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In search of necessary and sufficient conditions to solve the parabolic Anderson model with fractional Gaussian noises

Shuhui Liu* † ‡ Yaozhong Hu§ ¶ Xiong Wang ‖

Abstract

This paper attempts to obtain necessary and sufficient conditions to solve the parabolic Anderson model with fractional Gaussian noises: $\frac{\partial}{\partial t}u(t,x)=\frac{1}{2}\Delta u(t,x)+u(t,x)\dot{W}(t,x)$, where $W(t,x)$ is the fractional Brownian field with temporal Hurst parameter $H_0\in[1/2,1)$ and spatial Hurst parameters $H=(H_1,\cdots,H_d)\in(0,1)^d$, and $\dot{W}(t,x)=\frac{\partial^{d+1}}{\partial t\partial x_1\cdots\partial x_d}W(t,x)$. When $d=1$ and when $(H_0,H)\in(\frac{1}{2},1)\times(\frac{1}{20},\frac{1}{2})$ we show that the condition $2H_0+H>5/2$ is necessary and sufficient to ensure the existence of a unique solution for the parabolic Anderson Model. When $d\geq 2$, we find the necessary and sufficient condition on the Hurst parameters so that each chaos of the solution candidate is square integrable.

Keywords: parabolic Anderson model; fractional Gaussian noise; chaos expansion; solvability; necessary and sufficient condition; Hölder-Young-Brascamp-Lieb inequality; Hardy-Littlewood-Sobolev inequality.
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1 Introduction and main results

In this work, we study the solvability (i.e., existence and uniqueness) problem of the following stochastic heat equation on the Euclidean space $\mathbb{R}^d$, also known as the

*Department of Applied Mathematics, The Hong Kong Polytechnic University, HungHom, Kowloon, Hong Kong.
†School of Mathematics, Shandong University, Jinan, Shandong 250100, China.
E-mail: shuhuiliusdu@gmail.com
‡SL was visiting University of Alberta supported by the China Scholarship Council.
§Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada. E-mail: yaozhong@ualberta.ca
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‖Department of Mathematics, Johns Hopkins University, Baltimore, 21218, USA. E-mail: xiongwang@jhu.edu (corresponding author)

parabolic Anderson model (PAM):

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{1}{2}\Delta u(t,x) + u(t,x)\dot{W}(t,x)\,,\ t>0, x\in\mathbb{R}^d\,; \\ u(0,x) = u_0(x)\,, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\mathbb{R}^d$, the process $\{W(t,x)\,, t\ge 0\,, x\in\mathbb{R}^d\}$ is a centered fractional Gaussian field of Hurst parameters $(H_0, H_1, \cdots, H_d)$ and $\dot{W}(t,x) = \frac{\partial^{1+d}}{\partial t\partial x_1\cdots\partial x_d}W(t,x)$. This means that formally the mean of $\dot{W}(t,x)$ is zero and its covariance function is given by

$$\mathrm{Cov}(\dot{W}(t,x), \dot{W}(s,y)) = \gamma_0(s-t)\gamma(x-y) \tag{1.2}$$

with

$$\gamma_0(t) = c_{H_0}|t|^{2H_0-2}\,, \quad \gamma(x) = \prod_{j=1}^d c_{H_j}|x_j|^{2H_j-2}\,, \quad t\in\mathbb{R}, x=(x_1,\cdots,x_d)\in\mathbb{R}^d \tag{1.3}$$

where $c_{H_j} = H_j(2H_j-1)$ for $j=0,\cdots,d$. Throughout this work we assume that the Hurst parameters $(H_0, H_1, \cdots, H_d)$ satisfy

$$H_0\in[1/2,1)\,, \text{ and } H_j\in(0,1) \quad \forall j=1,\cdots,d.$$

The case $H_0 = 1/2$ corresponds to $\gamma_0(t) = \delta(t)$ and in this case the noise is called time white; the case $H_1 = \cdots = H_d = 1/2$ corresponds to $\gamma(x) = \delta(x)$ and in this case the noise is called space white. When the noise is space time white the square integrable solution exists only when the space dimension is one. In this case (and when the initial condition is the Dirac delta function) the seminal paper [18] connects the equation (1.1) with the Kardar-Parisi-Zhang (KPZ) equation via the Hopf-Cole transformation $h(t,x) = \log(u(t,x))$ and develops the theory of regularity structures. In the last decades, the parabolic Anderson model has received great attention partly due to its connection with the KPZ equation. Many sharp properties of the solution to (1.1) have been obtained for general Gaussian noises, including the space time white noise. For further reading, we recommend the works of [3, 11, 16, 26, 30, 32, 39] and the references therein.

Equation (1.1) depends only on the initial condition and the covariance structure of noise $\dot{W}$. If we assume that the initial condition is as nice as needed (e.g. $u_0\equiv 1$), then the solvability and the properties of the solution to (1.1) are completely determined by the covariance structure of $\dot{W}$. It is interesting to ask under what conditions on the covariance structure of the noise, the equation (1.1) has a unique solution. Several progresses have been made along this direction, mostly in the form of sufficient condition, which will be recalled in the next subsection. Now it is natural to ask if such condition is also necessary or not. If not, it is interesting to find conditions that are both necessary and sufficient. In this paper we shall present some partial results for the above problem.

Usually there are two different interpretations of the product $u(t,x)\dot{W}(t,x)$ in (1.1) used so far in literature. One is in the Stratonovich sense (or pathwise sense), and the other one is in the Itô/Skorohod sense. We chose the latter one for (1.1) since it enables us to immediately express the formal solution candidate through its Itô-Wiener chaos expansion:

$$u(t,x) = \sum_{n=0}^\infty u_n(t,x) = \sum_{n=0}^\infty \frac{1}{n!}I_n(f_n(\cdot,t,x))\,, \quad \text{for any } (t,x)\in\mathbb{R}_+\times\mathbb{R}^d \tag{1.4}$$

where f_n is given by (2.12) in the next section through the heat kernel associated with (1.1) and $I_n(f_n(\cdot,t,x))$ is the multiple Itô-Wiener integral with respect to $f_n(\cdot,t,x)$.

The advantage of using the chaos expansion lies in the fact that multiple Itô-Wiener integrals of different orders are orthogonal. It is easy to verify (e.g. [15]) that *the PAM* (1.1) *has a unique square integrable solution if and only if the above chaos expansion is convergent in* $L^2(\Omega)$ *for all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. When the chaos expansion is convergent, one has

$$\mathbb{E}\left[u(t,x)^2\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[u_n(t,x)^2\right] \,. \tag{1.5}$$

From the above discussion we see that the solvability problem of (1.1) is decomposed into the following two consecutive questions.

(Q1) What are the necessary and sufficient conditions for the finiteness of $\mathbb{E}\left[u_n^2(t,x)\right]$ for all positive integer n? This involves the challenging problem of finding necessary and sufficient conditions for the integrability of singular multiple integrals presented in Proposition 3.4 of Section 3.

(Q2) Under what necessary and sufficient conditions Equation (1.5) is convergent? In Section 5, we answer this question in one dimensional case by deriving proper growth bounds for $\mathbb{E}\left[u_n^2(t,x)\right]$ as a function of n.

1.1 Main results

In the course of completing the above two tasks, we will see that the cases of Hurst parameters greater or less than $3/4$ need to be treated differently. This is a bit contrary to the conventional division which usually divides the region of Hurst parameters into the region that the Hurst parameter is greater than $1/2$ and region that the Hurst parameter is less than $1/2$. For this reason, and without loss of generality, we assume that the Hurst parameters are so arranged that there is an integer d_* between 0 and d such that $H_k < \frac{3}{4}$ for $k = 1, 2, \cdots, d_*$ and $H_k \geq \frac{3}{4}$ for $k = d_*+1, \cdots, d$. The case $d_* = 0$ means that all Hurst parameters are greater than or equal to $3/4$ and $d_* = d$ means that all Hurst parameters are less than $3/4$. We also denote by $|H|$ the sum of all spatial Hurst parameters, by H_* the sum of all Hurst parameters that are less than $3/4$, and by H^* the sum of all Hurst parameters that are greater than or equal to $3/4$. Namely, we denote

$$\begin{cases} H_1, \cdots, H_{d_*} < 3/4, \quad H_{d_*+1}, \cdots, H_d \geq 3/4\,, \\ H_* := H_1 + \cdots + H_{d_*}, \quad H^* := H_{d_*+1} + \cdots + H_d\,, \\ d^* = d - d_*\,, \quad |H| := H_1 + \cdots + H_d. \end{cases} \tag{1.6}$$

Concerning the above first question (Q1), we have the following result.

Theorem 1.1. *Suppose* $u_0 \equiv 1$ *and* $H_0 \geq \frac{1}{2}$. *Let* u_n *be the* n*-th chaos candidate of the solution to* (1.1)*, defined by* (2.10)-(2.12) *in the next section. Then* $\mathbb{E}[u_n(t,x)^2] < +\infty$ *for any* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ *and any* $n \geq 1$ *if and only if all of the following conditions hold:*

$$\begin{cases} H_* > \frac{3}{4} d_* - 1, & \text{(1.7a)} \\ |H| + 2H_0 > d, & \text{(1.7b)} \\ H_* + 2H_0 > \frac{3}{4} d_* + \frac{1}{2}, & \text{(1.7c)} \\ |H| + 2H_* + 4H_0 > d + \frac{3}{2} d_*. & \text{(1.7d)} \end{cases}$$

The initial condition $u_0 \equiv 1$ may be extended to any initial condition that is uniformly bounded below and above. Using the Itô isometry $\mathbb{E}[u_n(t,x)^2]$ is expressed as a multiple integral involving the spatial and temporal variables. Bounding this multiple integral by integrating the space variables will yield a multiple integral of the form (3.40) involving multiple integral of powers of some singular kernels. This multiple integral appears elementary. However, determining the necessary and sufficient conditions for the exponents to ensure the integral's finiteness is a highly complex problem. There are some studies about similar integrals in analysis (e.g., [37, 40, 41]). However, they seem bard to be applied to the above case. We shall divide the integral (4.7) associated with $\mathbb{E}\left[u_n(t,x)^2\right]$ into two distinct cases determined by a new mechanism (4.9). Then, the proof relies on applications of both the Hardy-Littlewood-Sobolev and the Hölder-Young-Brascamp-Lieb inequalities. The detailed proof will be given in Section 4.

If $H_k < 3/4$ for all $k = 1, \cdots, d$, then we have $|H| = H_*$ and $d = d_*$. The conditions (1.7a)-(1.7d) in (1.7) can be slightly simplified to:

Corollary 1.2. *If $u_0 \equiv 1$, $H_0 \ge \frac{1}{2}$ and $H_k < 3/4$ for all $k = 1, \cdots, d$, then the necessary and sufficient condition so that $\mathbb{E}[u_n(t,x)^2] < +\infty$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and any $n \ge 1$ becomes*

$$\begin{cases} H_* > \frac{3}{4}d - 1, \\ H_* + 2H_0 > (d \vee \frac{3d+2}{4}), \\ 3H_* + 4H_0 > \frac{5}{2}d. \end{cases} \tag{1.8}$$

When $d = 1$, $H_0 \ge \frac{1}{2}$ and $H = H_1 < \frac{1}{2}$, the first condition (1.7a) is trivial. The second and the last are implied by the third. Thus, we reduce Theorem 1.1 to the following theorem.

Theorem 1.3. *If $d = 1$, $u_0 \equiv 1$ and $H_0 \ge \frac{1}{2}$, $0 < H < \frac{1}{2}$, then the necessary and sufficient condition so that $\mathbb{E}[u_n(t,x)^2] < +\infty$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and any $n \ge 1$ is*

$$2H_0 + H > 5/4\,. \tag{1.9}$$

Example 1.4. It is easy to verify that when $d = 3$, $H_0 = 1$, $H_1 = H_2 = H_3 = 1/2$, the condition (1.8) is satisfied. Thus, if $d = 3$ and when the noise is time independent and space white, then $\mathbb{E}[u_n(t,x)^2] < \infty$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and any n (which is known in [19, Theorem 4.1]). However, also in [19, Theorem 4.1] it is shown that in this case $\sum_{n=0}^{\infty} \mathbb{E}[u_n(t,x)^2] = \infty$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ if the initial condition $u_0(x) \ge c$ for some constant $c > 0$. This stresses the point that $\mathbb{E}[u_n(t,x)^2] < \infty$ for all n does not automatically imply $\sum_{n=0}^{\infty} \mathbb{E}[u_n(t,x)^2] < \infty$.

The following results are byproducts while proving Theorem 1.1.

Proposition 1.5. *Suppose $u_0 \equiv 1$, $H_0 \ge \frac{1}{2}$ and suppose all the conditions in* (1.7) *hold.*

(i) If $H_0 + H_* > \frac{3d_*}{4}$ and if $|H| > d - 1$, then the solution is in $L^p(\Omega)$ for any $p \ge 1$. Moreover, there are two positive constants $C_{1,H}$ and $c_{2,H}$, independent of p, t and x such that

$$\mathbb{E}\left[|u(t,x)|^p\right] \le C_{1,H} \exp\left[c_{2,H}\, p^{\frac{|H|-d+2}{|H|-d+1}}\, t^{\frac{|H|+2H_0-d}{|H|-d+1}}\right] \tag{1.10}$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

(ii) If $H_0 + H_* > \frac{3d_*}{4}$ and if $|H| = d - 1$, then there is a $t_0 > 0$ such that when $t < t_0$, $\mathbb{E}[u(t,x)^2] = \sum_{n=0}^{\infty} \mathbb{E}[u_n(t,x)^2] < +\infty$ for $x \in \mathbb{R}^d$.

The critical time t_0 in Proposition 1.5 (ii) is usually called the blow-up time, see for example [19]. In [13, (1.9)], the authors show that $t_0 = t_0(p) = \frac{C_{H_0,H}}{(p-1)^{2H_0-1}}$ under the critical condition [13, (1.8)]. Quastel, Ramirez, and Virag [36] use an elementary version of the Skorokhod integral to define the solution at all times, including $t > t_0$. They construct u as a randomized shift or as the free energy of an undirected polymer in a random environment.

When $H_0 = 1/2$, the first condition (1.7a) and the fourth condition (1.7d) are entailed by the second condition (1.7b) and the third one (1.7c). Therefore, combining Theorem 1.1 and Proposition 1.5, we have the following proposition.

Proposition 1.6. *Suppose $u_0 \equiv 1$ and $H_0 = \frac{1}{2}$, namely, the noise is white in time. Then the necessary and sufficient condition for $\mathbb{E}[u(t,x)^2] < +\infty$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ is*

$$|H| > d - 1 \quad \text{and} \quad H_* > \frac{3}{4} d_* - \frac{1}{2}. \tag{1.11}$$

This means that when the noise is white in time, the convergence of the Itô-Wiener chaos expansion is equivalent to the finiteness of each chaos.

In general the above second question (Q2) is more difficult to answer since we need more precise bound of $\mathbb{E}\left[u_n(t,x)^2\right]$ in terms of n so that the series (1.5) is summable (Namely $\sum_{n=0}^{\infty} \mathbb{E}\left[u_n(t,x)^2\right] < \infty$). Proposition 1.5 provides a sufficient condition for the convergence of the Itô-Wiener chaos expansion (2.13). It is interesting to know if this condition is necessary or not. Unfortunately, as we shall see in this manuscript, it is not necessary. Next, we enlarge the known range of Hurst parameters for which the Itô-Wiener chaos expansion converges. We shall focus on $d = 1$ since the problem in general dimension case is much more difficult.

It is known (from subsection 1.2) that when $d = 1$ and $H_0 > 1/2$, a sufficient condition for the convergence of the Itô-Wiener chaos expansion is $H_0 + H_1 > 3/4$ and a necessary condition is $2H_0 + H_1 > 5/4$. These two conditions do not coincide so we don't know which one of them is both necessary and sufficient, or none of them is. It is natural to seek a condition that is both necessary and sufficient. This problem seems hard. First, let us point out that the condition (1.7) may not be sufficient for the convergence of the Itô-Wiener chaos expansion. In fact, assuming the initial condition $u_0(x) = 1$ and $d = 3$, it is proved in [19] (see also Example 1.4) that if the noise is time independent ($H_0 = 1$) and space white ($H_1 = H_2 = H_3 = 1/2$) then $\mathbb{E}\left[u_n(t,x)^2\right] < \infty$ for each n (the conditions in Theorem 1.1 are satisfied), but $\sum_{n=0}^{\infty} \mathbb{E}\left[u_n(t,x)^2\right] = \infty$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$. The following second main result of this paper is to establish that the existing necessary condition ($2H_0 + H_1 > 5/4$) for the convergence of the Itô-Wiener chaos expansion is also sufficient under some circumstances.

Theorem 1.7. *Let $u(t,x)$ be the solution candidate given by (1.4). Suppose $d = 1$, $u_0 \equiv 1$ and $H_0 > \frac{1}{2}$, $H = H_1 < \frac{1}{2}$. If $(H_0, H) \in \mathcal{A}_1 \cup \mathcal{A}_2$, where*

$$\mathcal{A}_1 = \{(H_0, H) \in (1/2, 1) \times (1/20, 1/4) : \ 2H_0 + H > 5/4\} , \tag{1.12}$$

$$\mathcal{A}_2 = \{(H_0, H) \in (1/2, 1) \times (0, 1/20) : \ 4H_0 + 12H > 3\} , \tag{1.13}$$

then for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$\mathbb{E}[u(t,x)^2] = \sum_{n=1}^{\infty} \mathbb{E}[u_n(t,x)^2] < +\infty. \tag{1.14}$$

Compared with Theorem 1.1, the proof of the above theorem needs some sharp uniform bounds of $\mathbb{E}[u_n(t,x)^2]$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ as $n \to \infty$. The technique in the proof of Theorem 1.1 is insufficient. The details of the proof are given in Section 5.

In Figure 1, the domain (1.12) is shaded in light yellow and domain (1.13) is shaded in dark yellow. Since $2H_0 + H > 5/4$ is a necessary condition, Theorem 1.7 implies immediately the following corollary.

Corollary 1.8. *Suppose $d = 1$, $u_0 \equiv 1$. On the region $(H_0, H) \in (1/2, 1) \times (1/20, 1/4)$ the condition $2H_0 + H > 5/4$ in (1.9) serves as a necessary and sufficient condition for the solvability of (1.1).*

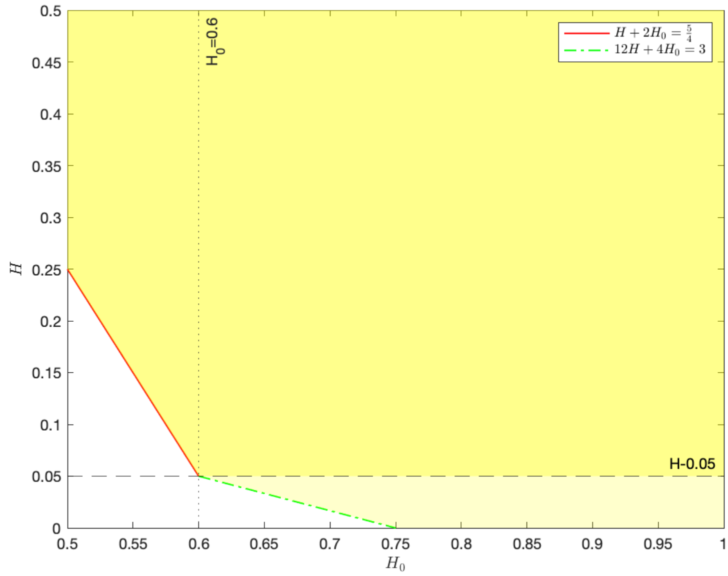

Figure 1: The dark yellow region is for region $\mathcal{A}_1$ and the light yellow region is for region $\mathcal{A}_2$

The condition $4H_0 + 12H > 3$ can be written as

$$2H_0 + H > \frac{1}{4} + \frac{5}{3}H_0\,,$$

which is implied by $2H_0+H > \frac{5}{4}$ and $H_0 \le 3/5$. Similarly, $4H_0+12H > 3$ is also implied by $2H_0 + H > \frac{5}{4}$ and $H \ge 1/20$. This combined with the above corollary implies immediately that

Corollary 1.9. *Suppose* $d = 1$. *If* $(H_0, H) \in (\frac{1}{2}, 1) \times (\frac{1}{20}, \frac{1}{2})$ *or* $(H_0, H) \in (\frac{1}{2}, \frac{3}{5}) \times (0, \frac{1}{2})$*, then the necessary and sufficient condition for the Itô-Wiener chaos expansion* (2.13) *to converge (namely for Equation* (1.1) *to be solvable) is* $2H_0 + H > \frac{5}{4}$.

1.2 Comparison with existing results

The solvability of equation (1.1) depends completely on the covariance kernels $\gamma_0(\cdot)$ and $\gamma(\cdot)$ when the initial condition is given and is assumed to be nice. When the temporal kernel $\gamma_0(\cdot) = \delta_0(\cdot)$ is the Dirac delta function, and when the spatial kernel is positive and $\gamma(x) = \int_{\mathbb{R}^d} e^{ix\xi}\mu(d\xi)$, $x \in \mathbb{R}^d$, is represented by the Fourier transform of a positive measure μ on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty \quad \text{(Dalang's condition)}\,, \tag{1.15}$$

the equation (1.1) is solvable (e.g., [17]). In the literature to obtain the necessary and sufficient condition, one usually takes the temporal kernel as $\gamma_0(t) = c_{\alpha_0}|t|^{-\alpha_0}$ with some $\alpha_0 > 0$ and with some constant $c_{\alpha_0} \in \mathbb{R}$ (one identifies the Dirac delta function case as $\alpha_0 = 1$). As for the spatial covariance function $\gamma(\cdot)$, three cases are commonly studied: (i) Riesz kernel, i.e., $\gamma(x) = c_{\alpha,d}|x|^{-\alpha}$ or equivalently $\mu(d\xi) = C_{\alpha,d}|\xi|^{-(d-\alpha)}\prod_{j=1}^d d\xi_j$ for some $\alpha \in \mathbb{R}$ and some positive constants $c_{\alpha,d}, C_{\alpha,d} \in \mathbb{R}$; (ii) fractional kernel, i.e., $\gamma(x) = \prod_{j=1}^d c_{\alpha_j}|x_j|^{-\alpha_j}$ or $\mu(d\xi) = \prod_{j=1}^d C_{\alpha_j}|\xi_j|^{-(1-\alpha_j)}d\xi_j$ for some $\alpha_j > 0$ and some positive constants $c_{\alpha_j}, C_{\alpha_j} \in \mathbb{R}$.

For the fractional noise, the relation between α_i and the Hurst parameters is given by $\alpha_i = 2-2H_i$, $i=0,1,\cdots,d$. In the above particular cases of Riesz or fractional kernels, Dalang's condition becomes $\alpha_0 = 1$ and

$$\begin{cases} 0<\alpha<2\,, & \text{Riesz kernel noise;} \\ \alpha_1,\cdots,\alpha_d \in (0,1)\,,\ \sum_{j=1}^d \alpha_j < 2\,, & \text{fractional noise}\,. \end{cases} \tag{1.16}$$

When $\gamma_0(\cdot)$ is locally integrable, the Dalang's condition (1.15) is also proved (e.g. [24]) to be sufficient for the solvability of (1.1) in the general Gaussian noise setting. Moreover, when $\gamma_0(t) = c_\alpha |t|^{-\alpha_0}$ and the spatial covariance $\gamma(\cdot)$ is given by the Riesz potential, the necessity of (1.15) has been verified in [2]. Notice that in terms of Hurst parameters, the second one in Dalang's condition (1.16) becomes $H_1,\cdots,H_d \in (1/2,1)\,,\ \sum_{j=1}^d H_j > d-1$. As a comparison, we draw attention to another random evolution model, the hyperbolic Anderson model (HAM) (i.e., we replace $\frac{\partial}{\partial t}$ by $\frac{\partial^2}{\partial t^2}$ in (1.1)). It is known that Dalang's condition (1.15) is a sufficient condition for the solvability of HAM. The readers are referred to [3] and references therein for details. During the preparation of this article, Chen, Deya, Song, and Tindel in [14] obtain that

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^{3-\alpha_0}}\mu(d\xi) < \infty \tag{1.17}$$

is the necessary and sufficient condition for HAM to admit a unique Skorohod solution when $\alpha_0 \in (0,1)$. Moreover, in the case of Riesz kernel or (regular) fractional noise, (1.17) is equivalent to $\alpha_0 + \alpha < 3$ or $\alpha_0 + \sum_{j=1}^d \alpha_j < 3$.

Let us emphasize that for the above mentioned Gaussian noises, it is critical to assume all $\alpha_j \in (0,1)$, and the solvability under this condition is now clear. Conventionally, the case that all the Hurst parameters exceed $1/2$ (namely, $\alpha_j \in (0,1)$ for all $j=1,\cdots,d$) is referred to the *regular* case. Otherwise (namely, $\alpha_j > 1$ for some $j=1,\cdots,d$), it is termed *rough* case. The *rough case* is more intriguing and poses significantly greater challenges due to the fact that the spatial covariance function $\gamma(\cdot)$ is no longer locally integrable and is no longer positive.

There are very limited results in the case when time is rough, i.e., $\alpha_0 > 1$ (or equivalently $H_0 < 1/2$). In the case of fractional noise, let us mention the work [28], where it is proved that all chaoses of the solution candidate to (1.1) exist in $L^2(\Omega)$ when $d=1$, $H_1 = 1/2$ and $3/8 < H_0 < 1/2$. This result is further strengthened by Chen, where the domain of solvability [12, (1.19)] is established as ($d=1$):

$$\begin{cases} H_0 \geq \frac{1}{2} \text{ and } H_1 < \frac{1}{2}: \text{ solvable if } H_0 + H_1 > \frac{3}{4}\,; \\ H_0 < \frac{1}{2} \text{ and } H_1 \geq \frac{1}{2}: \text{ solvable if } 4H_0 + H_1 > 2\,; \\ H_0 < \frac{1}{2} \text{ and } H_1 < \frac{1}{2}: \text{ solvable if } 2H_0 + H_1 > \frac{5}{4}\,. \end{cases} \tag{1.18}$$

Notice that when $H_1 = 1/2$, the above second condition is exactly $H_0 > 3/8$. Furthermore, if the noise is more smooth in the spatial variable, then H_0 can be arbitrarily between 0 and 1 (see, e.g., [10, 25]).

When $H_0 \geq 1/2$, and when some of the spatial Hurst parameters $(H_1,\cdots,H_d)$ are less than $1/2$ while others are greater than $1/2$, the solvability problem becomes much more complex.

Some notable achievements have been made when $H_0 = \frac{1}{2}$ (i.e., the noise is white in time) and $d=1$. In this case, it is known that $H_1 > 1/4$ is necessary and sufficient for the parabolic Anderson model (or hyperbolic Anderson model) to be solvable. We refer to the works of [4, 5, 22, 23, 31, 34] and references therein for detailed discussions.

For the more general case when the noise is rough in space, the best results up to date seem to be in [15], where both sufficient and necessary conditions are obtained for some special cases. More specifically, it is proved

(i) When $H_0 = 1/2$, namely when the noise is time white, the necessary and sufficient condition for the solvability of (1.1) (e.g., [15, Theorem 1], see also our previous Proposition 1.6) is $|H| := \sum_{i=1}^d H_i > d-1$ and $H_* > \frac{3}{4}d_* - \frac{1}{2}$. Notice that the condition $H_* > \frac{3}{4}d_* - \frac{1}{2}$, which is missing in [15, Theorem 1], is addressed in Section 4 of the present paper. In fact, when $d=1$ the above condition becomes $H := H_1 > 1/4$.

(ii) If $H_0 > 1/2$, then a sufficient condition for (1.1) to be solvable (e.g. [15, Theorem 3 (i)]) is

$$\begin{cases} H + H_0 > \frac{3}{4} & \text{if } d = 1\,, \\ |H| > d-1 & \text{if } d \geq 2\,. \end{cases} \tag{1.19}$$

(iii) If $H_0 > 1/2$, then a necessary condition for (1.1) to be solvable (e.g. [15, Theorem 3 (ii)]) is as follows

$$\begin{cases} H + 2H_0 > \frac{5}{4} & \text{if } d = 1\,, \\ |H| + 2H_0 > \frac{3d+2}{4} & \text{if } d \geq 2\,. \end{cases} \tag{1.20}$$

On the other hand, when $H_0 = 1/2$ and $d > 1$, it is pointed out in [15] that for (1.1) to be solvable, the number of Hurst parameters $(H_1, \cdots, H_d)$ that are less than $1/2$ can be at most 1.

One notices that the sufficient condition (1.19) and the necessary condition (1.20) are different, and this poses an intriguing open question: In situations where $H_0 > \frac{1}{2}$, is (1.19) a necessary condition, or is (1.20) a sufficient? Put it differently, is it possible to find a condition that is both necessary and sufficient? This current research is driven by the pursuit of an answer to this problem. More specifically, our first main result, as stated in Theorem 1.1, gives a necessary and sufficient condition ensuring that $\mathbb{E}\left[u_n(t,x)^2\right] < \infty$ holds for all values of n and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Here, u_n denotes the n-th chaos expansion of the solution candidate to (1.1). To obtain the necessary and sufficient condition for (1.5) to be convergent, we focus on $d=1$. Our second primary result, Theorem 1.7 asserts that the condition $H + 2H_0 > \frac{5}{4}$ specified in the first line of (1.20) serves as both necessary and sufficient criterion for the solvability of (1.1) in two pieces of domains: when $H_0 < \frac{3}{5}$ or when $H > \frac{1}{20}$. We believe that the above additional condition $H_0 < \frac{3}{5}$ or $H > \frac{1}{20}$ is due to some technicality and conjecture that the condition (1.9) is both necessary and sufficient for (1.1) to be solvable in the case of $d=1$ and $H \leq \frac{1}{2}$.

In the literature, one usually uses the powerful Hardy-Littlewood-Sobolev inequality to obtain a sufficient condition to ensure that (1.5) is convergent. However, our results demonstrate that the application of this inequality alone cannot give a condition that is both necessary and sufficient. We need to combine both the Hardy-Littlewood-Sobolev and the Hölder-Young-Brascamp-Lieb inequalities to arrive at our main results (Theorems 1.1 and 1.7). See Figure 1 for an explanation. We hope this methodology sheds light on the search of a condition that is both necessary and sufficient.

Let us point out that there is a necessary and sufficient condition ([27, Theorem 1.1]) for the solvability of the stochastic heat equation with additive noise: $\frac{\partial}{\partial t}u_{\text{add}}(t,x) = \frac{1}{2}\Delta u_{\text{add}}(t,x) + \dot{W}(t,x)$, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. When the noise is fractional as in our case here, the condition becomes ([27, Equation (5.2)])

$$2H_0 + \sum_{i=1}^d H_i > d\,,$$

which is exactly (1.7b). In terms of our notation we see that $\mathbb{E}[u_{\text{add}}(t,x)^2] = \mathbb{E}[u_1(t,x)^2]$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Hence, Theorem 1.1 means that $\mathbb{E}[u_{\text{add}}(t,x)^2] = \mathbb{E}[u_1(t,x)^2] < \infty$ does not automatically imply $\mathbb{E}[u_n(t,x)^2] < \infty$ for all n and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

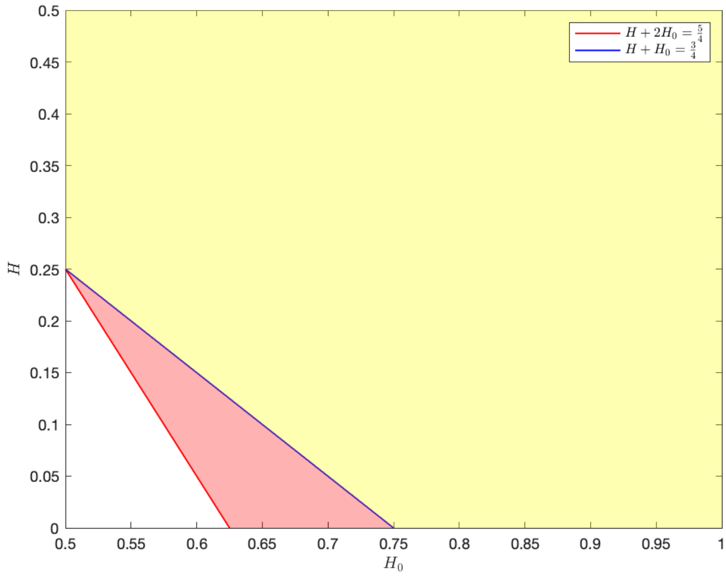

Figure 2: Hölder-Young-Brascamp-Lieb region (red) and Hardy-Littlewood-Sobolev region (yellow)

1.3 Structure of the paper

After some preparations in Section 2, we aim to bound $\mathbb{E}\left[u_n(t,x)^2\right]$ which amounts to compute some complicated multiple integrals with both space and time variables. In Section 3, we use the Fourier transforms to compute the spatial integral so to reduce the calculation of $\mathbb{E}\left[u_n(t,x)^2\right]$ to multiple integrals with respect to time variables on the simplex. We deal with the latter one carefully to obtain the necessary and sufficient condition so that $\mathbb{E}\left[u_n(t,x)^2\right] < \infty$ in Section 4. Section 5 focuses on the solvability of (1.1) in dimension one, the well-accepted condition $H + H_0 > \frac{3}{4}$ is improved to $H + 2H_0 > \frac{5}{4}$, which is shown to be necessary and sufficient in some regions as we discussed earlier.

In order to make the paper more readable, we delay some of the detailed computations to appendix, where we also recall briefly the Hardy-Littlewood-Sobolev inequality and the Hölder-Young-Brascamp-Lieb inequality since they are the main tools in this work.

2 Preliminaries

In this section, we begin by introducing the notations and facts that will be used throughout this paper. Let $\mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ be the space of infinitely differentiable functions with compact support on $\mathbb{R}_+ \times \mathbb{R}^d$. Given that we will be dealing with the fractional Brownian noise whose Hurst parameters can be greater than $1/2$ for some coordinates while being less than $1/2$ for others, and since we are only concerned with the parabolic Anderson model, it is more convenient for us to introduce the scalar product by using the Fourier transform with respect to the spatial variables.

We denote the Fourier transform $\widehat{f}$ with respect to the d spatial variables $x_1, \cdots, x_d$ as follows:

$$\widehat{f}(\xi) := \mathcal{F}[f](\xi) = \int_{\mathbb{R}^d} e^{-\iota \xi x} f(x) dx, \quad \text{where } \iota = \sqrt{-1}.$$

Let $H_0 \in (1/2, 1)$ and $H_1, \cdots, H_d \in (0,1)$. Also, let us denote $\gamma_0(r) = \gamma_{H_0}(r) := H_0(2H_0 -$

$1)|r|^{2H_0-2}\,, r\in\mathbb{R}$. We introduce a scalar product on $\mathcal{C}_0^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ as defined below:

$$\langle\varphi,\psi\rangle_{\mathfrak{H}}=\int_{\mathbb{R}_+^2\times\mathbb{R}^d}\hat{\varphi}(r,\boldsymbol{\xi})\overline{\hat{\psi}}(s,\boldsymbol{\xi})\prod_{k=1}^d|\xi_k|^{1-2H_k}\gamma_0(r-s)drdsd\boldsymbol{\xi}, \tag{2.1}$$

where $\boldsymbol{\xi}:=(\xi_1,\cdots,\xi_d)$, $d\boldsymbol{\xi}:=d\xi_1\cdots d\xi_d$. We denote $\mathfrak{H}$ as the Hilbert space obtained from the completion of $\mathcal{C}_0^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{H}}$.

The noise in the present paper is given by an isonormal Gaussian process $W=\{W(\varphi);\varphi\in\mathfrak{H}\}$ with covariance

$$\mathbb{E}[W(\varphi)W(\psi)]=\langle\varphi,\psi\rangle_{\mathfrak{H}}.$$

It is routine to prove that the function $\mathbf{1}_{[0,t)\times\prod_{k=1}^d[0,x_k]}$ belongs to $\mathfrak{H}$ for any $t>0$ and $x=(x_1,\cdots,x_d)\in\mathbb{R}^d$ (here $\mathbf{1}_{[b,a]}=-\mathbf{1}_{[a,b]}$ if $a<b$). We denote $W(t,x):=W(\mathbf{1}_{[0,t)\times\prod_{k=1}^d[0,x_k]})$. The Gaussian random field $\{W(t,x):t\ge0,x\in\mathbb{R}^d\}$ has mean zero and covariance given by

$$\mathbb{E}[W(t,x)W(s,y)]=C_{H_0,H}\gamma_{H_0}(t,s)\prod_{k=1}^d\gamma_{H_k}(x_k,y_k)\,,\quad s,t\ge0\,,x,y\in\mathbb{R}^d\,, \tag{2.2}$$

where $C_{H_0,H}$ is a constant that depends on H_0 and $H=(H_1,\cdots,H_d)$. It should be noticed that this constant $C_{H_0,H}$ may differ from those used in other literature, as we have set the constant in (2.1) to be 1.

Since the Gaussian field W is not a martingale in time (due to $H_0\neq1/2$), we cannot use the classical method of martingale measure to define the stochastic integral. Therefore, we shall use the chaos expansion to deal with our problem, and the most effective way to do this is to introduce the stochastic integral via multiple chaos Itô-Wiener integrals (see [20, 21]).

Let $e_1,\cdots,e_n,\cdots\in\mathcal{C}_0^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ be an orthogonal basis of $\mathfrak{H}$. Then $\{\tilde{e}_n=W(e_n),n=1,2,\cdots\}$ are independent standard normal random variables. We denote the symmetric tensor product by $\otimes$. The tensor product $\mathfrak{H}^{\otimes n}$ is completion of the linear span of $e_{\ell_1}\otimes\cdots\otimes e_{\ell_n}$ with respect to the scalar product generated by

$$\langle e_{\ell_1}\otimes\cdots\otimes e_{\ell_n}\,,e_{j_1}\otimes\cdots\otimes e_{j_n}\rangle=\frac{1}{n!}\sum_{\sigma\in\Sigma_n}\langle e_{\ell_1}\,,e_{j_{\sigma(1)}}\rangle\cdots\langle e_{\ell_n}\,,e_{j_{\sigma(n)}}\rangle$$

where Σ_n denotes the set of all permutations of $\{1,\cdots,n\}$. Let $\tilde{H}_m(x)=(-1)^me^{\frac{x^2}{2}}\frac{d^m}{dx^m}(e^{-\frac{x^2}{2}})$ be the m-th Hermite polynomial. Let $e_{i_1},\cdots,e_{i_k}$ be different and $n_1,\cdots,n_k$ are positive integers such that $n_1+\cdots+n_k=n$. We define the multiple integral as follows:

$$I_n(e_{i_1}^{\otimes n_1}\otimes\cdots\otimes e_{i_k}^{\otimes n_k})=\tilde{H}_{n_1}(\tilde{e}_{i_1})\cdots\tilde{H}_{n_k}(\tilde{e}_{i_k})\,.$$

Any element in $\mathfrak{H}^{\otimes n}$ can be approximated by $f_n=\sum_{0\le n_1,\cdots,n_k\le n}a_{i_1,\cdots,i_k}e_{i_1}^{\otimes n_1}\otimes\cdots\otimes e_{i_k}^{\otimes n_k}$, whose multiple Itô-Wiener integral is defined as:

$$\begin{aligned}I_n(f_n)&=\sum_{0\le n_1,\cdots,n_k\le n}a_{i_1,\cdots,i_k}I_n(e_{i_1}^{\otimes n_1}\otimes\cdots\otimes e_{i_k}^{\otimes n_k})\\&=\sum_{0\le n_1,\cdots,n_k\le n}a_{i_1,\cdots,i_k}\tilde{H}_{n_1}(\tilde{e}_{i_1})\cdots\tilde{H}_{n_k}(\tilde{e}_{i_k})\,.\end{aligned}$$

The Wiener chaos expansion theorem (e.g., [20, 35]) states that any random variable $F\in L^2(\Omega,\mathcal{F},\mathbb{P})$ admits a chaos expansion

$$F=\mathbb{E}[F]+\sum_{n=1}^\infty I_n(f_n), \tag{2.3}$$

where the series converges in $L^2(\Omega)$, and the elements $f_n \in \mathfrak{H}^{\otimes n}$ $(n \geq 1)$ depend on F, and

$$\mathbb{E}[|F|^2] = (\mathbb{E}[F])^2 + \sum_{n=1}^{\infty} \mathbb{E}[|I_n(f_n)|^2] = (\mathbb{E}[F])^2 + \sum_{n=1}^{\infty} n! \|f_n\|_{\mathfrak{H}^{\otimes n}} . \tag{2.4}$$

Let us recall that $\mathfrak{H}$ is a space of (possibly generalized) functions with $d+1$ variables. $f_n \in \mathfrak{H}^{\otimes n}$ is a (possibly generalized) function with $(d+1) \times n$ variables and can be approximated by smooth functions with compact support from $(\mathbb{R}_+ \times \mathbb{R}^d)^n$ to $\mathbb{R}$ with respect to the norm of $\mathfrak{H}^{\otimes n}$. Additionally, the multiple integral $I_n(f_n)$ is identified as follows:

$$I_n(f_n) = \int_{\mathbb{R}_+^n \times \mathbb{R}^{nd}} f_n(t_1, x_1, \cdots, t_n, x_n) W(dt_1, dx_1) \cdots W(dt_n, dx_n) .$$

The Fourier transform of f with respect to the spatial variables is defined as:

$$\hat{f}_n(t_1, \xi_1, \cdots, t_n, \xi_n) = \int_{\mathbb{R}^{nd}} f_n(t_1, x_1, \cdots, t_n, x_n) e^{-\iota \sum_{i=1}^n \xi_i x_i} dx_1 \cdots dx_n ,$$

where $\xi_i x_i = \sum_{k=1}^d \xi_{ik} x_{ik}$ is the Euclidean product of $\xi_i = (\xi_{i1}, \cdots, \xi_{id})^T$, $x_i = (x_{i1}, \cdots, x_{id})^T$ and $dx_i = dx_{i1} \cdots dx_{id}$. Using the above notation, we can express the $\mathfrak{H}^{\otimes n}$ norm of f as:

$$\begin{aligned} \|f_n\|^2_{\mathfrak{H}^{\otimes n}} \quad = \quad & \int_{\mathbb{R}_+^{2n} \times \mathbb{R}^{nd}} \hat{f}_n(r_1, \xi_1, \cdots, r_n, \xi_n) \bar{\hat{f}}_n(s_1, \xi_1, \cdots, s_n, \xi_n) \\ & \prod_{i=1}^n \prod_{k=1}^d |\xi_{ik}|^{1-2H_k} \prod_{i=1}^n \gamma_0(r_i - s_i) dr_1 \cdots dr_n ds_1 \cdots ds_n d\boldsymbol{\xi} . \end{aligned} \tag{2.5}$$

Thus, any square integrable nonlinear functional of W can be written as

$$F = \sum_{n=0}^{\infty} I_n(f_n) = \sum_{n=0}^{\infty} \int_{\mathbb{R}_+^n \times \mathbb{R}^{nd}} f_n(t_1, x_1, \cdots, t_n, x_n) W(dt_1, dx_1) \cdots W(dt_n, dx_n) ,$$

for some sequence $f_n \in \mathfrak{H}^{\otimes n}$. The expectation of F^2 can be expressed as:

$$\begin{aligned} \mathbb{E}[F^2] = C^n_{H_0,H} \sum_{n=0}^{\infty} n! \int_{\mathbb{R}_+^{2n} \times \mathbb{R}^{nd}} & \hat{f}_n(r_1, \xi_1, \cdots, r_n, \xi_n) \bar{\hat{f}}_n(s_1, \xi_1, \cdots, s_n, \xi_n) \\ & \times \prod_{i=1}^n \prod_{k=1}^d |\xi_{ik}|^{1-2H_k} \prod_{i=1}^n \gamma_0(r_i - s_i) dr_1 \cdots dr_n ds_1 \cdots ds_n d\boldsymbol{\xi} . \end{aligned}$$

Consider a random field $f(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^d \times \Omega$, where f is square integrable with $\mathbb{E}[f(t,x)^2] < \infty$. The field $f(t,x)$ can be expressed as a chaos expansion:

$$f(t,x) = f_0(t,x) + \sum_{n=1}^{\infty} I_n(f_n(t,x)) ,$$

where $f_n(t,x,\cdot)(t_1, x_1, \cdots, t_n, x_n) = f_n(t, x; t_1, x_1, \cdots, t_n, x_n)$ is an element in $\mathfrak{H}^{\otimes n}$ for any fixed t and x. The symmetrization of f_n, denoted as $\tilde{f}_n$, is defined as

$$\begin{aligned} \tilde{f}_n(t_1, x_1, \cdots, t_{n+1}, x_{n+1}) = & \frac{1}{(n+1)!} \sum_{\sigma} f_n(t_{\sigma(1)}, x_{\sigma(1)}, t_{\sigma(2)}, x_{\sigma(2)}, \cdots, t_{\sigma(i)}, x_{\sigma(i)} \\ & t_{\sigma(i+1)}, x_{\sigma(i+1)}, \cdots, t_{\sigma(n+1)}, x_{\sigma(n+1)}) \end{aligned} \tag{2.6}$$

where the summation is taken over all permutation σ on $\{1, \cdots, n+1\}$.

Definition 2.1. *For such f as above, $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$ we say that f is integrable if, for every $n \geq 0$, $\tilde{f}_n \in \mathfrak{H}^{\otimes(n+1)}$ and the series $\sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$ converges in $L^2(\Omega)$. We define the stochastic integral of $f(t,x)$ as*

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} f(t,x) W(dt,dx) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) \,. \tag{2.7}$$

We now define the concept of a strong (random field) solution to equation (1.1).

Definition 2.2. *A real-valued adapted stochastic process $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is said to be a* (global) random field solution (or mild solution) *of* (1.1) *if for all $t \in [0,\infty)$ and $x \in \mathbb{R}^d$, $\{G_{t-\cdot}(x-\cdot)u(\cdot,\cdot)\}$ is integrable and the following equality holds almost surely:*

$$u(t,x) = \int_{\mathbb{R}^d} G_t(x,y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)u(s,y)W(ds,dy), \tag{2.8}$$

where $G_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$ is the heat kernel, and the stochastic integral is understood in the sense of Definition 2.1. If equation (2.8) *holds up to some time instant $t < t_0$ for a positive t_0, then we say that equation* (1.1) *has a local random field solution.*

If $u(t,x)$ satisfies equation (2.8), then $u(s,y)$ has a similar representation. By substituting this expression into (2.8), we derive a new equation for u. Carrying this procedure repeatedly, we observe that if u is a strong solution of (1.1), for each integer $N \geq 1$, we have

$$u(t,x) = \sum_{n=0}^{N} u_n(t,x) + \mathfrak{R}_N(t,x), \tag{2.9}$$

where

$$u_n(t,x) = I_n(\tilde{f}_n(t,x)) \tag{2.10}$$

and $\mathfrak{R}_N(t,x) = I_{N+1}(g_N(t,x))$. The element $\tilde{f}_n(t,x)$ is the symmetric extension with respect to $(s_1,x_1),\cdots,(s_n,x_n)$ of the product of Heat kernels

$$\begin{aligned} & \prod_{i=0}^{n} G_{s_{n+1}-s_n}(x_{n+1}-x_n) \\ = \quad & G_{t-s_n}(x-x_n)G_{s_n-s_{n-1}}(x_n-x_{n-1})\cdots G_{s_2-s_1}(x_2-x_1)G_{s_1}u_0(x_1)\,, \end{aligned} \tag{2.11}$$

where $0 = s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} = t$. More precisely,

$$\begin{aligned} \tilde{f}_n(t,x) &:= \tilde{f}_n(t,x;s_1,x_1,\ldots,s_n,x_n) \\ &= \frac{1}{n!}\sum_{\sigma}\prod_{i=0}^{n} G_{s_{\sigma(i+1)}-s_{\sigma(i)}}(x_{\sigma(i+1)}-x_{\sigma(i)})\mathbf{1}_{\{0=s_0<s_{\sigma(1)}<\cdots<s_{\sigma(n)}<s_{n+1}=t\}}\,. \end{aligned} \tag{2.12}$$

The summation above is taken over all permutation σ on $\{1,\cdots,n\}$, which can be identified as a permutation on $\{0,1,\cdots,n,n+1\}$ such that $\sigma(0)=0$ and $\sigma(n+1)=n+1$. The function $g_N(t,x)$ is given by

$$g_N(t,x) = G_{t-s_{N+1}}(x-x_{N+1})G_{s_N-s_N}(x_{N+1}-x_N)\cdots G_{s_2-s_1}(x_2-x_1)u(s_1,x_1)\,.$$

It is evident that $\mathfrak{R}_N(t,x)$ is orthogonal to any multiple integral, with a deterministic kernel, of order less than or equal to N. Therefore, if $u(t,x)$ belongs to $L^2(\Omega)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, it must admit the chaos expansion (e.g., [20, 35]):

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x) = \sum_{n=0}^{\infty} I_n(\tilde{f}_n(t,x)), \tag{2.13}$$

where $\tilde{f}_n(t,x)$ represents the symmetric extension in (2.12). Conversely, if the series (2.13) converges in L^2, it can be easily verified that $\{G_{t-\cdot}(x-\cdot)u(\cdot,\cdot)\}$ is integrable and (2.8) holds true, establishing u as a mild solution of (1.1). Therefore, to study the solvability of (1.1) we need only to investigate the mean square convergence of the series (2.13).

Throughout the paper, the notation $A \lesssim B$ (respectively $A \gtrsim B$ and $A \simeq B$) indicates the existence of strict positive universal constants C_1 and C_2, such that $A \le C_1 B$ (respectively $A \ge C_2 B$ and $C_2 B \le A \le C_1 B$). We write constants c_a and C_b that depend only on a and b and write constants c and C for absolute constants. Their values may change from line to line.

3 Spatial integration

To compute the n-th chaos $\mathbb{E}[u_n(t,x)^2]$, we shall use the Fourier transform for the Hilbert scalar product (2.1). In this section, we aim to find sharp bounds for the integral with respect to $d\boldsymbol{\xi}$ that appears in $\mathbb{E}[u_n(t,x)^2]$. To simplify our notation, we introduce $\vec{\mathbf{s}}_j := (s_j, \cdots, s_n)$ for $1 \le j \le n$, and $\vec{\mathbf{s}} := \vec{\mathbf{s}}_1 = (s_1, \cdots, s_n)$. Conventionally, we will use

$$\begin{cases} d\vec{\mathbf{s}}_j := ds_j ds_{j+1} \cdots ds_n, \ \forall\, 1 \le j \le n, \\ d\vec{\mathbf{r}}_j := dr_j dr_{j+1} \cdots dr_n, \ \forall\, 1 \le j \le n, \end{cases} \tag{3.1}$$

as well as $d\vec{\mathbf{s}} = d\vec{\mathbf{s}}_1$, and $d\vec{\mathbf{r}} = d\vec{\mathbf{r}}_1$. Moreover, for any permutation σ on $\{j, j+1, \cdots, n\}$ for $1 \le j \le n$, we denote

$$\{\vec{\mathbf{s}}_j \in \mathbb{T}_t^\sigma\} := \{(s_j, s_{j+1}, \cdots, s_n) : 0 = s_0 < s_{\sigma(j)} < \cdots < s_{\sigma(n)} < s_{n+1} = t\}, \tag{3.2}$$

and $\{\vec{\mathbf{s}} \in \mathbb{T}_t^\sigma\} := \{\vec{\mathbf{s}}_1 \in \mathbb{T}_t^\sigma\}$. If σ is the natural permutation, then $\mathbb{T}_t^\sigma$ is abbreviated as $\mathbb{T}_t$. This is,

$$\{\vec{\mathbf{s}}_j \in \mathbb{T}_t\} := \{(s_j, s_{j+1}, \cdots, s_n) : 0 = s_0 < s_j < \cdots < s_n < s_{n+1} = t\}, \tag{3.3}$$

and $\{\vec{\mathbf{s}} \in \mathbb{T}_t\} := \{\vec{\mathbf{s}}_1 \in \mathbb{T}_t\}$.

It is easy to see that the Fourier transform of the symmetric function $f_n(t,x)$ in (2.12) with respect to spatial variables $x_1, x_2, \cdots, x_n$ is given by

$$\widehat{f}_n(t,x; s_1, \xi_1, \cdots, s_n, \xi_n) = \frac{1}{n!} \sum_\sigma \widehat{f}_{n,\sigma}^{(t,x)}(s_1, \xi_1, \cdots, s_n, \xi_n) \mathbf{1}_{\{0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t\}}$$

where

$$\widehat{f}_{n,\sigma}^{(t,x)}(s_1, \xi_1, \cdots, s_n, \xi_n) = \prod_{i=1}^n e^{-\frac{1}{2}(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_{\sigma(i)} + \cdots + \xi_{\sigma(1)}|^2} e^{-\iota x(\xi_n + \cdots + \xi_1)}. \tag{3.4}$$

The readers can observe that symmetrization enables us to apply classical results such as (2.5). However, the calculations and estimations related to time variables become significantly intricate.

We assume $u_0 \equiv 1$ on $\mathbb{R}^d$ throughout the remaining part of this paper. For $n \ge 1$, $\vec{\mathbf{s}} \in \mathbb{T}_t^\varsigma$ and $\vec{\mathbf{r}} \in \mathbb{T}_t^\sigma$, where ς and σ are two permutations of $\{1, 2, \cdots, n\}$, we can use equation (2.1), the definition of the tensor product norm and (3.4) to obtain that

$$\begin{aligned}
&\mathbb{E}[u_n(t,x)^2] \\
=&\frac{1}{n!}\sum_{\varsigma,\sigma}\int_{\substack{0<s_1,\cdots,s_n<t\\0<r_1,\cdots,r_n<t}}\int_{\mathbb{R}^{nd}}\prod_{i=1}^n e^{-\frac{1}{2}(s_{\varsigma(i+1)}-s_{\varsigma(i)})|\sum_{j=1}^{i}\xi_{\varsigma(j)}|^2-\frac{1}{2}(r_{\sigma(i+1)}-r_{\sigma(i)})|\sum_{j=1}^{i}\xi_{\sigma(j)}|^2} \\
&\qquad\prod_{i=1}^n\prod_{k=1}^d|\xi_{ik}|^{1-2H_k}d\xi_i\times\prod_{i=1}^n\gamma_0(s_i-r_i)d\vec{\mathbf{s}}d\vec{\mathbf{r}} \\
=&\sum_{\sigma}\int_{\substack{0<s_1<\cdots<s_n<t\\0<r_1,\cdots,r_n<t}}\int_{\mathbb{R}^{nd}}\prod_{i=1}^n e^{-\frac{1}{2}(s_{i+1}-s_i)|\sum_{j=1}^{i}\xi_j|^2-\frac{1}{2}(r_{\sigma(i+1)}-r_{\sigma(i)})|\sum_{j=1}^{i}\xi_{\sigma(j)}|^2} \\
&\qquad\prod_{i=1}^n\prod_{k=1}^d|\xi_{ik}|^{1-2H_k}d\xi_i\times\prod_{i=1}^n\gamma_0(s_i-r_i)d\vec{\mathbf{s}}d\vec{\mathbf{r}}\,,
\end{aligned}\tag{3.5}$$

where the second equality follows by restricting $\{0<s_1,\cdots,s_n<t\}$ to $\{0<s_1<\cdots<s_n<t\}$ and by the symmetry. When $\vec{\mathbf{s}}\in\mathbb{T}_t$ we define $h_{k,n}(\vec{\mathbf{s}})$ as:

$$\begin{aligned}
h_{k,n}(\vec{\mathbf{s}}) :=& \int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\frac{1}{2}(s_{i+1}-s_i)|\xi_{ik}+\cdots+\xi_{1k}|^2}\prod_{i=1}^n|\xi_{ik}|^{1-2H_k}d\xi_{1k}\cdots d\xi_{nk} \\
=&\int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\frac{1}{2}(s_{i+1}-s_i)|\eta_i|^2}\prod_{i=1}^n|\eta_i-\eta_{i-1}|^{1-2H_k}d\eta_i\,,
\end{aligned}\tag{3.6}$$

where the substitution $\eta_i:=\xi_{ik}+\cdots+\xi_{1k}$ is used in (3.6) and $\eta_0:=0$ by convention. Moreover, for any permutation σ of $\{1,2,\cdots,n\}$, we define similarly with the notation $\eta_i^\sigma=\xi_{\sigma(i)k}+\cdots+\xi_{\sigma(1)k}$:

$$\begin{aligned}
h_{k,n}(\vec{\mathbf{s}},\vec{\mathbf{r}}^\sigma) =& \int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\frac{1}{2}(s_{i+1}-s_i)|\xi_{ik}+\cdots+\xi_{1k}|^2-\frac{1}{2}(r_{\sigma(i+1)}-r_{\sigma(i)})|\xi_{\sigma(i)k}+\cdots+\xi_{\sigma(1)k}|^2} \\
&\qquad\times\prod_{i=1}^n|\xi_{ik}|^{1-2H_k}d\xi_{1k}\cdots d\xi_{nk} \\
=&\int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\frac{1}{2}(s_{i+1}-s_i)|\eta_i|^2-\frac{1}{2}(r_{\sigma(i+1)}-r_{\sigma(i)})|\eta_i^\sigma|^2}\prod_{i=1}^n|\eta_i-\eta_{i-1}|^{1-2H_k}d\eta_i\,.
\end{aligned}\tag{3.7}$$

Then by applying Hölder's inequality, we obtain

$$h_{k,n}(\vec{\mathbf{s}},\vec{\mathbf{r}}^\sigma)\le C_H^n h_{k,n}^{1/2}(\vec{\mathbf{s}})h_{k,n}^{1/2}(\vec{\mathbf{r}})\,,\quad \vec{\mathbf{s}},\vec{\mathbf{r}}\in\mathbb{T}_t\,.\tag{3.8}$$

Furthermore, we can simplify (3.5) as

$$\mathbb{E}\left[u_n(t,x)^2\right]=\sum_{\sigma}\int_{\substack{0<s_1<\cdots<s_n<t\\0<r_1,\cdots,r_n<t}}\prod_{k=1}^d h_{k,n}(\vec{\mathbf{s}},\vec{\mathbf{r}}^\sigma)\prod_{i=1}^n\gamma_0(s_i-r_i)d\vec{\mathbf{s}}d\vec{\mathbf{r}}\,.\tag{3.9}$$

In order to obtain a sharp bound for $h_{k,n}(\vec{\mathbf{s}})$, we can rewrite equation (3.6) as an expectation of normal variables. Let $X_0=0$ and let $\{X_1,\cdots,X_n\}$ be i.i.d. standard Gaussian random variables. Denote

$$w_0=1\quad\text{and}\quad w_i:=s_{i+1}-s_i\quad\text{for } 1\le i\le n.\tag{3.10}$$

Then for $0 < s_1 < \cdots < s_n < t$, for some positive constants c_{H_k} depending on Hurst parameters H_k we can reformulate $h_{k,n}(\vec{\mathbf{s}})$ in (3.6) as

$$\begin{aligned} h_{k,n}(\vec{\mathbf{s}}) &= c_{H_k}^n \left(\prod_{i=1}^n w_i^{-1/2}\right) \mathbb{E}\left[\prod_{i=1}^n \left|\frac{X_i}{\sqrt{w_i}} - \frac{X_{i-1}}{\sqrt{w_{i-1}}}\right|^{1-2H_k}\right] \\ &= c_{H_k}^n \left(\prod_{i=1}^n w_i^{-1/2}\right) \left(\prod_{i=1}^n (w_i w_{i-1})^{H_k-1/2}\right) \\ &\qquad \times \mathbb{E}\left[|X_1|^{1-2H_k} \prod_{i=2}^n \left|\sqrt{w_{i-1}}X_i - \sqrt{w_i}X_{i-1}\right|^{1-2H_k}\right] \\ &= c_{H_k}^n w_n^{H_k-1} \left(\prod_{i=1}^{n-1} w_i^{2H_k-3/2}\right) \left(\prod_{i=2}^n (w_i + w_{i-1})^{\frac{1}{2}-H_k}\right) \\ &\qquad \times \mathbb{E}\left[|X_1|^{1-2H_k} \prod_{i=2}^n \left|\sqrt{\frac{w_{i-1}}{w_{i-1}+w_i}}X_i - \sqrt{\frac{w_i}{w_{i-1}+w_i}}X_{i-1}\right|^{1-2H_k}\right] . \end{aligned} \tag{3.11}$$

Let us introduce the following notations to simplify the expectation in (3.11). We set

$$\lambda_1 = 1 \text{ and } \lambda_i := \sqrt{\frac{w_{i-1}}{w_{i-1}+w_i}} \in (0,1)\,, \quad i \geq 2\,, \tag{3.12}$$

and

$$\theta_k := 2H_k - 1\,, \quad \zeta_n := \prod_{i=1}^n \left|\lambda_i X_i - \sqrt{1-\lambda_i^2}\, X_{i-1}\right|^{-\theta_k} . \tag{3.13}$$

Let us denote the expectation in (3.11) as $\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n)$. In particular, we amend the expectation in (3.11) to

$$\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) := \mathbb{E}\left[\zeta_n\right] = \mathbb{E}\left[\prod_{i=1}^n \left|\lambda_i X_i - \sqrt{1-\lambda_i^2} X_{i-1}\right|^{1-2H_k}\right], \tag{3.14}$$

with the conventions $\prod_{i=m}^n a_i := 1$ if $m > n$, and the conventions $\lambda_1 = 1$, $X_0 = 1$. Recall that our objective is to obtain a sharp bound for $h_{k,n}(\vec{\mathbf{s}})$. To achieve this, it is adequate to establish a precise upper limit for $\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n)$, as presented below.

Lemma 3.1. *We have the following upper bounds for* $\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n)$*.*

(1) If $H_k < 3/4$, then there is a positive constant C_{H_k} such that for all $\lambda_1, \cdots, \lambda_n$

$$\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) \leq C_{H_k}^n . \tag{3.15}$$

(2) If $H_k \geq 3/4$, then for any $\beta_k \in (4H_k - 3, 2H_k - 1)$, there is a positive constant C_{β_k} such that for all $\lambda_1, \cdots, \lambda_n$

$$\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) \leq C_{H_k,\beta_k}^n \prod_{i=2}^n \lambda_i^{-\beta_k} . \tag{3.16}$$

Proof. We divide the proof into three steps based on the value of H_k: $(0, 1/2)$, $[1/2, 3/4)$, and $[3/4, 1)$.

Step 1: *The case* $0 < H_k < 1/2$, *i.e.,* $\theta_k < 0$. This case has been proven in [15]. More precisely, we have

$$
\begin{aligned}
\mathfrak{I}_{k,n}(\lambda_1,\cdots,\lambda_n) &\le C_{H_k}\mathbb{E}\left[|X_1|^{-\theta_k}\left(\prod_{i=2}^n(|X_i|\vee|X_{i-1}|)^{-\theta_k}\right)\right]\\
&\le C_{H_k}\mathbb{E}\left[|X_1|^{-\theta_k}\left(\prod_{i=2}^n(|X_i|+|X_{i-1}|)^{-\theta_k}\right)\right]\\
&\le C_{H_k}^n\,. \qquad (3.17)
\end{aligned}
$$

Step 2: *The case* $3/4 \le H_k < 1$. To bound $\mathfrak{I}_{k,n}(\lambda_1,\cdots,\lambda_n)$ in this case, we shall use the following estimation for a standard Gaussian random variable X (see Lemma A.1 in [29] for details): for any $0<\alpha<1$, $\lambda>0$ and $b\in\mathbb{R}$, there is a constant $C_\alpha>0$ independent of λ and b so that

$$
\mathbb{E}\left[|\lambda X+b|^{-\alpha}\right]\le C_\alpha(\lambda\vee|b|)^{-\alpha}\simeq(\lambda+|b|)^{-\alpha}. \qquad (3.18)
$$

Denote by $\mathbb{E}^Y$ the expectation with respect to the random variable Y while considering other random variables as constants. Thus, it is clear that for $\theta_k\ge\frac12$ (θ_k is defined in (3.13)) and $\beta_k\in(4H_k-3,2H_k-1)$

$$
\begin{aligned}
\mathfrak{I}_{k,n}(\lambda_1,\cdots,\lambda_n) &= \mathbb{E}\left[\mathbb{E}^{X_n}\left[\left|\lambda_nX_n-\sqrt{1-\lambda_n^2}\,X_{n-1}\right|^{-\theta_k}\zeta_{n-1}\right]\right]\\
&\le C_{H_k}\mathbb{E}\left[\left(\sqrt{1-\lambda_n^2}\,|X_{n-1}|+\lambda_n\right)^{-\theta_k}\zeta_{n-1}\right]\\
&\le C_{H_k}\lambda_n^{-\beta_k}\mathbb{E}\Bigg\{\mathbb{E}^{X_{n-1}}\Bigg[\left(\sqrt{1-\lambda_n^2}\,|X_{n-1}|+\lambda_n\right)^{-\theta_k+\beta_k}\\
&\qquad\left|\lambda_{n-1}X_{n-1}-\sqrt{1-\lambda_{n-1}^2}\,X_{n-2}\right|^{-\theta_k}\Bigg]\zeta_{n-2}\Bigg\}\\
&\le C_{H_k}\lambda_n^{-\beta_k}\mathbb{E}\Bigg\{\left[\mathbb{E}^{X_{n-1}}\left(\sqrt{1-\lambda_n^2}\,|X_{n-1}|+\lambda_n\right)^{-(\theta_k-\beta_k)p}\right]^{1/p}\\
&\qquad\left[\mathbb{E}^{X_{n-1}}\left|\lambda_{n-1}X_{n-1}-\sqrt{1-\lambda_{n-1}^2}\,X_{n-2}\right|^{-\theta_kq}\right]^{1/q}\zeta_{n-2}\Bigg\}, \qquad (3.19)
\end{aligned}
$$

by applying Hölder inequality with $\frac1p+\frac1q=1$ in the last inequality.

It is easy to see that under our assumption $4H_k-3<\beta_k<2H_k-1$, we have $2\theta_k-1<\beta_k<\theta_k$. This enables us to find p and q such that $0<(\theta_k-\beta_k)p<1$ and $0<\theta_kq<1$ hold. Therefore,

$$
\begin{aligned}
&\mathbb{E}^{X_{n-1}}\left[\left(\sqrt{1-\lambda_n^2}\,|X_{n-1}|+\lambda_n\right)^{-(\theta_k-\beta_k)p}\right]\\
&\qquad\le C_{\beta_k,\theta_k}(\sqrt{1-\lambda_n^2}+\lambda_n)^{-(\theta_k-\beta_k)p}\le C_{\beta_k,\theta_k}<\infty\,, \qquad (3.20)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}^{X_{n-1}}\left[\left|\lambda_{n-1}X_{n-1}-\sqrt{1-\lambda_{n-1}^2}\,X_{n-2}\right|^{-\theta_kq}\right]\\
&\qquad\le C_{\theta_k}(\lambda_{n-1}+\sqrt{1-\lambda_{n-1}^2}\,|X_{n-2}|)^{-\theta_kq}\,. \qquad (3.21)
\end{aligned}
$$

Substituting (3.20) and (3.21) into (3.19) we have

$$
\mathfrak{I}_{k,n}(\lambda_1,\cdots,\lambda_n)\le C_{H_k,\beta_k}\lambda_n^{-\beta_k}\mathbb{E}\left[(\lambda_{n-1}+\sqrt{1-\lambda_{n-1}^2}\,|X_{n-2}|)^{-\theta_k}\zeta_{n-2}\right]. \qquad (3.22)
$$

Continuing this way we obtain that for any $2\theta_k - 1 < \beta_k < \theta_k$, (3.16) holds.
Step 3: *The case* $\frac{1}{2} \le H_k < \frac{3}{4}$. When $H_k = 1/2$, it is easy to see that $\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) \equiv 1$. So we can assume $\frac{1}{2} < H_k < \frac{3}{4}$. The proof is similar to the case $H_k > \frac{3}{4}$ except that we take $\beta_k = 0$ now and the proof is simpler. In fact, the Hölder inequality is valid since we can still find p and q such that $0 < \theta_k p < 1$ and $0 < \theta_k q < 1$ hold. In conclusion, we have

$$\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) \le C^n_{H_k}. \tag{3.23}$$

This proves the lemma. □

To deal with the necessary condition in our main Theorem 1.1 in the next section, we also need a lower bound for $\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n)$. We can obtain the lower bound when $H_k < 3/4$. However, we are still not clear about the lower bound of $\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n)$ when $H_k \ge 3/4$ except when $n = 2$. But this result is sufficient for our purpose.

Lemma 3.2. *We have the following lower bounds for $\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n)$ defined by* (3.14)*.*

(1) If $H_k < 3/4$, then there is a positive constant c_{H_k}, such that for all $\lambda_1, \cdots, \lambda_n$

$$\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) \ge c^n_{H_k}. \tag{3.24}$$

(2) If $H_k \ge 3/4$, then there is a positive constant c_{H_k}, independent of λ_2 such that

$$\mathfrak{I}_{k,2}(\lambda_1, \lambda_2) \ge c_{H_k} \lambda_2^{-(4H_k-3)}, \tag{3.25}$$

where λ_2 is defined by (3.12)*.*

Proof. First, we prove statement (1). Set

$$A_n := \begin{cases} \{X_1 \le 0, X_2 \ge 0, \cdots, X_{n-1} \le 0, X_n \ge 0\} & \text{when } n \text{ is even}; \\ \{X_1 \le 0, X_2 \ge 0, \cdots, X_{n-1} \ge 0, X_n \le 0\} & \text{when } n \text{ is odd}. \end{cases}$$

When $0 < H_k \le \frac{1}{2}$, namely, $-1 \le \theta_k < 0$ for θ_k defined by (3.13), we have

$$\begin{aligned} \mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) :&= \mathbb{E}\left[|X_1|^{-\theta_k} \prod_{i=2}^n \left|\lambda_i X_i - \sqrt{1-\lambda_i^2} X_{i-1}\right|^{-\theta_k}\right] \\ &\ge \mathbb{E}\left[|X_1|^{-\theta_k} \prod_{i=2}^n \left|\lambda_i X_i - \sqrt{1-\lambda_i^2} X_{i-1}\right|^{-\theta_k} \cdot \mathbf{1}_{A_n}\right] \\ &\ge c^n_{H_k}\, \mathbb{E}\left[|X_1|^{-\theta_k} \left(\prod_{i=2}^n (|X_i| \wedge |X_{i-1}|)^{-\theta_k}\right) \cdot \mathbf{1}_{A_n}\right] \\ &\ge c^n_{H_k}\, \mathbb{E}\left[(|X_1| \wedge |X_2| \wedge \cdots \wedge |X_n|)^{-n\theta_k} \cdot \mathbf{1}_{A_n}\right], \end{aligned}$$

where we used the fact that $\lambda_i + \sqrt{1-\lambda_i^2} \ge 1$ in the above second inequality. Now we let

$$B_n := \begin{cases} \{X_1 \le -1, X_2 \ge 1, \cdots, X_{n-1} \le -1, X_n \ge 1\} & \text{when } n \text{ is even}; \\ \{X_1 \le -1, X_2 \ge 1, \cdots, X_{n-1} \ge 1, X_n \le -1\} & \text{when } n \text{ is odd}, \end{cases}$$

which is contained in A_n. We proceed to get

$$\begin{aligned} \mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) &\ge c^n_{H_k}\, \mathbb{E}\left[(|X_1| \wedge |X_2| \wedge \cdots \wedge |X_n|)^{-n\theta_k} \cdot \mathbf{1}_{B_n}\right] \\ &\ge c^n_{H_k} \left[\mathbb{P}(X_1 \le -1)\right]^n = c^n_{H_k} > 0, \end{aligned}$$

where we recall that c_{H_k} is a generic positive constant which may be different in different places.

When $\frac{1}{2} < H_k \leq \frac{3}{4}$, i.e., $0 < \theta_k \leq 1/2$, recalling $0 \leq \lambda_i \leq 1$, we have

$$\begin{aligned}\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) :&= \mathbb{E}\left[|X_1|^{-\theta_k}\prod_{i=2}^{n}\left|\lambda_i X_i - \sqrt{1-\lambda_i^2}X_{i-1}\right|^{-\theta_k}\right] \\ &\geq \mathbb{E}\left[|X_1|^{-\theta_k}\left(\prod_{i=2}^{n}(|X_i|+|X_{i-1}|)\right)^{-\theta_k}\right].\end{aligned}$$

Notice that the factors $|X_i|^2$ may appear in the product $\prod_{i=2}^{n}(|X_i|+|X_{i-1}|)$. But the expectation in the second line is bounded thanks to the assumption $0 > -\theta_k \geq -1/2$. Thus, we have

$$\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) \geq c_{H_k}^n > 0\,.$$

Now, we prove statement (2). In this case $H_k > \frac{3}{4}$, namely $\theta_k > 1/2$. We shall consider the lower bound only for $\mathfrak{I}_{k,2}(\lambda_1, \lambda_2)$, defined in (3.14) with $n = 2$ there. Remember that $\lambda_1 = 1$, $\mathfrak{I}_{k,2}(\lambda_1, \lambda_2)$ is a function of λ_2. It is clear that $\mathfrak{I}_{k,2}(1, \cdot)$ is continuous on $(0, 1]$. If $\lambda_2 \to 1$, then $\mathfrak{I}_{k,2}(1, \lambda_2) \to \mathbb{E}\left[|X_1|^{-\theta_k}|X_2|^{-\theta_k}\right] > 0$. We want to understand the behaviour of this function as λ_2 approaches 0. For simplicity, we assume $0 < \lambda_2 < \frac{1}{2}$. Then

$$\begin{aligned}\mathfrak{I}_{k,2}(\lambda_1, \lambda_2) &= \mathbb{E}\left[|X_1|^{-\theta_k}|\lambda_2 X_2 - X_1|^{-\theta_k}\right] \\ &= \frac{1}{\sqrt{2\pi}}\mathbb{E}^{X_2}\left[\int_{-\infty}^{\infty}|x_1|^{-\theta_k}|\lambda_2 X_2 - x_1|^{-\theta_k}e^{-\frac{x_1^2}{2}}dx_1\right] \\ &\geq c\mathbb{E}^{X_2}\left[\int_0^1 |x_1|^{-\theta_k}|\lambda_2 X_2 - x_1|^{-\theta_k}dx_1\right] \\ &= c\mathbb{E}^{X_2}\left[|\lambda_2 X_2|^{-2\theta_k+1}\cdot\int_0^{1/(\lambda_2|X_2|)}|\tilde{x}_1|^{-\theta_k}|1-\tilde{x}_1|^{-\theta_k}d\tilde{x}_1\right],\end{aligned}$$

where in the last equality we make a change of variables $x_1 \to \lambda_2|X_2|\tilde{x}_1$. Then, by the conditions $0 < \lambda_2 < \frac{1}{2}$ and $1/2 < \theta_k < 1$ we get

$$\begin{aligned}\mathfrak{I}_{k,2}(\lambda_1, \lambda_2) &\geq \mathbb{E}^{X_2}\left[|\lambda_2 X_2|^{-2\theta_k+1}\cdot\int_0^{2/|X_2|}|\tilde{x}_1|^{-\theta_k}|1-\tilde{x}_1|^{-\theta_k}d\tilde{x}_1\mathbf{1}_{\{|X_2|\leq 1\}}\right] \\ &\geq \mathbb{E}^{X_2}\left[|\lambda_2 X_2|^{-2\theta_k+1}\mathbf{1}_{\{|X_2|\leq 1\}}\cdot\int_0^2|\tilde{x}_1|^{-\theta_k}|1-\tilde{x}_1|^{-\theta_k}d\tilde{x}_1\right] \\ &\geq c_{\theta_k}\mathbb{E}^{X_2}\left[|\lambda_2 X_2|^{-2\theta_k+1}\mathbf{1}_{\{|X_2|\leq 1\}}\right]\,. \end{aligned} \tag{3.26}$$

Moreover, it is not hard to see

$$\begin{aligned}\mathbb{E}^{X_2}\left[|\lambda_2 X_2|^{-2\theta_k+1}\mathbf{1}_{\{|X_2|\leq 1\}}\right] &\geq c_{\theta_k}\lambda_2^{-2\theta_k+1}\cdot\int_0^1 x_2^{-2\theta_k+1}dx_2 \\ &\geq c_{\theta_k}\lambda_2^{-2\theta_k+1}.\end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.26) implies

$$\mathfrak{I}_{k,2}(\lambda_1, \lambda_2) \geq c\lambda_2^{-2\theta_k+1} = c\lambda_2^{3-4H_k}. \tag{3.28}$$

This completes the proof. □

Before stating Lemma 3.3, we introduce a set of indices $\mathcal{D}_n$, which consists of all indices $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ such that

$$\alpha_i \in \left\{0, \frac{1}{2}d - |H| + \frac{1}{2}\beta^*, d - 2|H| + \beta^*\right\}, \quad \alpha_i + \alpha_{i+1} \neq 0, \tag{3.29}$$

$$\text{and} \qquad |\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i = (n-1)\left(\frac{1}{2}d - |H| + \frac{1}{2}\beta^*\right).$$

Lemma 3.3. *Let* $h_n(\vec{\mathbf{s}}) := \prod_{k=1}^d h_{k,n}(\vec{\mathbf{s}})$*, where* $h_{k,n}(\vec{\mathbf{s}})$ $(k = 1, \cdots, d)$ *are defined by* (3.6)*. Take any* $\beta_k \in (4H_k - 3, 2H_k - 1)$ *for* $k = d_* + 1, \cdots, d$ *and set*

$$\beta^* := \beta_{d_*+1} + \cdots + \beta_d. \tag{3.30}$$

Then, we have the following estimations for $h_n(\vec{\mathbf{s}})$*.*

(i) If $\frac{1}{2}d - |H| + \frac{1}{2}\beta^* > 0$*, then*

$$h_n(\vec{\mathbf{s}}) \leq C_H^n w_n^{|H|-d+\alpha_n} \sum_{\boldsymbol{\alpha} \in \mathcal{D}_n} \left(\prod_{i=1}^{n-1} w_i^{\alpha_i + 2|H| - \frac{3}{2}d - \frac{1}{2}\beta^*}\right), \tag{3.31}$$

where w_i $(1 \leq i \leq n)$ *are given by* (3.10)*. Moreover, if* $\alpha_i = d - 2|H| + \beta^*$*, then both of* α_{i-1} *and* α_{i+1} *cannot be* $d - 2|H| + \beta^*$*.*

(ii) If $\frac{1}{2}d - |H| + \frac{1}{2}\beta^* \leq 0$*, then*

$$h_n(\vec{\mathbf{s}}) \leq C_H^n \prod_{i=1}^n w_i^{|H|-d}. \tag{3.32}$$

Proof. By Lemma 3.1, when $0 < H_k < \frac{3}{4}$,

$$\mathfrak{I}_{k,n}(\lambda_1, \cdots, \lambda_n) \leq C_{H_k}^n.$$

Then, from (3.11) we obtain when $k \leq d_*$,

$$h_{k,n}(\vec{\mathbf{s}}) \leq C_{H_k}^n w_n^{H_k - 1} \left(\prod_{i=1}^{n-1} w_i^{2H_k - 3/2}\right) \tag{3.33}$$

When $\frac{3}{4} \leq H_k < 1$, substituting (3.16) into (3.11) we get for $d_* < k \leq d$,

$$h_{k,n}(\vec{\mathbf{s}}) \leq C_{H_k,\beta_k}^n w_n^{H_k-1} \left(\prod_{i=1}^{n-1} w_i^{2H_k - 3/2}\right) \left(\prod_{i=2}^n (w_i + w_{i-1})^{\frac{1}{2} - H_k}\right) \times \left(\prod_{i=2}^n \lambda_i^{-\beta_k}\right). \tag{3.34}$$

Recall that $\lambda_i = \sqrt{\frac{w_{i-1}}{w_{i-1} + w_i}}$ $(i \geq 2)$ are defined in (3.12). Combining (3.33) and (3.34), we have

$$\begin{aligned} h_n(\vec{\mathbf{s}}) &:= \prod_{k=1}^d h_{k,n}(\vec{\mathbf{s}}) \\ &\leq C_H^n \prod_{k=1}^d \left[w_n^{H_k - 1} \left(\prod_{i=1}^{n-1} w_i^{2H_k - 3/2}\right) \left(\prod_{i=2}^n (w_i + w_{i-1})^{\frac{1}{2} - H_k}\right)\right] \prod_{k=d_*+1}^d \prod_{i=2}^n \lambda_i^{-\beta_k} \\ &= C_H^n w_n^{|H|-d} \left(\prod_{i=1}^{n-1} w_i^{2|H| - \frac{3}{2}d - \frac{1}{2}\beta^*}\right) \left(\prod_{i=2}^n (w_i + w_{i-1})^{\frac{1}{2}d - |H| + \frac{1}{2}\beta^*}\right). \end{aligned} \tag{3.35}$$

If $\frac{1}{2}d-|H|+\frac{1}{2}\beta^*>0$, then it follows from (3.35) by expanding the product $\prod_{i=2}^n(w_i+w_{i-1})^{\frac{1}{2}d-|H|+\frac{1}{2}\beta^*}$ that

$$h_n(\vec{\mathbf{s}})\le C_H^n w_n^{|H|-d+\alpha_n}\sum_{\boldsymbol{\alpha}\in\mathcal{D}_n}\left(\prod_{i=1}^{n-1}w_i^{\alpha_i+2|H|-\frac{3}{2}d-\frac{1}{2}\beta^*}\right),$$

where $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_n)$ is in $\mathcal{D}_n$. If $\frac{1}{2}d-|H|+\frac{1}{2}\beta^*\le 0$, then we can see from (3.35) that

$$h_n(\vec{\mathbf{s}})\le C_H^n w_n^{|H|-d}\left(\prod_{i=1}^{n-1}w_i^{2|H|-\frac{3}{2}d-\frac{1}{2}\beta^*}\right)\left(\prod_{i=2}^{n}w_{i-1}^{\frac{1}{2}d-|H|+\frac{1}{2}\beta^*}\right)=C_H^n\prod_{i=1}^n w_i^{|H|-d}.$$

We have finished the proof. □

Finally, we have our main result in this section.

Proposition 3.4. *The second moment of n-th chaos $u_n(t,x)$ given by* (2.10)-(2.12) *can be bounded as follows.*
(1) If $\frac{1}{2}d-|H|+\frac{1}{2}\beta^*>0$, then for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$

$$\mathbb{E}[u_n(t,x)^2]\le C^n t^{n(|H|+2H_0-d)}n!\cdot\sum_{\boldsymbol{\alpha}\in\mathcal{D}_n}\int_{\mathbb{T}_1\times\mathbb{T}_1}(1-s_n)^{-\rho_n}(1-r_n)^{-\rho_n}\times\prod_{i=1}^{n-1}|s_{i+1}-s_i|^{-\rho_i}\prod_{i=1}^{n-1}|r_{i+1}-r_i|^{-\rho_i}\prod_{i=1}^n\gamma_0(s_i-r_i)d\vec{\mathbf{s}}d\vec{\mathbf{r}},\qquad(3.36)$$

with $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_n)\in\mathcal{D}_n$ is given by (3.29) *and the dependence of $\{\rho_i\}$ on $\boldsymbol{\alpha}$ is determined by*

$$\begin{cases}\rho_i=\frac{1}{2}(\frac{3}{2}d+\frac{1}{2}\beta^*-2|H|-\alpha_i),1\le i\le n-1,\\ \rho_n=\frac{1}{2}(d-|H|-\alpha_n).\end{cases}\qquad(3.37)$$

(2) If $\frac{1}{2}d-|H|+\frac{1}{2}\beta^*\le 0$, then

$$\mathbb{E}[u_n(t,x)^2]\le C^n t^{n(|H|+2H_0-d)}n!\cdot\int_{\mathbb{T}_1\times\mathbb{T}_1}\prod_{i=1}^n|r_{i+1}-r_i|^{-\frac{1}{2}(d-|H|)}\times\prod_{i=1}^n|s_{i+1}-s_i|^{-\frac{1}{2}(d-|H|)}\prod_{i=1}^n\gamma_0(s_i-r_i)d\vec{\mathbf{s}}d\vec{\mathbf{r}}.\qquad(3.38)$$

Proof. By (3.8) and (3.9), we see that

$$\mathbb{E}\left[u_n(t,x)^2\right]\le C^n\int_{\substack{0<s_1<\cdots<s_n<t\\0<r_1,\cdots,r_n<t}}h_n^{1/2}(\vec{\mathbf{s}})h_n^{1/2}(\vec{\mathbf{r}})\prod_{i=1}^n\gamma_0(s_i-r_i)d\vec{\mathbf{s}}d\vec{\mathbf{r}},\qquad(3.39)$$

By substituting the bound for h_n as derived in Lemma 3.3 into the inequality above, we obtain

$$\mathbb{E}\left[u_n(t,x)^2\right]\le C^n\sum_{\boldsymbol{\alpha}\in\mathcal{D}_n}n!\int_{\mathbb{T}_t\times\mathbb{T}_t}(t-s_n)^{\frac{1}{2}(\alpha_n+|H|-d)}(t-r_n)^{\frac{1}{2}(\alpha_n+|H|-d)}\times\prod_{i=1}^n\gamma_0(s_i-r_i)\prod_{i=1}^{n-1}|s_{i+1}-s_i|^{\frac{1}{2}(\alpha_i+2|H|-\frac{3}{2}d-\frac{1}{2}\beta^*)}\times\prod_{i=1}^{n-1}|r_{i+1}-r_i|^{\frac{1}{2}(\alpha_i+2|H|-\frac{3}{2}d-\frac{1}{2}\beta^*)}d\vec{\mathbf{s}}d\vec{\mathbf{r}}.\qquad(3.40)$$

Now a change of variables $s_i\to t\cdot s_i$ and $r_i\to t\cdot r_i$ $(1\le i\le n)$ yields (3.36). The estimate (3.38) can be proved similarly. □

Remark 3.5. We make a remark on the condition $\beta_k \in (4H_k - 3, 2H_k - 1)$. It is used in the equation (3.36), where we wish ρ_i defined in (3.37) or equivalently β^* to be as small as possible. Thus, during the proofs in the future we may take $\beta_k = \{(4H_k - 3) \vee 0\} + \varepsilon$ and $\beta^* = 4H^* - 3d^* + \varepsilon$ for any arbitrarily small $\varepsilon > 0$. However, the presence of this ε would distract the main idea of the proof. Since our final conditions in (1.7) are in strict inequality, we can let $\varepsilon = 0$, namely, we will take $\beta_k = \{(4H_k - 3) \vee 0\}$ and $\beta^* = 4H^* - 3d^*$, in the following proofs to simplify the presentation. This will not cause problem. For instance, let us take a look of the condition $\rho_1 < 1$ in equation (4.18), which is equivalent to $\frac{1}{2}(\frac{3}{2}d + \frac{1}{2}\beta^* - 2|H|) < 1$. If this is true, then it is easy to see that one can find a (sufficiently small) ε such that $\frac{1}{2}(\frac{3}{2}d + \frac{1}{2}(\beta^* + \varepsilon) - 2|H|) < 1$. In another word, the condition that there is a $\varepsilon > 0$ such that $\frac{1}{2}(\frac{3}{2}d + \frac{1}{2}(\beta^* + \varepsilon) - 2|H|) < 1$ is equivalent to the condition $\frac{1}{2}(\frac{3}{2}d + \frac{1}{2}\beta^* - 2|H|) < 1$. This means that if we use β^* to replace $\beta^* + \varepsilon$, we will obtain the same set of conditions since we have only finite set of them.

4 Necessary conditions and sufficient conditions

In this section, we give a proof of Theorem 1.1.

4.1 Proof of Theorem 1.1: Sufficiency

Now we begin to prove the sufficiency of the conditions in Theorem 1.1. Proposition 1.5 and Proposition 1.6 will be by-products of our proof. We only consider $H_0 > 1/2$. The case when $H_0 = 1/2$ can be proved in the same way, and conditions (1.7) becomes (1.11) in this case.

We shall divide the proof into two cases: (I) $\frac{1}{2}d_* - H_* \le d^* - H^*$ and (II) $\frac{1}{2}d_* - H_* > d^* - H^*$.

The case $\frac{1}{2}d_* - H_* \le d^* - H^*$. By (3.38), we have for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$\begin{aligned}\mathbb{E}[u_n(t,x)^2] \le & C^n n! t^{n(|H|+2H_0-d)} \int_{\mathbb{T}_1\times\mathbb{T}_1} \prod_{i=1}^n (s_{i+1}-s_i)^{\frac{|H|-d}{2}} \\ & \cdot \prod_{i=1}^n (r_{i+1}-r_i)^{\frac{|H|-d}{2}} |s_i - r_i|^{2H_0-2} d\vec{\mathbf{s}} d\vec{\mathbf{r}}\,, \end{aligned} \tag{4.1}$$

here in this case, we recall that the conventions $\mathbb{T}_1$ is an ordered set defined in (3.3) and $s_{n+1} = r_{n+1} = 1$. Let $\tilde{h}_n(\vec{\mathbf{s}})$ be the symmetric extension of $\prod_{i=1}^n (s_{i+1}-s_i)^{\frac{|H|-d}{2}} \mathbf{1}_{\{0<s_1<\cdots<s_n<1\}}$ to $[0,1]^n$. The application of the Hardy-Littlewood-Sobolev inequality (see Lemma A.2 in Appendix) yields

$$\begin{aligned}\mathbb{E}[u_n(t,x)^2] \le & C^n (n!)^{-1} t^{n(|H|+2H_0-d)} \int_{[0,1]^{2n}} \tilde{h}_n(\vec{\mathbf{s}})\tilde{h}_n(\vec{\mathbf{r}}) \cdot |s_i - r_i|^{2H_0-2} d\vec{\mathbf{s}} d\vec{\mathbf{r}} \\ = & C^n (n!)^{-1} t^{n(|H|+2H_0-d)} \left[\int_{[0,1]^n} \tilde{h}_n(\vec{\mathbf{s}})^{\frac{1}{H_0}} d\vec{\mathbf{s}}\right]^{2H_0} \\ \le & C^n (n!)^{2H_0-1} t^{n(|H|+2H_0-d)} C_H^n \left(\int_{\mathbb{T}_1} \prod_{i=1}^n (s_{i+1}-s_i)^{\frac{|H|-d}{2H_0}} d\vec{\mathbf{s}}\right)^{2H_0}.\end{aligned}$$

Then, by [24, Lemma 4.5] we get

$$\begin{aligned}\mathbb{E}[u_n(t,x)^2] \le & C^n (n!)^{2H_0-1} C_H^n t^{n(|H|+2H_0-d)} \left[\Gamma\left(\left(\frac{|H|-d}{2H_0}+1\right)n+1\right)\right]^{-2H_0} \\ = & C^n (n!)^{-(|H|-d+1)} C_H^n t^{n(|H|+2H_0-d)}\,. \end{aligned} \tag{4.2}$$

We rewrite the statement (4.2) as, for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$

$$\|u_n(t,x)\|_2 \le C_H^n (n!)^{-\frac{|H|-d+1}{2}} t^{\frac{n(|H|+2H_0-d)}{2}} \tag{4.3}$$

under the condition

$$\frac{|H|-d}{2H_0} > -1 \Leftrightarrow |H|+2H_0>d,$$

which is exactly the second condition (1.7b). Furthermore, by the hypercontractivity inequality, we have for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$

$$\|u_n(t,x)\|_p \le p^{n/2} C_H^n (n!)^{-\frac{|H|-d+1}{2}} t^{\frac{n(|H|+2H_0-d)}{2}}\,. \tag{4.4}$$

Now, if

$$|H|>d-1\,, \tag{4.5}$$

then the chaos expansion of the solution is summable in $L^p(\Omega)$. By the Stirling formula, we have $\Gamma(an+1)\sim (n!)^a$, and then by the evaluation of Mittag-Leffler summation (see, e.g., [6, Lemma A.3]), it holds that for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$

$$\begin{aligned}
\mathbb{E}\left[|u(t,x)|^p\right] &\le \left(\sum_{n=0}^{\infty}\|u_n(t,x)\|_p\right)^p \\
&\le \left(\sum_{n=0}^{\infty} p^{n/2} C_H^n (n!)^{-\frac{|H|-d+1}{2}} t^{\frac{n(|H|+2H_0-d)}{2}}\right)^p \\
&\le \left(\sum_{n=0}^{\infty} \frac{p^{n/2}C_H^n t^{\frac{n(|H|+2H_0-d)}{2}}}{\Gamma(\frac{|H|-d+1}{2}n+1)}\right)^p \\
&\le C_H \exp\left[c_H p^{\frac{|H|-d+2}{|H|-d+1}} t^{\frac{|H|+2H_0-d}{|H|-d+1}}\right]\,,
\end{aligned}$$

where the generic positive constants C_H and c_H may differ from line to line.

Remark 4.1. It is interesting to note that the right-hand side of (4.1), as a function of $H_0\in[\frac{1}{2},1]$, takes the minimum value at $H_0=1$, which is exactly equal to

$$\begin{aligned}
&n! t^{n(|H|+2-d)} \left[\int_{\mathbb{T}_1}\prod_{i=1}^n (s_{i+1}-s_i)^{\frac{|H|-d}{2}} d\vec{\mathbf{s}}\right]^2 \\
&\quad = C^n t^{n(|H|+2-d)} (n!)^{1-2\left(1+\frac{|H|-d}{2}\right)} = (n!)^{-(|H|-d+1)} t^{n(|H|+2-d)}\,.
\end{aligned} \tag{4.6}$$

In terms of the exponent of $n!$, it is the same as the estimate (4.2). This means that our approach to using the Hardy-Littlewood-Sobolev inequality is “sharp” if we want $u(t,x)$ to be summable. Moreover, H_0 has no contribution to the exponent of $n!$ on the right-hand side of (4.2), which means that it plays no role in guaranteeing the summability of the right-hand side of (4.2).

The case $\frac{1}{2}d_* - H_* > d^* - H^*$. This case is more complicated. From (3.36) in Proposition 3.4, it suffices to consider the following integral:

$$\begin{aligned}
\mathcal{I}_{\vec{\boldsymbol{\rho}},\gamma_0} := \int_{\mathbb{T}_1\times\mathbb{T}_1} &(1-s_n)^{-\rho_n}(1-r_n)^{-\rho_n}\prod_{i=1}^n \gamma_0(s_i-r_i) \\
&\times \prod_{i=1}^{n-1}|s_{i+1}-s_i|^{-\rho_i}\prod_{i=1}^{n-1}|r_{i+1}-r_i|^{-\rho_i} d\vec{\mathbf{s}}d\vec{\mathbf{r}}\,,
\end{aligned} \tag{4.7}$$

where we denote $\vec{\boldsymbol{\rho}}$ by $(\rho_1,\cdots,\rho_n)$ and ρ_i, $i=1,\cdots,n$ are given by (3.37).

If we use the Hardy-Littlewood-Sobolev inequality as for (4.2) and by [24, Lemma 4.5], we obtain when $\rho_i < H_0$ for $i = 1, \cdots, n$

$$\begin{aligned}\mathbb{E}[u_n(t,x)^2] \leq &(n!)^{2H_0-1} t^{n(|H|+2H_0-d)} C_H^n \left(\int_{\mathbb{T}_1} \prod_{i=1}^n (s_{i+1}-s_i)^{-\frac{\rho_i}{H_0}} d\vec{\mathbf{s}} \right)^{2H_0} \\ =&(n!)^{2H_0-1} C_H^n t^{n(|H|+2H_0-d)} \left[\Gamma\left(\sum_{i=1}^n (1-\frac{\rho_i}{H_0}) \right) \right]^{-2H_0} \\ =&(n!)^{-(|H|-d+1)} C_H^n t^{n(|H|+2H_0-d)}, \end{aligned} \tag{4.8}$$

where the last step is followed by the Stirling formula. From the definition of ρ_i's, the condition $\rho_i < H_0$ for $i = 1, \cdots, n$ is equivalent to

$$H_0 + |H| > \frac{3}{4}d + \frac{\beta^*}{4} \quad \text{or} \quad H_0 + H_* > \frac{3}{4}d_* .$$

By the similar arguments as in the case $\frac{1}{2}d_* - H_* \leq d^* - H^*$ below (4.2), Proposition 1.5 is proved.

Now we return to the question concerning the finiteness of $\mathbb{E}\left[u_n^2(t,x)\right]$. It is self-evident that $|H| \geq d-1$ is sufficient. But it is certainly not necessary since H_0 may play a role now. We would like to seek the necessary and sufficient conditions.

The right-hand side of (3.36) is a multiple integrals with some simple integrands. At first glance, we may think that the integrability problem of these kernels is simple. However, it can be a complicated problem in analysis. There are some studies about similar integrals (e.g., [37, 40, 41]). However, we cannot find results which are directly applicable. The difficulty is that the $\rho_i' s$ appeared in the integrand in (4.7) are different. To solve this integrability problem, our idea is to use both the Hardy-Littlewood-Sobolev inequality and the Hölder-Young-Brascamp-Lieb inequality obtained in [7], which we shall recall in Appendix A.

Take arbitrary $\boldsymbol{\alpha} := (\alpha_1, \cdots, \alpha_n) \in \mathcal{D}_n$. Notice that $\alpha_i + \alpha_{i+1} \neq 0$. We divide our discussion on the integration $\mathcal{I}_{\vec{\boldsymbol{\rho}},\gamma_0}$ according to the following two cases.

$$\begin{cases} \textbf{Case 1:} \quad \alpha_1 \neq 0\,, & \text{we use the Hardy-Littlewood} \\ & \qquad\qquad \text{-Sobolev inequality (A.2)}\,; \\ \textbf{Case 2:} \quad \alpha_1 = 0\,, \alpha_2 \neq 0\,, & \text{we use the Hölder-Young} \\ & \qquad\qquad \text{-Brascamp-Lieb inequality (A.3)}\,. \end{cases} \tag{4.9}$$

Remark 4.2. As mentioned previously in introduction, it is widely believed that the Hardy-Littlewood-Sobolev inequality is the best tool to handle the integral (4.7) over the simplex $\mathbb{T}_1 \times \mathbb{T}_1$. However, for the specific case of $d = 1$ with H_1 abbreviated as H, this approach yields the sufficient condition $H + H_0 \geq \frac{3}{4}$ which is, as we see now, not a necessary one. This approach was first introduced in [22, 23] and then used in some related references [3, 6, 15, 38], just to name a few. In particular, it utilizes the Hardy-Littlewood-Sobolev inequality for all $\boldsymbol{\alpha} \in \mathcal{D}_n$ simultaneously, overlooking the nuanced yet crucial distinctions we've highlighted in the algorithm (4.9), especially Case 2.

Let us discuss the cases listed in (4.9) separately.

Case 1: When $\alpha_1 \neq 0$, we integrate s_1 and r_1 in (4.7) first. This means we write

$$\begin{aligned}\mathcal{I}_{\vec{\boldsymbol{\rho}},\gamma_0} := &\int_{\substack{0<s_2<s_3<\cdots<s_n<t \\ 0<r_2<r_3<\cdots<r_n<t}} (1-s_n)^{-\rho_n} (1-r_n)^{-\rho_n} \prod_{i=2}^n \gamma_0(s_i - r_i) \\ &\times \prod_{i=2}^{n-1} |s_{i+1}-s_i|^{-\rho_i} \prod_{i=2}^{n-1} |r_{i+1}-r_i|^{-\rho_i} \mathcal{I}_1(s_2, r_2) d\vec{\mathbf{s}}_2 d\vec{\mathbf{r}}_2\,, \end{aligned} \tag{4.10}$$

where

$$\mathcal{I}_1(s_2, r_2) := \int_{\substack{0<s_1<s_2 \\ 0<r_1<r_2}} \gamma_0(s_1 - r_1)|s_2 - s_1|^{-\rho_1}|r_2 - r_1|^{-\rho_1} dr_1 ds_1 . \tag{4.11}$$

Noticing that under the second inequality (1.7b): $|H| + 2H_0 > d$, it holds

$$H_0 - \rho_1 = H_0 - \frac{1}{2}\left[\frac{3}{2}d + \frac{1}{2}\beta^* - 2|H| - \alpha_1\right] > 0, \tag{4.12}$$

since $\alpha_1 \in \left\{\frac{1}{2}d - |H| + \frac{1}{2}\beta^*, d - 2|H| + \beta^*\right\}$. Then an application of the Hardy-Littlewood-Sobolev inequality (Lemma A.2 in Appendix) yields

$$\begin{aligned}\mathcal{I}_1(s_2, r_2) \leq & C_{H_0,\rho_1}\left(\int_{0<s_1<s_2} |s_2 - s_1|^{-\frac{\rho_1}{H_0}} ds_1\right)^{H_0}\left(\int_{0<r_1<r_2} |r_2 - r_1|^{-\frac{\rho_1}{H_0}} dr_1\right)^{H_0} \\ = & C_{H_0,\rho_1} B\left(1, 1 - \frac{\rho_1}{H_0}\right)^{2H_0} s_2^{H_0-\rho_1} r_2^{H_0-\rho_1} \leq C_{H_0,\rho_1} t^{2(H_0-\rho_1)} ,\end{aligned}$$

where we recall $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function.

Case 2: When $\alpha_1 = 0$, $\alpha_2 \neq 0$, we integrate s_1, s_2, r_1, r_2 in (4.7) first. Namely, we write now

$$\begin{aligned}\mathcal{I}_{\vec{\rho},\gamma_0} := & \int_{\substack{0<s_3<\cdots<s_n<t \\ 0<r_3<\cdots<r_n<t}} (1 - s_n)^{-\rho_n}(1 - r_n)^{-\rho_n} \prod_{i=3}^{n} \gamma_0(s_i - r_i) \\ & \times \prod_{i=3}^{n-1} |s_{i+1} - s_i|^{-\rho_i} \prod_{i=3}^{n-1} |r_{i+1} - r_i|^{-\rho_i} \mathcal{I}_2(s_3, r_3) d\vec{\mathbf{s}}_3 d\vec{\mathbf{r}}_3 ,\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}\mathcal{I}_2(s_3, r_3) := & \int_{\substack{0<s_1<s_2<s_3 \\ 0<r_1<r_2<r_3}} \prod_{i=1}^{2} \gamma_0(s_i - r_i)|s_3 - s_2|^{-\rho_2}|r_3 - r_2|^{-\rho_2} \\ & |s_2 - s_1|^{-\rho_1}|r_2 - r_1|^{-\rho_1} ds_1 ds_2 dr_1 dr_2 .\end{aligned} \tag{4.14}$$

We claim that

$$\sup_{0\leq s_3,r_3\leq t} \mathcal{I}_2(s_3, r_3) \leq C_t < +\infty .$$

We shall prove it via the so-called non-homogeneous Hölder-Young-Brascamp-Lieb inequality (see Theorem A.3 in Appendix). This inequality aims at the following multilinear functional

$$\Lambda(f_1, \ldots, f_m) = \int \prod_{j=1}^{m} f_j(l_j(\mathbf{x})) \prod_{j=1}^{n} d\mu(x_j) , \tag{4.15}$$

where $l_j(\cdot)$'s are linear functions. In the seminal paper [8, Theorem 2.2], the authors established the condition (A.5) that is both necessary and sufficient for the finiteness of $\Lambda(f_1, \ldots, f_m)$. To use this theorem, we introduce the following linear transformations $l_j : \mathbb{R}^4 \to \mathbb{R}$

$$\begin{aligned}& l_1(s_1, s_2, r_1, r_2) = r_2 - r_1 ,\ l_3(s_1, s_2, r_1, r_2) = r_2 ,\ l_5(s_1, s_2, r_1, r_2) = s_1 - r_1 , \\ & l_2(s_1, s_2, r_1, r_2) = s_2 - s_1 ,\ l_4(s_1, s_2, r_1, r_2) = s_2 ,\ l_6(s_1, s_2, r_1, r_2) = s_2 - r_2 ,\end{aligned}$$

and the nonnegative functions $f_j : \mathbb{R}^4 \to \mathbb{R}_+$

$$\begin{aligned}& f_1(x) = |x|^{-\rho_1}\mathbf{1}_{\{0<x<r_3\}} ,\ f_3(x) = |r_3 - x|^{-\rho_2}\mathbf{1}_{\{0<x<r_3\}} ,\ f_5(x) = |x|^{-\gamma_0}\mathbf{1}_{\{|x|<1\}} , \\ & f_2(x) = |x|^{-\rho_1}\mathbf{1}_{\{0<x<s_3\}} ,\ f_4(x) = |s_3 - x|^{-\rho_2}\mathbf{1}_{\{0<x<s_3\}} ,\ f_6(x) = |x|^{-\gamma_0}\mathbf{1}_{\{|x|<1\}} ,\end{aligned}$$

where $\gamma_0 = 2 - 2H_0$ is the fractional power in the kernel $\gamma_0(x) = |x|^{-\gamma_0} = |x|^{2H_0-2}$ and

$$\begin{cases} \rho_1 = \frac{1}{2}\left(\frac{3}{2}d + \frac{1}{2}\beta^* - 2|H|\right), \\ \rho_2 = \frac{1}{2}\left(\frac{3}{2}d + \frac{1}{2}\beta^* - 2|H| - \alpha_2\right). \end{cases}$$

With these notations we can first bound $\mathcal{I}_2(s_3, r_3)$

$$\begin{aligned} \mathcal{I}_2(s_3, r_3) \le & \int_{\substack{0<s_1,s_2<1 \\ 0<r_1,r_2<1}} \prod_{i=1}^{2} \gamma_0(s_i - r_i)|s_3 - s_2|^{-\rho_2}\mathbf{1}_{\{0<s_2<s_3\}}|r_3 - r_2|^{-\rho_2}\mathbf{1}_{\{0<r_2<r_3\}} \\ & \times |s_2 - s_1|^{-\rho_1}\mathbf{1}_{\{0<s_2-s_1<s_3\}}|r_2 - r_1|^{-\rho_1}\mathbf{1}_{\{0<r_2-r_1<r_3\}} ds_1 ds_2 dr_1 dr_2 \\ =: & \int_{[0,1]^4} \prod_{j=1}^{6} f_j(l_j(s_1, s_2, r_1, r_2)) ds_1 ds_2 dr_1 dr_2 =: \mathcal{J}(f_1, \cdots, f_6). \end{aligned}$$

Now, the above integral $\mathcal{J}(f_1, \cdots, f_6)$ is in a form that the Hölder-Young-Brascamp-Lieb theorem may apply. We want to show that under the condition (1.7) we can appropriately choose $p_1, \cdots, p_6$ such that the dimension condition (A.5) of Theorem A.3 in the appendix is verified so that we can apply this Hölder-Young-Brascamp-Lieb theorem to obtain

$$\mathcal{I}_2(s_3, r_3) \le \mathcal{J}(f_1, \cdots, f_6) \le \prod_{j=1}^{6} \|f_j\|_{L^{p_j}(\mathbb{R})} \le C < \infty. \tag{4.16}$$

To verify the dimension condition (A.5) of Theorem A.3, we want to show that there exist $p_1, \cdots, p_6 \in [1, +\infty)$ so that

$$\begin{aligned} & f_j \in L^{p_j}(\mathbb{R}) \text{ and for every subspace } V \subseteq \mathbb{R}^4: \\ & co\dim_{\mathcal{H}}(V) \ge \sum_{j=1}^{6} \frac{1}{p_j} \cdot co\dim_{\mathcal{H}_j}(l_j(V)). \end{aligned} \tag{4.17}$$

Denote $z_j = p_j^{-1} \in (0, 1]$ for $j = 1, \cdots, 6$. Take $\beta^* = \beta_{d_*+1} + \cdots + \beta_d = 4H^* - 3d^*$ in Lemma 3.3. Then the integrability conditions $f_j \in L^{p_j}(\mathbb{R})$, $j = 1, \cdots, 6$ are equivalent to

$$\begin{cases} 1 \ge z_1, z_2 > \rho_1 = \frac{1}{2}(\frac{3}{2}d + \frac{1}{2}\beta^* - 2|H|) = \frac{1}{2}\left(\frac{3}{2}d_* - 2H_*\right), \\ 1 \ge z_3, z_4 > \rho_2 = \frac{1}{2}(\frac{3}{2}d + \frac{1}{2}\beta^* - 2|H| - \alpha_2) = \frac{1}{2}\left(\frac{3}{2}d_* - 2H_* - \alpha_2\right), \\ 1 \ge z_5, z_6 > \gamma_0 = 2 - 2H_0. \end{cases} \tag{4.18}$$

By the assumption $H_0 > \frac{1}{2}$ we can choose z_5, z_6 so that the third line in (4.18) holds true. As for the first and second lines, since $\alpha_2 > 0$, we only need to explain that there are z_1 and z_2 such that the first line in (4.18) holds. Note that

$$\begin{aligned} \rho_1 < 1 \Leftrightarrow & \frac{3}{2}d_* - 2H_* < 2 \\ \Leftrightarrow & H_* > \frac{3}{4}d_* - 1. \end{aligned}$$

Therefore, the integrability conditions (4.18) are implied by the assumption $H_0 > \frac{1}{2}$ and the first inequality (1.7a):

$$H_* > \frac{3}{4}d_* - 1. \tag{4.19}$$

On the other hand, we need to verify the dimensional conditions in (4.17). For fixed co-dimension of V, our strategy is to select V such that $co\dim_{\mathcal{H}_j}(l_j(V))=1$ as many j as possible, i.e. $\dim_{\mathcal{H}_j}(l_j(V))=0$ as many j as possible. It is not difficult to see that

$$\dim_{\mathcal{H}_j}(l_j(V))=0 \text{ if } V=\ker(l_j).$$

Therefore, we only need to take into account those V's which are given by $\ker(l_j)$ ($1\le j\le 6$) or their intersections. Thus, we shall select $J\subseteq\{1,\cdots,6\}$ such that $V=\cap_{j\in J}\ker(l_j)$ and in this case

$$\begin{aligned}\dim_{\mathcal{H}}(V) &= \dim(\cap_{j\in J}\ker(l_j))\\ &= \dim(\mathcal{H})-\dim\left(span\{l_j, j\in J\}\right)\\ &= \dim(\mathcal{H})-\operatorname{rank}([l_j]_{j\in J})\,. \end{aligned}\tag{4.20}$$

In the following, we shall discuss the cases $co\dim_{\mathcal{H}}(V)=4,3,2,1,0$. It is trivial to see that the dimension condition (4.17) holds when $co\dim_{\mathcal{H}}(V)=0$. Therefore, we just have to verify (4.17) for $co\dim_{\mathcal{H}}(V)=4,3,2,1$ step by step. This is done in Lemma A.4 in Appendix under conditions (1.7). This means that (4.16) is achieved under conditions (1.7) by using the Hölder-Young-Brascamp-Lieb inequality.

To summarize **Case 1** and **Case 2**, we obtain that $\mathcal{I}(\rho,\gamma)$ defined by (4.7) can be bounded as

$$\mathcal{I}_{\vec{\boldsymbol{\rho}},\gamma_0}\le\begin{cases} B_1\cdot\displaystyle\int_{\substack{0<s_2<s_3<\cdots<s_n<t\\0<r_2<r_3<\cdots<r_n<t}}(1-s_n)^{-\rho_n}(1-r_n)^{-\rho_n}\prod_{i=2}^n\gamma_0(s_i-r_i)\\ \qquad\times\displaystyle\prod_{i=2}^{n-1}|s_{i+1}-s_i|^{-\rho_i}\prod_{i=2}^{n-1}|r_{i+1}-r_i|^{-\rho_i}d\vec{\mathbf{s}}_2 d\vec{\mathbf{r}}_2\quad\alpha_1\neq 0; & (4.21)\\ C\cdot\displaystyle\int_{\substack{0<s_3<\cdots<s_n<t\\0<r_3<\cdots<r_n<t}}(1-s_n)^{-\rho_n}(1-r_n)^{-\rho_n}\prod_{i=3}^n\gamma_0(s_i-r_i)\\ \qquad\times\displaystyle\prod_{i=3}^{n-1}|s_{i+1}-s_i|^{-\rho_i}\prod_{i=3}^{n-1}|r_{i+1}-r_i|^{-\rho_i}d\vec{\mathbf{s}}_3 d\vec{\mathbf{r}}_3\,,\quad\alpha_1=0\,,\alpha_2\neq 0. & (4.22)\end{cases}$$

where $B_1:=B\left(1,1-\frac{\rho_1}{H_0}\right)^{2H_0}$. The remaining integration has the same form as the original one but with strictly less multiplicity. We can then use the same argument as above to prove

$$\mathcal{I}_{\vec{\boldsymbol{\rho}},\gamma_0}\le C^n<\infty.$$

Thus, we have for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$

$$\mathbb{E}[u_n(t,x)^2]\le C^n\cdot n!\cdot t^{n(|H|+2H_0-d)}<\infty\,.\tag{4.23}$$

As a result, we have proven (4.23) under the conditions (1.7) and we finish the proof of sufficiency in Theorem 1.1.

4.2 Sufficient condition of Theorem 1.1: an alternative proof

In this subsection, we give another proof of the sufficient condition using some elementary computations in the Appendix A. Let us stress that Lemma A.7 is the key for the proofs here.

As explained in the previous section, we need to show that $\mathcal{I}_1(s_2,r_2)$ defined by (4.11) is bounded under the condition $\alpha_1\neq 0$ and $\mathcal{I}_2(s_3,r_3)$ defined by (4.14) is bounded under the condition $\alpha_1=0$ but $\alpha_2\neq 0$. Recall that

$$\begin{aligned}\rho_i =&\frac{1}{2}(\frac{3}{2}d+\frac{1}{2}\beta^*-2|H|-\alpha_i)\\ =&\begin{cases}\frac{3}{4}d_*-H_*, & \text{if } \alpha_i=0\\ \frac{d-|H|}{2}, & \text{if } \alpha_i=\frac{1}{2}d-|H|+\frac{1}{2}\beta^*.\end{cases}\end{aligned}$$

Case 1: $\alpha_1\neq 0$. In this case, we apply Lemma A.7 to $\mathcal{I}_1(s_2,r_2)$ with

$$\begin{aligned}\beta &= 0\,,\quad \gamma_0=2-2H_0\,,\quad a=b=0\,,\quad A=s_2\,,\quad B=r_2\,,\\ \alpha &= \rho_1=\frac{1}{2}(\frac{3}{2}d+\frac{1}{2}\beta^*-2|H|-\alpha_1)\,,\end{aligned}$$

and $\alpha_1\in\left\{\frac{1}{2}d-|H|+\frac{1}{2}\beta^*, d-2|H|+\beta^*\right\}$. In order to use this lemma, we should require the following condition

$$\gamma_0<2-\alpha \Leftrightarrow H_*+2H_0>\frac{3}{4}d_*,$$

which is implied by the third inequality (1.7c). Then,

$$\mathcal{I}_1(s_2,r_2)\lesssim s_2^{1-\rho_1-\frac{\gamma_0}{2}}r_2^{1-\rho_1-\frac{\gamma_0}{2}}+s_2^{1-\rho_1-q_2}r_2^{1-\rho_1-q_2}|r_2-s_2|^{-\gamma_0+2q_2}. \tag{4.24}$$

The right-hand side of (4.24) is bounded if

$$\begin{cases}1-\rho_1-\frac{\gamma_0}{2}>0\,;\\ -\gamma_0+2q_2\geq 0\,,\end{cases} \tag{4.25}$$

for some $q_2\in(0,(1-\rho_1)\wedge\frac{\gamma_0}{2}]$. Since $\alpha_1\neq 0$, it follows that $\rho_1\leq\frac{1}{2}(d-|H|)$. The first condition of (4.25) is then equivalent to $|H|+2H_0>d$, which is the second inequality (1.7b). Since the first condition of (4.25) holds true, we have $\frac{\gamma_0}{2}<1-\rho_1$. Then we can take $q_2=\frac{\gamma_0}{2}$ and we see easily that the second condition of (4.25) is true.

Case 2: $\alpha_1=0$ **and** $\alpha_2\neq 0$. We shall use Lemma A.7 to bound $\mathcal{I}_2(s_3,r_3)$ given by (4.14). Notice that in this case we should also integrate $\mathcal{I}_1(s_2,r_2)$ first to obtain (4.24). But in this case since $\alpha_1=0$, it follows

$$\rho_1=\frac{1}{2}(\frac{3}{2}d+\frac{1}{2}\beta^*-2|H|)=\frac{1}{2}\left(\frac{3}{2}d_*-2H_*\right).$$

In order to use Lemma A.7, we need $\rho_1<1$. This is equivalent to

$$H_*>\frac{3}{4}d_*-1 \tag{4.26}$$

which is the first inequality (1.7a). Under this condition we can estimate $\mathcal{I}_1(s_2,r_2)$ as (4.24).

If $\frac{\gamma_0}{2}\leq 1-\rho_1$, then we can choose $q_2=\frac{\gamma_0}{2}$ and $\mathcal{I}_1(s_2,r_2)$ is also bounded. Hence, in this case we may proceed as in the case $\alpha_1\neq 0$.

If $\frac{\gamma_0}{2}>1-\rho_1$, namely,

$$H_0+H_*<\frac{3}{4}d_*\,, \tag{4.27}$$

then in this case we choose $q_2 = 1-\rho_1$ and we have to consider the integral $\mathcal{I}_2(s_3, r_3)$ given by (4.14). We integrate s_1 and r_1 first. By (4.24), we have

$$\begin{aligned}\mathcal{I}_2(s_3,r_3) \le & \int_{\substack{0<s_2<s_3\\0<r_2<r_3}} |s_3-s_2|^{-\rho_2}|r_3-r_2|^{-\rho_2}(s_2r_2)^{1-\rho_1-\frac{\gamma_0}{2}}|s_2-r_2|^{-\gamma_0}ds_2dr_2\\ &+\int_{\substack{0<s_2<s_3\\0<r_2<r_3}} |s_3-s_2|^{-\rho_2}|r_3-r_2|^{-\rho_2}|s_2-r_2|^{-2\gamma_0+2-2\rho_1}ds_2dr_2\\ &=: \mathcal{I}_{2,1}(s_3,r_3)+\mathcal{I}_{2,2}(s_3,r_3)\,, \end{aligned} \tag{4.28}$$

where we recall that $\gamma_0 = 2-2H_0$,

$$\rho_2 = \frac{1}{2}(\frac{3}{2}d+\frac{1}{2}\beta^*-2|H|-\alpha_2) \quad \text{and} \quad \alpha_2 \in \left\{\frac{1}{2}d-|H|+\frac{1}{2}\beta^*, d-2|H|+\beta^*\right\}.$$

To use Lemma A.7 for $\mathcal{I}_{2,1}(s_3, r_3)$, we need

$$1-\rho_1-\frac{\gamma_0}{2} > -1 \Leftrightarrow H_*+H_0 > \frac{3}{4}d_*-1\,,$$

which is implied by the first condition (1.7a). Similarly, for the term $\mathcal{I}_{2,2}(s_3, r_3)$, we need

$$-2\gamma_0+2-2\rho_1 > -1\,.$$

This condition is equivalent to

$$2H_0+H_* > \frac{3}{4}d_*+\frac{1}{2}\,, \tag{4.29}$$

which is the third condition (1.7c).

Now we apply Lemma A.7 to $\mathcal{I}_{2,1}(s_3, r_3)$ with

$$\alpha=\rho_2,\ \beta=-(1-\rho_1-\frac{\gamma_0}{2}),\ a=b=0,\ A=s_3 \text{ and } B=r_3.$$

It is clear that the following condition

$$\begin{aligned}&\gamma_0 < 2-\beta-(\beta\vee\alpha) = 2+1-\rho_1-\frac{\gamma_0}{2}-(-1+\rho_1+\frac{\gamma_0}{2})\vee\rho_2\\ &\Leftrightarrow\ H_*+2H_0 > \frac{3}{4}d_* \text{ and } |H|+2H_*+6H_0 > d+\frac{3}{2}d_*,\end{aligned}$$

holds under the third and fourth inequalities (1.7d). Thus, we have for $\bar{q}_2 \in [\beta+\gamma_0-1, (1-\alpha)\wedge\frac{\gamma_0}{2}] = [-2+\rho_1+\frac{3}{2}\gamma_0, (1-\rho_2)\wedge\frac{\gamma_0}{2}]$,

$$\begin{aligned}\mathcal{I}_{2,1}(s_3,r_3) \lesssim & |s_3|^{-\rho_2+(1-\rho_1-\frac{\gamma_0}{2})+1-\frac{\gamma_0}{2}}|r_3|^{-\rho_2+(1-\rho_1-\frac{\gamma_0}{2})+1-\frac{\gamma_0}{2}}\\ &+|s_3|^{-\rho_2+(1-\rho_1-\frac{\gamma_0}{2})+1-\bar{q}_2}|r_3|^{-\rho_2+(1-\rho_1-\frac{\gamma_0}{2})+1-\bar{q}_2}|s_3-r_3|^{-\gamma_0+2\bar{q}_2}.\end{aligned} \tag{4.30}$$

Noticing that under the second inequality $|H|+2H_0 > d$ of condition (1.7), we have $1-\rho_2 > \frac{\gamma_0}{2}$. Taking $\bar{q}_2 = \frac{\gamma_0}{2}$, $\mathcal{I}_{2,1}(s_3, r_3)$ is bounded if

$$-\rho_2+(1-\rho_1-\frac{\gamma_0}{2})+1-\frac{\gamma_0}{2} > 0 \Leftrightarrow |H|+2H_*+4H_0 > d+\frac{3}{2}d_*. \tag{4.31}$$

We can see that the above first inequality is exactly (1.7d).

Next, we apply Lemma A.7 to $\mathcal{I}_{2,2}(s_3, r_3)$ with

$$\alpha=\rho_2,\ \beta=0,\ \tilde{\gamma}=2\gamma_0-2(1-\rho_1),\ a=b=0,\ A=s_3 \text{ and } B=r_3.$$

Thus, under the condition

$$\tilde\gamma < 2-\beta-(\beta\vee\alpha) = 2-\rho_2 \\ \Leftrightarrow\ |H|+4H_*+8H_0 > d+3d_*,$$

which is implied by the third and fourth inequalities of (1.7), we have

$$\mathcal{I}_{2,2}(s_3,r_3) \lesssim |s_3|^{-\rho_2+1-\frac{\tilde\gamma}{2}}|r_3|^{-\rho_2+1-\frac{\tilde\gamma}{2}} \\ + |s_3|^{-\rho_2+1-\tilde q_2}|r_3|^{-\rho_2+1-\tilde q_2}|s_3-r_3|^{-\tilde\gamma+2\tilde q_2}, \tag{4.32}$$

where $\tilde q_2 \in [\beta+\tilde\gamma-1, (1-\alpha)\wedge\frac{\tilde\gamma}{2}] = [2\gamma+2\rho_1-3, (1-\rho_2)\wedge\frac{\tilde\gamma}{2}]$. Notice that under the fourth inequality (1.7d), we have $1-\rho_2 > \frac{\tilde\gamma}{2}$. Thus, letting $\tilde q_2 = \frac{\tilde\gamma}{2}$, $\mathcal{I}_{2,2}(s_3,r_3)$ is bounded.

As a result, by Lemma A.7 in Appendix, we have proved that under conditions (1.7), $\mathcal{I}_1(s_2,r_2)$ and $\mathcal{I}_2(s_3,r_3)$ are bounded. Then, in a similar argument to that in Section 4.1, we see that the conditions (1.7) are sufficient for $\mathbb{E}[u_n(t,x)^2] < +\infty$ for any $n \ge 1$.

4.3 Proof of Theorem 1.1: necessity

In this subsection we shall show the necessity of condition (1.7). Focusing on $H_0 > 1/2$ firstly, we shall prove the four inequalities in (1.7) separately.

The necessity of condition (1.7b)**.** This is in fact a consequence of [27]. But we shall give a simpler proof due to our special situation. We only need to focus on the first chaos expansion $u_1(t,x)$. Namely, for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$

$$\begin{aligned}\mathbb{E}[u_1(t,x)^2] &= \mathbb{E}[I_1(f_1)^2] \\ &= \int_{[0,t]^2}\int_{\mathbb{R}} e^{-\frac12(t-s_1+t-r_1)|\xi_1|^2}\times\prod_{k=1}^d|\xi_{1k}|^{1-2H_k}\gamma_0(s_1-r_1)d\xi_1ds_1dr_1 \\ &= \int_{[0,t]^2}\prod_{k=1}^d h_k(s_1,r_1)\gamma_0(s_1-r_1)ds_1dr_1\,,\end{aligned}$$

where $h_k(s_1,r_1)=\int_{\mathbb{R}} e^{-\frac12(t-s_1+t-r_1)|\xi_{1k}|^2}\times|\xi_{1k}|^{1-2H_k}d\xi_{1k}$. It is not difficult to verify that

$$h_k(s_1,r_1) \ge C(t-s_1+t-r_1)^{H_k-1}.$$

Thus, taking $\widetilde s_1 = t-s_1$, $\widetilde r_1 = t-r_1$, $\widetilde v = \widetilde r_1-\widetilde s_1$ and $\widetilde w = \widetilde s_1+\widetilde r_1$

$$\begin{aligned}\mathbb{E}[u_1(t,x)^2] &\ge C\int_{[0,t]^2}(\widetilde s_1+\widetilde r_1)^{|H|-d}\cdot|\widetilde s_1-\widetilde r_1|^{2H_0-2}ds_1dr_1 \\ &\ge C\int_0^t\int_0^{\widetilde w}|\widetilde w|^{|H|-d}\cdot|\widetilde v|^{2H_0-2}d\widetilde v d\widetilde w + \int_t^{2t}\int_0^{2t-\widetilde w}|\widetilde w|^{|H|-d}\cdot|\widetilde v|^{2H_0-2}d\widetilde v d\widetilde w \\ &\ge C\int_0^t|\widetilde w|^{|H|-d+2H_0-1}d\widetilde w.\end{aligned}$$

Accordingly, $\mathbb{E}[u_1(t,x)^2] < +\infty$ happens only if

$$|H|-d+2H_0-1 > -1 \Leftrightarrow |H|+2H_0 > d.$$

The necessity of conditions (1.7c) **and** (1.7d)**.** We shall show the necessity by means of the finiteness of $\mathbb{E}[u_2(t,x)^2]$ for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$.

By definition of $h_{k,n}(\vec{\mathbf{s}},\vec{\mathbf{r}})$ with $n=2$ in (3.7), it follows that

$$\begin{aligned}
h_{k,2}(\vec{\mathbf{s}},\vec{\mathbf{r}}) &= c_{H_k}^n\left(\prod_{i=1}^2 \widetilde{w}_i^{-1/2}\right)\mathbb{E}\left[\prod_{i=1}^2\left|\frac{X_i}{\sqrt{\widetilde{w}_i}}-\frac{X_{i-1}}{\sqrt{\widetilde{w}_{i-1}}}\right|^{1-2H_k}\right],\\
&= c_{H_k}^n \widetilde{w}_2^{H_k-1}\widetilde{w}_1^{2H_k-3/2}(\widetilde{w}_2+\widetilde{w}_1)^{\frac12-H_k}\\
&\quad\times\mathbb{E}\left[|X_1|^{1-2H_k}\left|\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_1+\widetilde{w}_2}}X_2-\sqrt{\frac{\widetilde{w}_2}{\widetilde{w}_1+\widetilde{w}_2}}X_1\right|^{1-2H_k}\right]
\end{aligned}$$

where $\widetilde{w}_1=s_2-s_1+r_2-r_1$ and $\widetilde{w}_2=t-s_2+t-r_2$. Recalling $\mathcal{J}_{k,2}(\widetilde{\lambda}_1,\widetilde{\lambda}_2)$ defined in (3.14) with $n=2$, and letting $X_0=0$, $\widetilde{\lambda}_1=1$ and $\widetilde{\lambda}_2:=\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_1+\widetilde{w}_2}}$, one has

$$\begin{aligned}
\mathcal{J}_{k,2}(\widetilde{\lambda}_1,\widetilde{\lambda}_2) &= \mathbb{E}\left[\prod_{i=1}^2\left|\widetilde{\lambda}_iX_i-\sqrt{1-\widetilde{\lambda}_i^2}X_{i-1}\right|^{1-2H_k}\right]\\
&= \mathbb{E}\left[|X_1|^{1-2H_k}\left|\sqrt{\frac{\widetilde{w}_1}{\widetilde{w}_1+\widetilde{w}_2}}X_2-\sqrt{\frac{\widetilde{w}_2}{\widetilde{w}_1+\widetilde{w}_2}}X_1\right|^{1-2H_k}\right].
\end{aligned}$$

It follows from (3.25) that

$$\mathcal{J}_{k,2}(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\geq c_{H_k}\widetilde{\lambda}_2^{-(4H_k-3)}.$$

Since $2H_*-\frac32 d_*<0$ and $(\tilde{w}_2+\tilde{w}_1)^{|H|-d}>\tilde{w}_1^{2H_*-\frac32 d_*}(\tilde{w}_2+\tilde{w}_1)^{|H|-d-2H_*+\frac32 d_*}$, we have

$$\begin{aligned}
&\mathbb{E}[u_2(t,x)^2]\geq c_{H_k}\int_{\substack{0<s_1<s_2<t\\0<r_1<r_2<t}}\prod_{k=1}^d h_{k,2}(\vec{\mathbf{s}},\vec{\mathbf{r}})\prod_{i=1}^2\gamma_0(s_i-r_i)ds_1dr_1ds_2dr_2\\
&= c_{H_k}\int_{\substack{0<s_1<s_2<t\\0<r_1<r_2<t}}|t-s_2+t-r_2|^{|H|-d}|t-s_1+t-r_1|^{|H|-d}\prod_{i=1}^2\gamma_0(s_i-r_i)ds_idr_i\\
&\geq c_{H_k}\int_{\substack{0<s_1<s_2<t\\0<r_1<r_2<t}}|t-s_2+t-r_2|^{|H|-d}|s_2-s_1+r_2-r_1|^{2H_*-\frac32 d_*}\\
&\quad\times|t-s_1+t-r_1|^{|H|-2H_*-d+\frac32 d_*}\times|s_1-r_1|^{2H_0-2}|s_2-r_2|^{2H_0-2}ds_1dr_1ds_2dr_2. \qquad (4.33)
\end{aligned}$$

Denote the integral domain $\mathbf{I_1}:=\left\{(s_1,r_1,s_2,r_2):0<s_1<r_1<\frac{t}{2}<s_2<r_2<t\right\}$. Taking $\tilde{v}=r_1-s_1$, $\tilde{x}=s_2-r_1$, $\tilde{y}=r_2-s_2$ and $\tilde{z}=t-r_2$ yields

$$\begin{aligned}
\mathbb{E}[u_2(t,x)^2] &\geq c_H\int_{\mathbf{I_1}}|t-s_2+t-r_2|^{|H|-d}|s_2-s_1+r_2-r_1|^{2H_*-\frac32 d_*}\\
&\qquad\times|s_1-r_1|^{2H_0-2}|s_2-r_2|^{2H_0-2}ds_1dr_1ds_2dr_2\\
&\geq c_H\int_{0<\tilde{v},\tilde{x},\tilde{y},\tilde{z}<\frac{t}{2}}|\tilde{y}+2\tilde{z}|^{|H|-d}|\tilde{v}+2\tilde{x}+\tilde{y}|^{2H_*-\frac32 d_*}\\
&\qquad\times|\tilde{v}|^{2H_0-2}|\tilde{y}|^{2H_0-2}d\tilde{v}d\tilde{x}d\tilde{y}d\tilde{z}\\
&\geq c_H\int_{0<\tilde{x},\tilde{y},\tilde{z}<\frac{t}{2}}|\tilde{y}+2\tilde{z}|^{|H|-d}|2\tilde{x}+\tilde{y}|^{2H_*-\frac32 d_*+2H_0-1}|\tilde{y}|^{2H_0-2}d\tilde{x}d\tilde{y}d\tilde{z},
\end{aligned}$$

where in the last inequality, we used Lemma A.1. By integrating $\tilde{z}$, we have

$$\begin{aligned}
\mathbb{E}[u_2(t,x)^2]\geq c_H\int_{0<\tilde{x},\tilde{y}<\frac{t}{2}}&\left(|\tilde{y}+t|^{|H|-d+1}-|\tilde{y}|^{|H|-d+1}\right)\\
&\cdot|2\tilde{x}+\tilde{y}|^{2H_*-\frac32 d_*+2H_0-1}|\tilde{y}|^{2H_0-2}d\tilde{x}d\tilde{y}. \qquad (4.34)
\end{aligned}$$

From the above, we can see that for the two cases $|H|-d+1>0$ and $|H|-d+1<0$, the terms that really matter are different. Hence, to study the boundedness of the above integral, we shall discuss the two cases $|H|-d+1>0$ and $|H|-d+1<0$ separately.

When $|H|-d+1>0$, it holds that $|\tilde{y}+t|^{|H|-d+1}\geq|\tilde{y}+2\tilde{y}|^{|H|-d+1}\geq 3^{|H|-d+1}|\tilde{y}|^{|H|-d+1}$. So there exists a constant which may depend on H: $c_H\in(0,1)$ such that $|\tilde{y}+t|^{|H|-d+1}-|\tilde{y}|^{|H|-d+1}\geq c_H|\tilde{y}+t|^{|H|-d+1}$. Then from (4.34) we have

$$\begin{aligned}\mathbb{E}[u_2(t,x)^2]&\geq c_H\int_{0<\tilde{x},\tilde{y}<\frac{t}{2}}|\tilde{y}+t|^{|H|-d+1}|2\tilde{x}+\tilde{y}|^{2H_*-\frac{3}{2}d_*+2H_0-1}|\tilde{y}|^{2H_0-2}d\tilde{x}d\tilde{y}\\&\geq c_H t^{|H|-d+1}\int_{0<\tilde{x},\tilde{y}<\frac{t}{2}}|2\tilde{x}+\tilde{y}|^{2H_*-\frac{3}{2}d_*+2H_0-1}|\tilde{y}|^{2H_0-2}d\tilde{x}d\tilde{y}\\&\geq c_H\int_{0<\tilde{x}<\frac{t}{2}}|\tilde{x}|^{2H_*-\frac{3}{2}d_*+4H_0-2}d\tilde{x},\end{aligned}$$

where the last inequality is due to Lemma A.1. Thus, $\mathbb{E}[u_2(t,x)^2]<+\infty$ happens in this case only if

$$2H_*-\frac{3}{2}d_*+4H_0-2>-1\Leftrightarrow H_*+2H_0>\frac{3d_*}{4}+\frac{1}{2}.$$

This implies the third inequality in (1.7). We add $|H|\geq d-1$ and $2H_*+4H_0>\frac{3d_*}{2}+1$ to obtain $|H|+2H_*+4H_0>d+\frac{3}{2}d_*$. This is the fourth condition in (1.7).

When $|H|-d+1<0$, it holds that $|\tilde{y}+t|^{|H|-d+1}\leq 3^{|H|-d+1}|\tilde{y}|^{|H|-d+1}$. So there exists a constant depending on H: $c_H\in(0,1)$ such that $|\tilde{y}|^{|H|-d+1}-|\tilde{y}+t|^{|H|-d+1}\geq c_H|\tilde{y}+t|^{|H|-d+1}$. Then from (4.34) and Lemma A.1 it follows

$$\begin{aligned}\mathbb{E}[u_2(t,x)^2]&\geq c_H\int_{0<\tilde{x},\tilde{y}<\frac{t}{2}}|\tilde{y}|^{|H|-d+1}|2\tilde{x}+\tilde{y}|^{2H_*-\frac{3}{2}d_*+2H_0-1}|\tilde{y}|^{2H_0-2}d\tilde{x}d\tilde{y}\\&\geq c_H\int_{0<\tilde{x}<\frac{t}{2}}|\tilde{x}|^{|H|-d+1+2H_*-\frac{3}{2}d_*+4H_0-2}d\tilde{x}.\end{aligned}$$

Therefore, $\mathbb{E}[u_2(t,x)^2]<+\infty$ happens in this case $|H|<d-1$ only if

$$|H|-d+1+2H_*-\frac{3}{2}d_*+4H_0-2>-1\Leftrightarrow|H|+2H_*+4H_0>d+\frac{3}{2}d_*.$$

Subtracting $|H|<d-1$ from $|H|+2H_*+4H_0>d+\frac{3}{2}d_*$ implies that $H_*+2H_0>\frac{3d_*}{4}+\frac{1}{2}$. This is the third condition in (1.7).

The necessity of condition (1.7a)**.** We shall still use (4.34):

$$\begin{aligned}\mathbb{E}[u_2(t,x)^2]\geq c_H\int_{\mathbf{I_1}}&|t-s_2+t-r_2|^{|H|-d}|s_2-s_1+r_2-r_1|^{2H_*-\frac{3}{2}d_*}\\&\times|s_1-r_1|^{2H_0-2}|s_2-r_2|^{2H_0-2}ds_1dr_1ds_2dr_2.\end{aligned}$$

Assuming $H_*\leq(\frac{3}{4}d_*-1)$, we shall derive that $\mathbb{E}[u_2^2(t,x)]=\infty$, which contradicts with $\mathbb{E}[u_n(t,x)^2]<\infty$ when $n\geq 1$. Let us consider the integral domain $\mathbf{I_2}=\{\frac{t}{4}<r_1<r_2<\frac{t}{3}<\frac{3}{4}t<s_1<s_2<\frac{5}{6}t\}$. We have

$$\mathbb{E}[u_2(t,x)^2]\geq c_H\int_{\mathbf{I_1}\cap\mathbf{I_2}}|s_2-s_1+r_2-r_1|^{2H_*-\frac{3}{2}d_*}ds_1dr_1ds_2dr_2.$$

Integrating r_1 first and then integrating r_2 yield

$$\begin{aligned}
&\int_{\frac{t}{4}<r_1<r_2<\frac{t}{3}} |s_2-s_1+r_2-r_1|^{2H_*-\frac{3}{2}d_*}dr_1dr_2\\
&\qquad =-\frac{t}{12}(2H_*-\frac{3}{2}d+1)^{-1}|s_2-s_1|^{2H_*-\frac{3}{2}d_*+1}\\
&\qquad\quad +(2H_*-\frac{3}{2}d_*+1)^{-1}(2H_*-\frac{3}{2}d_*+2)^{-1}|s_2-s_1+\frac{t}{12}|^{2H_*-\frac{3}{2}d_*+2}\\
&\qquad\quad -(2H_*-\frac{3}{2}d_*+1)^{-1}(2H_*-\frac{3}{2}d_*+2)^{-1}|s_2-s_1|^{2H_*-\frac{3}{2}d_*+2}\\
&\qquad \geq -\frac{t}{12}(2H_*-\frac{3}{2}d+1)^{-1}|s_2-s_1|^{2H_*-\frac{3}{2}d_*+1}\\
&\qquad\quad -(2H_*-\frac{3}{2}d_*+1)^{-1}(2H_*-\frac{3}{2}d_*+2)^{-1}|s_2-s_1|^{2H_*-\frac{3}{2}d_*+2}\,,
\end{aligned}$$

where the inequality holds because we drop a positive term. Since $H_*\leq(\frac{3}{4}d_*-1)$, we have $2H_*-\frac{3}{2}d_*+1\leq-1$. Thus, we see

$$\int_{\frac{3}{4}t<s_1<s_2<\frac{5}{6}t}|s_2-s_1|^{2H_*-\frac{3}{2}d_*+1}ds_1ds_2=\infty,$$

which contradicts with $\mathbb{E}[u_n(t,x)^2]<\infty$ ($n\geq1$). This says that the condition $H_*>(\frac{3}{4}d_*-1)$ is necessary.

When $H_0=1/2$, the lower bound of $\mathbb{E}[u_2(t,x)^2]$ in (4.33) becomes

$$\begin{aligned}
\mathbb{E}[u_2(t,x)^2]\geq c_H\int_{0<s_1<s_2<t}&|t-s_2|^{|H|-d}|s_2-s_1|^{2H_*-\frac{3}{2}d_*}\\
&\times|t-s_1|^{|H|-2H_*-d+\frac{3}{2}d_*}ds_1ds_2\\
=c_H\int_{0<\tilde{x},\tilde{z}<\frac{t}{2}}&|\tilde{z}|^{|H|-d}|\tilde{x}|^{2H_*-\frac{3}{2}d_*}|\tilde{x}+\tilde{z}|^{|H|-2H_*-d+\frac{3}{2}d_*}d\tilde{x}d\tilde{z}.
\end{aligned}\tag{4.35}$$

The right-hand side of (4.35) is integrable only if

$$\begin{cases}|H|-d>-1,\\ 2H_*-\frac{3}{2}d_*>-1,\\ |H|-2H_*-d+\frac{3}{2}d_*>-1,\end{cases}$$

which amounts to

$$\begin{cases}|H|-d>-1,\\ H_*>\frac{3}{4}d_*-\frac{1}{2}.\end{cases}$$

Moreover, applying Lemma A.1 to (4.35), we have

$$\mathbb{E}[u_2(t,x)^2]\geq c_H\int_{0<\tilde{z}<\frac{t}{2}}|\tilde{z}|^{|H|-d}|\tilde{z}|^{|H|-d+1}d\tilde{z}.$$

Thus, $\mathbb{E}[u_2(t,x)^2]<+\infty$ implies the integral on the right-hand side of the above inequality is finite, which happens only if

$$|H|-d+|H|-d+1>-1\Leftrightarrow|H|>d-1.$$

Therefore, when $H_0=1/2$, we proved that the sufficient condition (1.11) is also necessary.

As a result, we prove that all the conditions in (1.7) are necessary for $\mathbb{E}[u_n(t,x)^2]<\infty$ for fixed $n>1$ and $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$. This completes the proof of Theorem 1.1.

5 Convergence of the Itô-Wiener chaos expansion

In this section, we prove Theorem 1.7. The result is visualized in Figure 3. More precisely, the existing condition $H + H_0 > \frac{3}{4}$ (yellow shaded region in the left figure) is expanded to encompass the yellow shaded region in the right figure.

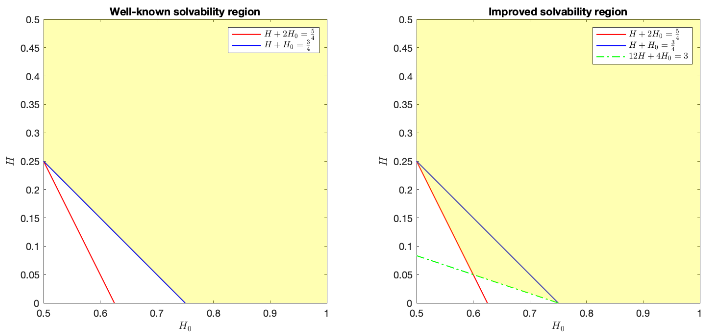

Figure 3: Solvability regions

Proof of Theorem 1.7. For any permutation σ of $\{1, \cdots, n\}$, recall the notation $\mathbb{T}_1^\sigma$ given by (3.2) and

$$\widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{s}}, \boldsymbol{\eta}) := \widehat{f}_{n,\sigma}^{(t,x)}(s_1, \xi_1, \cdots, s_n, \xi_n) = \prod_{i=1}^{n} e^{-\frac{1}{2}(s_{\sigma(i+1)} - s_{\sigma(i)})|\eta_i^\sigma|^2} e^{-\iota x \eta_n^\sigma} \mathbf{1}_\sigma(\vec{\mathbf{s}}) \,, \tag{5.1}$$

with the notations $\boldsymbol{\eta} := (\eta_1, \cdots, \eta_n)$, $\mathbf{1}_\sigma(\vec{\mathbf{s}}) := \mathbf{1}_{\{\vec{\mathbf{s}} \in \mathbb{T}_1^\sigma\}} = \mathbf{1}_{\{0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < 1\}}(\vec{\mathbf{s}})$, $\eta_i^\sigma := \xi_{\sigma(1)} + \cdots + \xi_{\sigma(i)}$. And we set $\widehat{f}_n^{(t,x)}(\cdot) := \widehat{f}_{n,\sigma}^{(t,x)}(\cdot)$ if σ is the natural permutation. The notations involved with the variable $\vec{\mathbf{r}}$ are defined similarly.

By applying change of variables $\eta_j = \xi_1 + \cdots + \xi_j$ and triangle inequality, we have by expanding the product

$$\begin{aligned}
\prod_{j=1}^{n} |\xi_j|^{1-2H} &= \prod_{j=1}^{n} |\eta_j - \eta_{j-1}|^{1-2H} \\
&\leq \prod_{j=1}^{n} (|\eta_j| + |\eta_{j-1}|)^{1-2H} \leq C_H \sum_{\boldsymbol{\pi}} \prod_{j=1}^{n} |\eta_j|^{(1-2H)\pi_j} \,,
\end{aligned}$$

where $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_n)$ with $\pi_j \in \{0, 1, 2\}$ and $|\boldsymbol{\pi}| := \sum_{i=1}^n \pi_i = n - 1$ can be thought as one dimensional simplified version of (3.29). Then, it is not hard to see that by the change of variables described before

$$\begin{aligned}
I_n :=& \| \frac{1}{n!} \sum_{\sigma} f_{n,\sigma}^{(t,x)}(\cdot) \|_{\mathfrak{H}^{\otimes n}}^2 \\
=& \frac{1}{(n!)^2} \int_{\mathbb{R}_+^{2n}} \int_{\mathbb{R}^n} \sum_{\sigma} \widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{r}}, \boldsymbol{\xi}) \cdot \sum_{\varsigma} \overline{\widehat{f}_{n,\varsigma}^{(t,x)}(\vec{\mathbf{s}}, \boldsymbol{\xi})} \prod_{j=1}^{n} |\xi_j|^{1-2H} \prod_{i=1}^{n} |s_i - r_i|^{-\gamma_0} d\boldsymbol{\xi} d\vec{\mathbf{s}} d\vec{\mathbf{r}} \\
\leq& C_H \sum_{\boldsymbol{\pi}} \int_{\mathbb{R}_+^{2n}} \int_{\mathbb{R}^n} \frac{1}{n!} \sum_{\sigma} \widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{r}}, \boldsymbol{\eta}) \cdot \frac{1}{n!} \sum_{\varsigma} \overline{\widehat{f}_{n,\varsigma}^{(t,x)}(\vec{\mathbf{s}}, \boldsymbol{\eta})} \\
& \qquad \times \prod_{j=1}^{n} |\eta_j|^{(1-2H)\pi_j} \prod_{i=1}^{n} |s_i - r_i|^{-\gamma_0} d\boldsymbol{\eta} d\vec{\mathbf{s}} d\vec{\mathbf{r}} \,,
\end{aligned}$$

where $d\boldsymbol{\eta} = d\eta_1 \cdots d\eta_n$ and σ, ς are two permutations. Let us set

$$\psi_n(\vec{\mathbf{r}}, \vec{\mathbf{s}}; \boldsymbol{\eta}) := \frac{1}{n!} \sum_{\sigma} \widehat{f}^{(t,x)}_{n,\sigma}(\vec{\mathbf{r}}, \boldsymbol{\eta}) \cdot \frac{1}{n!} \sum_{\varsigma} \overline{\widehat{f}^{(t,x)}_{n,\varsigma}(\vec{\mathbf{s}}, \boldsymbol{\eta})}$$

and

$$\Psi_{n,\boldsymbol{\tau}}(\vec{\mathbf{r}}, \vec{\mathbf{s}}) := \int_{\mathbb{R}^n} \psi_n(\vec{\mathbf{r}}, \vec{\mathbf{s}}; \boldsymbol{\eta}) \prod_{j=1}^{n} |\eta_j|^{\tau_j} d\boldsymbol{\eta}$$

with indices $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ depending on $\boldsymbol{\pi}$ to be chosen later. By Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$, $k + k' = 1$ and $m + m' = 1$, we have

$$\begin{aligned} I_n \leq & C_H \sum_{\boldsymbol{\pi}} \left| \int_{\mathbb{R}_+^{2n}} \Psi_{n,\boldsymbol{\tau}}(\vec{\mathbf{r}}, \vec{\mathbf{s}}) \prod_{i=1}^{n} |s_i - r_i|^{-pm\gamma_0} d\vec{\mathbf{s}} d\vec{\mathbf{r}} \right|^{\frac{1}{p}} \\ & \times \left| \int_{\mathbb{R}_+^{2n}} \Psi_{n,\boldsymbol{\tau}'}(\vec{\mathbf{r}}, \vec{\mathbf{s}}) \prod_{i=1}^{n} |s_i - r_i|^{-qm'\gamma_0} d\vec{\mathbf{s}} d\vec{\mathbf{r}} \right|^{\frac{1}{q}} \\ =: & C_H \sum_{\boldsymbol{\pi}} \mathcal{J}_1(p, m, k, \boldsymbol{\pi}) \times \mathcal{J}_2(q, m', k', \boldsymbol{\pi}) \, , \end{aligned} \tag{5.2}$$

with $\mathcal{J}_1, \mathcal{J}_2$ depend on the $\boldsymbol{\tau}$, which is determined by $\boldsymbol{\pi}$ as given by

$$\tau_j = kp(1 - 2H)\pi_j \quad \text{and} \quad \tau_j' = k'q(1 - 2H)\pi_j$$

depending on p, q, k, k' will be chosen later. We will apply the combination of the Hölder-Young-Brascamp-Lieb inequality and the Hardy-Littlewood-Sobolev inequality to $\mathcal{J}_1(p, m, k, \boldsymbol{\pi})$ as in Section 4.1, and will apply solely the Hardy-Littlewood-Sobolev inequality to $\mathcal{J}_2(q, m', k', \boldsymbol{\pi})$. As we noticed in Section 4.1, the condition of using Hölder-Young-Brascamp-Lieb inequality is weaker than the one of using Hardy-Littlewood-Sobolev. But the bound for I_n obtained by applying the combination of the Hölder-Young-Brascamp-Lieb inequality and the Hardy-Littlewood-Sobolev inequality is less precise in terms of n, which ensures the finiteness under more general condition. The application of Hardy-Littlewood-Sobolev inequality to the multiple integral, on the other hand, will produce a more explicit decay factor of the form $\frac{1}{(n!)^\kappa}$ with certain $\kappa > 0$, which is key for guaranteeing the convergence of (1.14).

Step 1: Estimate $\mathcal{J}_1(p, m, k, \boldsymbol{\pi})$. We shall use the Hölder-Young-Brascamp-Lieb inequality to show $\mathcal{J}_1 := \mathcal{J}_1(p, m, k, \boldsymbol{\pi}) \leq C^n$. First, we have

$$\psi_n(\vec{\mathbf{r}}, \vec{\mathbf{s}}; \boldsymbol{\eta}) \leq \frac{1}{(n!)^2} \sum_{\sigma, \varsigma} |\widehat{f}^{(t,x)}_{n,\sigma}(\vec{\mathbf{r}}, \boldsymbol{\eta}) \cdot \overline{\widehat{f}^{(t,x)}_{n,\varsigma}(\vec{\mathbf{s}}, \boldsymbol{\eta})}| \, ,$$

for any permutations (of $\{1, \cdots, n\}$) σ, ς and consequently we get

$$\begin{aligned} |\mathcal{J}_1|^p = & \int_{\mathbb{R}_+^{2n}} \Psi_{n,\boldsymbol{\tau}}(\vec{\mathbf{r}}, \vec{\mathbf{s}}) \prod_{j=1}^{n} |s_j - r_j|^{-pm\gamma_0} d\vec{\mathbf{s}} d\vec{\mathbf{r}} \\ \leq & \frac{1}{(n!)^2} \sum_{\sigma, \varsigma} \int_{\mathbb{R}_+^{2n}} \int_{\mathbb{R}^n} |\widehat{f}^{(t,x)}_{n,\sigma}(\vec{\mathbf{r}}, \boldsymbol{\eta}) \cdot \overline{\widehat{f}^{(t,x)}_{n,\varsigma}(\vec{\mathbf{s}}, \boldsymbol{\eta})}| \prod_{j=1}^{n} |\eta_j|^{\tau_j} |s_j - r_j|^{-pm\gamma_0} d\boldsymbol{\eta} d\vec{\mathbf{s}} d\vec{\mathbf{r}} \, . \end{aligned} \tag{5.3}$$

An application of the Cauchy-Schwarz inequality to (5.3) yields

$$\begin{aligned}|\mathcal{J}_1|^p \le& \frac{1}{(n!)^2}\sum_{\sigma,\varsigma}\left(\int_{\mathbb{R}_+^{2n}}\int_{\mathbb{R}^n}|\widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{r}},\boldsymbol{\eta})\overline{\widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})}|\prod_{j=1}^n|\eta_j|^{\tau_j}|s_j-r_j|^{-pm\gamma_0}d\boldsymbol{\eta}d\vec{\mathbf{s}}d\vec{\mathbf{r}}\right)^{\frac12}\\ &\times\left(\int_{\mathbb{R}_+^{2n}}\int_{\mathbb{R}^n}|\widehat{f}_{n,\varsigma}^{(t,x)}(\vec{\mathbf{r}},\boldsymbol{\eta})\overline{\widehat{f}_{n,\varsigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})}|\prod_{j=1}^n|\eta_j|^{\tau_j}|s_j-r_j|^{-pm\gamma_0}d\boldsymbol{\eta}d\vec{\mathbf{s}}d\vec{\mathbf{r}}\right)^{\frac12}\\ =&\frac{1}{(n!)^2}\sum_{\sigma,\varsigma}\int_{\mathbb{R}_+^{2n}}\int_{\mathbb{R}^n}|\widehat{f}_{n}^{(t,x)}(\vec{\mathbf{r}},\boldsymbol{\eta})\overline{\widehat{f}_{n}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})}|\prod_{j=1}^n|\eta_j|^{\tau_j}|s_j-r_j|^{-pm\gamma_0}d\boldsymbol{\eta}d\vec{\mathbf{s}}d\vec{\mathbf{r}},\end{aligned}$$

where the equality holds since the integrals in line 1 and line 2 are actually independent of permutations σ,ς. We introduce the following notations to simplify the presentation:

$$\varrho_j := \frac{1+\tau_j}{2} = \begin{cases}\frac12\,, & \pi_j=0\,;\\ \frac12+\frac{kp}{2}(1-2H)\,, & \pi_j=1\,;\\ \frac12+kp(1-2H)\,, & \pi_j=2\,.\end{cases}$$

Then noticing the definition of $\widehat{f}_n^{(t,x)}(\vec{\mathbf{r}},\boldsymbol{\eta})$ in (5.1) with σ being natural permutation, the integration with respect to η gives

$$\begin{aligned}|\mathcal{J}_1|^p \le& C^n\int_{\mathbb{T}_1\times\mathbb{T}_1}\int_{\mathbb{R}^n}\prod_{j=1}^n e^{-\frac12(s_{j+1}-s_j+r_{j+1}-r_j)|\eta_j|^2}\prod_{j=1}^n|\eta_j|^{\tau_j}|s_j-r_j|^{-pm\gamma_0}d\boldsymbol{\eta}d\vec{\mathbf{s}}d\vec{\mathbf{r}}\\ \le& C^n\int_{\mathbb{T}_1\times\mathbb{T}_1}\int_{\mathbb{R}^n}\prod_{j=1}^n|s_{j+1}-s_j+r_{j+1}-r_j|^{-\frac{\varrho_j}{2}}\times|s_j-r_j|^{-pm\gamma_0}d\boldsymbol{\eta}d\vec{\mathbf{s}}d\vec{\mathbf{r}}\\ \le& C^n\int_{\mathbb{T}_1\times\mathbb{T}_1}\prod_{j=1}^n|s_{j+1}-s_j|^{-\frac{\varrho_j}{2}}|r_{j+1}-r_j|^{-\frac{\varrho_j}{2}}\times|s_j-r_j|^{-pm\gamma_0}d\vec{\mathbf{s}}d\vec{\mathbf{r}}\le C^n\,,\end{aligned}\tag{5.4}$$

under the *dimension conditions* of Theorem A.3.

Similarly to the argument in the proof of Theorem 1.1, the *dimension conditions* will hold if the following conditions are satisfied with $H_0^{(1)} := 1-pm(1-H_0)$:

$$2\Big[\frac14+\frac{kp}{2}(1-2H)\Big]+2(2-2H_0^{(1)})<3 \quad\text{and}\quad pm\gamma_0<1$$

which is equivalent to

$$1<p<\frac{5}{2k(1-2H)+8m(1-H_0)} \quad\text{and}\quad 1<p<\frac{1}{2m(1-H_0)}\,.\tag{5.5}$$

Step 2: Estimate $\mathcal{J}_2(q,m',k',\boldsymbol{\pi})$. We shall use the Hardy-Littlewood-Sobolev inequality solely to show

$$\mathcal{J}_2 := \mathcal{J}_2(q,m',k',\boldsymbol{\pi}) \le C^n(n!)^{-\frac2q+\frac{1}{2q}+\frac{k'}{2}(1-2H)}\,,$$

with $\mathcal{J}_2(q,m',k',\boldsymbol{\pi})$ defined by (5.2). By the Cauchy-Schwarz inequality, we have

$$\Psi_{n,\boldsymbol{\tau}'}(\vec{\mathbf{r}},\vec{\mathbf{s}}) \le \sqrt{\Psi_{n,\boldsymbol{\tau}'}(\vec{\mathbf{r}},\vec{\mathbf{r}})}\sqrt{\Psi_{n,\boldsymbol{\tau}'}(\vec{\mathbf{s}},\vec{\mathbf{s}})}\,.$$

We substitute this into $\mathcal{J}_2$ and then apply the Hardy-Littlewood-Sobolev inequality to obtain

$$\begin{aligned}\mathcal{J}_2 =&\Big(\int_{\mathbb{R}_+^{2n}}\Psi_{n,\boldsymbol{\tau'}}(\vec{\mathbf{r}},\vec{\mathbf{s}})\prod_{i=1}^n|s_i-r_i|^{-qm'\gamma_0}d\vec{\mathbf{s}}d\vec{\mathbf{r}}\Big)^{\frac{1}{q}}\\ \le&\Big(\int_{\mathbb{R}_+^{2n}}\sqrt{\Psi_{n,\boldsymbol{\tau'}}(\vec{\mathbf{r}},\vec{\mathbf{r}})}\sqrt{\Psi_{n,\boldsymbol{\tau'}}(\vec{\mathbf{s}},\vec{\mathbf{s}})}\prod_{i=1}^n|s_i-r_i|^{-qm'\gamma_0}d\vec{\mathbf{s}}d\vec{\mathbf{r}}\Big)^{\frac{1}{q}}\\ \le&C^n\Big(\int_{\mathbb{R}_+^{n}}\Big|\Psi_{n,\boldsymbol{\tau'}}(\vec{\mathbf{s}},\vec{\mathbf{s}})\Big|^{\frac{1}{2H_0^{(2)}}}d\vec{\mathbf{s}}\Big)^{\frac{2H_0^{(2)}}{q}},\end{aligned}$$

where $H_0^{(2)}:=1-qm'(1-H_0)$ and we should require $qm'\gamma_0<1$. Recall that $\gamma_0=2-2H_0$, then

$$qm'\gamma_0<1\Leftrightarrow 1<q<\frac{1}{2m'(1-H_0)}\,. \tag{5.6}$$

Notice that

$$\begin{aligned}\Psi_{n,\boldsymbol{\tau'}}(\vec{\mathbf{s}},\vec{\mathbf{s}})=&\int_{\mathbb{R}^n}\Big|\frac{1}{n!}\sum_{\sigma}\widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})\cdot\frac{1}{n!}\sum_{\varsigma}\overline{\widehat{f}_{n,\varsigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})}\Big|\prod_{j=1}^n|\eta_j|^{\tau'_j}d\boldsymbol{\eta}\\ =&\int_{\mathbb{R}^n}\Big|\frac{1}{(n!)^2}\sum_{\sigma,\varsigma}\widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})\cdot\overline{\widehat{f}_{n,\varsigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})}\Big|\prod_{j=1}^n|\eta_j|^{\tau'_j}d\boldsymbol{\eta}\\ =&\int_{\mathbb{R}^n}\frac{1}{(n!)^2}\sum_{\sigma}|\widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})|^2\prod_{j=1}^n|\eta_j|^{\tau'_j}d\boldsymbol{\eta}\,,\end{aligned} \tag{5.7}$$

where the last equality follows from

$$\mathbf{1}_\sigma(\vec{\mathbf{s}})\cdot\mathbf{1}_\varsigma(\vec{\mathbf{s}})=\begin{cases}0\,, & \sigma\neq\varsigma\,;\\ \mathbf{1}_\sigma(\vec{\mathbf{s}})\,, & \sigma=\varsigma\,.\end{cases}$$

Hence, we have by (5.7)

$$\begin{aligned}\int_{\mathbb{R}_+^n}\Big|\Psi_{n,\boldsymbol{\tau'}}(\vec{\mathbf{s}},\vec{\mathbf{s}})\Big|^{\frac{1}{2H_0^{(2)}}}d\vec{\mathbf{s}}=&\frac{1}{(n!)^{\frac{1}{H_0^{(2)}}}}\int_{\mathbb{R}_+^n}\Big|\int_{\mathbb{R}^n}\sum_{\sigma}|\widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})|^2\prod_{j=1}^n|\eta_j|^{\tau'_j}d\boldsymbol{\eta}\Big|^{\frac{1}{2H_0^{(2)}}}d\vec{\mathbf{s}}\\ =&\frac{1}{(n!)^{\frac{1}{H_0^{(2)}}}}\sum_{\sigma'}\int_{\mathbb{T}_1^{\sigma'}}\Big|\int_{\mathbb{R}^n}\sum_{\sigma}|\widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})|^2\prod_{j=1}^n|\eta_j|^{\tau'_j}d\boldsymbol{\eta}\Big|^{\frac{1}{2H_0^{(2)}}}d\vec{\mathbf{s}}\\ =&\frac{1}{(n!)^{\frac{1}{H_0^{(2)}}}}\sum_{\sigma}\int_{\mathbb{T}_1^{\sigma}}\Big|\int_{\mathbb{R}^n}|\widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})|^2\prod_{j=1}^n|\eta_j|^{\tau'_j}d\boldsymbol{\eta}\Big|^{\frac{1}{2H_0^{(2)}}}d\vec{\mathbf{s}}\,.\end{aligned}$$

Similar to (5.4), we get

$$\int_{\mathbb{R}_+^n}\Big|\Psi_{n,\boldsymbol{\tau'}}(\vec{\mathbf{s}},\vec{\mathbf{s}})\Big|^{\frac{1}{2H_0^{(2)}}}d\vec{\mathbf{s}}\le\frac{C^n}{(n!)^{\frac{1}{H_0^{(2)}}}}\sum_{\sigma}\int_{\mathbb{T}_1^{\sigma}}\prod_{i=1}^n|s_{\sigma(i+1)}-s_{\sigma(i)}|^{-\frac{\varrho'_j}{2H_0^{(2)}}}d\vec{\mathbf{s}} \tag{5.8}$$

with ϱ'_j being defined by

$$\varrho'_j=\frac{1+\tau'_j}{2}=\begin{cases}\frac{1}{2}\,, & \pi_j=0\,;\\ \frac{1}{2}+\frac{k'q}{2}(1-2H)\,, & \pi_j=1\,;\\ \frac{1}{2}+k'q(1-2H)\,, & \pi_j=2\,.\end{cases}$$

The finiteness of (5.8) requires that for $j=1,\cdots,n$, $\frac{\varrho_j'}{2H_0^{(2)}}$ satisfies the following *Hardy-Littlewood-Sobolev conditions*:

$$\begin{aligned}\frac{\varrho_j'}{2H_0^{(2)}}<1 \Leftrightarrow \frac{\frac{1}{2}+k'q(1-2H)}{2H_0^{(2)}}<1\\ \Leftrightarrow\ 1<q<\frac{3}{2k'(1-2H)+4m'(1-H_0)}\,.\end{aligned} \tag{5.9}$$

Under the above condition we have from (5.8)

$$\begin{aligned}\left[\int_{\mathbb{R}_+^n}\left|\Psi_{n,\boldsymbol{\tau}'}^{(q)}(\vec{\mathbf{s}},\vec{\mathbf{s}})\right|^{\frac{1}{2H_0^{(2)}}}d\vec{\mathbf{s}}\right]^{\frac{2H_0^{(2)}}{q}} &\le\frac{C^n}{(n!)^{\frac{2}{q}}}\left[n!\cdot\frac{\prod_{i=1}^n\Gamma(1-\varrho_j'/2H_0^{(2)})}{\Gamma(n-\sum_{j=1}^n\varrho_j'/2H_0^{(2)}+1)}\right]^{\frac{2H_0^{(2)}}{q}}\\ &\le C^n(n!)^{-\frac{2}{q}+[\frac{1}{2q}+\frac{k'}{2}(1-2H)]}\,,\end{aligned} \tag{5.10}$$

where the last inequality follows from the Stirling formula and the fact that

$$\sum_{j=1}^n\varrho_j'=n\Big[\frac{1}{2}+\frac{k'q}{2}(1-2H)\Big]\,.$$

Incorporating (5.4) from Step 1 and (5.10) from Step 2 into (5.2), we obtain for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$

$$\mathbb{E}[u(t,x)^2]\le\sum_{n=0}^\infty n!\sum_{\boldsymbol{\pi}}\mathcal{J}_1\times\mathcal{J}_2\le\sum_{n=0}^\infty C^n(n!)^{1-\frac{2}{q}+[\frac{1}{2q}+\frac{k'}{2}(1-2H)]}\,.$$

This proves $\mathbb{E}[u(t,x)^2]<\infty$ if

$$1-\frac{2}{q}+\Big[\frac{1}{2q}+\frac{k'}{2}(1-2H)\Big]<0\Leftrightarrow 1<q<\frac{3}{k'(1-2H)+2}\,. \tag{5.11}$$

Step 3: Constraints on p (q), m (m'), and k (k'). The conditions (5.6) and (5.9) are summarized as

$$1<q<\frac{3}{2k'(1-2H)+4m'(1-H_0)}\quad\text{and}\quad 1<q<\frac{1}{2m'(1-H_0)}\,. \tag{5.12}$$

Remember that in **Step 2** and **Step 3**, in order to ensure $\mathbb{E}[u(t,x)^2]<+\infty$, it is necessary to have

$$1<p<\frac{5}{2k(1-2H)+8m(1-H_0)}\wedge\frac{1}{2m(1-H_0)}\,; \tag{5.13}$$

$$1<q<\frac{3}{2k'(1-2H)+4m'(1-H_0)}\wedge\frac{1}{2m'(1-H_0)}\,; \tag{5.14}$$

$$1<q<\frac{3}{k'(1-2H)+2}\,. \tag{5.15}$$

To determine the conditions on H and H_0 under which (5.13)-(5.15) are met, we initially get rid of the wedge symbol "$\wedge$" in the above first two inequalities (5.13) and (5.14). To this end, we consider the following four cases:

$$\begin{cases}\text{Case 1}: \frac{5}{2k(1-2H)+8m(1-H_0)}<\frac{1}{2m(1-H_0)}\,,\quad \frac{3}{2k'(1-2H)+4m'(1-H_0)}<\frac{1}{2m'(1-H_0)}\,;\\ \text{Case 2}: \frac{5}{2k(1-2H)+8m(1-H_0)}<\frac{1}{2m(1-H_0)}\,,\quad \frac{3}{2k'(1-2H)+4m'(1-H_0)}>\frac{1}{2m'(1-H_0)}\,;\\ \text{Case 3}: \frac{5}{2k(1-2H)+8m(1-H_0)}>\frac{1}{2m(1-H_0)}\,,\quad \frac{3}{2k'(1-2H)+4m'(1-H_0)}<\frac{1}{2m'(1-H_0)}\,;\\ \text{Case 4}: \frac{5}{2k(1-2H)+8m(1-H_0)}>\frac{1}{2m(1-H_0)}\,,\quad \frac{3}{2k'(1-2H)+4m'(1-H_0)}>\frac{1}{2m'(1-H_0)}\,.\end{cases}$$

We will provide a detailed treatment of Case 1, and the approach for the other cases is analogous.

Case 1: In this case, the constraints (5.13)-(5.15) on $p,\ q$ become the following conditions:

$$\begin{cases} 1 < q < \frac{3}{k'(1-2H)+2}\,; \\ 1 < q < \frac{3}{2k'(1-2H)+4m'(1-H_0)} < \frac{1}{2m'(1-H_0)}\,; \\ 1 < p < \frac{5}{2k(1-2H)+8m(1-H_0)} < \frac{1}{2m(1-H_0)}\,. \end{cases} \tag{5.16}$$

Denote the region

$$\mathcal{S} := \left\{ (H, H_0) \in (0, \frac{1}{4}) \times (\frac{1}{2}, \frac{3}{4}) :\ H + 2H_0 > \frac{5}{4},\ H + H_0 \le \frac{3}{4},\ 12H + 4H_0 > 3 \right\}. \tag{5.17}$$

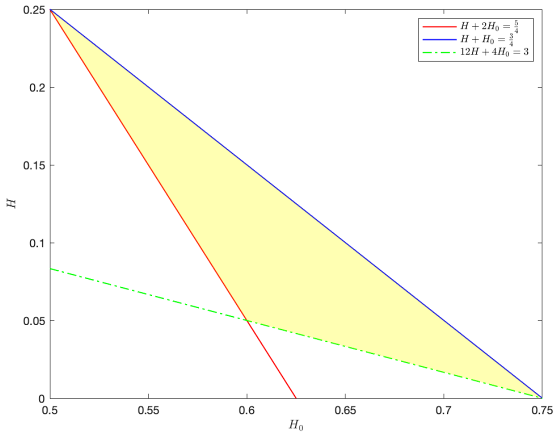

Figure 4: The region $\mathcal{S}$ in (5.17)

We shall show that for any $(H, H_0) \in \mathcal{S}$, there exist $p, q \in [1, \infty)$, and $m, m' \in [0, 1],\ k, k' \in [0, 1]$ such that conditions in. (5.16) hold. For any arbitrarily small $\varepsilon > 0$, let us choose

$$q = \frac{3}{k'(1-2H)+2} - \varepsilon, \tag{5.18}$$

$$p = \frac{q}{q-1} = \frac{3-\varepsilon}{1-k'(1-2H)-\varepsilon}, \tag{5.19}$$

$$m' = \frac{2-k'(1-2H)}{4(1-H_0)}. \tag{5.20}$$

Thus it suffices to show that for any $(H, H_0) \in \mathcal{S}$, there exists $k' \in [0, 1]$ such that the parameters (p, q, m') given by (5.18)-(5.20) satisfy conditions (5.16) for sufficiently small $\varepsilon > 0$.

Firstly, we need to ensure $p, q \ge 1$ and $0 \le m' \le 1$. Note that

$$q \ge 1 \Leftrightarrow k' \le \frac{1-2\varepsilon}{(1-2H)(1+\varepsilon)} \sim \frac{1}{1-2H}\,,$$

which is implied by $k' \le 1$ and $H > 0$. Now $p \ge 1$ is equivalent to $q \ge 1$. Besides,

$$0 \le m' \le 1 \Leftrightarrow k' \ge \frac{4H_0 - 2}{1-2H}\,. \tag{5.21}$$

In order to be able to find a $k' \in [0,1]$ so that the above inequality (5.21) holds true, we need

$$\frac{4H_0-2}{1-2H} \le 1 \Leftrightarrow H+2H_0 \le \frac{3}{2},$$

which holds for $(H,H_0) \in \mathcal{S}$ since $H+H_0 \le \frac{3}{4}$ and then $H_0 < \frac{3}{4}$.

Next, we shall verify the conditions in (5.16) with the p, q and m' given by (5.18)-(5.20). It is clear that we can take $\varepsilon = 0$.

(i) The inequality in the first line of (5.16) holds obviously with the choice of q in (5.18).

(ii) For the q defined by (5.18) we now show the inequality in the second line of (5.16):

$$q < \frac{3}{2k'(1-2H)+4m'(1-H_0)},$$

which is equivalent to

$$\frac{3}{k'(1-2H)+2} \le \frac{3}{2k'(1-2H)+4m'(1-H_0)}$$
$$\Leftrightarrow \quad m' \le \frac{2-k'(1-2H)}{4(1-H_0)}.$$

This holds with the choice of m' in (5.20).

(iii) Recall that the third inequality in the second line of (5.16) is

$$\frac{3}{2k'(1-2H)+4m'(1-H_0)} < \frac{1}{2m'(1-H_0)}$$

which is equivalent to

$$m'(1-H_0) < k'(1-2H).$$

By using the definition (5.20) of m', the constraint placed on k' as mentioned above is equivalent to

$$\frac{2}{5(1-2H)} < k' < 1. \tag{5.22}$$

(iv) The second inequality of the third line of (5.16), by choice of p in (5.19), is equivalent to

$$25-12(H+2H_0) < k'(1-2H)+24m'(1-H_0).$$

With m' defined in (5.20) this can be rewritten as

$$0 < k' < \frac{12(H+2H_0)-13}{5(1-2H)}. \tag{5.23}$$

(v) Now we consider the third inequality in the last line of (5.16). Noticing that $k=1-k'$ and $m=1-m'$ and substituting the value of m', the last line of (5.16) becomes

$$0 < k' < \frac{2+4(H_0-2H)}{5(1-2H)}. \tag{5.24}$$

In summary, when k' fulfills the requirements specified in (5.22), (5.23), and (5.24), it ensures (5.16). The conditions met by k' can be concisely summarized as follows.

$$\left\{\frac{2}{5(1-2H)} \vee \frac{4H_0-2}{1-2H}\right\} < k' < \left\{\frac{12(H+2H_0)-13}{5(1-2H)} \wedge \frac{2+4(H_0-2H)}{5(1-2H)} \wedge 1\right\}. \tag{5.25}$$

Notice that

$$12(H+2H_0)-13 \le 2+4(H_0-2H) \Leftrightarrow H+H_0 \le \frac{3}{4}$$

and

$$\frac{12(H+2H_0)-13}{5(1-2H)} \le 1 \Leftrightarrow 11H+12H_0 \le 9,$$

which holds under the condition $H+H_0 \le \frac{3}{4}$. Thus, on the set $\mathcal{S}$ the condition (5.25) is reduced to

$$\left\{\frac{2}{5(1-2H)} \vee \frac{4H_0-2}{1-2H}\right\} < k' < \frac{12(H+2H_0)-13}{5(1-2H)}. \tag{5.26}$$

Finally, it remains to show that when $(H,H_0) \in \mathcal{S}$, there is k' satisfying (5.26) which amounts to

$$\left\{\frac{2}{5(1-2H)} \vee \frac{4H_0-2}{1-2H}\right\} \le \frac{12(H+2H_0)-13}{5(1-2H)}. \tag{5.27}$$

We shall prove the above inequality assuming $H_0 \le 3/5$ and $H_0 > 3/5$ respectively. First, let us assume $H_0 \le \frac{3}{5}$. In this case, we have $\frac{2}{5(1-2H)} \le \frac{4H_0-2}{1-2H}$. Then (5.27) becomes

$$\frac{2}{5(1-2H)} < \frac{12(H+2H_0)-13}{5(1-2H)} \Leftrightarrow H+2H_0 > \frac{5}{4}$$

which corresponds to the red solid line in Figure 4. Next, we assume $H_0 > \frac{3}{5}$ which means $\frac{2}{5(1-2H)} > \frac{4H_0-2}{1-2H}$. In this case, (5.27) becomes

$$\frac{4H_0-2}{1-2H} < \frac{12(H+2H_0)-13}{5(1-2H)} \Leftrightarrow 12H+4H_0 > 3\,,$$

which aligns with the green dashed line in Figure 4. Therefore, for any given $(H,H_0) \in \mathcal{S}$ (as illustrated in Figure 4) we can select an appropriate k' such that condition (5.26) is satisfied.

Case 2-Case 4: These cases can be approached with the same methodology as Case 1. Nevertheless, we find that it does not allow for an extension of the solvability region. Consequently, we will omit the details.

As a consequence, on the region $\mathcal{S}$ (refer to Figure 4), we have for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$\mathbb{E}[u(t,x)^2] = \sum_{n=1}^{\infty} \mathbb{E}[u_n(t,x)^2] < +\infty.$$

Since the above convergence was proved in [15] when $H_0+H > 3/4$, we complete the proof. □

A Some lemmas

Lemma A.1. *Let $0<\alpha,\beta<1$ with $\alpha+\beta>1$. Then there is a constant $c \ge 1$ independent of $\varepsilon>0$ such that for all $x \in (0,3\varepsilon)$,*

$$c^{-1}x^{1-(\alpha+\beta)} \le \int_0^{\varepsilon} u^{-\alpha}(u+x)^{-\beta}du \le cx^{1-(\alpha+\beta)}. \tag{A.1}$$

This lemma can be found in [15], and the following lemma can be found in [29].

Lemma A.2 (Hardy-Littlewood-Sobolev inequality)**.** *For any $\varphi \in L^{1/H_0}(\mathbb{R}^n)$, it holds*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(\vec{\mathbf{r}})\varphi(\vec{\mathbf{s}})\prod_{i=1}^{n}|s_i-r_i|^{2H_0-2}d\vec{\mathbf{r}}d\vec{\mathbf{s}} \le C_{H_0}^n\left(\int_{\mathbb{R}^n}|\varphi(\vec{\mathbf{r}})|^{1/H_0}d\vec{\mathbf{r}}\right)^{2H_0}, \tag{A.2}$$

where $C_{H_0}>0$.

The Hölder-Young-Brascamp-Lieb type inequality has been studied and extended by various groups after the seminar work of Brascamp and Lieb [9]. We refer [1, Appendix A], [33] and the references therein to the readers on this interesting topic. Below, we present the homogeneous and non-homogeneous Hölder-Young-Brascamp-Lieb inequalities that we are going to use in this work. Let $\mathcal{H} = \mathbb{R}^n$, $\mathcal{H}_1 = \mathbb{R}^{n_1}, \dots, \mathcal{H}_m = \mathbb{R}^{n_m}$ be finite dimensional Euclidean spaces equipped with the corresponding Lebesgue measure $\mu(\cdot)$. We remark that $\mathcal{H}$, $\mathcal{H}_1, \dots, \mathcal{H}_m$ can be Hilbert space in general (e.g., [7]). For $j = 1, \cdots, m$, let $f_j : \mathcal{H}_j \to \mathbb{R}_+$ be nonnegative function satisfying $f_j \in L^{p_j}(\mu)$ for $p_1, \cdots, p_m \in [1, \infty)$ and let $l_j : \mathcal{H} \to \mathcal{H}_j$ be surjective linear transformations.

We recall the (local) multilinear functional

$$\Lambda(f_1, \dots, f_m) := \int \prod_{j=1}^{m} f_j(l_j(\mathbf{x})) \prod_{j=1}^{n} d\mu(x_j)\,,$$

and the Hölder-Young-Brascamp-Lieb type inequality

$$|\Lambda(f_1, \dots, f_m)| \le K_{\mathcal{H}} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mu)}\,, \tag{A.3}$$

for some positive finite constant $K_{\mathcal{H}}$. There are many works focusing on the conditions under which (A.3) holds true. We shall use a specific condition, the nonhomogeneous Hölder-Young-Brascamp-Lieb inequality, as presented in [8, Theorem 2.2] (also discussed in [7]).

Theorem A.3 (Nonhomogeneous Hölder-Young-Brascamp-Lieb inequality)**.** *Let $\mathcal{H}$, $\mathcal{H}_1, \dots, \mathcal{H}_m$, be finite dimensional Hilbert spaces equipped with the corresponding Lebesgue measure $\mu(\cdot)$ and $f_j \in L^{p_j}(\mu)$. Consider a local version of* (4.15)

$$\Lambda_{loc}(f_1, \dots, f_m) := \int_{\{\mathbf{x} \in \mathcal{H} : |\mathbf{x}| \le 1\}} \prod_{j=1}^{m} f_j(l_j(\mathbf{x})) \prod_{j=1}^{n} d\mu(x_j)\,. \tag{A.4}$$

A necessary and sufficient condition that Hölder-Young-Brascamp-Lieb type inequality (A.3) *holds for Λ_{loc}, with $0 < K_{\mathcal{H}} < \infty$ for all nonnegative measurable functions f_j, is that every subspace $V \subseteq \mathcal{H}$ satisfies the following dimension condition:*

$$codim_{\mathcal{H}}(V) \ge \sum_j p_j^{-1} codim_{\mathcal{H}_j}(l_j(V))\,. \tag{A.5}$$

Using the above theorem, we can show an important ingredient in the proof of Theorem 1.1.

Lemma A.4. *Under the condition $H_0 \ge 1/2$ and the conditions* (1.7a)-(1.7d) *of Theorem 1.1, the following dimensional conditions hold when $co\dim_{\mathcal{H}}(V) = 1, 2, 3, 4$:*

$$co\dim_{\mathcal{H}}(V) \ge \sum_{j=1}^{6} z_j \cdot co\dim_{\mathcal{H}_j}(l_j(V))\,.$$

Proof. We show the lemma for $co\dim = 1, 2, 3, 4$ separately.

Case 1: $co\dim_{\mathcal{H}}(V) = 1$, i.e. $\dim_{\mathcal{H}}(V) = 3$. In this case, we only need to consider $\mathrm{rank}([l_j]_{j \in J}) = 1$, i.e. $|J| = 1$. From (4.18), $z_j \le 1$ ($j = 1, \cdots, 6$) under the condition $H_0 \ge 1/2$ and the condition (1.7a). Thus, the dimension condition (4.17) holds.

Case 2: $co\dim_{\mathcal{H}}(V) = 2$, i.e. $\dim_{\mathcal{H}}(V) = 2$. We shall choose J such that $\mathrm{rank}([l_j]_{j \in J}) = 2$.

Note that we only have to deal with $|J| = 3$ since the dimension condition (4.17) automatically holds for $|J| = 2$ and it is impossible to have $\mathrm{rank}([l_j]_{j \in J}) = 3$ for $|J| = 4, 5, 6$.

The only possible case is $\mathrm{rank}([l_3, l_4, l_6]) = 2$. Then, the condition (4.17) in this case is equivalent to

$$\begin{aligned} 2 \ &\geq z_3 + z_4 + z_6 > \rho_2 + \rho_2 + \gamma_0 \\ &= (\frac{3}{2}d + \frac{1}{2}\beta^* - 2|H| - \alpha_2) + 2 - 2H_0 \\ \Leftrightarrow \qquad &|H| + 2H_0 > d + (\frac{1}{2}d - |H| + \frac{1}{2}\beta^*) - \alpha_2 \,. \end{aligned} \tag{A.6}$$

Since $\alpha_2 \in \{\frac{1}{2}d - |H| + \frac{1}{2}\beta^*, d - 2|H| + \beta^*\}$, it is obvious that (A.6) holds under the condition (1.7b):

$$|H| + 2H_0 > d. \tag{A.7}$$

Case 3: $co\dim_{\mathcal{H}}(V) = 3$, i.e. $\dim_{\mathcal{H}}(V) = 1$. We shall choose J such that $\mathrm{rank}([l_j]_{j\in J}) = 3$.

For $|J| = 3$, the dimension condition (4.17) holds automatically since $z_i \leq 1$ for $i = 1, \cdots, 6$. Moreover, it is impossible to get $\mathrm{rank}([l_j]_{j\in J}) = 3$ for $|J| = 5, 6$ by computation of the relevant ranks. Thus, we only need to consider $|J| = 4$. By simple analysis, we know that there are four cases:

$$\begin{aligned} &\mathrm{rank}([l_1, l_2, l_5, l_6]) = \mathrm{rank}([l_1, l_3, l_4, l_6]) \\ &= \mathrm{rank}([l_2, l_3, l_4, l_6]) = \mathrm{rank}([l_3, l_4, l_6, l_5]) = 3\,. \end{aligned}$$

We treat $J = \{1, 2, 5, 6\}$ as an example. The rest is similar and we omit the details.

Now, we have $\mathrm{rank}([l_1, l_2, l_5, l_6]) = 3$, $\dim(\cap_{j=1,2,5,6} \ker(l_j)) = 1$ and $co\dim_{\mathcal{H}}(V) = 3$. In fact, we can obtain that $V = span\{(1, 1, 1, 1)\} = \cap_{j=1,2,5,6} \ker(l_j)$ and

$$\begin{aligned} &co\dim_{\mathcal{H}_1}(l_1(V)) = 1\,, co\dim_{\mathcal{H}_3}(l_3(V)) = 0\,, co\dim_{\mathcal{H}_5}(l_5(V)) = 1\,, \\ &co\dim_{\mathcal{H}_2}(l_2(V)) = 1\,, co\dim_{\mathcal{H}_4}(l_4(V)) = 0\,, co\dim_{\mathcal{H}_6}(l_6(V)) = 1\,. \end{aligned}$$

Then, in this case the condition (4.17) is analogous to

$$\begin{aligned} 3 \ &\geq z_1 + z_2 + z_5 + z_6 \\ &> \rho_1 + \rho_1 + \gamma_0 + \gamma_0 \\ &= (\frac{3}{2}d_* - 2H_*) + 4 - 4H_0\,. \end{aligned} \tag{A.8}$$

It is easy to see that (A.8) holds under the third condition (1.7c) of (1.7):

$$H_* + 2H_0 > \frac{3}{4}d_* + \frac{1}{2}. \tag{A.9}$$

Case 4: $co\dim_{\mathcal{H}}(V) = 4$, i.e. $\dim_{\mathcal{H}}(V) = 0$. We shall choose J such that $\mathrm{rank}([l_j]_{j\in J}) = 4$.

Obviously, we only need to consider $|J| = 4, 5, 6$. We first deal with $|J| = 6$. This means

$$V = \bigcap_{j=1}^{6} \ker(l_j) = \{0, 0, 0, 0\}\,.$$

Thus, the dimensional condition (4.17) in this case is equivalent to the following:

$$\begin{aligned} 4 \ &\geq z_1 + z_2 + z_3 + z_4 + z_5 + z_6 \\ &> \rho_1 + \rho_1 + \rho_2 + \rho_2 + \gamma_0 + \gamma_0 \\ &= (\frac{3}{2}d_* - 2H_*) + 2(2 - 2H_0) + (\frac{3}{2}d + \frac{1}{2}\beta^* - 2|H| - \alpha_2) \end{aligned}$$

which amounts to

$$|H| + 2H_* + 4H_0 > d + \frac{3}{2}d_* + (\frac{1}{2}d - |H| + \frac{1}{2}\beta^* - \alpha_2). \tag{A.10}$$

Since $\alpha_2 \in \{\frac{1}{2}d - |H| + \frac{1}{2}\beta^*, d - 2|H| + \beta^*\}$, we know (A.10) holds under the condition (1.7d):

$$|H| + 2H_* + 4H_0 > d + \frac{3}{2}d_*. \tag{A.11}$$

When $|J| = 4, 5$, the corresponding dimension conditions (4.17) can be implied by (A.10). Thus, the proof of the lemma is complete. □

Next, we give some estimations for the alternative proof of the sufficiency of Theorem 1.1.

Lemma A.5. *For any* $\alpha, \beta, \gamma \in (0, 1)$ *and* $a, b \in \mathbb{R}$ *denote*

$$\begin{cases} \mathfrak{B}_1(a, b) := \int_0^{\frac{1}{2}} u^{-\beta}(1-u)^{-\alpha}|au - b|^{-\gamma}du\,; & \text{(A.12)} \\ \mathfrak{B}_2(a, b) := \int_{\frac{1}{2}}^1 u^{-\beta}(1-u)^{-\alpha}|au - b|^{-\gamma}du. & \text{(A.13)} \end{cases}$$

Then

$$\mathfrak{B}_1(a, b) \lesssim \begin{cases} |a|^{-q}|b|^{-\gamma+q} \quad \text{for any } q \in [0, 1-\beta] \quad \text{if } \beta + \gamma > 1; \\ (|a| + |b|)^{-\gamma} \lesssim |a|^{-q_1}|b-a|^{-q_2}|b|^{-\gamma+q_1+q_2} \\ \qquad \text{for any } q_1, q_2, q_1 + q_2 \in [0, \gamma] \quad \text{if } \beta + \gamma < 1, \end{cases} \tag{A.14}$$

and

$$\mathfrak{B}_2(a, b) \lesssim \begin{cases} |a|^{-q}|b-a|^{-\gamma+q} \quad \text{for any } q \in [0, 1-\alpha] \quad \text{if } \alpha + \gamma > 1; \\ (|a| + |b-a|)^{-\gamma} \lesssim |a|^{-q_1}|b-a|^{-q_2}|b|^{-\gamma+q_1+q_2} \\ \qquad \text{for any } q_1, q_2, q_1 + q_2 \in [0, \gamma] \quad \text{if } \alpha + \gamma < 1. \end{cases} \tag{A.15}$$

Proof. First let $\beta + \gamma > 1$. Since for $x \geq 0$, $\beta \in (0, 1)$, $\gamma \in (0, 1)$ it holds that

$$\int_0^x |\xi|^{-\beta}|1-\xi|^{-\gamma}d\xi \lesssim |x|^{-\beta+1} \wedge 1\,. \tag{A.16}$$

Then for any $\rho_1 \in [0, 1]$,

$$\begin{aligned} \int_0^{\frac{1}{2}} w^{-\beta}(1-w)^{-\alpha}|aw - b|^{-\gamma}dw &\lesssim \int_0^{\frac{1}{2}} w^{-\beta}|aw - b|^{-\gamma}dw \\ &\lesssim |b|^{-\gamma}\int_0^{\frac{1}{2}} w^{-\beta}\left|\frac{|a|}{|b|}w - 1\right|^{-\gamma}dw \\ &\lesssim |b|^{-\gamma-\beta+1} \cdot |a|^{\beta-1} \times \int_0^{\frac{|a|}{2|b|}} \xi^{-\beta}|1-\xi|^{-\gamma}d\xi\,. \end{aligned}$$

We can apply (A.16) in the last integral to get

$$\begin{aligned} \int_0^{\frac{1}{2}} w^{-\beta}(1-w)^{-\alpha}|aw - b|^{-\gamma}dw &\lesssim |b|^{-\gamma-\beta+1} \cdot |a|^{\beta-1} \cdot \left(\frac{|a|}{|b|} \wedge 1\right)^{-\beta+1} \\ &\lesssim |b|^{-\gamma-\beta+1} \cdot |a|^{\beta-1} \cdot \left(\frac{|a|}{|b|}\right)^{(-\beta+1)\rho_1} \\ &= |a|^{-q_1}|b|^{-\gamma+q_1}\,, \end{aligned} \tag{A.17}$$

where $q_1 = (1-\beta)(1-\rho_1) \in [0, 1-\beta]$ and we make substitution $\xi = \frac{|a|}{|b|}w$ in the third inequality. Here we need that $\beta,\ \gamma \in (0,1),\ \beta+\gamma>1$.

Make substitution $\widetilde{w} = 1-w$. Then for $0 \le \rho_2 \le 1$,

$$\int_{\frac{1}{2}}^{1} |1-w|^{-\alpha} w^{-\beta} |w-b_v|^{-\gamma} dw = \int_0^{\frac{1}{2}} \widetilde{w}^{-\alpha} |1-\widetilde{w}|^{-\beta} |a-b-a\widetilde{w}|^{-\gamma} d\widetilde{w}\,. \tag{A.18}$$

This can be used to show the first inequality in (A.15).

Now if $\beta \in (0,1)$, $\gamma \in (0,1)$ and $\beta+\gamma<1$, then

$$\int_0^x |\xi|^{-\beta}|1-\xi|^{-\gamma} d\xi \lesssim \frac{|x|^{-\beta+1}(1+|x|^{1-\beta-\gamma})}{1+|x|^{-\beta+1}} \qquad \forall\, x>0\,.$$

With the same argument as above we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} w^{-\beta}(1-w)^{-\beta}|aw-b|^{-\gamma} dw \\
&\quad \lesssim |b|^{-\gamma-\beta+1} \cdot |a|^{\beta-1} \times \int_0^{\frac{|a|}{2|b|}} \xi^{-\beta}|1-\xi|^{-\gamma} d\xi \\
&\quad \lesssim |b|^{-\gamma-\beta+1} \cdot |a|^{\beta-1} \cdot \frac{|\frac{a}{b}|^{-\beta+1}(1+|\frac{a}{b}|^{1-\beta-\gamma})}{1+|\frac{a}{b}|^{-\beta+1}}\,. \\
&\quad \lesssim (|a|+|b|)^{-\gamma}\,.
\end{aligned} \tag{A.19}$$

This shows the first part of the second inequality in (A.14). Since $|a|+|b| \ge |a| \vee |b-a| \vee |b|$ implies the following inequalities:

$$\begin{aligned}
(|a|+|b|)^{-\gamma} &\le [|a| \vee |b-a| \vee |b|]^{-q_1}\, [|a| \vee |b-a| \vee |b|]^{-q_2}\, [|a| \vee |b-a| \vee |b|]^{-\gamma+q_1+q_2} \\
&\le |a|^{-q_1}|b-a|^{-q_2}|b|^{-\gamma+q_1+q_2}\,.
\end{aligned}$$

This shows the second part of the second inequality in (A.14). The second inequality in (A.15) can be proved similarly since $|a|+|b-a| \ge |a| \vee |b-a| \vee |b|$. This completes the proof of the lemma. □

We immediately have the following Corollary.

Corollary A.6. *The following conclusion holds true.*

$$\begin{cases} \mathfrak{B}_1(a,b) \lesssim |a|^{-q}|b|^{-\gamma+q} & \text{for any } q \in [0, (1-\beta)\wedge\gamma]; \\ \mathfrak{B}_2(a,b) \lesssim |a|^{-q}|b-a|^{-\gamma+q} & \text{for any } q \in [0, (1-\alpha)\wedge\gamma]. \end{cases} \tag{A.20}$$

Now we give a lemma that indicates the upper bound of the integrations with respect to $\vec{\mathrm{s}}$ and $\vec{\mathrm{r}}$ in Proposition 3.4.

Lemma A.7. *Let $0 \le a < A < \infty$, $0 \le b < B < \infty$ and let $\alpha,\beta,\gamma \in [0,1)$, $\gamma < 2-\beta-(\beta\vee\alpha)$. Denote*

$$\mathcal{I}_{\alpha,\beta,\gamma}(A,a;B,b) := \int_{\substack{a<s<A\\ b<r<B}} |A-s|^{-\alpha}|s-a|^{-\beta}|s-r|^{-\gamma}|B-r|^{-\alpha}|r-b|^{-\beta} ds dr\,. \tag{A.21}$$

Then for any $q_2 \in (0 \vee (\beta+\gamma-1), (1-\alpha)\wedge\frac{\gamma}{2}]$,

$$\mathcal{I}_{\alpha,\beta,\gamma}(A,a;B,b) \lesssim |A-a|^{-\alpha-\beta+1-\frac{\gamma}{2}}|B-b|^{-\alpha-\beta+1-\frac{\gamma}{2}} \tag{A.22}$$

$$+\, |A-a|^{-\alpha-\beta+1-q_2}|B-b|^{-\alpha-\beta+1-q_2}|A-B|^{-\gamma+2q_2}\,. \tag{A.23}$$

Proof. Making substitution $s = a+(A-a)w,\ r = b+(B-b)v$ and denoting $\bar{a} := A-a,\ b_v := (B-b)v-(a-b)$ and $M := |A-a|^{-\alpha-\beta+1}\cdot|B-b|^{-\alpha-\beta+1}$, we have

$$\begin{aligned}\mathcal{I}_{\alpha,\beta,\gamma}(A,a;B,b) =& M\times\int_0^1\int_0^1 |1-w|^{-\alpha}w^{-\beta}|1-v|^{-\alpha}v^{-\beta}|\bar{a}w-b_v|^{-\gamma}dwdv\\ =& M\times\int_0^1 |1-v|^{-\alpha}v^{-\beta}\tilde{\mathcal{I}}(v)dv\,.\end{aligned} \tag{A.24}$$

where by Corollary A.6 $\tilde{\mathcal{I}}(v)$ is defined and bounded as follows.

$$\begin{aligned}\tilde{\mathcal{I}}(v) =& (\int_0^{\frac12}+\int_{\frac12}^1)\left[|1-w|^{-\alpha}w^{-\beta}|\bar{a}w-b_v|^{-\gamma}\right]dw\\ \le& \bar{a}^{-q_1}|b_v|^{-\gamma+q_1}+\bar{a}^{-q_2}|b_v-\bar{a}|^{-\gamma+q_2}\,,\end{aligned}$$

where $q_1\in(0,(1-\beta)\wedge\gamma)$ and $q_2\in(0,(1-\alpha)\wedge\gamma)$. Thus, we have

$$\begin{aligned}\mathcal{I}_{\alpha,\beta,\gamma}(A,a;B,b) =& M\bar{a}^{-q_1}\times\int_0^1|1-v|^{-\alpha}v^{-\beta}|b_v|^{-\gamma+q_1}dv\\ &+M\bar{a}^{-q_2}\int_0^1|1-v|^{-\alpha}v^{-\beta}|b_v-\bar{a}|^{-\gamma+q_2}dv\\ =:& M\bar{a}^{-q_1}\times\mathfrak{B}_1+M\bar{a}^{-q_2}\mathfrak{B}_2\,,\end{aligned} \tag{A.25}$$

where $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are defined and dealt with as follows. By Lemma A.5 we obtain

$$\begin{aligned}\mathfrak{B}_1 :=& \int_0^1|1-v|^{-\alpha}|v|^{-\beta}\cdot|(B-b)v-(a-b)|^{-\gamma+q_1}dv\\ \lesssim&\int_0^{1/2}|v|^{-\beta}|(B-b)v-(a-b)|^{-\gamma+q_1}dv\\ &+\int_0^{1/2}|v|^{-\alpha}|(B-b)v-(B-a)|^{-\gamma+q_1}dv\\ \lesssim&|B-b|^{-\gamma+q_1}+(|B-b|+|B-a|)^{-\gamma+q_1}\\ \lesssim&|B-b|^{-\gamma+q_1}\,,\end{aligned} \tag{A.26}$$

where in obtaining the bound for the above first integral we need

$$\begin{cases} q_1>\gamma+\beta-1 & \text{and hence}\quad \gamma+\beta-1<q_1<(1-\beta)\wedge\gamma\\ q_1>\gamma+\alpha-1 & \text{and hence}\quad \gamma+\alpha-1<q_1<(1-\beta)\wedge\gamma\end{cases}$$

which is possible by the assumption $\gamma<2-\beta-(\beta\vee\alpha)$.

Now we turn to bound $\mathfrak{B}_2$.

$$\begin{aligned}\mathfrak{B}_2 :=& \int_0^1|1-v|^{-\alpha}|v|^{-\beta}|(B-b)v-(A-b)|^{-\gamma+q_2}dv\\ \lesssim&\int_0^{1/2}|v|^{-\beta}|(B-b)v-(A-b)|^{-\gamma+q_2}dv\\ &+\int_0^{1/2}|v|^{-\alpha}|(B-b)v-(B-A)|^{-\gamma+q_2}dv\\ \lesssim&(|B-b|+|A-b|)^{-\gamma+q_2}+|B-b|^{-q_3}|B-A|^{-\gamma+q_2+q_3}\\ \lesssim&|B-b|^{-q_3}|B-A|^{-\gamma+q_2+q_3}\,,\end{aligned} \tag{A.27}$$

where $\gamma+\beta-1 < q_2 \le (1-\alpha)\wedge\gamma$, $q_3 \in [0,(1-\alpha)\wedge(\gamma-q_2)]$. Substituting the bounds (A.26)-(A.27) for $\mathfrak{B}_1$ and $\mathfrak{B}_2$ into (A.25), we have

$$\begin{aligned}\mathcal{I}_{\alpha,\beta,\gamma}(A,a;B,b)\lesssim & M\bar{a}^{-q_1}|B-b|^{-\gamma+q_1}\\ &+M\bar{a}^{-q_2}|B-b|^{-q_3}|B-A|^{-\gamma+q_2+q_3}\\ \lesssim & |A-a|^{-\alpha-\beta+1-q_1}|B-b|^{-\alpha-\beta+1-\gamma+q_1}\\ &+|A-a|^{-\alpha-\beta+1-q_2}|B-b|^{-\alpha-\beta+1-q_3}|A-B|^{-\gamma+q_2+q_3}.\end{aligned}$$

If we integrate r first, then we will have

$$\begin{aligned}\mathcal{I}_{\alpha,\beta,\gamma}(A,a;B,b)\lesssim & |B-b|^{-\alpha-\beta+1-q_1}|A-a|^{-\alpha-\beta+1-\gamma+q_1}\\ &+|B-b|^{-\alpha-\beta+1-q_2}|A-a|^{-\alpha-\beta+1-q_3}|A-B|^{-\gamma+q_2+q_3}.\end{aligned}$$

Thus, by choosing $q_2=q_3\in(0\vee(\gamma+\beta-1),(1-\alpha)\wedge\frac{\gamma}{2}]$, it holds

$$\begin{cases}\mathcal{I}_{\alpha,\beta,\gamma}(A,a;B,b)\lesssim & |B-b|^{-\alpha-\beta+1-q_1}|A-a|^{-\alpha-\beta+1-\gamma+q_1}\\ & +|B-b|^{-\alpha-\beta+1-q_2}|A-a|^{-\alpha-\beta+1-q_2}|A-B|^{-\gamma+2q_2}\,; \quad \text{and}\\ \mathcal{I}_{\alpha,\beta,\gamma}(A,a;B,b)\lesssim & |A-a|^{-\alpha-\beta+1-q_1}|B-b|^{-\alpha-\beta+1-\gamma+q_1}\\ & +|A-a|^{-\alpha-\beta+1-q_2}|B-b|^{-\alpha-\beta+1-q_2}|A-B|^{-\gamma+2q_2}\,.\end{cases}$$

Denote

$$\begin{aligned}\mu:=&\min(|A-a|^{-\alpha-\beta+1-q_1}|B-b|^{-\alpha-\beta+1-\gamma+q_1},\\ &\quad |B-b|^{-\alpha-\beta+1-q_1}|A-a|^{-\alpha-\beta+1-\gamma+q_1})\\ =&|A-a|^{-\alpha-\beta+1}|B-b|^{-\alpha-\beta+1}\\ &\quad \min(|A-a|^{-q_1}|B-b|^{-\gamma+q_1},|B-b|^{-q_1}|A-a|^{-\gamma+q_1})\\ \le&|A-a|^{-\alpha-\beta+1}|B-b|^{-\alpha-\beta+1}\\ &\quad \left(|A-a|^{-q_1}|B-b|^{-\gamma+q_1}\right)^{1/2}\left(|B-b|^{-q_1}|A-a|^{-\gamma+q_1}\right)^{1/2}\\ =&|A-a|^{-\alpha-\beta+1-\frac{\gamma}{2}}|B-b|^{-\alpha-\beta+1-\frac{\gamma}{2}}\,.\end{aligned}$$

Then

$$\begin{aligned}\mathcal{I}_{\alpha,\beta,\gamma}(A,a;B,b)\lesssim&\mu+|A-a|^{-\alpha-\beta+1-q_2}|B-b|^{-\alpha-\beta+1-q_2}|A-B|^{-\gamma+2q_2}\\ \le&|A-a|^{-\alpha-\beta+1-\frac{\gamma}{2}}|B-b|^{-\alpha-\beta+1-\frac{\gamma}{2}}\\ &+|A-a|^{-\alpha-\beta+1-q_2}|B-b|^{-\alpha-\beta+1-q_2}|A-B|^{-\gamma+2q_2}\,.\end{aligned}$$

This completes the proof of the lemma. □

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