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On possibilities of improving the accuracy of the geocentric gravitational constant GM by combining SLR and atomic clocks measurements

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Abstract: Nowadays, the geocentric gravitational constant GM is determined by solving equations of motion for trajectories of artificial satellites measured by Satellite Laser Ranging (SLR). The estimated value of GM and its uncertainty $398600441.8 \pm 0.8 \times 10^6$ m^{3}s^{-2} are currently adopted by the International Astronomical Union. In this study, we investigate possibility of improving the accuracy of GM by integrating atomic clocks measurements with SLR. The functional model defines GM in terms of geopotential differences observed by atomic clocks at two points in space and their distance measured by SLR. Two types of observation equations are established. The first equation defines geopotential differences with respect to the geoidal geopotential value W_0. The second equation defines distances with respect to the geocentric position of ground-based station determined from GNSS measurements. With the improving stability of atomic clocks to 10^{-18}, it will be possible to measure geopotential differences with the accuracy ± 0.1 m^{2}s^{-2} (equivalent to ± 1 cm in terms of the geoidal heights), while SLR measurements can currently be carried out with sub-centimetre accuracy under optimal conditions and applying advanced corrections and numerical procedures. Taking into consideration both, accuracy characteristics and their expected improvement, we conduct sensitivity analysis to assess accuracy requirements needed to improve the accuracy GM ($\pm 0.8 \times 10^6$ m^{3}s^{-2}). Error analysis indicates that combination of relativistic measurements with SLR cannot currently improve the accuracy of GM due to insufficient stability of atomic clocks. Nevertheless, the accuracy improvement by an order of magnitude might be feasible in the feature if relativistic measurements are carried out by atomic clocks with stability 10^{-20} (or better), while also achieving sub-millimetre accuracy of SLR. In this way, integration of relativistic measurements with SLR could improve the current accuracy of GM, while the critical aspect is determination of the geoidal geopotential value W_0 with sub-millimetre accuracy in terms of geoidal heights that could be achievable.

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1. Introduction

The geocentric gravitational constant GM is defined as the product of the total mass of the Earth M (including the atmosphere) and Newton's gravitational constant G. This constant is commonly used in physical geodesy and gravimetric geophysics in computations dealing with the Earth's gravity field. The value GM $= 398600441.8 \pm 0.000000008 \times 10^6\ \mathrm{m^3\,s^{-2}}$ (that is compatible with the barycentric coordinate time), currently adopted by the Integrational Astronomical Union (IAU) and published in its 2009 System of Astronomical Constants, was estimated by *Ries et al. (1992).* The value $3.986004415 \pm 0.000000008 \times 10^{14}\ \mathrm{m^3\,s^{-2}}$ can be found in the IERS 2003 conventions (*McCarthy and Petit, 2004*). More recently, *Amin et al. (2019)* and *Cherrier et al. (2022)* reported the improved estimates $398600460.55 \pm 0.03 \times 10^6\ \mathrm{m^3\,s^{-2}}$ and $398600441.9 \pm 0.2 \times 10^6\ \mathrm{m^3\,s^{-2}}$ respectively, but both estimates have not yet been officially adopted by IAU.

The GM value has a better relative accuracy than individual values of Newton's gravitational constant G and the total mass of the Earth M that are currently defined by the values G $= 6.67428 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ and M $= 5.9722 \pm 0.0006 \times 10^{24}$ kg, respectively; see the IAU 2009 System of Astronomical Constants. It is worth noting that Newton's gravitational constant is one of the most fundamental physical constants adopted, for instance, in definition of Planck units (Planck time, length, mass, change, and temperature) that are used in the quantum theory of gravity and cosmology (e.g., *Kisak, 2015*). Despite advances in experimental physics (Cavendish experiment), the universal gravitational constant G remains one of the least accurately estimated constants in physics. On the other hand, the geocentric (or generally any planeto-centric) constant GM can be estimated more accurately by analysing trajectories of artificial as well as natural satellites (*Lerch et al., 1978; Cherrier et al., 2022; Ries et al., 1992*).

A practical determination of GM is based on solving the equations of motion with trajectories of artificial satellites measured by Satellite Laser Ranging (SLR). SLR is a highly precise geodetic technique used to measure the distance between a ground-based station and a satellite equipped with retroreflectors. The accuracy of SLR measurements has significantly improved over the years, reaching now the mm-level precision (*Pearlman et al., 2019*). This advancement is largely due to technological improvements in laser systems, timing devices, and data processing strategies. One of the key factors contributing to the high accuracy of SLR is the use of short-pulse lasers, which allow for precise time-of-flight measurements. According to *Pearlman et al. (2019)*, modern SLR systems can achieve range accuracies better than ± 1 cm, with some systems reaching sub-millimetre precision under optimal conditions. This level of accuracy is crucial for applications such as monitoring the Earth's gravitational field, tectonic plate movements, and sea level rise. Furthermore, advancements in atmospheric modelling have enhanced the accuracy of SLR observations by correcting for atmospheric delays. The integration of improved models for tropospheric and ionospheric refraction, as discussed by *Kehm et al. (2018)*, has reduced systematic errors in SLR data. SLR's accuracy is also bolstered by international collaboration and standardization efforts, such as those led by the International Laser Ranging Service (ILRS) of the International Association of Geodesy (IAG). These efforts ensure consistent data quality and facilitate the integration of SLR data with other geodetic techniques, enhancing the overall understanding of Earth's dynamics (*Noll et al., 2019*).

It is worth noting that the accuracy of atomic clocks depends on many aspects, particularly the type of atomic clock and the specific atom used. For more details about these aspects, we refer readers to *Petit et al. (2014)* and citations therein. Another aspect that limits the use of atomic clocks is the reproducibility (see *Bagherbandi et al., 2023*). It is expected that stability of atomic clocks at the level 10^{-18} could be achieved by optical atomic clocks which utilize strontium or ytterbium atoms. Even better stability (potentially by orders of magnitude) might be achieved in the future based on quantum logic clocks that will utilize entangled ions to reduce quantum noise (e.g., *Chou et al., 2010*). Alternative options have been proposed based on facilitating nuclear transitions (e.g., thorium-229) which could be even more stable and less susceptible to external perturbations than electron-based transitions (e.g., *Stellmer et al. 2016*).

Since GM is a fundamental constant defining the Earth's gravity field, measurements of gravity field parameters could potentially be used together with SLR observables to determine GM with a better accuracy than that currently obtained solely from SLR data. Values of the gravity potential cannot be measured by classical methods in physics. However, its differences between two points in space can be observed by atomic clocks based on adopting the relativistic theory of time dilatation (cf. *Bjerhammar, 1985; 1986*). Einstein's general theory of relativity predicts that clocks near a gravitating body tick slower compared to clocks at zero gravitation. Comparing frequencies of two clocks provides an observable that can be related to the difference in the gravity potential between them (e.g., *Denker et al., 2018*). Currently, the most stable laboratory clocks have been evaluated to frequency uncertainties 10^{-18} that corresponds to the accuracy of the respective geopotential differences $\pm 0.1\,\mathrm{m}^2\,\mathrm{s}^{-2}$. A practical use of these measurements for chronometric levelling in the context of vertical datum unification and other applications, such as the determination of GM, has until now been limited by current accuracy restrictions and many other technical considerations (see, e.g., *Mai, 2013; Kopeikin et al., 2018; Mehlstäubler et al. 2018; Puetzfeld and Lämmerzahl, 2019; Shen et al., 2011; 2019*). The main problem is maintaining a highly accurate transfer between two clocks. The noise and feedback properties of the link must be reduced to an extent that overcomes limits imposed by the laser phase delay and noise sources on the fiber link (*Schioppo et al., 2022*).

Assuming a rapid improvement of SLR's accuracy and stability of atomic clocks in the foreseeable future, we examine the possibility of improving the current accuracy of GM by integrating atomic clocks measurements with SLR. Taking into consideration status of both techniques as well as anticipated accuracy and stability improvements, we establish the error propagation model to find a functional relation that describes the dependence of the accuracy of GM on errors of geopotential differences (as measured by atomic clocks) and distances (as measured by SLR). We then conduct a sensitivity analysis to assess the accuracy criteria necessary to improve the GM estimates.

The study is organized into four sections. Fundamental definitions of the Earth's gravity potential are briefly recapitulated in Section 2. A theoretical model for the determination of GM from SLR and atomic clocks measurements is postulated in Section 3. The error analysis is conducted in Section 4. Summary of results and concluding remarks are given in Section 5.

2. Theory

This section provides basic definitions of the Earth's gravity potential that are then used to define the functional model to determine GM from SLR and atomic clocks measurements in Section 3.

2.1. Spatial representation of the Earth's gravity potential

The Earth's gravitational potential V at the external computation point (r,Ω), i.e., $V(r,\Omega)$ is defined by Newton's volume integral in the following form (*Kellogg, 1967*):

$$V = V(r,\Omega) = \mathrm{G} \iint_0^{\mathrm{R}+H'} \rho\,(r',\Omega')\,\ell^{-1}(r,\psi,r')\,r'^{\,2}\,\mathrm{d}r'\mathrm{d}\Omega', \tag{1}$$

where r is the geocentric radius, and the pair $\Omega = (\theta,\lambda)$ represents the geocentric direction of the computation point with θ and λ denoting spherical co-latitude and longitude, respectively. The volume integral on the right-hand side of Eq. (1) is evaluated for the 3D mass density distribution function $\rho(r',\Omega')$ within the Earth's interior limited by the Earth's surface described by the geocentric radius r'. In the Earth's spherical approximation, the geocentric radius is defined approximately as the mean Earth's radius R plus the topographic height $H = H(\Omega')$, i.e., $r' = \mathrm{R} + H'$. The term $\mathrm{d}\Omega' = \sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\lambda'$ denotes an infinitesimal surface element of a unit sphere, $\{\Omega' = (\theta',\lambda') : \theta' \in [0,\pi] \wedge \lambda' \in [0,2\pi)\}$ represents the full spatial angle, and $\int \mathrm{d}\Omega' = \iint \sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\lambda'$. The Euclidean spatial distance ℓ between the computation and integration (running) points in Eq. (1) reads:

$$\ell(r,\psi,r') = \sqrt{r^2 + r'^2 - 2rr'\cos\psi}\,, \tag{2}$$

where $\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda'-\lambda)$ is cosine of the spherical distance ψ between the two points.

Since the definition of the gravitational potential in Eq. (1) does not explicitly involve the GM parameter, we first define the total mass of the Earth M and then describe the gravitational potential V in terms of GM.

The total mass of the Earth M is defined as the volumetric integral of mass density distribution within the Earth:

$$\mathrm{M} = \iint_0^{\mathrm{R}+H'} \rho\,(r',\Omega')\,r'^{\,2}\,\mathrm{d}r'\mathrm{d}\Omega' =$$

$$= \int_{\lambda'=0}^{2\pi}\int_{\theta'=0}^{\pi}\int_{r=0}^{\mathrm{R}+H'} \rho\,(r',\Omega')\, r'^{\,2}\,\mathrm{d}r'\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\lambda' \,. \tag{3}$$

From Eq. (3), GM is given by:

$$\mathrm{GM} = \mathrm{G}\int_{\lambda'=0}^{2\pi}\int_{\theta'=0}^{\pi}\int_{r=0}^{\mathrm{R}+H'} \rho\,(r',\Omega')\, r'^{\,2}\,\mathrm{d}r'\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\lambda' \,. \tag{4}$$

It is important to note that the actual values of M and GM are practically determined for the whole mass of the Earth including the atmosphere, while both parameters in Eqs. (3) and (4) are defined only for the total mass of the solid Earth. This approximation is, nevertheless, permissible because the total mass of the Earth's atmosphere ($\sim 5.15\times10^{18}$ kg; see *Lide, 2013*) is roughly one millionth of the Earth's mass ($\mathrm{M} = 5.9722 \pm 0.0006 \times 10^{24}$ kg).

The Earth's centrifugal potential Φ is defined by:

$$\Phi = \frac{1}{2}\,\overline{\varpi}^2\, r^2 \sin^2\theta \,, \tag{5}$$

where $\overline{\varpi}$ denotes the mean Earth's angular velocity.

The Earth's gravity potential W (i.e., the geopotential), defined as the sum of the Earth's gravitational and centrifugal potentials V and Φ, is then given by:

$$\begin{aligned} W &= V + \Phi = \\ &= \mathrm{G}\int_{\lambda'=0}^{2\pi}\int_{\theta'=0}^{\pi}\int_{r=0}^{\mathrm{R}+H'} \rho\,(r',\Omega')\,\ell^{-1}(r,\psi,r')\, r'^{\,2}\,\mathrm{d}r'\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\lambda' \;+ \\ &\quad + \frac{1}{2}\,\overline{\varpi}^2\, r^2\sin^2\theta \,. \end{aligned} \tag{6}$$

2.2. Spherical representation of the Earth's gravity potential

Substituting from Eq. (4) to Eq. (1) and solving the Laplace equation, the Earth's gravitational potential V can be represented in terms of the spherical harmonics in the following form (e.g., *Heiskanen and Moritz, 1967*):

$$V\,(r,\Omega) = \frac{\mathrm{GM}}{\mathrm{R}}\sum\nolimits_{n=0}^{\infty}\left(\frac{\mathrm{R}}{r}\right)^{n+1}\sum\nolimits_{m=-n}^{n} \mathrm{V}_{n,m} Y_{n,m}\,(\Omega)\,, \tag{7}$$

where $\mathrm{V}_{n,m}$ are (fully-normalized) numerical coefficients of the gravitational potential V of degree n and order m, and (fully-normalized) surface spherical functions $Y_{n,m}$ read:

$$Y_{n,m}(\Omega) = P_{n,m}(\sin\varphi)\begin{cases}\cos m\lambda\,, & m \geq 0\\ \sin|m|\lambda\,, & m < 0\,,\end{cases} \tag{8}$$

where $P_{n,m}$ are (fully-normalized) associated Legendre functions.

From Eqs. (5) and (7), the spherical harmonic representation of the Earth's gravity potential W is given by:

$$W(r,\Omega) = \frac{\mathrm{GM}}{\mathrm{R}}\sum\nolimits_{n=0}^{\infty}\left(\frac{\mathrm{R}}{r}\right)^{n+1}\sum\nolimits_{m=-n}^{n}\mathrm{V}_{n,m}Y_{n,m}(\Omega) + \frac{1}{2}\,\varpi^2\,r^2\sin^2\theta\,. \tag{9}$$

3. Functional model for determination of the GM value

Analysing orbital motions of artificial satellites observed by SLR, *Ries et al. (1992)* determined the geocentric gravitational constant GM with the accuracy $\pm\,0.8\times10^6\,\mathrm{m}^3\,\mathrm{s}^{-2}$ (in terms of the standard deviation). *Cherrier et al. (2022)* reported a better accuracy $\pm\,0.2\times10^6\,\mathrm{m}^3\,\mathrm{s}^{-2}$, but this value has not yet officially been adopted by IAU. To inspect the possibility of improving the accuracy of GM based on combining SLR and atomic clocks measurements, the functional model was established for the geopotential differences ΔW observed by atomic clocks at ground station and satellite positions, and distances between them measured by SLR. To simplify the model, we consider only vertical (radial) distances Δr. As stated above, the current stability of atomic clocks (10^{-18}) allows measuring the geopotential differences ΔW with the accuracy $\pm\,0.1\,\mathrm{m}^2\,\mathrm{s}^{-2}$, while mm-level accuracy of SLR could be achieved under optimal conditions.

The geopotential differences measured by atomic clocks at ground station and satellite positions must be referenced to the geoidal geopotential value W_0. A ground station should preferably be located near the coast to readily establish the relation between the geopotential value of the ground station (such as a tide-gauge station) and the geopotential value W_0 at the geoid. Alternatively, long-term GNSS measurements of the vertical position (i.e., the geocentric radius) of ground stations must be conducted. Both schemes adopted in deriving observations equations for determination of the GM value are explained below.

To simplify the mathematical model, we disregarded the centrifugal potential. The functional relation is then formulated only for the Earth's gravitational potential, so that:

$$V(r,\Omega) = \frac{\mathrm{GM}}{\mathrm{R}} \sum\nolimits_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \sum\nolimits_{m=-n}^{n} \mathrm{V}_{n,m} Y_{n,m}(\Omega) = \\ = \frac{\mathrm{GM}}{\mathrm{R}} \sum\nolimits_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \mathrm{V}_n(\Omega)\,, \tag{10}$$

where $\mathrm{V}_n(\Omega)$ are the Laplace coefficients of the gravitational potential of degree n. Separating the zero-degree spherical harmonic from higher-degree terms in Eq. (10), we get:

$$V = \frac{\mathrm{GM}}{\mathrm{R}} + \frac{\mathrm{GM}}{\mathrm{R}} \sum\nolimits_{n=1}^{\infty} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \mathrm{V}_n(\Omega)\,. \tag{11}$$

Disregarding the first- and higher-degree spherical harmonics of the gravitational potential, derivations are then done only for the first term on the right-hand side of Eq. (11), i.e.:

$$V \approx \frac{\mathrm{GM}}{\mathrm{R}}\,. \tag{12}$$

Adopted approximations in a functional model that is used (in the next section) to derive the error propagation based on disregarding the centrifugal potential and higher-degree spherical harmonics of the gravitational potential are permissible. For the nominal mean angular velocity of the Earth $\overline{\omega} = 7.292115 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$ (*Groten, 2000*), the largest value of the centrifugal potential ($\Phi \approx 108159.5\ \mathrm{m^2\,s^{-2}}$) at the equator is two orders of magnitude smaller than the geoidal geopotential value ($\mathrm{W}_0 = 62636854.2\,\mathrm{m^2\,s^{-2}}$), e.g., *Dayoub et al. (2012).* Similarly, the gravitational contribution of higher-degree spherical harmonics of potential is several orders of magnitude smaller than the zero-degree term $V = \mathrm{GM/R}$. We, therefore, can conduct a theoretical error analysis for a simplified functional model (in Eq. 12) without depreciating numerical findings.

3.1. Observation equation for the geopotential differences

We first define the observation equation for the geopotential difference ΔW measured by atomic clocks. After disregarding the centrifugal potential and non-zero spherical harmonics of the gravitational potential, the functional relation between GM and the gravitational potential difference $\Delta V = V_1 - V_2$ measured at two positions with the geocentric radii r_1 and r_2 is written in the following form:

$$\Delta W \approx \Delta V = V_1 - V_2 = \frac{\mathrm{GM}}{r_1} - \frac{\mathrm{GM}}{r_2}\,. \tag{13}$$

In our numerical experiment, we assumed that the geocentric radius r_1 of a ground station is determined from GNSS measurements, while the geocentric radius r_2 of a satellite is obtained from SLR measurements Δr, hence $r_2 = r_1 + \Delta r$. In this way, we can treat both measurements individually based on specific assumptions on their accuracy. The functional relation in Eq. (13) then becomes:

$$\Delta V = \frac{\mathrm{GM}}{r_1} - \frac{\mathrm{GM}}{r_1 + \Delta r} = \mathrm{GM}\,\frac{\Delta r}{r_1(r_1 + \Delta r)}\,. \tag{14}$$

Separating GM on the left-hand side of Eq. (14), we get:

$$\mathrm{GM} = \Delta V\,\frac{r_1(r_1 + \Delta r)}{\Delta r} = \left(\frac{r_1^2}{\Delta r} + r_1\right)\Delta V\,. \tag{15}$$

3.2. Observation equation for SLR vertical distances

From Eq. (12), we can define the observation equation for the vertical distance Δr measured by SLR. We first define the geocentric radii r_1 and r_2 as a function of GM. We then write:

$$r_1 = \frac{\mathrm{GM}}{V_1}\,, \qquad r_2 = \frac{\mathrm{GM}}{V_2}\,. \tag{16}$$

The vertical distance Δr is obtained from Eq. (16) in the following form:

$$\Delta r = r_2 - r_1 = \frac{\mathrm{GM}}{V_2} - \frac{\mathrm{GM}}{V_1}\,. \tag{17}$$

Note that the vertical distance Δr in Eq. (17) is defined as the difference between r_2 and r_1 to get its positive value, i.e., $\Delta r = r_2 - r_1$. A positive value of ΔV in Eq. (13) is obtained for $\Delta V = V_1 - V_2$ because according to the geodetic convention $V_1 \geq V_2$.

The expression in Eq. (17) is rewritten as:

$$\Delta r = \frac{\mathrm{GM}}{V_2 V_1}(V_1 - V_2) = \frac{\mathrm{GM}}{V_1 V_2}\,\Delta V = \frac{\mathrm{GM}}{V_1(V_1 + \Delta V)}\,\Delta V\,, \tag{18}$$

where V_2 was defined as the corresponding value V_1 and the potential difference ΔV.

From Eq. (18), we write:

$$\mathrm{GM} = \frac{V_1(V_1 + \Delta V)}{\Delta V}\,\Delta r = \left(\frac{V_1^2}{\Delta V} + V_1\right)\Delta r\,. \tag{19}$$

If we consider that $V_1 \approx \mathrm{W}_0$, Eq. (19) becomes:

$$\mathrm{GM} \cong \left(\frac{\mathrm{W}_0^2}{\Delta V} + \mathrm{W}_0\right)\Delta r\,. \tag{20}$$

In the observation equation given in Eq. (20), the (geo)potential difference ΔV is defined with respect to the nominal value of the gravitational potential V_1. Optimally $V_1 = \mathrm{W}_0$, or V_1 is linked with W_0.

4. Error analysis

The observation equations in Eqs. (15) and (20) are used to derive the corresponding error propagation models. GNSS uncertainties in the measured geocentric radius r_1 are considered for Eq. (15), while uncertainties in the estimated value W_0 are considered for Eq. (20).

4.1. Error analysis for Eq. (15)

From Eq. (15), the total differential of the potential difference is obtained in the following form:

$$\mathrm{d}_{\mathrm{GM}} = \left(\frac{r_1^2}{\Delta r} + r_1\right)\mathrm{d}_{\Delta V} + \Delta V\left(\frac{2r_1}{\Delta r} + 1\right)\mathrm{d}_{r_1} - \Delta V\left(\frac{r_1}{\Delta r}\right)^2 \mathrm{d}_{\Delta r}\,, \tag{21}$$

where d_{GM}, $\mathrm{d}_{\Delta V}$, d_{r_1}, and $\mathrm{d}_{\Delta r}$ are, respectively, differentials of GM, ΔV, r_1, and Δr.

We can simplify Eq. (21) by setting $r_1 \approx \mathrm{R}$, without affecting the accuracy of results. We then write:

$$\mathrm{d}_{\mathrm{GM}} \cong \mathrm{R}\left(\frac{\mathrm{R}}{\Delta r} + 1\right)\mathrm{d}_{\Delta V} + \Delta V\left(\frac{2\mathrm{R}}{\Delta r} + 1\right)\mathrm{d}_{r_1} - \Delta V\left(\frac{\mathrm{R}}{\Delta r}\right)^2 \mathrm{d}_{\Delta r}\,, \tag{22}$$

From Eq. (22), we can derive the variance of GM in the following form:

$$\sigma_{\rm GM}^2 \cong {\rm R}^2 \left(\frac{\rm R}{\Delta r}+1\right)^2 \sigma_{\Delta V}^2 + \Delta V^2 \left(\frac{2{\rm R}}{\Delta r}+1\right)^2 \sigma_r^2 + \Delta V^2 \left(\frac{\rm R}{\Delta r}\right)^4 \sigma_{\Delta r}^2 \,, \qquad (23)$$

where $\sigma_{\rm GM}$, $\sigma_{\Delta V}$, σ_r, and $\sigma_{\Delta r}$ are, respectively, the standard deviations of GM, ΔV, r_1, and Δr. Assuming that GNSS, SLR, and geopotential difference measurements are uncorrelated, unknown co-variances in Eq. (23) are disregarded.

As seen in Eq. (23), the GM's accuracy depends on errors in measured values of (geo)potential differences by atomic clocks, GNSS-determined vertical position of ground station, and the vertical (radial) distance measured by SLR. Using the error propagation in Eq. (23), rough estimates of GM errors can be provided for expected errors in ΔV, r_1, and Δr measurements. For the frequency stability of atomic clocks 10^{-18}, we have $\sigma_{\Delta V} \cong \sigma_{\Delta W} \approx \pm\, 0.1\,{\rm m^2\,s^{-2}}$. We further assume mm-level accuracy of SLR measurements, i.e., $\sigma_{\Delta r} \approx \pm 0.001$ m, and cm-level GNSS vertical positioning errors; hence $\sigma_r \approx \pm 0.01$ m. If we consider that SLR measurements are conducted for the low orbit satellites (300 − 1200 km), we can set $\Delta r \approx 1 \times 10^6$ m and ${\rm R} \approx 6371 \times 10^3$ m. Inserting these values to Eq. (23), we get:

$$\begin{aligned} \sigma_{\rm GM} &\cong \sqrt{{\rm R}^2 \left(\frac{\rm R}{\Delta r}+1\right)^2 \sigma_{\Delta V}^2 + \Delta V^2 \left(\frac{2{\rm R}}{\Delta r}+1\right)^2 \sigma_r^2 + \Delta V^2 \left(\frac{\rm R}{\Delta r}\right)^4 \sigma_{\Delta r}^2} \approx \\ &\approx \sqrt{2.2 \times 10^{13} + 1.8 \times 10^{12} + 1.6 \times 10^{11}} \approx \\ &\approx \pm 4.9 \times 10^6 \; [{\rm m^3\,s^{-2}}] \,. \end{aligned} \qquad (24)$$

As seen in Eq. (24), the accuracy of GM is mainly affected by uncertainties in measured geopotential differences, i.e., $\sigma_{\Delta V} \cong \sigma_{\Delta W} \approx \pm\, 0.1\,{\rm m^2\,s^{-2}}$. For the stability of atomic clocks 10^{-18}, the error in estimated value of GM reaches $\sigma_{\rm GM} \approx \pm 4.9 \times 10^6\,{\rm m^3\,s^{-2}}$. Taking into consideration the accuracy of GM according to the officially adopted value by IAU: $\sigma_{\rm GM} \approx \pm 0.8 \times 10^6\,{\rm m^3\,s^{-2}}$ (*Ries et al., 1992*) or $\pm 0.2 \times 10^6\,{\rm m^3\,s^{-2}}$ (*Cherrier et al., 2022*), combination of atomic clocks measurements with SLR actually worsens the GM accuracy. When taking only errors in SLR measurements into consideration in Eq. (24), their contribution $\sigma_{\rm GM} \approx \pm 0.4 \times 10^6\,{\rm m^3\,s^{-2}}$ is very similar to the current accuracy of GM. Hence, relativistic measurements of the geopotential differences by atomic clocks with the stability 10^{-18} cannot improve the accuracy of GM. Furthermore, the current level of GM's accuracy ($\sigma_{\mathrm{GM}} \approx \pm 0.8 \times 10^6\,\mathrm{m}^3\,\mathrm{s}^{-2}$) could only be achieved with the improved stability of atomic clocks by an order of magnitude (i.e., 10^{-19}) while also considering mm-level accuracy for both, SLR as well as GNSS measurements, i.e., $\sigma_{\Delta r} \approx \pm 0.001$ m and $\sigma_r \approx \pm 0.001$ m. In this scenario, errors in atomic clocks and SLR measurements affect the accuracy of GM at the same level. To improve the current accuracy of GM by an order of magnitude would require a sub-millimetre accuracy of SLR and GNSS measurements, i.e., $\sigma_{\Delta r} \approx \pm 0.0001$ m and $\sigma_r \approx \pm 0.0001$ m, and the stability of atomic clocks 10^{-20}, i.e. $\sigma_{\Delta V} \cong \sigma_{\Delta W} \approx \pm 0.001\,\mathrm{m}^2\,\mathrm{s}^{-2}$. Such improvement, especially in GNSS (absolute) vertical positioning, is obviously hard to achieve.

A simulation of this numerical experiment for the laboratory conditions with laser and atomic clocks measurements conducted on a short vertical baseline, e.g., 10 m, are also not convincing. Despite laser measurements could reach nanometre accuracy and atomic clock measurements could be carried out under optimal conditions, no improvement could be expected.

4.2. Error analysis for Eq. (20)

The total differential of Eq. (20) is obtained in the following form:

$$\mathrm{d}_{\mathrm{GM}} \cong \mathrm{W}_0 \left(\frac{\mathrm{W}_0}{\Delta V} + 1 \right) \mathrm{d}_{\Delta r} - \left(\frac{\mathrm{W}_0}{\Delta V} \right)^2 \Delta r\, \mathrm{d}_{\Delta V} + \left(1 + 2 \frac{\mathrm{W}_0}{\Delta V} \right) \Delta r\, \mathrm{d}_{\mathrm{W}_0} . \quad (25)$$

From Eq. (25), the variance σ^2_{GM} of GM is obtained in the following form:

$$\sigma^2_{\mathrm{GM}} \cong \mathrm{W}_0^2 \left(\frac{\mathrm{W}_0}{\Delta V} + 1 \right)^2 \sigma^2_{\Delta r} + \left(\frac{\mathrm{W}_0}{\Delta V} \right)^4 \Delta r^2 \sigma^2_{\Delta V} + \left(1 + 2 \frac{\mathrm{W}_0}{\Delta V} \right)^2 \Delta r^2 \sigma^2_{\mathrm{W}_0}, \quad (26)$$

where $\sigma_V \approx \sigma_{\mathrm{W}_0}$ is the standard deviation of W_0. We again assume that SLR and the geopotential difference measurements are uncorrelated as well as the estimated value W_0.

As seen in Eq. (26), the accuracy of GM in this case depends on errors in measured values of (geo)potential differences by atomic clocks and the vertical distances measured by SLR. In addition, the accuracy of W_0 must be taken into consideration.

The estimation of the reference geopotential value W_0 at the geoid has been the subject of several studies (e.g., *Burša et al., 1999; 2007; Sánchez, 2007; Dayoub et al., 2012*) based on analysis of satellite altimetry data over the (ice-free) oceans. According to these studies, the effect of Earth's gravitational coefficients $V_{n,m}$ above degree $n > 120$ on W_0 is negligible. Moreover, the value W_0 is invariant with respect to the tide system. *Burša et al. (2007)* provided the estimate $W_0 = 62636856.0 \pm 0.5\,\mathrm{m}^2\,\mathrm{s}^{-2}$ that was adopted by IAU. *Sánchez (2007)* determined W_0 from different mean sea level models and global geopotential models. They demonstrated that the choice of the models is not essential for estimating W_0, while the latitudinal domain of the altimetric mean sea level data plays a major role. In a more recent study, *Dayoub et al. (2012)* inspected previous estimates using various methods and datasets. They confirmed the above conclusions but reported and recommended a new value $W_0 = 62636854.2 \pm 0.5\,\mathrm{m}^2\,\mathrm{s}^{-2}$. They also acquired that the dependency of W_0 on the latitude domain is merely due to the mean dynamic topography (i.e., difference between time-averaged ocean surface and the geoid). It is worth noting that estimates of W_0 by *Sánchez (2007)* and *Dayoub et al. (2012)* based on theory published by *Sacerdote and Sansò (2004)* are biased due to disregarding the Earth's mass transport, such as sea level rise. Moreover, these estimates relay only on data over oceans, while ignoring land mass data coverage. *Amin et al. (2019)* proposed an unbiased method to estimate W_0 that uses global data coverage over land and oceans by utilizing satellite altimetry as well as global gravitational model. Unlike previous estimates, their results revealed that that it is not sufficient to use only the satellite-component of a quasi-stationary GGM to estimate W_0. They confirmed a high sensitivity of applied method to the altimetry-based geoid heights, i.e., mean sea surface and mean dynamic topography models. According to their estimate, $W_0 = 62636848.102 \pm 0.004\,\mathrm{m}^2\,\mathrm{s}^{-2}$.

We again consider the frequency stability of atomic clocks 10^{-18}, i.e. $\sigma_{\Delta V} \cong \sigma_{\Delta W} \approx \pm 0.1\,\mathrm{m}^2\,\mathrm{s}^{-2}$, and mm-level accuracy of SLR measurements, i.e., $\sigma_{\Delta r} \approx \pm 0.001\mathrm{m}$. Moreover, we set $\Delta r \approx 1 \times 10^6\mathrm{m}$. Taking into consideration current uncertainties in estimated value W_0, i.e., $\sigma_{W_0} \approx \pm 0.5\ \mathrm{m}^2\,\mathrm{s}^{-2}$, the accuracy of GM according to Eq. (26) becomes:

$$\sigma_{\mathrm{GM}} \cong \sqrt{\mathrm{W}_0^2\left(\frac{\mathrm{W}_0}{\Delta V}+1\right)^2\sigma_{\Delta r}^2+\left(\frac{\mathrm{W}_0}{\Delta V}\right)^4\Delta r^2\sigma_{\Delta V}^2+\left(1+2\frac{\mathrm{W}_0}{\Delta V}\right)^2\Delta r^2\sigma_{\mathrm{W}_0}^2} \approx$$

$$\approx \sqrt{2.1\times10^{11}+1.7\times10^{13}+4.7\times10^{13}} \approx$$

$$\approx \pm 8.0\times10^{6}\ [\mathrm{m^3\,s^{-2}}]. \tag{27}$$

As seen in Eq. (27), the accuracy of GM is mainly affected by uncertainties in measured geopotential differences and the estimated value W_0. For the stability of atomic clocks 10^{-18}, the error in GM reaches $\sigma_{\mathrm{GM}} \approx \pm 8.0\times10^6\ \mathrm{m^3\,s^{-2}}$. This uncertainty is about two times larger than corresponding uncertainty in Eq. (24). The main reason is that uncertainties ($\pm 0.5\ \mathrm{m^2\,s^{-2}}$) in W_0 estimates (equivalent to ± 5 cm in terms of geoidal heights) are much larger than errors in GNSS measurements ($\sigma_r \approx \pm 0.01$ m), see Eq. (24). If the stability of atomic clock improves by an order of magnitude (i.e., 10^{-19}) and the accuracy of W_0 reaches mm-level in terms of geoidal heights, the current level of GM's accuracy could be achieved (with our estimated uncertainty of $\sim 0.6\times10^6\ \mathrm{m^3\,s^{-2}}$ being similar to value $\pm 0.8\times10^6\ \mathrm{m^3\,s^{-2}}$ reported by *Ries et al. (1992)*. A better accuracy could only be accomplished, if the accuracy of all measurements improves. For a sub-millimetre accuracy of SLR measurements, i.e., $\sigma_{\Delta r} \approx \pm 0.0001$ m, the stability of atomic clocks 10^{-20}, i.e. $\sigma_{\Delta V} \cong \sigma_{\Delta W} \approx \pm 0.001\ \mathrm{m^2\,s^{-2}}$, and $\sigma_{\mathrm{W}_0} \approx \pm 0.001\ \mathrm{m^2\,s^{-2}}$, the accuracy of GM could be improved by an order magnitude, i.e., $\sigma_{\mathrm{GM}} \approx \pm 0.06\times10^6\ \mathrm{m^3\,s^{-2}}$. A significant improvement of the accuracy of W_0 achieved by *Amin et al. (2019)*, i.e. $\sigma_{\mathrm{W}_0} \approx \pm 0.004\ \mathrm{m^2\,s^{-2}}$, is thus very important step towards improvement of GM. It is worth noting that a repetition of this numerical experiment for the laboratory conditions (i.e., 10 m vertical baseline) does not show any benefit.

5. Conclusions

We have investigated the possibility of improving the accuracy of the geocentric gravitational constant GM by combining relativistic measurements of the geopotential differences by atomic clocks with SLR observations. For this purpose, we established two types of observations equations that link the GM value with the geopotential differences (measured by atomic clocks) and vertical distances (measured by SLR) in terms of the gravitational potential, while disregarding the centrifugal force and non-zero spherical harmonics of the gravitational potential, both having negligible effect on numerical findings in the error analysis.

According to our results, the current stability of atomic clocks (10^{-18} or lower) does not provide any meaningful use of relativistic measurements to estimate GM. A possible benefit of integrating atomic clock with SLR measurements to increase the accuracy of GM becomes with improved stability of atomic clocks by two orders of magnitude (10^{-20}) and SLR measurements realized with a sub-millimetre accuracy, both likely achievable in the foreseeable future when taking into consideration the latest technological progress. A crucial aspect is then an accurate determination of the geoidal geopotential value W_0 from satellite-altimetry data with a sub-millimetre level of accuracy in terms of geoidal heights. *Amin et al. (2019)* demonstrated that mm-level accuracy (or better) could be achieved.

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