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On the Boltzmann equation with strong kinetic singularity and its grazing limit from a new perspective

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Abstract

For the inverse power law potential $U(r) = r^{-p}$, the Boltzmann kernel has the asymptotic behavior $B(v - v_*, \sigma) \sim \theta^{-2-2s}|v - v_*|^\gamma$ as the deviation angle tends to 0. Global well-posedness of the Boltzmann equation with such singular kernels has been built in the parameter range $\gamma > -3, 0 < s < 1$ independently by Gressman and Strain (J Am Math Soc 24:771–847, 2011), Alexandre et al. (J Funct Anal 262:915–1010, 2012), triggering many other theoretical developments thereafter. In this work, we consider stronger kinetic singularity and extend the global well-posedness theory to the range $\gamma > -2s - 3, 0 < s < 1$. This range is optimal by recalling that the dominant part of the Boltzmann operator behaves like the factional Laplace operator $(-\Delta)^s$ which allows a singularity with exponent $-2s - 3$ in 3-dimensional space. Based on the global well-posedness result, we prove the grazing limit of the Boltzmann equation to the Landau equation as $s \to 1^-$ from a new perspective for any $\gamma > -5$ that includes the Coulomb potential $\gamma = -3$. As a byproduct, the Landau equation is globally well-posed for any $\gamma > -5$.

Contents

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1 Introduction

The Boltzmann and Landau equations are the two most classical kinetic equations. Regarding to the Cauchy problem, there has been extensive work in different frameworks, e.g. [4, 5, 8, 9, 13, 14, 17, 19–23, 26, 31–38]. In fact, the Landau equation was derived by Landau in 1936 from the Boltzmann equation with cutoff Rutherford cross section. Mathematical justification of the grazing collision limit has proved to be successful since 1990s by adding a cutoff on the deviation angle together with suitable scaling parameter to the Boltzmann cross-section, cf. [6, 7, 10–12, 16, 18, 24, 25, 27].

In this work, we first extend the global well-posedness results [4, 19] on the Boltzmann equation with non-cutoff kernels $B(v-v_*,\sigma)\sim\theta^{-2-2s}|v-v_*|^\gamma$ to the wider parameter range $\gamma>-2s-3$(*allowing more kinetic singularity*). We then justify the grazing limit to the Landau equation as $s\to1^-$ for any $\gamma>-5$ only with a simple scaling factor $1-s$(*without introducing angular cutoff*).

1.1 A natural scaling

Consider the Cauchy problem of the Landau equation

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q_L^\gamma(F, F), \quad t > 0, x \in \mathbb{T}^3, v \in \mathbb{R}^3, \\ F|_{t=0} = F_0. \end{cases} \tag{1.1}$$

Here the Landau operator $Q_L^\gamma(g, h)$ is defined by

$$Q_L^\gamma(g, h)(v) := \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} a^\gamma(v - v_*)[g(v_*)\nabla_v h(v) - \nabla_{v_*} g(v_*) h(v)] \mathrm{d}v_* \right\}, \tag{1.2}$$

where the symmetric matrix a^γ is given by

$$a^\gamma(z) := \Lambda |z|^{\gamma+2} \Pi(z), \quad \Pi(z) := \left(I_3 - \frac{z \otimes z}{|z|^2} \right). \tag{1.3}$$

Here, I_3 is the 3×3 identity matrix and Λ is a constant.

We will show that the solution F_L^γ of (1.1) can be derived from that of the Boltzmann equation with a natural scaling. Let $F_B^{s,\gamma}$ be the solution to the Boltzmann equation with a scaled initial datum

$$\partial_t F + v \cdot \nabla_x F = Q_B^{s,\gamma}(F, F), \quad F|_{t=0} = (1 - s) F_0, \tag{1.4}$$

where $Q_B^{s,\gamma}$ is the Boltzmann operator defined by

$$Q_B^{s,\gamma}(g, h)(v) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B^{s,\gamma}(v - v_*, \sigma) \left(g'_* h' - g_* h \right) \mathrm{d}\sigma \mathrm{d}v_*. \tag{1.5}$$

Here, $B^{s,\gamma}$ is the angular non-cutoff kernel derived from inverse power law potentials, given by

$$B^{s,\gamma}(v - v_*, \sigma) = \left(\sin \frac{\theta}{2} \right)^{-2-2s} |v - v_*|^\gamma, \tag{1.6}$$

where θ is the deviation angle and $\cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma$. The main result in this article is to rigorously prove for $\gamma > -5$,

$$F_B^{s,\gamma}/(1 - s) \to F_L^\gamma, \quad s \to 1^-. \tag{1.7}$$

It is commonly believed in the kinetic theory that the Boltzmann equation tends to the Landau equation as the singularity parameter s tends 1. Our result (1.7) mathematically validates this belief for any $\gamma > -5$.

Recall that (1.6) is derived from the inverse power law potential $U(r) = r^{-p}$ and the parameters are determined through $s = 1/p, \gamma = 1 - 4s$. The Coulomb potential corresponds to $p = s = 1, \gamma = -3$. To concentrate on the effect $s \to 1^-$, in the article we will regard γ as a fixed constant in the range $\gamma > -5$ for the Landau equation and $\gamma > -3 - 2s$ for the Boltzmann equation. In view of Coulomb potential as a special case of the inverse power law, the result also holds for the case when $\gamma = 1 - 4s$ with the limit $s \to 1^-$. This is because (1.7) holds for any $\gamma > -5$ and it is uniform with respect to γ around -3.

Introducing an angular cutoff $\theta \geq \theta_{\min} > 0$ in the Rutherford cross section from the Coulomb potential $s = 1, \gamma = -3$, Landau in 1936 derived the Landau operator (1.2) from the Boltzmann operator (1.5) by using Taylor expansion and keeping the terms up to the second order. On one hand, the cutoff gives rise to a finite momentum transfer so that the Boltzmann operator is well defined. On the other hand, the cutoff generates a physical constant $\ln(1/\theta_{\min})$ (known as Coulomb logarithm) in the Landau operator. The constant is closely related to the screen effect in an electrically neutral plasma interacting through the Coulomb force among microscopic charged particles.

The Landau equation turns out to be an effective equation and has been widely used as a fundamental model in plasma physics. But rigorous mathematical analysis on the limit appeared much later until the 1990s. The first rigorous result was obtained by Desvillettes. Roughly speaking, Desvillettes in [12] introduced a cutoff and scaling procedure such that grazing collisions dominate. He then established the limit in the operator level with error analysis and also for the linearized equation. The result in [12] can be applied to the kernel (1.6) with $\gamma > -3, 0 < s < 1$ for the inverse power law potentials. Precisely, when one applies this argument to the kernel (1.6), the cutoff and scaling procedure yield

$$B^{\epsilon,s,\gamma}(v - v_*, \sigma) = \epsilon^{2s-2} 1_{\theta \leq \epsilon} \left(\sin\frac{\theta}{2}\right)^{-2-2s} |v - v_*|^{\gamma}. \tag{1.8}$$

The angular cutoff $\theta \leq \epsilon$ keeps only grazing collisions in the limit, and the scaling factor ϵ^{2s-2} increases the effect of such collisions to have a finite mean momentum transfer. The result in [12] shows for any fixed $\gamma > -3, 0 < s < 1$ that

$$Q_B^{\epsilon,s,\gamma} = Q_L^{\gamma} + O(\epsilon). \tag{1.9}$$

Here $Q_B^{\epsilon,s,\gamma}$ is the Boltzmann operator associated to the kernel $B^{\epsilon,s,\gamma}$.

As for Landau's original derivation of the Landau operator from the Boltzmann operator with cutoff $\theta \geq \theta_{\min} > 0$ for the Coulomb potential, Degond and Lucquin-Desreux [11] first proved the limit in L^1 space with a logarithmic error estimate. Moreover, they interpreted the threshold $\theta_{\min}$ as the "plasma parameter". Precisely, for the scaled kernel

$$B^{\epsilon}(v - v_*, \sigma) = |\ln \epsilon|^{-1} 1_{\theta \geq \epsilon} \left(\sin\frac{\theta}{2}\right)^{-4} |v - v_*|^{-3}, \tag{1.10}$$

the result in [11] implies the following expansion in L^1 space,

$$Q_B^{\epsilon} = Q_L^{-3} + O(|\ln \epsilon|^{-1}). \tag{1.11}$$

Here Q_B^{ϵ} is the Boltzmann operator associated to the kernel B^{ϵ}.

Later on, there have been fruitful results on the grazing limits on the two scaled kernels (1.8) and (1.10) and other more general kernels of the same types. Moreover, the limit was justified from operator level to equation level. In details, the grazing limits have been established in different settings: homogeneous and inhomogeneous cases, weak and classical solutions, cf. [6, 7, 16, 18, 24, 25, 27, 38] and the references therein. See also [10] from the gradient flow perspective. In particular, a general mathematical framework for grazing limit was proposed in [38] which includes both (1.8) and (1.10). We will come back to discuss the work [38] later.

Now let us compare the approach presented in this paper with (1.10) and (1.8). Note that both the kernels (1.10) and (1.8) contain an angular cutoff together with a suitable scaling factor. However, our approach relies only on a simple scaling without introducing angular cutoff. Note that the scalings on initial datum (1.4) and on the solution (1.7) are equivalent to that on the kernel by observing that the Boltzmann operator is quadratic. To be clear, let $\tilde{F}_B^{s,\gamma} = F_B^{s,\gamma}/(1-s)$. Then $\tilde{F}_B^{s,\gamma}$ is the solution to

$$\partial_t F + v \cdot \nabla_x F = \tilde{Q}_B^{s,\gamma}(F, F), \quad F|_{t=0} = F_0. \tag{1.12}$$

Here $\tilde{Q}_B^{s,\gamma}$ is the Boltzmann operator with the kernel defined by

$$\tilde{B}^{s,\gamma}(v - v_*, \sigma) = (1-s) B^{s,\gamma}(v - v_*, \sigma). \tag{1.13}$$

In order to prove the limit (1.7), it is equivalent to show that

$$\tilde{F}_B^{s,\gamma} \rightarrow F_L^{\gamma}, \quad s \rightarrow 1^-. \tag{1.14}$$

The factor $(1-s)$ appears naturally in (1.13) that corresponds to the grazing limit. We remark that the family (1.13) indexed by s also satisfies the general mathematical framework proposed in [38].

For clear presentation, we summarize our method and the previous approaches for the grazing limit in Fig. 1. The cutoff plus scaling method for general potentials ($B^{\epsilon,s,\gamma}$ in (1.8) with $\gamma > -3$ with fixed $0 < s < 1$) is listed in the left column in Fig. 1, where one can replace $B^{\epsilon,s,\gamma}$ by B^{ϵ} (given in (1.10) with $\gamma = -3, s = 1$ for the Coulomb potential). The middle column shows that the Landau equation with initial datum F_0 yields a solution F_L^{γ}. Our approach corresponds to the right column in Fig. 1. That is, firstly the Boltzmann equation with the scaled initial datum $(1-s)F_0$ gives a solution $F_B^{s,\gamma}$. Then the inverse scaling $F_B^{s,\gamma}/(1-s)$ with limit $s \rightarrow 1^-$ gives the solution F_L^{γ} of the Landau equation. Note that the Boltzmann equation is associated to the physical cross-section (1.6) with parameter s, γ. We also remark that in the previous kernels (1.10) and (1.8) for the grazing limit, the parameter s is fixed while the angular cutoff

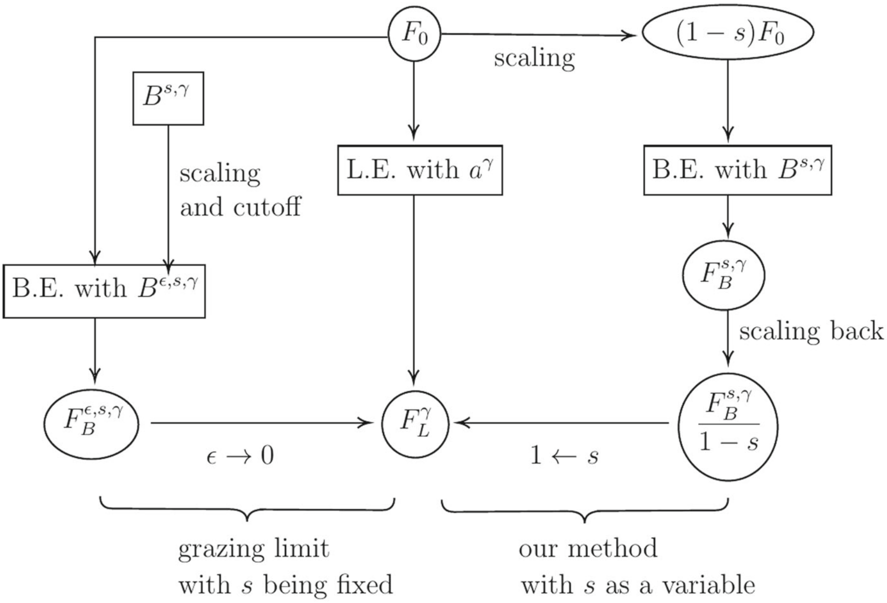

Fig. 1 Grazing limit

parameter ϵ plays a role in the limit. In this article, the limit of s to 1 also captures the grazing limit of the Boltzmann equation to Landau equation.

In this work, we will investigate the above limit in the near-equilibrium framework where unique global classical solution can be constructed. We first recall some relevant results on the well-posedness theories in this framework. For the non-cutoff kernel (1.6), Gressman-Strain in [19] established global well-posedness of the Boltzmann equation in the following parameter range

$$\gamma > -3, \quad 0 < s < 1. \tag{1.15}$$

Independently, Alexandre–Morimoto–Ukai–Xu–Yang [4] proved the same result in the range (1.15) with a constraint $\gamma + 2s > -\frac{3}{2}$ to obtain stronger estimates on the nonlinear operators. In order to consider the grazing limit for Coulomb potential $\gamma = -3$, we need to obtain some uniform estimates for stronger kinetic singularity $\gamma < -3$. There are some discussions about this in several previous works. For instance, weak solutions were constructed for $\gamma \geq -3$ in the classical work [38] by Villani in which a remark says that "one could take $\gamma > -4$". As for Landau equation, Guo [20] firstly proved global well-posedness of for $\gamma \geq -3$ and pointed out that "our theorem is still valid for certain γ even below -3".

Motivated by the above studies, we will consider stronger kinetic singularity and more negative γ in this article. A new contribution to the Boltzmann equation is to extend the parameters range for γ and s to include the triangle $0 < s < 1$, $-3-2s < \gamma \leq -3$, see the region formed by the three red dash lines in Fig. 2. On one hand, we justify the well-posedness of the Boltzmann equation in a region below $\gamma = -3$.

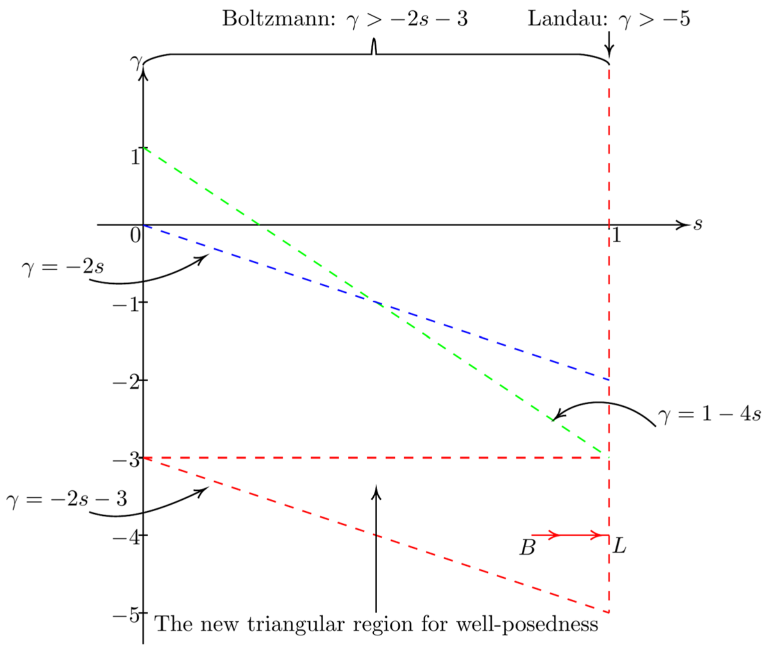

Fig. 2 The parameter domain

On the other hand, more importantly, uniform estimates are obtained to the left of the vertical line $s = 1$ so that the limit $s \to 1^-$ can be considered for any $\gamma > -5$. This then obviously includes the Coulomb potential $\gamma = -3$ and the cases mentioned in the previous literature. Hence, as a byproduct, we show that the Landau equation is well-posed for $-5 < \gamma < -3$.

Now let us explain why $\gamma = -2s - 3$ is a critical value in three-dimensional space. Actually, As shown in the papers [1–4, 19, 23, 26, 30, 32], it is now well known that the Boltzmann operator $Q_B^{s,\gamma}(g, \cdot)$ in (1.5) behaves like the fractional Laplace operator $-C_g(-\Delta)^s$. Recall that $-(-\Delta)^s$ in three dimensions can be defined by a singular integral

$$-(-\Delta)^s f(v) := C_s \lim_{r \to 0^+} \int_{\mathbb{R}^3 \setminus B(v,r)} \frac{f(v) - f(v_*)}{|v - v_*|^{3+2s}} \mathrm{d}v_*. \tag{1.16}$$

This implies that $\gamma > -2s - 3$ is necessary for suitably general function spaces. Moreover, there exists a universal constant $c > 0$ such that

$$\lim_{s \to 1^-} \frac{C_s}{1 - s} = c.$$

This is consistent with the scaling factor $1 - s$ appeared in (1.7) and (1.13). Then as $s \to 1^-$, $-(-\Delta)^s \to \Delta$ which is the main part of the Landau operator (1.2).

Now let us briefly discuss the relation of our approach with the general mathematical framework for grazing limit proposed in [38]. According to [38], a family $\{b_n\}_n$ of angular collisions kernels concentrates on the grazing collisions if

$$\begin{cases} \int_{\mathbb{S}^2} b_n(\cos\theta)\sin^2\frac{\theta}{2}\mathrm{d}\sigma \to_{n\to\infty} \mu_\infty \in (0,+\infty), \\ \forall\theta_0 > 0, \quad \sup_{\theta\geq\theta_0} b_n(\cos\theta) \to_{n\to\infty} 0 \end{cases} \tag{1.17}$$

Note that this assumption is so general that it includes the three cases of kenerls discussed above (1.8), (1.10) and (1.13), with $n = 1/\epsilon$ or $n = 1/(1-s)$. Therefore, one can directly apply [38] and [6] to obtain corresponding results in the spatially homogeneous and inhomogeneous cases respectively in the limit $s \to 1^-$ for weak solutions. However, for the inhomogeneous case the work [6] requires $\gamma \geq -3$ and how to study weak solution for $\gamma < -3$ in the grazing limit remains unsolved. In order to consider $\gamma < -3$, we choose to work in the close-to-equilibrium framework where some regularity of the solutions can be used to absorb the extra singularity in the relative velocity near $|v - v_*| = 0$. To consider larger range of negative γ is not only for a mathematical consistency with the fractional Laplace operator $-(-\Delta)^s \to \Delta$ but also for some physical applications. As pointed out in [5], cross-sections which present non-integrable kinetic singularity but no angular singularity are also used by physicists. Our results show that the index γ of kinetic singularity can be below the critical non-integrable value -3 until $-3-2s$ at least in perturbative framework for global classical solutions.

1.2 Main results

We will state the main results in this subsection. Consider the following Cauchy problem of the Boltzmann equation

$$\begin{cases} \partial_t F + v\cdot\nabla_x F = Q_B^{s,\gamma}(F,F), \quad t>0, x\in\mathbb{T}^3, v\in\mathbb{R}^3, \\ F|_{t=0} = F_0. \end{cases} \tag{1.18}$$

Here $F(t,x,v) \geq 0$ is the density function of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, position $x \in \mathbb{T}^3 := [-\pi,\pi]^3$. The Boltzmann operator is defined as

$$Q_B^{s,\gamma}(g,h)(v) := \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} B^{s,\gamma}(v-v_*,\sigma)\left(g'_*h' - g_*h\right)\mathrm{d}\sigma\mathrm{d}v_*. \tag{1.19}$$

Here, $h = h(v)$, $g_* = g(v_*)$, $h' = h(v')$, $g'_* = g(v'_*)$ where v', v'_* are given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \sigma\in\mathbb{S}^2. \tag{1.20}$$

Recalling (1.6) and (1.13), from now on we take

$$B^{s,\gamma}(v-v_*,\sigma)=(1-s)\left(\sin\frac{\theta}{2}\right)^{-2-2s}1_{0\le\theta\le\pi/2}|v-v_*|^{\gamma}. \tag{1.21}$$

That is, the angular function is

$$b^{s}(\theta)=(1-s)\left(\sin\frac{\theta}{2}\right)^{-2-2s}1_{0\le\theta\le\pi/2}. \tag{1.22}$$

Note that the angle variable is restricted to $0\le\theta\le\pi/2$ by symmetry as in the literature.

Thanks to the factor $1-s$, the mean moment transfer is finite, that is,

$$\int_{\mathbb{S}^2} b^{s}(\theta)\sin^2\frac{\theta}{2}\mathrm{d}\sigma=4\pi\times 2^{s-1}. \tag{1.23}$$

In accordance with (1.23), we take $\Lambda=\pi$ in (1.3). In fact, we will show that

$$Q_B^{s,\gamma}=2^{s-1}Q_L^{\gamma}+O(1-s),$$

and so $Q_B^{s,\gamma}\to Q_L^{\gamma}$ as $s\to 1^-$. See (5.1) for details.

To construct the global-in-time classical solution in the spatially inhomogeneous case, one usually considers the near equilibrium framework. Recall that solutions of (1.1) and (1.18) conserve mass, momentum and energy. We assume F_0 is a small perturbation of the equilibrium μ where $\mu(v):=(2\pi)^{-3/2}e^{-|v|^2/2}$. Let us recall the linearized versions of (1.18) and (1.1). Set $F=\mu+\mu^{1/2}f$, then (1.18) is reduced to

$$\begin{cases}\partial_t f+v\cdot\nabla_x f+\mathcal{L}_B^{s,\gamma}f=\Gamma_B^{s,\gamma}(f,f), \quad t>0, x\in\mathbb{T}^3, v\in\mathbb{R}^3,\\ f|_{t=0}=f_0.\end{cases} \tag{1.24}$$

Here the linearized Boltzmann operator $\mathcal{L}_B^{s,\gamma}$ and the nonlinear term $\Gamma_B^{s,\gamma}$ are defined by

$$\begin{aligned}\Gamma_B^{s,\gamma}(g,h)&:=\mu^{-1/2}Q_B^{s,\gamma}(\mu^{1/2}g,\mu^{1/2}h),\\ \mathcal{L}_B^{s,\gamma}g&:=-\Gamma_B^{s,\gamma}(\mu^{1/2},g)-\Gamma_B^{s,\gamma}(g,\mu^{1/2}).\end{aligned} \tag{1.25}$$

With the same decomposition $F=\mu+\mu^{1/2}f$, (1.1) becomes

$$\begin{cases}\partial_t f+v\cdot\nabla_x f+\mathcal{L}_L^{\gamma}f=\Gamma_L^{\gamma}(f,f), \quad t>0, x\in\mathbb{T}^3, v\in\mathbb{R}^3,\\ f|_{t=0}=f_0.\end{cases} \tag{1.26}$$

The linearized Landau operator $\mathcal{L}_L^{\gamma}$ and the nonlinear term Γ_L^{γ} are defined by

$$\Gamma_L^{\gamma}(g,h):=\mu^{-1/2}Q_L^{\gamma}(\mu^{1/2}g,\mu^{1/2}h),$$

$$\mathcal{L}_L^{\gamma} g := -\Gamma_L^{\gamma}(\mu^{1/2}, g) - \Gamma_L^{\gamma}(g, \mu^{1/2}). \tag{1.27}$$

Note that the conservation laws imply that for all $t \geq 0$,

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} F(t,x,v)\phi(v)\mathrm{d}x\mathrm{d}v = \int_{\mathbb{T}^3\times\mathbb{R}^3} F(0,x,v)\phi(v)\mathrm{d}x\mathrm{d}v,$$
$$\phi(v) = 1, v_j, |v|^2, \quad j = 1,2,3. \tag{1.28}$$

Without loss of generality, we assume F_0 and the equilibrium μ have the same conserved quantities so that the initial datum f_0 satisfies

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} \sqrt{\mu} f_0 \phi \mathrm{d}x\mathrm{d}v = 0, \quad \phi(v) = 1, v_j, |v|^2, \quad j = 1,2,3. \tag{1.29}$$

Then for all $t \geq 0$,

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} \sqrt{\mu} f(t) \phi \mathrm{d}x\mathrm{d}v = 0, \quad \phi(v) = 1, v_j, |v|^2, \quad j = 1,2,3. \tag{1.30}$$

Physical solutions are non-negative and so we consider $F_0 = \mu + \mu^{\frac{1}{2}} f_0 \geq 0$.

The case for hard potential with $\gamma + 2s \geq 0$ is relatively easy because the linearized Boltzmann operator has a spectrum gap. This corresponds to the region above the blue line in Fig. 2. Therefore, we only consider the soft potentials in this article when

$$0 < s < 1, \quad -3 < \gamma + 2s \leq 0. \tag{1.31}$$

Note that this corresponds to the region inside the parallelogram in Fig. 2. To overcome the lack of spectrum gap, the following weighted Sobolev space was introduced by Guo as energy space for global well-posedness

$$\mathcal{E}_{N,l}^{s,\gamma}(f) := \sum_{j=0}^{N} \|f\|^2_{H_x^{N-j}\dot{H}^j_{l+j(\gamma+2s)}}. \tag{1.32}$$

If $s = 1$, we sometimes write $\mathcal{E}_{N,l}^{\gamma}(f) = \mathcal{E}_{N,l}^{1,\gamma}(f)$ which is the energy space for the Landau equation, cf. subsect. 1.3 for notations on function spaces.

There are three main results given in the following theorem. The first one is the global well-posedness of the Boltzmann equation (1.24) in the parameter range (1.31). The second one is the global well-posedness of the Landau equation (1.26) for $-5 < \gamma < -2$. The last one is about the grazing limit of the Boltzmann equation (1.24) to the Landau equation (1.26) by proving a global-in-time asymptotic formula for the limit $s \to 1^-$.

Theorem 1.1 [Well-posedness of the Boltzmann equation] *Let* $0 < s < 1, -3 < \gamma + 2s \leq 0$. *Let* $N \geq 4, l \geq -N(\gamma + 2s)$. *There is a constant* $\delta_{s,\gamma,N,l} > 0$ *such that,* *if*

$$\mathcal{E}^{s,\gamma}_{N,l}(f_0) \leq \delta_{s,\gamma,N,l}, \tag{1.33}$$

then (1.24) *admits a unique global solution* $f^{s,\gamma}$ *satisfying* $\mu + \mu^{\frac{1}{2}} f^{s,\gamma} \geq 0$ *and*

$$\sup_{t\geq 0} \mathcal{E}^{s,\gamma}_{N,l}(f^{s,\gamma}(t)) \leq Z_{s,\gamma,N,l}\mathcal{E}^{s,\gamma}_{N,l}(f_0), \tag{1.34}$$

for some constant $Z_{s,\gamma,N,l}$. *Here* $\delta_{s,\gamma,N,l} = \frac{1}{2}\eta^2_{s,\gamma,N,l}$ *where* $\eta_{s,\gamma,N,l}$ *is given in Theorem* 4.1. *See* (4.8) *for the definition of* $Z_{s,\gamma,N,l}$. *For any fixed* N, l, *there are two functions* $\delta_{N,l}, Z_{N,l} : (0, 1] \times (0, 3] \to (0, \infty)$ *satisfying*

$$\delta_{s,\gamma,N,l} = \delta_{N,l}(s, \gamma + 2s + 3), \quad Z_{s,\gamma,N,l} = Z_{N,l}(s, \gamma + 2s + 3), \tag{1.35}$$

and

$$\begin{aligned} &\delta_{N,l}(x_1, x_2) \text{ is non-decreasing w.r.t. each argument,} \\ &\quad \text{and vanishes as } x_1 \to 0^+ \text{ or } x_2 \to 0^+; && (1.36) \\ &Z_{N,l}(x_1, x_2) \text{ is non-increasing w.r.t. each argument} \\ &\quad \text{and tends to infinity as } x_1 \to 0^+ \text{ or } x_2 \to 0^+. && (1.37) \end{aligned}$$

[Well-posedness of the Landau equation] *Taking* $s = 1$ *in the above, the global well-posedness of* (1.26) *holds true. We state this result in details for later discussion. Let* $-3 < \gamma+2 < 0$. *Let* $N \geq 4, l \geq -N(\gamma+2)$. *Let* $\delta_{\gamma,N,l} = \delta_{1,\gamma,N,l}$, $Z_{\gamma,N,l} = Z_{1,\gamma,N,l}$. *If*

$$\mathcal{E}^{\gamma}_{N,l}(f_0) \leq \delta_{\gamma,N,l}, \tag{1.38}$$

then (1.26) *admits a unique global solution* f^{γ} *satisfying* $\mu + \mu^{\frac{1}{2}} f^{\gamma} \geq 0$ *and*

$$\sup_{t\geq 0} \mathcal{E}^{\gamma}_{N,l}(f^{\gamma}(t)) \leq Z_{\gamma,N,l}\mathcal{E}^{\gamma}_{N,l}(f_0). \tag{1.39}$$

[Asymptotic formula of the grazing limit] *Fix* $-5 < \gamma < -2$. *Let* $N \geq 4, l \geq -N(\gamma + 2)$. *Let* $s_* := \frac{1}{2}(1 - \frac{\gamma+3}{2})$. *Assume*

$$\mathcal{E}^{\gamma}_{N+3,l+2N-3\gamma+5}(f_0) \leq \delta_{s_*,\gamma,N+3,l+2N-3\gamma+5}. \tag{1.40}$$

Since for any $s_* \leq s \leq 1$,

$$\begin{aligned} \mathcal{E}^{s,\gamma}_{N+3,l+2N-3\gamma+5}(f_0) &\leq \mathcal{E}^{\gamma}_{N+3,l+2N-3\gamma+5}(f_0) \\ &\leq \delta_{s_*,\gamma,N+3,l+2N-3\gamma+5} \leq \delta_{s,\gamma,N+3,l+2N-3\gamma+5}, \end{aligned}$$

then by the above well-posedness result, (1.24) has a unique solution $f^{s,\gamma}$ for $s_* \le s < 1$, and (1.26) has a unique solution f^{γ}. Moreover, the family of solutions $\{f^{s,\gamma}\}_{s_* \le s < 1}$ satisfies

$$\sup_{t\ge 0} \mathcal{E}^{\gamma}_{N,l}(f^{s,\gamma}(t) - f^{\gamma}(t)) \le (1-s)^2 \exp \left(C_{N,l} Z^3_{s_*,\gamma,N,l} C^2_{\gamma} \mathcal{E}^{1,\gamma}_{N+3,l+2N-3\gamma+5}(f_0) \right). \tag{1.41}$$

Here $C_\gamma = (\gamma+5)^{-1}$.

In the following we will give some remarks. First of all, we will keep track of the dependence on the parameters s, γ. This kind of dependence gives the precise condition on the parameters for the global well-posedness theory of the Boltzmann and Landau equation. In particular, we have the explicit relation between δ, Z and s, γ in (1.35). On one hand, the region of parameters for the well-posedness of the Boltzmann equation is non-empty as long as $0 < s < 1, \gamma > -2s - 3$. On the other hand, according to (1.36), the region indicated by the "radius" $\delta_{s,\gamma,N,l} > 0$ may shrink as $s \to 0^+$ or $\gamma \to (-2s-3)^+$.

Recently, the limit from inverse power law to hard sphere model was established in [29] for the homogeneous Boltzmann equation in the context of weak solutions. Notably, the authors start directly from the kernel derived from inverse power law and then study the limit $s \to 0^+$. The kernel considered in [29] is $|v - v_*|^{1-4s} b_s(\theta)$ where the angular function b_s is very complicated and can only be given in an implicit way.

The estimate (1.41) implies that

$$F^{s,\gamma}_B = F^{\gamma}_L + (1-s) F^{s,\gamma}_R, \tag{1.42}$$

where $F^{s,\gamma}_B$ and F^{γ}_L are solutions to (1.18) and (1.1) respectively. Here the error term $F^{s,\gamma}_R$ is uniformly bounded in some function space. The asymptotic formula validates our approach shown in Fig. 1.

A few more remarks are given as follows.

Remark 1.1 For the potential function $U(r) \sim r^{-p}$, the above theorem justifies the natural limit $p \to 1^+$ from the Boltzmann equation with inverse power law to the Landau equation with the Coulomb potential.

Remark 1.2 If s and γ are regarded as two independent parameters, the well-posedness of the Boltzmann equation holds when $\gamma > -3 - 2s$ and $0 < s < 1$. In fact, the angular singularity $s \to 1^-$ is the essential reason of possible ill-posedness for inverse power law potentials.

Remark 1.3 Global well-posedness in some lower regularity function spaces, such as the one introduced in [17], may be obtained for the parameters γ, s in the optimal range $0 < s < 1, \gamma > -3 - 2s$. We may also further consider time decay rates as in [17, 33, 34]. For brevity and to focus on the key points, we will not pursue these analyses in this work.

Remark 1.4 In the previous studies, the condition $\gamma > -3$ is needed due to the following estimate. For any $v \in \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} \mu(v_*)|v - v_*|^\gamma \mathrm{d}v_* \le C_\gamma \langle v\rangle^\gamma, \tag{1.43}$$

where C_γ is a constant that blows up as $\gamma \to (-3)^+$. To understand why $\gamma > -3-2s$ is sufficient for well-posedness, intuitively, we know that the angular singularity in the cross-section leads to a fractional derivative of order $2s$. Thus, we can expect to recover an extra $2s$ in the range for γ below $\gamma = -3$ by sacrificing some regularity of the solution.

1.3 Notations

We list some notations that are used in this work.

Common notations. The length of a multi-index $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$ is denoted by $|\beta| = \beta_1 + \beta_2 + \beta_3$. $a \lesssim b$ means that there is a generic constant C, which may be different in different places, such that $a \le Cb$. We use the notation $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. The bracket $\langle\cdot\rangle$ is defined by $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$. The weight function W_l is defined by $W_l(v) := \langle v\rangle^l$. We denote by $C(\lambda_1, \lambda_2, \cdots, \lambda_n)$ or $C_{\lambda_1,\lambda_2,\cdots,\lambda_n}$ a constant depending on parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$. We use $|\cdot|_{L^2}, |\cdot|_{L^2_x}, \|\cdot\|_{L^2_xL^2}$ to distinguish the L^2-norms in the three spaces $L^2(\mathbb{R}^3)$, $L^2(\mathbb{T}^3)$, $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$, and $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_x$, $(\cdot,\cdot)$ to denote the corresponding inner products. As usual, 1_A is the characteristic function of the set A. If A, B are two operators, then $[A, B] := AB - BA$.

Function spaces. For $\alpha, \beta \in \mathbb{N}^3$, set $\partial^\alpha := \partial^\alpha_x$, $\partial_\beta := \partial^\beta_v$, $\partial^\alpha_\beta := \partial^\alpha_x\partial^\beta_v$ for simplicity. We will use the following function spaces.

- For real number n, l,

$$H^n_l := \left\{ f(v) \big| |f|^2_{H^n_l} := |\langle D\rangle^n W_l f|^2_{L^2} = \int_{\mathbb{R}^3} |(\langle D\rangle^n W_l f)(v)|^2 \mathrm{d}v < +\infty \right\}.$$

  Here $a(D)$ is a pseudo-differential operator with the symbol $a(\xi)$ defined by

$$\big(a(D)f\big)(v) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(v-y)\cdot\xi} a(\xi) f(y)\mathrm{d}y\mathrm{d}\xi.$$

- For $n \in \mathbb{N}, l \in \mathbb{R}$,

$$H^n_l := \left\{ f(v) \big| |f|^2_{H^n_l} := \sum_{|\beta|\le n} |\partial_\beta f|^2_{L^2_l} < \infty \right\},$$

  where $|f|_{L^2_l} := |W_l f|_{L^2}$ is the usual L^2 norm with weight W_l.

- For $n \in \mathbb{N}, l \in \mathbb{R}$,

$$\dot{H}^n_l := \left\{ f(v) \middle| |f|^2_{\dot{H}^n_l} := \sum_{|\beta|=n} |\partial_\beta f|^2_{L^2_l} < \infty \right\}. \tag{1.44}$$

- For $m \in \mathbb{N}$,

$$H^m_x := \left\{ f(x) \middle| |f|^2_{H^m_x} := \sum_{|\alpha|\le m} |\partial^\alpha f|^2_{L^2_x} < \infty \right\}.$$

- For $m, n \in \mathbb{N}, l \in \mathbb{R}$,

$$H^m_x H^n_l := \left\{ f(x,v) \middle| \|f\|^2_{H^m_x H^n_l} := \sum_{|\alpha|\le m, |\beta|\le n} ||\partial^\alpha_\beta f|_{L^2_l}|^2_{L^2_x} < \infty \right\}.$$

We write $\|f\|_{H^m_x L^2_l} := \|f\|_{H^m_x L^2_l}$ if $n = 0$ and $\|f\|_{L^2_x L^2_l} := \|f\|_{H^0_x H^0_l}$ if $m = n = 0$. The space $H^m_x \dot{H}^n_l$ can be defined similarly.

Next, let us recall the dissipation norm of the linearized operators $\mathcal{L}^{s,\gamma}_B$ and $\mathcal{L}^\gamma_L$. For $l \in \mathbb{R}$, set

$$|f|^2_{s,l} := |W_s((-\Delta_{\mathbb{S}^2})^{1/2}) W_l f|^2_{L^2} + |W_s(D) W_l f|^2_{L^2} + |W_s W_l f|^2_{L^2}. \tag{1.45}$$

Here $W_s(D)$ is the pseudo-differential operator with symbol W_s. The operator $W_s((-\Delta_{\mathbb{S}^2})^{1/2})$ is defined as follows. If $v = r\sigma, r \ge 0, \sigma \in \mathbb{S}^2$, then

$$\left(W_s((-\Delta_{\mathbb{S}^2})^{1/2}) f\right)(v) := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (1 + l(l+1))^{\frac{s}{2}} Y^m_l(\sigma) f^m_l(r), \tag{1.46}$$

where $f^m_l(r) = \int_{\mathbb{S}^2} Y^m_l(\sigma) f(r\sigma) \mathrm{d}\sigma$, and $\{Y^m_l\}_{l\ge 0, -l\le m\le l}$ are the real spherical harmonics satisfying $(-\Delta_{\mathbb{S}^2}) Y^m_l = l(l+1) Y^m_l$. Note that the function W_s is the common weight in the three individual norms. The dissipation norm $|\cdot|_{s,\gamma/2}$ characterizes $\mathcal{L}^{s,\gamma}_B$, see Theorem 2.2 and Theorem 3.2 for details. Similarly, the dissipation norm $|\cdot|_{1,\gamma/2}$ characterizes $\mathcal{L}^\gamma_L$.

Sometimes, we also write $|f|_{L^2_{s,l}} = |f|_{s,l}$. For functions defined on $\mathbb{T}^3 \times \mathbb{R}^3$, the space $H^m_x H^n_{s,l}$ with $m, n \in \mathbb{N}$ is defined by

$$H^m_x H^n_{s,l} := \left\{ f(x,v) \middle| \|f\|^2_{H^m_x H^n_{s,l}} := \sum_{|\alpha|\le m, |\beta|\le n} ||\partial^\alpha_\beta f|_{L^2_{s,l}}|^2_{L^2_x} < \infty \right\}.$$

Set $\|f\|_{H^m_x L^2_{s,l}} := \|f\|_{H^m_x H^0_{s,l}}$ if $n = 0$ and $\|f\|_{L^2_x L^2_{s,l}} := \|f\|_{H^0_x H^0_{s,l}}$ if $m = n = 0$. Again, the space $H^m_x \dot{H}^n_{s,l}$ can be defined accordingly.

We sometimes omit the range of some frequently used variables in the integrals. Usually, $\sigma \in \mathbb{S}^2$, $v, v_*, u, \xi \in \mathbb{R}^3$. For example, $\int(\cdots)\mathrm{d}\sigma := \int_{\mathbb{S}^2}(\cdots)\mathrm{d}\sigma$, $\int(\cdots)\mathrm{d}\sigma \mathrm{d}v\mathrm{d}v_* := \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3}(\cdots)\mathrm{d}\sigma\mathrm{d}v\mathrm{d}v_*$. Integration w.r.t. other variables is understood in a similar way. Whenever a new variable appears, we will specify its range.

When there is no confusion, we drop the subscripts B and L in the Boltzmann and Landau operators for brevity.

1.4 Organization of the article

In Sect. 2, we derive upper bound estimates of the linear operator $\mathcal{L}^{s,\gamma}$ and the non-linear term $\Gamma^{s,\gamma}$. We prove coercivity estimate of the linear operator in Sect. 3. Theorem 1.1 is proved in Sect. 4. In the Appendix 5, for completeness, we prove the operator convergence stated in Proposition 4.2.

2 Upper bound estimates

In this section, we will derive the upper bound estimates on the collision operators based on the following operator splitting method.

We divide the Boltzmann operator with respect to the relative velocity into two parts

$$B(v - v_*, \sigma) = B^{s,\gamma}_\eta(v - v_*, \sigma) + B^{s,\gamma,\eta}(v - v_*, \sigma),$$

where

$$\begin{aligned} B^{s,\gamma}_\eta(v - v_*, \sigma) &:= \psi_\eta(|v - v_*|)B(v - v_*, \sigma), \\ B^{s,\gamma,\eta}(v - v_*, \sigma) &:= (1 - \psi_\eta(|v - v_*|))B(v - v_*, \sigma). \end{aligned} \tag{2.1}$$

Here $\psi_\eta(\cdot) = \psi(\cdot/\eta)$ where ψ is a non-increasing smooth function on $\mathbb{R}_+$ given in (2.12) and satisfies

$$\begin{aligned} &\psi = 1 \text{ on } [0, 3/4], \quad \psi \text{ is strictly decreasing on } [3/4, 4/3], \\ &\psi = 0 \text{ on } [4/3, \infty), \quad |\psi'| \le 4. \end{aligned} \tag{2.2}$$

Note that $B^{s,\gamma}_\eta$ is supported in $|v - v_*| \le 4\eta/3$ so that it is singular when $|v - v_*| \to 0$, while $B^{s,\gamma,\eta}$ is supported in $|v - v_*| \ge 3\eta/4$ without any singularity. From now on, We always assume $0 < \eta \le 1$.

Let $Q^{s,\gamma}_\eta$ and $Q^{s,\gamma,\eta}$ be the Boltzmann operators defined with kernel $B^{s,\gamma}_\eta$ and $B^{s,\gamma,\eta}$ respectively. And then let $\Gamma^{s,\gamma}_\eta$ and $\Gamma^{s,\gamma,\eta}$ be the nonlinear terms defined with kernel $B^{s,\gamma}_\eta$ and $B^{s,\gamma,\eta}$ respectively.

Recall that the nonlinear term Γ for a general kernel B is defined by

$$\Gamma(g,h) = \mu^{-1/2}Q(\mu^{1/2}g,\mu^{1/2}h) = \int B(v-v_*,\sigma)\mu_*^{1/2}\big(g'_*h' - g_*h\big)\mathrm{d}\sigma\mathrm{d}v_*$$
$$= \int B(v-v_*,\sigma)\big((\mu^{1/2}g)'_*h' - (\mu^{1/2}g)_*h\big)\mathrm{d}\sigma\mathrm{d}v_*$$
$$+ \int B(v-v_*,\sigma)\big(\mu_*^{1/2} - (\mu^{1/2})'_*\big)g'_*h'\mathrm{d}\sigma\mathrm{d}v_* = Q(\mu^{1/2}g,h) + I(g,h),$$

where for brevity,

$$I(g,h) := \int B(v-v_*,\sigma)\big(\mu_*^{1/2} - (\mu^{1/2})'_*\big)g'_*h'\mathrm{d}\sigma\mathrm{d}v_*. \tag{2.3}$$

Let $I^{s,\gamma}$, $I_\eta^{s,\gamma}$ and $I^{s,\gamma,\eta}$ be the bi-linear operators defined according to (2.3) with kernel $B^{s,\gamma}$, $B_\eta^{s,\gamma}$ and $B^{s,\gamma,\eta}$ respectively. Other operators with such subscripts and superscripts are understood in the same way.

To implement the energy estimates for the nonlinear equations, we need to take derivatives. By binomial expansion,

$$\partial_\beta^\alpha \Gamma^{s,\gamma}(g,h) = \sum_{\beta_0+\beta_1+\beta_2=\beta,\alpha_1+\alpha_2=\alpha} C_\beta^{\beta_0,\beta_1,\beta_2} C_\alpha^{\alpha_1,\alpha_2} \Gamma^{s,\gamma}(\partial_{\beta_1}^{\alpha_1}g, \partial_{\beta_2}^{\alpha_2}h;\beta_0), \tag{2.4}$$

where

$$\Gamma^{s,\gamma}(g,h;\beta)(v) := \int B^{s,\gamma}(v-v_*,\sigma)(\partial_\beta\mu^{1/2})_*(g'_*h' - g_*h)\mathrm{d}\sigma\mathrm{d}v_*. \tag{2.5}$$

Note that

$$\Gamma^{s,\gamma}(g,h;\beta) = Q^{s,\gamma}(g\partial_\beta\mu^{1/2},h) + I^{s,\gamma}(g,h;\beta), \tag{2.6}$$

where

$$I^{s,\gamma}(g,h;\beta) := \int B^{s,\gamma}(v-v_*,\sigma)\big((\partial_\beta\mu^{1/2})_* - (\partial_\beta\mu^{1/2})'_*\big)g'_*h'\mathrm{d}\sigma\mathrm{d}v_*. \tag{2.7}$$

Thus, in general we need to consider $I^{s,\gamma}(g,h;\beta)$. This is again divided into two parts: $I^{s,\gamma,\eta}(g,h;\beta)$ and $I_\eta^{s,\gamma}(g,h;\beta)$.

Recall

$$\mathcal{L}^{s,\gamma}f = -\Gamma^{s,\gamma}(\mu^{\frac{1}{2}},f) - \Gamma^{s,\gamma}(f,\mu^{\frac{1}{2}}).$$

By binomial expansion,

$$\partial_\beta^\alpha\mathcal{L}^{s,\gamma}f = \sum_{\beta_0+\beta_1+\beta_2=\beta} C_\beta^{\beta_0,\beta_1,\beta_2}\mathcal{L}^{s,\gamma}(\partial_{\beta_2}^\alpha f;\beta_0,\beta_1), \tag{2.8}$$

where

$$\mathcal{L}^{s,\gamma}(f;\beta_0,\beta_1) := -\Gamma^{s,\gamma}(\partial_{\beta_1}\mu^{1/2}, f;\beta_0) - \Gamma^{s,\gamma}(f, \partial_{\beta_1}\mu^{1/2};\beta_0). \tag{2.9}$$

We also define

$$\begin{aligned}\mathcal{L}_1^{s,\gamma}(f;\beta_0,\beta_1) &:= -\Gamma^{s,\gamma}(\partial_{\beta_1}\mu^{1/2}, f;\beta_0),\\ \mathcal{L}_2^{s,\gamma}(f;\beta_0,\beta_1) &:= -\Gamma^{s,\gamma}(f, \partial_{\beta_1}\mu^{1/2};\beta_0).\end{aligned} \tag{2.10}$$

In the same way, we can define $\mathcal{L}^{s,\gamma,\eta}(\cdot;\beta_0,\beta_1)$, $\mathcal{L}_1^{s,\gamma,\eta}(\cdot;\beta_0,\beta_1)$, $\mathcal{L}_2^{s,\gamma,\eta}(\cdot;\beta_0,\beta_1)$ with kernel $B^{s,\gamma,\eta}$ and $\mathcal{L}_\eta^{s,\gamma}(\cdot;\beta_0,\beta_1)$, $\mathcal{L}_{\eta,1}^{s,\gamma}(\cdot;\beta_0,\beta_1)$, $\mathcal{L}_{\eta,2}^{s,\gamma}(\cdot;\beta_0,\beta_1)$ with kernel $B_\eta^{s,\gamma}$.

The rest of this section is organized as follows. Some preliminary knowledge is given in subsect. 2.1. We deal with the singular region $|v-v_*| \lesssim \eta$ and regular region $|v-v_*| \gtrsim \eta$ in subsects. 2.2 and 2.3 respectively. Estimates in the full region are given in subsect. 2.4. In the last subsect. 2.5, commutator and weighted upper bound estimates are collected for later energy estimate. Note that subsect. 2.2 is the core part of this section.

2.1 Preliminaries

We give some preliminary knowledge in this subsection.

2.1.1 Dyadic decomposition

Let φ be a smooth function on $\mathbb{R}_+$ satisfying

$$\begin{gathered}\varphi = 0 \text{ on } [0,3/4], \quad \varphi \text{ is strictly increasing on } [3/4,4/3], \quad \varphi = 1 \text{ on } [4/3,3/2],\\ \varphi \text{ is strictly decreasing on } [3/2,8/3], \quad \varphi = 0 \text{ on } [8/3,\infty), \quad |\varphi'| \le 4.\end{gathered}$$

Moreover, φ is chosen such that the functions $\{\varphi_k(\cdot) := \varphi(\cdot/2^k)\}_{k\in\mathbb{Z}}$ is a partition of unit on $(0,\infty)$. That is,

$$\sum_{k=-\infty}^{\infty} \varphi_k = 1 \text{ on } (0,\infty). \tag{2.11}$$

We set

$$\psi(r) = \sum_{k=-\infty}^{-1} \varphi_k(r) \text{ for } r > 0 \text{ and } \psi(0) = 1. \tag{2.12}$$

Then we have the identity

$$\psi + \sum_{k=0}^{\infty} \varphi_k \equiv 1 \text{ on } [0, \infty). \tag{2.13}$$

With a little abuse of notation, we define radial functions $\varphi(v) := \varphi(|v|)$, $\psi(v) := \psi(|v|)$, $\varphi_k(v) := \varphi_k(|v|)$ for $v \in \mathbb{R}^3$.

Given a general Boltzmann kernel $B = B(v - v_*, \sigma) = B(|v - v_*|, \cos\theta)$ where $\cos\theta = \frac{v-v_*}{|v-v_*|} \cdot \sigma$, let Q be the Boltzmann operator with kernel B. That is,

$$Q(g, h) := \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} B(|v - v_*|, \cos\theta)(g'_* h' - g_* h) \mathrm{d}\sigma \mathrm{d}v_*. \tag{2.14}$$

Let $\mathcal{U}_k := \sum_{j \le k} \varphi_j$, $\tilde{\varphi}_k := \sum_{|j-k| \le N_0 - 1} \varphi_j$ for $k \in \mathbb{Z}$ for some fixed integer $N_0 \ge 4$. Suppose the relative velocity satisfies $|v - v_*| \le \frac{4}{3}$. Let $v_* \in \{\frac{3}{4} \times 2^j \le |v_*| \le \frac{8}{3} \times 2^j\}$. Then the fact that

$$\frac{\sqrt{2}}{2}|v - v_*| \le |v' - v_*| \le |v - v_*|,$$

implies that

- If $j \ge N_0 - 1$, then $|v|, |v'| \in [(\frac{3}{4} - \frac{8}{3} \times 2^{-N_0})2^j, \frac{8}{3}(1 + 2^{-N_0})2^j] \subset \mathrm{Supp}\tilde{\varphi}_j$;
- If $j \le N_0 - 2$, then $|v|, |v'| \le \frac{3}{2} \times 2^{N_0 - 1} \subset \mathrm{Supp}\mathcal{U}_{N_0-1}$.

Hence, if Q is localized in $|v - v_*| \le \frac{4}{3}$, then

$$\begin{aligned} \langle Q(g, h), f\rangle = &\sum_{j \ge N_0 - 1} \langle Q(\varphi_j g, \tilde{\varphi}_j h), \tilde{\varphi}_j f\rangle \\ &+ \langle Q(\mathcal{U}_{N_0-2} g, \mathcal{U}_{N_0-1} h), \mathcal{U}_{N_0-1} f\rangle. \end{aligned} \tag{2.15}$$

Let us recall the Bobylev formula which is about the Fourier transform of the Boltzmann operator. For (2.14), the Bobylev formula reads

$$\begin{aligned} \mathcal{F}(Q(g, h))(\xi) = &\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \left(\hat{B}\left(|\eta_* - \xi^-|, \frac{\xi}{|\xi|} \cdot \sigma\right) - \hat{B}\left(|\eta_*|, \frac{\xi}{|\xi|} \cdot \sigma\right)\right) \\ &\mathcal{F}g(\eta_*)\mathcal{F}h(\xi - \eta_*) \mathrm{d}\sigma \mathrm{d}\eta_*, \end{aligned}$$

where $\xi^- = \frac{\xi - |\xi|\sigma}{2}$ and

$$\hat{B}(|\xi|, \cos\theta) := \int_{\mathbb{R}^3} B(|q|, \cos\theta) e^{-iq\cdot\xi} \mathrm{d}q.$$

Here $\mathcal{F}$ is the Fourier transform operator. As usual, denote $\hat{f} = \mathcal{F} f$, then

$$\langle Q(g, h), f\rangle = \langle \widehat{Q(g, h)}, \hat{f}\rangle$$

$$= \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\hat{B}\left(|\eta_* - \xi^-|, \frac{\xi}{|\xi|} \cdot \sigma \right) \right.$$
$$\left. - \hat{B}\left(|\eta_*|, \frac{\xi}{|\xi|} \cdot \sigma \right) \right) \hat{g}(\eta_*) \hat{h}(\xi - \eta_*) \bar{\hat{f}}(\xi) \mathrm{d}\sigma \mathrm{d}\eta_* \mathrm{d}\xi. \tag{2.16}$$

Note that

$$||\xi| - |\eta_*|| \le |\xi - \eta_*| \le |\xi| + |\eta_*|.$$

Fix $j, p \in \mathbb{Z}$, suppose $\frac{3}{4} \times 2^p \le |\eta_*| \le \frac{8}{3} \times 2^p$ and $\frac{3}{4} \times 2^j \le |\xi - \eta_*| \le \frac{8}{3} \times 2^j$. Since $N_0 \ge 4$, then

- If $p \le j - N_0$, then $|\xi| \in [(\frac{3}{4} - \frac{8}{3} \times 2^{-N_0})2^j, \frac{8}{3}(1 + 2^{-N_0})2^j] \subset \mathrm{Supp}\tilde{\varphi}_j$;
- If $p \ge j + N_0$, then $|\xi| \in [(\frac{3}{4} - \frac{8}{3} \times 2^{-N_0})2^p, \frac{8}{3}(1 + 2^{-N_0})2^p] \subset \mathrm{Supp}\tilde{\varphi}_p$;
- If $|p - j| < N_0$, then $|\xi| \in [0, \frac{3}{2} \times 2^{p+N_0}] \cap [0, \frac{3}{2} \times 2^{j+N_0}] \subset \mathrm{Supp}\mathcal{U}_{p+N_0} \cap \mathrm{Supp}\mathcal{U}_{j+N_0}$.

Define

$$\mathfrak{F}_{-1} f(x) := \psi(D) f, \quad \mathfrak{F}_j f(x) := \varphi_j(D) f, \ j \ge 0. \tag{2.17}$$

Observe $\mathfrak{F}_{-1}$ and $\mathfrak{F}_j$ localize the frequency of function f in the region $|\xi| \lesssim 1$ and $|\xi| \sim 2^j$ respectively. By (2.13), the dyadic decomposition in frequency space reads

$$f = \sum_{j \ge -1} \mathfrak{F}_j f.$$

Note that $\mathfrak{F}_{-1}$ has symbol ψ instead of φ_{-1}. Set $\tilde{\mathfrak{F}}_k := \tilde{\varphi}_k(D)$.
Then we have

$$\begin{aligned} \langle Q(g,h), f \rangle = & \sum_{j \ge N_0 - 1} \sum_{-1 \le p \le j - N_0} \langle Q(\mathfrak{F}_p g, \mathfrak{F}_j h), \tilde{\mathfrak{F}}_j f \rangle \\ & + \sum_{p \ge N_0 - 1} \sum_{-1 \le j \le p - N_0} \langle Q(\mathfrak{F}_p g, \mathfrak{F}_j h), \tilde{\mathfrak{F}}_p f \rangle \\ & + \sum_{p, j \ge -1, |p-j| < N_0} \sum_{q \le j + N_0} \langle Q(\mathfrak{F}_p g, \mathfrak{F}_j h), \mathfrak{F}_q f \rangle. \end{aligned} \tag{2.18}$$

We now recall the definition of symbol class $S^m_{1,0}$.

Definition 2.1 A smooth function $W(v, \xi)$ is said to be a symbol of type $S^m_{1,0}$ if for any $\alpha, \beta \in \mathbb{N}^3$,

$$|(\partial^\alpha_\xi \partial^\beta_v W)(v, \xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m - |\alpha|},$$

where $C_{\alpha,\beta}$ is a constant depending only on α and β.

Lemma 2.1 *Let $l, m, r \in \mathbb{R}$, $M(v,\xi) = M(\xi) \in S^r_{1,0}$ and $\Phi(v,\xi) = \Phi(\xi) \in S^l_{1,0}$. Then there exists a constant C such that*

$$|[M(D), \Phi] f|_{H^m} \le C |f|_{H^{m+r-1}_{l-1}}.$$

See Lemma 5.3 in [26] for the proof of Lemma 2.1. Based on Lemma 2.1, one directly has

$$|f|^2_{H^m_l} \sim \sum_{j,k=-1}^{\infty} 2^{2mk} 2^{2lj} |\varphi_k(D)\varphi_j f|^2_{L^2} \sim \sum_{j,k=-1}^{\infty} 2^{2mk} 2^{2lj} |\varphi_j \varphi_k(D) f|^2_{L^2}. \tag{2.19}$$

Here for simplicity, we take $\varphi_{-1} := \psi$. By (2.19), we use both $|W_l f|_{H^{m+s}}$ and $|W_l W_s(D) f|_{H^m}$ in the rest of this article.

2.1.2 Taylor expansion and symmetry

When evaluating the difference $f' - f$ (or $f'_* - f_*$) before and after collision, Taylor expansion is applied. We first denote the 1st-order expansion by

$$\begin{aligned} f' - f &= \int_0^1 (\nabla f)(v(\kappa)) \cdot (v' - v) \mathrm{d}\kappa, \\ f'_* - f_* &= \int_0^1 (\nabla f)(v_*(\iota)) \cdot (v'_* - v_*) \mathrm{d}\iota, \end{aligned} \tag{2.20}$$

where for $\kappa, \iota \in [0, 1]$,

$$v(\kappa) = \kappa v' + (1-\kappa) v, \quad v_*(\iota) = \iota v'_* + (1-\iota) v_*. \tag{2.21}$$

To cancel the angular singularity, the second order expansion is needed:

$$f' - f = (\nabla f)(v) \cdot (v' - v) + \int_0^1 (1-\kappa)(\nabla^2 f)(v(\kappa)) : (v'-v) \otimes (v'-v) \mathrm{d}\kappa, \tag{2.22}$$

$$f' - f = (\nabla f)(v') \cdot (v' - v) - \int_0^1 \kappa (\nabla^2 f)(v(\kappa)) : (v'-v) \otimes (v'-v) \mathrm{d}\kappa, \tag{2.23}$$

where $A : B := trace(AB)$ for two matrices A, B. Thanks to the symmetry property of σ-integral,

$$\int B\left(|v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma\right)(v' - v)\mathrm{d}\sigma = \int B\left(|v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma\right) \sin^2\frac{\theta}{2}(v_* - v)\mathrm{d}\sigma, \tag{2.24}$$

$$\int B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)(v'-v)h(v')\mathrm{d}\sigma\mathrm{d}v = 0. \tag{2.25}$$

Here, the formula (2.24) holds for fixed v, v_* and (2.25) holds for fixed v_*.

We now give a useful formula on the change of variables $v \to v(\kappa)$ and $v_* \to v_*(\iota)$.

Lemma 2.2 *For $a \in [0, 2]$, let us define*

$$\psi_a(\theta) := \left(\cos^2\frac{\theta}{2} + (1-a)^2\sin^2\frac{\theta}{2}\right)^{-1/2}. \tag{2.26}$$

For any $0 \le \kappa, \iota \le 1$, it holds that

$$\begin{aligned} &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2_+} B(|v-v_*|, \cos\theta)g(v_*(\iota))f(v(\kappa))\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2_+} B(|v-v_*|\psi_{\kappa+\iota}(\theta), \cos\theta)g(v_*)f(v)\psi^3_{\kappa+\iota}(\theta)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma. \end{aligned} \tag{2.27}$$

Here $\mathbb{S}^2_+$ stands for $(v-v_*)\cdot\sigma \ge 0$.

Before giving the proof of this lemma, we firstly note that the above formula is general as it simultaneously deals with the two changes $v \to v(\kappa)$ and $v_* \to v_*(\iota)$. This will be used in the proof of Proposition 4.2. If $\kappa = \iota = 0$, then $\psi_{\kappa+\iota}(\theta) = \psi_0(\theta) = 1$ and it corresponds to the identity transformation. If $\kappa = \iota = 1$, then $\psi_{\kappa+\iota}(\theta) = \psi_2(\theta) = 1$ and it corresponds to the change of velocities for pre-post collision: $(v, v_*, \sigma) \to (v', v'_*, \sigma' = (v-v_*)/|v-v_*|)$. If $\kappa = 1, \iota = 0$ or $\kappa = 0, \iota = 1$, then $\psi_{\kappa+\iota}(\theta) = \psi_1(\theta) = \cos^{-1}\frac{\theta}{2}$ and it corresponds to $v \to v'$ or $v_* \to v'_*$ respectively. This is consistent with the cancellation lemma given in [1]. If $\iota = 0$ or $\kappa = 0$, then it corresponds to the individual change $v \to v(\kappa)$ or $v_* \to v_*(\iota)$ respectively.

Note that $1 \le \psi_a(\theta) \le \sqrt{2}$ for $a \in [0, 2], \theta \in [0, \pi/2]$. Thanks to Lemma 2.2 and $1 \le \psi_a(\theta) \le \sqrt{2}$, considering the kernel (1.21), we can skip the details regarding the above mentioned change of variables. As a result, in most part of this article, $v(\kappa)$ and $v_*(\iota)$ will be replaced by v and v_* respectively at the cost of a multiplicative constant.

Proof of Lemma 2.2 The case $\kappa = \iota = 1$ is obviously given by the standard change of variable $(v, v_*, \sigma) \to (v', v'_*, \sigma')$ where $\sigma' = (v-v_*)/|v-v_*|$. This change has unit Jacobian.

Now we deal with the case $\kappa + \iota < 2$. Recalling (2.21), it is direct to check

$$|v-v_*| = |v(\kappa) - v_*(\iota)|\psi_{\kappa+\iota}(\theta). \tag{2.28}$$

Let β be the angle between $v(\kappa) - v_*(\iota)$ and σ, then $\cos\beta = \varphi_{\kappa+\iota}\left(\sin\frac{\theta}{2}\right)$ where

$$\varphi_a(x) := \frac{1-x^2+(a-1)x^2}{\left(1-x^2+(1-a)^2x^2\right)^{1/2}}.$$

Let $\delta_a := \arccos(\frac{\sqrt{2}}{2}\frac{a}{\sqrt{1+(1-a)^2}})$. If $\kappa+\iota<2$, then $\delta_{\kappa+\iota}>0$ and the function: $\theta\in[0,\frac{\pi}{2}]\to\beta_{\kappa+\iota}\in[0,\delta_{\kappa+\iota}]$ is a bijection. It holds that

$$\det\left(\frac{\partial(v(\kappa),v_*(\iota))}{\partial(v,v_*)}\right)=\alpha_{\kappa+\iota}(\theta), \tag{2.29}$$

where for $0\leq a\leq 2$,

$$\alpha_a(\theta):=\left(1-\frac{a}{2}\right)^2\left(\left(1-\frac{a}{2}\right)+\frac{a}{2}\cos\theta\right). \tag{2.30}$$

By (2.28) and (2.29), with $\mathrm{d}\sigma=\sin\beta\mathrm{d}\beta\mathrm{d}\mathbb{S}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2_+}B(|v-v_*|,\cos\theta)g(v_*(\iota))f(v(\kappa))\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma\\&=2\pi\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^{\delta_{\kappa+\iota}}B(|v-v_*|\psi_{\kappa+\iota}(\theta),\cos\theta)g(v_*)f(v)\alpha^{-1}_{\kappa+\iota}(\theta)\sin\beta\mathrm{d}v\mathrm{d}v_*\mathrm{d}\beta.\end{aligned}$$

It is elementary to check that

$$\begin{aligned}\alpha^{-1}_{\kappa+\iota}(\theta)\sin\beta\mathrm{d}\beta&=-\alpha^{-1}_{\kappa+\iota}(\theta)\mathrm{d}\cos\beta=-\frac{1}{4}\varphi'_{\kappa+\iota}\left(\sin\frac{\theta}{2}\right)\sin^{-1}\frac{\theta}{2}\alpha^{-1}_{\kappa+\iota}(\theta)\sin\theta\mathrm{d}\theta\\&=\psi^3_{\kappa+\iota}(\theta)\sin\theta\mathrm{d}\theta.\end{aligned}$$

Then we go back from β to θ and use $\mathrm{d}\sigma=\sin\theta\mathrm{d}\theta\mathrm{d}\mathbb{S}$ to get (2.27). □

2.2 Upper bound in the singular region

In this subsection, we will derive the upper bound estimates on the collision operators in the singular region $|v-v_*|\lesssim\eta$. Throughout this subsection, $0<s<1,\gamma>-2s-3, 0<\eta\leq 1$.

2.2.1 Upper bound of $Q^{s,\gamma}_\eta$

We give the upper bound of $Q^{s,\gamma}_\eta$ in the following proposition.

Proposition 2.1 *Let $l_1,l_2,l_3\in\mathbb{R}$ satisfying $l_1+l_2+l_3=0$. For any fixed $0<\delta\leq 1/2$, for any combination $a_1,a_2,a_3\geq 0$, $a_1+a_2\geq s$, $a_1+a_3\geq 2s$, $a_2+a_3\geq 2s$ satisfying the constraint $a_1+a_2+a_3=2s+\frac{3}{2}+\delta$, we have*

$$|\langle Q^{s,\gamma}_\eta(g,h),f\rangle|\lesssim C_{\delta,s,\gamma,\eta}|g|_{H^{a_1}_{l_1}}|h|_{H^{a_2}_{l_2}}|f|_{H^{a_3}_{l_3}},$$

where

$$C_{\delta,s,\gamma,\eta}:=\delta^{-1/2}C_{s,\gamma,\eta},\quad C_{s,\gamma,\eta}:=\frac{1}{s}\frac{\eta^{\gamma+2s+3}}{\gamma+2s+3}. \tag{2.31}$$

The constant associated to $\lesssim$ in the above inequality depends only on $|l_1|, |l_2|, |l_3|$.

Proof Recalling the decomposition in frequency space (2.18), we have

$$\begin{aligned}\langle Q_\eta^{s,\gamma}(g,h), f\rangle = &\sum_{j\le k-N_0}\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_k f\rangle\\ &+\sum_{|j-k|<N_0}\sum_{l\le k+N_0}\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \mathfrak{F}_l f\rangle\\ &+\sum_{j\ge k+N_0}\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_j f\rangle.\end{aligned}$$

We estimate the second quantity in details for illustration. That is, the sum over $|j-k| < N_0, l \le k + N_0$. Note that

$$\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \mathfrak{F}_l f\rangle = \int B_\eta^{s,\gamma}(\mathfrak{F}_j g)_*\mathfrak{F}_k h((\mathfrak{F}_l f)' - \mathfrak{F}_l f)\mathrm{d}V,$$

where for brevity of notation, $\mathrm{d}V = \mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma$. Motivated by (2.25), we write

$$\begin{aligned}\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \mathfrak{F}_l f\rangle &= \int B_\eta^{s,\gamma}(\mathfrak{F}_j g)_*(\mathfrak{F}_k h)'((\mathfrak{F}_l f)' - \mathfrak{F}_l f)\mathrm{d}V\\ &\quad+\int B_\eta^{s,\gamma}(\mathfrak{F}_j g)_*(\mathfrak{F}_k h - (\mathfrak{F}_k h)')((\mathfrak{F}_l f)' - \mathfrak{F}_l f)\mathrm{d}V\\ &:= \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

For $\mathcal{I}_1$, let $E := \sin\frac{\theta}{2} \le 2^{-l}|v-v_*|^{-1} \wedge \sqrt{2}/2$ and write

$$\begin{aligned}\mathcal{I}_1 &= \mathcal{I}_{1,\le}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f) + \mathcal{I}_{1,\ge}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f),\\ \mathcal{I}_{1,\le}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f) &:= \int B_\eta^{s,\gamma}1_E(\mathfrak{F}_j g)_*(\mathfrak{F}_k h)'((\mathfrak{F}_l f)' - \mathfrak{F}_l f)\mathrm{d}V,\\ \mathcal{I}_{1,\ge}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f) &:= \int B_\eta^{s,\gamma}1_{E^c}(\mathfrak{F}_j g)_*(\mathfrak{F}_k h)'((\mathfrak{F}_l f)' - \mathfrak{F}_l f)\mathrm{d}V.\end{aligned}$$

For term $\mathcal{I}_{1,\le}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f)$, we apply (2.23) and (2.25) to get

$$\begin{aligned}\mathcal{I}_{1,\le}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f) = -\int B_\eta^{s,\gamma}1_E(\mathfrak{F}_j g)_*(\mathfrak{F}_k h)'\bigg(\int_0^1 \kappa(\nabla^2\mathfrak{F}_l f)(v(\kappa))\\ : (v'-v)\otimes(v'-v)\mathrm{d}\kappa\bigg)\mathrm{d}V.\end{aligned}$$

By using $|\nabla^2\mathfrak{F}_l f|_{L^\infty} \lesssim 2^{\frac{7}{2}l}|\mathfrak{F}_l f|_{L^2}$, the change of variable $v \to v'$, and $\int \sin^2\frac{\theta}{2}b^s(\theta)1_E\mathrm{d}\sigma \lesssim 2^{2sl-2l}|v-v_*|^{2s-2}$, we have

$$|\mathcal{I}_{1,\le}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f)| \lesssim 2^{2sl}2^{\frac{3}{2}l}|\mathfrak{F}_l f|_{L^2}\int |v-v_*|^{\gamma+2s}1_{|v-v_*|\le 4\eta/3}|(\mathfrak{F}_j g)_*(\mathfrak{F}_k h)|\mathrm{d}v\mathrm{d}v_*.$$

By using the fact that

$$\| \cdot |^{\gamma+2s} 1_{|\cdot| \leq 4\eta/3} |_{L^1} \lesssim \frac{\eta^{\gamma+2s+3}}{\gamma+2s+3}, \tag{2.32}$$

we get

$$|\mathcal{I}_{1,\leq}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f)| \lesssim \frac{\eta^{\gamma+2s+3}}{\gamma+2s+3} 2^{2sl+\frac{3}{2}l} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\mathfrak{F}_l f|_{L^2}.$$

Since $\int b^s(\theta) 1_{E^c} \mathrm{d}\sigma \lesssim s^{-1} 2^{2sl} |v - v_*|^{2s}$, then

$$\begin{aligned} |\mathcal{I}_{1,\geq}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f)| &\lesssim C_{s,\gamma,\eta} 2^{2sl} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\mathfrak{F}_l f|_{L^\infty} \\ &\lesssim C_{s,\gamma,\eta} 2^{2sl+\frac{3}{2}l} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\mathfrak{F}_l f|_{L^2}. \end{aligned}$$

Combining the estimates on $\mathcal{I}_{1,\leq}$ and $\mathcal{I}_{1,\geq}$, we get

$$|\mathcal{I}_1| \lesssim C_{s,\gamma,\eta} 2^{2sl+\frac{3}{2}l} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\mathfrak{F}_l f|_{L^2}.$$

Now we estimate $\mathcal{I}_2$. Let $F := \sin\frac{\theta}{2} \leq 2^{-l/2-k/2} |v - v_*|^{-1} \wedge \sqrt{2}/2$. We write

$$\begin{aligned} \mathcal{I}_2 &= \mathcal{I}_{2,\leq}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f) + \mathcal{I}_{2,\geq}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f), \\ \mathcal{I}_{2,\leq}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f) &:= -\int B_\eta^{s,\gamma} 1_F (\mathfrak{F}_j g)_* ((\mathfrak{F}_k h)' - \mathfrak{F}_k h)((\mathfrak{F}_l f)' - \mathfrak{F}_l f) \mathrm{d}V, \\ \mathcal{I}_{2,\geq}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f) &:= -\int B_\eta^{s,\gamma} 1_{F^c} (\mathfrak{F}_j g)_* ((\mathfrak{F}_k h)' - \mathfrak{F}_k h)((\mathfrak{F}_l f)' - \mathfrak{F}_l f) \mathrm{d}V. \end{aligned}$$

By the 1st-order Taylor expansion (2.20), using $|\nabla \mathfrak{F}_l f|_{L^\infty} \lesssim 2^{\frac{5}{2}l} |\mathfrak{F}_l f|_{L^2}$, the change of variable in Lemma 2.2,
and $\int \sin^2\frac{\theta}{2} b^s(\theta) 1_F \mathrm{d}\sigma \lesssim 2^{sl+sk-l-k} |v - v_*|^{2s-2}$, we get

$$|\mathcal{I}_{2,\leq}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f)| \lesssim 2^{sl+sk-k} 2^{\frac{3}{2}l} |\mathfrak{F}_l f|_{L^2} \int |v - v_*|^{\gamma+2s} 1_{|v-v_*| \leq 4\eta/3} |(\mathfrak{F}_j g)_* (\nabla \mathfrak{F}_k h)| \mathrm{d}v \mathrm{d}v_*.$$

Then (2.32) implies

$$\begin{aligned} |\mathcal{I}_{2,\leq}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f)| &\lesssim \frac{\eta^{\gamma+2s+3}}{\gamma+2s+3} 2^{sl+sk-k+\frac{3}{2}l} |\mathfrak{F}_j g|_{L^2} |\nabla \mathfrak{F}_k h|_{L^2} |\mathfrak{F}_l f|_{L^2} \\ &\lesssim \frac{\eta^{\gamma+2s+3}}{\gamma+2s+3} 2^{sl+sk+\frac{3}{2}l} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\mathfrak{F}_l f|_{L^2}. \end{aligned}$$

Since $\int b^s(\theta) 1_{F^c} \mathrm{d}\sigma \lesssim s^{-1} 2^{sl+sk} |v - v_*|^{2s}$, then

$$|\mathcal{I}_{2,\geq}(\mathfrak{F}_j g, \mathfrak{F}_k h, \mathfrak{F}_l f)| \lesssim C_{s,\gamma,\eta} 2^{sl+sk} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\mathfrak{F}_l f|_{L^\infty}$$

$$\lesssim C_{s,\gamma,\eta} 2^{sl+sk+\frac{3}{2}l} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\mathfrak{F}_l f|_{L^2}.$$

Combining the estimates on $\mathcal{I}_{2,\leq}$ and $\mathcal{I}_{2,\geq}$, we have

$$|\mathcal{I}_2| \lesssim C_{s,\gamma,\eta} 2^{sl+sk+\frac{3}{2}l} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\mathfrak{F}_l f|_{L^2}.$$

Therefore, for $l \leq k + N_0$, we obtain

$$|\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \mathfrak{F}_l f\rangle| \lesssim C_{s,\gamma,\eta} 2^{sl+sk+\frac{3}{2}l} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\mathfrak{F}_l f|_{L^2}. \tag{2.33}$$

From this, we have

$$\sum_{|j-k|<N_0} \sum_{l\leq k+N_0} |\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \mathfrak{F}_l f\rangle| \lesssim C_{s,\gamma,\eta} \sum_{|j-k|<N_0} 2^{sk} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} \left(\sum_{l\leq k+N_0} 2^{sl+\frac{3}{2}l} |\mathfrak{F}_l f|_{L^2} \right).$$

Let $a + b = s + \frac{3}{2}$ for $a \geq 0$, for any fixed k, the sum over $-1 \leq l \leq k + N_0$ can be estimated by using Cauchy–Schwarz inequality as

$$\begin{aligned} \sum_{l\leq k+N_0} 2^{sl+\frac{3}{2}l} |\mathfrak{F}_l f|_{L^2} &\leq \left(\sum_{l\leq k+N_0} 2^{2al} |\mathfrak{F}_l f|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_{l\leq k+N_0} 2^{2bl} \right)^{\frac{1}{2}} \\ &\lesssim |f|_{H^a} C_{b,k}, \end{aligned} \tag{2.34}$$

where for $b \neq 0$,

$$C_{b,k}^2 = \frac{2^{2b(k+N_0+1)} - 2^{-2b}}{2^{2b} - 1}.$$

For b close to 0, by allowing an extra δ-order regularity, we conclude that

$$\sum_{|j-k|<N_0} \sum_{l\leq k+N_0} \langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \mathfrak{F}_l f\rangle \lesssim \delta^{-1/2} C_{s,\gamma,\eta} |g|_{H^{a_1}} |h|_{H^{a_2}} |f|_{H^{a_3}}, \tag{2.35}$$

where $a_1, a_2, a_3 \geq 0$, $a_1 + a_2 \geq s$ satisfying the constraint $a_1 + a_2 + a_3 = 2s + \frac{3}{2} + \delta$ for any fixed small $\delta > 0$. Indeed, recalling (2.34), in which we can take $a + b = s + \frac{3}{2} + \delta$ and $b \geq \frac{\delta}{2}$, then $C_{b,k} \lesssim \delta^{-1/2} 2^{k\delta/2}$. Since $|j - k| < N_0$, we get (2.35) for $a_1 + a_2 \geq s + \frac{\delta}{2}$. In (2.34) we can also take $= s + \frac{3}{2} + \frac{\delta}{2}$, $b = -\frac{\delta}{2}$ and so $C_{b,k} \lesssim \delta^{-1/2}$, then we get (2.35) for $a_1 + a_2 = s$, $a_3 = s + \frac{3}{2} + \frac{\delta}{2}$. This obviously implies that (2.35) holds for $s \leq a_1 + a_2 \leq s + \frac{\delta}{2}$.

Similar argument can be applied to $\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_k f\rangle$ for $j \leq k - N_0$ to obtain

$$|\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_k f\rangle| \lesssim C_{s,\gamma,\eta} 2^{2sk+\frac{3}{2}j} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\tilde{\mathfrak{F}}_k f|_{L^2}. \tag{2.36}$$

Here we take L^∞ on $\mathfrak{F}_j g$ so that there is a factor $2^{\frac{3}{2}j}$. We also apply Taylor expansions to $\mathfrak{F}_k h$ and $\tilde{\mathfrak{F}}_k f$ to obtain the factor 2^{2sk}. Let $0 < \delta \leq \frac{1}{2}$, for any combination $a_1, a_2, a_3 \geq 0, a_2 + a_3 \geq 2s$ satisfying the constraint $a_1 + a_2 + a_3 = 2s + \frac{3}{2} + \delta$, it holds that

$$\left| \sum_{j \leq k - N_0} \langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_k f\rangle \right| \lesssim \delta^{-1/2} C_{s,\gamma,\eta} |g|_{H^{a_1}} |h|_{H^{a_2}} |f|_{H^{a_3}}.$$

Again similar argument can be applied to $\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_j f\rangle$ for $j \geq k + N_0$ to get

$$|\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_j f\rangle| \lesssim C_{s,\gamma,\eta} 2^{2sj+\frac{3}{2}k} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_k h|_{L^2} |\tilde{\mathfrak{F}}_j f|_{L^2}. \tag{2.37}$$

Here we also apply Taylor expansions to $\mathfrak{F}_k h$ and $\tilde{\mathfrak{F}}_j f$ to get the factor 2^{2sj} or $2^{sk+sj} \leq 2^{2sj}$. Here we take L^∞ on $\mathfrak{F}_k h$ or $\nabla \mathfrak{F}_k h$ so that at the end there is a factor $2^{\frac{3}{2}k}$. Let $0 < \delta \leq \frac{1}{2}$, for any combination $a_1, a_2, a_3 \geq 0, a_1 + a_3 \geq 2s$ satisfying the constraint $a_1 + a_2 + a_3 = 2s + \frac{3}{2} + \delta$, it holds that

$$\left| \sum_{j \geq k + N_0} \langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_j f\rangle \right| \lesssim \delta^{-1/2} C_{s,\gamma,\eta} |g|_{H^{a_1}} |h|_{H^{a_2}} |f|_{H^{a_3}}.$$

In summary, we prove the desired estimate with $l_1 = l_2 = l_3 = 0$.

By (2.15) and (2.19), we can freely transfer weight among g, h, f so that the estimate in the proposition holds for $l_1 + l_2 + l_3 = 0$. □

Based on the proof of the above proposition, we will derive another version of cancellation lemma introduced in [1].

The idea is to gain $|v - v_*|^{2s}$ at the price of $2s$-order derivatives on the functions.

Lemma 2.3 [Revised cancellation lemma for relative velocity near origin] *Let $a_1, a_2, l_1, l_2 \in \mathbb{R}$ satisfying $a_1 + a_2 = 2s, l_1 + l_2 = 0$, then*

$$\left| \int B_\eta^{s,\gamma} g_*(f' - f) \mathrm{d}V \right| \lesssim C_{s,\gamma,\eta} |g|_{H^{a_1}_{l_1}} |f|_{H^{a_2}_{l_2}}.$$

Proof Recalling the decomposition in frequency space, we have

$$\int B_\eta^{s,\gamma} g_*(f' - f) \mathrm{d}V = \langle Q_\eta^{s,\gamma}(g, 1), f\rangle = \sum_{|j-l|<N_0} \langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, 1), \mathfrak{F}_l f\rangle.$$

This is because frequency of g and f lies in the same region when $h = 1$. Following the estimate on $\mathcal{I}_1$ in the proof of Proposition 2.1, we have

$$|\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, 1), \mathfrak{F}_l f\rangle| \lesssim C_{s,\gamma,\eta} 2^{2sl} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_l f|_{L^2}.$$

Since $|j - l| < N_0$, by the Cauchy–Schwarz inequality, we can estimate the sum as

$$\sum_{|j-l|<N_0} 2^{2sl} |\mathfrak{F}_j g|_{L^2} |\mathfrak{F}_l f|_{L^2} \lesssim |g|_{H^{a_1}} |f|_{H^{a_2}}.$$

By (2.15) and (2.19), we can freely transfer weight among g, f so that the proof of the proposition is completed. □

2.2.2 Upper bound of $I_\eta^{s,\gamma}$

We now estimate $\langle I_\eta^{s,\gamma}(g, h; \beta), f\rangle$ where $I_\eta^{s,\gamma}(g, h; \beta)$ is defined by (2.7) with $B_\eta^{s,\gamma}$.

Proposition 2.2 *Let* $l \geq 0$. *Let* $(a_1, a_2) = (\frac{3}{2} + \delta, s)$ *or* $(0, \frac{3}{2} + \delta)$. *Then*

$$\langle I_\eta^{s,\gamma}(g, h; \beta), f\rangle \lesssim_l C_{\delta,s,\gamma,\eta} |g|_{H^{a_1}_{-l}} |h|_{H^{a_2}_{-l}} |f|_{H^s_{-l}}.$$

Proof We only consider the case when $\beta = 0$ because the following argument also works with a slight change when we replace $\mu^{1/2}$ by $\partial_\beta \mu^{1/2}$. Recall

$$\langle I_\eta^{s,\gamma}(g, h), f\rangle = \int B_\eta^{s,\gamma}((\mu^{1/2})'_* - \mu_*^{1/2}) g_* h f' \mathrm{d}V = \mathcal{I}_1 + \mathcal{I}_2, \tag{2.38}$$

where

$$\mathcal{I}_1 := \int B_\eta^{s,\gamma}((\mu^{1/2})'_* - \mu_*^{1/2}) g_* h (f' - f) \mathrm{d}V,$$
$$\mathcal{I}_2 := \int B_\eta^{s,\gamma}((\mu^{1/2})'_* - \mu_*^{1/2}) g_* h f \mathrm{d}V.$$

Firstly, for $\mathcal{I}_1$, by the Cauchy–Schwarz inequality, we have $|\mathcal{I}_1| \leq \mathcal{I}_{1,1}^{1/2} \mathcal{I}_{1,2}^{1/2}$, where

$$\mathcal{I}_{1,1} := \int B_\eta^{s,\gamma}((\mu^{1/4})'_* + \mu_*^{1/4})^2 (f' - f)^2 \mathrm{d}V,$$
$$\mathcal{I}_{1,2} := \int B_\eta^{s,\gamma}((\mu^{1/4})'_* - \mu_*^{1/4})^2 g_*^2 h^2 \mathrm{d}V.$$

Using $((\mu^{1/4})'_* + \mu_*^{1/4})^2 \leq 2((\mu^{1/2})'_* + \mu_*^{1/2})$, by the change of variable $(v, v_*, \sigma) \to (v', v'_*, \sigma')$, we have

$$\mathcal{I}_{1,1} \leq 4 \int B_\eta^{s,\gamma} \mu_*^{1/2} (f' - f)^2 \mathrm{d}V = 4\mathcal{N}_\eta^{s,\gamma}(\mu^{1/4}, f),$$

where

$$\mathcal{N}_\eta^{s,\gamma}(g,h) := \int B_\eta^{s,\gamma} g_*^2 (h'-h)^2 \mathrm{d}V. \tag{2.39}$$

Using $(f'-f)^2 = (f^2)' - f^2 - 2f(f'-f)$, we get

$$\mathcal{N}_\eta^{s,\gamma}(\mu^{1/4}, f) = \int B_\eta^{s,\gamma} \mu_*^{1/2}((f^2)' - f^2)\mathrm{d}V - 2\langle Q_\eta^{s,\gamma}(\mu^{1/2}, f), f\rangle. \tag{2.40}$$

By (2.23) and (2.25), the change of variable in Lemma 2.2, we have

$$\begin{aligned}\left|\int B_\eta^{s,\gamma}\mu_*^{1/2}((f^2)'-f^2)\mathrm{d}V\right| &= \left|\int B_\eta^{s,\gamma}((\mu^{1/2})'-\mu^{1/2})f_*^2\mathrm{d}V\right| \\ &\lesssim \int 1_{|v-v_*|\le 4\eta/3}|v-v_*|^{\gamma+2}\mu_*^{\frac{1}{8}} f_*^2 \mathrm{d}v_*\mathrm{d}v \\ &\lesssim \frac{\eta^{\gamma+5}}{\gamma+5}|\mu^{1/16}f|_{L^2}^2.\end{aligned}$$

Here we have used $\mu(v(\kappa)) \lesssim \mu^{1/2}(v_*(\iota))$ for $|v-v_*|\lesssim 1$. By Proposition 2.1, we have

$$|\langle Q_\eta^{s,\gamma}(\mu^{1/2}, f), f\rangle| \lesssim_l C_{s,\gamma,\eta}|f|_{H^s_{-l}}^2.$$

Since $\gamma+5\ge\gamma+2s+3$,

$$\mathcal{I}_{1,1} \lesssim \mathcal{N}_\eta^{s,\gamma}(\mu^{1/4}, f) \lesssim_l C_{s,\gamma,\eta}|f|_{H^s_{-l}}^2. \tag{2.41}$$

It is straightforward to see

$$\begin{aligned}\mathcal{I}_{1,2} &\lesssim \int 1_{|v-v_*|\le 4\eta/3}|v-v_*|^{\gamma+2}\mu^{1/8}\mu_*^{1/8}g_*^2h^2\mathrm{d}v_*\mathrm{d}v \\ &\lesssim \delta^{-1}\frac{\eta^{\gamma+5}}{\gamma+5}|\mu^{1/16}g|_{H^{a_1}}^2|\mu^{1/16}h|_{H^{a_2}}^2,\end{aligned}$$

where $a_1+a_2=\frac{3}{2}+\delta$. Combining the estimates on $\mathcal{I}_{1,1}$ and $\mathcal{I}_{1,2}$ gives

$$\mathcal{I}_1 \lesssim_l C_{\delta,s,\gamma,\eta}|\mu^{1/16}g|_{H^{a_1}}|\mu^{1/16}h|_{H^{a_2}}|f|_{H^s_{-l}}.$$

We now turn to estimate $\mathcal{I}_2$. By Proposition 2.1, we directly have

$$\begin{aligned}|\mathcal{I}_2| &= \left|\int B_\eta^{s,\gamma}((\mu^{1/2})'-\mu^{1/2})g(hf)_*\mathrm{d}V\right| = \left|\langle Q_\eta^{s,\gamma}(hf,g),\mu^{1/2}\rangle\right| \\ &\lesssim C_{s,\gamma,\eta}|hf|_{H^s_{-2l}}|g|_{L^2_{-l}}|\mu^{\frac{1}{2}}|_{H^{s+2}_{3l}} \lesssim_l C_{\delta,s,\gamma,\eta}|g|_{L^2_{-l}}|h|_{H^{\frac{3}{2}+\delta}_{-l}}|f|_{H^s_{-l}}.\end{aligned}$$

By Lemma 2.4 to be proved below, we have

$$|\mathcal{I}_2| = \left|\langle Q_\eta^{s,\gamma}(hf, g), \mu^{1/2}\rangle\right| \lesssim_l C_{\delta,s,\gamma,\eta}|g|_{H_{-l}^{\frac{3}{2}+\delta}}|h|_{H_{-l}^s}|f|_{H_{-l}^s}.$$

Combining the estimates on $\mathcal{I}_1$ and $\mathcal{I}_2$ completes the proof of the proposition. □

Lemma 2.4 *It holds that*

$$\left|\langle Q_\eta^{s,\gamma}(f_2 f_3, f_1), \mu^{1/2}\rangle\right| \lesssim_l C_{\delta,s,\gamma,\eta}|f_1|_{H_{-l}^{\frac{3}{2}+\delta}}|f_2|_{H_{-l}^s}|f_3|_{H_{-l}^s}.$$

Proof We will follow the proof of Proposition 2.1 by taking $g = f_2 f_3, h = f_1, f = \mu^{1/2}$.

Recall (2.33) for $l \leq k + N_0$ that

$$|\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \mathfrak{F}_l f\rangle| \lesssim C_{s,\gamma,\eta}2^{sl+sk+\frac{3}{2}l}|\mathfrak{F}_j g|_{L^2}|\mathfrak{F}_k h|_{L^2}|\mathfrak{F}_l f|_{L^2}.$$

Note that

$$|\mathfrak{F}_j g|_{L^2} = |\varphi_j(\hat{f}_2 * \hat{f}_3)|_{L^2} \leq |\varphi_j|_{L^r}|\hat{f}_2|_{L^q}|\hat{f}_3|_{L^q} \lesssim 2^{\frac{3}{r}j}|\hat{f}_2|_{L^q}|\hat{f}_3|_{L^q}. \quad (2.42)$$

Here $r \geq 2 \geq q$ satisfy

$$\frac{1}{r} + \frac{2}{q} = 1 + \frac{1}{2}, \quad \frac{3}{r} = \frac{3}{2} - s.$$

For the chosen r, when $|j-k| < N_0$, we have $2^{\frac{3}{r}j}2^{sk} \lesssim 2^{\frac{3}{2}k}$.

For $1/q = 1/2 + 1/p$, we have

$$|\hat{f}_2|_{L^q} \leq |W_s \hat{f}_2|_{L^2}|W_{-s}|_{L^p} \lesssim |f_2|_{H^s},$$

because $sp = 6$. Then we obtain

$$\begin{aligned}
&\sum_{|j-k|<N_0}\sum_{l\leq k+N_0}|\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \mathfrak{F}_l f\rangle| \lesssim C_{s,\gamma,\eta} \\
&\quad \sum_{|j-k|<N_0}\sum_{l\leq k+N_0} 2^{sl+\frac{3}{2}l+\frac{3}{2}k}|f_2|_{H^s}|f_3|_{H^s}|\mathfrak{F}_k h|_{L^2}|\mathfrak{F}_l f|_{L^2} \\
&\quad \lesssim C_{s,\gamma,\eta}|f_2|_{H^s}|f_3|_{H^s}|f|_{H^{s+2}}\sum_{|j-k|<N_0} 2^{\frac{3}{2}k}|\mathfrak{F}_k h|_{L^2} \\
&\quad \lesssim C_{\delta,s,\gamma,\eta}|f_2|_{H^s}|f_3|_{H^s}|f|_{H^{s+2}}|h|_{H^{3/2+\delta}} \lesssim C_{\delta,s,\gamma,\eta}|f_1|_{H^{3/2+\delta}}|f_2|_{H^s}|f_3|_{H^s}.
\end{aligned}$$

Recalling (2.36) for $j \leq k - N_0$, and using (2.42) for $r = q = 2$, we have

$$|\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_k f\rangle| \lesssim C_{s,\gamma,\eta}2^{2sk+3j}|f_2|_{L^2}|f_3|_{L^2}|\mathfrak{F}_k h|_{L^2}|\tilde{\mathfrak{F}}_k f|_{L^2},$$

which yields

$$\sum_{j\le k-N_0} |\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_k f\rangle| \lesssim C_{s,\gamma,\eta}|f_2|_{L^2}|f_3|_{L^2} \sum_{j\le k-N_0} 2^{2sk+3j}|\mathfrak{F}_k h|_{L^2}|\tilde{\mathfrak{F}}_k f|_{L^2}$$
$$\lesssim C_{s,\gamma,\eta}|f_2|_{L^2}|f_3|_{L^2}\sum_k 2^{2sk+3k}|\mathfrak{F}_k h|_{L^2}|\tilde{\mathfrak{F}}_k f|_{L^2} \lesssim C_{s,\gamma,\eta}|f_2|_{L^2}|f_3|_{L^2}|h|_{L^2}|f|_{H^{2s+3}}$$
$$\lesssim C_{s,\gamma,\eta}|f_1|_{L^2}|f_2|_{L^2}|f_3|_{L^2}.$$

Recalling (2.37) for $j \ge k + N_0$, and using (2.42) for $r = q = 2$, we have

$$|\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_j f\rangle| \lesssim C_{s,\gamma,\eta}2^{2sj+\frac{3}{2}j+\frac{3}{2}k}|f_2|_{L^2}|f_3|_{L^2}|\mathfrak{F}_k h|_{L^2}|\tilde{\mathfrak{F}}_j f|_{L^2},$$

which yields

$$\sum_{j\ge k+N_0} |\langle Q_\eta^{s,\gamma}(\mathfrak{F}_j g, \mathfrak{F}_k h), \tilde{\mathfrak{F}}_j f\rangle| \lesssim C_{s,\gamma,\eta}|f_2|_{L^2}|f_3|_{L^2} \sum_{j\ge k+N_0} 2^{2sj+\frac{3}{2}j+\frac{3}{2}k}|\mathfrak{F}_k h|_{L^2}|\tilde{\mathfrak{F}}_k f|_{L^2}$$
$$\lesssim C_{s,\gamma,\eta}|f_2|_{L^2}|f_3|_{L^2}|h|_{L^2}\sum_j 2^{2sj+3j}|\tilde{\mathfrak{F}}_j f|_{L^2}$$
$$\lesssim C_{s,\gamma,\eta}|f_2|_{L^2}|f_3|_{L^2}|h|_{L^2}|f|_{H^{2s+4}} \lesssim C_{s,\gamma,\eta}|f_1|_{L^2}|f_2|_{L^2}|f_3|_{L^2}.$$

Combining the above estimates gives

$$|\langle Q_\eta^{s,\gamma}(f_2 f_3, f_1), \mu^{1/2}\rangle| \lesssim C_{\delta,s,\gamma,\eta}|f_1|_{H^{\frac{3}{2}+\delta}}|f_2|_{H^s}|f_3|_{H^s}.$$

By (2.15) and (2.19), since we can freely transfer weight among $f_1, f_2, f_3, \mu^{\frac{1}{2}}$, then we put the negative order $-l$ weight on each of f_1, f_2, f_3 and the positive order $3l$ weight on $\mu^{\frac{1}{2}}$ to obtain the final estimate. □

2.2.3 Upper bound of $\Gamma_\eta^{s,\gamma}$

Similarly to (2.6),

$$\Gamma_\eta^{s,\gamma}(g, h; \beta) = Q_\eta^{s,\gamma}(g\partial_\beta\mu^{1/2}, h) + I_\eta^{s,\gamma}(g, h; \beta). \tag{2.43}$$

By Propositions 2.1 and 2.2, we have the following proposition.

Proposition 2.3 *Let* $(a_1, a_2) = (\frac{3}{2} + \delta, s)$ *or* $(s, \frac{3}{2} + \delta)$. *Then it holds that*

$$\langle \Gamma_\eta^{s,\gamma}(g, h; \beta), f\rangle \lesssim_l C_{\delta,s,\gamma,\eta}|g|_{H^{a_1}_{-l}}|h|_{H^{a_2}_{-l}}|f|_{H^s_{-l}}.$$

2.2.4 Upper bound of $\mathcal{L}_\eta^{s,\gamma}$

Recall that $\mathcal{L}_\eta^{s,\gamma}(f; \beta_0, \beta_1) = \mathcal{L}_{\eta,1}^{s,\gamma}(f; \beta_0, \beta_1) + \mathcal{L}_{\eta,2}^{s,\gamma}(f; \beta_0, \beta_1)$ where

$$\mathcal{L}_{\eta,1}^{s,\gamma}(f; \beta_0, \beta_1) := -\Gamma_\eta^{s,\gamma}(\partial_{\beta_1}\mu^{1/2}, f; \beta_0),$$

$$\mathcal{L}^{s,\gamma}_{\eta,2}(f;\beta_0,\beta_1) := -\Gamma^{s,\gamma}_{\eta}(f,\partial_{\beta_1}\mu^{1/2};\beta_0).$$

If $|\beta_0| = |\beta_1| = 0$, the operators are reduced to

$$\mathcal{L}^{s,\gamma}_{\eta} f = \mathcal{L}^{s,\gamma}_{\eta}(f;0,0), \quad \mathcal{L}^{s,\gamma}_{\eta,1} f = \mathcal{L}^{s,\gamma}_{\eta,1}(f;0,0), \quad \mathcal{L}^{s,\gamma}_{\eta,2} f = \mathcal{L}^{s,\gamma}_{\eta,2}(f;0,0).$$

By Proposition 2.3, we have the following proposition.

Proposition 2.4 *It holds that*

$$|\langle \mathcal{L}^{s,\gamma}_{\eta,1}(f;\beta_0,\beta_1), h\rangle| \lesssim_l C_{s,\gamma,\eta} |f|_{H^s_{-l}} |h|_{H^s_{-l}}.$$

Note that for any $0 \le \kappa, \iota \le 1$,

$$\left(1 - \frac{\sqrt{2}}{2}\right)(|v|^2 + |v_*|^2) \le |v(\kappa)|^2 + |v_*(\iota)|^2 \le \left(1 + \frac{\sqrt{2}}{2}\right)(|v|^2 + |v_*|^2),$$

which yields

$$\mu^2(v)\mu^2(v_*) \le \mu(v(\kappa))\mu(v_*(\iota)) \le \mu^{\frac{1}{4}}(v)\mu^{\frac{1}{4}}(v_*). \tag{2.44}$$

Thus, this estimate keeps a μ-type weight in the upper bound of $\mathcal{L}^{s,\gamma}_{\eta,2}$.

Proposition 2.5 *It holds that*

$$|\langle \mathcal{L}^{s,\gamma}_{\eta,2}(f;\beta_0,\beta_1), h\rangle| \lesssim C_{s,\gamma,\eta} |\mu^{1/32} f|_{H^s} |\mu^{1/32} h|_{H^s}.$$

Proof For simplicity, we only consider $|\beta_0| = |\beta_1| = 0$. Note that

$$\begin{aligned}
\langle -\mathcal{L}^{s,\gamma}_{\eta,2} f, h\rangle &= \langle \mu^{-1/2} Q^{s,\gamma}_{\eta}(\mu^{1/2} f, \mu), h\rangle \\
&= \int B^{s,\gamma}_{\eta} (\mu^{1/2} f)_* \mu ((\mu^{-1/2} h)' - \mu^{-1/2} h) \mathrm{d}V \\
&= \int B^{s,\gamma}_{\eta} (\mu^{1/2} f)_* \mu^{1/2} (h' - h) \mathrm{d}V \\
&\quad + \int B^{s,\gamma}_{\eta} f_* \mu^{1/2} h' ((\mu^{1/2})'_* - \mu^{1/2}_*) \mathrm{d}V := Y_1 + Y_2.
\end{aligned}$$

We first estimate Y_1. Observe that

$$\begin{aligned}
Y_1 &= \int B^{s,\gamma}_{\eta} (\mu^{1/2} f)_* \mu^{1/2} (h' - h) \mathrm{d}V \\
&= \int B^{s,\gamma}_{\eta} (\mu^{1/2} f)_* ((\mu^{1/2} h)' - \mu^{1/2} h) \mathrm{d}V \\
&\quad + \int B^{s,\gamma}_{\eta} (\mu^{1/2} f)_* (\mu^{1/2} - (\mu^{1/2})') h' \mathrm{d}V := Y_{1,1} + Y_{1,2}.
\end{aligned}$$

Lemma 2.3 implies

$$|Y_{1,1}| \lesssim C_{s,\gamma,\eta} |\mu^{1/2} f|_{H^s} |\mu^{1/2} h|_{H^s}.$$

By (2.23) and (2.25), using (2.44), and the change of variable $v \to v'$, we have

$$|Y_{1,2}| \lesssim \int 1_{|v-v_*|\le 4\eta/3} |v-v_*|^{\gamma+2} \mu_*^{1/12} f_* \mu^{1/12} h \mathrm{d}v_* \mathrm{d}v \lesssim \frac{\eta^{\gamma+5}}{\gamma+5} |\mu^{1/16} f|_{L^2} |\mu^{1/16} h|_{L^2}.$$

We now turn to Y_2. Note that

$$\begin{aligned} Y_2 &= \int B_\eta^{s,\gamma} f_* \mu^{1/2} h' ((\mu^{1/2})'_* - \mu_*^{1/2}) \mathrm{d}V \\ &= \int B_\eta^{s,\gamma} f_* (\mu^{1/2} - (\mu^{1/2})') h' ((\mu^{1/2})'_* - \mu_*^{1/2}) \mathrm{d}V \\ &\quad + \int B_\eta^{s,\gamma} f_* (\mu^{1/2})' h' ((\mu^{1/2})'_* - \mu_*^{1/2}) \mathrm{d}V \\ &:= Y_{2,1} + Y_{2,2}. \end{aligned}$$

Observe $Y_{2,2} = \int B_\eta^{s,\gamma} (\mu^{1/2} h)_* (\mu^{1/2} - (\mu^{1/2})') f' \mathrm{d}V$. Similarly to the estimate on $Y_{1,2}$, we get

$$|Y_{2,2}| \lesssim \frac{\eta^{\gamma+5}}{\gamma+5} |\mu^{1/16} f|_{L^2} |\mu^{1/16} h|_{L^2}.$$

By the Cauchy–Schwarz inequality, we get

$$\begin{aligned} |Y_{2,1}| \le & \left(\int B_\eta^{s,\gamma} f_*^2 (\mu^{1/4} - (\mu^{1/4})')^2 ((\mu^{1/4})'_* + \mu_*^{1/4})^2 \mathrm{d}V \right)^{1/2} \\ & \times \left(\int B_\eta^{s,\gamma} (h^2)' ((\mu^{1/4})'_* - \mu_*^{1/4})^2 (\mu^{1/4} + (\mu^{1/4})')^2 \mathrm{d}V \right)^{1/2} \\ = & \left(\int B_\eta^{s,\gamma} f_*^2 (\mu^{1/4} - (\mu^{1/4})')^2 ((\mu^{1/4})'_* + \mu_*^{1/4})^2 \mathrm{d}V \right)^{1/2} \\ & \times \left(\int B_\eta^{s,\gamma} h_*^2 (\mu^{1/4} - (\mu^{1/4})')^2 ((\mu^{1/4})'_* + \mu_*^{1/4})^2 \mathrm{d}V \right)^{1/2}. \end{aligned}$$

By Taylor expansion, using (2.44) to obtain the μ-type weight, we get

$$\int B_\eta^{s,\gamma} f_*^2 (\mu^{1/4} - (\mu^{1/4})')^2 ((\mu^{1/4})'_* + \mu_*^{1/4})^2 \mathrm{d}V \lesssim \frac{\eta^{\gamma+5}}{\gamma+5} |\mu^{1/32} f|_{L^2}^2,$$

which gives $|Y_{2,1}| \lesssim (\gamma+5)^{-1} \eta^{\gamma+5} |\mu^{1/32} f|_{L^2} |\mu^{1/32} h|_{L^2}$. Combining the above estimates completes the proof of the proposition. □

By Propositions 2.4 and 2.5, we also have the following proposition.

Proposition 2.6 *It holds that*

$$|\langle \mathcal{L}_{\eta}^{s,\gamma}(f;\beta_0,\beta_1),h\rangle| \lesssim_l C_{s,\gamma,\eta}|f|_{H^s_{-l}}|h|_{H^s_{-l}}.$$

2.3 Upper bound estimate in the regular region

In this subsection, we will derive some upper bound estimates in the regular region $|v-v_*| \gtrsim \eta$. This region is relatively easy by observing

$$|v-v_*|^{\gamma}\psi^{\eta}(|v-v_*|) \lesssim \eta^{\gamma}\langle v-v_*\rangle^{\gamma}. \tag{2.45}$$

So we briefly illustrate the key ideas and give relevant references instead of reproducing the proof. Throughout this subsection, $0 < s < 1, -5 \leq \gamma \leq 0, 0 < \eta \leq 1$.

2.3.1 Upper bound of $Q^{s,\gamma,\eta}$

Set $u = v - v_*$, then $v = v_* + u, v' = v_* + u^+$ where $u^+ = \frac{|u|\sigma+u}{2}$. Define the translation operator T_{v_*} by $(T_{v_*}f)(v) = f(v_*+v)$. We recall the geometric decomposition into radial and spherical parts

$$\begin{aligned} f(v')-f(v) &= \left(\left((T_{v_*}f)(u^+)-(T_{v_*}f)\left(|u|\frac{u^+}{|u^+|}\right)\right)\right) \\ &\quad + \left((T_{v_*}f)\left(|u|\frac{u^+}{|u^+|}\right)-(T_{v_*}f)(u)\right) \\ &= \text{radial part} + \text{ spherical part}. \end{aligned} \tag{2.46}$$

We first analyze the radial part. Note that $|u^+ - |u|\frac{u^+}{|u^+|}| = |u|\left(1-\cos\frac{\theta}{2}\right) = 2|u|\sin^2\frac{\theta}{4}$, which yields an order-2 cancellation in the angular singularity. By using the localization technique in [28], the radial part in (2.46) can be controlled by gain of W_s in the phase and frequency spaces, namely the two Sobolev norms L^2_s and H^s. Similarly to Lemma 2.6 in [28], we have

Lemma 2.5 *Set* $\mathcal{Y}^{s,\gamma,\eta}(h,f) := \int b^s(\frac{u}{|u|}\cdot\sigma)|u|^{\gamma}\psi^{\eta}(u)h(u)[f(u^+)-f(|u|\frac{u^+}{|u^+|})]\mathrm{d}\sigma\mathrm{d}u$, *where we use* $b^s(\cos\theta)$ *to denote* $b^s(\theta)$ *with a little abuse of notation, then*

$$|\mathcal{Y}^{s,\gamma,\eta}(h,f)| \lesssim s^{-1}\eta^{\gamma}(|W_{\gamma/2}h|_{L^2_s}+|W_{\gamma/2}h|_{H^s})(|W_{\gamma/2}f|_{L^2_s}+|W_{\gamma/2}f|_{H^s}).$$

The following lemma deals with the quadratic term in the case $\gamma = 0$. Similarly to Lemma 2.2 in [28], we have

Lemma 2.6 *Let* $\mathcal{Z}^s(f) := \int b^s(\frac{u}{|u|}\cdot\sigma)|f(|u|\frac{u^+}{|u^+|})-f(u^+)|^2\mathrm{d}\sigma\mathrm{d}u$. *Then*

$$\mathcal{Z}^s(f) \lesssim s^{-1}(|W_s(D)f|^2_{L^2}+|W_sf|^2_{L^2}).$$

We now analyze the spherical part. In the following lemma, we recall a preliminary result on the characterization of norm $|\left(1-\Delta_{\mathbb{S}^2}\right)^{s/2} f|_{L^2(\mathbb{S}^2)}$. The proof of the lemma can be found in Lemma 5.5 of [26]. Note that we add a factor $s(1-s)$ for consideration of the limit $s \to 0^+$ or $s \to 1^-$.

Lemma 2.7 *Let f be a smooth function defined on $\mathbb{S}^2$. Then*

$$|f|^2_{L^2(\mathbb{S}^2)} + s(1-s)\iint \frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}} \mathrm{d}\sigma \mathrm{d}\tau \sim |\left(1-\Delta_{\mathbb{S}^2}\right)^{s/2} f|^2_{L^2(\mathbb{S}^2)}. \tag{2.47}$$

The constant associated to $\sim$ is independent of s.

By Lemma 2.7, it is direct to derive the following lemma. One can refer to Proposition 2.3 in [28].

Lemma 2.8 *Let $\mathcal{A}^s(f) := s\int b^s(\frac{u}{|u|}\cdot\sigma)|f(u)-f(|u|\frac{u^+}{|u^+|})|^2 \mathrm{d}\sigma \mathrm{d}u$ where $u^+ = \frac{u+|u|\sigma}{2}$, then*

$$\mathcal{A}^s(f) + |f|^2_{L^2} \sim |W_s((-\Delta_{\mathbb{S}^2})^{1/2}) f|^2_{L^2}. \tag{2.48}$$

Then similarly to Proposition 2.6 in [28], by the decomposition (2.46), using (2.45), then by Lemmas 2.5 and 2.8, we can derive the following upper bound of $Q^{s,\gamma,\eta}$.

Proposition 2.7 *It holds that*

$$|\langle Q^{s,\gamma,\eta}(g,h), f\rangle| \lesssim s^{-1}\eta^\gamma |g|_{L^1_{|\gamma|+2s}} |h|_{s,\gamma/2} |f|_{s,\gamma/2}.$$

2.3.2 Upper bound of $I^{s,\gamma,\eta}$

We now study $\langle I^{s,\gamma,\eta}(g,h;\beta), f\rangle$ where $I^{s,\gamma,\eta}(g,h;\beta)$ is defined by (2.7) with $B^{s,\gamma,\eta}$. To this end, we first derive a revised version of the cancellation lemma introduced in [1] for the kernel $B^{s,\gamma,\eta}$.

Lemma 2.9 [Revised cancellation lemma for relative velocity away from origin] *For $|a| \leq -\gamma$, we have*

$$\left|\int B^{s,\gamma,\eta}(|v-v_*|,\cos\theta) g_*(h'-h)\mathrm{d}V\right| \lesssim \eta^\gamma |g|_{L^1_a} |h|_{L^1_{-a}}. \tag{2.49}$$

Proof By the cancellation lemma in [1], recalling $B^{s,\gamma,\eta}(|v-v_*|,\cos\theta) = |v-v_*|^\gamma b^s(\theta)\psi^\eta(|v-v_*|)$ where $\psi^\eta = 1-\psi_\eta$, we have

$$\int B^{s,\gamma,\eta}(|v-v_*|,\cos\theta) g_*(h'-h)\mathrm{d}V = \int S^{s,\gamma,\eta}(v-v_*) g_* h \mathrm{d}v_* \mathrm{d}v, \tag{2.50}$$

where

$$S^{s,\gamma,\eta}(v-v_*) = \int |v-v_*|^\gamma b^s(\theta)\left(\cos^{-\gamma-3}\frac{\theta}{2}\psi^\eta(\frac{|v-v_*|}{\cos\frac{\theta}{2}}) - \psi^\eta(|v-v_*|)\right)\mathrm{d}\sigma. \quad (2.51)$$

By using (1.23) and the property of ψ^η, we get

$$|S^{s,\gamma,\eta}(v-v_*)| \lesssim 1_{|v-v_*|/\eta \geq 3/(4\sqrt{2})}|v-v_*|^\gamma.$$

Observe that

$$\begin{aligned} 1_{|v-v_*|/\eta\geq 3/(4\sqrt{2})}|v-v_*|^\gamma \lesssim\ & 1_{1\geq|v-v_*|\geq\eta\times 3/(4\sqrt{2})}\eta^\gamma\langle v_*\rangle^a\langle v\rangle^{-a} \\ & +1_{|v-v_*|\geq 1}\langle v_*\rangle^a\langle v\rangle^{-a}, \end{aligned}$$

which yields (2.49) and this completes the proof of the lemma. □

Recalling the decomposition (2.40), by Lemma 2.9 and Proposition 2.7, we have the following upper bound estimate on $\mathcal{N}^{s,\gamma,\eta}$ which is defined by

$$\mathcal{N}^{s,\gamma,\eta}(g,h) := \int B^{s,\gamma,\eta} g_*^2 (h'-h)^2 \mathrm{d}V. \quad (2.52)$$

Proposition 2.8 *It holds uniformly for* $1/4 \leq a \leq 1/2$ *that*

$$\mathcal{N}^{s,\gamma,\eta}(\mu^a, f) \lesssim s^{-1}\eta^\gamma |f|^2_{s,\gamma/2}.$$

The following lemma is about an integral over the sphere $\mathbb{S}^2$.

Lemma 2.10 *Recall* $b^s(\theta)$ *from (1.22). Denote* $A_s(\xi) := \int b^s(\theta)\min\{|\xi|^2\sin^2(\theta/2), 1\}\,\mathrm{d}\sigma$. *Then*

$$A_s(\xi)+1 \gtrsim \langle\xi\rangle^{2s}, \quad A_s(\xi) \lesssim 1_{|\xi|\leq\sqrt{2}}|\xi|^2 + 1_{|\xi|>\sqrt{2}}\max\{\frac{1-s}{s}, 1\}|\xi|^{2s} \lesssim s^{-1}\langle\xi\rangle^{2s}.$$

As Proposition 4.2 in [27] and Proposition 3.2 in [16], by applying Proposition 2.7 and Lemma 2.9, we can derive the upper bound of $\langle I^{s,\gamma,\eta}(g,h;\beta), f\rangle$ stated in the following proposition. Note that the factor s^{-1} comes from Lemma 2.10 and the factor η^γ comes from (2.45).

Proposition 2.9 *It holds that*

$$|\langle I^{s,\gamma,\eta}(g,h;\beta), f\rangle| \lesssim s^{-1}\eta^\gamma |g|_{L^2}|h|_{s,\gamma/2}|W_s f|_{L^2_{\gamma/2}}.$$

The constant associated to $\lesssim$ *may depend on* $|\beta|$ *but not on* s, γ.

2.3.3 Upper bound of $\Gamma^{s,\gamma,\eta}$

Similarly to (2.6),

$$\Gamma^{s,\gamma,\eta}(g,h;\beta) = Q^{s,\gamma,\eta}(g\partial_\beta\mu^{1/2},h) + I^{s,\gamma,\eta}(g,h;\beta). \tag{2.53}$$

By Propositions 2.7 and 2.9, we have the following estimate.

Proposition 2.10 *It holds that*

$$|\langle\Gamma^{s,\gamma,\eta}(g,h;\beta),f\rangle| \lesssim s^{-1}\eta^\gamma|g|_{L^2}|h|_{s,\gamma/2}|f|_{s,\gamma/2}.$$

2.3.4 Upper bound of $\mathcal{L}^{s,\gamma,\eta}$

Recall $\mathcal{L}^{s,\gamma,\eta}(f;\beta_0,\beta_1) = \mathcal{L}_1^{s,\gamma,\eta}(f;\beta_0,\beta_1) + \mathcal{L}_2^{s,\gamma,\eta}(f;\beta_0,\beta_1)$ where

$$\mathcal{L}_1^{s,\gamma,\eta}(f;\beta_0,\beta_1) := -\Gamma^{s,\gamma,\eta}(\partial_{\beta_1}\mu^{1/2},f;\beta_0),$$
$$\mathcal{L}_1^{s,\gamma,\eta}(f;\beta_0,\beta_1) := -\Gamma^{s,\gamma,\eta}(f,\partial_{\beta_1}\mu^{1/2};\beta_0).$$

If $|\beta_0| = |\beta_1| = 0$, the operators are reduced to

$$\mathcal{L}^{s,\gamma,\eta}f = \mathcal{L}^{s,\gamma,\eta}(f;0,0), \quad \mathcal{L}_1^{s,\gamma,\eta}f = \mathcal{L}_1^{s,\gamma,\eta}(f;0,0), \quad \mathcal{L}_2^{s,\gamma,\eta}f = \mathcal{L}_2^{s,\gamma,\eta}(f;0,0).$$

By using Proposition 2.10, we have the following estimates.

Proposition 2.11 *It holds that*

$$|\langle\mathcal{L}_1^{s,\gamma,\eta}(f;\beta_0,\beta_1),h\rangle| \lesssim s^{-1}\eta^\gamma|f|_{s,\gamma/2}|h|_{s,\gamma/2}.$$

For $\mathcal{L}_2^{s,\gamma,\eta}$, we have the following estimate.

Proposition 2.12 *It holds that*

$$|\langle\mathcal{L}_2^{s,\gamma,\eta}(f;\beta_0,\beta_1),h\rangle| \lesssim \eta^\gamma|\mu^{1/32}f|_{L^2}|\mu^{1/32}h|_{L^2}. \tag{2.54}$$

Proof Since the proof is similar to that of Proposition 2.5, we omit the details. Here, we do not have the factor s^{-1} because of the cancellation Lemma 2.9. □

Propositions 2.11 and 2.12 give the following estimate.

Proposition 2.13 *It holds that*

$$|\langle\mathcal{L}^{s,\gamma,\eta}(f;\beta_0,\beta_1),h\rangle| \lesssim s^{-1}\eta^\gamma|f|_{s,\gamma/2}|h|_{s,\gamma/2}.$$

2.4 Upper bounds of $\Gamma^{s,\gamma}$, $\mathcal{L}^{s,\gamma}$

We combine the the estimates in subsect. 2.2 and 2.3 to obtain several operator estimates for later use.

Propositions 2.3 and 2.10 with $\eta = 1$ give the following upper bound estimate on $\Gamma^{s,\gamma}$.

Theorem 2.1 *For* $0 < \delta \le \frac{1}{2}$, *let* $(a_1, a_2) = (\frac{3}{2} + \delta, s)$ *or* $(s, \frac{3}{2} + \delta)$. *Let* $C_{\delta,s,\gamma} = \delta^{-1/2} s^{-1} (\gamma + 2s + 3)^{-1}$. *Then*

$$|\langle \Gamma^{s,\gamma}(g, h; \beta), f\rangle| \lesssim_l C_{\delta,s,\gamma} |g|_{H^{a_1}_{-l}} |h|_{H^{a_2}_{-l}} |f|_{H^s_{-l}} + s^{-1} |g|_{L^2} |h|_{s,\gamma/2} |f|_{s,\gamma/2}.$$

Propositions 2.6 and 2.13 with $\eta = 1$ give the following upper bound estimate on $\mathcal{L}^{s,\gamma}$.

Theorem 2.2 *Set*

$$C_{s,\gamma} = s^{-1}(\gamma + 2s + 3)^{-1}. \tag{2.55}$$

It holds that

$$|\langle \mathcal{L}^{s,\gamma}(f; \beta_0, \beta_1), h\rangle| \lesssim C_{s,\gamma} |f|_{s,\gamma/2} |h|_{s,\gamma/2}.$$

The following result will be used in Sect. 4 to obtain dissipation estimate on the macroscopic component.

Proposition 2.14 *Let* P *be a polynomial function. For any combination* $a_1, a_2 \ge 0$ *satisfying the constraint* $a_1 + a_2 = s$, *it holds that*

$$|\langle \Gamma^{s,\gamma}(g, h), \mu^{\frac{1}{2}} P\rangle| \lesssim C_{s,\gamma} |\mu^{\frac{1}{4}} g|_{H^{a_1}} |\mu^{\frac{1}{4}} h|_{H^{a_2}} + s^{-1} |\mu^{\frac{1}{4}} g|_{L^2} |\mu^{\frac{1}{4}} h|_{L^2}. \tag{2.56}$$

Proof First note that

$$\begin{aligned}\langle \Gamma^{s,\gamma}(g, h), \mu^{\frac{1}{2}} P\rangle &= \langle Q^{s,\gamma}(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), P\rangle \\ &= \langle Q^{s,\gamma}_1(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), P\rangle + \langle Q^{s,\gamma,1}(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), P\rangle.\end{aligned}$$

Applying Proposition 2.1 with $a_3 = s + \frac{3}{2} + \delta$ and taking l_3 negatively small enough relative to the degree of P, we get

$$\left|\langle Q^{s,\gamma}_1(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), P\rangle\right| \lesssim C_{s,\gamma} |\mu^{\frac{1}{4}} g|_{H^{a_1}} |\mu^{\frac{1}{4}} h|_{H^{a_2}}.$$

By using (2.22) to $P' - P$ and (2.24), thanks to the factor $\mu^{\frac{1}{2}} \mu_*^{\frac{1}{2}}$, we have

$$\left|\langle Q^{s,\gamma,1}(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), P\rangle\right| \lesssim s^{-1} |\mu^{\frac{1}{4}} g|_{L^2} |\mu^{\frac{1}{4}} h|_{L^2}.$$

Combining the two parts completes the proof of the proposition. □

2.5 Commutator estimates and weighted estimates

Similarly to the commutator estimates in [28] and [16], we have the following commutator estimate for $\Gamma^{s,\gamma}$.

Proposition 2.15 *Let* $l, l_1 \geq 0$. *Let* $0 < \delta < 1/2$ *and* $(a_1, a_2) = (\frac{3}{2}+\delta, s)$ *or* $(s, \frac{3}{2}+\delta)$, *then*

$$\begin{aligned}&|\langle \Gamma^{s,\gamma}(g, W_l h; \beta) - W_l \Gamma^{s,\gamma}(g, h; \beta), f\rangle| \\ &\quad \lesssim_{l,l_1} C_{\delta,s,\gamma} |\mu^{1/64} g|_{H^{a_1}} |h|_{H^{a_2}_{-l_1}} |f|_{s,\gamma/2} + s^{-1} |g|_{L^2} |h|_{L^2_{l+\gamma/2}} |f|_{s,\gamma/2}.\end{aligned}$$

Theorem 2.1 and Proposition 2.15 together give the following weighted upper bound estimate.

Corollary 2.1 *Let* $l \geq 0$. *Let* $0 < \delta < 1/2$ *and* $(a_1, a_2) = (\frac{3}{2}+\delta, s)$ *or* $(s, \frac{3}{2}+\delta)$, *then*

$$\begin{aligned}|\langle \Gamma^{s,\gamma}(g, h; \beta), W_{2l} f\rangle| &\lesssim_l C_{\delta,s,\gamma} |g|_{H^{a_1}_{-5/2}} |h|_{H^{a_2}_{-5/2}} |f|_{s,l+\gamma/2} \\ &\quad + s^{-1} |g|_{L^2} |h|_{s,l+\gamma/2} |f|_{s,l+\gamma/2}.\end{aligned}$$

As an application of Proposition 2.15, we have the following corollary.

Corollary 2.2 *Let* $l, l_1 \geq 0$, *then*

$$\begin{aligned}|\langle W_l \mathcal{L}^{s,\gamma}(g; \beta_0, \beta_1) - \mathcal{L}^{s,\gamma}(W_l g; \beta_0, \beta_1), f\rangle| &\lesssim_{l,l_1} C_{s,\gamma} |g|_{H^s_{-l_1}} |f|_{s,\gamma/2} \\ &\quad + s^{-1} |g|_{L^2_{l+\gamma/2}} |f|_{s,\gamma/2}.\end{aligned}$$

In the special case $\beta_0 = \beta_1 = 0$ and $g = f$, thanks to some symmetry structure, similarly to Lemma 3.10 in [16], we have

$$|\langle [W_l, \mathcal{L}^{s,\gamma}] f, W_l f\rangle| \lesssim_l (\gamma + 5)^{-1} |f|^2_{L^2_{l+\gamma/2}}. \tag{2.57}$$

3 Coercivity estimate

In this section, we will prove coercivity estimate of the linear operator $\mathcal{L}^{s,\gamma,\eta}$ for some $\eta > 0$. This is a linear counterpart of the famous Boltzmann's H-theorem near Maxwellian. Unless otherwise specified, the parameter range is $-5 \leq \gamma \leq 0, 0 < s < 1$. The parameter γ actually can tend to $-\infty$ because we consider the regular domain $|v - v_*| \gtrsim \eta > 0$.

The proof contains two parts. One is a rough coercivity estimate capturing the norm $|\cdot|_{s,\gamma/2}$ with a lower order correction norm $|\cdot|_{L^2_{\gamma/2}}$. The other is a spectrum-gap type estimate to recover the lower order norm $|\cdot|_{L^2_{\gamma/2}}$. Accordingly, we divide this section into two subsections.

3.1 Rough coercivity estimate

In this subsection, we will prove the rough coercivity estimate of $\mathcal{L}^{s,\gamma,\eta}$ for small $\eta > 0$ in Theorem 3.1. The strategy relies on the following relation (see the proof of Theorem 3.1):

$$\langle \mathcal{L}^{s,\gamma,\eta} f, f\rangle + \eta^{\gamma}|f|^2_{L^2_{\gamma/2}} \gtrsim \mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f) + \mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2}), \tag{3.1}$$

where the functional $\mathcal{N}^{s,\gamma,\eta}$ is defined by (2.52). If $\eta = 0$, then $\psi^{\eta} = 1$ and we write $\mathcal{N}^{s,\gamma} = \mathcal{N}^{s,\gamma,0}$. If $\gamma = \eta = 0$, we write $\mathcal{N}^{s} = \mathcal{N}^{s,0,0}$ for simplicity. Thanks to (3.1), to obtain the coercivity estimate of $\mathcal{L}^{s,\gamma,\eta}$, it suffices to estimate from below the two functionals $\mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f)$ and $\mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2})$.

3.1.1 Gain of weight from $\mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2})$

The functional $\mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2})$ produces weight W_s in the phase space.

Proposition 3.1 *Let* $0 \le \eta \le 1$*, then*

$$\mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2}) + |f|^2_{L^2_{\gamma/2}} \ge C|f|^2_{L^2_{\gamma/2+s}},$$

where $C > 0$ *is a generic constant.*

Proof Let $0 < \delta \le 1$. We consider the set $A(\delta) := \{(v_*, v, \sigma) : |v| \le 2, |v_*| \ge 4, \sin\frac{\theta}{2} \le \delta|v_*|^{-1}\}$. Since $|v - v_*| \ge 2 \ge 4/3$ in the set $A(\delta)$, we get

$$\mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2}) \ge \int B^{s,\gamma} 1_{A(\delta)} f_*^2((\mu^{1/2})' - \mu^{1/2})^2 \mathrm{d}V. \tag{3.2}$$

Note that $\nabla\mu^{1/2} = -\frac{\mu^{1/2}}{2}v$ and $\nabla^2\mu^{1/2} = \frac{\mu^{1/2}}{4}(-2I_3 + v \otimes v)$. By Taylor expansion (2.22), using the basic inequality $(a-b)^2 \ge a^2/2 - b^2$, we have

$$\left(\mu^{1/2}(v') - \mu^{1/2}(v)\right)^2 \ge \frac{\mu(v)}{8}|v\cdot(v'-v)|^2 - \int_0^1 \left|(\nabla^2\mu^{1/2})(v(\kappa))\right|^2\left|v'-v\right|^4 \mathrm{d}\kappa.$$

Plugging this into (3.2), we get

$$\begin{aligned}\mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2}) \ge\ & \frac{1}{8}\int B^{s,\gamma} 1_{A(\delta)} f_*^2 \mu(v)|v\cdot(v'-v)|^2 \mathrm{d}V \\ & - \int B^{s,\gamma} 1_{A(\delta)} f_*^2 |(\nabla^2\mu^{1/2})(v(\kappa))|^2 |v'-v|^4 \mathrm{d}V\mathrm{d}\kappa \\ :=\ & \frac{1}{8}\mathcal{I}_1^{s,\gamma}(\delta) - \mathcal{I}_2^{s,\gamma}(\delta).\end{aligned} \tag{3.3}$$

To estimate $\mathcal{I}_1^{s,\gamma}(\delta)$, for fixed v, v_*, choose an orthonormal basis $(h^1_{v,v_*}, h^2_{v,v_*}, \frac{v-v_*}{|v-v_*|})$ such that $\mathrm{d}\sigma = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$. Then one has

$$\frac{v'-v}{|v'-v|} = \cos\frac{\theta}{2}\cos\phi h^1_{v,v_*} + \cos\frac{\theta}{2}\sin\phi h^2_{v,v_*} - \sin\frac{\theta}{2}\frac{v-v_*}{|v-v_*|},$$

and

$$\frac{v}{|v|} = c_1 h^1_{v,v_*} + c_2 h^2_{v,v_*} + c_3\frac{v-v_*}{|v-v_*|},$$

where $c_3 = \frac{v}{|v|}\cdot\frac{v-v_*}{|v-v_*|}$ and c_1, c_2 are constants independent of θ and ϕ. Then we have

$$\frac{v}{|v|}\cdot\frac{v'-v}{|v'-v|} = c_1\cos\frac{\theta}{2}\cos\phi + c_2\cos\frac{\theta}{2}\sin\phi - c_3\sin\frac{\theta}{2}.$$

Thus

$$\begin{aligned}|\frac{v}{|v|}\cdot\frac{v'-v}{|v'-v|}|^2 &= c_1^2\cos^2\frac{\theta}{2}\cos^2\phi + c_2^2\cos^2\frac{\theta}{2}\sin^2\phi + c_3^2\sin^2\frac{\theta}{2}\\ &+2c_1c_2\cos^2\frac{\theta}{2}\cos\phi\sin\phi - 2c_3\cos\frac{\theta}{2}\sin\frac{\theta}{2}(c_1\cos\phi + c_2\sin\phi).\end{aligned}$$

Since $|v'-v| = |v-v_*|\sin\frac{\theta}{2}$ and $\cos^2\frac{\theta}{2} \geq \frac{1}{2}$, by integrating with respect to σ and recalling $b^s(\theta) = (1-s)\sin^{-2-2s}\frac{\theta}{2}1_{0\leq\theta\leq\pi/2}$, we have

$$\begin{aligned}\int b^s(\theta)1_{A(\delta)}|v\cdot(v'-v)|^2\mathrm{d}\sigma &= 4\int_0^\pi\int_0^{2\pi} b^s(\theta)\sin\frac{\theta}{2}1_{A(\delta)}|v\cdot(v'-v)|^2\mathrm{d}\phi\mathrm{d}\sin\frac{\theta}{2}\\ &\geq 2\pi(c_1^2+c_2^2)|v|^2|v-v_*|^2\int_0^\pi b^s(\theta)\sin^3\frac{\theta}{2}1_{A(\delta)}\mathrm{d}\sin\frac{\theta}{2}\\ &\gtrsim \delta^{2-2s}(c_1^2+c_2^2)|v_*|^{2s-2}|v|^2|v-v_*|^2 1_{B(\delta)},\end{aligned}$$

where $B(\delta) = \{(v_*, v) : |v_*| \geq 4, |v| \leq 2\}$. Note that

$$(c_1^2+c_2^2)|v-v_*|^2 = (1-c_3^2)|v-v_*|^2 = \left(1-\left(\frac{v}{|v|}\cdot\frac{v_*}{|v_*|}\right)^2\right)|v_*|^2$$

gives

$$\int b^s(\theta)1_{A(\delta)}|v\cdot(v'-v)|^2\mathrm{d}\sigma \gtrsim \delta^{2-2s}\left(1-\left(\frac{v}{|v|}\cdot\frac{v_*}{|v_*|}\right)^2\right)|v_*|^{2s}|v|^2 1_{B(\delta)}.$$

Plugging this estimate in the definition of $\mathcal{I}_1^{s,\gamma}(\delta)$, we get

$$\mathcal{I}_1^{s,\gamma}(\delta) \gtrsim \int \delta^{2-2s}\left(1-\left(\frac{v}{|v|}\cdot\frac{v_*}{|v_*|}\right)^2\right)|v_*|^{2s}|v|^2 1_{B(\delta)}|v-v_*|^\gamma\mu(v)f_*^2\mathrm{d}v\mathrm{d}v_*.$$

Note that in the region $B(\delta)$, one has

$$\frac{1}{2}|v_*| \leq |v - v_*| \leq \frac{3}{2}|v_*|. \tag{3.4}$$

We then obtain

$$\begin{aligned}
\mathcal{I}_1^{s,\gamma}(\delta) &\gtrsim \delta^{2-2s} \int \left(1 - \left(\frac{v}{|v|} \cdot \frac{v_*}{|v_*|}\right)^2\right) |v_*|^{\gamma+2s} |v|^2 1_{B(\delta)} \mu(v) f_*^2 \mathrm{d}v \mathrm{d}v_* \\
&\gtrsim \delta^{2-2s} \int |v_*|^{\gamma+2s} 1_{|v_*|\geq 4} f_*^2 \mathrm{d}v_*,
\end{aligned}$$

where we have used the fact that $\int \left(1 - \left(\frac{v}{|v|} \cdot \frac{v_*}{|v_*|}\right)^2\right) |v|^2 \mu(v) 1_{|v|\leq 2} \mathrm{d}v > 0$ which is independent of v_*.

We now turn to estimate $\mathcal{I}_2^{s,\gamma}(\delta)$. By (3.4) and $|v(\kappa) - v_*| \leq |v - v_*|$, we have

$$1_{A(\delta)} \leq 1_{|v_*|\geq 4} 1_{\sin \frac{\theta}{2} \leq \frac{3}{2}\delta |v - v_*|^{-1}}.$$

Recalling $B^{s,\gamma} = |v - v_*|^\gamma b^s(\theta)$, $|v' - v| = |v - v_*| \sin \frac{\theta}{2}$, by (3.4), and by using $|\nabla^2 \mu^{1/2}| \lesssim \mu^{1/4}$ and the change of variable $v \to v(\kappa)$ in Lemma 2.2, we have

$$\begin{aligned}
\mathcal{I}_2^{s,\gamma}(\delta) &= \int |v - v_*|^\gamma b^s(\theta) 1_{A(\delta)} |(\nabla^2 \mu^{1/2})(v(\kappa))|^2 |v' - v|^4 f_*^2 \mathrm{d}V \mathrm{d}\kappa \\
&\lesssim \int b^s(\theta) \sin^4 \frac{\theta}{2} 1_{|v_*|\geq 4} 1_{\sin \frac{\theta}{2} \leq \frac{3}{2}\delta |v - v_*|^{-1}} |v - v_*|^{4-2s} \\
&\quad \mu^{1/2}(v(\kappa)) |v_*|^{\gamma+2s} f_*^2 \mathrm{d}V \mathrm{d}\kappa \\
&= \int b^s(\theta) \sin^4 \frac{\theta}{2} 1_{|v_*|\geq 4} 1_{\sin \frac{\theta}{2} \leq \frac{3}{2}\delta |v - v_*|^{-1} \psi_\kappa^{-1}(\theta)} |v - v_*|^{4-2s} \psi_\kappa^{7-2s}(\theta) \\
&\quad \mu^{1/2}(v) |v_*|^{\gamma+2s} f_*^2 \, \mathrm{d}V \mathrm{d}\kappa \\
&\lesssim \delta^{4-2s} \int 1_{|v_*|\geq 4} \mu^{1/2}(v) |v_*|^{\gamma+2s} f_*^2 \mathrm{d}v \mathrm{d}v_* \lesssim \delta^{4-2s} \left(\int 1_{|v_*|\geq 4} |v_*|^{\gamma+2s} f_*^2 \mathrm{d}v_*\right).
\end{aligned}$$

Combining the estimate on $\mathcal{I}_1^{s,\gamma}(\delta)$ and $\mathcal{I}_2^{s,\gamma}(\delta)$ gives

$$\mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2}) \geq \left(C_1 - C_2\delta^2\right) \delta^{2-2s} \int 1_{|v_*|\geq 4} |v_*|^{\gamma+2s} f_*^2 \mathrm{d}v_*,$$

for some generic constants $C_1, C_2 > 0$. By choosing δ such that $C_2\delta^2 = C_1/2$, and observing $|v_*|^{\gamma+2s} \sim \langle v_* \rangle^{\gamma+2s}$ for $|v_*| \geq 4$, we get

$$\begin{aligned}
\mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2}) &\gtrsim \int 1_{|v_*|\geq 4} \langle v_* \rangle^{\gamma+2s} f_*^2 \mathrm{d}v_* \\
&= \left(|f|_{L^2_{\gamma/2+s}} - \int 1_{|v_*|<4} \langle v_* \rangle^{\gamma+2s} f_*^2 \mathrm{d}v_*\right).
\end{aligned} \tag{3.5}$$

If $|v_*| \leq 4$, then $\langle v_*\rangle^{\gamma+2s} \sim \langle v_*\rangle^{\gamma}$. Then the proof of the proposition is completed. □

In the following, we focus on gain of regularity from $\mathcal{N}^{s,\gamma,\eta}(\mu^{\frac{1}{2}}, f)$. The strategy can be stated as follows.

(1) Gain of regularity from $\mathcal{N}^{s}(g, f)$;
(2) Gain of regularity from $\mathcal{N}^{s,0,\eta}(g, f)$ by reducing to $\mathcal{N}^{s}(g, f)$;
(3) Gain of regularity from $\mathcal{N}^{s,\gamma,\eta}(g, f)$ by reducing to $\mathcal{N}^{s,0,\eta}(g, f)$.

3.1.2 Gain of regularity from $\mathcal{N}^{s}(g, f)$.

We derive Sobolev regularity from $\mathcal{N}^{s}(g, f)$ by the following result obtained in [1]. For $g \geq 0$ with $|g|_{L^1} \geq \delta > 0$ and $|g|_{L^1_1} \leq \lambda < \infty$, there exists a constant $C(\delta, \lambda) > 0$ such that

$$\int b(\cos\theta)g_*(f' - f)^2 \mathrm{d}V + |f|^2_{L^2} \geq C(\delta, \lambda)|a(D)f|^2_{L^2}, \tag{3.6}$$

where $a(\xi) := \int b(\frac{\xi}{|\xi|} \cdot \sigma) \min\{|\xi|^2 \sin^2(\theta/2), 1\}\mathrm{d}\sigma + 1$. By applying (3.6) to the angular function b^s and using Lemma 2.10, we have the following lemma.

Lemma 3.1 *Let g be a function such that $|g|_{L^2} \geq \delta > 0$, $|g|_{L^2_{1/2}} \leq \lambda < \infty$, then there is a constant $C(\delta, \lambda) > 0$ such that*

$$\mathcal{N}^{s}(g, f) + |f|^2_{L^2} \geq C(\delta, \lambda)|f|^2_{H^s}.$$

We now extract the anisotropic norm $|W_s((-\Delta_{\mathbb{S}^2})^{1/2})f|^2_{L^2_{\gamma/2}}$ from $\mathcal{N}^{s}(g, f)$ by Bobylev's formula and the upper bound of the radial part.

Lemma 3.2 *It holds that*

$$\mathcal{N}^{s}(g, f) + |g|^2_{L^2_s}(|W_s(D)f|^2_{L^2} + |W_s f|^2_{L^2}) \gtrsim |g|^2_{L^2}|W_s((-\Delta_{\mathbb{S}^2})^{1/2})f|^2_{L^2}.$$

Proof By Bobylev's formula, we have

$$\begin{aligned}\mathcal{N}^{s}(g, f) &= \frac{1}{(2\pi)^3}\int b^s\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\Big(\widehat{g^2}(0)|\hat{f}(\xi) - \hat{f}(\xi^+)|^2 \\ &\quad +2\Re((\widehat{g^2}(0) - \widehat{g^2}(\xi^-))\hat{f}(\xi^+)\bar{\hat{f}}(\xi)\Big)\mathrm{d}\sigma\mathrm{d}\xi \\ &:= \frac{|g|^2_{L^2}}{(2\pi)^3}\mathcal{I}_1 + \frac{2}{(2\pi)^3}\mathcal{I}_2,\end{aligned}$$

where $\xi^+ = \frac{\xi+|\xi|\sigma}{2}$ and $\xi^- = \frac{\xi-|\xi|\sigma}{2}$. Note that $\widehat{g^2}(0) - \widehat{g^2}(\xi^-) = \int(1 - \cos(v\cdot\xi^-))g^2(v)\mathrm{d}v$ and $1-\cos(v\cdot\xi^-) \lesssim \min\{|v|^2|\xi|^2|\frac{\xi}{|\xi|}-\sigma|^2, 1\} \sim \min\{|v|^2|\xi^+|^2|\frac{\xi^+}{|\xi^+|}-$

$\sigma|^2, 1\}$. By the Cauchy–Schwarz inequality and the change of variable $\xi \to \xi^+$, using Lemma 2.10 and the fact that $W_s(|v||\xi|) \lesssim W_s(|v|)W_s(|\xi|)$, we have

$$|\mathcal{I}_2| \lesssim s^{-1} \int (W_s)^2(|v||\xi|)|\hat{f}(\xi)|^2 g^2(v)\mathrm{d}v\mathrm{d}\xi \lesssim s^{-1}|W_s g|^2_{L^2}|W_s(D)f|^2_{L^2}. \quad (3.7)$$

Now we turn to estimate $\mathcal{I}_1$. By the geometric decomposition

$$\hat{f}(\xi) - \hat{f}(\xi^+) = \hat{f}(\xi) - \hat{f}\left(|\xi|\frac{\xi^+}{|\xi^+|}\right) + \hat{f}\left(|\xi|\frac{\xi^+}{|\xi^+|}\right) - \hat{f}(\xi^+), \quad (3.8)$$

and using $(a+b)^2 \geq a^2/2 - b^2$, we have $\mathcal{I}_1 \geq \frac{1}{2}\mathcal{I}_{1,1} - \mathcal{I}_{1,2}$ where

$$\mathcal{I}_{1,1} := \int b^s\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left|\hat{f}(\xi) - \hat{f}\left(|\xi|\frac{\xi^+}{|\xi^+|}\right)\right|^2 \mathrm{d}\sigma\mathrm{d}\xi,$$
$$\mathcal{I}_{1,2} := \int b^s\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left|\hat{f}\left(|\xi|\frac{\xi^+}{|\xi^+|}\right) - \hat{f}(\xi^+)\right|^2 \mathrm{d}\sigma\mathrm{d}\xi.$$

By Lemma 2.8,

$$\mathcal{I}_{1,1} + s^{-1}|f|^2_{L^2} \sim s^{-1}|W_s((-\Delta_{\mathbb{S}^2})^{1/2})f|^2_{L^2}. \quad (3.9)$$

By Lemma 2.6,

$$\mathcal{I}_{1,2} \lesssim s^{-1}\Big(|W_s(D)\hat{f}|^2_{L^2} + |W_s\hat{f}|^2_{L^2}\Big) = s^{-1}\Big(|W_s(D)f|^2_{L^2} + |W_s f|^2_{L^2}\Big). \quad (3.10)$$

Combining (3.7), (3.9) and (3.10) completes the proof. □

3.1.3 Gain of regularity from $\mathcal{N}^{s,0,\eta}(g, f)$

We first introduce some notations. Recall $\psi_R(v) := \psi(v/R)$. Let $\psi_{r,u}(v) := \psi_r(v-u)$ and $\phi_{R,r,u} := \psi_{14R} - \psi_{4r,u}$ for some $r, R > 0$ and $u \in \mathbb{R}^3$. The following lemma bounds $\mathcal{N}^{s,0,\eta}(g, f)$ by $\mathcal{N}^s(g, f)$ from below provided the distance between supports of g and f is suitably large.

Lemma 3.3 *For $0 \leq \eta \leq 1 \leq R$, we have*

$$\mathcal{N}^{s,0,\eta}(g, f) + |g|^2_{L^2}|f|^2_{L^2} \gtrsim \mathcal{N}^s(\psi_R g, (1-\psi_{4R})f). \quad (3.11)$$

For $0 \leq \eta \leq r \leq 1 \leq R$, $u \in B_{6R}$, we have

$$\mathcal{N}^{s,0,\eta}(g, f) + r^{-2}R^2|g|^2_{L^2}|f|^2_{L^2} \gtrsim \mathcal{N}^s(\phi_{R,r,u}g, \psi_{r,u}f). \quad (3.12)$$

Proof We proceed in the spirit of [3, 25]. Note that ψ_R is supported in $|v| \leq \frac{4}{3}R$ and equals to 1 in $|v| \leq \frac{3}{4}R$. $1-\psi_R$ is supported in $|v| \geq \frac{3}{4}R$ and equals to 1 in $|v| \geq \frac{4}{3}R$. If $|v_*| \leq \frac{4}{3}R$ and $|v| \geq 3R$, then $|v-v_*| \geq \frac{5}{3}R \geq \frac{4}{3}\eta$, which gives $\psi_R(v_*)(1-\psi_{4R}(v)) \leq 1_{|v-v_*|\geq 4\eta/3}$. Hence,

$$\begin{aligned}\mathcal{N}^{s,0,\eta}(g,f) &\geq \int b^s(\theta)1_{|v-v_*|\geq 4\eta/3}g_*^2(f'-f)^2\mathrm{d}V\\ &\geq \int b^s(\theta)(\psi_R g)_*^2(f'-f)^2(1-\psi_{4R})^2\mathrm{d}V\\ &\geq \frac{1}{2}\int b^s(\theta)(\psi_R g)_*^2(((1-\psi_{4R})f)'-(1-\psi_{4R})f)^2\mathrm{d}V\\ &\quad -\int b^s(\theta)(\psi_R g)_*^2(f')^2(\psi'_{4R}-\psi_{4R})^2\mathrm{d}V := \frac{1}{2}\mathcal{I}_1-\mathcal{I}_2.\end{aligned}$$

We now estimate $\mathcal{I}_2$. Since $|\nabla\psi_{4R}|_{L^\infty} \lesssim R^{-1}|\nabla\psi|_{L^\infty} \lesssim R^{-1}$, we get $(\psi_R)_*^2(\psi'_{4R}-\psi_{4R})^2 \lesssim R^{-2}|v'-v|^2 = R^{-2}|v-v_*|^2\sin^2(\theta/2)$. If $|v_*| \leq \frac{4}{3}R \leq 2R$, $|v| \geq 20R$, $0 \leq \theta \leq \pi/2$, we have

$$|v'-v_*| = \cos(\theta/2)|v-v_*| \geq \cos(\theta/2)(|v|-|v_*|) \geq 9\sqrt{2}R.$$

Then we have $|v'| \geq |v'-v_*|-|v_*| \geq 9\sqrt{2}R-2R \geq 6R$, which gives $\psi_{4R}(v') = 0 = \psi_{4R}(v)$. That is, $(\psi_R)_*^2(\psi'_{4R}-\psi_{4R})^2$ is supported in $|v| \leq 20R, |v_*| \leq 2R$. Thus, we have

$$(\psi_R)_*^2(\psi'_{4R}-\psi_{4R})^2 \leq 1_{|v|\leq 20R,|v_*|\leq 2R}R^{-2}|v-v_*|^2\sin^2(\theta/2) \lesssim \sin^2(\theta/2).$$

By the change of variable $v \to v'$ and using (1.23), we get

$$\mathcal{I}_2 \lesssim \int g_*^2 f^2\mathrm{d}v\mathrm{d}v_* \lesssim |g|_{L^2}^2|f|_{L^2}^2.$$

This together with the fact that $\mathcal{I}_1 = \mathcal{N}^s(\psi_R g,(1-\psi_{4R})f)$ give (3.11).

If $v \in \mathrm{supp}\psi_{r,u}$, $v_* \in \mathrm{supp}\phi_{R,r,u}$, we claim $|v-v_*| \geq r \geq \eta$. In fact, if $v \in \mathrm{supp}\psi_{r,u}$, then $|v-u| \leq \frac{4}{3}r$. If $|v_*-u| \leq 3r = \frac{3}{4}\times 4r$, then $\psi_{4r,u}(v_*) = 1$. Moreover, $|v_*| \leq |u|+|u-v_*| \leq 6R+3r \leq 9R \leq \frac{3}{4}\times 14R$, then $\psi_{14R}(v_*) = 1$. As a result, $\phi_{R,r,u}(v_*) = 0$. From this if $v_* \in \mathrm{supp}\phi_{R,r,u}$, then $|v_*-u| \geq 3r$. Therefore, if $v \in \mathrm{supp}\psi_{r,u}$, $v_* \in \mathrm{supp}\phi_{R,r,u}$, then $|v-v_*| \geq |v_*-u|-|v-u| \geq \frac{5}{3}r \geq \frac{4}{3}\eta$. Thus,

$$\begin{aligned}\mathcal{N}^{s,0,\eta}(g,f) &= \int b^s(\theta)1_{|v-v_*|\geq 4\eta/3}g_*^2(f'-f)^2\mathrm{d}V\\ &\geq \int b^s(\theta)(\phi_{R,r,u}g)_*^2(f'-f)^2\psi_{r,u}^2\mathrm{d}V\\ &\geq \frac{1}{2}\int b^s(\theta)(\phi_{R,r,u}g)_*^2((\psi_{r,u}f)'-\psi_{r,u}f)^2\mathrm{d}V\end{aligned}$$

$$-\int b^s(\theta)(\phi_{R,r,u}g)_*^2(f')^2(\psi'_{r,u}-\psi_{r,u})^2\mathrm{d}V := \frac{1}{2}\mathcal{J}_1 - \mathcal{J}_2.$$

We now estimate $\mathcal{J}_2$. Since $|\nabla\psi_{r,u}(v)| \lesssim r^{-1}|\nabla\psi|_{L^\infty}1_{3r/4\le|v-u|\le 4r/3}$, by Taylor expansion, we get

$$|\psi'_{r,u}-\psi_{r,u}|^2 = \left|\int_0^1 \nabla\psi_{r,u}(v(\kappa))\cdot(v'-v)\mathrm{d}\kappa\right|^2 \lesssim r^{-2}|v-v_*|^2\sin^2(\theta/2) \int_0^1 1_{3r/4\le|v(\kappa)-u|\le 4r/3}\mathrm{d}\kappa.$$

For $u\in B_{6R}$, $|v_*|\le 20R$, $3r/4\le|v(\kappa)-u|\le 4r/3$, we have

$$|v-v_*| \le \sqrt{2}|v(\kappa)-v_*| \le \sqrt{2}|v(\kappa)-u| + \sqrt{2}|u-v_*| \le 4\sqrt{2}r/3+\sqrt{2}(6R+20R) \le 28\sqrt{2}R,$$

and

$$(\phi_{R,r,u})_*^2(\psi'_{r,u}-\psi_{r,u})^2 \lesssim r^{-2}R^2\sin^2(\theta/2).$$

By the change of variable $v\to v'$ and (1.23), we get

$$\mathcal{J}_2 \lesssim r^{-2}R^2\int g_*^2 f^2\mathrm{d}v\mathrm{d}v_* \lesssim r^{-2}R^2|g|_{L^2}^2|f|_{L^2}^2.$$

This together with the fact that $\mathcal{J}_1 = \mathcal{N}^s(\phi_{R,r,u}g, \psi_{r,u}f)$ give (3.12). □

3.1.4 Gain of regularity from $\mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f)$

We first establish a relation between $\mathcal{N}^{s,\gamma,\eta}$ and $\mathcal{N}^{s,0,\eta}$.

Lemma 3.4 *For $\gamma\le 0\le\eta$, one has*

$$\mathcal{N}^{s,\gamma,\eta}(g,f) + |g|^2_{L^2_{|\gamma/2|+1}}|f|^2_{L^2_{\gamma/2}} \gtrsim \mathcal{N}^{s,0,\eta}(W_{\gamma/2}g, W_{\gamma/2}f).$$

Proof Set $F = W_{\gamma/2}f$. If $\gamma\le 0$, then $|v-v_*|^\gamma \ge \langle v-v_*\rangle^\gamma$ and thus

$$\mathcal{N}^{s,\gamma,\eta}(g,f) \ge \int b^s(\theta)\psi^\eta(|v-v_*|)\langle v-v_*\rangle^\gamma g_*^2((W_{-\gamma/2}F)'-W_{-\gamma/2}F)^2\mathrm{d}V.$$

We apply the following decomposition

$$(W_{-\gamma/2}F)'-W_{-\gamma/2}F = (W_{-\gamma/2})'(F'-F) + F(W'_{-\gamma/2}-W_{-\gamma/2}).$$

Using $(a+b)^2 \geq a^2/2 - b^2$, we get $\mathcal{N}^{s,\gamma,\eta}(g, f) \geq \frac{1}{2}\mathcal{I}_1 - \mathcal{I}_2$ where

$$\mathcal{I}_1 := \int b^s(\theta)\psi^\eta(|v-v_*|)\langle v-v_*\rangle^\gamma g_*^2 W'_{-\gamma}(F'-F)^2 \mathrm{d}V,$$
$$\mathcal{I}_2 := \int b^s(\theta)\psi^\eta(|v-v_*|)\langle v-v_*\rangle^\gamma g_*^2 F^2 (W'_{-\gamma/2} - W_{-\gamma/2})^2 \mathrm{d}V.$$

Since $\langle v_*\rangle^\gamma \lesssim \langle v_* - v'\rangle^\gamma \langle v'\rangle^{-\gamma} \sim \langle v_* - v\rangle^\gamma \langle v'\rangle^{-\gamma}$, we get $\mathcal{I}_1 \gtrsim \mathcal{N}^{s,0,\eta}(W_{\gamma/2}g, W_{\gamma/2}f)$. Taylor expansion implies that

$$(W'_{-\gamma/2} - W_{-\gamma/2})^2 \lesssim |v-v_*|^2 \sin^2(\theta/2) \int \langle v(\kappa)\rangle^{-\gamma-2} \mathrm{d}\kappa.$$

Note that

$$\langle v-v_*\rangle^\gamma |v-v_*|^2 \langle v(\kappa)\rangle^{-\gamma-2} \lesssim \langle v(\kappa) - v_*\rangle^{\gamma+2} \langle v(\kappa)\rangle^{-\gamma-2} \lesssim \langle v_*\rangle^{|\gamma|+2}.$$

By the above estimate and (1.23), we get

$$\mathcal{I}_2 \lesssim \int g_*^2 \langle v_*\rangle^{|\gamma|+2} F^2 \mathrm{d}v\mathrm{d}v_* \lesssim |g|^2_{L^2_{|\gamma/2|+1}} |F|^2_{L^2}.$$

Combining the estimates on $\mathcal{I}_1$ and $\mathcal{I}_2$ completes the proof of the lemma. □

We are now ready to derive gain of regularity from $\mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f)$.

Proposition 3.2 *For* $-5 \leq \gamma \leq 0 \leq \eta \leq \eta_1 := \frac{9}{64}(2\pi)^{1/2}(\frac{1}{4}W_{-5}(3/4)\mu(3/4))^{1/3}$, *it holds that*

$$\mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f) + |W_s W_{\gamma/2} f|^2_{L^2} \gtrsim |W_s((-\Delta_{\mathbb{S}^2})^{1/2}) W_{\gamma/2} f|^2_{L^2} + |W_s(D) W_{\gamma/2} f|^2_{L^2}.$$

Proof Taking $g = \mu^{1/2}$ in Lemma 3.4, we have

$$\begin{aligned}\mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f) + |f|^2_{L^2_{\gamma/2}} &\gtrsim \mathcal{N}^{s,0,\eta}(W_{\gamma/2}\mu^{1/2}, W_{\gamma/2} f) \\ &\geq \mathcal{N}^{s,0,\eta}(W_{-5/2}\mu^{1/2}, W_{\gamma/2} f).\end{aligned}$$

Taking $g = W_{-5/2}\mu^{1/2}$, $f = F := W_{\gamma/2} f$ in Lemma 3.3, we have for $\eta \leq r \leq 1 \leq R$, $u \in B_{6R}$ that

$$\mathcal{N}^{s,0,\eta}(W_{-5/2}\mu^{1/2}, F) + |F|^2_{L^2} \gtrsim \mathcal{N}^s(\psi_R W_{-5/2}\mu^{1/2}, (1-\psi_{4R})F), \quad (3.13)$$
$$\mathcal{N}^{s,0,\eta}(W_{-5/2}\mu^{1/2}, F) + r^{-2}R^2|F|^2_{L^2} \gtrsim \mathcal{N}^s(\phi_{R,r,u} W_{-5/2}\mu^{1/2}, \psi_{r,u} F). \quad (3.14)$$

From now on, take $R = 1$, then $\psi_R(v) = \psi(v) = 1$ for $|v| \leq 3/4$, we get

$$|\psi_R W_{-5/2}\mu^{1/2}|^2_{L^2} = \int \psi^2 W_{-5}\mu \mathrm{d}v \geq \frac{4\pi}{3}(3/4)^3 W_{-5}(3/4)\mu(3/4) := \delta_*^2.$$

Recalling $\phi_{R,r,u} = \psi_{14R} - \psi_{4r,u}$ and $\psi_{14R} \geq \psi_R$, we have

$$\int \phi^2_{R,r,u} W_{-5}\mu \mathrm{d}v \geq \frac{1}{2}\int \psi^2_{14R} W_{-5}\mu \mathrm{d}v - \int \psi^2_{4r,u} W_{-5}\mu \mathrm{d}v$$
$$\geq \frac{1}{2}\delta_*^2 - \int \psi^2_{4r,u} W_{-5}\mu \mathrm{d}v.$$

Note that $\int \psi^2_{4r,u} W_{-5}\mu \mathrm{d}v \leq \frac{4\pi}{3}(\frac{16}{3}r)^3(2\pi)^{-\frac{3}{2}} := Cr^3$. By choosing r such that $Cr^3 = \frac{1}{4}\delta_*^2$, we get

$$|\phi_{R,r,u} W_{-5/2}\mu^{1/2}|^2_{L^2} \geq \delta_*^2/4.$$

Note that $r = \frac{9}{64}(2\pi)^{1/2}(\frac{1}{4}W_{-5}(3/4)\mu(3/4))^{1/3}$. Therefore, we have

$$\min\{|\phi_{R,r,u} W_{-5/2}\mu^{1/2}|_{L^2}, |\psi_R W_{-5/2}\mu^{1/2}|_{L^2}\} \geq \delta_*/2. \tag{3.15}$$

On the other hand, note that

$$\max\{|\psi_R W_{-5/2}\mu^{1/2}|_{L^2_1}, |\phi_{R,r,u} W_{-5/2}\mu^{1/2}|_{L^2_1}\} \leq |\mu|^{1/2}_{L^1} = 1. \tag{3.16}$$

Thanks to (3.15) and (3.16), by Lemma 3.1, as δ_* is a generic constant, we get

$$\mathcal{N}^s(\psi_R W_{-3/2}\mu^{1/2}, (1-\psi_{4R})F) + |(1-\psi_{4R})F|^2_{L^2} \gtrsim |(1-\psi_{4R})F|^2_{H^s}, \tag{3.17}$$
$$\mathcal{N}^s(\phi_{R,r,u} W_{-3/2}\mu^{1/2}, \psi_{r,u}F) + |\psi_{r,u}F|^2_{L^2} \gtrsim |\psi_{r,u}F|^2_{H^s}. \tag{3.18}$$

Note that $1-\psi_{4R}(v) = 1$ if $|v| \geq 6R \geq \frac{16}{3}R$. There is a finite cover of B_{6R} with open ball $B_r(u_j)$ for $u_j \in B_{6R}$. More precisely, there exists $\{u_j\}_{j=1}^N \subset B_{6R}$ such that $B_{6R} \subset \cup_{j=1}^N B_r(u_j)$, where $N \sim \frac{1}{r^3}$ is a generic constant. We then have $\psi_{4R} \leq \sum_{j=1}^N \psi_{r,u_j}$ and thus $|\psi_{4R}F|^2_{H^s} \leq N\sum_{j=1}^N |\psi_{r,u_j}F|^2_{H^s}$. From this together with (3.13), (3.14), (3.17), (3.18), since r is a generic constant, we get for any $0 \leq \eta \leq r$,

$$\mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f) + |f|^2_{L^2_{\gamma/2}} \gtrsim r^8|W_{\gamma/2}f|^2_{H^s} \gtrsim |W_s(D)W_{\gamma/2}f|^2_{L^2}. \tag{3.19}$$

Thanks to (3.15) and (3.16), by Lemma 3.2, we get

$$\mathcal{N}^s\big(\psi_R W_{-3/2}\mu^{1/2}, (1-\psi_{4R})F\big) + \big(|(1-\psi_{4R})F|^2_{H^s} + |(1-\psi_{4R})F|^2_{L^2_s}\big)$$
$$\gtrsim \big|W_s((-\Delta_{\mathbb{S}^2})^{1/2})(1-\psi_{4R})F\big|^2_{L^2},$$
$$\mathcal{N}^s\big(\phi_{R,r,u} W_{-3/2}\mu^{1/2}, \psi_{r,u}F\big) + \big(|\psi_{r,u}F|^2_{H^s} + |\psi_{r,u}F|^2_{L^2_s}\big)$$
$$\gtrsim \big|W_s((-\Delta_{\mathbb{S}^2})^{1/2})\psi_{r,u}f\big|^2_{L^2}.$$

By applying the same finite cover argument, we also have

$$\mathcal{N}^{s,\gamma,\eta}\big(\mu^{1/2}, f\big) + \big|W_s(D)W_{\gamma/2}f\big|^2_{L^2} + \big|W_s W_{\gamma/2}f\big|^2_{L^2} \gtrsim \big|W_s((-\Delta_{\mathbb{S}^2})^{1/2})W_{\gamma/2}f\big|^2_{L^2}. \tag{3.20}$$

Finally a suitable combination of (3.19) and (3.20) completes the proof. □

3.1.5 Rough coercivity estimate of $\langle \mathcal{L}^{s,\gamma,\eta} f, f\rangle$

By Propositions 3.1 and 3.2, we have the following estimate.

Proposition 3.3 *Let $-5 \le \gamma \le 0 \le \eta \le \eta_1$ where η_1 is the constant in Proposition 3.2. Then*

$$\mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f) + \mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f) + |f|^2_{L^2_{\gamma/2}} \gtrsim |f|^2_{s,\gamma/2}.$$

Now we are ready to prove the following rough coercivity estimate.

Theorem 3.1 *Let $-5 \le \gamma \le 0 < \eta \le \eta_1$ where η_1 is the constant in Proposition 3.2. Then*

$$\langle \mathcal{L}^{s,\gamma,\eta} f, f\rangle + \eta^\gamma |f|^2_{L^2_{\gamma/2}} \gtrsim |f|^2_{s,\gamma/2}. \tag{3.21}$$

Proof We recall that $\mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f) + \mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2})$ corresponds to the anisotropic norm $|||\cdot|||$ introduced in [4]. By the proof of Proposition 2.16 in [4],

$$\langle \mathcal{L}_1^{s,\gamma,\eta} f, f\rangle \ge \frac{1}{10}(\mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f) + \mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2})) - \frac{3}{10}\left|\int B^{s,\gamma,\eta}\mu_*((f^2)' - f^2)\mathrm{d}V\right|.$$

By the cancellation Lemma 2.9, we have

$$\left|\int B^{s,\gamma,\eta}\mu_*((f^2)' - f^2)\mathrm{d}V\right| \lesssim \eta^\gamma |f|^2_{L^2_{\gamma/2}}.$$

Therefore, we have

$$\langle \mathcal{L}_1^{s,\gamma,\eta} f, f\rangle \ge \frac{1}{10}\Big(\mathcal{N}^{s,\gamma,\eta}(\mu^{1/2}, f) + \mathcal{N}^{s,\gamma,\eta}(f, \mu^{1/2})\Big) - C\eta^\gamma |f|^2_{L^2_{\gamma/2}}. \tag{3.22}$$

By Proposition 2.12, we get

$$|\langle \mathcal{L}_2^{s,\gamma,\eta} f, f\rangle| \lesssim \eta^\gamma |\mu^{1/8} f|^2_{L^2} \lesssim \eta^\gamma |f|^2_{L^2_{\gamma/2}}. \tag{3.23}$$

Note that (3.22) and (3.23) imply (3.1).

Then by applying Proposition 3.3, we finish the proof. □

3.2 Spectrum-gap type estimate

In this subsection, we consider the coercivity estimates of $\mathcal{L}^{s,\gamma,\eta}$ in the microscopic space. This is also referred to as the "spectral gap" estimate.

Recall $\ker(\mathcal{L}_B^{s,\gamma}) = \ker(\mathcal{L}_L^{\gamma}) = \mathrm{span}\{\sqrt{\mu}, \sqrt{\mu}v_1, \sqrt{\mu}v_2, \sqrt{\mu}v_3, \sqrt{\mu}|v|^2\} := \ker$. An orthonormal basis of ker can be chosen as

$$\{\sqrt{\mu}, \sqrt{\mu}v_1, \sqrt{\mu}v_2, \sqrt{\mu}v_3, \sqrt{\mu}(|v|^2-3)/\sqrt{6}\} := \{e_j\}_{1\le j\le 5}.$$

The projection operator $\mathbf{P}$ on the kernel space is defined as follows:

$$\mathbf{P}f := \sum_{j=1}^{5}\langle f, e_j\rangle e_j = (a + b\cdot v + c|v|^2)\sqrt{\mu}, \tag{3.24}$$

where for $1 \le i \le 3$,

$$a = \int\left(\frac{5}{2} - \frac{|v|^2}{2}\right)\sqrt{\mu}f\mathrm{d}v; \quad b_i = \int v_i\sqrt{\mu}f\mathrm{d}v;$$
$$c = \int\left(\frac{|v|^2}{6} - \frac{1}{2}\right)\sqrt{\mu}f\mathrm{d}v. \tag{3.25}$$

We will show that the lower order term $|f|^2_{L^2_{\gamma/2}}$ in (3.21) can be dropped for $f \in \ker^{\perp}$.

The idea is to firstly consider the special case $\gamma = 0$ and then to use mathematical induction for the general case $\gamma < 0$.

3.2.1 The case $\gamma = 0$

This case is clear, cf. the explicit spectrum computation by Wang-Chang [39], in which the authors showed that the smallest positive eigenvalue is bounded from below by $\int b(\cos\theta)\sin^2\frac{\theta}{2}\mathrm{d}\sigma$ up to a multiplicative factor. Recalling (1.23), it holds for $f \in \ker^{\perp}$ that

$$\langle \mathcal{L}^{s,0,0}f, f\rangle \ge \lambda_1|f|^2_{L^2}.$$

By the proof of Theorem 3.1 for the case of $\gamma = 0$, we can also take $\eta = 0$ to get

$$\langle \mathcal{L}^{s,0,0}f, f\rangle + |f|^2_{L^2} \gtrsim |f|^2_{s,0}.$$

Hence, there exists a generic constant $\lambda_2 > 0$ such that

$$\langle \mathcal{L}^{s,0,0}f, f\rangle \ge \lambda_2|f|^2_{s,0}. \tag{3.26}$$

We now show that $\mathcal{L}^{s,0,\eta}$ also satisfies the above estimate if η is small enough. For this, we prove smallness of $\langle \mathcal{L}^{s,0}_{\eta}f, f\rangle$ when η is small.

Lemma 3.5 *Let $0 \le \eta \le 1$, then it holds for $f \in \ker^{\perp}$ that*

$$\langle \mathcal{L}_{\eta}^{s,0} f, f\rangle \lesssim \eta^3 |f|_{H^s}^2.$$

Proof Firstly, note that $\langle \mathcal{L}_{\eta}^{s,0} f, f\rangle \le 2\mathcal{N}_{\eta}^{s}(\mu^{1/2}, f) + 2\mathcal{N}_{\eta}^{s}(f, \mu^{1/2})$, where the functional $\mathcal{N}_{\eta}^{s,\gamma}$ is defined in (2.39). By (2.41), we have

$$\mathcal{N}_{\eta}^{s}(\mu^{1/2}, f) \lesssim \eta^{2s+3} |f|_{H^s}^2.$$

Recall

$$\mathcal{N}_{\eta}^{s}(f, \mu^{1/2}) = \int b^s(\theta)\psi_{\eta}(|v - v_*|) f_*^2((\mu^{1/2})' - \mu^{1/2})^2 \mathrm{d}V.$$

By using $|(\mu^{1/2})' - \mu^{1/2}| \lesssim |v - v_*|\theta$ and (1.23), we have

$$\begin{aligned}\mathcal{N}_{\eta}^{s}(f, \mu^{1/2}) &\lesssim \int b^s(\theta)\psi_{\eta}(|v - v_*|) f_*^2 |v - v_*|^2 \theta^2 \mathrm{d}V \\ &\lesssim \int 1_{|v-v_*|\le 4\eta/3} f_*^2 |v - v_*|^2 \mathrm{d}v\mathrm{d}v_* \lesssim \eta^5 |f|_{L^2}^2.\end{aligned}$$

Combining the above estimates completes the proof.

□

From (3.26), by taking η small enough in Lemma 3.5, we get the following coercivity estimate.

Lemma 3.6 *There is a generic constant $\eta_2 > 0$ such that for $f \in \ker^{\perp}$, it holds that*

$$\langle \mathcal{L}^{s,0,\eta_2} f, f\rangle \ge \lambda_0 |f|_{s,0}^2.$$

3.2.2 The general case $\gamma < 0$

The coercivity estimate of $\mathcal{L}^{s,\gamma,\eta}$ in the space $\ker^{\perp}$ for $\gamma < 0$ can be stated as follows.

Theorem 3.2 *For $-5 \le \gamma \le 0$, with the constant η_1 defined in Proposition 3.2 and the constants η_2, λ_0 defined in Lemma 3.6, let $\eta = \min\{\eta_1, \eta_2\}$. There is a generic constant $0 < c < 1$ such that for any $\gamma \in [-5, 0]$ satisfying $-ks \le \gamma < -(k-1)s$ for some integer $k \ge 0$, it holds for $f \in \ker^{\perp}$ that*

$$\langle \mathcal{L}^{s,\gamma,\eta} f, f\rangle \ge c^{2^k - 1} s^{2^k - 1} \lambda_0^{2^k} |f|_{s,\gamma/2}^2. \tag{3.27}$$

Remark 3.1 Theorem 3.2 indeed holds for any $\gamma \le 0$ even though we only need it for $-5 \le \gamma \le 0$.

The analysis can also be applied to the case when $\gamma > 0$.

For later use, set $\lambda_{s,\gamma} := c^{2^k-1}s^{2^k-1}\lambda_0^{2^k}$. The following remark is about the lower bound of $\lambda_{s,\gamma}$.

Remark 3.2 For $\gamma > -2s-3$, as $-\gamma/s \le 2+3/s$, then

$$\lambda_{s,\gamma} \ge c^{2^{\lceil 2+3/s\rceil}-1}s^{2^{\lceil 2+3/s\rceil}-1}\lambda_0^{2^{\lceil 2+3/s\rceil}} := \lambda_s. \tag{3.28}$$

Here $\lceil a\rceil$ is the smallest integer no less than a. Note that λ_s is non-decreasing with respect to s.

Motivated by [4, 27] about exchanging the kinetic component in the cross-section with a weight of velocity on the function, we can apply an induction argument based on the estimate for the case $\gamma = 0$ obtained in Lemma 3.6 and the gain of moment of order s. As the first step, we reduce the case $-s \le \gamma < 0$ to the special case $\gamma = 0$, and then by induction to cover the whole range $-5 \le \gamma \le 0$. To this end, we need a scaled weight function

$$U_\delta(v) := W(\delta v) = (1+\delta^2|v|^2)^{1/2} \ge \max\{\delta|v|, 1\}. \tag{3.29}$$

Here δ is a sufficiently small parameter to be chosen later. We now give two technical lemmas on some integrals involving U_δ.

Lemma 3.7 *Let $-5 \le \alpha, \beta < 0 < s, \delta < 1$. Set $X(\beta,\delta) := \delta^{-\beta}\left((U_\delta^{\beta/2})'(U_\delta^{\beta/2})'_* - U_\delta^{\beta/2}(U_\delta^{\beta/2})_*\right)^2$. Recall (1.22) for $b^s(\theta)$. Then for $v \in \mathbb{R}^3$,*

$$\int b^s(\theta)|v-v_*|^\alpha\psi^\eta(|v-v_*|)X(\beta,\delta)\mu_*\mathrm{d}\sigma\mathrm{d}v_* \lesssim s^{-1}\delta^{2s}\eta^\alpha\langle v\rangle^{\alpha+\beta+2s}.$$

Proof First, it is straightforward to check

$$|v-v_*|^\alpha\psi^\eta(|v-v_*|)\mu_* \lesssim \eta^\alpha\langle v-v_*\rangle^\alpha\mu_* \lesssim \eta^\alpha\langle v\rangle^\alpha\langle v_*\rangle^{|\alpha|}\mu_* \lesssim \eta^\alpha\langle v\rangle^\alpha\mu_*^{\frac{1}{2}}.$$

Note that

$$\begin{aligned} X(\beta,\delta) \lesssim\ & \delta^{-\beta}(U_\delta^\beta)'_*\left((U_\delta^{\beta/2})' - (U_\delta^{\beta/2})\right)^2 \\ & +\delta^{-\beta}U_\delta^\beta\left((U_\delta^{\beta/2})'_* - (U_\delta^{\beta/2})_*\right)^2 := A_1 + A_2. \end{aligned}$$

We only estimate A_1 because A_2 can be estimated similarly.

For $a \le 0$, one has

$$|\nabla U_\delta^a| \lesssim |a|\delta U_\delta^a, \tag{3.30}$$

which gives

$$\left((U_\delta^a)' - (U_\delta^a)\right)^2 = \left|\int_0^1 (\nabla U_\delta^a)(v(\kappa))\cdot(v'-v)\mathrm{d}\kappa\right|^2$$

$$\lesssim a^2\delta^2 \sin^2\frac{\theta}{2}|v-v_*|^2\int_0^1 U_\delta^{2a}(v(\kappa))\mathrm{d}\kappa.$$

Thanks to $|v_*'|^2+|v(\kappa)|^2 \sim |v|^2+|v_*|^2$, we have

$$\delta^{-2a}(U_\delta^{2a})_*' U_\delta^{2a}(v(\kappa)) \lesssim \langle v\rangle^{2a}, \tag{3.31}$$

which gives

$$\delta^{-2a}(U_\delta^{2a})_*'\big((U_\delta^a)'-(U_\delta^a)\big)^2 \lesssim a^2\delta^2\sin^2\frac{\theta}{2}|v-v_*|^2\langle v\rangle^{2a}. \tag{3.32}$$

Divide the integral $\int b^s(\theta)|v-v_*|^\alpha A_1\mu_*\mathrm{d}\sigma\mathrm{d}v_*$ into two parts: $\mathcal{I}_\leq$ and $\mathcal{I}_\geq$ corresponding to $\delta|v-v_*|\leq 1$ and $\delta|v-v_*|\geq 1$. When $\delta|v-v_*|\leq 1$, using (3.32) with $a=\beta/2$, we have

$$\mathcal{I}_\leq \lesssim \eta^\alpha\delta^2\langle v\rangle^{\alpha+\beta}\int 1_{|v-v_*|\leq\delta^{-1}}b^s(\theta)\sin^2\frac{\theta}{2}|v-v_*|^2\mu_*^{\frac{1}{2}}\mathrm{d}\sigma\mathrm{d}v_* \lesssim \eta^\alpha\delta^{2s}\langle v\rangle^{\alpha+\beta+2s}.$$

For $\delta|v-v_*|\geq 1$, we further divide the integral $\mathcal{I}_\geq$ into two parts: $\mathcal{I}_{\geq,\leq}$ and $\mathcal{I}_{\geq,\geq}$ corresponding to $\sin\frac{\theta}{2}\leq\delta^{-1}|v-v_*|^{-1}$ and $\sin\frac{\theta}{2}\geq\delta^{-1}|v-v_*|^{-1}$ respectively. By using (3.32) with $a=\beta/2$, we have

$$\begin{aligned}\mathcal{I}_{\geq,\leq} &\lesssim \eta^\alpha\delta^2\langle v\rangle^{\alpha+\beta}\int 1_{\sin\frac{\theta}{2}\leq\delta^{-1}|v-v_*|^{-1}}b^s(\theta)\sin^2\frac{\theta}{2}|v-v_*|^2\mu_*^{\frac{1}{2}}\mathrm{d}\sigma\mathrm{d}v_*\\ &\lesssim \eta^\alpha\delta^{2s}\langle v\rangle^{\alpha+\beta+2s}.\end{aligned}$$

For the remainder with $\sin\frac{\theta}{2}\geq\delta^{-1}|v-v_*|^{-1}$, it holds from (3.31) that $A_1\lesssim\langle v\rangle^\beta$ and

$$\mathcal{I}_{\geq,\geq} \lesssim \eta^\alpha\langle v\rangle^{\alpha+\beta}\int 1_{\sin\frac{\theta}{2}\geq\delta^{-1}|v-v_*|^{-1}}b^s(\theta)\mu_*^{\frac{1}{2}}\mathrm{d}\sigma\mathrm{d}v_* \lesssim s^{-1}\eta^\alpha\delta^{2s}\langle v\rangle^{\alpha+\beta+2s}.$$

Combining the above estimates completes the proof of the lemma. □

Lemma 3.8 *Let* $-5\leq\alpha,\beta<0<s,\eta,\delta<1$. *Set* $\varphi_{\beta,\delta}:=(1-U_\delta^{\beta/2})\mu^{\frac{1}{2}}$, *then*

$$\mathcal{I}:=\int b^s(\theta)|v-v_*|^\alpha\psi^\eta(|v-v_*|)(\varphi_{\beta,\delta}'-\varphi_{\beta,\delta})^2\mathrm{d}\sigma\mathrm{d}v \lesssim s^{-1}\delta^2\eta^\alpha\langle v_*\rangle^{\alpha+2s}.$$

Proof By (3.30), we get

$$|\varphi_{\beta,\delta}|\lesssim|\beta|\delta\mu^{\frac{1}{4}}, \quad |\nabla\varphi_{\beta,\delta}|\lesssim|\beta|\delta\mu^{\frac{1}{4}}. \tag{3.33}$$

From this, we first have

$$(\varphi_{\beta,\delta}'-\varphi_{\beta,\delta})^2 \lesssim \delta^2((\mu^{1/2})'+\mu^{1/2}). \tag{3.34}$$

By 1st-order Taylor expansion, we get

$$(\varphi'_{\beta,\delta} - \varphi_{\beta,\delta})^2 \lesssim \delta^2 \sin^2\frac{\theta}{2}|v - v_*|^2 \int_0^1 \mu^{1/2}(v(\kappa))\mathrm{d}\kappa. \tag{3.35}$$

Combing these two estimates gives

$$(\varphi'_{\beta,\delta} - \varphi_{\beta,\delta})^2 \lesssim \delta^2 \min\left\{1, \sin^2\frac{\theta}{2}|v - v_*|^2\right\} \int_0^1 ((\mu^{1/2})' + \mu^{1/2} + \mu^{1/2}(v(\kappa)))\mathrm{d}\kappa. \tag{3.36}$$

By Lemma 2.2 and Lemma 2.10, we get

$$\begin{aligned}\mathcal{I} &\lesssim \delta^2 \int b^s(\theta)|v - v_*|^\alpha \psi^\eta(|v - v_*|) \min\left\{1, \sin^2\frac{\theta}{2}|v - v_*|^2\right\} \mu^{\frac{1}{2}} \mathrm{d}\sigma \mathrm{d}v \\ &\lesssim s^{-1}\delta^2\eta^\alpha \int \langle v - v_*\rangle^{\alpha+2s} \mu^{\frac{1}{2}} \mathrm{d}v \lesssim s^{-1}\delta^2\eta^\alpha \langle v_*\rangle^{\alpha+2s},\end{aligned}$$

which ends the proof of the lemma. □

We are now ready to prove the coercivity estimate of $\mathcal{L}^{s,\gamma,\eta}$ for $-5 \le \gamma \le 0$ by induction.

Proof of Theorem 3.2 In the proof, $0 < \eta = \min\{\eta_1, \eta_2\} < 1$ is a fixed constant. Hence, $1 \le \eta^\gamma \le \eta^{-5}$ for $-5 \le \gamma \le 0$. For brevity, set

$$J^{s,\gamma,\eta}(f) := 4\langle \mathcal{L}^{s,\gamma,\eta} f, f\rangle, \quad \mathbb{A}(g, h) := (g_* h + g h_* - g'_* h' - g' h'_*),$$
$$\mathbb{F}(g, h) := \mathbb{A}^2(g, h).$$

With these notations, we have $J^{s,\gamma,\eta}(f) = \int B^{s,\gamma,\eta}\mathbb{F}(\mu^{1/2}, f)\mathrm{d}V$. We divide the proof into five steps.

Step 1: Localization of $\mathcal{J}^{s,\gamma,\eta}(f)$. By (3.29) and if $a \le 0$, we get

$$\begin{aligned}|v - v_*|^{-a} &\le \max\{1, 2^{-a-1}\}\delta^a((\delta|v|)^{-a} + (\delta|v_*|)^{-a}) \\ &\le 2\max\{1, 2^{-a-1}\}\delta^a U_\delta^{-a}(v) U_\delta^{-a}(v_*),\end{aligned}$$

which gives

$$|v - v_*|^a \ge C_a \delta^{-a} U_\delta^a(v) U_\delta^a(v_*),$$

where $C_a = \frac{1}{2}\min\{1, 2^{a+1}\}$. With $\gamma = \alpha + \beta$, $\gamma \le \alpha$, $\beta \le 0$, we have

$$\mathcal{J}^{s,\gamma,\eta}(f) \ge C_\beta \delta^{-\beta} \int B^{s,\alpha,\eta} U_\delta^\beta (U_\delta^\beta)_* \mathbb{F}(\mu^{1/2}, f)\mathrm{d}V.$$

By setting $h = U_\delta^{\beta/2}$, $\phi = \mu^{\frac{1}{2}}$ and putting the weight function $U_\delta^\beta (U_\delta^\beta)_*$ inside $\mathbb{F}(\cdot, \cdot)$, we have

$$\begin{aligned}
&U_\delta^\beta (U_\delta^\beta)_* \mathbb{F}(\mu^{1/2}, f) \\
&= h_*^2 h^2 \mathbb{F}(\phi, f) = \left(hh_* (\phi_* f + \phi f_*) - hh_* \left(\phi'_* f' + \phi' f'_*\right)\right)^2 \\
&= \left(hh_* (\phi_* f + \phi f_*) - h'h'_* \left(\phi'_* f' + \phi' f'_*\right) + \left(h'h'_* - hh_*\right)\left(\phi'_* f' + \phi' f'_*\right)\right)^2 \\
&\geq \frac{1}{2}\left(hh_* (\phi_* f + \phi f_*) - h'h'_* \left(\phi'_* f' + \phi' f'_*\right)\right)^2 - \left(h'h'_* - hh_*\right)^2 \left(\phi'_* f' + \phi' f'_*\right)^2 \\
&= \frac{1}{2}\mathbb{F}(h\phi, hf) - \left(h'h'_* - hh_*\right)^2 \left(\phi'_* f' + \phi' f'_*\right)^2 .
\end{aligned} \tag{3.37}$$

Thus,

$$\begin{aligned}
\mathcal{J}^{s,\gamma,\eta}(f) \geq \frac{1}{2} C_\beta \delta^{-\beta} \int B^{s,\alpha,\eta} \mathbb{F}(U_\delta^{\beta/2}\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f) \mathrm{d}V \\
- C_\beta \delta^{-\beta} \int B^{s,\alpha,\eta} \left(h'h'_* - hh_*\right)^2 \left(\phi'_* f' + \phi' f'_*\right)^2 \mathrm{d}V.
\end{aligned} \tag{3.38}$$

We further reduce
$\mathbb{F}(U_\delta^{\beta/2}\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f)$ to $\mathbb{F}(\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f)$ with some correction terms. That is,

$$\begin{aligned}
\mathbb{F}(U_\delta^{\beta/2}\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f) &= \mathbb{A}^2(U_\delta^{\beta/2}\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f) \\
&= \Big(\mathbb{A}(\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f) - \mathbb{A}\big((1 - U_\delta^{\beta/2})\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f\big)\Big)^2 \\
&\geq \frac{1}{2}\mathbb{A}^2(\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f) - \mathbb{A}^2\big((1 - U_\delta^{\beta/2})\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f\big) \\
&= \frac{1}{2}\mathbb{F}(\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f) - \mathbb{F}\big((1 - U_\delta^{\beta/2})\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f\big).
\end{aligned} \tag{3.39}$$

By symmetry and recalling $\phi = \mu^{\frac{1}{2}}$,

$$\int B^{s,\alpha,\eta} \left(h'h'_* - hh_*\right)^2 \left(\phi'_* f' + \phi' f'_*\right)^2 \mathrm{d}V \leq 4 \int B^{s,\alpha,\eta} \left(h'h'_* - hh_*\right)^2 \mu_* f^2 \mathrm{d}V. \tag{3.40}$$

By (3.38), (3.39) and (3.40), we get

$$\begin{aligned}
\mathcal{J}^{s,\gamma,\eta}(f) \;\geq\; & \frac{1}{4} C_\beta \delta^{-\beta} \int B^{s,\alpha,\eta} \mathbb{F}\Big(\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f\Big) \mathrm{d}V \\
& - \frac{1}{2} C_\beta \delta^{-\beta} \int B^{s,\alpha,\eta} \mathbb{F}\Big((1 - U_\delta^{\beta/2})\mu^{\frac{1}{2}}, U_\delta^{\beta/2} f\Big) \mathrm{d}V \\
& - 4 C_\beta \delta^{-\beta} \int B^{s,\alpha,\eta} \left(h'h'_* - hh_*\right)^2 \mu_* f^2 \mathrm{d}V \\
:= & \frac{1}{4} C_\beta J_1^{\alpha,\beta} - \frac{1}{2} C_\beta J_2^{\alpha,\beta} - 4 C_\beta J_3^{\alpha,\beta}.
\end{aligned}$$

We always choose β in the range $-1 \le -s \le \beta \le 0$ and so $C_\beta = \frac{1}{2}$. Noting that $J_1^{\alpha,\beta} = \delta^{-\beta}\mathcal{J}^{s,\alpha,\eta}(U_\delta^{\beta/2} f)$, we have

$$\mathcal{J}^{s,\gamma,\eta}(f) \ge \frac{1}{8}\delta^s \mathcal{J}^{s,\alpha,\eta}(U_\delta^{\beta/2} f) - \frac{1}{4}J_2^{\alpha,\beta} - 2J_3^{\alpha,\beta}. \tag{3.41}$$

Step 2: Upper bound of $J_2^{\alpha,\beta}$. For simplicity of notation, set $\varphi_{\beta,\delta} = (1 - U_\delta^{\beta/2})\mu^{\frac{1}{2}}$, $\psi_{\beta,\delta} = U_\delta^{\beta/2} f$. Then

$$J_2^{\alpha,\beta} = \delta^{-\beta}\int B^{s,\alpha,\eta}\mathbb{F}(\varphi_{\beta,\delta}, \psi_{\beta,\delta})\mathrm{d}V \lesssim \delta^{-\beta}\mathcal{N}^{s,\alpha,\eta}(\varphi_{\beta,\delta}, \psi_{\beta,\delta}) + \delta^{-\beta}\mathcal{N}^{s,\alpha,\eta}(\psi_{\beta,\delta}, \varphi_{\beta,\delta}). \tag{3.42}$$

By (3.30), for $-5/2 \le a \le 0$,

$$0 \le 1 - U_\delta^a(v) = U_\delta^a(0) - U_\delta^a(v) \lesssim \delta|v|. \tag{3.43}$$

By (3.43), we have

$$(\varphi_{\beta,\delta}^2)_* = \big((1 - U_\delta^{\beta/2})\mu^{\frac{1}{2}}\big)_*^2 \lesssim \delta^2\mu_*^{\frac{1}{2}}. \tag{3.44}$$

From this and Proposition 2.8, by using the fact that $\delta^{-\beta/2}U_\delta^{\beta/2} \in S_{1,0}^{\beta/2}$ is a radial symbol of order $\beta/2$, we obtain

$$\delta^{-\beta}\mathcal{N}^{s,\alpha,\eta}(\varphi_{\beta,\delta}, \psi_{\beta,\delta}) \lesssim \delta^2\delta^{-\beta}\mathcal{N}^{s,\alpha,\eta}(\mu^{1/4}, \psi_{\beta,\delta}) \lesssim s^{-1}\delta^2\delta^{-\beta}|U_\delta^{\beta/2} f|_{s,\alpha/2}^2 \lesssim s^{-1}\delta^2|f|_{s,\gamma/2}^2. \tag{3.45}$$

By Lemma 3.8, we then have

$$\delta^{-\beta}\mathcal{N}^{s,\alpha,\eta}(\psi_{\beta,\delta}, \varphi_{\beta,\delta}) \lesssim s^{-1}\delta^2\eta^\alpha\delta^{-\beta}\int (U_\delta^{\beta/2} f)_*^2\langle v_*\rangle^{\alpha+2s}\mathrm{d}v_* \lesssim s^{-1}\delta^2|W_{\gamma/2+s} f|_{L^2}^2 \lesssim s^{-1}\delta^2|f|_{s,\gamma/2}^2. \tag{3.46}$$

Putting the estimates (3.45) and (3.46) into (3.42), we get

$$J_2^{\alpha,\beta} \lesssim s^{-1}\delta^2|f|_{s,\gamma/2}^2. \tag{3.47}$$

Step 3: Upper bound of $J_3^{\alpha,\beta}$. Lemma 3.7 yields

$$J_3^{\alpha,\beta} = \int B^{s,\alpha,\eta}X(\beta,\delta)\mu_* f^2\mathrm{d}V \lesssim s^{-1}\delta^{2s}|f|_{L^2_{\gamma/2+s}}^2. \tag{3.48}$$

Step 4: The case $-s \le \gamma < 0$. We take $\alpha = 0$, $\beta = \gamma$. Recall $\mathcal{J}^{s,\alpha,\eta}(U_\delta^{\beta/2} f) = 4\langle \mathcal{L}^{s,\alpha,\eta} U_\delta^{\gamma/2} f, U_\delta^{\gamma/2} f\rangle$. By Lemma 3.6, we have

$$\mathcal{J}^{s,0,\eta}(U_\delta^{\beta/2} f) \ge 4\lambda_0 |(\mathbf{I}-\mathbf{P})U_\delta^{\gamma/2} f|_{s,0}^2 \ge 4\lambda_0 |(\mathbf{I}-\mathbf{P})U_\delta^{\gamma/2} f|_{L^2_s}^2. \tag{3.49}$$

We claim that there exists $\delta_1 > 0$ such that if $0 < \delta \le \delta_1$, then for any $-5/2 \le a \le 0$,

$$|(\mathbf{I}-\mathbf{P})U_\delta^a f|_{L^2_s}^2 \ge \frac{1}{4}|f|_{L^2_{s+a}}^2. \tag{3.50}$$

This yields

$$\mathcal{J}^{s,0,\eta}(U_\delta^{\beta/2} f) \ge \lambda_0 |f|_{L^2_{s+\gamma/2}}^2. \tag{3.51}$$

We now prove (3.50).
Note that

$$|(\mathbf{I}-\mathbf{P})(U_\delta^a f)|_{L^2_s}^2 \ge \frac{1}{2}|U_\delta^a f|_{L^2_s}^2 - |\mathbf{P}(U_\delta^a f)|_{L^2_s}^2.$$

Since $\delta \le 1$ and $a \le 0$, $U_\delta^a \ge W_a$. Hence,

$$|U_\delta^a f|_{L^2_s}^2 \ge |f|_{L^2_{s+a}}^2. \tag{3.52}$$

We now estimate $|\mathbf{P}(U_\delta^a f)|_{L^2}$ for $f \in \ker^\perp$.
Since

$$\mathbf{P}(U_\delta^a f) = \sum_{i=1}^5 e_i \int e_i U_\delta^a f \mathrm{d}v = \sum_{i=1}^5 e_i \int e_i (U_\delta^a - 1) f \mathrm{d}v,$$

then

$$\left| \int e_i (U_\delta^a - 1) f \mathrm{d}v \right| \lesssim \delta |\mu^{\frac{1}{8}} f|_{L^2}.$$

Therefore,

$$|\mathbf{P}(U_\delta^a f)|_{L^2_s}^2 \lesssim \delta^2 |\mu^{\frac{1}{8}} f|_{L^2}^2 \lesssim \delta^2 |f|_{L^2_{s+a}}^2. \tag{3.53}$$

By (3.52) and (3.53), choosing δ_1 small enough, we obtain (3.50).

By plugging the estimates (3.51), (3.47), (3.48) into (3.41), for any $-s \le \gamma < 0$ and $0 < \delta \le \delta_1$, for some universal constants $0 < C_1$ and $1 \le C_2$, we have

$$\mathcal{J}^{s,\gamma,\eta}(f) \ge C_1 \delta^s |f|_{L^2_{\gamma/2+s}}^2 - C_2 s^{-1} \delta^{2s} |f|_{s,\gamma/2}^2. \tag{3.54}$$

It is straightforward to check from above that $C_1 = \lambda_0/8$. Recalling Theorem 3.1, for some universal constants $0 < C_3 \le 1 \le C_4$, we have

$$\mathcal{J}^{s,\gamma,\eta}(f) \ge C_3 |f|^2_{s,\gamma/2} - C_4 |f|^2_{L^2_{\gamma/2+s}}. \tag{3.55}$$

We can assume $\frac{C_1 C_3}{2C_4 C_2 s^{-1}} \le \delta_1^s$ by taking a larger C_4 if necessary.

The combination (3.55) $\times C_5 \delta^{2s}$+(3.54) gives

$$(1 + C_5\delta^{2s})\mathcal{J}^{s,\gamma,\eta}(f) \ge (C_1 - C_4C_5\delta^s)\delta^s |f|^2_{L^2_{\gamma/2+s}} \\ +(C_3C_5 - C_2 s^{-1})\delta^{2s}|f|^2_{s,\gamma/2}. \tag{3.56}$$

We can then take C_5 large enough such that $C_3C_5 - C_2 s^{-1} \ge C_2 s^{-1}$, for example $C_5 = 2C_2 s^{-1}/C_3 \ge 2$. And then we choose δ small enough such that $C_1 - C_4C_5\delta^s \ge 0$, for example $\delta^s = \frac{C_1}{C_4C_5} = \frac{\lambda_0 C_3}{16 C_4 C_2 s^{-1}} \le \delta_1^s$. Note that we can assume $C_5\delta^{2s} = \frac{C_1^2 C_3}{2C_4^2 C_2 s^{-1}} \le \delta_1^s \frac{C_1}{C_4} \le 1$. Otherwise, we can take an even larger C_4. Thus, we get

$$\mathcal{J}^{s,\gamma,\eta}(f) \ge \frac{1}{2} C_2 s^{-1}\delta^{2s}|f|^2_{s,\gamma/2} = \frac{1}{2}\left(\frac{C_3}{16(C_2 s^{-1})^{1/2}C_4}\right)^2 \lambda_0^2 |f|^2_{s,\gamma/2}.$$

Recalling $\mathcal{J}^{s,\gamma,\eta}(f) = 4\langle \mathcal{L}^{s,\gamma,\eta} f, f\rangle$, we get for $-s \le \gamma < 0$ that

$$\langle \mathcal{L}^{s,\gamma,\eta} f, f\rangle \ge c_s \lambda_0^2 |f|^2_{s,\gamma/2},$$

where

$$c_s = \frac{1}{8}\left(\frac{C_3}{16 C_2^{1/2} C_4}\right)^2 s = 2^{-11} C_3^2 C_2^{-1} C_4^{-2} s.$$

Step 5: The case $-ks \le \gamma < -(k-1)s$ for $k \ge 2$. In the previous step, starting from the case $\gamma = 0$ by using Lemma 3.6 where the constant is λ_0, to derive the $-s \le \gamma < 0$ case, we have a new constant $c_s\lambda_0^2$. For $-ks \le \gamma < -(k-1)s$, we can choose $\alpha = -(k-1)s$ and $\beta = \gamma + (k-1)s$ to apply the result of $\langle \mathcal{L}^{s,\alpha,\eta} f, f\rangle$. Note that the constants C_2, C_3, C_4 are universal with respect to α, β satisfying $\alpha + \beta = \gamma, -s \le \beta \le 0, -5 \le \gamma \le \alpha \le 0$. Then $\lambda_n = c_s \lambda_{n-1}^2$ implies that

$$\lambda_k = c_s^{2^k-1} \lambda_0^{2^k}.$$

For $-ks \le \gamma < (k-1)s$, by induction we will have

$$\mathcal{J}^{s,\gamma,\eta}(f) \ge \lambda_k |f|^2_{s,\gamma/2}. \tag{3.57}$$

This completes the proof of the theorem by taking $c = 2^{-11} C_3^2 C_2^{-1} C_4^{-2}$. ◻

4 Well-posedness and grazing limit

In this section, we will prove Theorem 1.1. We divide the proof into three subsections. The first subsection is about the *a priori* estimates for a linear equation with a general source. In subsect. 4.2, we prove the global well-posedness result in Theorem 1.1. In subsect. 4.3, we derive the global asymptotic formula (1.41) in Theorem 1.1. Throughout this section, $C_N, C_l, C_{N,l}$ are some large constants depending only on the corresponding indices. Moreover, they could change from line to line.

4.1 A priori estimate

We consider the following linear equation with a general source g,

$$\partial_t f + v \cdot \nabla_x f + \mathcal{L}^{s,\gamma} f = g. \tag{4.1}$$

A temporal energy functional $\mathcal{I}^N(f)$ satisfying for some generic constant C_1 that

$$|\mathcal{I}^N(f)| \leq C_1 \|\mu^{1/8} f\|^2_{H^N_x L^2} \tag{4.2}$$

is used to capture the dissipation of the macro components $\mathrm{M}(t,x) := (a(t,x), b(t,x), c(t,x))$ of the solution f, where (a,b,c) is defined in (3.25).

Lemma 4.1 *There exist two generic constants $C_2, c_0 > 0$ such that for any $N \geq 2$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}^N(f) + c_0|\mathrm{M}|^2_{H^N_x} \leq C_2(C^2_{s,\gamma}\|\mu^{1/8} f_2\|^2_{H^N_x L^2} + \mathrm{NL}^N(g)), \tag{4.3}$$

where

$$\mathrm{NL}^N(g) := \sum_{j=1}^{13} |\langle \partial^\alpha g, \mu^{1/2} P_j\rangle|^2_{H^{N-1}_x}.$$

Here, the standard thirteen moment polynomials P_j are defined by

$$P_1 = 1,\ P_2 = v_1,\ P_3 = v_2,\ P_4 = v_3,\ P_5 = v_1^2,\ P_6 = v_2^2,\ P_7 = v_3^2,$$
$$P_8 = v_1 v_2,\ P_9 = v_2 v_3,\ P_{10} = v_3 v_1,\ P_{11} = |v|^2 v_1,\ P_{12} = |v|^2 v_2,\ P_{13} = |v|^2 v_3.$$

We refer readers to [15, 21, 28] for the detailed proof of Lemma 4.1. The constant $C_{s,\gamma}$ in (4.3) comes from Theorem 2.2.

We now use Theorem 3.2 to derive the microscopic dissipation.

Lemma 4.2 *Let $|\alpha| + |\beta| \leq N$, $q \geq 0$, then*

$$(\mathcal{L}^{s,\gamma} W_q \partial^\alpha_\beta f, W_q \partial^\alpha_\beta f) \geq (7/8)\lambda_s \|W_q \partial^\alpha_\beta f_2\|^2_{L^2_x L^2_{s,\gamma/2}}$$

$$-C_{q,|\beta|}\Big(\|\mu^{1/8}\partial^\alpha f_2\|^2_{L^2_xL^2}+|\partial^\alpha \mathbf{M}|^2_{L^2_x}\Big).$$

Proof By Theorem 3.2 and recalling the constant λ_s in (3.28),

$$(\mathcal{L}^{s,\gamma}W_q\partial^\alpha_\beta f, W_q\partial^\alpha_\beta f)\geq \lambda_s\|(\mathbf{I}-\mathbf{P})W_q\partial^\alpha_\beta f\|^2_{L^2_xL^2_{s,\gamma/2}}.$$

It is straightforward to check for any $0\leq\alpha<1$ that

$$|x|^2\geq \alpha|y|^2-\frac{\alpha}{1-\alpha}|x-y|^2. \tag{4.4}$$

By the macro–micro decomposition $f=f_1+f_2$, we deduce that

$$\begin{aligned}(\mathcal{L}^{s,\gamma}W_q\partial^\alpha_\beta f, W_q\partial^\alpha_\beta f)&\geq \lambda_s\|(\mathbf{I}-\mathbf{P})W_q\partial^\alpha_\beta (f_1+f_2)\|^2_{L^2_xL^2_{s,\gamma/2}}\\ &\geq (7/8)\lambda_s\|W_q\partial^\alpha_\beta f_2\|^2_{L^2_xL^2_{s,\gamma/2}}\\ &\quad -C_{q,|\beta|}\Big(\|\mu^{1/8}\partial^\alpha f_2\|^2_{L^2}+|\partial^\alpha \mathbf{M}|^2_{L^2_x}\Big),\end{aligned}$$

where we have used (4.4) to take out $W_q\partial^\alpha_\beta f_2$ as the leading term. Moreover, we have used $f=f_1+f_2$ and the definition of (a,b,c). This completes the proof of the lemma.

□

We now apply the commutator estimate obtained in Corollary 2.2 to derive the following lemma.

Lemma 4.3 *Let* $|\alpha|+|\beta|\leq N$, $\beta_0+\beta_1+\beta_2=\beta$, $q\geq 0$, *then for any* $0<\delta\leq 1$, *we have*

$$\begin{aligned}&\Big|\Big(W_q\mathcal{L}^{s,\gamma}(\partial^\alpha_{\beta_2}f;\beta_0,\beta_1)-\mathcal{L}^{s,\gamma}(W_q\partial^\alpha_{\beta_2}f;\beta_0,\beta_1), W_q\partial^\alpha_\beta f\Big)\Big|\\ &\leq \delta\|\partial^\alpha_\beta f_2\|^2_{L^2_xL^2_{s,q+\gamma/2}}+\delta^{-1}C_qC^2_{s,\gamma}\left(\|\partial^\alpha_{\beta_2}f_2\|^2_{L^2_xL^2_{q+\gamma/2}}+\|\partial^\alpha_{\beta_2}f_2\|^2_{H^s_{\gamma/2}}\right)\\ &\quad+\delta^{-1}C_{q,|\beta|}C^2_{s,\gamma}|\partial^\alpha\mathbf{M}|^2_{L^2_x}.\end{aligned}$$

Proof By taking $l_1=-\gamma/2$ in Corollary 2.2 and by using the decomposition $f=f_1+f_2=\mathbf{P}f+f_2$, for any $0<\delta<1$, we have

$$\begin{aligned}&\Big|\Big(W_q\mathcal{L}^{s,\gamma}(\partial^\alpha_{\beta_2}f;\beta_0,\beta_1)-\mathcal{L}^{s,\gamma}(W_q\partial^\alpha_{\beta_2}f;\beta_0,\beta_1), W_q\partial^\alpha_\beta f\Big)\Big|\\ &\lesssim_q s^{-1}\|\partial^\alpha_{\beta_2}f\|_{L^2_xL^2_{q+\gamma/2}}\|\partial^\alpha_\beta f\|_{L^2_xL^2_{s,q+\gamma/2}}+C_{s,\gamma}\|\partial^\alpha_{\beta_2}f\|_{H^s_{\gamma/2}}\|\partial^\alpha_\beta f\|_{L^2_xL^2_{s,q+\gamma/2}}\\ &\leq \delta\|\partial^\alpha_\beta f\|^2_{L^2_xL^2_{s,q+\gamma/2}}+\delta^{-1}C_qC^2_{s,\gamma}\Big(\|\partial^\alpha_{\beta_2}f\|^2_{L^2_xL^2_{q+\gamma/2}}+\|\partial^\alpha_{\beta_2}f\|^2_{H^s_{\gamma/2}}\Big)\\ &\leq \delta\|\partial^\alpha_\beta f_2\|^2_{L^2_xL^2_{s,q+\gamma/2}}+\delta^{-1}C_qC^2_{s,\gamma}\Big(\|\partial^\alpha_{\beta_2}f_2\|^2_{L^2_xL^2_{q+\gamma/2}}+\|\partial^\alpha_{\beta_2}f_2\|^2_{H^s_{\gamma/2}}\Big)\end{aligned}$$

$$+\delta^{-1}C_{q,|\beta|}C_{s,\gamma}^2|\partial^\alpha \mathrm{M}|^2_{L^2_x}.$$

This completes the proof of the lemma. □

The following lemma is about the commutator $[\partial_\beta, \mathcal{L}^{s,\gamma}]$.

Lemma 4.4 *Let* $|\alpha|+|\beta|\le N$, $|\beta|\ge 1$, $q\ge 0$, *then*

$$\begin{aligned}|(W_q[\partial_\beta,\mathcal{L}^{s,\gamma}]\partial^\alpha f, W_q\partial^\alpha_\beta f)| \le \delta\|\partial^\alpha_\beta f_2\|^2_{L^2_xL^2_{s,q+\gamma/2}} &+ \delta^{-1}C_{q,N}C_{s,\gamma}^2|\partial^\alpha \mathrm{M}|^2_{L^2_x}\\ &+\delta^{-1}C_{q,N}C_{s,\gamma}^2\sum_{\beta_2<\beta}\Big(\|\partial^\alpha_{\beta_2}f_2\|^2_{L^2_xL^2_{q+\gamma/2}}\\ &+1_{q>0}\|\partial^\alpha_{\beta_2}f_2\|^2_{H^s_{\gamma/2}}\Big).\end{aligned}$$

Proof By (2.8), $[\partial_\beta,\mathcal{L}^{s,\gamma}]g=\sum_{\beta_2<\beta}C_\beta^{\beta_0,\beta_1,\beta_2}\mathcal{L}^{s,\gamma}(\partial_{\beta_2}g;\beta_0,\beta_1)$. Thus

$$\begin{aligned}W_q[\partial_\beta,\mathcal{L}^{s,\gamma}]\partial^\alpha f &= W_q\sum_{\beta_2<\beta}C_\beta^{\beta_0,\beta_1,\beta_2}\mathcal{L}^{s,\gamma}(\partial^\alpha_{\beta_2}f;\beta_0,\beta_1)\\ &=\sum_{\beta_2<\beta}C_\beta^{\beta_0,\beta_1,\beta_2}\mathcal{L}^{s,\gamma}(W_q\partial^\alpha_{\beta_2}f;\beta_0,\beta_1)\\ &\quad+\sum_{\beta_2<\beta}C_\beta^{\beta_0,\beta_1,\beta_2}[W_q,\mathcal{L}^{s,\gamma}(\cdot;\beta_0,\beta_1)]\partial^\alpha_{\beta_2}f.\end{aligned}$$

By upper bound estimate in Theorem 2.2, we get

$$\begin{aligned}|\mathcal{L}^{s,\gamma}(W_q\partial^\alpha_{\beta_2}f;\beta_0,\beta_1),W_q\partial^\alpha_\beta f)| &\lesssim C_{s,\gamma}\|\partial^\alpha_{\beta_2}f\|_{L^2_xL^2_{s,q+\gamma/2}}\|\partial^\alpha_\beta f\|_{L^2_xL^2_{s,q+\gamma/2}}\\ &\le \delta\|\partial^\alpha_\beta f_2\|^2_{L^2_xL^2_{s,q+\gamma/2}}\\ &\quad+\delta^{-1}C_{s,\gamma}^2C_{q,N}\|\partial^\alpha_{\beta_2}f_2\|^2_{L^2_xL^2_{s,q+\gamma/2}}\\ &\quad+\delta^{-1}C_{s,\gamma}^2C_{q,N}|\partial^\alpha \mathrm{M}|^2_{L^2_x}.\end{aligned}$$

We apply Lemma 4.3 to deal with $[W_q,\mathcal{L}^{s,\gamma}(\cdot;\beta_0,\beta_1)]$. Note that if $q=0$, the commutator $[W_q,\mathcal{L}^{s,\gamma}(\cdot;\beta_0,\beta_1)]=0$. Taking sum over $\beta_2<\beta$ completes the proof of the lemma. □

For any non-negative integers n, m, recall

$$\|f\|^2_{H^n_x\dot{H}^m_l}=\sum_{|\alpha|\le n,|\beta|=m}\|\partial^\alpha_\beta f\|^2_{L^2_xL^2_l},\quad \|f\|^2_{H^n_x\dot{H}^m_{s,l}}=\sum_{|\alpha|\le n,|\beta|=m}\|\partial^\alpha_\beta f\|^2_{L^2_xL^2_{s,l}}.$$

Let $N \geq 4, l \geq -N(\gamma+2s)$. For some generic constants M, L_j and K_j (which may depend on s, γ, N, l and will be determined later) for $0 \leq j \leq N$, we define

$$\Xi^{s,\gamma}_{N,l}(f) = M\mathcal{I}^N(f) + \sum_{j=0}^{N} L_j \|f\|^2_{H^{N-j}_x \dot{H}^j} + \sum_{j=0}^{N} K_j \|f\|^2_{H^{N-j}_x \dot{H}^j_{l+j(\gamma+2s)}}, \quad (4.5)$$

$$\tilde{\mathcal{D}}^{s,\gamma}_{N,l}(f) = c_0 M|\mathrm{M}|^2_{H^N_x} + \lambda_s \sum_{j=0}^{N} L_j \|f_2\|^2_{H^{N-j}_x \dot{H}^j_{s,\gamma/2}} + \lambda_s \sum_{j=0}^{N} K_j \|f_2\|^2_{H^{N-j}_x \dot{H}^j_{s,l+j(\gamma+2s)+\gamma/2}}. \quad (4.6)$$

We are now ready to prove the a priori estimate of (4.1).

Proposition 4.1 *Let $N \geq 4, l \geq -N(\gamma+2s)$. Suppose f is a solution to (4.1), then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Xi^{s,\gamma}_{N,l}(f) + \frac{1}{4}\tilde{\mathcal{D}}^{s,\gamma}_{N,l}(f) \leq MC_2\mathrm{NL}^N(g) + \sum_{j=0}^{N} 2L_j \sum_{|\alpha|\leq N-j, |\beta|=j} (\partial^\alpha_\beta g, \partial^\alpha_\beta f) + \sum_{j=0}^{N} 2K_j \sum_{|\alpha|\leq N-j, |\beta|=j} \left(W_{l+j(\gamma+2s)}\partial^\alpha_\beta g, W_{l+j(\gamma+2s)}\partial^\alpha_\beta f\right). \quad (4.7)$$

The constants in (4.5) and (4.6) satisfy

$$\max\{M, \{L_j\}_{0\leq j\leq N}, \{K_j\}_{0\leq j\leq N}\} = L_0 = Z_{s,\gamma,N,l} := X_{s,\gamma} Y_{s,\gamma,l} (U_{s,\gamma,N,l} W_{s,\gamma,N,l})^N, \quad (4.8)$$

$$\min\{c_0 M, \{\lambda_s L_j\}_{0\leq j\leq N}, \{\lambda_s K_j\}_{0\leq j\leq N}\} = \lambda_s, \quad (4.9)$$

where

$$X_{s,\gamma} = \lambda_s^{-1} C^2_{s,\gamma} C_2, \quad (4.10)$$

$$Y_{s,\gamma,l} = 8^{l/s}\lambda_s^{-l/s}\left(C_l(\gamma+2s+3)^{-1}\right)^{1+l/s}, \quad (4.11)$$

$$W_{s,\gamma,N,l} = \lambda_s^{-2} C^2_{s,\gamma} C_N, \quad (4.12)$$

$$U_{s,\gamma,N,l} = \max\{\lambda_s^{-2} C^2_{s,\gamma} C_{N,l}, 8^{l/s}(C_l(\gamma+2s+3)^{-1}\lambda_s^{-1})^{1+l/s}\}. \quad (4.13)$$

Here C_2 is the constant appeared in (4.3). Recalling the constants λ_s from (3.28) and $C_{s,\gamma}$ from (2.55), it is straightforward to check that for any fixed N, l, there is a function $(x_1, x_2) \in (0,1) \times (0,3] \to Z_{N,l}(x_1, x_2) \in (0,\infty)$ satisfying (1.35) and (1.37).

Proof We divide the proof into three steps to construct the energy functional $\Xi^{s,\gamma}_{N,l}(f)$ in (4.5).

Step 1: Propagation of $\|f\|^2_{H^N_x L^2}$. By applying ∂^α to equation (4.1), taking inner product with $\partial^\alpha f$, taking sum over $|\alpha| \le N$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{H^N_x L^2} + \sum_{|\alpha|\le N}(\mathcal{L}^{s,\gamma}\partial^\alpha f, \partial^\alpha f) = \sum_{|\alpha|\le N}(\partial^\alpha g, \partial^\alpha f). \tag{4.14}$$

By Theorem 3.2 and Remark 3.2, using $\partial^\alpha f_2 = (\partial^\alpha f)_2$, we have $(\mathcal{L}^{s,\gamma}\partial^\alpha f, \partial^\alpha f) \ge \lambda_s\|\partial^\alpha f_2\|^2_{L^2_x L^2_{s,\gamma/2}}$, which yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{H^N_x L^2} + \lambda_s\|f_2\|^2_{H^N_x L^2_{s,\gamma/2}} \le \sum_{|\alpha|\le N}(\partial^\alpha g, \partial^\alpha f). \tag{4.15}$$

Multiplying (4.15) by a large constant $2M_1$ and adding it to (4.3), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(M_1\|f\|^2_{H^N_x L^2} + \mathcal{I}^N(f)\big) + \big(c_0|\mathrm{M}|^2_{H^N_x} + M_1\lambda_s\|f_2\|^2_{H^N_x L^2_{s,\gamma/2}}\big)\\ &\quad\le 2M_1\sum_{|\alpha|\le N}(\partial^\alpha g, \partial^\alpha f) + C_2\mathrm{NL}^N(g).\end{aligned} \tag{4.16}$$

Here M_1 is large enough such that $M_1 \ge 2C_1$ and $M_1\lambda_s \ge C_2C^2_{s,\gamma}$ to insure from (4.2) that

$$\begin{aligned}\frac{1}{2}M_1\|f\|^2_{H^N_x L^2} &\le M_1\|f\|^2_{H^N_x L^2} + \mathcal{I}^N(f) \le \frac{3}{2}M_1\|f\|^2_{H^N_x L^2},\\ M_1\lambda_s\|f_2\|^2_{H^N_x L^2_{s,\gamma/2}} &\ge C_2C^2_{s,\gamma}\|\mu^{1/8}f_2\|^2_{H^N_x L^2}.\end{aligned}$$

Note that the term $C_2C^2_{s,\gamma}\|\mu^{1/8}f_2\|^2_{H^N_x L^2}$ in (4.3) is absorbed by the dissipation of the microscopic component $\|f_2\|^2_{H^N_x L^2_{s,\gamma/2}}$ in (4.15). We may assume $\lambda_s \le 1$ and $C_2 \gg C_1$. Then we can take $M_1 = X_{s,\gamma}$ defined in (4.10).

Step 2: Propagation of $\|f\|^2_{H^N_x L^2_l}$. By applying $W_l\partial^\alpha$ to Eq. (4.1), taking inner product with $W_l\partial^\alpha f$, taking sum over $|\alpha| \le N$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{H^N_x L^2_l} + \sum_{|\alpha|\le N}(W_l\mathcal{L}^{s,\gamma}\partial^\alpha f, W_l\partial^\alpha f) = \sum_{|\alpha|\le N}(W_l\partial^\alpha g, W_l\partial^\alpha f). \tag{4.17}$$

Using commutator to transfer weight gives

$$W_l\mathcal{L}^{s,\gamma}\partial^\alpha f = \mathcal{L}^{s,\gamma}W_l\partial^\alpha f + [W_l, \mathcal{L}^{s,\gamma}]\partial^\alpha f.$$

By Lemma 4.2, we get

$$(\mathcal{L}^{s,\gamma}W_l\partial^\alpha f, W_l\partial^\alpha f) \ge (7/8)\lambda_s\|\partial^\alpha f_2\|^2_{L^2_x L^2_{s,l+\gamma/2}} - C_l\left(\|\partial^\alpha f_2\|^2_{L^2_x L^2_{\gamma/2}} + |\partial^\alpha \mathrm{M}|^2_{L^2_x}\right).$$

By (2.57),

$$|([W_l, \mathcal{L}^{s,\gamma}]\partial^\alpha f, W_l \partial^\alpha f)| \leq C_l(\gamma+5)^{-1}\|\partial^\alpha f_2\|^2_{L^2_x L^2_{l+\gamma/2}} + C_l(\gamma+5)^{-1}|\partial^\alpha \mathrm{M}|^2_{L^2_x}.$$

Since $|h|^2_{L^2_q} \leq \delta |h|^2_{L^2_{q+s}} + \delta^{-q/s}|h|^2_{L^2}$ for any $1 > \delta > 0$, we have

$$\|\partial^\alpha f_2\|^2_{L^2_x L^2_{l+\gamma/2}} \leq \delta \|\partial^\alpha f_2\|^2_{L^2_x L^2_{s,l+\gamma/2}} + \delta^{-l/s}\|\partial^\alpha f_2\|^2_{L^2_x L^2_{s,\gamma/2}}. \tag{4.18}$$

By taking $\delta = \delta_{l,s,\gamma}$ where $\delta_{l,s,\gamma} C_l(\gamma+5)^{-1} = \lambda_s/8$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{H^N_x L^2_l} + \frac{3}{2}\lambda_s \|f_2\|^2_{H^N_x L^2_{s,l+\gamma/2}} &\leq C_{l,s,\gamma}\|f_2\|^2_{H^N_x L^2_{s,\gamma/2}} + C_{l,\gamma}|\mathrm{M}|^2_{H^N_x} \\ &+2\sum_{|\alpha|\leq N}(W_l\partial^\alpha g, W_l \partial^\alpha f),\end{aligned} \tag{4.19}$$

for some constants $C_{l,\gamma}$ and $C_{l,s,\gamma}$ satisfying

$$C_{l,\gamma} \leq C_l(\gamma+2s+3)^{-1}, \quad C_{l,s,\gamma} \leq \left(C_l(\gamma+2s+3)^{-1}\right)^{1+l/s}(\lambda_s/8)^{-l/s}. \tag{4.20}$$

We choose a constant M_2 large enough such that

$$c_0 M_2/2 \geq C_{l,\gamma}, \quad M_2 M_1 \lambda_s/2 \geq C_{l,s,\gamma}.$$

Recalling $M_1 = X_{s,\gamma}$ defined in (4.10), for simplicity, we can take $M_2 = Y_{s,\gamma,l}$ defined in (4.11). Then the combination (4.16)$\times M_2$+(4.19) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(M_2\mathcal{I}^N(f) + M_1M_2\|f\|^2_{H^N_x L^2} + \|f\|^2_{H^N_x L^2_l}) \\ &\quad+\frac{1}{2}\left(M_2 c_0|\mathrm{M}|^2_{H^N_x} + M_2M_1\lambda_s\|f_2\|^2_{H^N_x L^2_{s,\gamma/2}} + \lambda_s\|f_2\|^2_{H^N_x L^2_{s,l+\gamma/2}}\right) \\ &\leq M_2C_2\mathrm{NL}^N(g) + 2M_2M_1\sum_{|\alpha|\leq N}(\partial^\alpha g, \partial^\alpha f) + 2\sum_{|\alpha|\leq N}(W_l\partial^\alpha g, W_l\partial^\alpha f).\end{aligned} \tag{4.21}$$

Step 3: Propagation of $\|f\|^2_{H^{N-j}_x \dot{H}^j}$ and $\|f\|^2_{H^{N-j}_x \dot{H}^j_{l+j(\gamma+2s)}}$ for $j \geq 1$. For notation convenience, set

$$\mathcal{X}^i(f) := M^i\mathcal{I}^N(f) + \sum_{0\leq j\leq i} L^i_j\|f\|^2_{H^{N-j}_x \dot{H}^j} + \sum_{0\leq j\leq i} K^i_j\|f\|^2_{H^{N-j}_x \dot{H}^j_{l+j(\gamma+2s)}},$$

$$\mathcal{Y}^i(f) := M^i c_0 |\mathrm{M}|^2_{H^N_x} + \lambda_s \sum_{j=0}^{i} L^i_j \|f_2\|^2_{H^{N-j}_x \dot{H}^j_{s,\gamma/2}}$$
$$+\lambda_s \sum_{j=0}^{i} K^i_j \|f_2\|^2_{H^{N-j}_x \dot{H}^j_{s,l+j(\gamma+2s)+\gamma/2}},$$
$$\mathcal{Z}^i(g,f) := M^i C_2 \mathrm{NL}^N(g) + \sum_{j=0}^{i} 2L^i_j \sum_{|\alpha|\le N-j, |\beta|=j} (\partial^\alpha_\beta g, \partial^\alpha_\beta f)$$
$$+\sum_{j=0}^{i} 2K^i_j \sum_{|\alpha|\le N-j, |\beta|=j} (W_{l+j(\gamma+2s)} \partial^\alpha_\beta g, W_{l+j(\gamma+2s)} \partial^\alpha_\beta f).$$

We will use mathematical induction to prove that for any $0 \le i \le N$, there are some constants $M^i, L^i_j, K^i_j \ge 1, 0 \le j \le i$ satisfying

$$M^i c_0 \ge 1, \quad L^i_j \ge L^i_{j+1}, \quad K^i_j \ge K^i_{j+1}, \quad L^i_j \ge K^i_j,$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{X}^i(f) + 2^{-1-i/N} \mathcal{Y}^i(f) \le \mathcal{Z}^i(g,f). \tag{4.22}$$

It is easy to check that (4.22) is valid for $i = 0$ thanks to (4.21). More precisely, $M^0 = M_2, L^0_0 = M_1 M_2, K^0_0 = 1$.

We obtain (4.7) by taking $i = N$ in (4.22).

Assume (4.22) is valid for $i = k$ for some $0 \le k \le N - 1$. We now prove (4.22) is also valid for $i = k + 1$ by first considering $\|f\|^2_{H^{N-(k+1)}_x \dot{H}^{k+1}}$ and then $\|f\|^2_{H^{N-(k+1)}_x \dot{H}^{k+1}_{l+(k+1)(\gamma+2s)}}$.

Let α and β be multi-indices such that $|\alpha| \le N - (k+1)$ and $|\beta| = k + 1 \ge 1$. Applying ∂^α_β to both sides of (4.1) gives

$$\partial_t \partial^\alpha_\beta f + v \cdot \nabla_x \partial^\alpha_\beta f + \sum_{\beta_1 \le \beta, |\beta_1|=1} \partial^{\alpha+\beta_1}_{\beta-\beta_1} f + \partial^\alpha_\beta \mathcal{L}^{s,\gamma} f = \partial^\alpha_\beta g. \tag{4.23}$$

Taking inner product with $\partial^\alpha_\beta f$, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\partial^\alpha_\beta f\|^2_{L^2_x L^2} + \sum_{\beta_1 \le \beta, |\beta_1|=1} (\partial^{\alpha+\beta_1}_{\beta-\beta_1} f, \partial^\alpha_\beta f) + (\partial^\alpha_\beta \mathcal{L}^{s,\gamma} f, \partial^\alpha_\beta f)$$
$$= (\partial^\alpha_\beta g, \partial^\alpha_\beta f). \tag{4.24}$$

Estimate of $(\partial^{\alpha+\beta_1}_{\beta-\beta_1} f, \partial^\alpha_\beta f)$. By Cauchy–Schwarz inequality and using $f = f_1 + f_2$, we get

$$\begin{aligned}|(\partial^{\alpha+\beta_1}_{\beta-\beta_1} f, \partial^\alpha_\beta f)| &\le \|\partial^{\alpha+\beta_1}_{\beta-\beta_1} f\|_{L^2_x L^2_{-s-\gamma/2}} \|\partial^\alpha_\beta f\|_{L^2_x L^2_{s+\gamma/2}} \\ &\le \|\partial^{\alpha+\beta_1}_{\beta-\beta_1} f\|_{L^2_x L^2_{s,-(\gamma+2s)+\gamma/2}} \|\partial^\alpha_\beta f\|_{L^2_x L^2_{s,\gamma/2}} \\ &\le \delta \|\partial^\alpha_\beta f_2\|^2_{L^2_x L^2_{s,\gamma/2}} + C_\delta \|\partial^{\alpha+\beta_1}_{\beta-\beta_1} f_2\|_{L^2_x L^2_{s,l+|\beta-\beta_1|(\gamma+2s)+\gamma/2}} \\ &\quad + C_\delta C_{|\beta|} |\mathrm{M}|^2_{H^{N-k}_x}. \end{aligned} \tag{4.25}$$

Here $C_\delta \lesssim \delta^{-1}$.

Estimate of $(\partial^\alpha_\beta \mathcal{L}^{s,\gamma} f, \partial^\alpha_\beta f)$. Using $\partial^\alpha_\beta \mathcal{L}^{s,\gamma} f = \mathcal{L}^{s,\gamma} \partial^\alpha_\beta f + [\partial_\beta, \mathcal{L}^{s,\gamma}] \partial^\alpha f$, by Lemma 4.2 and Lemma 4.4 with $\delta = \lambda_s/8$, we have

$$\begin{aligned}(\partial^\alpha_\beta \mathcal{L}^{s,\gamma} f, \partial^\alpha_\beta f) &\ge (3/4)\lambda_s \|\partial^\alpha_\beta f_2\|^2_{L^2_x L^2_{s,\gamma/2}} - \lambda_s^{-1} C^2_{s,\gamma} C_N |\partial^\alpha \mathrm{M}|^2_{L^2_x} \\ &\quad - \lambda_s^{-1} C^2_{s,\gamma} C_N \sum_{\beta_2<\beta} \|\partial^\alpha_{\beta_2} f_2\|^2_{L^2_x L^2_{s,\gamma/2}}. \end{aligned} \tag{4.26}$$

By plugging (4.25) and (4.26) into (4.32), taking $\delta = \lambda_s/4N$ and taking sum over $|\alpha| \le N-(k+1)$, $|\beta| = k+1$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} \|f\|^2_{H^{N-k-1}_x \dot{H}^{k+1}} + \lambda_s \|f_2\|^2_{H^{N-k-1}_x \dot{H}^{k+1}_{s,\gamma/2}} \\ &\le 2 \sum_{|\alpha|\le N-k-1, |\beta|=k+1} (\partial^\alpha_\beta g, \partial^\alpha_\beta f) + \lambda_s^{-1} C^2_{s,\gamma} C_N \|f_2\|^2_{H^{N-k}_x \dot{H}^k_{s,l+k(\gamma+2s)+\gamma/2}} \\ &\quad + \lambda_s^{-1} C^2_{s,\gamma} C_N \|f_2\|^2_{H^{N-k}_x H^k_{s,\gamma/2}} + \lambda_s^{-1} C^2_{s,\gamma} C_N |\mathrm{M}|^2_{H^{N-k}_x}. \end{aligned} \tag{4.27}$$

By the induction assumption, (4.22) is true when $i = k$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{X}^k(f) + 2^{-1-k/N} \mathcal{Y}^k(f) \le \mathcal{Z}^k(g, f). \tag{4.28}$$

Note that $\mathcal{Y}^k(f)$ contains all the norms on the right hand side of (4.27).

We choose a constant W_k large enough such that

$$W_k 2^{-1-\frac{k}{N}-\frac{1}{2N}} (2^{\frac{1}{2N}} - 1) \lambda_s \ge \lambda_s^{-1} C^2_{s,\gamma} C_N.$$

Note that this also gives

$$W_k 2^{-1-\frac{k}{N}-\frac{1}{2N}} (2^{\frac{1}{2N}} - 1) M^k c_0 \ge \lambda_s^{-1} C^2_{s,\gamma} C_N.$$

That is, we can take

$$W_k = \lambda_s^{-2} C_{s,\gamma}^2 C_N. \tag{4.29}$$

Then (4.28)$\times W_k$+(4.27) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(W_k \mathcal{X}^k(f) + \|f\|^2_{H_x^{N-k-1}\dot{H}^{k+1}}\Big) + 2^{-1-k/N-1/2N} W_k \mathcal{Y}^k(f) \\
&\quad + \lambda_s \|f_2\|^2_{H_x^{N-k-1}\dot{H}^{k+1}_{s,\gamma/2}} \\
&\leq 2 \sum_{|\alpha|\leq N-k-1, |\beta|=k+1} (\partial^\alpha_\beta g, \partial^\alpha_\beta f) + W_k \mathcal{Z}^k(g, f).
\end{aligned} \tag{4.30}$$

Let α and β be multi-indices such that $|\alpha| \leq N-(k+1)$ and $|\beta| = k+1 \geq 1$. Let $q = l + (k+1)(\gamma + 2s)$. Applying $W_q \partial^\alpha_\beta$ to both sides of (4.1) gives

$$\begin{aligned}
&\partial_t W_q \partial^\alpha_\beta f + v \cdot \nabla_x W_q \partial^\alpha_\beta f + \sum_{\beta_1 \leq \beta, |\beta_1|=1} W_q \partial^{\alpha+\beta_1}_{\beta-\beta_1} f \\
&\quad + W_q \partial^\alpha_\beta \mathcal{L}^{s,\gamma} f = W_q \partial^\alpha_\beta g.
\end{aligned} \tag{4.31}$$

Taking inner product with $W_q \partial^\alpha_\beta f$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha_\beta f\|^2_{L^2_x L^2_q} + \sum_{\beta_1 \leq \beta, |\beta_1|=1} (W_q \partial^{\alpha+\beta_1}_{\beta-\beta_1} f, W_q \partial^\alpha_\beta f) \\
&\quad + (W_q \partial^\alpha_\beta \mathcal{L}^{s,\gamma} f, W_q \partial^\alpha_\beta f) = (W_q \partial^\alpha_\beta g, W_q \partial^\alpha_\beta f).
\end{aligned} \tag{4.32}$$

Estimate of $(W_q \partial^{\alpha+\beta_1}_{\beta-\beta_1} f, W_q \partial^\alpha_\beta f)$. Similarly to (4.25), we have

$$\begin{aligned}
\Big|(W_q \partial^{\alpha+\beta_1}_{\beta-\beta_1} f, W_q \partial^\alpha_\beta f)\Big| &\leq \delta \|\partial^\alpha_\beta f_2\|^2_{L^2_x L^2_{s,\gamma/2}} + C_\delta \|\partial^{\alpha+\beta_1}_{\beta-\beta_1} f_2\|_{L^2_x L^2_{s,l+k(\gamma+2s)+\gamma/2}} \\
&\quad + C_\delta C_{l,|\beta|} |\mathrm{M}|^2_{H_x^{N-k}}.
\end{aligned} \tag{4.33}$$

Estimate of $(W_q \partial^\alpha_\beta \mathcal{L}^{s,\gamma} f, W_q \partial^\alpha_\beta f)$. Observe that

$$W_q \partial^\alpha_\beta \mathcal{L}^{s,\gamma} f = \mathcal{L}^{s,\gamma} W_q \partial^\alpha_\beta f + [W_q, \mathcal{L}^{s,\gamma}] \partial^\alpha_\beta f + W_q [\partial_\beta, \mathcal{L}^{s,\gamma}] \partial^\alpha f.$$

By Lemma 4.2, (2.57) and Lemma 4.4, taking $\delta = \lambda_s/8$ in Lemma 4.4, we have

$$\begin{aligned}
&(\mathcal{L}^{s,\gamma} W_q \partial^\alpha_\beta f + [W_q, \mathcal{L}^{s,\gamma}] \partial^\alpha_\beta f + W_q [\partial_\beta, \mathcal{L}^{s,\gamma}] \partial^\alpha f, W_q \partial^\alpha_\beta f) \\
&\geq \frac{3}{4}\lambda_s \|W_q \partial^\alpha_\beta f_2\|^2_{L^2_x L^2_{s,\gamma/2}} + \lambda_s^{-1} C_{q,N} C^2_{s,\gamma} |\partial^\alpha \mathrm{M}|^2_{L^2_x} \\
&\quad + \lambda_s^{-1} C_{q,N} C^2_{s,\gamma} \sum_{\beta_2 < \beta} \Big(\|\partial^\alpha_{\beta_2} f_2\|^2_{L^2_x L^2_{q+\gamma/2}} + \|\partial^\alpha_{\beta_2} f_2\|^2_{H^s_{\gamma/2}}\Big)
\end{aligned}$$

$$+C_q(\gamma+5)^{-1}\|\partial^\alpha_\beta f\|^2_{L^2_x L^2_{q+\gamma/2}}.$$

By using the decomposition $f = f_1 + f_2$ and (4.18) for the last term, since $q \le l$, we get

$$\begin{aligned}
&(W_q\partial^\alpha_\beta \mathcal{L}^{s,\gamma} f, W_q\partial^\alpha_\beta f)\\
&\ge \frac{5}{8}\lambda_s\|W_q\partial^\alpha_\beta f_2\|^2_{L^2_x L^2_{s,\gamma/2}} + \lambda_s^{-1}C_{N,l}C^2_{s,\gamma}|\partial^\alpha \mathrm{M}|^2_{L^2_x}\\
&\quad+\lambda_s^{-1}C_{N,l}C^2_{s,\gamma}\sum_{\beta_2<\beta}\left(\|\partial^\alpha_{\beta_2} f_2\|^2_{L^2_x L^2_{q+\gamma/2}} + \|\partial^\alpha_{\beta_2} f_2\|^2_{H^s_{\gamma/2}}\right)\\
&\quad+C_{l,s,\gamma}\|\partial^\alpha_\beta f_2\|^2_{L^2_x L^2_{s,\gamma/2}}.
\end{aligned} \tag{4.34}$$

Plugging (4.33) and (4.34) into (4.32), taking $\delta = \lambda_s/8N$, and taking sum over $|\alpha| \le N-(k+1)$, $|\beta| = k+1$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{H^{N-k-1}_x \dot{H}^{k+1}_q} + \lambda_s\|f_2\|^2_{H^{N-k-1}_x \dot{H}^{k+1}_{s,q+\gamma/2}}\\
&\le 2\sum_{|\alpha|\le N-k-1,|\beta|=k+1}(W_q\partial^\alpha_\beta g, W_q\partial^\alpha_\beta f) + \lambda_s^{-1}C_N\|f_2\|^2_{H^{N-k}_x \dot{H}^k_{s,q+k(\gamma+2s)+\gamma/2}}\\
&\quad+\lambda_s^{-1}C_{N,l}C^2_{s,\gamma}\|f_2\|^2_{H^{N-k-1}_x H^k_{s,l+k(\gamma+2s)+\gamma/2}}\\
&\quad+\lambda_s^{-1}C_{N,l}C^2_{s,\gamma}|\partial^\alpha \mathrm{M}|^2_{L^2_x} + C_{l,s,\gamma}\|f_2\|^2_{H^{N-k-1}_x \dot{H}^{k+1}_{s,\gamma/2}}.
\end{aligned} \tag{4.35}$$

Note that $2^{-1-k/N-1/2N}\mathcal{Y}^k(f) + \lambda_s\|f_2\|^2_{H^{N-k-1}_x \dot{H}^{k+1}_{s,\gamma/2}}$ in (4.30) contains all the norms on the right hand side of (4.35). We choose a constant U_k large enough such that

$$U_k 2^{-1-\frac{k}{N}-\frac{1}{N}}(2^{\frac{1}{2N}}-1)\lambda_s \ge \lambda_s^{-1}C^2_{s,\gamma}C_{N,l}, \quad U_k\lambda_s/2 \ge C_{l,s,\gamma}.$$

By recalling (4.20), we choose

$$U_k = \max\{\lambda_s^{-2}C^2_{s,\gamma}C_{N,l}, 8^{l/s}(C_l(\gamma+5)^{-1}\lambda_s^{-1})^{1+l/s}\}. \tag{4.36}$$

Then (4.30)$\times U_k$+(4.35) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(U_k W_k \mathcal{X}^k(f) + U_k\|f\|^2_{H^{N-k-1}_x \dot{H}^{k+1}} + \|f\|^2_{H^{N-k-1}_x \dot{H}^{k+1}_q})\\
&\quad+2^{-1-k/N-1/N}U_k W_k\mathcal{Y}^k(f) + 2^{-1}U_k\lambda_s\|f_2\|^2_{H^{N-k-1}_x \dot{H}^{k+1}_{s,\gamma/2}}\\
&\quad+\lambda_s\|f_2\|^2_{H^{N-k-1}_x \dot{H}^{k+1}_{s,q+\gamma/2}}\\
&\le 2\sum_{|\alpha|\le N-k-1,|\beta|=k+1}(W_q\partial^\alpha_\beta g, W_q\partial^\alpha_\beta f)
\end{aligned}$$

$$+2U_k \sum_{|\alpha|\le N-k-1, |\beta|=k+1} (\partial^\alpha_\beta g, \partial^\alpha_\beta f) + U_k W_k \mathcal{Z}^k(g, f).$$

Hence (4.22) holds for $i = k+1$. Precisely, we set $M^{k+1} = U_k W_k M^k$, $L_j^{k+1} = U_k W_k L_j^k$, $K_j^{k+1} = U_k W_k K_j^k$ for $0 \le j \le k$ and $L_{k+1}^{k+1} = U_k$, $K_{k+1}^{k+1} = 1$. Note that $L_0^N = L_0^0 \prod_{j=0}^{N-1} U_j W_j = M_1 M_2 \prod_{j=0}^{N-1} U_j W_j$. By taking $i = N$ in (4.22) and $M = M^N$, $L_j = L_j^N$, $K_j = K_j^N$ for $0 \le j \le N$, we get (4.7). It is straightforward to check the constants satisfy (4.8)-(4.13). And this completes the proof of the proposition. □

4.2 Global well-posedenss

We first derive the following a priori estimate for solutions to the Cauchy problem (1.24).

Theorem 4.1 *Let $N \ge 4$, $l \ge -N(\gamma+2s)$. If $f^{s,\gamma}$ is a solution of the Cauchy problem (1.24) satisfying $\sup_{0\le t\le T} \|f^{s,\gamma}(t)\|_{H^N_{x,v}} \le \eta_{s,\gamma,N,l} := C_{N,l}^{-1} Z_{s,\gamma,N,l}^{-1} \lambda_s C_{s,\gamma}^{-1}$, then for any $t \in [0, T]$, the solution $f^{s,\gamma}$ satisfies*

$$\begin{aligned} &\mathcal{E}^{s,\gamma}_{N,l}(f^{s,\gamma}(t)) + \frac{1}{8}\lambda_s \int_0^t (|\mathrm{M}(\tau)|^2_{H^N_x} + \mathcal{D}^{s,\gamma}_{N,l}(f_2^{s,\gamma})(\tau)) \mathrm{d}\tau \\ &\le Z_{s,\gamma,N,l} \mathcal{E}^{s,\gamma}_{N,l}(f_0). \end{aligned} \tag{4.37}$$

Proof of Theorem 4.1 We apply Proposition 4.1 by taking $g = \Gamma^{s,\gamma}(f^{s,\gamma}, f^{s,\gamma})$ to have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \Xi^{N,l}(f^{s,\gamma}) + \frac{1}{4}\tilde{\mathcal{D}}^{s,\gamma}_{N,l}(f^{s,\gamma}) \le &\sum_{j=0}^N 2K_j \mathcal{A}^{N,j,l}_{s,\gamma}(f^{s,\gamma}, f^{s,\gamma}, f^{s,\gamma}) \\ &+ \sum_{j=0}^N 2L_j \mathcal{B}^{N,j}_{s,\gamma}(f^{s,\gamma}, f^{s,\gamma}, f^{s,\gamma}) \\ &+ MC_2 \mathcal{C}^N_{s,\gamma}(f^{s,\gamma}, f^{s,\gamma}), \end{aligned} \tag{4.38}$$

where

$$\mathcal{A}^{N,j,l}_{s,\gamma}(g, h, f) := \sum_{|\alpha|\le N-j, |\beta|=j} (W_{l+j(\gamma+2s)} \partial^\alpha_\beta \Gamma^{s,\gamma}(g, h), W_{l+j(\gamma+2s)} \partial^\alpha_\beta f), \tag{4.39}$$

$$\mathcal{B}^{N,j}_{s,\gamma}(g, h, f) := \sum_{|\alpha|\le N-j, |\beta|=j} (\partial^\alpha_\beta \Gamma^{s,\gamma}(g, h), \partial^\alpha_\beta f), \tag{4.40}$$

$$\mathcal{C}^N_{s,\gamma}(g, h) := \sum_{|\alpha|\le N-1} \sum_{j=1}^{13} \int_{\mathbb{T}^3} |\langle \partial^\alpha \Gamma^{s,\gamma}(g, h), \mu^{\frac{1}{2}} P_j \rangle|^2 \mathrm{d}x. \tag{4.41}$$

Recall from (1.32) the energy functional $\mathcal{E}^{s,\gamma}_{N,l}(f) = \sum_{j=0}^{N} \|f\|^2_{H^{N-j}_x \dot{H}^j_{l+j(\gamma+2s)}}$. Define the dissipation functional $\mathcal{D}^{s,\gamma}_{N,l}(f) = \sum_{j=0}^{N} \|f\|^2_{H^{N-j}_x \dot{H}^j_{s,l+j(\gamma+2s)+\gamma/2}}$. Set

$$\|f\|^2_{H^m_{x,v}} := \sum_{|\alpha|+|\beta|\le m} \|\partial^\alpha_\beta f\|^2_{L^2_x L^2}, \quad \|f\|^2_{D^m_{s,\gamma}} := \sum_{|\alpha|+|\beta|\le m} \|\partial^\alpha_\beta f\|^2_{L^2_x L^2_{s,\gamma/2}}.$$

We claim

$$|\mathcal{A}^{N,j,l}_{s,\gamma}(g,h,f)| \lesssim_{N,l} C_{s,\gamma} \left(\|g\|_{H^N_{x,v}} \left(\mathcal{D}^{s,\gamma}_{N,l}(h)\right)^{\frac{1}{2}} + \|g\|_{D^N_{s,\gamma}} \|h\|_{H^N_{x,v}} \right) \|f\|_{H^{N-j}_x \dot{H}^j_{s,l+j(\gamma+2s)+\gamma/2}} \tag{4.42}$$

$$|\mathcal{B}^{N,j}_{s,\gamma}(g,h,f)| \lesssim_N C_{s,\gamma} (\|g\|_{H^N_{x,v}} \|h\|_{D^N_{s,\gamma}} + \|g\|_{D^N_{s,\gamma}} \|h\|_{H^N_{x,v}}) \|f\|_{H^{N-j}_x \dot{H}^j_{s,\gamma/2}}, \tag{4.43}$$

$$\mathcal{C}^N_{s,\gamma}(g,f) \lesssim_N C^2_{s,\gamma} \|g\|^2_{H^N_x L^2} \|h\|^2_{H^N_x L^2_{s,\gamma/2}}. \tag{4.44}$$

With the above nonlinear estimates, by recalling (4.6), (4.8) and (4.9), if

$$C_{s,\gamma} \sup_{0\le t\le T} \|f^{s,\gamma}(t)\|_{H^N_{x,v}} \le 1, \tag{4.45}$$

then

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \Xi^{N,l}(f^{s,\gamma}) + \frac{1}{4}\lambda_s (|\mathrm{M}|^2_{H^N_x} + \mathcal{D}^{s,\gamma}_{N,l}(f^{s,\gamma}_2)) \\ &\le C_{N,l} Z_{s,\gamma,N,l} \left(C_{s,\gamma} \|f^{s,\gamma}\|_{H^N_{x,v}} + C^2_{s,\gamma} \|f^{s,\gamma}\|^2_{H^N_{x,v}} \right) \mathcal{D}^{s,\gamma}_{N,l}(f^{s,\gamma}) \\ &\le C_{N,l} Z_{s,\gamma,N,l} \lambda_s^{-1} C_{s,\gamma} \|f^{s,\gamma}\|_{H^N_{x,v}} \times \frac{1}{8}\lambda_s \left(|\mathrm{M}|^2_{H^N_x} + \mathcal{D}^{s,\gamma}_{N,l}(f^{s,\gamma}_2) \right), \end{aligned} \tag{4.46}$$

where we have used $\mathcal{D}^{s,\gamma}_{N,l}(f) \lesssim_{N,l} (|\mathrm{M}|^2_{H^N_x} + \mathcal{D}^{s,\gamma}_{N,l}(f_2))$ in the last inequality.

Now under the assumption

$$C_{N,l} Z_{s,\gamma,N,l} \lambda_s^{-1} C_{s,\gamma} \sup_{0\le t\le T} \|f^{s,\gamma}(t)\|_{H^N_{x,v}} \le 1, \tag{4.47}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \Xi^{N,l}(f^{s,\gamma}) + \frac{1}{8}\lambda_s (|\mathrm{M}|^2_{H^N_x} + \mathcal{D}^{s,\gamma}_{N,l}(f^{s,\gamma}_2)) \le 0, \tag{4.48}$$

which gives

$$\Xi^{N,l}(f^{s,\gamma}(t)) + \frac{1}{8}\lambda_s \int_0^t \left(|\mathrm{M}(\tau)|^2_{H^N_x} + \mathcal{D}^{s,\gamma}_{N,l}(f^{s,\gamma}_2)(\tau) \right) \mathrm{d}\tau \le \Xi^{N,l}(f_0). \tag{4.49}$$

Recalling (4.8), we have

$$\mathcal{E}^{s,\gamma}_{N,l}(f) \le \Xi^{s,\gamma}_{N,l}(f) \le Z_{s,\gamma,N,l} \mathcal{E}^{s,\gamma}_{N,l}(f). \tag{4.50}$$

Table 1 Index choice

$\vert\alpha_1\vert+\vert\beta_1\vert$	$\vert\alpha_2\vert+\vert\beta_2\vert$	(a_1,a_2,b_1,b_2)	$\vert\alpha_1\vert+a_1+\vert\beta_1\vert+b_1$	$\vert\alpha_2\vert+a_2+\vert\beta_2\vert+b_2$
0	$\le\vert\alpha\vert+\vert\beta\vert$	(2,0,2,s)	4	$\le\vert\alpha\vert+\vert\beta\vert+s$
1	$\le\vert\alpha\vert+\vert\beta\vert-1$	(1,1,2,s)	4	$\le\vert\alpha\vert+\vert\beta\vert+s$
2	$\le\vert\alpha\vert+\vert\beta\vert-2$	(0,2,2,s)	4	$\le\vert\alpha\vert+\vert\beta\vert+s$
3	$\le\vert\alpha\vert+\vert\beta\vert-3$	(1,1,s,2)	4 + s	$\le\vert\alpha\vert+\vert\beta\vert$
$\vert\alpha_1\vert+\vert\beta_1\vert\ge 4$	$\le\vert\alpha\vert+\vert\beta\vert-4$	(0,2,s,2)	N + s	$\le\vert\alpha\vert+\vert\beta\vert$

Note that (4.47) implies (4.45). Therefore, we obtain (4.37) under the condition (4.47).

Now it remains to prove (4.42), (4.43) and (4.44). We first consider $\mathcal{B}^{N,j}_{s,\gamma}(g,h,f)$ defined in (4.40). By the binomial expansion (2.4), we have

$$\partial^\alpha_\beta\Gamma^{s,\gamma}(g,h)=\sum C(\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)\Gamma^{s,\gamma}(\partial^{\alpha_1}_{\beta_1}g,\partial^{\alpha_2}_{\beta_2}h;\beta_0),$$

where the sum is taken over $\alpha_1+\alpha_2=\alpha$, $\beta_0+\beta_1+\beta_2=\beta$.

By taking $\delta=\frac{1}{2}$ in Theorem 2.1, for $(b_1,b_2)=(2,s)$ or $(s,2)$, we have

$$\begin{aligned}|\langle\Gamma^{s,\gamma}(\partial^{\alpha_1}_{\beta_1}g,\partial^{\alpha_2}_{\beta_2}h;\beta_0),\partial^\alpha_\beta f\rangle|\lesssim C_{s,\gamma}\Big(&|\partial^{\alpha_1}_{\beta_1}g|_{H^{b_1}_{\gamma/2}}|\partial^{\alpha_2}_{\beta_2}h|_{H^{b_2}_{\gamma/2}}\\&+|\partial^{\alpha_1}_{\beta_1}g|_{L^2}|\partial^{\alpha_2}_{\beta_2}h|_{s,\gamma/2}\Big)|\partial^\alpha_\beta f|_{s,\gamma/2}.\end{aligned}$$

Using the fact that $\int|ghf|\mathrm{d}x\lesssim|g|_{H^{a_1}_x}|h|_{H^{a_2}_x}|f|_{L^2_x}$ for $a_1+a_2=2$, $a_1,a_2\ge 0$, we have

$$\begin{aligned}\Big|\Big(\Gamma^{s,\gamma}(\partial^{\alpha_1}_{\beta_1}g,\partial^{\alpha_2}_{\beta_2}h;\beta_0),\partial^\alpha_\beta f\Big)\Big|\lesssim{}& C_{s,\gamma}\|\partial^{\alpha_1}_{\beta_1}g\|_{H^{a_1}_xH^{b_1}_{\gamma/2}}\|\partial^{\alpha_2}_{\beta_2}h\|_{H^{a_2}_xH^{b_2}_{\gamma/2}}\|\partial^\alpha_\beta f\|_{L^2_xL^2_{s,\gamma/2}}\\&+C_{s,\gamma}\|\partial^{\alpha_1}_{\beta_1}g\|_{H^{a_1}_xL^2}\|\partial^{\alpha_2}_{\beta_2}h\|_{H^{a_2}_xL^2_{s,\gamma/2}}\|\partial^\alpha_\beta f\|_{L^2_xL^2_{s,\gamma/2}}.\end{aligned}$$

The second term in the above inequality is directly bounded by $C_{s,\gamma}\|g\|_{H^N_{x,v}}\|h\|_{D^N_{s,\gamma}}\|\partial^\alpha_\beta f\|_{L^2_xL^2_{s,\gamma/2}}$ by suitably choosing a_1,a_2. Next we will give the choices of a_1,a_2,b_1,b_2 for the first term.

In the following, we choose $a_1,a_2\in\{0,1,2\}$ with $a_1+a_2=2$ and $b_1,b_2\in\{s,2\}$ with $b_1+b_2=2+s$. For $N\ge 4$ and multi-indices α,β with $|\alpha|+|\beta|\le N$, we consider all the combinations of $\alpha_1,\alpha_2,\beta_1,\beta_2$ such that $\alpha_1+\alpha_2=\alpha$, $\beta_1+\beta_2\le\beta$ in Table 1 for the choices of a_1,a_2,b_1,b_2.

With this, the part containing s is bounded by dissipation functional $D^N_{s,\gamma}$, and the other part is bounded by energy functional $H^N_{x,v}$. As a result,

$$\Big|\Big(\Gamma^{s,\gamma}(\partial^{\alpha_1}_{\beta_1}g,\partial^{\alpha_2}_{\beta_2}h;\beta_0),\partial^\alpha_\beta f\Big)\Big|\lesssim C_{s,\gamma}(\|g\|_{H^N_{x,v}}\|h\|_{D^N_{s,\gamma}}+\|g\|_{D^N_{s,\gamma}}\|h\|_{H^N_{x,v}})\|\partial^\alpha_\beta f\|_{L^2_xL^2_{s,\gamma/2}}.$$

Taking sum yields (4.43).

Similarly, we can use Corollary 2.1 to derive (4.42) and use Proposition 2.14 to derive (4.44). This completes the proof of the theorem. □

Proof of Theorem 1.1 *(Global well-posedness)* Local well-posedness of the Cauchy problem (1.24) and non-negativity of $\mu + \mu^{\frac{1}{2}} f$ can be proved by standard iteration. From this together with Theorem 4.1, by taking $\delta_{s,\gamma,N,l} = \frac{1}{2}\eta^2_{s,\gamma,N,l}$, the standard continuity argument yields the global well-posedness result (1.34) for the Boltzmann equation. Recalling the constants λ_s from (3.28), $C_{s,\gamma}$ from (2.55), $Z_{s,\gamma,N,l}$ from (4.8) and the constant $\eta_{s,\gamma,N,l}$ from Theorem 4.1, it is straightforward to check that for any fixed N, l, there is a function $(x_1, x_2) \in (0, 1) \times (0, 3] \to \delta_{N,l}(x_1, x_2) \in (0, \infty)$ satisfying (1.35) and (1.36). Moreover, since all the estimates are uniform for $s \to 1^-$, the global well-posedness result (1.39) for the Landau equation follows by a similar argument. □

4.3 Asymptotic formula for the limit

We prove (1.41) in this subsection. Let $f^{s,\gamma}$ and f^γ be the solutions to (1.24) and (1.26) respectively. Observe $F_R^{s,\gamma} := (1-s)^{-1}(f^{s,\gamma} - f^\gamma)$ solves

$$\begin{aligned}\partial_t F_R^{s,\gamma} + v\cdot\nabla_x F_R^{s,\gamma} + \mathcal{L}_L^\gamma F_R^{s,\gamma} &= (1-s)^{-1}[(\mathcal{L}_L^\gamma - \mathcal{L}_B^{s,\gamma}) f^{s,\gamma}\\ &+(\Gamma_B^{s,\gamma} - \Gamma_L^\gamma)(f^{s,\gamma}, f^\gamma)] + \Gamma_B^{s,\gamma}(f^{s,\gamma}, F_R^{s,\gamma}) + \Gamma_L^\gamma(F_R^{s,\gamma}, f^\gamma). \quad (4.51)\end{aligned}$$

We will apply Proposition 4.1 to the above equation for $F_R^{s,\gamma}$. For brevity, we set

$$G_1 = (1-s)^{-1}[(\mathcal{L}_L^\gamma - \mathcal{L}_B^{s,\gamma}) f^{s,\gamma} + (\Gamma_B^{s,\gamma} - \Gamma_L^\gamma)(f^{s,\gamma}, f^\gamma)], \qquad (4.52)$$

$$G_2 = \Gamma_B^{s,\gamma}(f^{s,\gamma}, F_R^{s,\gamma}), \quad G_3 = \Gamma_L^\gamma(F_R^{s,\gamma}, f^\gamma). \qquad (4.53)$$

By applying Proposition 4.1 with $s = 1$, $g = G_1 + G_2 + G_3$, since $|\langle \partial^\alpha g, \mu^{\frac{1}{2}} P_j\rangle|^2 \le 3\sum_{i=1}^3 |\langle \partial^\alpha G_i, \mu^{\frac{1}{2}} P_j\rangle|^2$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Xi^{1,\gamma}_{N,l}(F_R^{s,\gamma}) + \frac{1}{4}\tilde{\mathcal{D}}^{1,\gamma}_{N,l}(F_R^{s,\gamma}) \le &\sum_{i=1}^3\sum_{j=0}^N 2K_j\\ &\sum_{|\alpha|\le N-j, |\beta|=j} (W_{l+j(\gamma+2)}\partial^\alpha_\beta G_i, W_{l+j(\gamma+2)}\partial^\alpha_\beta F_R^{s,\gamma}) \quad (4.54)\\ &+\sum_{i=1}^3\sum_{j=0}^N 2L_j \sum_{|\alpha|\le N-j, |\beta|=j} (\partial^\alpha_\beta G_i, \partial^\alpha_\beta F_R^{s,\gamma})\\ &+3MC_2\sum_{i=1}^3 \mathrm{NL}^N(G_i).\end{aligned}$$

Let us first estimate the terms containing G_1. Recalling (4.52) and (4.54), we need to estimate the following quantities

$$\mathcal{I}_{1,i} := \sum_{|\alpha|\le N-j, |\beta|=j} (W_{l+j(\gamma+2)}\partial^\alpha_\beta G_{1,i}, W_{l+j(\gamma+2)}\partial^\alpha_\beta F_R^{s,\gamma}), \tag{4.55}$$

$$\mathcal{I}_{2,i} := \sum_{|\alpha|\le N-j, |\beta|=j} (\partial^\alpha_\beta G_{1,i}, \partial^\alpha_\beta F_R^{s,\gamma}), \quad \mathcal{I}_{3,i} := \mathrm{NL}^N(G_{1,i}). \tag{4.56}$$

Here, for $i = 1, 2$,

$$G_{1,1} = (1-s)^{-1}(\mathcal{L}_L^\gamma - \mathcal{L}_B^{s,\gamma}) f^{s,\gamma},$$
$$G_{1,2} = (1-s)^{-1}(\Gamma_B^{s,\gamma} - \Gamma_L^\gamma)(f^{s,\gamma}, f^\gamma). \tag{4.57}$$

These terms contain operator difference. We now establish $Q_B^{s,\gamma} \to Q_L^\gamma, \Gamma_B^{s,\gamma} \to \Gamma_L^\gamma, \mathcal{L}_B^{s,\gamma} \to \mathcal{L}_L^\gamma$ as $s \to 1^-$. The results can be given in weighted L^2-norm by using the estimates from [12] and [25]. We obtain the results for any $-5 < \gamma \le 0$ by allowing higher regularity.

Proposition 4.2 *Let $-5 < \gamma \le 0$. Fix $l \ge 0$. Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ satisfying $a_1 + a_2 = \gamma + 6$ and $b_1 + b_2 = \gamma + 2$. If $-9/2 < \gamma \le 0$, then*

$$\left|\langle Q_B^{s,\gamma}(g,h) - Q_L^\gamma(g,h), W_l\psi\rangle\right| \lesssim_{l,a_1,a_2,b_1,b_2} (1-s)|g|_{H^2_{|\gamma+2|+2}}|h|_{H^2_{l+b_1}}|\psi|_{L^2_{b_2}}$$
$$+ (1-s)(\gamma+\frac{9}{2})^{-1}|g|_{H^3_{l+|a_1|+|a_2|+2}}|h|_{H^3_{l+a_1}}|\psi|_{L^2_{a_2}}. \tag{4.58}$$

If $-5 < \gamma \le 0$, then

$$\left|\langle Q_B^{s,\gamma}(g,h) - Q_L^\gamma(g,h), W_l\psi\rangle\right| \lesssim_{l,a_1,a_2,b_1,b_2} (1-s)(\gamma+5)^{-1}|g|_{H^2_{|\gamma+2|+2}}|h|_{H^2_{l+b_1}}|\psi|_{L^2_{b_2}}$$
$$+ (1-s)|g|_{H^{3+s_1}_{l+|a_1|+|a_2|+2}}|h|_{H^{3+s_2}_{l+a_1}}|\psi|_{L^2_{a_2}}, \tag{4.59}$$

where $s_1, s_2 \ge 0$ satisfying $s_1 + s_2 = 1$.

For completeness, the proof of Proposition 4.2 will be given in the Appendix. Here, we only concern about the dependence on the two physical parameters γ, s and do not pursue the precise dependence on l, a_1, a_2, b_1, b_2. Roughly speaking, the dependence on l, a_1, a_2, b_1, b_2 is of the form $c^l, c^{|a_1|}, c^{|a_2|}, c^{|b_1|}, c^{|b_2|}$ for some generic constant $c > 1$.

We can also get similar results for the non-linear terms $\Gamma_B^{s,\gamma}(g,h)$ and $\Gamma_L^\gamma(g,h)$ by slightly revising the proof of Proposition 4.2. In this situation, we can get rid of the weight on g.

Proposition 4.3 *Let $-5 < \gamma \le 0$. Fix $l \ge 0$. Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ satisfying $a_1 + a_2 = \gamma + 6$ and $b_1 + b_2 = \gamma + 2$. If $-9/2 < \gamma \le 0$, then*

$$\left|\langle \Gamma_B^{s,\gamma}(g,h) - \Gamma_L^\gamma(g,h), W_l\psi\rangle\right| \lesssim_{l,a_1,a_2,b_1,b_2} (1-s)|g|_{H^2}|h|_{H^2_{l+b_1}}|\psi|_{L^2_{b_2}} \tag{4.60}$$

$$+ (1-s)(\gamma+\frac{9}{2})^{-1}|g|_{H^3}|h|_{H^3_{l+a_1}}|\psi|_{L^2_{a_2}}.$$

If $-5<\gamma\leq 0$, *then*

$$\left|\langle \Gamma_B^{s,\gamma}(g,h)-\Gamma_L^{\gamma}(g,h), W_l\psi\rangle\right| \lesssim_{l,a_1,a_2,b_1,b_2} (1-s)(\gamma+5)^{-1}|g|_{H^2}|h|_{H^2_{l+b_1}}|\psi|_{L^2_{b_2}} \\ + (1-s)|g|_{H^{3+s_1}}|h|_{H^{3+s_2}_{l+a_1}}|\psi|_{L^2_{a_2}}, \tag{4.61}$$

where $s_1, s_2 \geq 0$ *satisfying* $s_1+s_2=1$.

Recalling $\mathcal{L}_B^{s,\gamma}f = -\Gamma_B^{s,\gamma}(\mu^{\frac{1}{2}},f)-\Gamma_B^{s,\gamma}(f,\mu^{\frac{1}{2}}), \mathcal{L}_L^{\gamma}f = -\Gamma_L^{\gamma}(\mu^{\frac{1}{2}},f)-\Gamma_L^{\gamma}(f,\mu^{\frac{1}{2}})$, as an application of Proposition 4.3, we can put the higher regularity on $\mu^{\frac{1}{2}}$ to obtain the following proposition.

Proposition 4.4 *Let* $-5<\gamma\leq 0$. *Fix* $l\geq 0$. *Let* $a_1,a_2\in\mathbb{R}$ *satisfying* $a_1+a_2=\gamma+6$. *Then*

$$\left|\langle \mathcal{L}_B^{s,\gamma}f-\mathcal{L}_L^{\gamma}f, W_l\psi\rangle\right| \lesssim_{l,a_1,a_2} (1-s)(\gamma+5)^{-1}|f|_{H^3_{(l+a_1)^+}}|\psi|_{L^2_{a_2}}. \tag{4.62}$$

Let $C_\gamma := (\gamma+5)^{-1}$. By (4.61) and (4.62),

$$(1-s)^{-1}\left|\langle \Gamma_B^{s,\gamma}(g,h)-\Gamma_L^{\gamma}(g,h), W_l\psi\rangle\right| \lesssim_l C_\gamma|g|_{H^3}|h|_{H^4_{l+5+\gamma/2}}|\psi|_{L^2_{1+\gamma/2}}, \tag{4.63}$$

$$(1-s)^{-1}\left|\langle \mathcal{L}_B^{s,\gamma}f-\mathcal{L}_L^{\gamma}f, W_l\psi\rangle\right| \lesssim_l C_\gamma|f|_{H^3_{l+5+\gamma/2}}|\psi|_{L^2_{1+\gamma/2}}. \tag{4.64}$$

Recall (4.55) and (4.57) for $\mathcal{I}_{1,1}$ and $\mathcal{I}_{1,2}$. We now estimate these two terms in details. By (4.64), we have

$$\begin{aligned}|\mathcal{I}_{1,1}| &\lesssim_{N,l} C_\gamma \sum_{|\alpha|\leq N-j,|\beta|=j}\sum_{\beta_1\leq\beta}\|\partial^\alpha_{\beta_1}f^{s,\gamma}\|_{L^2_xH^3_{l+j(\gamma+2)+5+\gamma/2}}\|W_{l+j(\gamma+2)}\partial^\alpha_\beta F_R^{s,\gamma}\|_{L^2_xL^2_{1+\gamma/2}} \\ &\leq C_{N,l}C_\gamma(\mathcal{D}^{s,\gamma}_{N+3,l+5-3(\gamma+2s)+N(2-2s)}(f^{s,\gamma}))^{1/2}\|F_R^{s,\gamma}\|_{H^{N-j}_x\dot{H}^j_{1,l+j(\gamma+2)+\gamma/2}},\end{aligned}$$

where we have used for any $0\leq k\leq N+3$,

$$\mathcal{D}^{s,\gamma}_{N+3,l+5-3(\gamma+2s)+N(2-2s)}(f^{s,\gamma}) \geq \|f^{s,\gamma}\|^2_{H^{N+3-k}_x\dot{H}^k_{s,l+5-3(\gamma+2s)+N(2-2s)+k(\gamma+2s)+\gamma/2}}.$$

In particular, taking $k=j+3$ gives

$$\mathcal{D}^{s,\gamma}_{N+3,l+5-3(\gamma+2s)+N(2-2s)}(f^{s,\gamma}) \geq \|f^{s,\gamma}\|^2_{H^{N-j}_x\dot{H}^{j+3}_{s,l+5+j(\gamma+2)+\gamma/2}}.$$

Similarly, by (4.63), we have

$$
\begin{aligned}
|\mathcal{I}_{1,2}| \lesssim_{N,l} C_\gamma & \sum_{|\alpha|\le N-j,|\beta|=j} \\
& \sum_{\alpha_1+\alpha_2=\alpha,\beta_1\le\beta} \int |\partial^{\alpha_1}_{\beta_1} f^{s,\gamma}|_{H^3} |\partial^{\alpha_2}_{\beta_2} f^\gamma|_{H^4_{l+j(\gamma+2)+5+\gamma/2}} |W_{l+j(\gamma+2)} \partial^\alpha_\beta F_R^{s,\gamma}|_{L^2_{1+\gamma/2}} \mathrm{d}x \\
\le \; & C_{N,l} C_\gamma \| f^{s,\gamma}\|_{H^{N+3}_{x,v}} \\
& \left(\sum_{|\alpha|+|\beta|\le N+3,|\beta|\le j+3} \|\partial^\alpha_\beta f^\gamma\|^2_{L^2_x L^2_{1,l+j(\gamma+2)+5+\gamma/2}} \right)^{1/2} \|F_R^{s,\gamma}\|_{H_x^{N-j}\dot{H}^j_{1,l+j(\gamma+2)+\gamma/2}} \\
\le \; & C_{N,l} C_\gamma \| f^{s,\gamma}\|_{H^{N+3}_{x,v}} (\mathcal{D}^{1,\gamma}_{N+3,l+5-3(\gamma+2)}(f^\gamma))^{1/2} \|F_R^{s,\gamma}\|_{H_x^{N-j}\dot{H}^j_{1,l+j(\gamma+2)+\gamma/2}}.
\end{aligned}
$$

Recalling (4.56) and (4.57) for $\mathcal{I}_{2,1}$ and $\mathcal{I}_{2,2}$, it is obvious that these two terms are also bounded by the previous upper bounds of $\mathcal{I}_{1,1}$ and $\mathcal{I}_{1,2}$. Similarly, by (4.64) and (4.63), we have

$$
\begin{aligned}
|\mathcal{I}_{3,1}| &\le C_{N,l} C_\gamma^2 \mathcal{D}^{s,\gamma}_{N+3,l+5-3(\gamma+2s)+N(2-2s)}(f^{s,\gamma}), \\
|\mathcal{I}_{3,2}| &\le C_{N,l} C_\gamma^2 \| f^{s,\gamma}\|^2_{H^{N+3}_{x,v}} \mathcal{D}^{1,\gamma}_{N+3,l+5-3(\gamma+2)}(f^\gamma).
\end{aligned}
$$

Let us next estimate the terms containing G_2. Recall (4.39), (4.40) and (4.41) that

$$
\begin{aligned}
& \sum_{|\alpha|\le N-j,|\beta|=j} (W_{l+j(\gamma+2)}\partial^\alpha_\beta G_2, W_{l+j(\gamma+2)}\partial^\alpha_\beta F_R^{s,\gamma}) = \mathcal{A}^{N,j,l+2j-2sj}_{s,\gamma}(f^{s,\gamma}, F_R^{s,\gamma}, F_R^{s,\gamma}), \\
& \sum_{|\alpha|\le N-j,|\beta|=j} (\partial^\alpha_\beta G_2, \partial^\alpha_\beta F_R^{s,\gamma}) = \mathcal{B}^{N,j}_{s,\gamma}(f^{s,\gamma}, F_R^{s,\gamma}, F_R^{s,\gamma}), \\
& \mathrm{NL}^N(G_2) = \mathcal{C}^N_{s,\gamma}(f^{s,\gamma}, F_R^{s,\gamma}).
\end{aligned}
$$

As for $-5 < \gamma \le -2$, $\frac{1}{4} \le \frac{1}{2}(1-\frac{\gamma+3}{2}) \le s \le 1$, it holds that

$$
C_{s,\gamma} = s^{-1}(\gamma+2s+3)^{-1} \le 8(\gamma+5)^{-1}.
$$

Therefore we can replace $C_{s,\gamma}$ with $C_\gamma = (\gamma+5)^{-1}$ in the rest of this section.

Recalling (4.42), we have

$$
\begin{aligned}
& \left| \mathcal{A}^{N,j,l+2j-2sj}_{s,\gamma}(f^{s,\gamma}, F_R^{s,\gamma}, F_R^{s,\gamma}) \right| \\
& \lesssim_{N,l} C_\gamma (\| f^{s,\gamma}\|_{H^N_{x,v}} \left(\mathcal{D}^{1,\gamma}_{N,l}(F_R^{s,\gamma}) \right)^{\frac{1}{2}} \\
& \quad + \| f^{s,\gamma}\|_{D^N_{s,\gamma}} \| F_R^{s,\gamma}\|_{H^N_{x,v}}) \| F_R^{s,\gamma}\|_{H_x^{N-j}\dot{H}^j_{s,l+j(\gamma+2)+\gamma/2}}.
\end{aligned}
\tag{4.65}
$$

Note that the estimate (4.65) takes account of the additional weight $2j - 2sj$ and is controlled by the dissipation norm of the linearized Landau operator. By (4.43) and

(4.44), we have

$$\left|\mathcal{B}^{N,j}_{s,\gamma}(f^{s,\gamma}, F^{s,\gamma}_R, F^{s,\gamma}_R)\right| \lesssim_N C_\gamma(\|f^{s,\gamma}\|_{H^N_{x,v}}\|F^{s,\gamma}_R\|_{D^N_{s,\gamma}} + \|f^{s,\gamma}\|_{D^N_{s,\gamma}}\|F^{s,\gamma}_R\|_{H^N_{x,v}})\|F^{s,\gamma}_R\|_{H^{N-j}_x \dot{H}^j_{s,\gamma/2}}, \tag{4.66}$$

$$\mathcal{C}^N_{s,\gamma}(f^{s,\gamma}, F^{s,\gamma}_R) \lesssim_N C^2_\gamma\|f^{s,\gamma}\|^2_{H^N_x L^2}\|F^{s,\gamma}_R\|^2_{H^N_x L^2_{s,\gamma/2}}. \tag{4.67}$$

Let us estimate the terms containing G_3. By taking $s = 1$ in (4.39), (4.40) and (4.41), and replacing $\Gamma^{1,\gamma}$ by Γ^γ_L, we can define $\mathcal{A}^{N,j,l}_{1,\gamma}(g, h, f), \mathcal{B}^{N,j}_{1,\gamma}(g, h, f), \mathcal{C}^N_{1,\gamma}(g, h)$ similarly. Then

$$\sum_{|\alpha|\leq N-j, |\beta|=j}(W_{l+j(\gamma+2)}\partial^\alpha_\beta G_3, W_{l+j(\gamma+2)}\partial^\alpha_\beta F^{s,\gamma}_R) = \mathcal{A}^{N,j,l}_{1,\gamma}(F^{s,\gamma}_R, f^\gamma, F^{s,\gamma}_R),$$

$$\sum_{|\alpha|\leq N-j, |\beta|=j}(\partial^\alpha_\beta G_3, \partial^\alpha_\beta F^{s,\gamma}_R) = \mathcal{B}^{N,j}_{1,\gamma}(F^{s,\gamma}_R, f^\gamma, F^{s,\gamma}_R), \quad \mathrm{NL}^N(G_3) = \mathcal{C}^N_{1,\gamma}(F^{s,\gamma}_R, f^\gamma).$$

Note that these quantities contain the nonlinear term Γ^γ_L of the Landau operator. By taking $s = 1$ in the estimates of the nonlinear term $\Gamma^{s,\gamma}_B$ in previous sections, we can obtain estimates for Γ^γ_L. As a result, similarly to (4.65), (4.66) and (4.67), we have

$$|\mathcal{A}^{N,j,l}_{1,\gamma}(F^{s,\gamma}_R, f^\gamma, F^{s,\gamma}_R)| \lesssim_{N,l} C_\gamma\left(\|F^{s,\gamma}_R\|_{H^N_{x,v}}\left(\mathcal{D}^{1,\gamma}_{N,l}(f^\gamma)\right)^{\frac{1}{2}} + \|F^{s,\gamma}_R\|_{D^N_{1,\gamma}}\|f^\gamma\|_{H^N_{x,v}}\right)\|F^{s,\gamma}_R\|_{H^{N-j}_x \dot{H}^j_{1,l+j(\gamma+2)+\gamma/2}},$$

$$|\mathcal{B}^{N,j}_{1,\gamma}(F^{s,\gamma}_R, f^\gamma, F^{s,\gamma}_R)| \lesssim_N C_\gamma\left(\|F^{s,\gamma}_R\|_{H^N_{x,v}}\|f^\gamma\|_{D^N_{1,\gamma}} + \|F^{s,\gamma}_R\|_{D^N_{1,\gamma}}\|f^\gamma\|_{H^N_{x,v}}\right)\|F^{s,\gamma}_R\|_{H^{N-j}_x \dot{H}^j_{1,\gamma/2}},$$

$$\mathcal{C}^N_{1,\gamma}(F^{s,\gamma}_R, f^\gamma) \lesssim_N C^2_\gamma\|F^{s,\gamma}_R\|^2_{H^N_x L^2}\|f^\gamma\|^2_{H^N_x L^2_{1,\gamma/2}}.$$

Plugging the above nonlinear estimates into (4.54), recalling (4.8) and (4.9), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Xi^{1,\gamma}_{N,l}(F^{s,\gamma}_R) + \frac{1}{4}\lambda_1\left(|\mathrm{M}|^2_{H^N_x} + \mathcal{D}^{1,\gamma}_{N,l}((F^{s,\gamma}_R)_2)\right) \\
&\leq C_{N,l}Z_{1,\gamma,N,l}\Big\{(C_\gamma\|f^{s,\gamma}\|_{H^N_{x,v}} + C_\gamma\|f^\gamma\|_{H^N_{x,v}} + C^2_\gamma\|f^{s,\gamma}\|^2_{H^N_x L^2})\mathcal{D}^{1,\gamma}_{N,l}(F^{s,\gamma}_R) \\
&\quad + \|f^{s,\gamma}\|_{D^N_{s,\gamma}}\|F^{s,\gamma}_R\|_{H^N_{x,v}}\left(\mathcal{D}^{1,\gamma}_{N,l}(F^{s,\gamma}_R)\right)^{\frac{1}{2}}\Big\} \\
&\quad + C_{N,l}Z_{1,\gamma,N,l}\Big\{C_\gamma\|F^{s,\gamma}_R\|_{H^N_{x,v}}\left(\mathcal{D}^{1,\gamma}_{N,l}(f^\gamma)\right)^{\frac{1}{2}}\left(\mathcal{D}^{1,\gamma}_{N,l}(F^{s,\gamma}_R)\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+C_\gamma^2 \|F_R^{s,\gamma}\|^2_{H^N_{x,v}} \mathcal{D}^{1,\gamma}_{N,l}(f^\gamma)\Big\} \\
&+C_{N,l} Z_{1,\gamma,N,l}\Big\{C_\gamma (\mathcal{D}^{s,\gamma}_{N+3,l+5-3(\gamma+2s)+N(2-2s)}(f^{s,\gamma}))^{1/2} \\
&+C_\gamma \|f^{s,\gamma}\|_{H^{N+3}_{x,v}} (\mathcal{D}^{1,\gamma}_{N+3,l+5-3(\gamma+2)}(f^\gamma))^{1/2}\Big\} \left(\mathcal{D}^{1,\gamma}_{N,l}(F_R^{s,\gamma})\right)^{\frac{1}{2}} \\
&+C_{N,l} Z_{1,\gamma,N,l}\Big\{C_\gamma^2 \mathcal{D}^{s,\gamma}_{N+3,l+5-3(\gamma+2s)+N(2-2s)}(f^{s,\gamma}) \\
&+C_\gamma^2 \|f^{s,\gamma}\|^2_{H^{N+3}_{x,v}} \mathcal{D}^{1,\gamma}_{N+3,l+5-3(\gamma+2)}(f^\gamma)\Big\}. \qquad (4.68)
\end{aligned}
$$

Recalling (3.28), λ_1 is a generic constant for any $-5 \le \gamma \le 0$. By using

$$
\mathcal{D}^{1,\gamma}_{N,l}(F_R^{s,\gamma}) \lesssim_{N,l} |\mathrm{M}|^2_{H^N_x} + \mathcal{D}^{1,\gamma}_{N,l}((F_R^{s,\gamma})_2),
$$

we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \Xi^{1,\gamma}_{N,l}(F_R^{s,\gamma}) + \frac{1}{8}\lambda_1 (|\mathrm{M}|^2_{H^N_x} + \mathcal{D}^{1,\gamma}_{N,l}((F_R^{s,\gamma})_2)) \\
&\le C_{N,l} Z_{1,\gamma,N,l} (C_\gamma \|f^{s,\gamma}\|_{H^N_{x,v}} + C_\gamma \|f^\gamma\|_{H^N_{x,v}} + C_\gamma^2 \|f^{s,\gamma}\|^2_{H^N_x L^2}) \mathcal{D}^{1,\gamma}_{N,l}(F_R^{s,\gamma}) \\
&\quad + C_{N,l} Z^2_{1,\gamma,N,l}\Big\{C_\gamma^2 \|F_R^{s,\gamma}\|^2_{H^N_{x,v}} \mathcal{D}^{s,\gamma}_{N,l}(f^{s,\gamma}) + C_\gamma^2 \|F_R^{s,\gamma}\|^2_{H^N_{x,v}} \mathcal{D}^{1,\gamma}_{N,l}(f^\gamma)\Big\} \\
&\quad + C_{N,l} Z^2_{1,\gamma,N,l}\Big\{C_\gamma^2 \mathcal{D}^{s,\gamma}_{N+3,l+5-3(\gamma+2s)+N(2-2s)}(f^{s,\gamma}) \\
&\quad + C_\gamma^2 \|f^{s,\gamma}\|^2_{H^{N+3}_{x,v}} \mathcal{D}^{1,\gamma}_{N+3,l+5-3(\gamma+2)}(f^\gamma)\Big\}. \qquad (4.69)
\end{aligned}
$$

By the assumption (1.40) and Theorem 4.1, the solutions $f^{s,\gamma}$ and f^γ satisfy

$$
\begin{aligned}
&\mathcal{E}^{s,\gamma}_{N+3,l_*}(f^{s,\gamma}(t)) + \frac{1}{8}\lambda_s \int_0^t (|\mathrm{M}(\tau)|^2_{H^{N+3}_x} + \mathcal{D}^{s,\gamma}_{N+3,l_*}(f_2^{s,\gamma})(\tau))\mathrm{d}\tau \\
&\quad \le Z_{s,\gamma,N,l} \mathcal{E}^{s,\gamma}_{N+3,l_*}(f_0), \qquad (4.70) \\
&\mathcal{E}^{1,\gamma}_{N+3,l_*}(f^{\gamma}(t)) + \frac{1}{8}\lambda_1 \int_0^t (|\mathrm{M}(\tau)|^2_{H^{N+3}_x} + \mathcal{D}^{1,\gamma}_{N+3,l_*}(f_2^{\gamma})(\tau))\mathrm{d}\tau \\
&\quad \le Z_{1,\gamma,N,l} \mathcal{E}^{1,\gamma}_{N+3,l_*}(f_0), \qquad (4.71)
\end{aligned}
$$

where $l_* = l + 2N - 3\gamma + 5 \ge l + 5 - 3(\gamma + 2s) + N(2 - 2s)$. By the smallness of the energy functional, the term containing $\mathcal{D}^{1,\gamma}_{N,l}(F_R^{s,\gamma})$ is absorbed by the left hand side in (4.69). Then the initial condition $F_R^{s,\gamma}(0) = 0$ implies

$$\sup_{t\geq 0} \Xi^{1,\gamma}_{N,l}(F^{s,\gamma}_R(t)) \leq \exp\left(C_{N,l} Z^3_{s,\gamma,N,l} C^2_\gamma \mathcal{E}^{1,\gamma}_{N+3,l_*}(f_0)\right) C_{N,l} Z^3_{s,\gamma,N,l} C^2_\gamma \mathcal{E}^{1,\gamma}_{N+3,l_*}(f_0)$$
$$\leq \exp\left(C_{N,l} Z^3_{s,\gamma,N,l} C^2_\gamma \mathcal{E}^{1,\gamma}_{N+3,l_*}(f_0)\right).$$

Recalling $F^{s,\gamma}_R := (1-s)^{-1}(f^{s,\gamma} - f^\gamma)$, (1.37) and (4.50), we get (1.41). This completes the proof of Theorem 1.1.

5 Appendix

We now prove Proposition 4.2.

Proof of Proposition 4.2 The proof is based on [12] and [25]. Recall the Boltzmann operator $Q^{s,\gamma}_B$ in (1.19) and the kernel $B^{s,\gamma}$ in (1.21). Following the proof in [12] and [25], we derive that

$$\begin{aligned} Q^{s,\gamma}_B(g,h) &= \int_{\mathbb{R}^3} \left(\nabla_v - \nabla_{v_*}\right) \cdot \left[U^{s,\gamma}_1(v-v_*)\left(\nabla_v - \nabla_{v_*}\right)(g_* h)\right] \mathrm{d}v_* \\ &+ \int_{\mathbb{R}^3} \left[U^{s,\gamma}_2(v-v_*) : \left(\nabla_v - \nabla_{v_*}\right)^2 (g_* h)\right] \mathrm{d}v_* \\ &+ \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} R_1(v,v_*,\sigma) B^{s,\gamma} \mathrm{d}v_* \mathrm{d}\sigma. \end{aligned}$$

where

$$\begin{aligned} U^{s,\gamma}_1(v-v_*) &:= \frac{1}{4}\Big[\,|v-v_*|^2 I_3 - (v-v_*)\otimes(v-v_*)\Big] \int \sin^2\frac{\theta}{2} B^{s,\gamma}\mathrm{d}\sigma, \\ U^{s,\gamma}_2(v-v_*) &:= \left(\frac{3}{4}(v-v_*)\otimes(v-v_*) - \frac{1}{4}|v-v_*|^2 I_3\right)\int \sin^4\frac{\theta}{2} B^{s,\gamma}\mathrm{d}\sigma. \end{aligned}$$

The function $R_1(v,v_*,\sigma)$ reads

$$\begin{aligned} R_1(v,v_*,\sigma) &= r_1(v,v_*,\sigma)\left(g(v_*) - \frac{1}{2}A\cdot\nabla g(v_*) + \frac{1}{8}A\otimes A : \nabla^2 g(v_*) + r_2(v,v_*,\sigma)\right) \\ &+ \frac{1}{8}A\otimes A : \nabla^2 h(v)\left(-\frac{1}{2}A\cdot\nabla g(v_*) + \frac{1}{8}A\otimes A : \nabla^2 g(v_*) + r_2(v,v_*,\sigma)\right) \\ &+ \frac{1}{2}A\cdot\nabla h(v)\left(\frac{1}{8}A\otimes A : \nabla^2 g(v_*) + r_2(v,v_*,\sigma)\right) + h(v) r_2(v,v_*,\sigma), \end{aligned}$$

where $A = 2(v'-v)$ and

$$r_1(v,v_*,\sigma) = \frac{1}{16}\sum_{1\leq i,j,k\leq 3}\int_0^1 (1-\kappa)^2 A_i A_j A_k \partial^3_{ijk} h\left(v + \kappa\left(v'-v\right)\right)\mathrm{d}\kappa,$$

$$r_2(v, v_*, \sigma) = -\frac{1}{16} \sum_{1 \le i,j,k \le 3} \int_0^1 (1-\iota)^2 A_i A_j A_k \partial^3_{ijk} g \left(v_* + \iota \left(v'_* - v_*\right)\right) \mathrm{d}\iota.$$

Note that $R_1(v, v_*, \sigma)$ contains $|A|^k$ for $k \ge 3$.

Recalling (1.23) and (1.3) with $\Lambda = \pi$, it is straightforward to check that

$$U_1^{s,\gamma}(z) = \left(\frac{1}{4} \int \sin^2 \frac{\theta}{2} b^s(\theta) \mathrm{d}\sigma\right) |z|^{\gamma+2} \Pi(z) = 2^{s-1} \pi |z|^{\gamma+2} \Pi(z) = 2^{s-1} a^\gamma(z).$$

Recall the Landau operator Q_L^γ given by (1.2) and (1.3) with $\Lambda = \pi$. In another form,

$$Q_L^\gamma(g, h) = \int_{\mathbb{R}^3} \left(\nabla_v - \nabla_{v_*}\right) \cdot \left[a^\gamma (v - v_*) \left(\nabla_v - \nabla_{v_*}\right) (g_* h)\right] \mathrm{d}v_*,$$

which gives

$$Q_B^{s,\gamma}(g, h) = 2^{s-1} Q_L^\gamma(g, h) + \int_{\mathbb{R}^3} \left[U_2^{s,\gamma}(v - v_*) : \left(\nabla_v - \nabla_{v_*}\right)^2 (g_* h)\right] \mathrm{d}v_* \\ + \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} R_1(v, v_*, \sigma) B^{s,\gamma} \mathrm{d}v_* \mathrm{d}\sigma. \tag{5.1}$$

We now have

$$Q_B^{s,\gamma}(g, h) - Q_L^\gamma(g, h) = (2^{s-1} - 1) Q_L^\gamma(g, h) \\ + \int_{\mathbb{R}^3} \left[U_2^{s,\gamma}(v - v_*) : \left(\nabla_v - \nabla_{v_*}\right)^2 (g_* h)\right] \mathrm{d}v_* \\ + \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} R_1(v, v_*, \sigma) B^{s,\gamma} \mathrm{d}v_* \mathrm{d}\sigma := \sum_{i=1}^3 E_i.$$

Note that for $0 < s < 1$,

$$|2^{s-1} - 1| \le 1 - s. \tag{5.2}$$

In order to show the result for $\gamma > -5$, we rewrite the Landau operator $Q_L^\gamma(g, h)$. Recall that

$$Q_L^\gamma(g, h) = \nabla \cdot \int_{\mathbb{R}^3} a^\gamma (v - v_*) (g_* \nabla h - (\nabla g)_* h) \mathrm{d}v_* \\ = \nabla \cdot [(a^\gamma * g) \nabla h - (a^\gamma * \nabla g) h].$$

In order not to have any derivatives on the kernel function a^γ, we write

$$Q_L^\gamma(g, h) = (a^\gamma * g) : \nabla^2 h - (a^\gamma * : \nabla^2 g) h.$$

More precisely,

$$Q_L^\gamma(g,h) = \sum_{i,j=1}^{3} (a_{ij}^\gamma * g)\partial_{ij}^2 h - \sum_{i,j=1}^{3} (a_{ij}^\gamma * \partial_{ij}^2 g)h.$$

Note that

$$|a^\gamma(v-v_*)| \lesssim |v-v_*|^{\gamma+2}. \tag{5.3}$$

To estimate $\left|\langle Q_L^\gamma(g,h), W_l\psi\rangle\right|$, it suffices to consider the following type of integral

$$\int |v-v_*|^{\gamma+2}|(\partial^{\alpha_1}g)_*\partial^{\alpha_2}hW_l\psi|\mathrm{d}v\mathrm{d}v_*, \tag{5.4}$$

where $(|\alpha_1|, |\alpha_2|) = (2,0)$ or $(0,2)$.

Note that

$$\begin{aligned}\int \sin^4\frac{\theta}{2}B^{s,\gamma}\mathrm{d}\sigma \le \int \sin^3\frac{\theta}{2}B^{s,\gamma}\mathrm{d}\sigma &= 8\pi(1-s)|v-v_*|^\gamma \int_0^{1/\sqrt{2}} t^{2-2s}dt \\ &\lesssim (1-s)|v-v_*|^\gamma,\end{aligned} \tag{5.5}$$

which gives

$$|U_2^{s,\gamma}(v-v_*)| \lesssim (1-s)|v-v_*|^{\gamma+2}. \tag{5.6}$$

This shows that in order to estimate $|\langle E_2, W_l\psi\rangle|$, it suffices to consider the integral (5.4) for $|\alpha_1|+|\alpha_2| = 2$. In general, we consider

$$\int |v-v_*|^{\gamma+2}|g_*h\psi|\mathrm{d}v\mathrm{d}v_*,$$

for $\gamma > -5$. Note that the integral has singularity as $\gamma \to (-5)^+$. It is obvious that $|v-v_*| \le W(v)W(v_*)$. If $\gamma+2 \ge 0$, then

$$|v-v_*|^{\gamma+2} \le W_{\gamma+2}(v)W_{\gamma+2}(v_*),$$

which gives

$$\int |v-v_*|^{\gamma+2}|g_*h\psi|\mathrm{d}v\mathrm{d}v_* \le |g|_{L^1_{\gamma+2}}|h|_{L^2_{b_1}}|\psi|_{L^2_{b_2}},$$

where $b_1, b_2 \in \mathbb{R}$ satisfying $b_1+b_2 = \gamma+2$. If $\gamma+2<0$, then

$$\begin{aligned}|v-v_*|^{\gamma+2} \lesssim\ & 1_{|v-v_*|\le 1}|v-v_*|^{\gamma+2}W_{\gamma+2}(v)W_{|\gamma+2|}(v_*) \\ &+1_{|v-v_*|\ge 1}W_{\gamma+2}(v)W_{|\gamma+2|}(v_*),\end{aligned}$$

which gives

$$\int |v-v_*|^{\gamma+2}|g_* h\psi|\mathrm{d}v\mathrm{d}v_* \lesssim \frac{1}{\gamma+5}|g|_{L^p_{|\gamma+2|}}|h|_{L^q_{b_1}}|\psi|_{L^2_{b_2}} + |g|_{L^1_{|\gamma+2|}}|h|_{L^2_{b_1}}|\psi|_{L^2_{b_2}},$$

where $2 \le p, q \le \infty$ satisfying $1/p + 1/q = 1/2$. Here we have used $\int 1_{|v-v_*|\le 1}|v-v_*|^{\gamma+2}\mathrm{d}v_* \lesssim \frac{1}{\gamma+5}$. In summary, by using the basic inequality $|g|_{L^1} \lesssim |g|_{L^2_2}$ and the embedding $H^2 \hookrightarrow L^\infty$ and $H^s \hookrightarrow L^p$ where $1/p = 1/2 - s/3$, for $-5 < \gamma \le 0$, we have

$$\int |v-v_*|^{\gamma+2}|g_* h\psi|\mathrm{d}v\mathrm{d}v_* \lesssim \frac{1}{\gamma+5}|g|_{H^{s_1}_{|\gamma+2|+2}}|h|_{H^{s_2}_{b_1}}|\psi|_{L^2_{b_2}}, \tag{5.7}$$

where $0 \le s_1, s_2 \le 2$ satisfying $s_1 + s_2 = 2$.

By applying (5.7) for estimation on (5.4), and by recalling (5.2) and (5.6), we obtain

$$\left|\left\langle (2^{s-1}-1)Q^\gamma_L(g,h), W_l\psi\right\rangle\right| + |\langle E_2, W_l\psi\rangle| \lesssim \frac{1-s}{\gamma+5}|g|_{H^2_{|\gamma+2|+2}}|h|_{H^2_{l+b_1}}|\psi|_{L^2_{b_2}}, \tag{5.8}$$

where $b_1 + b_2 = \gamma + 2$.

We now turn to estimate E_3. By the fact $|A| \lesssim \sin\frac{\theta}{2}|v-v_*|$, one has $\max\{|A|^3, |A|^4, |A|^5, |A|^6\} \lesssim \sin^3\frac{\theta}{2}|v-v_*|^3 W_3(v-v_*)$. Plugging this into the definition of $R_1(v, v_*, \sigma)$, one has

$$|R_1(v,v_*,\sigma)| \lesssim \sin^3\frac{\theta}{2}|v-v_*|^3 W_3(v-v_*)\sum_{i=1}^4 R_{1,i}(v,v_*,\sigma),$$

$$R_{1,1}(v,v_*,\sigma) = \sum_{i=0}^2\sum_{j=3-i}^2 |\nabla^i g(v_*)||\nabla^j h(v)|,$$

$$R_{1,2}(v,v_*,\sigma) = \sum_{i=0}^2 |\nabla^i g(v_*)|\int_0^1 (1-\kappa)^2\left|\nabla^3 h\,(v(\kappa))\right|\mathrm{d}\kappa,$$

$$R_{1,3}(v,v_*,\sigma) = \sum_{i=0}^2 |\nabla^i h(v)|\int_0^1 (1-\iota)^2\left|\nabla^3 g\,(v_*(\iota))\right|\mathrm{d}\iota,$$

$$R_{1,4}(v,v_*,\sigma) = \int_0^1 (1-\iota)^2\left|\nabla^3 g\,(v_*(\iota))\right|\mathrm{d}\iota\int_0^1 (1-\kappa)^2\left|\nabla^3 h\,(v(\kappa))\right|\mathrm{d}\kappa.$$

Then we have $|\langle E_3, W_l\psi\rangle| \lesssim \sum_{i=1}^4 J_i$, where

$$J_i = \int B^{s,\gamma} R_{1,i}(v,v_*,\sigma)\sin^3\frac{\theta}{2}|v-v_*|^3 W_3(v-v_*)W_l(v)|\psi(v)|\mathrm{d}V.$$

In general, for $0 \le \iota, \kappa \le 1$, we consider

$$\mathcal{I}(g,h) = \int B^{s,\gamma} \sin^3 \frac{\theta}{2} |v - v_*|^3 W_3(v - v_*) W_l(v) |g(v_*(\iota)) h(v(\kappa)) \psi(v)| \mathrm{d}V.$$

Let $a_1, a_2 \in \mathbb{R}$ satisfying $a_1 + a_2 = \gamma + 6$. If $\gamma + 3 \ge 0$, then

$$|v - v_*|^{\gamma+3} W_3(v - v_*) W_l(v) \lesssim_{l,a_1,a_2} W_{l+|a_1|+|a_2|}(v_*(\iota)) W_{l+a_1}(v(\kappa)) W_{a_2}(v). \tag{5.9}$$

If $\gamma + 3 < 0$, we have

$$\begin{aligned} &|vv_*|^{\ gamma+3} W_3(vv_*) W_l(v) \\ &\lesssim_{l,a_1,a_2} 1_{|v-v_*|\le 1} |vv_*|^{\ gamma+3} W_{l+|a_1|+|a_2|}(v_*(\iota)) W_{l+a_1}(v(\kappa)) W_{a_2}(v) \\ &+ 1_{|v-v_*|\ge 1} W_{l+|a_1|+|a_2|}(v_*(\iota)) W_{l+a_1}(v(\kappa)) W_{a_2}(v). \end{aligned} \tag{5.10}$$

By the above estimates, we have

$$\begin{aligned} |\mathcal{I}(g,h)| \lesssim_{l,a_1,a_2} & \int b^s(\theta) \sin^3 \frac{\theta}{2} |\tilde{g}(v_*(\iota)) \tilde{h}(v(\kappa)) \tilde{\psi}(v)| \mathrm{d}V \\ & + 1_{\gamma+3<0} \int 1_{|v-v_*|\le 1} |v - v_*|^{\gamma+3} b^s(\theta) \sin^3 \frac{\theta}{2} |\tilde{g}(v_*(\iota)) \tilde{h}(v(\kappa)) \tilde{\psi}(v)| \mathrm{d}V \\ & := \mathcal{I}_1(g,h) + \mathcal{I}_2(g,h), \end{aligned}$$

where $\tilde{g} = W_{l+|a_1|+|a_2|} g$, $\tilde{h} = W_{l+a_1} h$, $\tilde{\psi} = W_{a_2} \psi$.

We now consider the functional $\mathcal{I}_1(g,h)$ where there is no singularity. By Cauchy–Schwarz inequality, applying the change of variable (2.27), using the fact $1 \le \psi_a(\theta) \le \sqrt{2}$ and (5.5), we have

$$\begin{aligned} \mathcal{I}_1(g,h) &\lesssim \left(\int b^s(\theta) \sin^3 \frac{\theta}{2} |\tilde{g}(v_*(\iota)) \tilde{h}^2(v(\kappa))| \mathrm{d}V \right)^{1/2} \\ &\quad \left(\int b^s(\theta) \sin^3 \frac{\theta}{2} (\psi_{\kappa+\iota}(\theta))^3 |\tilde{g}(v_*(\iota)) \tilde{\psi}^2(v)| \mathrm{d}V \right)^{1/2} \\ &= \left(\int b^s(\theta) \sin^3 \frac{\theta}{2} (\psi_{\kappa+\iota}(\theta))^3 |\tilde{g}(v_*) \tilde{h}^2(v)| \mathrm{d}V \right)^{1/2} \\ &\quad \left(\int b^s(\theta) \sin^3 \frac{\theta}{2} (\psi_{\iota}(\theta))^3 |\tilde{g}(v_*) \tilde{\psi}^2(v)| \mathrm{d}V \right)^{1/2} \\ &\lesssim \left(\int b^s(\theta) \sin^3 \frac{\theta}{2} |\tilde{g}(v_*) \tilde{h}^2(v)| \mathrm{d}V \right)^{1/2} \\ &\quad \left(\int b^s(\theta) \sin^3 \frac{\theta}{2} |\tilde{g}(v_*) \tilde{\psi}^2(v)| \mathrm{d}V \right)^{1/2} \end{aligned}$$

$$\lesssim (1-s)\left(\int |\tilde{g}(v_*)\tilde{h}^2(v)|\mathrm{d}v\mathrm{d}v_*\right)^{1/2}\left(\int |\tilde{g}(v_*)\tilde{\psi}^2(v)|\mathrm{d}v\mathrm{d}v_*\right)^{1/2}$$
$$\lesssim (1-s)|\tilde{g}|_{L^1}|\tilde{h}|_{L^2}|\tilde{\psi}|_{L^2}.$$

We now consider the functional $\mathcal{I}_2(g,h)$ where there is singularity as $|v-v_*|\to 0$. By Cauchy–Schwarz inequality, applying the change of variable (2.27), using the fact $1\le\psi_\kappa(\theta)\le\sqrt{2}$ and (5.5), we have

$$\mathcal{I}_2(g,h)\lesssim (1-s)\left(\int 1_{|v-v_*|\le 1}|v-v_*|^{\gamma+3}|\tilde{g}(v_*)\tilde{h}^2(v)|\mathrm{d}v\mathrm{d}v_*\right)^{1/2}$$
$$\times\left(\int 1_{|v-v_*|\le 1}|v-v_*|^{\gamma+3}|\tilde{g}(v_*)\tilde{\psi}^2(v)|\mathrm{d}v\mathrm{d}v_*\right)^{1/2}.$$

If $\gamma>-9/2$, using $\int 1_{|v-v_*|\le 1}|v-v_*|^{\gamma+3}|g_*|\mathrm{d}v_*\lesssim\frac{1}{\gamma+9/2}|g|_{L^2}$ to obtain

$$\mathcal{I}_2(g,h)\lesssim (1-s)(\gamma+\frac{9}{2})^{-1}|\tilde{g}|_{L^2}|\tilde{h}|_{L^2}|\tilde{\psi}|_{L^2}. \tag{5.11}$$

If $\gamma>-11/2$, we will have

$$\mathcal{I}_2(g,h)\lesssim (1-s)\left(\gamma+\frac{11}{2}\right)^{-5/6}|\tilde{g}|_{L^p}|\tilde{h}|_{L^q}|\tilde{\psi}|_{L^2}, \tag{5.12}$$

where $2\le p,q\le 6$ satisfying $1/p+1/q=2/3$. Indeed, putting together $\tilde{g}$ and $\tilde{h}$, we can get

$$\mathcal{I}_2(g,h)\lesssim (1-s)\left(\int 1_{|v-v_*|\le 1}|v-v_*|^{\frac{4}{5}(\gamma+3)}|\tilde{g}^2(v_*)\tilde{h}^2(v)|\mathrm{d}v\mathrm{d}v_*\right)^{1/2}$$
$$\times\left(\int 1_{|v-v_*|\le 1}|v-v_*|^{\frac{6}{5}(\gamma+3)}|\tilde{\psi}^2(v)|\mathrm{d}v\mathrm{d}v_*\right)^{1/2}.$$

Using $\int 1_{|v-v_*|\le 1}|v-v_*|^{\frac{6}{5}(\gamma+3)}\mathrm{d}v_*\lesssim\frac{1}{\gamma+11/2}$, the latter integral is bounded by $(\gamma+11/2)^{-1}|\tilde{\psi}|^2_{L^2}$. Let $k(z)=1_{|z|\le 1}|z|^{\frac{4}{5}(\gamma+3)}$, then $|k|_{L^{3/2}}\lesssim(\gamma+11/2)^{-2/3}$. Thus, the first integral is bounded by

$$|(k*\tilde{g}^2)\tilde{h}^2|_{L^1}\lesssim|(k*\tilde{g}^2)|_{L^r}|\tilde{h}^2|_{L^{r'}}\lesssim|k|_{L^{3/2}}|\tilde{g}^2|_{L^{q'}}|\tilde{h}^2|_{L^{r'}}$$
$$\lesssim|k|_{L^{3/2}}|\tilde{g}|^2_{L^{2q'}}|\tilde{h}|^2_{L^{2r'}}\lesssim(\gamma+11/2)^{-2/3}|\tilde{g}|^2_{L^{2q'}}|\tilde{h}|^2_{L^{2r'}},$$

where $1/r+1/r'=1$, $1+1/r=2/3+1/q'$. Then we get $\frac{1}{2q'}+\frac{1}{2r'}=\frac{2}{3}$. Combining the two estimates yields (5.12).

We conclude that if $-9/2 < \gamma \le 0$,

$$|\mathcal{I}(g,h)| \lesssim (1-s)\left(\gamma+\frac{9}{2}\right)^{-1} |\tilde{g}|_{L^2_2}|\tilde{h}|_{L^2}|\tilde{\psi}|_{L^2};$$

if $-11/2 < \gamma \le 0$, by the Sobolev embedding $H^s \hookrightarrow L^p$ where $1/p = 1/2 - s/3$,

$$|\mathcal{I}(g,h)| \lesssim (1-s)\left(\gamma+\frac{11}{2}\right)^{-5/6} |\tilde{g}|_{H^{s_1}_2}|\tilde{h}|_{H^{s_2}}|\tilde{\psi}|_{L^2}.$$

Therefore, if $-9/2 < \gamma \le 0$,

$$|\langle E_3, W_l\psi\rangle| \lesssim (1-s)\left(\gamma+\frac{9}{2}\right)^{-1} |g|_{H^3_{l+|a_1|+|a_2|+2}}|h|_{H^3_{l+a_1}}|\psi|_{L^2_{a_2}}; \tag{5.13}$$

if $-11/2 < \gamma \le 0$,

$$|\langle E_3, W_l\psi\rangle| \lesssim (1-s)\left(\gamma+\frac{11}{2}\right)^{-5/6} |g|_{H^{3+s_1}_{l+|a_1|+|a_2|+2}}|h|_{H^{3+s_2}_{l+a_1}}|\psi|_{L^2_{a_2}}. \tag{5.14}$$

By combining (5.8), (5.13) and (5.14), the proof of Proposition 4.2 is completed. □

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