' willingness to pay increases), by the same logic, Case II is more likely to

[8] The larger the value of T, the lower the risk of credit expiration, and thus, the lower the consumer's incentive to make a purchase in state T. Hence, in order to attract the consumer to make a purchase in state T, the firm must decrease the price to a sufficiently low level.

be optimal.[9] Moreover, comparing Figure 2(a) and Figure 2(b) shows that as T increases, Case III is more likely to be optimal, which is consistent with Theorem 1. Finally, as expected, Case IV is never optimal.

4.2 Demand Induction

Proposition 1(b)-(c) shows that when low-valuation consumers' credit is close to expiration, they may be induced to make a purchase in order to mitigate the risk of credit expiration. This section illustrates this demand induction phenomenon under the optimal credit refund policy through a representative numerical example.

t	0	1	2	3	4
v_H	1	1	1	1	1
v_L	0	1	1	1	1

t	0	1	2	3	4	5	6	7
v_H	1	1	1	1	1	1	1	1
v_L	0	1	1	1	1	1	0	0

Table 1 Consumers' optimal purchase decisions when $T=4$ (left) and $T=7$ (right). The values 1 and 0 denote purchase and no-purchase, respectively.

Table 1 shows consumers' optimal purchase decisions in each state under the firm's optimal credit refund policy with the exogenous expiration terms $T=4$ and $T=7$, respectively. Here, we take $(\lambda, \alpha, \gamma, v_H, v_L) = (0.2, 0.4, 0.15, 1, 0.4)$. In this example, a high-valuation consumer always makes a purchase upon arrival, even without a credit. In contrast, a low-valuation consumer does not make a purchase when she does not have a credit ($t=0$). Under a short expiration term ($T=4$), a low-valuation consumer makes a purchase whenever she has a credit on hand. However, under a longer expiration term ($T=7$), a low-valuation consumer only makes a purchase when the credit is within 5 periods from expiration ($t \leq 5$). When there is still a long time for the credit to expire (i.e., $t > 5$), a low-valuation consumer does not make a purchase. This result is also consistent with our finding in Theorem 1; when v_L is not close to v_H and α is not too small, Case II (Case III) is optimal for a relatively small (large) expiration term.

5. The Firm's Decision Problem under an Endogenous Expiration Term

This section analyzes the firm's decision problem under an endogenous expiration term T. Section 5.1 derives the firm's optimal decisions. Section 5.2 discusses the intra-consumer price discrimination achieved by the credit refund policy. We also conduct a sensitivity analysis to see how the optimal expiration term changes with different problem parameters; the results are relegated to Appendix G.

[9] One can verify that T_0 decreases in α. Hence, as α increases, it is more likely that $T > T_0$. Therefore, Case III is more likely to be optimal. Similarly, one can verify that T_0 increases in v_L for any fixed v_H. Thus, as v_L increases, it is more likely that $T < T_0$. Therefore, Case II is more likely to be optimal.

5.1 The Firm's Optimal Decisions

Our analysis assumes, without loss of generality, that there is a finite upper-bound $\bar{T} < \infty$ such that $T \le \bar{T}$. The analysis leverages our understanding of the problem under an exogenous expiration term. One challenge in the analysis is that the firm's profit depends on the threshold τ, which does not admit an explicit characterization for a given T. Therefore, directly analyzing how the firm's profit changes with respect to T is difficult. To address this challenge, we show that the firm's profit first increases and then decreases in T when the distance between τ and T is *fixed*. This structural property allows us to construct a candidate set for the optimal expiration term.

Similar to the analysis under an exogenous expiration term, we consider the four cases outlined in Section 4.1. Propositions B.1 and B.4 in Appendix B.1 show that the optimal refund scheme is not unique and the expiration term can take any positive integer in Cases I and IV. Therefore, there is no need to find the optimal expiration term for Cases I and IV. For Cases II and III, we analyze, for each *fixed* $i \in \{0, 1, \ldots, T-1\}$, how the optimal profit changes with respect to T when $\tau = T - i$. For each fixed i, the optimization problem can be written as

$$\max_{p,c} \quad \lambda \left\{ \alpha q_0 p + \sum_{j=1}^{T-i} q_j (p-c) + \sum_{j=T-i+1}^{T} \alpha q_j (p-c) \right\} \tag{12}$$

$$s.t. \quad p \le \gamma J(T) + (1-\gamma) v_H, \tag{13}$$

$$p \le \gamma J(T) + (1-\gamma) v_L + c - J(T-i-1), \tag{14}$$

$$p \ge \gamma J(T) + (1-\gamma) v_L + c - J(T-i), \tag{15}$$

$$p \ge c. \tag{16}$$

The problem (12)–(16) is almost the same as the problem (7)–(11) except that τ is replaced by $T-i$ for each *fixed* i. The objective function and constraints can be similarly interpreted. The complete solution to the problem (12)–(16) is quite involved and is provided in Lemma A.5 in Appendix A. We provide a summary of the analysis below.

A critical quantity in our analysis is the threshold value T_i, which is defined as

$$T_i = \max\{T \in \mathbb{N} : A(T-i) \le B(T, T-i)\}, \quad \forall 0 \le i \le \bar{T} - 1.$$

Note that the definition reduces to (5) when $i = 0$. The following lemma establishes the existence of T_i's and the ordering relations among them.

LEMMA 1. *(a) For each $1 \le i \le \bar{T} - 1$, T_i exists and is finite;*
(b) $T_0 \le T_1 \le \cdots \le T_{\bar{T}-1}$.

The next Lemma presents some properties of the optimal profit in the problem (12)–(16). It has two parts corresponding to $T \le T_i$ and $T \ge T_i + 1$, respectively.

Lemma 2. *For each fixed* $i \in \{0,1,\ldots,\bar{T}-1\}$*, let* $\pi^*_{3,T-i}$ *denote the optimal profit in the problem* (12)–(16)*. We have the following results:*

(a) If $T \le T_i$*, then* $\pi^*_{3,T-i}$ *is increasing in* T*. Moreover,*

$$\pi^*_{3,T-i} = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\frac{\alpha(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}v_L; \tag{17}$$

(b) Suppose $T \ge T_i + 1$*. If* $C(T, T-i) \ge 0$*, then* $\pi^*_{3,T-i}$ *is decreasing in* T*. Otherwise, there is no feasible solution to the problem* (12)–(16)*.*

Lemma 2 establishes that, for each fixed $i \in \{0,1,\ldots,\bar{T}-1\}$, when $\tau = T-i$, the optimal profit first increases and then decreases in T, attaining the maximum at either T_i or T_i+1. This result can be used to construct an efficient algorithm to find the optimal expiration term. Note that T_i may exceed $\bar{T}$ for a large value of i. Let

$$\tilde{i} = \max\{i : T_i \le \bar{T}\}. \tag{18}$$

To find the optimal expiration term, it suffices to compare the profits $\pi^*_{3,T-i}$ for $T \in \{T_i : 0 \le i \le \tilde{i}\} \cup \{T_i + 1 : 0 \le i \le \tilde{i}\}$. The next theorem summarizes this result.

Theorem 2. *The optimal expiration term* $T^* \in \{T_0, T_0+1, T_1, T_1+1, \ldots, T_{\tilde{i}}, T_{\tilde{i}}+1, \bar{T}\}$*.*

i \ T	1	2	3	4	5	6	7	8	9	10	T_i
0	1	2	3	4	5	6	7	8	9	10	5
1	–	1	2	3	4	5	6	7	8	9	6
2	–	–	1	2	3	4	5	6	7	8	8
3	–	–	–	1	2	3	4	5	6	7	9
4	–	–	–	–	1	2	3	4	5	6	10
5	–	–	–	–	–	1	2	3	4	5	11
6	–	–	–	–	–	–	1	2	3	4	12
7	–	–	–	–	–	–	–	1	2	3	13
8	–	–	–	–	–	–	–	–	1	2	14
9	–	–	–	–	–	–	–	–	–	1	15

Table 2 The values of $\tau = T - i$ for different i and T. The last column shows the value of T_i.

We illustrate Theorem 2 with a numerical example in Table 2. The parameter values are the same as Table 1 and we take $\bar{T} = 10$. Using the definition of $\tilde{i}$ in (18), $\tilde{i} = 4$. When $i = 0$, $T_0 = 5$. Following Lemma 2, the optimal profit first increases when $T \le 5$ and then decreases when $T \ge 6$, achieving the maximum when T is either 5 or 6. The situation is similar for $1 \le i \le 4$. However, for $i \ge 5$, $T_i > 10$ and the optimal profit increases in T when $T \le 10$. Moreover, it can be verified that for any $T \le 10$, the profit for all $i \ge 5$ is less than that when $i = 4$ because $\pi^*_{3,T-i}$ in (17) is decreasing in i. Consequently, the optimal expiration term T^* must be in the set $\{T_0, T_0+1, T_1, T_1+1, T_2, T_2+1, T_3, T_3+1, T_4, \bar{T}\} = \{5,6,7,8,9,10\}$. We comment here that $T_4+1 > \bar{T}$, so it is removed from the set. To find the optimal expiration term, it suffices to compare the optimal profit for each T in the set, which gives $T^* = T_0 = 5$.

5.2 Intra-Consumer Discrimination

Following Lemma 2, this section discusses the mechanism of a credit refund policy. We can show that a credit refund policy achieves price discrimination between high- and low-valuation consumers, which is a form of intra-consumer discrimination toward the same consumer with stochastic valuation.

It suffices for us to discuss the mechanism of a credit refund policy for any exogenous $T \le \bar{T}$. For a given T, there always exists an $i \in \{0, 1, \dots, \bar{T}-1\}$ such that $T_{i-1} < T \le T_i$. Following Lemma 2 and Lemma A.5 in Appendix A (which includes the complete solution to the problem (12)–(16)), the firm can set the following price to ensure that a high-valuation consumer purchases in each state, while a low-valuation consumer purchases in states $\{1, \dots, T-i\}$ and does not make a purchase in states $\{0, T-i+1, \dots, T\}$:

$$\underbrace{\gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c}_{\text{the value of credit upon a product return}} + \underbrace{(1-\gamma)\Big\{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\Big\}v_H}_{\text{the utility from purchase and consumption when valuation is high}}$$
$$+ \underbrace{(1-\gamma)\Big\{\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\Big\}v_L}_{\text{the utility from purchase and consumption when valuation is low}}.$$

The first term denotes the expected value of the earned credit upon a product return, while the second and third terms denote the expected utility that a consumer obtains from making a purchase and then not returning the product when her valuation is high and low, respectively. Therefore, the price is equal to the expected utility of making a purchase by a consumer with stochastic valuation.

Under this credit refund policy, the firm can achieve the profit

$$\lambda(1-\gamma)\alpha v_H + \lambda \sum_{j=1}^{T-i} q_j (1-\gamma)(1-\alpha) v_L, \tag{19}$$

where q_j is the stationary probability of state j. The composition of the price implies that the firm's long-run average profit equals a consumer's long-run average utility when she makes a purchase and does not return the product. The probability that a high-valuation consumer arrives, makes a purchase, and then does not return the product is $\lambda(1-\gamma)\alpha$; hence, the expected profit is

$\lambda(1-\gamma)\alpha v_H$, which is also an upper bound on the profit that the firm can collect from a high-valuation consumer. Meanwhile, the probability for a low-valuation consumer to do the same is $\lambda\sum_{j=1}^{T-i} q_j(1-\gamma)(1-\alpha)$. Thus, the expected profit collected from a low-valuation consumer equals the second term in (19). It indicates that a high-valuation consumer pays an effective price v_H, while a low-valuation pays an effective price v_L. Moreover, a high-valuation consumer has a higher purchase probability than a low-valuation consumer, given that a high-valuation consumer makes a purchase in each state but a low-valuation consumer only makes a purchase in states $\{1,\ldots,T-i\}$. This allows the firm to achieve intra-consumer discrimination toward the same consumer with stochastic valuation. Note that a low-valuation consumer does not make a purchase without a credit (in state 0); however, a high-valuation consumer makes a purchase in each state, and the earned credit sustains her in the market when her valuation is low and motivates her to make a purchase in states $\{1,\ldots,T-i\}$. In the end, a low-valuation consumer pays a price $p-c$ for each purchase, while a high-valuation consumer pays a price p in state 0 and $p-c$ in other states, thus leading to distinct effective prices paid by low- and high-valuation consumers.

6. Comparison between Cash Refund and Credit Refund Policies

This section compares a credit refund policy with a cash refund policy. The cash refund policy is formally introduced in Section 6.1. Section 6.2 reports the comparison results and discusses the managerial insights.

6.1 The Cash Refund Policy

Suppose the firm offers a cash refund policy (p,r), where p is the price and r is the refund. A high-valuation consumer who purchases the product obtains a utility of $(1-\gamma)v_H+\gamma r$. Similarly, a low-valuation consumer who purchases the product obtains a utility of $(1-\gamma)v_L+\gamma r$. The firm chooses (p,r) such that either only high-valuation consumers make a purchase or all consumers make a purchase. For a given r, the maximum price is given by $(1-\gamma)v_H+\gamma r$ and $(1-\gamma)v_L+\gamma r$, respectively. The corresponding net revenue per consumer is $(1-\gamma)v_H$ and $(1-\gamma)v_L$, respectively. Hence, the optimal per period profit, denoted by π^c, is

$$\max\{\lambda(1-\gamma)v_L,\lambda(1-\gamma)\alpha v_H\}=\begin{cases}\lambda\alpha(1-\gamma)v_H, & \text{if } \alpha v_H\geq v_L,\\ \lambda(1-\gamma)v_L, & \text{if } \alpha v_H<v_L.\end{cases} \tag{20}$$

That is, the effective price paid by consumers is either v_H where only high-valuation consumers make a purchase, or v_L where all consumers make a purchase. For ease of reference, we call it cash refund policy with effective price v_H or v_L. Note that the optimal price-refund pair (p,r) is not unique.

One may think that the firm's revenue can be improved by customizing the price and refund for high- and low-valuation consumers. The discussion below shows that this is not the case. Suppose the firm can offer a menu (p_H, r_H) and (p_L, r_L) such that high-valuation consumers choose (p_H, r_H) and low-valuation consumers choose (p_L, r_L). From incentive compatibility, we must have

$$(1-\gamma)v_H + \gamma r_H - p_H \geq (1-\gamma)v_H + \gamma r_L - p_L, \;\; (1-\gamma)v_L + \gamma r_L - p_L \geq (1-\gamma)v_L + \gamma r_H - p_H.$$

It follows that $\gamma(r_H - r_L) \geq p_H - p_L \geq \gamma(r_H - r_L)$. Therefore, we must have $p_H - p_L = \gamma(r_H - r_L)$, or alternatively $p_H = p_L + \gamma(r_H - r_L)$. In order to guarantee a nonnegative consumer surplus, we must have $(1-\gamma)v_H + \gamma r_H - p_H \geq 0$ and $(1-\gamma)v_L + \gamma r_L - p_L \geq 0$. It follows that $p_L \leq (1-\gamma)v_L + \gamma r_L$. In order to maximize the profit, the firm chooses the highest prices possible, which are given by $p_L = (1-\gamma)v_L + \gamma r_L$ and $p_H = (1-\gamma)v_L + \gamma r_H$. The corresponding profit is $\lambda(1-\gamma)v_L$. Therefore, offering different prices and refunds for high- and low-valuation consumers does not improve the seller's profit.

6.2 Comparison

This section compares a credit refund policy with a cash refund policy in terms of both the firm profit and consumer welfare.

Our first result states that a credit refund policy with any exogenous expiration term at least weakly dominates a cash refund policy in terms of the firm profit. This result immediately implies that a credit refund policy with an optimally chosen expiration term also dominates a cash refund policy.

THEOREM 3. *(a) The firm's profit under a credit refund policy with any exogenous expiration term is at least as high as that of a cash refund policy; strict domination happens when* $v_L/v_H \leq \alpha < 1$.

(b) If the expiration term is endogenously chosen and $T^* = T_0$,[10] *then the profit ratio between a credit refund policy and a cash refund policy increases in* γ, *and increases in* α *for* $\alpha < v_L/v_H$ *and decreases otherwise.*

In a cash refund policy, the firm sets the effective price at either v_L, at which all consumers make a purchase, or v_H, at which only high-valuation consumers make a purchase. A credit refund policy allows the firm to achieve finer market segmentation. If the firm uses a credit refund policy (p, c, T) such that low-valuation consumers always make a purchase, regardless of whether they have a

[10] The numerical studies reveal that $T^* = T_0$ in most regions of the parameter set.

credit on hand (Case I), then the corresponding profit is $\lambda(1-\gamma)v_L$, which equals the profit under the cash refund policy with effective price v_L. The firm can also set (p, c, T) such that low-valuation consumers never make a purchase, regardless of whether they have a credit on hand (Case IV), then the corresponding profit is $\lambda\alpha(1-\gamma)v_H$, which equals the profit under the cash refund policy with effective price v_H. However, the firm also has the option to set the policy parameters such that low-valuation consumers make a purchase only in states $\{1, \ldots, T-i\}$ for some $0 \le i < T$ and collects the profit in (19); see also the relevant discussion in Section 5.2. A credit refund policy allows the firm to collect the maximum revenue v_H from high-valuation consumers, even when the price p is lower than v_H. Moreover, the credit can motivate consumers to make a purchase, even when they have a low valuation, leading to an increase in the profit.

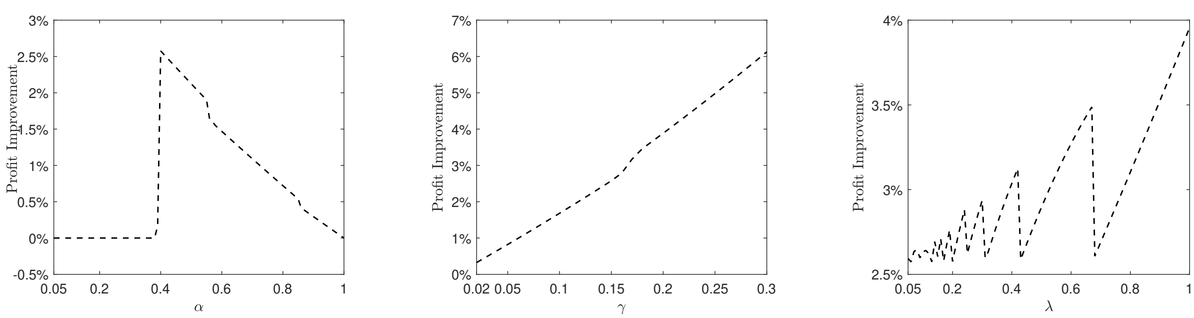
The superiority of a credit refund policy over a cash refund policy arises from its ability to facilitate intra-consumer discrimination. Therefore, the stronger the power of intra-consumer discrimination, the more superior the credit refund policy. Note that the optimal profit by setting $T^* = T_0$ can be written as

$$\lambda(1-\gamma)\max\left\{\alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}v_L, v_L\right\}, \tag{21}$$

while the profit of the cash refund policy is $\lambda(1-\gamma)\max\{\alpha v_H, v_L\}$. The power of intra-consumer discrimination hinges on low-valuation consumers' purchase probability in the first term within the curly brackets in equation (21). How do the problem parameters, including the fraction of high-valuation consumers, consumers' return rate, and arrival rate, impact intra-consumer discrimination and the profitability of credit refund policies? We focus on the more interesting case when $v_L/v_H \le \alpha < 1$.[11] Theorem 3(a) shows that when $v_L/v_H \le \alpha < 1$, the credit refund policy can strictly dominate the cash refund policy. However, as α continues to increase, although the firm's profit increases, the advantage of the credit refund policy over the cash refund policy diminishes. Note that when α increases, the fraction of low-valuation consumer decreases, and the value of T_0 also decreases in α, so the extra profit collected from the low-valuation consumers decreases in α. In the extreme case when α reaches one (which means that all consumers in the market are high-valuation consumers), the credit refund policy cannot earn more profit than the cash refund policy. This means that if there are too few low-valuation consumers, adopting credit refund policies brings little benefits over cash refund policies. Hence, to enjoy a substantial benefit associated

[11] It is easy to see when α is sufficiently small, both equation (21) and the profit of the cash refund policy reduce to $\lambda(1-\gamma)v_L$, and thus the profit ratio becomes 1. As α increases but is still smaller than v_L/v_H, equation (21) reduces to $\lambda(1-\gamma)\left\{\alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}v_L\right\}$, while the profit of the cash refund policy is still $\lambda(1-\gamma)v_L$. One can check that the profit ratio is increasing in both α and γ.

with credit refund policies, a certain amount of low-valuation consumers is a necessity. In this regard, compared with expensive and large airlines where most consumers have a high-valuation, it is more appropriate for small regional airline to offer credit refund when consumers cancel their flights. Second, as the return rate γ increases, the firm's profit decreases because the firm can earn a profit only on the products that are not returned. However, as the return rate increases, the credit earned by high-valuation consumers is more likely to be utilized by low-valuation consumers, leading to an increase in low-valuation consumers' purchase probability. This is confirmed by the first term within the curly brackets in equation (21). Moreover, the value of T_0 increases in γ, which also increases low-valuation consumers' purchase probability. Therefore, as consumers' return rate increases, the power of intra-consumer discrimination is enhanced, and the credit refund policy enjoys a bigger advantage over the cash refund policy. This means that the products with a relatively higher return rate are more suitable to adopt credit refund policies. Hence, for business passengers who have a higher return rate than leisure passengers, the airline firms should offer credit refund rather than cash refund. Lastly, the impact of arrive rate on the profit ratio is more involved. As the arrival rate λ increases, on the one hand, the low-valuation consumers arrives at the market more frequently, which makes it less likely for a credit to expire. On the other hand, the firm would decrease the value of T_0 accordingly, which may increase the likelihood of credit expiration. Overall, it is not clear how the arrival rate λ affects the low-valuation consumer's purchase probability and thus the power of intra-consumer discrimination.



(a)$(\lambda,\gamma)=(0.2,0.15)$ (b)$(\alpha,\lambda)=(0.4,0.2)$ (c)$(\alpha,\gamma)=(0.4,0.15)$

Figure 3 Sensitivity analysis of the profit improvement of credit refund policy over cash refund policy with respect to α, γ, and λ.

We illustrate the sensitivity results using the following numerical example. Figure 3 takes $(v_H, v_L)=(10,4)$ and characterizes how the profit improvement percentage changes with respect to the different parameters. Figure 3(a) sets $(\lambda,\gamma)=(0.2,0.15)$ and varies α from 0.05 to 1 with a step size of 0.01. Consistent with Theorem 3(b), Figure 3(a) shows that when α is small (few high-valuation consumers), the profit improvement is zero; this is because both credit and cash refund policies induce low-valuation consumers to purchase whenever they visit the market, yielding the same profit. When α crosses a threshold, credit refund policies can price discriminate high- and low-valuation consumers, leading to a higher profit than cash refund policies. As α continues to increase, the profit improvement tends to decrease because the fraction of low-valuation consumers becomes smaller and discriminating between high- and low-valuation consumers is less valuable. Figure 3(b) takes $(\alpha, \lambda) = (0.4, 0.2)$ and varies γ from 0.02 to 0.3 with a step size of 0.01. Unsurprisingly, Figure 3(b) shows that the profit improvement is increasing in γ, indicating that products with higher return rates are more likely to benefit from credit refund policies. Finally, Figure 3(c) takes $(\alpha, \gamma) = (0.4, 0.15)$ and varies λ from 0.05 to 1 with a step size of 0.01. Figure 3(c) shows that the impact of arrival rate on the profit improvement is more nuanced. However, in general, as the arrival rate increases, the profit improvement tends to increase, indicating that it is more appropriate to adopt credit refund policies for products with higher purchase frequency.

Admittedly, the result in Theorem 3 is strong and seems to be at odds with the popularity of cash refund policies in practice. Indeed, credit refund policies and cash refund policies co-exist in the market. Their adoption varies from industry to industry and across firms in the same industry. Our result here lends credence to the wide use of credit refund policies. However, our analysis is abstracted away from several factors that may have influenced the comparison with cash refund policies. First, our analysis assumes that consumers maximize their average surplus and do not discount future surplus. Focusing on the average surplus retains the tractability of the problem and we believe that it still captures the essence of the problem. However, one can readily see that if consumers discount future surplus, a credit refund policy would be less appealing to the firm. This is because consumers' perceived value of the credit goes down when they discount future surplus, leading to lower consumer surplus that the firm can collect. Second, we do not consider the cost of operating a credit refund policy. A cash refund policy is relatively easy to operate and does not require record-keeping over time, while credit refunds have to be tracked and maintained in a central repository. Incorporating the cost into the model will also render the credit refund policy less effective. Third, normally, credit refund is less appealing than cash refund which does not impose any restrictions to consumers, so when the firm switches from cash refund to credit refund, the effective market size may shrink. However, there are also factors that would make a credit refund policy even more appealing. One such factor is consumer forgetfulness — consumers may forget to use credit that they have earned, thereby benefiting the firm. Our ultimate goal here is not to claim that a credit refund policy always dominates a cash refund policy in all practical situations, but rather offer an explanation for its wide use in practice.

Proposition 2. *(a) When the firm switches from a cash refund policy with effective price v_H to a credit refund policy, both the consumer surplus and social welfare increase.*

(b) When the firm switches from a cash refund policy with effective price v_L to a credit refund policy, both the consumer surplus and social welfare decrease.

Proposition 2(a) shows that the switch from a cash refund policy with effective price v_H to a credit refund policy can lead to a win-win outcome for the firm and consumers. When the firm sets the effective price to v_H in a cash refund policy, only high-valuation consumers make a purchase; hence, the aggregate consumer surplus is 0. Because the aggregate consumer surplus is nonnegative for a credit refund policy, it is at least as high as that under a cash refund policy. It follows immediately that the social welfare (the sum of the firm's profit and consumer surplus) increases as well.

However, the switch from a cash refund policy with effective price v_L to a credit refund policy can hurt the consumer surplus. When the firm sets the effective price to v_L in a cash refund policy, both high- and low-valuation consumers make a purchase; hence, the aggregate consumer surplus is $\lambda\alpha(1-\gamma)(v_H - v_L)$. In Case II or III of a credit refund policy, the average price paid by the consumer lies between v_L and v_H; hence, the consumer surplus is less than $\lambda\alpha(1-\gamma)(v_H - v_L)$. One can check that the social welfare for a cash refund policy in this case is $\lambda(1-\gamma)(\alpha v_H + (1-\alpha)v_L)$, which is the maximum social welfare possible. Although the firm can achieve intra-consumer price discrimination via a credit refund policy, low-valuation consumers do not make a purchase in each state (their purchase probabilities are less than 1), so the sum of the firm's profit collected from low-valuation consumers and consumer surplus is less than $\lambda(1-\gamma)(1-\alpha)v_L$; hence, the social welfare is less than that in a cash refund policy.

7. Extensions

This section considers several extensions that generalize several assumptions in our base model.

7.1 Endogenous Return Rate

Our model assumes that consumers' return rate is exogenous. This assumption fits well with travel products, such as flight tickets. Travelers with prior experience on a flight usually have a pretty good idea about the flight experience before making a purchase. However, a traveler may need to change the travel plan and cancel the ticket due to exogenous factors such as emergencies or changes in business meetings, etc. Much of the existing literature on consumer returns focuses on the situation in which returns are driven by endogenous valuation heterogeneity. Modeling work in this literature stream typically assumes that consumers have valuation uncertainty toward the product before making a purchase, and this uncertainty is resolved after receiving the product. In such models, consumer return rates are endogenous. Whether return rates should be exogenous or endogenous, therefore, depends on specific applications. In online retail, consumers can only assess products without physically touching or feeling them before purchase; it is reasonable to expect that consumers have considerable valuation uncertainty when making a purchase. Retail in physical stores, on the other hand, often allows consumers to assess products more accurately, and as a result, their valuation uncertainty is substantially reduced. Nevertheless, consumers may still choose to return their purchases for various reasons. In practice, consumer returns may be driven by both exogenous and endogenous factors.

In Section 5.2, we point out that a credit refund policy helps the firm achieve intra-consumer discrimination. It works well when return rates are exogenous, as assumed in our analysis. When return rates are endogenous (e.g., driven by valuation uncertainty before purchase), we expect that a credit refund continues to be effective in a different model setup. Suppose the firm implements dynamic pricing strategy. Intuitively, the consumer chooses to return the product only when the realized valuation is low. Upon a product return, the consumer receives a credit with an expiration term. As the credit approaches expiration, the consumer may make a purchase at a higher retailing price in order to use the credit. On the contrary, she will never make a purchase at the higher price without a credit on hand. This is another form of demand induction effect. Because the consumer always makes a purchase before knowing her valuation towards the product, it seems that the credit refund policy cannot discriminate the consumer with a high-valuation realization and the consumer with a low-valuation realization. However, it is possible that the firm takes advantage of consumers' demand induction effect; that is, given that the consumer has a very high incentive to use the credit when it is close to expiration, the firm may charge a slightly higher effective price than the consumer's valuation expectation, leading to a profit improvement than cash refund policy. Anyway, it is definitely worthwhile to investigate this interesting question in future work.

7.2 Consumer Discounting

Our main analysis assumes that consumers maximize their long-run average surplus. This means that consumers do not discount their future surplus. An alternative assumption is that consumers discount their future surplus with a per-period discount factor $\delta \in (0,1)$. Under consumer discounting, the optimality equations of consumers' decision problem can be written as follows:

$$
\begin{aligned}
J(t) = &\alpha\lambda \max\left\{\gamma\delta J(T) + (1-\gamma)(\delta J(0) + v_H) - p + c, \delta J(t-1)\right\} \\
&+ (1-\alpha)\lambda \max\left\{\gamma\delta J(T) + (1-\gamma)(\delta J(0) + v_L) - p + c, \delta J(t-1)\right\}
\end{aligned}
$$

$$+ (1-\lambda)\delta J(t-1), \qquad \forall t = 1, \ldots, T, \tag{22}$$

$$J(0) = \alpha\lambda \max\left\{\gamma\delta J(T) + (1-\gamma)(\delta J(0) + v_H) - p, \delta J(0)\right\}$$
$$+ (1-\alpha)\lambda \max\left\{\gamma\delta J(T) + (1-\gamma)(\delta J(0) + v_L) - p, \delta J(0)\right\} + (1-\lambda)\delta J(0). \tag{23}$$

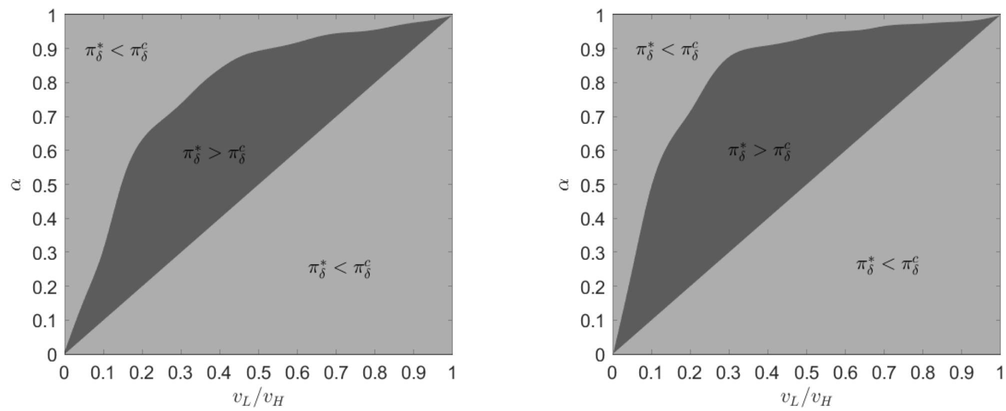
Explicit solutions of the equations are possible but are substantially more complicated than the average reward case. Moreover, analyzing the firm's profit maximization problem (taking consumers' responses into account) is extremely challenging, if not impossible. Despite this difficulty, we are still able to obtain some analytical results regarding the comparison between credit and cash refund policies in this setting. In particular, we show that a credit refund policy can be worse than a cash refund policy. Let π^*_δ and π^c_δ denote the profits of a credit refund policy and a cash refund policy, respectively.

Proposition 3. *(a) If v_L is sufficiently close to v_H or α is sufficient small, then $\pi^*_\delta \le \pi^c_\delta$.*

(b) If $v_L/v_H \le \alpha < 1$ and δ is sufficiently close to 1, then $\pi^*_\delta \ge \pi^c_\delta$.

Recall that, when consumers maximize their average surplus, as assumed in our main analysis, Figure 2 shows that when v_L is sufficiently close to v_H or α is sufficient small, Case I is optimal and the firm's profit from a credit refund policy is $\lambda(1-\gamma)v_L$, which is equal to that of a cash refund policy. When consumers discount the future surplus, the expected value of a credit is discounted as well; however, it makes no difference to a cash refund. Hence, the optimal profit of a credit refund policy with consumer discounting is less than that of a cash refund policy when v_L is sufficiently close to v_H or α is sufficient small. That is, consumer discounting makes a credit refund policy less appealing and makes it possible that a cash refund policy outperforms a credit refund policy. However, when the impact of consumer discounting is limited (i.e., δ is sufficiently close to 1) and the effect of price discrimination is more pronounced (e.g., when α takes an intermediate value), a credit refund policy can still be more profitable than a cash refund policy.

Figures 4(a) and (b) illustrate the comparison results for $\delta = 0.96$ and 0.98, respectively. All other parameter values are the same as those in Figure 2(a). In Figures 4(a) and (b), a cash refund policy is better than a credit refund policy in the lower-right region, consistent with Proposition 3(a). Interestingly, a cash refund policy also dominates in the upper-left region where v_L/v_H is small and α is large. When α is large, the faction of low-valuation consumers is small, limiting the benefit of intra-consumer discrimination achieved by a credit refund policy. On top of it, a lower value of v_L further mitigates the benefit. Comparing Figures (a) and (b), the region where the credit refund policy dominates is enlarged when δ is greater. This is not too surprising since the discounting effect is less prominent for a larger value of δ.



(a) $\delta = 0.96$ (b) $\delta = 0.98$

Figure 4 Credit refund policies vs. cash refund policies in the presence of consumer discounting.

Overall, consumer discounting dilutes the power of intra-consumer price discrimination, but the credit refund policy can still perform better than the cash refund policy under certain conditions. In addition, our numerical results reveal that the demand induction phenomena are more pronounced when consumers discount their future surplus. That is, as the credit on hand gets closer to expiration, consumers are more likely to purchase than the case without discounting. For instance, with the same parameter values and the same credit refund policy as in Table 1 ($T = 7$), low-valuation consumers start to purchase in state 6 (7) when $\delta = 0.97$ (0.95). When there is no discounting ($\delta = 1$), low-valuation consumers only start to purchase in state 5. This observation can be explained by the fact that consumers gain more benefits from early purchases in the presence of discounting.

8. Concluding Remarks

This paper considers the optimal design of credit refund policies, together with pricing decisions, in the presence of forward-looking consumers. We first analyze a consumer's decision problem under a given price and credit refund policy. We find that a consumer may be induced to make a purchase as the credit approaches expiration, which we call the *demand induction* effect. Note that this demand induction effect is purely driven by economic benefits instead of psychological or behavioral considerations. We then characterize the firm's optimal pricing and credit refund policy with either exogenous or endogenous expiration terms. We show that there exists an intermediate optimal expiration term, under which a high-valuation consumer always makes a purchase, while a low-valuation consumer may make a purchase only when she has a credit on hand. Importantly, such consumer behavior facilitates intra-consumer price discrimination. The credit refund earned by a high-valuation consumer sustains her in the market when her valuation is low. Consequently, in the long run, a high-valuation consumer pays a higher price than that when her valuation is low. Finally, we compare credit refund policies with cash refund policies, and conclude that credit refund policies can be more profitable. Moreover, the switch from cash refund policies to credit refund policies can lead to a win-win outcome for both the firm and consumers under certain conditions. Our work sheds light on the rationale of credit refund policies and provides guidance for the optimal design of such policies.

Upon consumer return, another alternative to cash refund is product replacement. Our model may be adapted to situations where replacement products rather than credit refunds are offered. Like credit refunds, replacement products are more restrictive than cash refunds. Furthermore, there are often time windows during which replacement products are offered and restocking fees might be charged. One difference is the possible valuation asymmetry between the firm and consumers. It is possible that consumers value a replacement product more than the cost to the firm, which is not the case for credit refunds.

There are several future research directions for this study. First, our analysis assumes that consumers are fully rational and forward-looking. As a result, they can strategically anticipate the value of the credit they earn. In practice, consumers may have bounded rationality; thus, it would be interesting to consider models where consumers are restricted to simpler strategies. Second, we assume that consumers' valuation uncertainty is fully resolved before making a purchase. In contrast, much of the existing literature on consumer returns hinges on the assumption that valuation uncertainty is only resolved after making a purchase. It would be worthwhile to investigate the efficacy of credit refund policies under such an assumption. Third, our consumer model assumes that the probability of product return is exogenous. Such an assumption is appropriate for some products but not for others. Investigating credit refund policies under alternative assumptions is a worthwhile topic for future research. Finally, our results imply some empirically testable hypotheses regarding consumer behavior and firms' adoption of credit refund policies. It would be interesting to conduct an empirical analysis with data collected from relevant industries.

References

Abbamonte, K. 2018. How to reduce returns and sell more with store credit. URL `https://www.shopify.com/retail/store-credit-for-customer-retention`. Last retrieved on January 4, 2023.

Abdulla, H., M. Ketzenberg, J. D. Abbey. 2019. Taking stock of consumer returns: A review and classification of the literature. Forthcoming in *Journal of Operations Management*.

Aviv, Y., A. Pazgal. 2008. Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing & Service Operations Management* **10**(3) 339–359.

Aydin, N., S. Birbil, J. Frenk, N. Noyan. 2013. Single-let airline revenue management with overbooking. *Transportation Science* **47**(4) 560–583.

Bertsimas, D., I. Popescu. 2003. Revenue management in a dynamic network environment. *Transportation Science* **37**(3) 257–277.

Besanko, D., W. L. Winston. 1990. Optimal price skimming by a monopolist facing rational consumers. *Management Science* **36**(5) 555–567.

Besbes, O., I. Lobel. 2015. Intertemporal price discrimination: Structure and computation of optimal policies. *Management Science* **61**(1) 92–110.

Chen, J., P. Bell. 2009. The impact of customer returns on pricing and order decisions. *European Journal of Operational Research* **195**(1) 280–295.

Dana, J. 1998. Advance-purchase discounts and price discrimination in competitive markets. *Journal of Political Economy* **106**(2) 395–422.

Deb, R. 2014. Intertemporal price discrimination with stochastic values. Working paper, University of Toronto.

Elmaghraby, W., A. Gulcu, P. Keskinocak. 2008. Designing optimal preannounced markdowns in the presence of rational customers with multiunit demands. *Manufacturing and Service Operations Management* **10**(1) 126–148.

Golfnow. 2023. Support FAQ. URL `https://www.golfnow.com/support`. Last retrieved on January 4, 2023.

Guo, L. 2009. Service cancellation and competitive refund policy. *Marketing Science* **28**(5) 901–917.

Heim, G. R., J. M. Field. 2007. Process drivers of e-service quality: Analysis of data from an online rating site. *Journal of Operations Management* **25**(5) 962–984.

Heiman, A., D. R. Just, B. P. McWilliams, D. Zilberman. 2015. A prospect theory approach to assessing changes in parameters of insurance contracts with an application to money-back guarantees. *Journal of Behavioral and Experimental Economics* **54**(1) 105–117.

Hsiao, L., Y.-j. Chen. 2012. Returns policy and quality risk in e-business. *Production and Operations Management* **21**(3) 489–503.

Huang, X., D. Zhang. 2019. Service product design and consumer refund policies. Forthcoming in *Marketing Science*.

Iliescu, D. C., L. A. Garrow, R. A. Parker. 2008. A hazard model of us airline passengers' refund and exchange behavior. *Transportation Research Part B* **42**(1) 229–242.

Inman, J. J., L. McAlister. 1994. Do coupon expiration dates affect consumer behavior? *Journal of Marketing Research* **31**(3) 423–428.

Khouja, M., H. Ajjan, X. Liu. 2019. The effect of return and price adjustment policies on a retailer's performance. *European Journal of Operational Research* **276**(1) 466–482.

Kivetz, R., O. Urminsky, Y. Zheng. 2006. The goal-gradient hypothesis resurrected: Purchase acceleration, illusionary goal progress, and customer retention. *Journal of Marketing Research* **43**(1) 39–58.

Liu, Y., W. L. Cooper. 2015. Optimal dynamic pricing with patient customers. *Operations Research* **63**(6) 1307–1319.

National Retail Federation. 2014. Consumer returns in the retail industry. `https://nfr.com/resources/retail-library/consumer-returns-the-retail-industry`.

Ofek, E., Z. Katona, M. Sarvary. 2011. "bricks and clicks": The impact of product returns on the strategies of multichannel retailers. *Marketing Science* **30**(1) 42–60.

Ross, S. 1983. *Introduction to stochastic dynamic programming*. Academic Press.

Shang, G., B. P. Ghosh, M. R. Galbreth. 2017. Optimal retail return polices with wardrobing. *Production and Operations Management* **26**(7) 1315–1332.

Shen, Z.-J. M., X. Su. 2007. Customer behavior modeling in revenue management and auctions: A review and new research opportunities. *Production and Operations Management* **16**(6) 713–728.

Shulman, J. D., A. T. Coughlan, R. Canan Savaskan. 2009. Optimal restocking fees and information provision in an integrated demand-supply model of products returns. *Manufacturing & Service Operations Management* **11**(4) 577–594.

Shulman, J. D., A. T. Coughlan, R. Canan Savaskan. 2011. Managing consumer returns in a competitive environment. *Management Science* **57**(2) 347–362.

Sider, A. 2018. Airline fight push to regulate ticket-change fees. *WSJ* Sep 21, 2018.

Sierag, D., G. Koole, R. van der Mei, J. van der Rest, B. Zwart. 2015. Revenue management under customer choice behavior with cancellations and overbooking. *European Journal of Operational Research* **246**(1) 170–185.

Stokey, N. 1979. Intertemporal price disrimination. *The Quarterly Journal of Economics* **93**(3) 355–371.

Su, X. 2007. Intertemporal pricing with strategic customer behavior. *Management Science* **53**(5) 726–741.

Su, X. 2009. Consumer returns policies and supply chain performance. *Manufacturing & Service Operations Management* **11**(4) 595–612.

Subramanian, J., S. Stidham, C. J. Lautenbacher. 1999. Airline yield management with overbooking, cancellations, and no-shows. *Transportation Science* **33**(2) 147–167.

Swinney, R. 2011. Selling to strategic consumers when product value is uncertain: the value of matching supply and demand. *Management Science* **57**(10) 1737–1751.

Talluri, K., G. van Ryzin. 2004. Revenue management under a general discrete choice model of consumer behavior. *Management Science* **50**(1) 15–33.

Ticketmaster. 2021. Updated information about event status, refunds, and options. URL `https://blog.ticketmaster.com/refund-credit-canceled-postponed-rescheduled-events/`. Last retrieved on January 4, 2023.

Toktay, L. B. 2003. *Forecasting Product Returns in Business Aspects of Closed-Loop Supply Chains*. Carnegie Bosch Institute, International Management Series.

Xie, J., E. Gerstner. 2007. Service escape: Profiting from customer cancellations. *Marketing Science* **26**(1) 18–30.

Online Appendices to "Intra-Consumer Price Discrimination with Credit Refund Policies"

These appendices provide supplemental materials for the paper and include seven sections:

- Appendix A includes proofs of the lemmas, propositions, and theorems in the main text.
- Appendix B provides a detailed analysis of the firm's problem in each case under an exogenous T and the proof of Theorem 1.
- Appendix C considers the problem with a general distribution of consumer valuations.
- Appendix D explores how the firm's optimal profit and credit refund policy change with the exogenous expiration term.
- Appendix E analyzes the problem when consumers have a deterministic valuation.
- Appendix F investigates the breakage rate, which is a managerially important quantity.
- Appendix G conducts a sensitivity analysis to examine how the optimal profit and expiration term change with the parameters.

Appendix A: Proofs of the Results in the Main Text

Before proving Proposition 1, we introduce two elementary yet useful properties of the bias function $J(\cdot)$ in Lemmas A.1 and A.2.

Lemma A.1. *The bias function $J(t)$ is increasing and concave in t.*

Intuitively, we expect that an on-hand credit is worth more if it is further from expiration, which implies that $J(t)$ is increasing in t. Lemma A.1 confirms this intuition.

Proof of Lemma A.1

Part 1: We show monotonicity by induction. We first show that $J(1) \geq J(0)$. From (1), we have

$$\begin{aligned}\rho^* + J(1) =& \alpha\lambda \max\{\gamma J(T) + (1-\gamma)(J(0)+v_H) - p + c, J(0)\} \\ &+ (1-\alpha)\lambda \max\{\gamma J(T) + (1-\gamma)(J(0)+v_L) - p + c, J(0)\} \\ &+ (1-\lambda)J(0). \end{aligned} \tag{A.1}$$

Note that the right-hand side of (A.1) is greater than the right-hand side of (2) due to the credit c in the two maximization terms. Therefore, we must have $J(1) \geq J(0)$.

For the inductive step, assume $J(t) \geq J(t-1)$ for $t = 1, \ldots, T-1$. We next show that $J(t+1) \geq J(t)$. From (1), we have

$$\rho^* + J(t+1) = \alpha\lambda \max\{\gamma J(T) + (1-\gamma)(J(0)+v_H) - p + c, J(t)\}$$

$$
\begin{aligned}
&+(1-\alpha)\lambda\max\{\gamma J(T)+(1-\gamma)(J(0)+v_L)-p+c,J(t)\}\\
&+(1-\lambda)J(t)\\
\geq&\alpha\lambda\max\{\gamma J(T)+(1-\gamma)(J(0)+v_H)-p+c,J(t-1)\}\\
&+(1-\alpha)\lambda\max\{\gamma J(T)+(1-\gamma)(J(0)+v_L)-p+c,J(t-1)\}\\
&+(1-\lambda)J(t-1)\\
=&\rho^*+J(t).
\end{aligned}
$$

In the above, the inequality follows from the inductive assumption that $J(t)\geq J(t-1)$. The last equality follows directly from (1). Hence, $J(t+1)\geq J(t)$.

Part 2: Now, we show the concavity. We expect to show that

$$J(t)-J(t-1)\leq J(t-1)-J(t-2).$$

According to (1), we have

$$
\begin{aligned}
\rho^*+J(t)=&\alpha\lambda\max\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-1)\}\\
&+(1-\alpha)\lambda\max\{\gamma J(T)+(1-\gamma)[J(0)+v_L]-p+c,J(t-1)\}\\
&+(1-\lambda)J(t-1).
\end{aligned}\tag{A.2}
$$

Similarly,

$$
\begin{aligned}
\rho^*+J(t-1)=&\alpha\lambda\max\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-2)\}\\
&+(1-\alpha)\lambda\max\{\gamma J(T)+(1-\gamma)[J(0)+v_L]-p+c,J(t-2)\}\\
&+(1-\lambda)J(t-2).
\end{aligned}\tag{A.3}
$$

Therefore, (A.2) − (A.3) yields

$$
\begin{aligned}
&J(t)-J(t-1)\\
=&\alpha\lambda\Big\{\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-1)\big\}\\
&\qquad-\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-2)\big\}\Big\}\\
&+(1-\alpha)\lambda\Big\{\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_L]-p+c,J(t-1)\big\}\\
&\qquad-\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_L]-p+c,J(t-2)\big\}\Big\}+(1-\lambda)\Big(J(t-1)-J(t-2)\Big).
\end{aligned}
$$

In order to show $J(t)-J(t-1)\leq J(t-1)-J(t-2)$, it suffices to show

$$
\begin{aligned}
&\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-1)\big\}\\
&\quad-\max\big\{\gamma J(T)+(1-\gamma)[J(0)+v_H]-p+c,J(t-2)\big\}\\
\leq&J(t-1)-J(t-2).
\end{aligned}\tag{A.4}
$$

Next, we consider three cases.

Case 1: Suppose

$$J(t-1) \geq J(t-2) \geq \gamma J(T) + (1-\gamma)[J(0) + v_H] - p + c.$$

In this case, the left hand side of the inequality (A.4) reduces to $J(t-1) - J(t-2)$, which is equal to the right hand side.

Case 2: Suppose

$$J(t-1) \geq \gamma J(T) + (1-\gamma)[J(0) + v_H] - p + c \geq J(t-2).$$

In this case, the left hand side reduces to

$$J(t-1) - \big(\gamma J(T) + (1-\gamma)[J(0) + v_H] - p + c\big) \leq J(t-1) - J(t-2).$$

Case 3: Suppose

$$\gamma J(T) + (1-\gamma)[J(0) + v_H] - p + c \geq J(t-1) \geq J(t-2).$$

In this case, the left hand side reduces to 0, which is clearly smaller than the right hand side. This completes the proof. ■

Lemma A.2 below to bound the difference between $J(t)$ and $J(0)$ for $t \geq 1$. This result is intuitive as we do not expect the value of an on-hand credit with any expiration term to exceed c.

LEMMA A.2. *We have $J(t) - J(0) \leq c$ for all $t = 1, \ldots, T$. That is, the value of a credit is bounded by c.*

Proof of Lemma A.2

According to Lemma A.1, $J(t)$ is increasing in t. Therefore, it suffices to show that $J(T) - J(0) \leq c$. The proof follows from a coupling argument. Consider two copies of the process, one starting in state T and the other one starting in state 0. Let the two processes follow the same decisions over time. Because the two processes are coupled, they will be in state T after a canceled purchase. Prior to matching in the state, the optimal reward differs by c at most. This completes the proof. ■

Proof of Proposition 1

Here, we provide an extended version of Proposition 1, which includes the optimal solution to the optimality equations (1)–(2).

Proposition A.1. *Fix $J(0)=0$. An optimal solution to the optimality equations (1)–(2) can be characterized as follows.*

(a) Suppose $c \le p \le \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L$, an optimal solution is given by

$$J(t) = [1-(1-\lambda)^t]c, \qquad \forall t = 1,\dots,T,$$
$$\rho^* = \lambda\big(\gamma[1-(1-\lambda)^T]c + (1-\gamma)[\alpha v_H + (1-\alpha)v_L] - p\big).$$

Consumers make a purchase in all states;

(b) Suppose $\max\{c, \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L\} < p \le \min\{p^C(T), p^D(T)\}$, an optimal solution is given by

$$J(1) = \frac{\lambda c - (1-\alpha)\lambda p + \lambda(1-\gamma)(1-\alpha)v_L}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]},$$
$$J(t) = \frac{[1-(1-\lambda)^t]J(1)}{\lambda}, \qquad \forall t = 2,\dots,T,$$
$$\rho^* = \alpha\lambda[\gamma J(T) + (1-\gamma)v_H - p].$$

High-valuation consumers make a purchase in all states and low-valuation consumers make a purchase whenever they have a credit on hand ($t \ge 1$);

(c) Suppose $\max\{c, p^D(\tau+1)\} < p \le \min\{p^C(\tau), p^D(\tau)\}$ for a fixed τ between 1 and $T-1$, an optimal solution is given by

$$J(1) = \lambda\Big(c + (1-\alpha)[\gamma J(T) + (1-\gamma)v_L - p]\Big),$$
$$J(t) = \frac{[1-(1-\lambda)^t]J(1)}{\lambda}, \qquad \forall t = 2,\dots,\tau,$$
$$J(t) = c[1-(1-\alpha\lambda)^{t-\tau}] + (1-\alpha\lambda)^{t-\tau}J(\tau), \qquad \forall t = \tau+1,\dots,T-1,$$
$$J(T) = \frac{c[1-(1-\alpha\lambda)^{T-\tau}] + (1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau][c + (1-\alpha)((1-\gamma)v_L - p)]}{1-(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau](1-\alpha)\gamma},$$
$$\rho^* = \alpha\lambda[\gamma J(T) + (1-\gamma)v_H - p].$$

High-valuation consumers make a purchase in all states and low-valuation consumers make a purchase whenever they have a credit within τ periods of expiration.

(d) Suppose $\max\{c, p^D(1)\} < p \le \gamma[1-(1-\alpha\lambda)^T]c + (1-\gamma)v_H$, an optimal solution is given by

$$J(t) = [1-(1-\alpha\lambda)^t]c, \qquad \forall t = 1,\dots,T,$$
$$\rho^* = \alpha\lambda\big(\gamma[1-(1-\alpha\lambda)^T]c + (1-\gamma)v_H - p\big).$$

High-valuation consumers make a purchase in all states and low-valuation consumers never make a purchase.

Proof. Part (a): Assume that a low-valuation consumer makes a purchase without a credit. That is,

$$\gamma J(T) + (1-\gamma)(J(0) + v_L) - p \geq J(0). \tag{A.5}$$

It follows (A.5) immediately that

$$\gamma J(T) + (1-\gamma)(J(0) + v_L) - p + c \geq J(0) + c \geq J(t-1),$$

for any $1 \leq t \leq T$, where the last inequality holds by Lemma A.2. This implies that the low-valuation consumer makes a purchase in all states. Of course, the high-valuation consumer makes a purchase in all states as well. Therefore, the dynamic programming equations (1) and (2) reduce to

$$\begin{aligned}
\rho^* + J(t) =& \alpha\lambda\Big(\gamma J(T) + (1-\gamma)(J(0) + v_H) - p + c\Big) + (1-\alpha)\lambda\Big(\gamma J(T) + (1-\gamma)(J(0) + v_L) - p + c\Big) \\
& + (1-\lambda)J(t-1), \qquad \forall t = 1, \ldots, T, \\
\rho^* + J(0) =& \alpha\lambda\Big(\gamma J(T) + (1-\gamma)(J(0) + v_H) - p\Big) + (1-\alpha)\lambda\Big(\gamma J(T) + (1-\gamma)(J(0) + v_L) - p\Big) \\
& + (1-\lambda)J(0).
\end{aligned}$$

Solving the above equations gives the solution of $J(\cdot)$ and ρ^*. The technical condition in the supposition can be obtained by using the expressions for $J(0)$ and $J(T)$ in (A.5). This completes the proof of Part (a).

Parts (b) and (c): Assume that only a high-valuation consumer makes a purchase without a credit.

In this case, only a high-valuation consumer makes a purchase without a credit, but a low-valuation consumer does not make a purchase without a credit. From (2), we must have

$$\gamma J(T) + (1-\gamma)(J(0) + v_H) - p \geq J(0), \tag{A.6}$$

$$\gamma J(T) + (1-\gamma)(J(0) + v_L) - p < J(0). \tag{A.7}$$

From (A.6) and Lemma A.2, we have

$$\gamma J(T) + (1-\gamma)(J(0) + v_H) - p + c \geq c + J(0) \geq J(t), \quad \forall t = 1, \ldots, T.$$

Using (1), this implies that a high-valuation consumer makes a purchase with a credit in all states. It is not immediately clear whether a low-valuation consumer makes a purchase with a credit. To proceed with our analysis, let

$$\tau = \max\{t \in \{1, \ldots, T\} : \gamma J(T) + (1-\gamma)(J(0) + v_L) - p + c \geq J(t-1)\}.$$

Note that because $J(t)$ is increasing in t by Lemma A.1, τ is a threshold such that a low-valuation consumer makes a purchase with a credit in state $t \leq \tau$, and does not make a purchase with a credit for $t > \tau$. Summarizing the discussion above, we have

$$\rho^* + J(t) = \alpha\lambda\Big(\gamma J(T) + (1-\gamma)(J(0)+v_H) - p + c\Big) + (1-\alpha\lambda)J(t-1), \qquad \forall t = \tau+1,\dots,T, \tag{A.8}$$

$$\rho^* + J(t) = \lambda\Big(\gamma J(T) + (1-\gamma)J(0) - p + c\Big) + \lambda(1-\gamma)\Big(\alpha v_H + (1-\alpha)v_L\Big) + (1-\lambda)J(t-1), \qquad \forall t = 1,\dots,\tau, \tag{A.9}$$

$$\rho^* + J(0) = \alpha\lambda\Big(\gamma J(T) + (1-\gamma)(J(0)+v_H) - p\Big) + (1-\alpha\lambda)J(0). \tag{A.10}$$

We can solve for ρ^* and $J(t)$ for $t = 0,1,\dots,T$. Note that there are $T+1$ equations and $T+2$ unknowns. Therefore, we fix $J(0) = 0$. The equations above simplify to

$$\rho^* + J(t) = \alpha\lambda\Big(\gamma J(T) + (1-\gamma)v_H - p + c\Big) + (1-\alpha\lambda)J(t-1), \qquad \forall t = \tau+1,\dots,T, \tag{A.11}$$

$$\rho^* + J(t) = \lambda\Big(\gamma J(T) - p + c\Big) + \lambda(1-\gamma)\Big(\alpha v_H + (1-\alpha)v_L\Big) + (1-\lambda)J(t-1), \quad \forall t = 1,\dots,\tau, \tag{A.12}$$

$$\rho^* = \alpha\lambda\Big(\gamma J(T) + (1-\gamma)v_H - p\Big). \tag{A.13}$$

We consider two subcases when solving the equations above.

<u>Subcase I: $\tau = T$.</u>

Note that $\tau = T$ means that

$$\gamma J(T) + (1-\gamma)v_L - p + c \geq J(T-1). \tag{A.14}$$

In this case, equation (A.11) vanishes. Using (A.13) in (A.12) leads to

$$J(t) = \lambda c + \lambda(1-\alpha)\Big(\gamma J(T) - p\Big) + \lambda(1-\gamma)(1-\alpha)v_L + (1-\lambda)J(t-1), \quad \forall t = 1,\dots,T. \tag{A.15}$$

Taking the difference in the above equation between t and $t-1$ for $t = 2,\dots,T$, we obtain

$$J(t) - J(t-1) = (1-\lambda)\Big(J(t-1) - J(t-2)\Big).$$

It follows that

$$J(t) = \sum_{i=1}^{t}[J(i) - J(i-1)] = \sum_{i=1}^{t}(1-\lambda)^{i-1}[J(1) - J(0)] = \sum_{i=1}^{t}(1-\lambda)^{i-1}J(1) = \frac{[1-(1-\lambda)^t]J(1)}{\lambda}, \quad \forall t = 2,\dots,T.$$

Using the expression for $J(T)$ in (A.15) and solving for $J(1)$, we obtain

$$J(1) = \frac{\lambda c - (1-\alpha)\lambda p + \lambda(1-\gamma)(1-\alpha)v_L}{1 - \gamma(1-\alpha)[1-(1-\lambda)^T]}.$$

Putting the expressions for $J(\cdot)$ in (A.6), (A.7), and (A.14) yields $p \leq p^C(T)$, $p > (1-\gamma)v_L + \gamma[1-(1-\lambda)^T]c$, and $p \leq p^D(T)$, respectively. This completes the proof of Part (b).

Subcase II: $\tau < T$.

Note that $\tau \leq T-1$ implies

$$J(\tau-1) \leq \gamma J(T) + (1-\gamma)v_L - p + c < J(\tau). \tag{A.16}$$

Using (A.13) in (A.11) and (A.12) leads to

$$J(t) = \alpha\lambda c + (1-\alpha\lambda)J(t-1), \qquad \forall t = \tau+1, \ldots, T, \tag{A.17}$$

$$J(t) = \lambda c + (1-\alpha)\lambda\Big(\gamma J(T) - p\Big) + \lambda(1-\gamma)(1-\alpha)v_L + (1-\lambda)J(t-1), \quad \forall t = 1, \ldots, \tau. \tag{A.18}$$

From (A.17), we have

$$J(t) = c[1-(1-\alpha\lambda)^{t-\tau}] + (1-\alpha\lambda)^{t-\tau}J(\tau), \qquad \forall t = \tau+1, \ldots, T. \tag{A.19}$$

Because $J(0) = 0$, taking $t = 1$ in (A.18) gives

$$\begin{aligned} J(1) &= \lambda c + (1-\alpha)\lambda\Big(\gamma J(T) - p\Big) + \lambda(1-\gamma)(1-\alpha)v_L. \\ &= \lambda\Big(c + (1-\alpha)(\gamma J(T) + (1-\gamma)v_L - p)\Big). \end{aligned} \tag{A.20}$$

Equation (A.18) can be rewritten as

$$J(t) = J(1) + (1-\lambda)J(t-1), \qquad \forall t = 1, \ldots, \tau. \tag{A.21}$$

Equation (A.21) implies that

$$J(t) = \frac{[1-(1-\lambda)^t]J(1)}{\lambda}, \qquad \forall t = 1, \ldots, \tau. \tag{A.22}$$

Using (A.22) for $t = \tau$ in (A.19), we obtain

$$J(t) = c[1-(1-\alpha\lambda)^{t-\tau}] + \frac{(1-\alpha\lambda)^{t-\tau}[1-(1-\lambda)^\tau]J(1)}{\lambda}, \qquad \forall t = \tau+1, \ldots, T, \tag{A.23}$$

Taking $t = T$ in (A.23) and using (A.20), we obtain

$$J(T) = c[1-(1-\alpha\lambda)^{T-\tau}] + (1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau][c + (1-\alpha)(\gamma J(T) + (1-\gamma)v_L - p)].$$

Solving the equation above gives

$$J(T) = \frac{c[1-(1-\alpha\lambda)^{T-\tau}] + (1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau][c + (1-\alpha)((1-\gamma)v_L - p)]}{1-(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau](1-\alpha)\gamma}.$$

Putting the expressions for $J(\cdot)$ in (A.6) and (A.16) yields $p \leq p^C(\tau)$ and $p^D(\tau+1) \leq p \leq p^D(\tau)$, respectively. Note that $\gamma J(T)+(1-\gamma)v_L - p + c < J(\tau)$ in (A.16) implies (A.7), so we do not need to derive the condition from constraint (A.7). This completes the proof of Part (c).

Part (d): Assume that a high-valuation consumer makes a purchase without a credit and a low-valuation consumer does not purchase in state 1. That is,

$$\gamma J(T)+(1-\gamma)(J(0)+v_H)-p \geq J(0), \tag{A.24}$$
$$\gamma J(T)+(1-\gamma)(J(0)+v_L)-p+c < J(0). \tag{A.25}$$

It follows (A.24) immediately that

$$\gamma J(T)+(1-\gamma)(J(0)+v_H)-p+c \geq J(0)+c \geq J(t-1),$$

for any $1 \leq t \leq T$, where the last inequality holds by Lemma A.2. This implies that the high-valuation consumer makes a purchase in all states. Moreover, it follows (A.25) immediately that

$$\gamma J(T)+(1-\gamma)(J(0)+v_L)-p < J(0),$$
$$\gamma J(T)+(1-\gamma)(J(0)+v_L)-p+c < J(t-1),$$

for any $1 \leq t \leq T$, where the last inequality holds by the monotonicity of $J(\cdot)$. This implies that the low-valuation consumer never makes a purchase, with and without a credit. Therefore, the dynamic programming equations (1) and (2) reduce to

$$\begin{aligned}
\rho^* + J(t) &= \alpha\lambda\Big(\gamma J(T)+(1-\gamma)(J(0)+v_H)-p+c\Big)+(1-\alpha)\lambda J(t-1)+(1-\lambda)J(t-1), \qquad \forall t=1,\ldots,T,\\
\rho^* + J(0) &= \alpha\lambda\Big(\gamma J(T)+(1-\gamma)(J(0)+v_H)-p\Big)+(1-\alpha)\lambda J(0)+(1-\lambda)J(0).
\end{aligned}$$

Solving the above equations gives the solution of $J(\cdot)$ and ρ^*. The technical condition in the supposition can be obtained by using the expressions for $J(0)$ and $J(T)$ in (A.24) and (A.25). This completes the proof of Part (d). ■

Lemma A.3. *The value T_0 defined in* (5) *exists and is finite.*

Proof of Lemma A.3

Both $A(T)$ and $B(T,T)$ are increasing in T. Moreover, one can verify that $A(1) < B(1,1)$ and $\lim_{T\to\infty} A(T) = \infty > \lim_{T\to\infty} B(T,T)$. Therefore, T_0 exists and is finite. ■

Proof of Theorem 1

The proof of Theorem 1 is relegated to Appendix B. ■

Proof of Lemma 1

Part (a): We show that T_1 exists and is finite. First,

$$
\begin{aligned}
&A(1)=(1-\gamma)(v_H-v_L),\\
&B(2,1)=(1-\gamma)\frac{v_H-(v_H-v_L)\gamma(1-\alpha)(1-\alpha\lambda)\lambda}{1-\gamma+\gamma(1-\alpha\lambda)(1-\lambda)},\\
&A(1)-B(2,1)=\frac{(1-\gamma)}{1-\gamma+\gamma(1-\alpha\lambda)(1-\lambda)}\Big\{(v_H-v_L)\big(1-\gamma[1-(1-\alpha\lambda)^2]\big)-v_H\Big\}<0,
\end{aligned}
$$

where the last inequality holds because $1-\gamma[1-(1-\alpha\lambda)^2]<1$.

Second,

$$
\begin{aligned}
\lim_{T\to\infty} A(T-1)&=\infty,\\
\lim_{T\to\infty} B(T,T-1)&=v_H-(v_H-v_L)\gamma(1-\alpha)(1-\alpha\lambda).
\end{aligned}
$$

Clearly,

$$
\lim_{T\to\infty} A(T-1)>\lim_{T\to\infty} B(T,T-1).
$$

Third,

$$
A(T-1)=(1-\gamma)(v_H-v_L)\Big\{\frac{\alpha}{(1-\lambda)^{T-2}+1-\alpha}\Big\}
$$

is increasing in T. Let $(1-\lambda)^{T-1}=x$, then

$$
B(T,T-1)=\frac{(1-\gamma)v_H-(1-\gamma)(v_H-v_L)\gamma(1-\alpha)(1-\alpha\lambda)(1-x)}{1-\gamma+\gamma(1-\alpha\lambda)x}.
$$

Taking derivative with respect to x yields

$$
\begin{aligned}
&(1-\gamma)(v_H-v_L)\gamma(1-\alpha)(1-\alpha\lambda)\Big(1-\gamma+\gamma(1-\alpha\lambda)x\Big)\\
&\quad-\Big\{(1-\gamma)v_H-(1-\gamma)(v_H-v_L)\gamma(1-\alpha)(1-\alpha\lambda)(1-x)\Big\}\gamma(1-\alpha\lambda)\\
=&(1-\gamma)\gamma(1-\alpha\lambda)\Big((1-\alpha)(v_H-v_L)-v_H\Big)-(1-\gamma)\gamma(v_H-v_L)\gamma(1-\alpha)(1-\alpha\lambda)\alpha\lambda\\
=&(1-\gamma)\gamma(1-\alpha\lambda)\Big((1-\alpha)(v_H-v_L)(1-\alpha\gamma\lambda)-v_H\Big)\\
<&0.
\end{aligned}
$$

Therefore, $B(T,T-1)$ is increasing in T.

The above three points imply that T_1 exists and is finite. The proof that T_i $(i\geq 2)$ exists and is finite follows a similar approach.

Part (b): We show $T_0+1\leq T_1$. The definition of T_0 implies that

$$
A(T_0)\leq B(T_0,T_0). \tag{A.26}
$$

According to the definition of T_1, it suffices to show $A(T_0+1-1) \leq B(T_0+1, T_0+1-1)$, that is, $A(T_0) \leq B(T_0+1, T_0)$. Given (A.26), it suffices to show $B(T_0, T_0) \leq B(T_0+1, T_0)$. One can check

$$\begin{aligned}
&B(T_0,T_0) - B(T_0+1,T_0)\\
=&\frac{v_H - (v_H - v_L)\gamma(1-\alpha)[1-(1-\lambda)^{T_0}]}{1-\gamma[1-(1-\lambda)^{T_0}]} - \frac{v_H - (v_H - v_L)\gamma(1-\alpha)(1-\alpha\lambda)[1-(1-\lambda)^{T_0}]}{1-\gamma[1-(1-\alpha\lambda)(1-\lambda)^{T_0}]}\\
=&\Big\{v_H - (v_H - v_L)\gamma(1-\alpha)[1-(1-\lambda)^{T_0}]\Big\}\Big\{1-\gamma[1-(1-\alpha\lambda)(1-\lambda)^{T_0}]\Big\}\\
&\quad - \Big\{v_H - (v_H - v_L)\gamma(1-\alpha)(1-\alpha\lambda)[1-(1-\lambda)^{T_0}]\Big\}\Big\{1-\gamma[1-(1-\lambda)^{T_0}]\Big\}\\
=&-v_H\gamma(1-\lambda)^{T_0}\alpha\lambda - (v_H - v_L)\gamma(1-\alpha)(1-\gamma)[1-(1-\lambda)^{T_0}]\alpha\lambda\\
<&0.
\end{aligned}$$

Consequently, $T_1 \geq T_0+1$. The proof of $T_i \leq T_{i+1}$ $(i \geq 1)$ follows a similar approach. ■

Proof of Lemma 2

Before proving Lemma 2, we first introduce a lemma.

LEMMA A.4. *We have*

$$\frac{1-\gamma+\gamma(1-\alpha\lambda)^i(1-\lambda)^{T-i}}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]} \cdot \frac{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}{(1-\gamma)[\alpha+(1-\alpha)(1-\lambda)^{T-i-1}] - (1-\lambda)^{T-i-1}[1-\gamma(1-\alpha\lambda)^{i+1}]}$$

is decreasing in T.

Proof. Let $(1-\lambda)^{T-i} = x$, then the first term

$$\frac{1-\gamma+\gamma(1-\alpha\lambda)^i(1-\lambda)^{T-i}}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]} = \frac{1-\gamma+\gamma(1-\alpha\lambda)^i x}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i + \gamma(1-\alpha)(1-\alpha\lambda)^i x}.$$

Taking derivative with respect to x yields

$$\begin{aligned}
&\frac{1}{\big(1-\gamma(1-\alpha)(1-\alpha\lambda)^i + \gamma(1-\alpha)(1-\alpha\lambda)^i x\big)^2}\cdot\\
&\Big\{\gamma(1-\alpha\lambda)^i\big(1-\gamma(1-\alpha)(1-\alpha\lambda)^i + \gamma(1-\alpha)(1-\alpha\lambda)^i x\big) - \gamma(1-\alpha)(1-\alpha\lambda)^i[1-\gamma+\gamma(1-\alpha\lambda)^i x]\Big\}\\
=&\frac{1}{\big(1-\gamma(1-\alpha)(1-\alpha\lambda)^i + \gamma(1-\alpha)(1-\alpha\lambda)^i x\big)^2}\Big\{\gamma(1-\alpha\lambda)^i\alpha + \gamma^2(1-\alpha)(1-\alpha\lambda)[1-(1-\alpha\lambda)^i]\Big\}\\
>&0.
\end{aligned}$$

Hence, the first term decreases in T.

While, the second term

$$\frac{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}{(1-\gamma)[\alpha+(1-\alpha)(1-\lambda)^{T-i-1}] - (1-\lambda)^{T-i-1}[1-\gamma(1-\alpha\lambda)^{i+1}]}$$

$$=\frac{1}{1-\gamma-\frac{(1-\lambda)^{T-i-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}[1-\gamma(1-\alpha\lambda)^{i+1}]}=\frac{1}{1-\gamma-\frac{1-\gamma(1-\alpha\lambda)^{i+1}}{\frac{\alpha}{(1-\lambda)^{T-i-1}}+1-\alpha}},$$

which, clearly, is decreasing in T. This completes the proof. ∎

Here, we provide an extended version of Lemma 2 which includes the complete solution to the problem (12)–(16), and then provide the proof of this extended version.

LEMMA A.5. *For each $i\in\{0,1,\dots,\bar{T}-1\}$, let $p^*_{3,T-i}$, $c^*_{3,T-i}$, and $\pi^*_{3,T-i}$ denote the optimal price, credit, and profit, respectively, to the problem (12)–(16). We have the following results:*

(a) Suppose $T\le T_i$, then

$$p^*_{3,T-i}=\gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c^*_{3,T-i}+(1-\gamma)\Big\{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\Big\}v_H$$
$$+(1-\gamma)\Big\{\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\Big\}v_L,$$

$$\pi^*_{3,T-i}=\lambda(1-\gamma)\alpha v_H+\lambda(1-\gamma)\frac{\alpha(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}v_L,$$

and $c^*_{3,T-i}$ takes any value between $A(T-i)$ and $\min\big\{B(T,T-i),A(T-i+1)\big\}$. Moreover, $\pi^*_{3,T-i}$ is increasing in T;

(b) Suppose $T\ge T_i+1$. If $C(T,T-i)\ge 0$, then

$$p^*_{3,T-i}=(1-\gamma)v_L+\gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c^*_{3,T-i}$$
$$+\frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}(1-\lambda)^{T-i-1}c^*_{3,T-i},$$

$$\pi^*_{3,T-i}=\lambda\alpha(1-\gamma)v_L\frac{1-\gamma+\gamma(1-\lambda)^{T-i}(1-\alpha\lambda)^i}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}\cdot$$
$$\frac{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}{(1-\gamma)[\alpha+(1-\alpha)(1-\lambda)^{T-i-1}]-(1-\lambda)^{T-i-1}[1-\gamma(1-\alpha\lambda)^{i+1}]},$$

$$c^*_{3,T-i}=C(T,T-i).$$

Moreover, $\pi^*_{3,T-i}$ is decreasing in T. Otherwise, there is no feasible solution to the problem (12)–(16).

Proof. Recall

$$p^C(T-i)=\gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c+(1-\gamma)\big\{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\big\}v_H$$
$$+(1-\gamma)\big\{\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\big\}v_L,$$

$$p^D(T-i)=(1-\gamma)v_L+\gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c+\frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}(1-\lambda)^{T-i-1}c.$$

Putting $J(\cdot)$ into the constraints (13) - (15) yields $p \le p^C(T-i)$, $p \le p^D(T-i)$, and $p \ge p^D(T-i+1)$, respectively. Summarizing the constraints gives

$$\max\left\{c, p^D(T-i+1)\right\} \le p \le \min\left\{p^C(T-i), p^D(T-i)\right\}.$$

One can verify that if $c \ge A(T-i)$, then $p^C(T-i) \le p^D(T-i)$, and vice versa. Next, we consider two cases and identify two possible solutions.

Case 1: Note that if $c \le A(T-i+1)$, then $p^C(T-i) \ge p^D(T-i+1)$, and vice versa. Moreover, if $c \le B(T,T-i)$, then $c \le p^C(T-i)$. Therefore, if $A(T-i) \le \min\left\{B(T,T-i), A(T-i+1)\right\}$, then one possible solution to the problem (12)–(16) is

$$\begin{aligned} p^*_{3,T-i} = p^C(T-i)|_{c=c^*_{3,T-i}} &= \gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c^*_{3,T-i} \\ &\quad + (1-\gamma)\left\{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\right\}v_H \\ &\quad + (1-\gamma)\left\{\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]\right\}v_L, \\ \pi^*_{3,T-i}(p^C(T-i)) &= \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\frac{\alpha(1-\alpha)\gamma(1-\alpha\gamma)^i[1-(1-\lambda)^{T-i}]}{1-(1-\alpha)\gamma(1-\alpha\gamma)^i[1-(1-\lambda)^{T-i}]}v_L, \end{aligned}$$

where $c^*_{3,T-i}$ takes any value between $A(T-i)$ and $\min\left\{B(T,T-i), A(T-i+1)\right\}$.

Case 2: Note that if $c \le C(T,T-i)$, then $c \le p^D(T-i)$. Moreover, $p^D(T-i) \ge p^D(T-i+1)$ holds automatically. Hence, the other possible solution is

$$\begin{aligned} c^*_{3,T-i} &= \min\{A(T-i), C(T,T-i)\}, \\ p^*_{3,T-i} &= p^D(T-i)|_{c=c^*_{3,T-i}} = (1-\gamma)v_L + \gamma[1-(1-\alpha\lambda)^i(1-\lambda)^{T-i}]c^*_{3,T-i} \\ &\qquad + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}(1-\lambda)^{T-i-1}c^*_{3,T-i}, \\ \pi^*_{3,T-i}(p^D(T-i)) &= \frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}\cdot \\ &\qquad \left\{(1-\gamma)v_L + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}(1-\lambda)^{T-i-1}c^*_{3,T-i}\right\}. \end{aligned}$$

Note that Case 1 requires $A(T-i) \le B(T,T-i)$,[12] hence, for any $T \le T_i$ such that $A(T-i) \le B(T,T-i)$, both Case 1 and Case 2 could occur if the firm sets the appropriate price and credit. However, since $c^*_{3,T-i} \le A(T-i)$ in Case 2, we have

$$\begin{aligned} &\pi^*_{3,T-i}(p^D(T-i)) \\ \le &\frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}\cdot \end{aligned}$$

[12] Note that $A(T-i) \le A(T-i+1)$ holds because of the monotonicity of $A(\cdot)$.

$$\Big\{(1-\gamma)v_L+\frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}(1-\lambda)^{T-i-1}A(T-i)\Big\}$$

$$=\lambda\alpha(1-\gamma)v_H+\lambda\alpha(1-\gamma)v_L\frac{(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{1-(1-\alpha)\gamma(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}$$

$$=\pi^*_{3,T-i}(p^C(T-i)).$$

Therefore, for any $T\le T_i$, Case 1 is optimal. One can verify that $\pi^*_{3,T-i}(p^C(T-i))$ increases in T. This completes the proof of Part (a).

For any $T\ge T_i+1$ such that $A(T-i)>B(T,T-i)$, only Case 2 is possible. Note that if $C(T,T-i)<0$, then $c>C(T,T-i)$, and thus $c>p^D(T-i)$. Therefore, there is no feasible solution to the problem (12)–(16). Hereafter, we assume $C(T,T-i)\ge 0$. Lemma B.5 implies that $A(T-i)>C(T,T-i)$. Hence, $c^*_{3,T-i}=C(T,T-i)$. Putting $C(T,T-i)$ into $\pi^*_{3,T-i}(p^D(T-i))$ yields

$$\pi^*_{3,T-i}(p^D(T-i))=\lambda\alpha(1-\gamma)v_L\frac{1-\gamma+\gamma(1-\lambda)^{T-i}(1-\alpha\lambda)^i}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}\cdot$$
$$\frac{\alpha+(1-\alpha)(1-\lambda)^{T-i-1}}{(1-\gamma)[\alpha+(1-\alpha)(1-\lambda)^{T-i-1}]-(1-\lambda)^{T-i-1}[1-\gamma(1-\alpha\lambda)^{i+1}]}.$$

Moreover, Lemma A.4 implies that $\pi^*_{3,T-i}(p^D(T-i))$ is decreasing in T. This completes the proof of Part (b). ■

Proof of Theorem 2

Lemma 2 indicates that the optimal profit when $\tau=T-i$ for each fixed i (where $0\le i\le\bar{T}-1$) is first increasing in T when $T\le T_i$ and then decreasing when $T\ge T_i+1$. Consequently,

$$T^*\in\big\{T_0,T_0+1,\ldots,T_{\bar{T}-1},T_{\bar{T}-1}+1,\bar{T}\big\}.$$

Note that for any $i\ge\tilde{i}+1$, we have $T_i>\bar{T}$, so

$$T^*\in\big\{T_0,T_0+1,\ldots,T_{\tilde{i}},T_{\tilde{i}}+1,\bar{T}\big\}.$$

This completes the proof. ■

Proof of Theorem 3

Part (a): Recall that

$$\pi^c=\max\{\lambda(1-\gamma)v_L,\lambda(1-\gamma)\alpha v_H\}.$$

Suppose $T\le T_0$. Theorem 1(a) implies that

$$\pi^*=\max\Big\{\lambda(1-\gamma)v_L,\lambda(1-\gamma)\alpha v_H+\lambda(1-\gamma)\alpha\frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}v_L\Big\}.$$

Clearly, $\pi^* \geq \pi^c$, and the strict inequality holds if $v_L/v_H \leq \alpha < 1$.

Suppose $T \geq T_0 + 1$. We know that there exists an $i \in \{1, \ldots, \bar{T} - 1\}$ such that $T_{i-1} < T \leq T_i$. Moreover, Theorem 1(b) implies that

$$\begin{aligned}\pi^* &\geq \max\left\{\lambda(1-\gamma)v_L, \pi_3^{T-i}\right\} \\ &= \max\left\{\lambda(1-\gamma)v_L, \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha \frac{\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]}{1-\gamma(1-\alpha)(1-\alpha\lambda)^i[1-(1-\lambda)^{T-i}]} v_L\right\},\end{aligned}$$

where the last equality holds because of Lemma 2(a). Clearly, $\pi^* \geq \pi^c$, and the strict inequality holds if $v_L/v_H \leq \alpha < 1$.

Part (b): Note that the optimal profit under the credit refund policy (when setting $T^* = T_0$) can be written as

$$\lambda(1-\gamma)\max\left\{\alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]} v_L, v_L\right\},$$

while the profit of the cash refund policy is

$$\lambda(1-\gamma)\max\{\alpha v_H, v_L\}.$$

When α is sufficiently small such that $\alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]} v_L < v_L$, the profit ratio is 1.

When $\alpha v_H \leq v_L \leq \alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]} v_L$, the profit under the credit refund policy reduces to $\lambda(1-\gamma)\left\{\alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]} v_L\right\}$, which increases in α. While, the profit under the cash refund policy reduces to $\lambda(1-\gamma)v_L$, which is independent of α. Hence, the profit ratio becomes

$$\frac{\alpha v_H + \frac{\alpha(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]} v_L}{v_L},$$

which increases in both α and γ.

When $v_L/v_H \leq \alpha < 1$, the profit ratio is

$$\frac{v_H + \frac{(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]}{1-(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]} v_L}{v_H}.$$

In order to see how this ratio changes with α and γ, it suffices to investigate how $(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]$ changes with α and γ. According to Proposition G.1, T_0 decreases in α, so $(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]$ decreases in α, and therefore the profit ratio also decreases in α. Similarly, according to Proposition G.1, T_0 increases in γ, so $(1-\alpha)\gamma[1-(1-\lambda)^{T_0}]$ increases in γ, and therefore the profit ratio increases in γ.

Combining the above three cases shows that the profit ratio increases in γ, and increases in α if $\alpha < v_L/v_H$ and decreases otherwise. This completes the proof. ■

Proof of Proposition 2

Part (a): In a cash refund policy, when the firm sets the effective price to v_H, only high-valuation consumers make a purchase; hence, the aggregate consumers' surplus is 0. In a credit refund policy, the forward-looking consumers will not purchase if the expected surplus is negative; hence, the aggregate consumers' surplus is nonnegative. Therefore, the consumer surplus increases. It follows immediately that the social welfare, which is the sum of the consumer surplus and firm profit, increases as well.

Part (b): In a cash refund policy, when the firm sets the effective price to v_L, both high- and low-valuation consumers make a purchase; hence, the aggregate consumers' surplus is $\lambda(1-\gamma)\alpha(v_H - v_L)$. As for a credit refund policy, the average price paid by the consumer in either case II or III lies between v_L and v_H; hence, the aggregate consumers' surplus is strictly less than $\lambda(1-\gamma)\alpha(v_H - v_L)$.

Note that the firm profit in a cash refund policy is $\lambda(1-\gamma)v_L$, so the social welfare is $\lambda(1-\gamma)(\alpha v_H + (1-\alpha)v_L)$, which is essentially the maximum social welfare possible. Hence, the social welfare also decreases when the firm switches from a cash refund policy with effective price v_L to a credit refund policy. This completes the proof. ■

Proof of Proposition 3

Before proving Proposition 3, we first introduce a lemma.

LEMMA A.6. *If $p \le \gamma \frac{\lambda\delta}{1-\delta+\lambda\delta}[1-(1-\lambda)^T\delta^T]c + (1-\gamma)v_L$, then the consumer always makes a purchase when she arrives at the market, independent of her valuation realization.*

Proof. Suppose the low-valuation consumer makes a purchase when she does not have a credit. That is,

$$\gamma\delta J(T) + (1-\gamma)[\delta J(0) + v_L] - p \ge \delta J(0). \tag{A.27}$$

Similar to Lemma A.2, one can show

$$J(t) - J(0) \le c.$$

Hence, it follows immediately that for any $t \ge 1$,

$$\gamma\delta J(T) + (1-\gamma)[\delta J(0) + v_L] - p + c \ge \delta J(t-1),$$

implying that the low-valuation consumer makes a purchase whenever she has a credit. Therefore, the low-valuation consumer always makes a purchase when she arrives at the market, and so does the high-valuation consumer.

Then, the consumer's dynamic programming equations reduce to

$$J(t) = \lambda\Big\{\gamma\delta J(T) + (1-\gamma)(\delta J(0) + E[v]) - p + c\Big\} + (1-\lambda)\delta J(t-1),$$
$$J(0) = \lambda\Big\{\gamma\delta J(T) + (1-\gamma)(\delta J(0) + E[v]) - p\Big\} + (1-\lambda)\delta J(0).$$

Solving the above set of equations yields

$$J(0) = \frac{\lambda}{1-\delta}[(1-\gamma)E[v] - p] + \frac{\lambda\gamma\delta}{(1-\delta)(1-\delta+\lambda\delta)}[1-(1-\lambda)^T\delta^T]\lambda c,$$
$$J(T) = \frac{\lambda}{1-\delta}[(1-\gamma)E[v] - p] + \frac{1-\delta+\lambda\gamma\delta}{(1-\delta)(1-\delta+\lambda\delta)}[1-(1-\lambda)^T\delta^T]\lambda c.$$

Putting $J(0)$ and $J(T)$ into (A.27) gives

$$p \le \gamma\frac{\lambda\delta}{1-\delta+\lambda\delta}[1-(1-\lambda)^T\delta^T]c + (1-\gamma)v_L.$$

This completes the proof. ■

Now, we are ready to prove Proposition 3.

Proof of Proposition 3. Note that the firm's profit π_δ^c in a cash refund policy is

$$\pi_\delta^c = \max\{\lambda(1-\gamma)v_L, \lambda(1-\gamma)\alpha v_H\} = \pi^c.$$

Next, we derive the firm's profit π_δ^* in credit refund policy. To derive the firm's optimal profit, we need to formulate and solve the firm's optimization problem, taking into account the consumer's optimal responses. Similar to the main model, consumers' optimal responses can be divided into four cases:

Case I: Consumers make a purchase in each state, independent of their valuation realization.

Case II: A low-valuation consumer makes a purchase whenever she has a credit on hand.

Case III: A low-valuation consumer makes a purchase when she has a credit and the credit is within τ periods of expiration, where $\tau < T$.

Case IV: A low-valuation consumer never makes a purchase in any state.

We first analyze the firm's profit in Case I. The consumer's state transition is the same as Case I in the main model, as depicted in Figure B.1 in Appendix B.1.1. Therefore, the firm's optimization problem can be written as follows,

$$\begin{aligned} \max_{p,c} \quad & \lambda\Big\{p - \gamma[1-(1-\lambda)^T]c\Big\} \\ s.t. \quad & p \le \gamma\frac{\lambda\delta}{1-\delta+\lambda\delta}[1-(1-\lambda)^T\delta^T]c + (1-\gamma)v_L \\ & p \ge c. \end{aligned}$$

Let $p^*_{1,\delta}$, $c^*_{1,\delta}$, and $\pi^*_{1,\delta}$ denote the optimal price, credit, and profit to the above problem. We obtain

$$p^*_{1,\delta} = \gamma \frac{\lambda\delta}{1-\delta+\lambda\delta}[1-(1-\lambda)^T\delta^T]c^*_{1,\delta} + (1-\gamma)v_L,$$

$$c^*_{1,\delta} \in \Big[0, \frac{(1-\gamma)v_L}{1-\gamma\frac{\lambda\delta}{1-\delta+\lambda\delta}[1-(1-\lambda)^T\delta^T]}\Big],$$

$$\pi^*_{1,\delta} = \lambda(1-\gamma)v_L + \lambda\gamma c^*_{1,\delta}\Big\{\frac{\lambda\delta}{1-(1-\lambda)\delta}[1-(1-\lambda)^T\delta^T] - [1-(1-\lambda)^T]\Big\}.$$

Note that

$$\begin{aligned}\pi^*_{1,\delta} =& \lambda(1-\gamma)v_L + \lambda\gamma c^*_{1,\delta}\Big\{\lambda\delta[1+(1-\lambda)\delta+\cdots+(1-\lambda)^{T-1}\delta^{T-1}] - \lambda[1+(1-\lambda)+\cdots+(1-\lambda)^{T-1}]\Big\}\\ \le& \lambda(1-\gamma)v_L = \pi^*_1, \end{aligned} \tag{A.28}$$

where the above inequality becomes equality when $\delta = 1$.

Let $\pi^*_{2,\delta}$, $\pi^*_{3,\delta}$, and $\pi^*_{4,\delta}$ denote the firm's profit in Cases II, III, and IV, respectively. Given that consumers discount their future surplus, the expected value of the credit is less than that in the average reward model. Thus,

$$\pi^*_{2,\delta} \le \pi^*_2, \quad \pi^*_{3,\delta} \le \pi^*_3, \quad \pi^*_{4,\delta} \le \pi^*_4. \tag{A.29}$$

Recall that when v_L is sufficiently close to v_H or α is sufficiently small,

$$\max\{\pi^*_2, \pi^*_3, \pi^*_4\} < \pi^*_1 = \lambda(1-\gamma)v_L. \tag{A.30}$$

Combining (A.29) and (A.30) yields

$$\max\{\pi^*_{2,\delta}, \pi^*_{3,\delta}, \pi^*_{4,\delta}\} < \lambda(1-\gamma)v_L.$$

Together with (A.28), we obtain

$$\pi^*_\delta = \max\{\pi^*_{1,\delta}, \pi^*_{2,\delta}, \pi^*_{3,\delta}, \pi^*_{4,\delta}\} < \lambda(1-\gamma)v_L \le \pi^c_\delta.$$

Therefore, if v_L is sufficiently close to v_H or α is sufficiently small, the cash refund policy brings more profit than the credit refund policy. This completes the proof of Part (a).

Note that

$$\pi^*_\delta = \max\{\pi^*_{1,\delta}, \pi^*_{2,\delta}, \pi^*_{3,\delta}, \pi^*_{4,\delta}\}, \quad \pi^* = \max\{\pi^*_1, \pi^*_2, \pi^*_3, \pi^*_4\}.$$

When δ is sufficiently close to 1, $\pi^*_{i,\delta} \approx \pi^*_i$ for each $1 \le i \le 4$. Hence, $\pi^*_\delta \approx \pi^*$. According to Theorem 3, if $v_L/v_H \le \alpha < 1$, then $\pi^* > \pi^c = \pi^c_\delta$. Therefore, $\pi^*_\delta \ge \pi^c_\delta$ if $v_L/v_H \le \alpha < 1$ and δ is sufficiently close to 1. This completes the proof of Part (b). ■

Appendix B: Detailed Analysis for the Firm's Problem under an Exogenous T

B.1 Analysis and Results

B.1.1 Case I: A low-valuation consumer makes a purchase without a credit In order to derive the revenue contribution by each consumer, we first analyze the consumer's state transition. Because low-valuation consumers make a purchase even without a credit, one can verify that they also make a purchase with a credit. Therefore, in this case, consumers make a purchase in each state, independent of their valuation realization. Figure B.1 depicts the consumer's state transition.

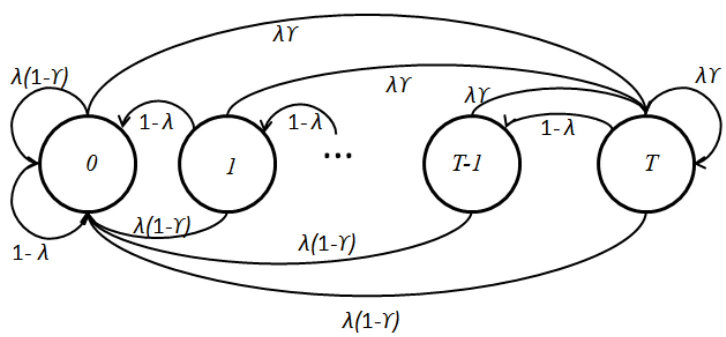


Figure B.1 State transition in Case I.

Lemma B.1 **(Stationary state probabilities in Case I)**. *Let q_i denote the stationary probability of state $i \in \{0, 1, \ldots, T\}$. We have*

$$q_0 = 1 - \gamma[1 - (1-\lambda)^T],$$
$$q_i = (1-\lambda)^{T-i}\lambda\gamma. \qquad \forall i = 1, 2 \ldots, T.$$

Note that both high- and low-valuation consumers make a purchase in each state, and each consumer pays a price $p - c$ in state $i \neq 0$ and a price p in state 0. Thus, the average profit contribution by each consumer is

$$\lambda\Big\{q_0 p + (1-q_0)(p-c)\Big\} = \lambda\Big\{p - (1-q_0)c\Big\} = \lambda\Big\{p - \gamma[1-(1-\lambda)^T]c\Big\}.$$

Therefore, the firm's optimization problem can be written as follows,

$$\max_{p,c} \quad \lambda\Big\{p - \gamma[1-(1-\lambda)^T]c\Big\} \tag{B.1}$$
$$s.t. \quad p \leq \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L \tag{B.2}$$
$$p \geq c. \tag{B.3}$$

Note that (B.2) ensures that a low-valuation consumer makes a purchase, even without a credit on hand. Let p_1^*, c_1^*, and π_1^* denote the optimal price, credit, and corresponding profit to the problem (B.1)–(B.3), respectively. We have the following result.

Proposition B.1. *The optimal solution to the problem* (B.1)–(B.3) *is given by*

$$p_1^* = \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L,$$

$$c_1^* \in \left[0, \frac{(1-\gamma)v_L}{1-\gamma[1-(1-\lambda)^T]}\right],$$

$$\pi_1^* = \lambda(1-\gamma)v_L.$$

Note that the optimal price follows Proposition 1(a). Recall that the first term in the optimal price denotes the expected value of an earned credit upon a product return, and the second term is the utility received by a low-valuation consumer for a product that is not returned. Given the composition of the price, the firm's long-run average profit stems from the utility $(1-\gamma)v_L$ that a low-valuation consumer receives when she does not return the product. Hence, the optimal profit includes only the second term of the optimal price, taking into account the consumer's arrival rate λ.

B.1.2 Case II: A low-valuation consumer makes a purchase whenever she has a credit

This section analyzes the case in which a low-valuation consumer does not make a purchase without a credit but makes a purchase whenever she has a credit on hand.

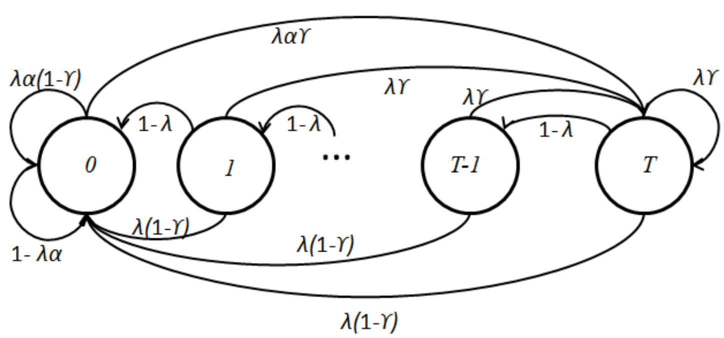


Figure B.2 State transition in Case II.

The state transition in this case is shown in Figure B.2. Notably, the only difference from Case I is that a low-valuation consumer does not make a purchase in state 0; thus, the transition from state 0 to state T occurs with a probability $\lambda\alpha\gamma$ rather than $\lambda\gamma$.

Lemma B.2 **(Stationary state probabilities in Case II)**. *Let* q_i *denote the stationary probability of state* $i \in \{0, 1, \ldots, T\}$. *We have*

$$\begin{aligned} q_0 &= \frac{1-\gamma[1-(1-\lambda)^T]}{\lambda\gamma\alpha} q_T, \\ q_i &= (1-\lambda)^{T-i} q_T, \qquad \forall i = 1, 2, \ldots, T-1. \\ q_T &= \frac{\lambda\gamma\alpha}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}. \end{aligned}$$

To understand the expression of q_T, consider the following event. For both types of consumers, the process to arrive at state T involves "arriving, making a purchase, and returning the product". First, a high-valuation consumer always makes a purchase whenever she arrives to the market, so the probability of "arriving, making a purchase, and returning" is simply $\lambda\gamma$. Hence, the probability that a consumer draws a high valuation and arrives at state T is simply $\alpha\lambda\gamma$. Second, a low-valuation consumer in state T could return to state T by making a purchase before her credit goes expired and then returning the product, the probability of which is $q_T[1-(1-\lambda)^T]\gamma$. Note that a low-valuation consumer who is in state $i \in \{1, \ldots, T-1\}$ could also arrive at state T by making a purchase and then returning, which has been included in $q_T[1-(1-\lambda)^T]\gamma$. Hence, the probability that a consumer draws a low valuation and arrives at state T is $(1-\alpha)q_T[1-(1-\lambda)^T]\gamma$. Because these two events are mutually exclusive, the total probability of both events is the sum, which should be equal to q_T. Therefore, we have the following equation:

$$\alpha\lambda\gamma + (1-\alpha)q_T[1-(1-\lambda)^T]\gamma = q_T.$$

Solving the equations gives the expression for q_T.

Note that both high- and low-valuation consumers make a purchase in state $i \in \{1, 2, \ldots, T\}$ and pay a price $p - c$; however, only a high-valuation consumer makes a purchase in state 0 and pays a price p. Hence, the average profit contribution by each consumer is

$$\lambda\Big\{q_0\alpha p + (1-q_0)(p-c)\Big\} = \frac{\lambda\alpha}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}\Big\{p - \gamma[1-(1-\lambda)^T]c\Big\}.$$

Therefore, the firm's optimization problem is as follows,

$$\max_{p,c} \quad \frac{\lambda\alpha}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}\Big\{p - \gamma[1-(1-\lambda)^T]c\Big\} \tag{B.4}$$

$$s.t. \quad p \le \gamma J(T) + (1-\gamma)v_H \tag{B.5}$$

$$p \le \gamma J(T) + (1-\gamma)v_L + c - J(T-1) \tag{B.6}$$

$$p \ge \gamma J(T) + (1-\gamma)v_L \tag{B.7}$$

$$p \ge c. \tag{B.8}$$

The expressions for $J(T)$ and $J(T-1)$ are given in Proposition A.1. Note that (B.5) implies that a high-valuation consumer makes a purchase without a credit on hand, (B.6) implies that a low-valuation consumer makes a purchase in state T (which implies that she always makes a purchase whenever she has a credit on hand), and (B.7) implies that a low-valuation consumer does not purchase without a credit on hand.

PROPOSITION B.2. *For any fixed T, the optimal solution to the problem* (B.4)–(B.8) *is as follows:*

(a) If $T \le T_0$, then the optimal price is $p^*_{2,a} = p^C(T)|_{c=c^*_{2,a}}$, where $c^*_{2,a}$ can take any value between $A(T)$ and $B(T,T)$. The optimal profit is given by

$$\pi^*_{2,a} = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha \frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} v_L;$$

(b) Otherwise, the optimal price is $p^*_{2,b} = p^D(T)|_{c=c^*_{2,b}}$, where $c^*_{2,b} = C(T,T)$. The optimal profit is given by

$$\pi^*_{2,b} = \lambda\alpha(1-\gamma)v_L \frac{1-\gamma[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} \cdot \frac{\alpha+(1-\alpha)(1-\lambda)^{T-1}}{\alpha[1-(1-\lambda)^{T-1}] - \alpha\gamma[1-(1-\lambda)^T]}.$$

Proposition B.2 characterizes the optimal solution in Case II. According to Proposition 1(b), the optimal price takes the minimum between $p^C(T)$ and $p^D(T)$. Recall that $p^C(T)$ is the highest price at which a high-valuation consumer purchases without a credit, while $p^D(T)$ is the highest price at which a low-valuation consumer purchases in state T (which implies that she would also purchase in states $i \in \{1,\dots,T-1\}$). The two parts of Proposition B.2 correspond to situations in which the optimal prices are $p^C(T)$ and $p^D(T)$, respectively.

Part (a) considers the case in which T is small (i.e., $T \le T_0$) and the risk of credit expiration is relatively large. In this case, a low-valuation consumer has a strong incentive to purchase in state T; hence, $p^D(T)$ is higher than $p^C(T)$. Therefore, the optimal price in this case is $p^C(T)$. To understand the expression of the profit $\pi^*_{2,a}$, note that the firm's long-run average profit stems from the utility that a consumer receives when she does not return the product. The probability that a high-valuation consumer shows up, makes a purchase, and then does not return, is $\lambda(1-\gamma)\alpha$. Hence, the expected profit collected from a high-valuation consumer is $\lambda(1-\gamma)\alpha v_H$. Similarly, the expected profit collected from a low-valuation consumer is

$$\lambda(1-q_0)(1-\gamma)(1-\alpha)v_L = \lambda(1-\gamma)\alpha \frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} v_L.$$

Combining these two parts gives the optimal profit $\pi^*_{2,a}$.

Part (b) considers the case in which T is large (i.e., $T > T_0$) and the risk of credit expiration is relatively small. In this case, a low-valuation consumer has a lower incentive to purchase; therefore, in order to ensure that a low-valuation consumer purchases in state T, the price $p^D(T)$ must be sufficiently low. Hence, $p^D(T)$ is smaller than $p^C(T)$, and thus the optimal price in this case is $p^D(T)$.

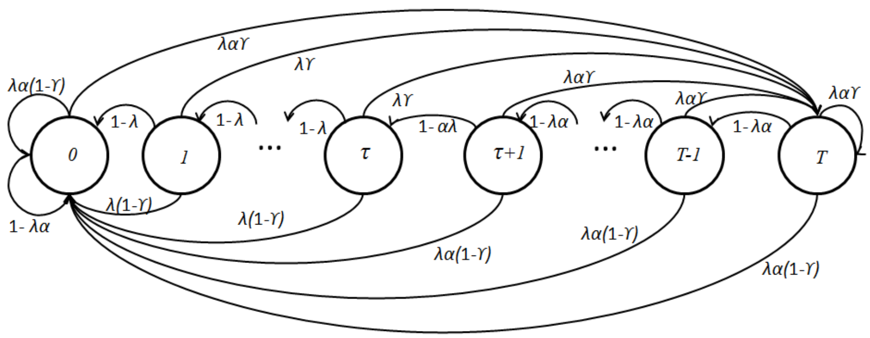


Figure B.3 State transition in Case III.

B.1.3 Case III: A low-valuation consumer makes a purchase when she has a credit and the credit is within τ periods of expiration ($\tau < T$) The state transition in this case is characterized in Figure B.3. The difference with Case II is that a low-valuation consumer does not make a purchase in state $\{\tau+1,\dots,T\}$. Therefore, the flow from state $i \in \{\tau+1,\dots,T\}$ to state T occurs with a probability $\lambda\alpha\gamma$ rather than $\lambda\gamma$, the flow from state $i \in \{\tau+1,\dots,T\}$ to state 0 occurs with a probability $\lambda\alpha(1-\gamma)$ rather than $\lambda(1-\gamma)$, and the flow from state $i \in \{\tau+1,\dots,T\}$ to state $i-1$ occurs with a probability $1-\lambda\alpha$ rather than $1-\lambda$.

Lemma B.3 **(Stationary state probabilities in Case III)**. *Let q_i denote the stationary probability of state $i \in \{0,1,\dots,T\}$. We have*

$$\begin{aligned}
q_0 &= \frac{(1-\gamma)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^T] + (1-\gamma)[1-(1-\alpha\lambda)^{T-\tau}] + (1-\lambda)^\tau(1-\alpha\lambda)^{T-\tau}}{\lambda\gamma\alpha} q_T, \\
q_i &= (1-\lambda)^{\tau-i}(1-\alpha\lambda)^{T-\tau} q_T, && \forall i = 1,2,\dots,\tau-1. \\
q_i &= (1-\alpha\lambda)^{T-i} q_T, && \forall i = \tau,\dots,T-1. \\
q_T &= \frac{\lambda\gamma\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}.
\end{aligned}$$

Note that both high- and low-valuation consumers make a purchase in state $i \in \{1,2,\dots,\tau-1,\tau\}$ and pay a price $p-c$. Only high-valuation consumers make a purchase in state $\{0,\tau+1,\dots,T\}$. They pay a price p in state 0 and a price $p-c$ in state $i \in \{\tau+1,\dots,T\}$. Hence, the average profit contribution by each consumer is

$$\begin{aligned}
&\lambda\Big\{q_0\alpha p + \sum_{i=1}^{\tau}(p-c)q_i + \sum_{i=\tau+1}^{T}\alpha(p-c)q_i\Big\} \\
=&\frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}\Big\{p-\gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^\tau]c\Big\}.
\end{aligned}$$

Therefore, the firm's optimization problem becomes

$$\max_{p,c} \quad \frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}\Big\{p-\gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^\tau]c\Big\} \qquad \text{(B.9)}$$

$$s.t.\quad p \le \gamma J(T) + (1-\gamma)v_H \tag{B.10}$$

$$p \le \gamma J(T) + (1-\gamma)v_L + c - J(\tau - 1) \tag{B.11}$$

$$p \ge \gamma J(T) + (1-\gamma)v_L + c - J(\tau) \tag{B.12}$$

$$p \ge c. \tag{B.13}$$

The expressions for $J(\cdot)$ are given in Proposition A.1. Note that (B.10) implies that a high-valuation consumer makes a purchase without a credit on hand, (B.11) implies that a low-valuation consumer makes a purchase when the credit is within τ periods of expiration, and (B.12) implies that a low-valuation consumer does not make a purchase when the credit is within $\tau + 1$ periods of expiration. Let p_3^*, c_3^*, and π_3^* denote the optimal price, credit, and corresponding profit to the problem (B.9)–(B.13).

Proposition B.3. *For any fixed T, the optimal solution to the problem* (B.9)–(B.13) *is as follows:*

$$\pi_3^* = \max_{1 \le \tau \le T-1} \pi_3^\tau,$$

$$\tau_3^* = \arg\max_{\tau \in \{1,\ldots,T-1\}} \pi_3^\tau,$$

$$p_3^* = p^D(\tau_3^*)|_{c=c_3^*},$$

$$c_3^* = \min\{A(\tau_3^*), C(T, \tau_3^*)\}.$$

In the above,

$$\pi_3^\tau = \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]} + \frac{\lambda\alpha(1-\lambda)^{\tau-1}\min\{A(\tau), C(T,\tau)\}}{\alpha + (1-\alpha)(1-\lambda)^{\tau-1}}, \quad \forall 1 \le \tau \le T.$$

In Proposition B.3, π_3^τ corresponds to the profit for a fixed τ. The key idea of the solution is to compare the profits associated with different values of τ and τ_3^* is the value of τ with the highest profit. At optimality, the firm sets the price p_3^* and credit refund c_3^* such that a low-valuation consumer makes a purchase only in state $\{1, \ldots, \tau_3^*\}$.

B.1.4 Case IV: A low-valuation consumer never makes a purchase in any state In order to derive the revenue contribution by each consumer, we first analyze the consumer's state transition. In this case, high-valuation consumers make a purchase in each state, while low-valuation consumers never make a purchase. Figure B.4 depicts the consumer's state transition.

Lemma B.4 **(Stationary state probabilities in Case IV)**. *Let q_i denote the stationary probability of state $i \in \{0, 1, \ldots, T\}$. We have*

$$q_0 = 1 - \gamma[1 - (1-\lambda\alpha)^T],$$

$$q_i = (1-\lambda\alpha)^{T-i}\lambda\alpha\gamma. \qquad \forall i = 1, 2 \ldots, T.$$

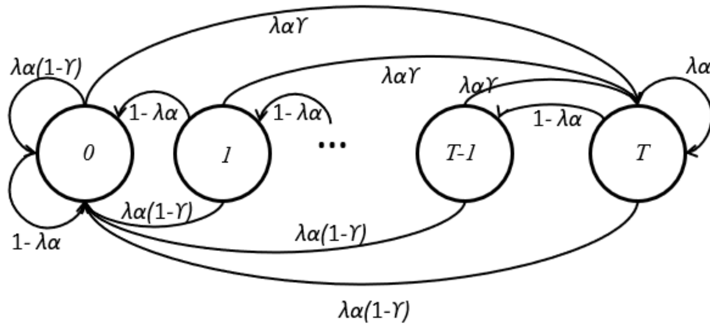


Figure B.4 State transition in Case IV.

Note that only high-valuation consumers make a purchase in each state and pays a price $p-c$ in state $i \neq 0$ and a price p in state 0. Thus, the average profit contribution by each consumer is

$$\lambda\alpha\Big\{q_0 p + (1-q_0)(p-c)\Big\} = \lambda\alpha\Big\{p - (1-q_0)c\Big\} = \lambda\alpha\Big\{p - \gamma[1-(1-\lambda\alpha)^T]c\Big\}.$$

Therefore, the firm's optimization problem can be written as follows,

$$\max_{p,c} \quad \lambda\alpha\Big\{p - \gamma[1-(1-\lambda\alpha)^T]c\Big\} \tag{B.14}$$
$$s.t. \quad p \leq \gamma[1-(1-\lambda\alpha)^T]c + (1-\gamma)v_H \tag{B.15}$$
$$p > (1-\gamma)v_L + \gamma[1-(1-\lambda\alpha)^T)]c + c \tag{B.16}$$
$$p \geq c. \tag{B.17}$$

Note that (B.15) ensures a high-valuation consumer makes a purchase even without credit on hand, and (B.16) ensures a low-valuation consumer does not make a purchase without credit on hand. Let p_4^*, c_4^*, and π_4^* denote the optimal price, credit, and the corresponding profit to the problem (B.14)–(B.17), respectively. We have the following result.

Proposition B.4. *The optimal solution to the problem* (B.14)–(B.17) *is given by*

$$p_4^* = \gamma[1-(1-\lambda\alpha)^T]c + (1-\gamma)v_H,$$

$$c_4^* \in \Big[0, \frac{(1-\gamma)v_H}{1-\gamma[1-(1-\lambda\alpha)^T]}\Big],$$

$$\pi_4^* = \lambda(1-\gamma)\alpha v_H.$$

To understand the profit expression, note that the optimal price in Case IV is given by $\gamma[1-(1-\lambda\alpha)^T]c + (1-\gamma)v_H$. The term $[1-(1-\lambda\alpha)^T]c$ denotes the expected value of an earned credit upon a product return,[13] the probability of which is γ. The term v_H is the utility a high-valuation

[13] In this case, only high-valuation consumers make a purchase, and the purchase is made whenever they arrive at the market. Hence, the probability that a credit will be used before expiration is $1-(1-\lambda\alpha)^T$, different from $1-(1-\lambda)^T$ in Case I.

consumer receives when she does not return the product, the probability of which is $1-\gamma$. Given the composition of the price, the firm's long-run average profit stems from the utility received by a high-valuation consumer when she does not return the product. Hence, the optimal profit includes only the second term of the optimal price, taking into account the arrival rate $\lambda\alpha$ of high-valuation consumers.

B.2 Proofs of Results in Appendix B.1 and Proof of Theorem 1

Proof of Lemma B.1

The balance equations can be written as follows,

$$\begin{aligned}
&\lambda\gamma q_0 = \lambda(1-\gamma)\sum_{i=1}^{T} q_i + (1-\lambda)q_1, \\
&q_i = (1-\lambda)q_{i+1}, && \forall i = 1,2\ldots,T-1. \\
&[\lambda(1-\gamma)+(1-\lambda)]q_T = \lambda\gamma\sum_{i=0}^{T-1} q_i, \\
&\sum_{i=0}^{T} q_i = 1.
\end{aligned}$$

Solving the above equations yields the stationary probabilities. ■

Proof of Proposition B.1

Constraint (B.2) indicates that the optimal price

$$p_1^* = \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L.$$

Putting p_1^* into constraint (B.3) yields that

$$c_1^* \leq \frac{(1-\gamma)v_L}{1-\gamma[1-(1-\lambda)^T]}.$$

Putting p_1^* into objective (B.1) gives the optimal profit. ■

Proof of Lemma B.2

The balance equations can be written as follows,

$$\begin{aligned}
&\lambda\alpha\gamma q_0 = \lambda(1-\gamma)\sum_{i=1}^{T} q_i + (1-\lambda)q_1, \\
&q_i = (1-\lambda)q_{i+1}, && \forall i = 1,2\ldots,T-1. \\
&[\lambda(1-\gamma)+(1-\lambda)]q_T = \lambda\gamma\sum_{i=1}^{T-1} q_i + \lambda\alpha\gamma q_0, \\
&\sum_{i=0}^{T} q_i = 1.
\end{aligned}$$

Solving the above equations yields the stationary probabilities. ■

Proof of Proposition B.2

Before proving Proposition B.2, we first introduce a lemma.

LEMMA B.5. *Suppose $C(T,k)\geq 0$ for fixed T and k. If $A(k)>(<)B(T,k)$, then $A(k)>(<)C(T,k)$.*

Proof. On the one hand,

$$\begin{aligned}&A(k)-B(T,k)\\=&\frac{1-\gamma}{(1-\lambda)^{k-1}\Big\{1-\gamma[1-(1-\alpha\lambda)^{T-k}(1-\lambda)^{k}]\Big\}}\cdot\\&\Big\{\big(\alpha-\alpha\gamma+(1-\gamma)(1-\alpha)(1-\lambda)^{k-1}+\gamma(1-\alpha\lambda)^{T-k+1}(1-\lambda)^{k-1}\big)(v_H-v_L)-(1-\lambda)^{k-1}v_H\Big\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&A(k)-C(T,k)\\=&\frac{1-\gamma}{(1-\lambda)^{k-1}\Big\{(1-\gamma)[\alpha+(1-\alpha)(1-\lambda)^{k-1}]-(1-\lambda)^{k-1}[1-\gamma(1-\alpha\lambda)^{T-k+1}]\Big\}}\cdot\\&\Big\{\big(\alpha-\alpha\gamma+(1-\gamma)(1-\alpha)(1-\lambda)^{k-1}+\gamma(1-\alpha\lambda)^{T-k+1}(1-\lambda)^{k-1}\big)(v_H-v_L)-(1-\lambda)^{k-1}v_H\Big\}.\end{aligned}$$

Note that the above denominator is positive, indicated by $C(T,k)\geq 0$. This completes the proof. ■

Proof of Proposition B.2. Recall that

$$p^C(T)=\gamma[1-(1-\lambda)^T]c+(1-\gamma)\Big\{1-\gamma(1-\alpha)[1-(1-\lambda)^T]\Big\}v_H+(1-\gamma)\gamma(1-\alpha)[1-(1-\lambda)^T]v_L,$$

$$p^D(T)=(1-\gamma)v_L+\frac{\gamma\alpha[1-(1-\lambda)^T]+(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-1}}c.$$

Putting $J(\cdot)$ into constraints (B.5) and (B.6) leads to $p\leq p^C(T)$ and $p\leq p^D(T)$, respectively. Note that (B.7) holds automatically if $p=\min\{p^C(T),p^D(T)\}$. Hence, the constraints for this optimization problem can be summarized as

$$c\leq p\leq \min\{p^C(T),p^D(T)\}.$$

One can verify that if $c\geq A(T)$, then $p^C(T)\leq p^D(T)$, and vice versa. We consider two cases and identify two possible solutions.

Case 1: One can check if $c\leq B(T,T)$, then $c\leq p^C(T)$. Therefore, if $A(T)\leq B(T,T)$, then one possible solution is

$$\begin{aligned}p^*_{2,a}=p^C(T)|_{c=c^*_{2,a}}=&\gamma[1-(1-\lambda)^T]c^*_{2,a}+(1-\gamma)\Big\{1-\gamma(1-\alpha)[1-(1-\lambda)^T]\Big\}v_H\\&+(1-\gamma)\gamma(1-\alpha)[1-(1-\lambda)^T]v_L,\end{aligned}$$

$$\pi^*_{2,a} = \pi^*_{2,a}(p^C(T)) = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha \frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} v_L,$$

where $c^*_{2,a}$ takes any value between $A(T)$ and $B(T,T)$.

Case 2: One can check if $c \leq C(T,T)$, then $c \leq p^D(T)$. Hence, the other possible solution is

$$c^*_{2,b} = \min\{A(T), C(T,T)\},$$

$$p^*_{2,b} = p^D(T)|_{c=c^*_{2,b}} = (1-\gamma)v_L + \frac{\gamma\alpha[1-(1-\lambda)^T] + (1-\lambda)^{T-1}}{\alpha + (1-\alpha)(1-\lambda)^{T-1}]} c^*_{2,b},$$

$$\pi^*_{2,b} = \pi^*_{2,b}(p^D(T)) = \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} + \frac{\lambda\alpha(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-1}} c^*_{2,b}.$$

Note that Case 1 requires $A(T) \leq B(T,T)$, hence, for any fixed $T \leq T_0$ such that $A(T) \leq B(T,T)$, both Case 1 and Case 2 could occur if the firm sets the appropriate price and credit. However,

$$\begin{aligned} \pi^*_{2,b}(p^D(T)) &\leq \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} + \frac{\lambda\alpha(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-1}} A(T) \\ &= \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha \frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} v_L \\ &= \pi^*_{2,a}(p^C(T)). \end{aligned}$$

Therefore, for any $T \leq T_0$, the firm should set the price and credit such that Case 1 occurs.

For any $T \geq T_0 + 1$, only Case 2 is possible. According to the definition of T_0, we have $A(T) > B(T,T)$. It follows Lemma B.5 that $A(T) > C(T,T)$. Therefore, $c^*_{2,b} = C(T,T)$. Putting $c^*_{2,b}$ into $\pi^*_{2,b}$ yields

$$\pi^*_{2,b} = \lambda\alpha(1-\gamma)v_L \frac{1-\gamma[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]} \cdot \frac{\alpha+(1-\alpha)(1-\lambda)^{T-1}}{\alpha[1-(1-\lambda)^{T-1}] - \alpha\gamma[1-(1-\lambda)^T]}.$$

This completes the proof. ∎

Proof of Lemma B.3

The balance equations can be written as follows,

$$\begin{aligned} &\lambda\alpha\gamma q_0 = \lambda(1-\gamma)\sum_{i=1}^{\tau} q_i + \lambda\alpha(1-\gamma)\sum_{i=\tau+1}^{T} q_i + (1-\lambda)q_1, \\ &q_i = (1-\lambda)q_{i+1}, && \forall i = 1, 2 \ldots, \tau-1. \\ &q_i = (1-\alpha\lambda)q_{i+1}, && \forall i = \tau, \tau+1, \ldots, T-1. \\ &[\lambda\alpha(1-\gamma) + (1-\alpha\lambda)]q_T = \lambda\alpha\gamma q_0 + \lambda\gamma\sum_{i=1}^{\tau} q_i + \lambda\alpha\gamma\sum_{i=\tau+1}^{T-1} q_i, \\ &\sum_{i=0}^{T} q_i = 1. \end{aligned}$$

Solving the above equations yields the stationary probabilities. ∎

Proof of Proposition B.3

Recall that

$$p^C(\tau) = \gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^\tau]c + (1-\gamma)\Big\{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]\Big\}v_H$$
$$+ (1-\gamma)\Big\{\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]\Big\}v_L,$$
$$p^D(\tau) = (1-\gamma)v_L + \gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^\tau]c + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}(1-\lambda)^{\tau-1}c.$$

Putting $J(\cdot)$ into constraints (B.10), (B.11), and (B.12) yields $p \le p^C(\tau)$, $p \le p^D(\tau)$, and $p \ge p^D(\tau+1)$, respectively. Hence, the constraints for the problem (B.9)–(B.13) can be summarized as follows,

$$\max\big\{c, p^D(\tau+1)\big\} \le p \le \min\big\{p^C(\tau), p^D(\tau)\big\}.$$

One can verify that if $c \ge A(\tau)$, then $p^C(\tau) \le p^D(\tau)$, and vice versa. We consider two cases regarding the relationship between $p^C(\tau)$ and $p^D(\tau)$.

Case 1: If $c \le A(\tau+1)$, then $p^C(\tau) \ge p^D(\tau+1)$, and vice versa. Moreover, if $c \le B(T,\tau)$, then $c \le p^C(\tau)$. Therefore, for any fixed $\tau \in \{1,\dots,T-1\}$, if $A(\tau) \le \min\big\{B(T,\tau), A(\tau+1)\big\}$, then one possible solution to the problem (7)–(11) is,

$$p_3^\tau = p^C(\tau)|_{c=c_3^\tau} = \gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^\tau]c_3^\tau + (1-\gamma)\Big\{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]\Big\}v_H$$
$$+ (1-\gamma)\Big\{\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]\Big\}v_L,$$
$$\pi_3^\tau(p^C(\tau)) = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\frac{\alpha(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}v_L,$$

and c_3^τ takes any value between $A(\tau)$ and $\min\big\{B(T,\tau), A(\tau+1)\big\}$.

Case 2: If $c \le C(T,\tau)$, then $c \le p^D(\tau)$. Moreover, $p^D(\tau) \ge p^D(\tau+1)$ holds automatically. Hence, for any fixed $\tau \in \{1,\dots,T-1\}$, the other possible solution is as follows,

$$p_3^\tau = p^D(\tau)|_{c=c_3^\tau} = (1-\gamma)v_L + \gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^\tau]c_3^\tau$$
$$+ \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}(1-\lambda)^{\tau-1}c_3^\tau,$$
$$\pi_3^\tau(p^D(\tau)) = \frac{\lambda\alpha}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}\cdot$$
$$\Big\{(1-\gamma)v_L + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}(1-\lambda)^{\tau-1}c_3^\tau\Big\},$$
$$c_3^\tau = \min\{A(\tau), C(T,\tau)\}.$$

Next, we show that the solution in Case 1 is a special case of that in Case 2. Note that the solution in Case 1 is possible only if $A(\tau) \leq B(T,\tau)$. According to Lemma B.5, if $A(\tau) \leq B(T,\tau)$, then $A(\tau) \leq C(T,\tau)$. One can check that if $c = A(\tau)$, then

$$p^D(\tau) = p^C(\tau),$$

and

$$\pi_3^\tau(p^D(\tau)) = \pi_3^\tau(p^C(\tau)).$$

Hence, we establish that the solution in Case 1 is a special case of Case 2.

Therefore, for any fixed $\tau \in \{1,2,\ldots,T-1\}$, the optimal price, credit, and profit are $p^D(\tau)|_{c=c_3^\tau}$, c_3^τ, and $\pi_3^\tau(p^D(\tau))$. Comparing the profit for each τ gives the solution to the problem (B.9)–(B.13). This completes the proof. ■

Proof of Lemma B.4

The balance equations can be written as follows,

$$\begin{aligned}
&\lambda\alpha q_0 = \lambda\alpha(1-\gamma)\sum_{i=0}^{T} q_i + (1-\lambda\alpha)q_1, \\
&q_i = (1-\lambda\alpha)q_{i+1}, && \forall i = 1,2\ldots,T-1. \\
&q_T = \lambda\alpha\gamma\sum_{i=0}^{T} q_i, \\
&\sum_{i=0}^{T} q_i = 1.
\end{aligned}$$

Solving the above equations yields the stationary probabilities. ■

Proof of Proposition B.4

Constraint (B.15) indicates that the optimal price

$$p_4^* = \gamma[1-(1-\lambda\alpha)^T]c + (1-\gamma)v_H.$$

Putting p_4^* into constraint (B.17) yields that

$$c_4^* \leq \frac{(1-\gamma)v_H}{1-\gamma[1-(1-\lambda\alpha)^T]}.$$

Putting p_4^* into objective (B.14) gives the optimal profit. ■

Proof of Theorem 1

To derive the firm's optimal decisions, it suffices to compare π_1^*, π_2^*, and π_3^*, because π_4^* is always smaller than π_2^* or π_3^* by Lemma 2(a). Recall that $\pi_1^* = \lambda(1-\gamma)v_L$. Observe that π_1^* is always included in π^*, so to complete the proof, it suffices to investigate/compare π_2^* and π_3^*.

Part (a): Suppose $T \le T_0$. For any $T \le T_0$, according to Proposition B.2(a), we have

$$\pi_{2,a}^* = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha\frac{\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}v_L.$$

Next, we derive π_3^*. If $T \le T_0$, then $T \le T_i$ for each $i \ge 1$, because $T_0 \le T_i$ by Lemma 1(b). Therefore, when $\tau = T - i$, we have $A(\tau) \le B(T,\tau)$. By Lemma B.5, it follows immediately that $A(\tau) \le C(T,\tau)$. Hence, $c_3^\tau = A(\tau)$. Therefore,

$$\begin{aligned}\pi_3^\tau &= \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]} + \frac{\lambda\alpha(1-\lambda)^{\tau-1}A(\tau)}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}\\ &= \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\frac{\alpha(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}v_L.\end{aligned}$$

Clearly, π_3^τ increases in τ. Thus,

$$\pi_3^* = \max_{\tau\in\{1,\dots,T-1\}}\pi_3^\tau = \pi_3^{T-1} = \lambda(1-\gamma)\alpha v_H + \lambda(1-\gamma)\alpha\frac{\gamma(1-\alpha)(1-\alpha\lambda)[1-(1-\lambda)^{T-1}]}{1-\gamma(1-\alpha)(1-\alpha\lambda)[1-(1-\lambda)^{T-1}]}v_L.$$

Because $1-(1-\lambda)^T > (1-\alpha\lambda)[1-(1-\lambda)^{T-1}]$, we have $\pi_{2,a}^* > \pi_3^*$. Hence, the optimal solution is either π_1^* or $\pi_{2,a}^*$, whichever is the largest.

Part (b): Suppose $T \ge T_0 + 1$. Here, we do not compare $\pi_{2,b}^*$ and π_3^*, rather, we show

$$\max\{\pi_{2,b}^*, \pi_3^*\} = \max_{\tau\in\{1,\dots,T\}}\pi_3^\tau.$$

Proposition B.3 shows that $\pi_3^* = \max_{\tau\in\{1,\dots,T-1\}}\pi_3^\tau$. To complete the proof, it suffices to show $\pi_{2,b}^*$ is equivalent to π_3^T.

For any $T \ge T_0 + 1$, the definition of T_0 indicates that $A(T) > B(T,T)$ and then it follows Lemma B.5 that $A(T) > C(T,T)$. Consequently, $\min\{A(T), C(T,T)\} = C(T,T)$. We have

$$\begin{aligned}\pi_3^T &= \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)(1-\alpha\lambda)^{T-T}[1-(1-\lambda)^T]} + \frac{\lambda\alpha(1-\lambda)^{T-1}C(T,T)}{\alpha+(1-\alpha)(1-\lambda)^{T-1}}\\ &= \frac{\lambda\alpha(1-\gamma)v_L}{1-\gamma(1-\alpha)[1-(1-\lambda)^T]}\cdot\frac{[\alpha+(1-\alpha)(1-\lambda)^{T-1}]\{1-\gamma[1-(1-\lambda)^T]\}}{\alpha[1-(1-\lambda)^{T-1}]-\alpha\gamma[1-(1-\lambda)^T]},\end{aligned}$$

which, according to Proposition B.2(b), is exactly the same as $\pi_{2,b}^*$. Moreover,

$$\begin{aligned}&p^D(T)|_{c=C(T,T)}\\ =&(1-\gamma)v_L + \gamma[1-(1-\alpha\lambda)^{T-\tau}(1-\lambda)^\tau]c + \frac{1-(1-\alpha)\gamma(1-\alpha\lambda)^{T-\tau}[1-(1-\lambda)^\tau]}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}(1-\lambda)^{\tau-1}c\end{aligned}$$

$$=(1-\gamma)v_L+\frac{\alpha\gamma[1-(1-\lambda)^T]+(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{\tau-1}}c$$
$$=(1-\gamma)v_L+\frac{\alpha\gamma[1-(1-\lambda)^T]+(1-\lambda)^{T-1}}{\alpha+(1-\alpha)(1-\lambda)^{T-1}}C(T,T),$$

which is exactly the same as $p^*_{2,b}$.

We have established that for any $T\geq T_0+1$, $\{\pi_3^\tau : 1\leq\tau\leq T\}$ is a set including the profits in Cases II and III. Thus, $\pi_3^{\hat{\tau}}$, where $\hat{\tau}=\arg\max_{\tau\in\{1,\ldots,T\}}\pi_3^\tau$, corresponds to the optimal profit in Cases II and III. Hence, the optimal solution for any $T\geq T_0+1$ is either π_1^* or $\pi_3^{\hat{\tau}}$, whichever is the largest. This completes the proof. ■

Appendix C: General Distribution of Consumer Valuations

Our main analysis assumes that consumers' valuation v for a product follows a two-point distribution. However, we expect that the benefit of a credit refund policy is preserved under more general valuation distributions as long as the firm can still achieve intra-consumer discrimination. Given that explicit characterizations of the optimal credit refund policy is infeasible, we use a numerical study to verify that our main result is robust to the assumption on the valuation distribution.

Suppose a consumer's valuation v for a product follows a general distribution with cdf $F(\cdot)$. Then, a consumer's decision problem can be written as

$$\rho^*+J(t)=\lambda\int_v \max\Big\{\gamma J(T)+(1-\gamma)(J(0)+v)-p+c, J(t-1)\Big\}dF(v)+(1-\lambda)J(t-1),\quad \forall 1\leq t\leq T, \quad \text{(C.1)}$$
$$\rho^*+J(0)=\lambda\int_v \max\Big\{\gamma J(T)+(1-\gamma)(J(0)+v)-p, J(0)\Big\}dF(v)+(1-\lambda)J(0). \quad \text{(C.2)}$$

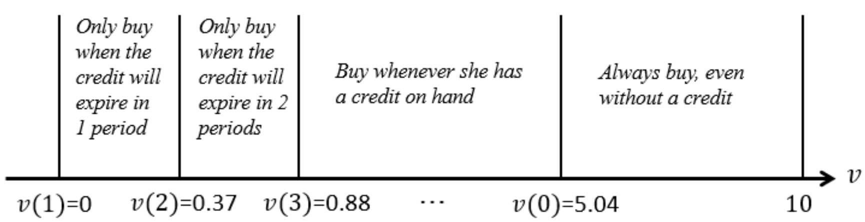


Figure C.1 Consumer behavior when the consumer's valuation has uniform distribution on $[0,10]$.

Intuitively, there exists a threshold $v(t)$ on the consumer's valuation such that, conditional on being in state t, the consumer will make a purchase if $v\geq v(t)$ and will not otherwise. In the numerical example, we assume that the consumer's valuation v follows a uniform distribution $U[0,10]$ and set $(\lambda,\gamma,T)=(0.15,0.15,3)$. On the consumer side, the numerical result shows that $v(t)$ is increasing in t for $t\geq 1$; in particular, $v(1)=0$, $v(2)=0.37$, $v(3)=0.88$, and $v(0)=5.04$. The values of $v(t)$ for $t = 0, 1, \ldots, 3$ indicate that, as shown in Figure C.1, a consumer with a valuation realization between $v(0)$ and 10 makes a purchase in each state, even when she does not have a credit on hand; a consumer with a valuation realization between $v(3)$ and $v(0)$ makes a purchase whenever she has a credit; a consumer with a valuation realization between $v(2)$ and $v(3)$ makes a purchase only when the credit will expire within 2 periods; and a consumer with a valuation realization between $v(1)$ and $v(2)$ makes a purchase only when the credit will expire within 1 period. Observe that as the credit is closer to expiration, the consumer is more likely to make a purchase. Moreover, the consumer with no credit makes a purchase only when her valuation realization is above $v(0)$, implying that the consumer without a credit is less likely to make a purchase.

Similar to the base model where the credit earned by the high-valuation consumers sustains the low-valuation consumers who never make a purchase without a credit, a consumer with a valuation realization between $v(0)$ and 10 makes a purchase without a credit, and the earned credit sustains the consumer with a valuation realization below $v(0)$ when she only makes a purchase with a credit. Furthermore, a consumer with a valuation realization between $v(0)$ and 10 pays a full price p^* in state 0 and a price $p^* - c^*$ in other states. A consumer with a valuation realization between $v(1)$ and $v(0)$ pays a price $p^* - c^*$ for each purchase but with different purchase probabilities (a consumer with a valuation realization between $v(1)$ and $v(2)$ has the lowest purchase probability because she makes a purchase only in state 1), leading to distinct effective prices or purchase probabilities for consumers with different valuation realizations. Perhaps unsurprisingly, the intra-consumer discrimination effect persists in such a general setting. Moreover, it can be shown that the firm's profit under the credit refund policy is 0.323, higher than 0.319, the profit under the cash refund policy. To summarize, under a general distribution of consumer valuation, both the demand induction and intra-consumer discrimination effects still exist.

Appendix D: The Effect of T

This section explores how the firm's optimal profit and credit refund policy change with respect to the exogenous expiration term T. Figure D.1 uses the same parameter values as Table 1.

Figure D.1(a) shows the profit of the optimal credit refund policy under exogenous expiration terms between 1 and 20. As a baseline, we also plot the profit of the optimal cash refund policy, which will be discussed in Section 6. Moreover, there exists an intermediate expiration term that maximizes the firm's profit. This observation can be explained as follows. On the one hand, if the expiration term is too long, then according to Theorem 1, a low-valuation consumer would delay her purchase until her credit is close to expiration, which decreases the consumer's purchase probability. On the other hand, if the expiration term is too short, the consumer faces a rather

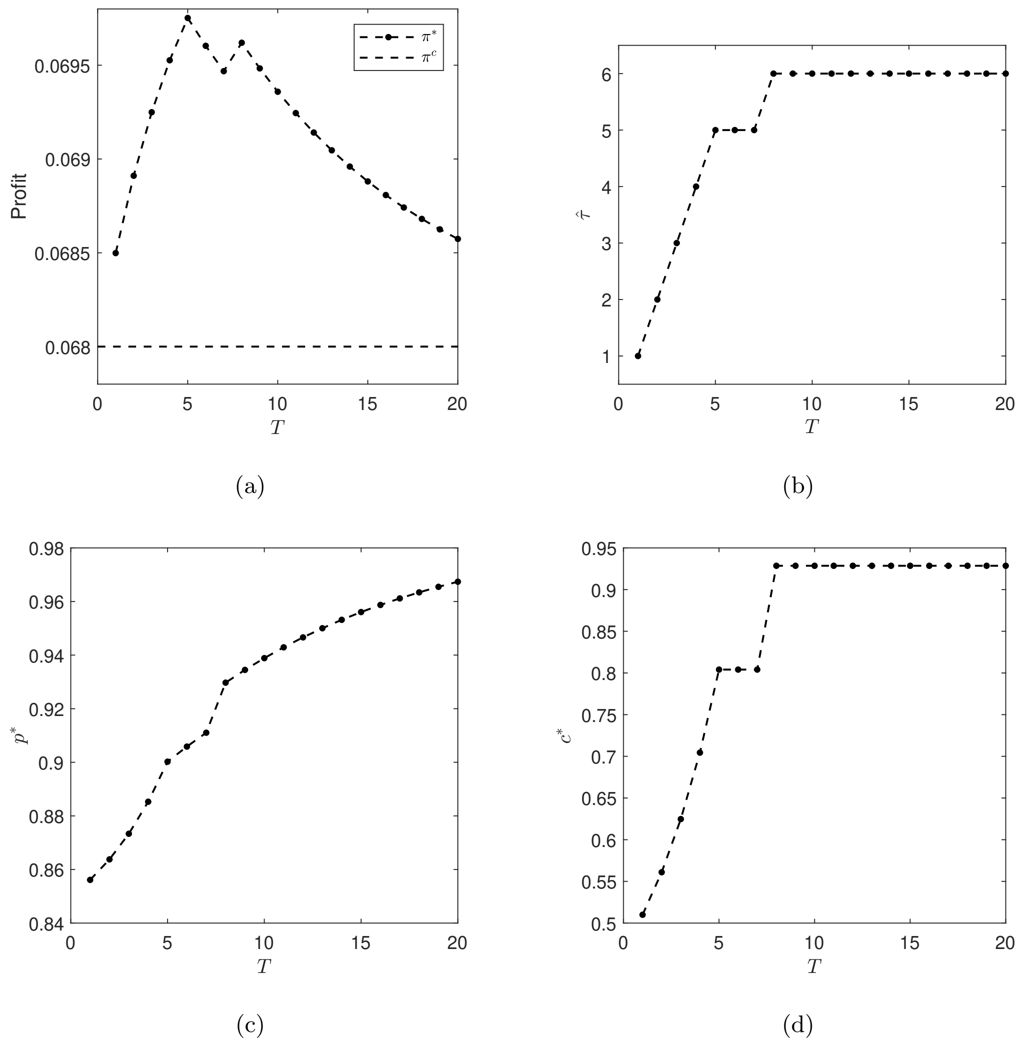


Figure D.1 The firm's optimal decisions vs T

high risk of credit expiration, which would make the credit less valuable to the consumer. Hence, an intermediate expiration term balancing the two effects is optimal.

Figure D.1(b) shows the threshold below which a low-valuation consumer is induced to make a purchase under each exogenous expiration term. For example, when $T = 7$ (8), a low-valuation consumer makes a purchase only when the credit is within 5 (6) periods from expiration, while for all $T \geq 10$, the value of the threshold $\hat{\tau}$ does not change. Notice that the threshold remains relatively small, even under a long expiration term. This is because low-valuation consumers make a purchase only when they face a high risk of credit expiration. Figure D.1(c) shows that the optimal price increases in the expiration term because a longer expiration term increases the value of a credit refund, and allows the firm to charge a higher price. A higher price is often paired with a higher credit, implying that the optimal credit increases in the expiration term as well, which is confirmed in Figure D.1(d).

Appendix E: Deterministic Consumer Valuation

In the main model, we assume that the consumer's valuation is stochastic, i.e., there is a probability α of taking a high valuation, and $1-\alpha$ of taking a low valuation. What if the consumer's valuation is deterministic? Will the credit refund policy still outperform the cash refund policy? To explore this question, we assume that each consumer has a fixed valuation for the product: a fraction α of the consumers takes a high valuation v_H, whereas the rest takes a low valuation v_L. As before, we first analyze the consumers' decision problem and then derive the firm's operational decisions.

The consumers' decision problem can be formulated as an infinite-horizon average reward dynamic program (see, e.g., Puterman 1994). For a generic consumer with a valuation $v \in \{v_H, v_L\}$, the optimality equations are given by

$$\rho^* + J(t) = \lambda \max\{\gamma J(T) + (1-\gamma)(J(0)+v) - p + c, J(t-1)\} + (1-\lambda)J(t-1), \quad \forall t = 1, \ldots, T, \tag{E.1}$$

$$\rho^* + J(0) = \lambda \max\{\gamma J(T) + (1-\gamma)(J(0)+v) - p, J(0)\} + (1-\lambda)J(0). \qquad t = 0. \tag{E.2}$$

Lemma E.1. *Suppose $\gamma[1-(1-\lambda)^T]c + (1-\gamma)v - p \geq 0$, then it is optimal for the consumer to make a purchase whenever she is in the market, and a solution to the optimality equations* (E.1)-(E.2) *is given by*

$$J(t) = [1-(1-\lambda)^t]c, \qquad \forall t = 0, 1, \ldots, T, \tag{E.3}$$

$$\rho^* = \lambda\big\{\gamma[1-(1-\lambda)^T]c + (1-\gamma)v - p\big\}. \tag{E.4}$$

Note that $[1-(1-\lambda)^T]c$ is the expected utility of an earned credit upon a product return, and v is the utility the consumer collects if she makes a purchase and does not return the product. Hence, $\gamma[1-(1-\lambda)^T]c + (1-\gamma)v$ is the expected utility of a purchase. Lemma E.1 indicates that a consumer always makes a purchase when she arrives to the market if the expected utility of a purchase is greater than the price. Therefore, the state transition, balance equations, and stationary probabilities are exactly the same as those in Case I in Appendix B.1.1.

Because $v \in \{v_H, v_L\}$, the optimal price

$$p^* \in \Big\{\gamma[1-(1-\lambda)^T]c + (1-\gamma)v_H, \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_L\Big\}.$$

If $p^* = \gamma[1-(1-\lambda)^T]c + (1-\gamma)v_H$, then only high-valuation consumers make a purchase. Note that each consumer pays a full price in state 0 and a price minus the credit in other states, so the corresponding revenue is

$$\pi = \lambda\alpha\{q_0 p^* + (1-q_0)(p^*-c)\} = \lambda(1-\gamma)\alpha v_H.$$

Otherwise, all consumers make a purchase, and the corresponding revenue is[14]

$$\pi = \lambda\{q_0 p^* + (1-q_0)(p^* - c)\} = \lambda(1-\gamma)v_L.$$

Therefore, the optimal revenue takes $\max\{\lambda(1-\gamma)v_L, \lambda(1-\gamma)\alpha v_H\}$. Clearly, in this deterministic setting, a credit refund policy cannot improve the firm's profit.

PROPOSITION E.1. *If the consumer's valuation is deterministic, then a credit refund policy, compared to a cash refund policy, cannot improve the firm's profit.*

E.1 Proofs

Before proving Lemma E.1, we first introduce a lemma.

LEMMA E.2. *If* $\gamma J(T) + (1-\gamma)[J(0)+v] - p \geq J(0)$, *then*

$$\gamma J(T) + (1-\gamma)[J(0)+v] - p + c \geq J(t-1)$$

for any $1 \leq t \leq T$.

Proof. Due to the monotonicity of $J(\cdot)$, it suffices to show

$$\gamma J(T) + (1-\gamma)[J(0)+v] - p + c \geq J(T-1).$$

According to (E.2),

$$\rho^* + J(0) = \lambda\big\{\gamma J(T) + (1-\gamma)[J(0)+v] - p\big\} + (1-\lambda)J(0) \geq J(0),$$

where the last inequality holds due to the supposition. Hence, we must have

$$\rho^* \geq 0. \tag{E.5}$$

Suppose for a contradiction that

$$\gamma J(T) + (1-\gamma)[J(0)+v] - p + c < J(T-1),$$

then according to (E.1),

$$\rho^* + J(T) = \lambda J(T-1) + (1-\lambda)J(T-1) = J(T-1),$$

implying $\rho^* < 0$ because $J(T) > J(T-1)$, which contradicts (E.5). This completes the proof. ■

[14] In Appendix B.1.1, we show that $q_0 = 1 - \gamma[1-(1-\lambda)^T]$.

Proof of Lemma E.1

According to Lemma E.2, if $\gamma J(T) + (1-\gamma)[J(0)+v] - p \geq J(0)$, then

$$\gamma J(T) + (1-\gamma)[J(0)+v] - p + c \geq J(t-1)$$

for any $1 \leq t \leq T$. That is, if the consumer makes a purchase in state 0, then she always makes a purchase in other states. Hence, the dynamic programming equations (E.1)-(E.2) can be rewritten as follows,

$$\begin{aligned} \rho^* + J(t) &= \lambda\big\{\gamma J(T) + (1-\gamma)[J(0)+v] - p + c\big\} + (1-\lambda)J(t-1), && \forall t = 1,\dots,T, \\ \rho^* + J(0) &= \lambda\big\{\gamma J(T) + (1-\gamma)[J(0)+v] - p\big\} + (1-\lambda)J(0), && t = 0. \end{aligned}$$

Solving the above set of equations gives

$$\begin{aligned} J(t) &= [1-(1-\lambda)^t]c, \\ \rho^* &= \lambda\big\{\gamma[1-(1-\lambda)^T]c + (1-\gamma)v - p\big\}. \end{aligned}$$

Putting $J(0)$ and $J(T)$ into the initial condition $\gamma J(T) + (1-\gamma)[J(0)+v] - p \geq J(0)$ yields

$$\gamma[1-(1-\lambda)^T]c + (1-\gamma)v - p \geq 0.$$

This completes the proof. ■

Appendix F: Breakage Rate

This section takes a closer look at a managerially important quantity, breakage rate, which receives considerable attention from practitioners. The breakage rate measures the proportion of the earned credit lost due to expiration. We let $B(\lambda,\gamma,\alpha,T^*)$ denote the breakage rate under an optimal credit refund policy with an endogenous expiration term.

To better understand the breakage rate[15], consider its complement, the redemption rate denoted by $E(\lambda,\gamma,\alpha,T^*)$, which is the ratio between the credit redeemed by the consumer and the credit issued by the firm. Suppose the optimal expiration term T^* takes T_i or T_i+1 for some $i \geq 0$. At optimality, a high-valuation consumer always makes a purchase, while a low-valuation consumer makes a purchase only in states $\{1,2,\dots,T^*-i\}$. The firm issues a credit when the consumer makes a purchase and then returns the product. Therefore, the total credit issued by the firm is

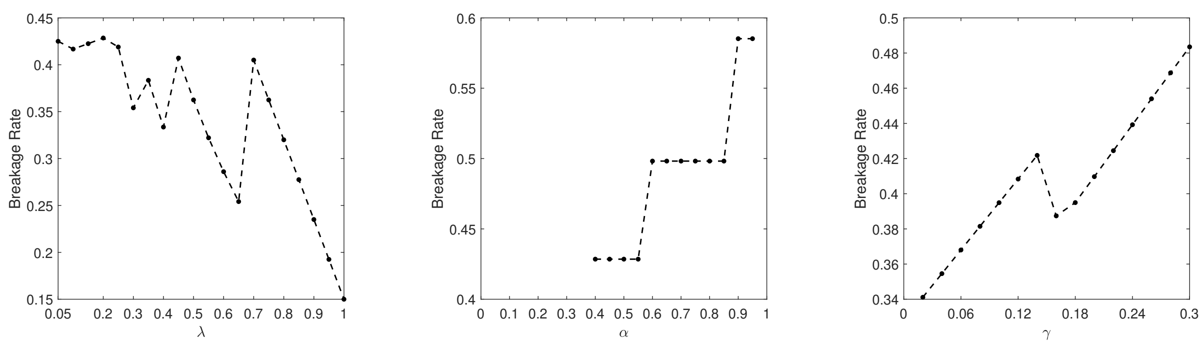
[15] We only discuss the breakage rate for Cases II and III. In Case I, the consumer makes a purchase in each state, regardless of her valuation realization, and thus the breakage rate is $(1-\lambda)^T$. However, the optimal expiration term can take any positive integers, so it is not meaningful to discuss the breakage rate for this case. Case IV is never optimal.

$\lambda\gamma\Big(\alpha q_0 + \sum_{j=1}^{T^*-i} q_j + \alpha \sum_{j=T^*-i+1}^{T^*} q_j\Big)$. The consumer utilizes a credit when she makes a purchase with a credit on hand and does not return the product, so the total credit used by the consumer is $\lambda(1-\gamma)\Big(\sum_{j=1}^{T^*-i} q_j + \alpha \sum_{j=T^*-i+1}^{T^*} q_j\Big)$. It follows that

$$E(\lambda,\gamma,\alpha,T^*) = \frac{\lambda(1-\gamma)\Big(\sum_{j=1}^{T^*-i} q_j + \alpha \sum_{j=T^*-i+1}^{T^*} q_j\Big)}{\lambda\gamma\Big(\alpha q_0 + \sum_{j=1}^{T^*-i} q_j + \alpha \sum_{j=T^*-i+1}^{T^*} q_j\Big)}$$
$$= \frac{(1-\gamma)[1-(1-\alpha\lambda)^i(1-\lambda)^{T^*-i}]}{1-(1-\gamma)(1-\alpha\lambda)^i(1-\lambda)^{T^*}+(1-\gamma)(1-\alpha\lambda)^i(1-\lambda)^{T^*-i}}.$$

Hence,

$$B(\lambda,\gamma,\alpha,T^*) = 1 - \frac{(1-\gamma)[1-(1-\alpha\lambda)^i(1-\lambda)^{T^*-i}]}{1-(1-\gamma)(1-\alpha\lambda)^i(1-\lambda)^{T^*}+(1-\gamma)(1-\alpha\lambda)^i(1-\lambda)^{T^*-i}}.$$



(a) (b) (c)

Figure F.1 Breakage Rate

Figure F.1 uses the same parameter values as in Figure G.1 and depicts how the breakage rate changes with λ, α, and γ, respectively, under an endogenous expiration term. One may expect that as the arrival rate λ increases, it is less likely for the credit to expire, so the breakage rate should decrease. However, Figure F.1(a) shows that the breakage rate is not monotonic with respect to λ. The reason is that as λ increases, the optimal expiration term decreases (as shown in Figure G.1(a)), which increases the chance of credit expiration. These two counteracting effects coexist, and it is not clear which effect dominates as the arrival rate varies.

One may also expect that as the consumer's willingness to pay increases, her purchase probability also increases; thus, the breakage rate should decrease. However, Figure F.1(b) shows that the breakage rate may increase in α. The reason is that, although a consumer's willingness to pay increases, the number of credit issued by the firm also increases. Moreover, as shown in Figure G.1(b), the optimal expiration term decreases in α, leading to an increase in the chance of credit expiration. As discussed in Figure G.1(b), the optimal expiration term is not fixed when α is small and $\alpha = 1$, so we do not discuss the breakage rate for these α values.

Figure F.1(c) shows that the breakage rate is not monotonic with respect to γ. On one hand, as the return rate increases, the probability of utilizing the credit decreases, leading to an increase in the breakage rate. On the other hand, the optimal expiration term may increase in the return rate, resulting in a declining breakage rate. Again, these two counteracting effects coexist, and thus the breakage rate is not monotonic in γ. As discussed in Figure G.1(c), the optimal expiration term is not fixed when $\gamma = 0$, so we do not discuss the breakage rate when $\gamma = 0$.

Appendix G: Sensitivity Analysis

For the firm's problem with endogenous expiration term, we are interested in how the optimal profit and expiration term change with the parameters λ, α, and γ. Our numerical investigation shows that the optimal expiration term often takes the value T_0. Therefore, we first discuss how T_0 changes with the parameters.

PROPOSITION G.1. *The value of T_0 decreases in λ and α, and increases in γ.*

Proof of Proposition G.1

Recall

$$T_0 = \max\{T \in \mathbb{N} : A(T) \leq B(T,T)\},$$

where

$$A(T) = (1-\gamma)(v_H - v_L)\Big\{\frac{\alpha}{(1-\lambda)^{T-1}} + 1 - \alpha\Big\},$$

$$B(T,T) = \frac{(1-\gamma)v_H - (1-\gamma)(v_H - v_L)\gamma(1-\alpha)[1-(1-\lambda)^T]}{1-\gamma[1-(1-\lambda)^T]}.$$

One can check

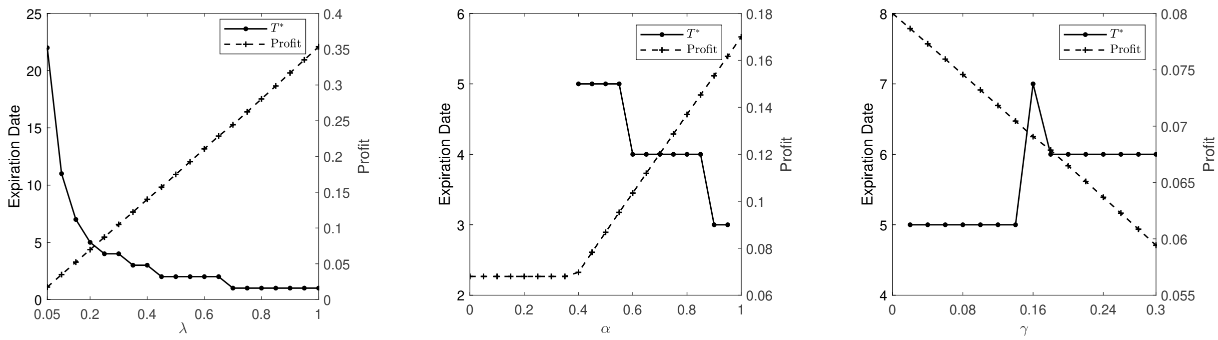
$$A(T) - B(T,T) = \frac{(1-\gamma)}{1-\gamma[1-(1-\lambda)^T]}\Big\{\alpha(1-\gamma)[1-(1-\lambda)^{T-1}](v_H - v_L) - (1-\lambda)^{T-1}v_L\Big\}.$$

Let

$$L = \alpha(1-\gamma)[1-(1-\lambda)^{T-1}](v_H - v_L) - (1-\lambda)^{T-1}v_L.$$

Note that L increases in T and T_0 is the maximum integer that keeps L negative, so in order to show T_0 decreases in λ and α and increases in γ, it suffices to show L increases in λ and α and decreases in γ. One can verify that L indeed increases in λ and α and decreases in γ. This completes the proof. ■

Proposition G.1 verifies that T_0 decreases in λ and α and increases in γ. Therefore, the optimal expiration term may decrease as the arrival rate increases, consumers' willingness to pay increases, or the return rate decreases. Intuitively, as the arrival rate increases, it is less likely for an earned credit to expire, and the firm should decrease the expiration term. A similar argument can be applied to consumers' willingness to pay and the return rate.



(a) (b) (c)

Figure G.1 The sensitivity of profit and expiration date with respect to α, γ, and λ.

We illustrate the sensitivity results using the following numerical example. Figure G.1(a) sets $(\alpha,\gamma,v_H,v_L) = (0.4, 0.15, 1, 0.4)$ and varies λ from 0.05 to 1, with a step size of 0.05. Unsurprisingly, the profit increases in λ. As the arrival rate increases, consumers' purchase probabilities increase, leading to an increase in the profit. The optimal expiration term T^* decreases in λ. We numerically verify that, in this particular example, the optimal expiration term under each λ is T_0, which decreases in λ according to the earlier discussion.

Figure G.1(b) sets $(\lambda,\gamma,v_H,v_L) = (0.2, 0.15, 1, 0.4)$ and varies α from 0 to 1, with a step size of 0.05. Recall that α is the probability that a consumer's valuation is v_H. As α increases, consumers' valuation increases, and thus, the profit increases. Moreover, in this particular example, when $\alpha \le 0.35$, Case I is optimal and the optimal expiration term can take any positive integers (see Proposition B.1 in Appendix B.1.1). Hence, we do not plot T^* in this range. When $0.4 \le \alpha \le 0.95$, Case II is optimal, and the optimal expiration term under each α is T_0. Thus, T^* decreases in α in this range. When $\alpha = 1$, all consumers are high-valuation consumers. The optimal profit is $\lambda(1-\gamma)v_H$ and the optimal expiration term can take any positive integers (see the analysis in Appendix E).

Figure G.1(c) sets $(\lambda,\alpha,v_H,v_L) = (0.2, 0.4, 1, 0.4)$ and varies γ from 0 to 0.3, with a step size of 0.02. The upper bound of 0.3 is informed by empirical evidence indicating that the product return rate is typically less than 30%. Note that the firm earns a lower profit if a consumer returns the product more frequently, so the profit decreases as the return rate increases. Surprisingly, T^* is not monotone in γ. In particular, T^* takes the value of T_1 when $\gamma = 0.16$, but takes the value of T_0 for all other values of γ. When $\gamma = 0$, consumers never return the product. According to Theorem 1, the optimal profit is $\max\{\lambda\alpha v_H, \lambda v_L\}$, and the optimal expiration term can take any positive integer no more than T_0. Hence, we do not plot T^* when $\gamma = 0$.