ANALYSIS OF FULLY DISCRETE FINITE ELEMENT METHODS FOR 2D NAVIER–STOKES EQUATIONS WITH CRITICAL INITIAL DATA

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Abstract. First-order convergence in time and space is proved for a fully discrete semi-implicit finite element method for the two-dimensional Navier–Stokes equations with L^2 initial data in convex polygonal domains, without extra regularity assumptions or grid-ratio conditions. The proof utilises the smoothing properties of the Navier–Stokes equations in the analysis of the consistency errors, an appropriate duality argument, and the smallness of the numerical solution in the discrete $L^2(0, t_m; H^1)$ norm when t_m is smaller than some constant. Numerical examples are provided to support the theoretical analysis.

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1. INTRODUCTION

We consider the initial and boundary value problem of the incompressible Navier–Stokes (NS) equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0 & \text{in } \Omega \times (0, T], \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T], \\ u = 0 & \text{on } \partial\Omega \times (0, T], \\ u = u^0 & \text{at } \Omega \times \{0\}, \end{cases} \tag{1.1}$$

in a convex polygon $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$, up to a given time $T > 0$. It is known that for any given initial value

$$u^0 \in \dot{L}^2 = \left\{ v \in L^2(\Omega)^2 : \nabla \cdot v = 0 \text{ and } v \cdot \nu = 0 \text{ on } \partial\Omega \right\}$$

(where ν denotes the unit normal vector on $\partial\Omega$), problem (1.1) has a unique weak solution $u \in L^2\left(0, T; \dot{H}_0^1\right) \cap H^1\left(0, T; \dot{H}^{-1}\right) \hookrightarrow C\left([0, T]; \dot{L}^2\right)$, where

$$\dot{H}_0^1 = \left\{ v \in H_0^1(\Omega)^2 : \nabla \cdot v = 0 \right\} \quad \text{and} \quad \dot{H}^{-1} \text{ denotes the dual space of } \dot{H}_0^1;$$

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see Theorem 3.2 of Chapter 3 from [1].

Stability and convergence of the numerical solution to the NS equations (1.1) were studied based on different regularity assumptions on the solution and initial data. In particular, if the initial data are sufficiently smooth, *i.e.*, $u^0 \in \dot{H}_0^1 \cap H^2(\Omega)^2$ or above, then the numerical solution was proved to be convergent with optimal order for finite element and spectral Galerkin methods with different time-stepping schemes, including the Crank–Nicolson method [2], the implicit-explicit Crank–Nicolson/Adams–Bashforth method [3–5], the semi-implicit Crank–Nicolson extrapolation method [6–9], the stabilization methods based on the Crank–Nicolson method [10], the Crank–Nicolson extrapolation method [11], the backward Euler method [12] and the BDF2 method [13], the projection methods [14–16], the projection-based variational multiscale methods based on the Crank–Nicolson method [17], the fractional-step methods [18], the three-step implicit-explicit backward extrapolating scheme [19, 20], the backward differentiation formulae [21–23], a second order energy- and helicity-preserving method [24], the implicit-explicit Euler method [3], and the implicit Euler methods with different spatial discretizations [23, 25–27]. The error estimates in the above-mentioned articles do not apply to nonsmooth initial data.

If the initial value u^0 is only in $\dot{H}_0^1$, the accuracy of second- or higher-order time-stepping schemes for the NS equations is often reduced. For the semidiscrete finite element method (FEM), it was shown in [28] that the L^2-norm error bound at time t is of $O(t^{-1/2}h^2)$, where h denotes the mesh size of the finite elements. For fully discrete FEMs, the linearized Crank–Nicolson scheme was proved 1.5th-order convergent in time [29], and the semi-implicit Euler scheme with spectral Galerkin method was proved first-order convergent under a CFL condition $\tau \ln(\lambda_m/\lambda_1) \le \kappa$ in [30], where τ is the time stepsize and λ_m is the maximal eigenvalue of the Stokes operator used by the spectral method (and κ is a positive constant).

The error estimates in the above-mentioned articles all require the initial value to be strictly smoother than $\dot{L}^2$, which is known to be a critical space for the 2D NS equations, a maximal Sobolev space on which well-posedness of the 2D NS equations is proved; see [31]. As a result, the error analysis in this case turns out to be much more challenging than that for smoother initial data. To the best of our knowledge, for $\dot{L}^2$ initial data, the only error estimate for the NS equations is in [32] for a semi-implicit Euler scheme with a spectral Galerkin method in space using the eigenfunctions and eigenvalues of the continuous Stokes operator. Under a CFL condition $\tau \le \kappa \lambda_M^{-1}$, it is shown in [32] that the backward Euler spectral Galerkin method with time stepsize τ and maximal eigenvalue λ_M has an error bound $O\left(\lambda_M^{-1/2} + \tau^{1/2}\right)$ on a bounded time interval. For the backward Euler scheme with finite element methods (FEMs) in space, several stability results were proved in [33] without error estimates. Overall, first-order convergence of the implicit or semi-implicit Euler methods and error estimates of fully discrete FEMs for the NS equations with $\dot{L}^2$ initial data still remains open.

In this article, we prove the first-order convergence of a fully discrete FEM for the 2D NS equations with $\dot{L}^2$ initial data, using semi-implicit Euler method in time and an inf-sup stable pair of finite element spaces with divergence-free velocity field, *i.e.*,

$$\|u_h^n - u(t_n)\|_{L^2} \le C\left(t_n^{-1}\tau_n + t_n^{-\frac{1}{2}}h\right) \quad \text{for } t_n \in (0, T],$$

where u_h^n denotes the numerical solution at time level $t = t_n$. The main difficulty in analysing numerical methods for the NS equations with $\dot{L}^2$ initial data is to control the nonlinear terms appearing in the error analysis by very weak bounds of the numerical solution, in the presence of singular consistency errors (see Lem. 3.3). We overcome these difficulties by utilising the $O(t^m)$-weighted L^2 estimates of the mth-order time derivative and $2m$th-order spatial derivatives (as shown in Lem. 3.2) and a duality argument with variable temporal stepsizes to resolve the initial singularity in the consistency errors (as shown in Sect. 3.3). It is known that variable stepsizes can help resolve the singularity in proving convergence of exponential integrators for semilinear parabolic equations with nonsmooth initial data; see [34]. However, the error analysis for the NS equations turns out to be completely different from the error analysis for the semilinear parabolic equation due to the lack of Lipschitz continuity of the nonlinearity and the critical nature of the $\dot{L}^2$ space. This leads to the critical difficulty in the use of duality argument – the lack of stronger norms than $L^2(0, T; H^1)$ to help control the nonlinear terms, as shown in (3.41), where the first term on the right-hand side of (3.41) has to be absorbed by the left-hand side. This difficulty

is overcome by proving the smallness of the numerical solution in the discrete $L^2(0, t_m; H^1)$ norm when t_m is smaller than some constant independent of the stepsize τ and mesh size h, as shown in Lemma 3.4.

The rest of this article is organised as follows. In Section 2, we describe the finite element method and time-stepping scheme to be analysed in this article, and present the main theorem of this article. The proof of the main theorem is presented in Section 3. The proof of two technical lemmas, including the temporally weighted regularity results and the strong and weak convergence of the numerical solution, are presented in appendices.

2. The main result

For $s \in \mathbb{R}$ and $1 \le p \le \infty$, we denote by $W^{s,p}(\Omega)$ the conventional Sobolev spaces of functions defined on Ω, with abbreviation $H^s(\Omega) = W^{s,2}(\Omega)$ and $L^p(\Omega) = W^{0,p}(\Omega)$. For the simplicity of notation, we denote by $\|\cdot\|_{W^{s,p}}$ the norm of the spaces $W^{s,p}(\Omega)$, $W^{s,p}(\Omega)^2$ and $W^{s,p}(\Omega)^{2 \times 2}$, omitting the dependence on Ω and dimension.

The natural function spaces associated to incompressible flow are the divergence-free subspaces of $L^2(\Omega)^2$ and $H_0^1(\Omega)^2$, denoted by

$$X = \dot{L}^2 \quad \text{and} \quad V = \dot{H}_0^1,$$

respectively, as defined in the introduction section. We denote by P_X the L^2-orthogonal projection from $L^2(\Omega)^2$ onto $\dot{L}^2$, and denote by

$$A = P_X \Delta$$

the Stokes operator on $\dot{L}^2$ with domain $D(A) = \dot{H}_0^1 \cap H^2(\Omega)^2$, which is a self-adjoint operator on $\dot{L}^2$ and bounded above by 0 (negative definite). The Stokes operator has an extension as a bounded operator $A : \dot{H}_0^1 \to \dot{H}^{-1}$ defined by

$$(Av, w) = -\int_\Omega \nabla v \cdot \nabla w \, \mathrm{d}x \quad \forall \, v, w \in \dot{H}_0^1. \tag{2.1}$$

Correspondingly, the NS equations (1.1) can be equivalently written into the abstract form:

$$\begin{cases} \partial_t u(t) + P_X(u(t) \cdot \nabla u(t)) - Au(t) = 0 & \text{for } t \in (0, T], \\ u(0) = u^0. \end{cases} \tag{2.2}$$

Henceforth we use the common notation $(\cdot, \cdot)$ to denote the inner products of the Hilbert spaces $L^2(\Omega)$, $L^2(\Omega)^2$ and $L^2(\Omega)^{2 \times 2}$, and define

$$L_0^2(\Omega) := \left\{ v \in L^2(\Omega) : \int_\Omega v \, \mathrm{d}x = 0 \right\}.$$

For the simplicity of notation, we denote by C a generic positive constant that may be different at different occurrences and may depend on u^0, Ω and T, but is independent of the stepsize τ and mesh size h (to be introduced below).

Let $V_h \times Q_h \subset H_0^1(\Omega)^2 \times L_0^2(\Omega)$ be a pair of finite element spaces defined with respect to a shape-regular and quasi-uniform triangulation of mesh size h, with the following properties:

(P1) There exists a linear projection operator $\Pi_h : H_0^1(\Omega)^2 \to V_h$ such that
    (i) $\nabla \cdot \Pi_h v = P_{Q_h} \nabla \cdot v$ for $v \in H_0^1(\Omega)^2$, where $P_{Q_h} : L_0^2(\Omega) \to Q_h$ denotes the L^2-orthogonal projection.
    (ii) The following approximation property holds for $v \in H_0^1(\Omega)^2 \cap H^m(\Omega)^2$:

$$\|v - \Pi_h v\|_{H^s(\Omega)} \le C h^{m-s} \|v\|_{H^m(\Omega)}, \quad 0 \le s \le 1, \quad 1 \le m \le 2. \tag{2.3}$$

(P2) $\nabla \cdot v_h \in Q_h$ for $v_h \in V_h$.

The two properties above guarantee the inf-sup condition for the pair $V_h \times Q_h$ (see [35]), *i.e.*,

$$\|q_h\|_{L^2(\Omega)} \leq \sup_{v_h \in V_h \setminus \{0\}} \frac{C(\nabla \cdot v_h, q_h)}{\|v_h\|_{H^1}} \quad \forall q_h \in Q_h. \tag{2.4}$$

Since $\nabla \cdot v_h \in Q_h$ for $v_h \in V_h$, it follows that the discrete divergence-free subspace of V_h coincides with its pointwise divergence-free subspace, *i.e.*,

$$X_h := \{v_h \in V_h : (\nabla \cdot v_h, q_h) = 0 \ \forall q_h \in Q_h\} = \{v_h \in V_h : \nabla \cdot v_h = 0\}. \tag{2.5}$$

Hence,

$$V_h \subset H_0^1(\Omega)^2 \quad \text{and} \quad X_h \subset \dot{H}_0^1 \subset \dot{L}^2 = X, \quad \text{but} \quad V_h \not\subset \dot{H}_0^1.$$

There exists several finite element spaces $V_h \times Q_h$ satisfying properties (P1)–(P2). An example was constructed in [35], where V_h consists of piecewise linear polynomials plus quadratic bubble functions, and Q_h is simply the space of piecewise constants with vanishing integral over Ω. Another example is the Scott–Vogelius element space proposed in [36], where V_h is the space of continuous piecewise polynomials of degree $k \geq 4$, and Q_h is the space of discontinuous piecewise polynomials of degree $k - 1$. The Scott–Vogelius element space was proved to be inf-sup stable in [37] (under certain additional mesh requirement), and therefore property (P1) is satisfied if Π_h is simply chosen to be the Stokes–Ritz projection; see Proposition 4.18 of [38].

We consider the following semidiscrete FEM for (1.1): for given $u^0 \in \dot{L}^2$, find $(u_h, p_h) \in C^1([0,T]; V_h) \times C([0,T]; Q_h)$ such that

$$\begin{cases} (\partial_t u_h, v_h) + (u_h \cdot \nabla u_h, v_h) + (\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\nabla \cdot u_h, q_h) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_h(0) = u_h^0 := P_{X_h} u^0, \end{cases} \tag{2.6}$$

for all test functions $(v_h, q_h) \in V_h \times Q_h$ and $t \in (0, T]$, where $P_{X_h} : \dot{L}^2 \to X_h$ denotes the L^2-orthogonal projection onto X_h. This is equivalent to computing $u_h^0 \in V_h$ by the weak formulation (with an auxiliary function $\eta_h \in Q_h$)

$$\begin{cases} (u^0 - u_h^0, v_h) - (\eta_h, \nabla \cdot v_h) = 0 \\ \qquad\qquad\qquad (\nabla \cdot u_h^0, q_h) = 0 \quad \text{for all test functions } (v_h, q_h) \in V_h \times Q_h. \end{cases} \tag{2.7}$$

If we denote by $A_h : X_h \to X_h$ the discrete Stokes operator defined by

$$(A_h v_h, w_h) = -(\nabla v_h, \nabla w_h), \quad \forall v_h, w_h \in X_h. \tag{2.8}$$

Then the semi-discrete problem (2.6) is equivalent to find $u_h \in C^1([0,T]; X_h)$ such that

$$\begin{cases} \partial_t u_h + P_{X_h}(u_h \cdot \nabla u_h) - A_h u_h = 0 \quad \text{for } t \in (0, T], \\ \qquad\qquad\qquad\qquad u_h(0) = u_h^0. \end{cases} \tag{2.9}$$

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the time interval $[0, T]$ with the following stepsizes:

$$\tau_1 \sim \tau_2 \quad \text{and} \quad \tau_n = t_n - t_{n-1} \sim (t_{n-1}/T)^\alpha \tau \quad \text{for } 2 \leq n \leq N, \tag{2.10}$$

where τ is the maximal stepsize, and $\alpha \in (0, 1)$ is a parameter which determines the local refinement of the temporal grids towards $t = 0$, with "$\sim$" denoting equivalent magnitude (up to a constant multiple). The stepsizes defined in this way has the following properties:

(1) $\tau_n \sim \tau_{n-1}$ for two consecutive stepsizes.
(2) $\tau_1 = \tau^{\frac{1}{1-\alpha}}$. Hence, the starting stepsize is much smaller than the maximal stepsize. This can resolve the solution's singularity at $t = 0$.
(3) The total number of time levels is $O(T/\tau)$. Hence, the total computational cost is equivalent to using a uniform stepsize τ.

With the nonuniform stepsizes defined above, we consider the following fully discrete semi-implicit Euler FEM: Find $(u_h^n, p_h^n) \in V_h \times Q_h$, $n = 1, \ldots, N$, such that

$$\begin{cases} \left(\dfrac{u_h^n - u_h^{n-1}}{\tau_n}, v_h\right) + (\nabla u_h^n, \nabla v_h) + \left(u_h^{n-1} \cdot \nabla u_h^n, v_h\right) - (p_h^n, \nabla \cdot v_h) = 0 & \forall\, v_h \in V_h, \\ \hspace{8cm} (\nabla \cdot u_h^n, q_h) = 0 & \forall\, q_h \in Q_h, \\ \hspace{9.5cm} u_h^0 = P_{X_h} u^0. \end{cases}$$

This is equivalent to finding $u_h^n \in X_h$, $n = 1, \ldots, N$, such that

$$\begin{cases} \left(\dfrac{u_h^n - u_h^{n-1}}{\tau_n}, v_h\right) + (\nabla u_h^n, \nabla v_h) + \left(u_h^{n-1} \cdot \nabla u_h^n, v_h\right) = 0 & \forall\, v_h \in X_h, \\ \hspace{7cm} u_h^0 = P_{X_h} u^0, \end{cases} \tag{2.11}$$

which can also be written into an abstract form by using the operator A_h defined in (2.8), *i.e.*,

$$\frac{u_h^n - u_h^{n-1}}{\tau_n} - A_h u_h^n + P_{X_h}\left(u_h^{n-1} \cdot \nabla u_h^n\right) = 0 \quad \text{for } 1 \le n \le N. \tag{2.12}$$

In view of (2.5), there holds $\nabla \cdot u_h^n = 0$ and therefore $\left(u_h^{n-1} \cdot \nabla u_h^n, u_h^n\right) = 0$ similarly as the continuous solution. As a result of this identity, substituting $v_h = u_h^n$ into (2.11) immediately yields unconditional energy stability of the semidiscrete FEM.

The main result of this article is the following theorem, which provides the convergence of the fully discrete method (2.11).

Theorem 2.1. *Let $u^0 \in \dot{L}^2$ and assume that the finite element space $X_h \times Q_h$ has properties* (P1)–(P2). *Then, when the time stepsizes satisfy* (2.10) *with a fixed constant $\alpha \in (\frac{1}{2}, 1)$, the fully discrete solution given by* (2.11) *has the following error bound:*

$$\|u_h^n - u(t_n)\|_{L^2} \le C\left(t_n^{-1}\tau_n + t_n^{-\frac{1}{2}}h\right), \tag{2.13}$$

where the constant C depends only on u^0, Ω and T (independent of $t_n \in (0, T]$, τ and h).

Remark 2.2. The error estimate in (2.13) indicates that the numerical solution can be accurate when t_n is away from zero (when τ_n and h are sufficiently small), but may not be accurate when t_n is close to zero. This phenomenon is known for fully discrete finite element solutions of linear and semilinear parabolic equations with nonsmooth initial data; see Theorem 14.8 of [39]. For the NS equations with critical initial data, it is the first time such results are obtained.

The proof of Theorem 2.1 is presented in Section 3.

3. Proof of Theorem 2.1

3.1. Preliminary results

In this subsection, we present some technical inequalities and regularity results that will be used in the error analysis for the NS equations.

First, the following interpolation inequalities, which hold in general convex polygons (*cf.* [40], Thms. 5.8, 5.9) and were often used in analysis of NS equations in the literature (*e.g.*, [1], Chap. III, Sect. 3.3) and will also be used in this article:

$$\|v\|_{L^4} \leq C\|v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}} \qquad\qquad \forall\, v \in H_0^1(\Omega), \tag{3.1}$$

$$\|\nabla v\|_{L^4} \leq C\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\Delta v\|_{L^2}^{\frac{1}{2}} \qquad\qquad \forall\, v \in H_0^1(\Omega) \cap H^2(\Omega), \tag{3.2}$$

$$\|v\|_{L^\infty} \leq C\|v\|_{L^2}^{\frac{1}{2}}\|v\|_{H^2}^{\frac{1}{2}} \qquad\qquad \forall\, v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{3.3}$$

Second, the following basic properties of finite element spaces and the finite element solution to the NS equations will be used.

(1) Approximation of X_h to $\dot{H}_0^1$: for $m = 1, 2$ there holds

$$\inf_{v_h \in X_h} (\|v - v_h\|_{L^2} + h\|v - v_h\|_{H^1}) \leq Ch^m\|v\|_{H^m} \quad \forall\, v \in \dot{H}_0^1 \cap H^m(\Omega)^2. \tag{3.4}$$

Since $\Pi_h v \in X_h$ for $v \in \dot{H}_0^1$, the inequality above follows from (2.3).

(2) H^1-stability of the L^2-orthogonal projection $P_{X_h} : \dot{H}_0^1 \to X_h$:

$$\|P_{X_h} v\|_{H^1} \leq C\|v\|_{H^1} \text{ for all } v \in \dot{H}_0^1. \tag{3.5}$$

Proof. By using the triangle inequality, inverse inequality and (3.4), we have

$$\|v - P_{X_h} v\|_{H^1} \leq \inf_{v_h \in X_h} (\|v - v_h\|_{H^1} + \|P_{X_h}(v - v_h)\|_{H^1})$$

$$\leq \inf_{v_h \in X_h} (\|v - v_h\|_{H^1} + Ch^{-1}\|v - v_h\|_{L^2}) \leq C\|v\|_{H^1}.$$

$\square$

(3) Error bound of the L^2-orthogonal projection $P_{X_h} : \dot{H}_0^1 \to X_h$:

$$\|v - P_{X_h} v\|_{L^2} + h\|v - P_{X_h} v\|_{H^1} \leq Ch^s\|v\|_{H^s} \quad \forall\, v \in \dot{H}_0^1 \cap H^2(\Omega)^2, \ s = 1, 2. \tag{3.6}$$

Proof. By using the triangle inequality, equations (3.4) and (3.5), we have

$$\|v - P_{X_h} v\|_{L^2} + h\|v - P_{X_h} v\|_{H^1}$$

$$\leq \inf_{v_h \in X_h} (\|v - v_h\|_{L^2} + h\|v - v_h\|_{H^1} + \|P_{X_h}(v - v_h)\|_{L^2} + h\|P_{X_h}(v - v_h)\|_{H^1})$$

$$\leq \inf_{v_h \in X_h} (\|v - v_h\|_{L^2} + h\|v - v_h\|_{H^1}) \leq Ch^s\|v\|_{H^s}.$$

$\square$

(4) The Stokes–Ritz projection and its error bound: Let $R_{X_h} : \dot{H}_0^1 \to X_h$ be the Stokes–Ritz projection, defined by

$$(\nabla(v - R_{X_h} v), \nabla w_h) = 0 \quad \forall\, w_h \in X_h, \ v \in \dot{H}_0^1. \tag{3.7}$$

which is equivalent to finding $(R_{X_h} v, \eta_h) \in V_h \times Q_h$ such that

$$(\nabla(v - R_{X_h} v), \nabla w_h) - (\eta_h, \nabla \cdot w_h) = 0 \qquad \forall\, w_h \in V_h,$$

$$(\nabla \cdot R_{X_h} v, q_h) = 0 \qquad \forall\, q_h \in Q_h.$$

The Stokes–Ritz projection has the following error bound (*cf.* [38], Lems. 2.44, 2.45 and [38], Prop. 4.18):

$$\|v - R_{X_h} v\|_{L^2} + h\|v - R_{X_h} v\|_{H^1} \leq Ch^s\|v\|_{H^s} \quad \forall\, v \in \dot{H}_0^1 \cap H^2(\Omega)^2, \ s = 1, 2. \tag{3.8}$$

(5) The numerical solution given by the fully discrete FEM in (2.11) satisfies the following basic energy estimate:

$$\max_{1 \leq n \leq N} \|u_h^n\|_{L^2}^2 + 2 \sum_{n=1}^N \tau_n \|\nabla u_h^n\|_{L^2}^2 \leq \|u_h^0\|_{L^2}^2. \tag{3.9}$$

This can be obtained by testing (2.12) with u_h^n.

Third, since inequality (3.2) cannot hold for finite element functions (which do not have second-order partial derivatives), we would need the following discrete analogy of (3.2).

Lemma 3.1. *The following inequality holds:*

$$\|\nabla \phi_h\|_{L^4} \leq C \|\nabla \phi_h\|_{L^2}^{\frac{1}{2}} \|A_h \phi_h\|_{L^2}^{\frac{1}{2}} \qquad \forall \phi_h \in X_h. \tag{3.10}$$

Proof. To obtain a bound for $\|\nabla \phi_h\|_{L^4}$, we let $\phi \in D(A) = \dot{H}_0^1 \cap H^2(\Omega)^2$ be the solution of

$$A\phi = A_h \phi_h \quad (\text{where } A_h \phi_h \in X_h \subset X), \tag{3.11}$$

which is equivalent to the linear Stokes equation

$$\begin{cases} -\Delta \phi + \nabla \eta = -A_h \phi_h & \text{in } \Omega, \\ \nabla \cdot \phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

By the standard H^2 estimate of the linear Stokes equation (*cf.* [41], Thm. 2), the solution $\phi \in D(A)$ satisfies

$$\|\phi\|_{H^2} \leq C \|A_h \phi_h\|_{L^2}. \tag{3.12}$$

This inequality and (3.2) imply that

$$\|\nabla \phi\|_{L^4} \leq C \|\nabla \phi\|_{L^2}^{\frac{1}{2}} \|A_h \phi_h\|_{L^2}^{\frac{1}{2}}. \tag{3.13}$$

On the one hand, testing (3.11) with $v_h = \Pi_h \phi - \phi_h \in X_h$ yields the following error estimate:

$$\|\nabla(\phi - \phi_h)\|_{L^2} \leq C \|\nabla(\phi - \Pi_h \phi)\|_{L^2} \leq \begin{cases} Ch\|\phi\|_{H^2} \\ C\|\phi\|_{H^1}. \end{cases}$$

Let $S_h \subset H_0^1(\Omega)$ be the piecewise linear Lagrange finite element space. Since the Scott–Zhang interpolation operator $\Pi_S : H_0^1(\Omega) \to S_h$ satisfies the basic error estimate:

$$\|\nabla(\phi - \Pi_S \phi)\|_{L^2} \leq \begin{cases} Ch\|\phi\|_{H^2} \\ C\|\phi\|_{H^1}, \end{cases}$$

by using the triangle inequality we obtain

$$\|\nabla(\phi_h - \Pi_S \phi)\|_{L^2} \leq \|\nabla(\phi_h - \phi)\|_{L^2} + \|\nabla(\phi - \Pi_S \phi)\|_{L^2} \leq Ch\|\phi\|_{H^2}.$$

By using the inverse inequality and the two estimates above, we have

$$\|\nabla(\phi_h - \Pi_S \phi)\|_{L^4} \leq Ch^{-\frac{1}{2}} \|\nabla(\phi_h - \Pi_S \phi)\|_{L^2} \leq \begin{cases} Ch^{\frac{1}{2}}\|\phi\|_{H^2} \\ Ch^{-\frac{1}{2}}\|\phi\|_{H^1}. \end{cases}$$

The geometric average of the two different estimates on the right-hand side yields

$$\|\nabla(\phi_h - \Pi_S \phi)\|_{L^4} \leq C\|\phi\|_{H^1}^{\frac{1}{2}}\|\phi\|_{H^2}^{\frac{1}{2}} \leq C\|\phi\|_{H^1}^{\frac{1}{2}}\|A_h\phi_h\|_{L^2}^{\frac{1}{2}}, \tag{3.14}$$

where the last inequality uses (3.12). Since

$$\|\nabla\Pi_S\phi\|_{L^4} \leq C\|\nabla\phi\|_{L^4} \qquad \text{(stability of } \Pi_S \text{ in } W^{1,4}(\Omega); \text{ see [42])}$$
$$\leq C\|\phi\|_{H^1}^{\frac{1}{2}}\|A_h\phi_h\|_{L^2}^{\frac{1}{2}} \quad \text{(here we have used (3.13)).} \tag{3.15}$$

Combining the two estimates above and using the triangle inequality, we obtain

$$\|\nabla\phi_h\|_{L^4} \leq C\|\phi\|_{H^1}^{\frac{1}{2}}\|A_h\phi_h\|_{L^2}^{\frac{1}{2}}. \tag{3.16}$$

It remains to prove the following inequality:

$$\|\phi\|_{H^1} \leq C\|\phi_h\|_{H^1}. \tag{3.17}$$

Then substituting (3.17) into (3.16) yields the desired result (3.10).

In fact, testing equation (3.11) by ϕ gives

$$\|\nabla\phi\|_{L^2}^2 = -(A_h\phi_h, \phi) = (\nabla\phi_h, \nabla P_{X_h}\phi) \leq C\|\nabla\phi_h\|_{L^2}\|P_{X_h}\phi\|_{H^1} \leq C\|\nabla\phi_h\|_{L^2}\|\phi\|_{H^1}.$$

where the last inequality uses the H^1 stability of the L^2 projection P_{X_h}, as shown in (3.5). Since $\phi \in H_0^1(\Omega)^2$, it follows that $\|\phi\|_{H^1} \leq C\|\nabla\phi\|_{L^2}$. Hence, the inequality above furthermore implies (3.17). This completes the proof of Lemma 3.1. $\qquad\square$

It is well known that the unique weak solution of the NS equations (1.1) satisfies the energy equality

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \|\nabla u\|_{L^2(0,t;L^2)}^2 = \frac{1}{2}\|u^0\|_{L^2}^2, \qquad \forall\, t > 0. \tag{3.18}$$

This can be obtained by testing (1.1) with u. In addition to this basic estimate, the following regularity result for the NS equations will be used in the error analysis.

Lemma 3.2. *For any given* $u^0 \in \dot{L}^2$, *the solution of* (1.1) *satisfies the following estimate:*

$$\|\partial_t^m u(t)\|_{L^2} + t^{\frac{1}{2}}\|\partial_t^m u(t)\|_{H^1} + t\|\partial_t^m u(t)\|_{H^2} \leq C_m t^{-m} \qquad \forall\, t > 0, \; m = 0, 1, 2\ldots \tag{3.19}$$

where the constant C_m *depends on* m *and* $\|u^0\|_{L^2}$.

Since we have not found a proof of Lemma 3.2 in the literature (the regularity results in [2,43] are for $\dot{H}_0^1 \cap H^2$ and $\dot{H}_0^1$ initial data, instead of $\dot{L}^2$ initial data), we present a proof of this lemma in Appendix A.

3.2. Estimates for the consistency errors

We denote $\hat{u}_h^n = P_{X_h}u(t_n)$. By testing equation (2.2) with $v_h \in X_h \subset X = \dot{L}^2$ and using the Stokes–Ritz projection operator defined in (3.7), we have

$$(\partial_t P_{X_h}u(t), v_h) + (u(t) \cdot \nabla u(t), v_h) + (\nabla R_{X_h}u(t), \nabla v_h) = 0 \qquad \forall\, v_h \in X_h.$$

which can be written into the abstract form

$$\partial_t P_{X_h}u(t) + P_{X_h}(u(t) \cdot \nabla u(t)) - A_h R_{X_h}u(t) = 0.$$

By considering this equation at $t = t_n$, we can write down the equation satisfied by $\hat{u}_h^n$, *i.e.*,

$$\frac{\hat{u}_h^n - \hat{u}_h^{n-1}}{\tau_n} + P_{X_h}(\hat{u}_h^{n-1} \cdot \nabla \hat{u}_h^n) - A_h \hat{u}_h^n = \mathcal{E}^n + \mathcal{F}_h^n \quad \text{for } n \geq 1, \tag{3.20}$$

where the truncation errors $\mathcal{E}^n$ and $\mathcal{F}_h^n$ are given by

$$\mathcal{E}^n = \left(\frac{\hat{u}_h^n - \hat{u}_h^{n-1}}{\tau_n} - \partial_t \hat{u}_h(t_n)\right) + P_{X_h}[(u(t_{n-1}) - u(t_n)) \cdot \nabla u(t_n)] =: \mathcal{E}_1^n + \mathcal{E}_2^n, \tag{3.21}$$

$$\mathcal{F}_h^n = -A_h(\hat{u}_h^n - R_{X_h} u(t_n)) + P_{X_h}[\hat{u}_h^{n-1} \cdot \nabla(\hat{u}_h^n - u(t_n))] + P_{X_h}[(\hat{u}_h^{n-1} - u(t_{n-1})) \cdot \nabla u(t_n)] \tag{3.22}$$
$$=: \mathcal{F}_{h,1}^n + \mathcal{F}_{h,2}^n + \mathcal{F}_{h,3}^n.$$

Lemma 3.3 (Consistency errors). *If $u^0 \in \dot{L}^2$ and the stepsize in* (2.10) *is used, then for all test functions $v_h \in X_h$, the consistency errors defined in* (3.21) *and* (3.22) *satisfy the following estimates:*

$$|(\mathcal{E}^n, v_h)| \leq C\tau_n t_n^{-\frac{3}{2}} \|\nabla v_h\|_{L^2} \quad \text{for } n \geq 1, \tag{3.23}$$

$$|(\mathcal{F}_h^n, v_h)| \leq \begin{cases} C h t_n^{-1} \|\nabla v_h\|_{L^2} & \text{for } n \geq 2, \\ C(h t_n^{-1} + \tau_1 t_1^{-\frac{3}{2}})\|\nabla v_h\|_{L^2} & \text{for } n = 1. \end{cases} \tag{3.24}$$

Proof. Testing (2.2) by $v_h \in X_h$ and integrating the result over the time interval (t_{n-1}, t_n), we obtain

$$(u(t_n) - u(t_{n-1}), v_h) = -\int_{t_{n-1}}^{t_n} (\nabla u(t), \nabla v_h)\, dt - \int_{t_{n-1}}^{t_n} (u(t) \cdot \nabla u(t), v_h)\, dt. \tag{3.25}$$

For $n = 1$, the truncation errors can be estimated by using (3.25) and (2.2), and the triangle inequality:

$$\begin{aligned}
|(\mathcal{E}_1^1, v_h)| &\leq \tau_1^{-1}|(u(t_1) - u(t_0), v_h)| + |(\partial_t u(t_1), v_h)| \\
&= \tau_1^{-1}\left|\int_0^{t_1} (\nabla u(t), \nabla v_h)\, dt - \int_0^{t_1} (u(t), u(t) \cdot \nabla v_h)\, dt\right| + |(\nabla u(t_1), \nabla v_h) - (u(t_1), u(t_1) \cdot \nabla v_h)| \\
&\leq C\tau_1^{-1}\int_0^{t_1} \left[\|u(t)\|_{H^1}\|\nabla v_h\|_{L^2} + \|u(t)\|_{L^4}^2\|\nabla v_h\|_{L^2}\right] dt + C\left(\|u(t_1)\|_{H^1} + \|u(t_1)\|_{L^4}^2\right)\|\nabla v_h\|_{L^2} \\
&\leq C\tau_1^{-1}\int_0^{t_1} [\|u(t)\|_{H^1}\|\nabla v_h\|_{L^2} + \|u(t)\|_{L^2}\|u(t)\|_{H^1}\|\nabla v_h\|_{L^2}]\, dt \\
&\quad + C(\|u(t_1)\|_{H^1} + \|u(t_1)\|_{L^2}\|u(t_1)\|_{H^1})\|\nabla v_h\|_{L^2} \quad \text{(here (3.1) is used)} \\
&\leq C\tau_1^{-\frac{1}{2}}\|u\|_{L^2(0,t_1;H^1)}\left(1 + \|u\|_{L^\infty(0,t_1;L^2)}\right)\|\nabla v_h\|_{L^2} + C\tau_1^{-\frac{1}{2}}\|\nabla v_h\|_{L^2} \\
&\quad \text{(here Lem. 3.2 is used)} \\
&\leq C\tau_1^{-\frac{1}{2}}\|\nabla v_h\|_{L^2} = C\tau_1 t_1^{-\frac{3}{2}}\|\nabla v_h\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
|(\mathcal{E}_2^1, v_h)| &= |(u(t_1), (u(t_0) - u(t_1)) \cdot \nabla v_h)| \\
&\leq C\|u(t_1)\|_{L^\infty}\|u(t_1) - u(t_0)\|_{L^2}\|\nabla v_h\|_{L^2} \\
&\leq C\|u(t_1)\|_{L^2}^{\frac{1}{2}}\|u(t_1)\|_{H^2}^{\frac{1}{2}}(\|u(t_0)\|_{L^2} + \|u(t_1)\|_{L^2})\|\nabla v_h\|_{L^2} \quad \text{(here (3.3) is used)} \\
&\leq C\tau_1^{-\frac{1}{2}}\|\nabla v_h\|_{L^2} \quad \text{(here Lem. 3.2 is used)} \\
&= C\tau_1 t_1^{-\frac{3}{2}}\|\nabla v_h\|_{L^2}.
\end{aligned}$$

By combining the two estimates above, we obtain $|(\mathcal{E}^n, v_h)| \leq C\tau_n t_n^{-\frac{3}{2}} \|\nabla v_h\|_{L^2}$ for $n = 1$.

In the case $n \geq 2$, by differentiating equation (2.2) in time and testing the result by $v_h \in X_h$, we obtain

$$(\partial_{tt} u(t), v_h) + (\partial_t \nabla u(t), \nabla v_h) + (\partial_t u(t) \cdot \nabla u(t), v_h) + (u(t) \cdot \nabla \partial_t u(t), v_h) = 0 \quad \forall\, v_h \in X_h,$$

which implies that

$$
\begin{aligned}
|(\mathcal{E}_1^n, v_h)| &= \left| \left(\frac{u(t_n) - u(t_{n-1})}{\tau_n} - \partial_t u(t_n), v_h \right) \right| = \left| \left(\int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{\tau_n} \partial_{tt} u(t) \mathrm{d}t, v_h \right) \right| \\
&= \left| \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{\tau_n} (\partial_{tt} u(t), v_h) \mathrm{d}t \right| \\
&\leq C\tau_n \max_{t \in [t_{n-1}, t_n]} |(\partial_{tt} u(t), v_h)| \\
&= C\tau_n \max_{t \in [t_{n-1}, t_n]} |(\partial_t \nabla u(t), \nabla v_h) - (u(t), \partial_t u(t) \cdot \nabla v_h) - (\partial_t u(t), u(t) \cdot \nabla v_h)| \\
&\leq C\tau_n \max_{t \in [t_{n-1}, t_n]} [\|\partial_t u(t)\|_{H^1} + \|u_h(t)\|_{L^4} \|\partial_t u(t)\|_{L^4}] \|\nabla v_h\|_{L^2} \\
&\leq C\tau_n \max_{t \in [t_{n-1}, t_n]} \left[\|\partial_t u(t)\|_{H^1} + \|u(t)\|_{L^2}^{\frac{1}{2}} \|u(t)\|_{H^1}^{\frac{1}{2}} \|\partial_t u(t)\|_{L^2}^{\frac{1}{2}} \|\partial_t u(t)\|_{H^1}^{\frac{1}{2}} \right] \|\nabla v_h\|_{L^2} \\
&\leq C\tau_n t_{n-1}^{-\frac{3}{2}} \|\nabla v_h\|_{L^2} \quad \text{(here Lem. 3.2 is used)},
\end{aligned}
$$

and

$$
\begin{aligned}
|(\mathcal{E}_2^n, v_h)| &= |(u(t_n), (u(t_{n-1}) - u(t_n)) \cdot \nabla v_h)| \\
&\leq C\|u(t_n)\|_{L^4} \|u(t_{n-1}) - u(t_n)\|_{L^4} \|\nabla v_h\|_{L^2} \\
&\leq C\tau_n \max_{t \in [t_{n-1}, t_n]} [\|u(t)\|_{L^4} \|\partial_t u(t)\|_{L^4}] \|\nabla v_h\|_{L^2} \\
&\leq C\tau_n \max_{t \in [t_{n-1}, t_n]} \left[\|u(t)\|_{L^2}^{\frac{1}{2}} \|u(t)\|_{H^1}^{\frac{1}{2}} \|\partial_t u(t)\|_{L^2}^{\frac{1}{2}} \|\partial_t u(t)\|_{H^1}^{\frac{1}{2}} \right] \|\nabla v_h\|_{L^2} \\
&\leq C\tau_n t_{n-1}^{-\frac{3}{2}} \|\nabla v_h\|_{L^2} \quad \text{(here Lem. 3.2 is used)}.
\end{aligned}
$$

Since $t_{n-1}^{-\frac{3}{2}} \sim t_n^{-\frac{3}{2}}$ for $n \geq 2$, the two estimates above imply $|(\mathcal{E}^n, v_h)| \leq C\tau_n t_n^{-\frac{3}{2}} \|\nabla v_h\|_{L^2}$ for $n \geq 2$. This completes the proof of (3.23).

To prove (3.24), we consider the expressions of $\mathcal{F}_{h,j}^n$, $j = 1, 2, 3$, defined in (3.22). By using the approximation properties of the projection operators P_{X_h} and R_{X_h} in (3.6) and (3.8), we have

$$
\begin{aligned}
\left| (\mathcal{F}_{h,1}^n, v_h) \right| &= |(\nabla (P_{X_h} u(t_n) - R_{X_h} u(t_n)), \nabla v_h)| \\
&\leq \left(\|\nabla (P_{X_h} u(t_n) - u(t_n))\|_{L^2} + \|\nabla (u(t_n) - R_{X_h} u(t_n))\|_{L^2} \right) \|\nabla v_h\|_{L^2} \\
&\leq Ch\|u(t_n)\|_{H^2} \|\nabla v_h\|_{L^2} \\
&\leq Ch t_n^{-1} \|\nabla v_h\|_{L^2} \quad \text{for } v_h \in X_h \text{ and } n \geq 1,
\end{aligned}
$$

where Lemma 3.2 in deriving the last inequality.

By applying the inverse inequality $\|v_h\|_{W^{1,4}} \leq Ch^{-\frac{1}{2}} \|v_h\|_{W^{1,2}}$ and (3.1), we obtain

$$
\begin{aligned}
\left| (\mathcal{F}_{h,2}^n, v_h) \right| &= |(P_{X_h} u(t_{n-1}) \cdot \nabla (P_{X_h} u(t_n) - u(t_n)), v_h)| \\
&= |(P_{X_h} u(t_n) - u(t_n), P_{X_h} u(t_{n-1}) \cdot \nabla v_h)| \\
&\leq C\|P_{X_h} u(t_n) - u(t_n)\|_{L^4} \|P_{X_h} u(t_{n-1})\|_{L^2} \|\nabla v_h\|_{L^4}
\end{aligned}
$$

$$\leq C\|P_{X_h}u(t_n) - u(t_n)\|_{L^2}^{\frac{1}{2}}\|P_{X_h}u(t_n) - u(t_n)\|_{H^1}^{\frac{1}{2}}\|u(t_{n-1})\|_{L^2}h^{-\frac{1}{2}}\|\nabla v_h\|_{L^2}$$
$$\leq C(h^2\|u(t_n)\|_{H^2})^{\frac{1}{2}}(h\|u(t_n)\|_{H^2})^{\frac{1}{2}}\|u(t_{n-1})\|_{L^2}h^{-\frac{1}{2}}\|\nabla v_h\|_{L^2}$$
$$\leq Ch\|u(t_n)\|_{H^2}\|u(t_{n-1})\|_{L^2}\|\nabla v_h\|_{L^2}$$
$$\leq Cht_n^{-1}\|\nabla v_h\|_{L^2} \quad \text{for } v_h \in X_h \text{ and } n \geq 1,$$

where Lemma 3.2 is used in deriving the last inequality.

Similarly as the estimate for $\left|\left(\mathcal{F}_{h,2}^n, v_h\right)\right|$, we have

$$\begin{aligned}
\left|\left(\mathcal{F}_{h,3}^n, v_h\right)\right| &= |((P_{X_h}u(t_{n-1}) - u(t_{n-1})) \cdot \nabla u(t_n), v_h)| \\
&= |(u(t_n), (P_{X_h}u(t_{n-1}) - u(t_{n-1})) \cdot \nabla v_h)| \\
&\leq C\|u(t_n)\|_{L^2}\|P_{X_h}u(t_{n-1}) - u(t_{n-1})\|_{L^4}\|\nabla v_h\|_{L^4} \\
&\leq C\|u(t_n)\|_{L^2}\|P_{X_h}u(t_{n-1}) - u(t_{n-1})\|_{L^2}^{\frac{1}{2}}\|P_{X_h}u(t_{n-1}) - u(t_{n-1})\|_{H^1}^{\frac{1}{2}}h^{-\frac{1}{2}}\|\nabla v_h\|_{L^2} \\
&\leq C\|u(t_n)\|_{L^2}(h^2\|u(t_{n-1})\|_{H^2})^{\frac{1}{2}}(h\|u(t_{n-1})\|_{H^2})^{\frac{1}{2}}h^{-\frac{1}{2}}\|\nabla v_h\|_{L^2} \\
&\leq Ch\|u(t_n)\|_{L^2}\|u(t_{n-1})\|_{H^2}\|\nabla v_h\|_{L^2} \\
&\leq Cht_n^{-1}\|\nabla v_h\|_{L^2} \quad \text{for } v_h \in X_h \text{ and } n \geq 2,
\end{aligned}$$

where Lemma 3.2 and $t_{n-1} \sim t_n$ are used for $n \geq 2$. For $n = 1$ there holds

$$\begin{aligned}
\left|\left(\mathcal{F}_{h,3}^n, v_h\right)\right| &= \left|\left(u(t_1), \left(P_{X_h}u^0 - u^0\right) \cdot \nabla v_h\right)\right| \\
&\leq C\|u(t_1)\|_{L^\infty}\left\|P_{X_h}u^0 - u^0\right\|_{L^2}\|\nabla v_h\|_{L^2} \\
&\leq C\|u(t_1)\|_{L^2}^{\frac{1}{2}}\|u(t_1)\|_{H^2}^{\frac{1}{2}}\|u^0\|_{L^2}\|\nabla v_h\|_{L^2} \\
&\leq Ct_1^{-\frac{1}{2}}\|\nabla v_h\|_{L^2} \quad (\text{here Lem. 3.2 is used}) \\
&\leq C\tau_1 t_1^{-\frac{3}{2}}\|\nabla v_h\|_{L^2} \quad \text{for } v_h \in X_h \text{ and } n = 1.
\end{aligned}$$

Collecting the above estimates of $\mathcal{F}_{h,j}^n$, $j = 1, 2, 3$, for $n \geq 2$ and $n = 1$, we obtain (3.24). $\qquad\square$

3.3. Error estimates in a sufficiently small time interval $[0, T_*]$

By subtracting (2.12) from (3.20), we obtain the following equation for the error function $e_h^n = \hat{u}_h^n - u_h^n$:

$$\frac{e_h^n - e_h^{n-1}}{\tau_n} - A_h e_h^n + P_{X_h}\left(\hat{u}_h^{n-1} \cdot \nabla \hat{u}_h^n - u_h^{n-1} \cdot \nabla u_h^n\right) = \mathcal{E}^n + \mathcal{F}_h^n. \tag{3.26}$$

We first estimate $\sum_{n=1}^m \tau_n\|e_h^n\|_{L^2}^2$ by a duality argument. To this end, we denote by $\phi_h^n \in X_h$ the solution of the backward problem

$$\begin{cases} -\dfrac{\phi_h^n - \phi_h^{n-1}}{\tau_{n-1}} - A_h\phi_h^{n-1} = e_h^{n-1} & \text{for } n = 2, \ldots, m+1, \\ \phi_h^{m+1} = 0, \end{cases} \tag{3.27}$$

which satisfies the following standard energy estimate:

$$\max_{1 \leq n \leq m}\|\phi_h^n\|_{H^1}^2 + \sum_{n=1}^m \tau_n\|A_h\phi_h^n\|_{L^2}^2 \leq C\sum_{n=1}^m \tau_n\|e_h^n\|_{L^2}^2. \tag{3.28}$$

This estimate can be obtained from testing (3.27) by $-A_h\phi_h^{n-1}$ and summing up the results for $n = 2, \ldots, m+1$.

Testing (3.27) by $\tau_{n-1}e_h^{n-1}$ and summing up the results for $n = 2, \ldots, m+1$, and using discrete integration by parts in time (with $e_h^0 = \phi_h^{m+1} = 0$), we obtain

$$
\begin{aligned}
\sum_{n=2}^{m+1} \tau_{n-1} \|e_h^{n-1}\|_{L^2}^2 &= \sum_{n=2}^{m+1} \tau_{n-1}\left(e_h^{n-1}, -\frac{\phi_h^n - \phi_h^{n-1}}{\tau_{n-1}} - A_h\phi_h^{n-1}\right) \\
&= \sum_{n=1}^{m}(e_h^n, \phi_h^n) - \sum_{n=1}^{m}(e_h^{n-1}, \phi_h^n) + \sum_{n=1}^{m}\tau_n(\nabla e_h^n, \nabla\phi_h^n) \\
&= \sum_{n=1}^{m}\tau_n\left(\frac{e_h^n - e_h^{n-1}}{\tau_n} - A_h e_h^n, \phi_h^n\right).
\end{aligned}
$$

By substituting (3.26) into the inequality above, we obtain

$$
\begin{aligned}
\sum_{n=2}^{m+1}\tau_{n-1}\|e_h^{n-1}\|_{L^2}^2 &= -\sum_{n=1}^{m}\tau_n\left(\hat{u}_h^{n-1}\cdot\nabla\hat{u}_h^n - u_h^{n-1}\cdot\nabla u_h^n, \phi_h^n\right) + \sum_{n=1}^{m}\tau_n(\mathcal{E}^n, \phi_h^n) + \sum_{n=1}^{m}\tau_n(\mathcal{F}_h^n, \phi_h^n) \\
&= -\sum_{n=1}^{m}\tau_n\left(e_h^{n-1}\cdot\nabla\hat{u}_h^n + u_h^{n-1}\cdot\nabla e_h^n, \phi_h^n\right) + \sum_{n=1}^{m}\tau_n(\mathcal{E}^n, \phi_h^n) + \sum_{n=1}^{m}\tau_n(\mathcal{F}_h^n, \phi_h^n) \\
&= \sum_{n=1}^{m}\tau_n\left[\left(\hat{u}_h^n, e_h^{n-1}\cdot\nabla\phi_h^n\right) + \left(e_h^n, u_h^{n-1}\cdot\nabla\phi_h^n\right)\right] + \sum_{n=1}^{m}\tau_n(\mathcal{E}^n, \phi_h^n) + \sum_{n=1}^{m}\tau_n(\mathcal{F}_h^n, \phi_h^n), \quad (3.29)
\end{aligned}
$$

where we have used integration by parts in deriving the last equality. The last term on the right-hand side of the above equation can be estimated as follows: In the case of $n \geq 2$ we use the decomposition $\mathcal{F}_h^n = \mathcal{F}_{h,1}^n + \mathcal{F}_{h,2}^n + \mathcal{F}_{h,3}^n$ with

$$
\begin{aligned}
\left|\left(\mathcal{F}_{h,1}^n, \phi_h^n\right)\right| &= |(P_{X_h}u(t_n) - R_{X_h}u(t_n), A_h\phi_h^n)| \\
&\leq \|P_{X_h}u(t_n) - R_{X_h}u(t_n)\|_{L^2}\|A_h\phi_h^n\|_{L^2} \\
&\leq Ch\|u(t_n)\|_{H^1}\|A_h\phi_h^n\|_{L^2} \quad \text{(here (3.6) and (3.8) are used)} \\
\left|\left(\mathcal{F}_{h,2}^n, \phi_h^n\right)\right| &= |(P_{X_h}u(t_n) - u(t_n), P_{X_h}u(t_{n-1})\cdot\nabla\phi_h^n)| \\
&\leq C\|P_{X_h}u(t_n) - u(t_n)\|_{L^2}\|P_{X_h}u(t_{n-1})\|_{L^4}\|\nabla\phi_h^n\|_{L^4} \\
&\leq C\|P_{X_h}u(t_n) - u(t_n)\|_{L^2}\|P_{X_h}u(t_{n-1})\|_{L^2}^{\frac{1}{2}}\|P_{X_h}u(t_{n-1})\|_{H^1}^{\frac{1}{2}}\|\nabla\phi_h^n\|_{L^2}^{\frac{1}{2}}\|A_h\phi_h^n\|_{L^2}^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad \text{(here (3.1) and Lem. 3.1 are used)} \\
&\leq Ch\|u(t_n)\|_{H^1}\|u(t_{n-1})\|_{L^2}^{\frac{1}{2}}\|u(t_{n-1})\|_{H^1}^{\frac{1}{2}}\|\nabla\phi_h^n\|_{L^2}^{\frac{1}{2}}\|A_h\phi_h^n\|_{L^2}^{\frac{1}{2}} \quad \text{(here (3.5) is used)} \\
\left|\left(\mathcal{F}_{h,3}^n, \phi_h^n\right)\right| &= |(u(t_n), (P_{X_h}u(t_{n-1}) - u(t_{n-1}))\cdot\nabla\phi_h^n)| \\
&\leq C\|u(t_n)\|_{L^4}\|P_{X_h}u(t_{n-1}) - u(t_{n-1})\|_{L^2}\|\nabla\phi_h^n\|_{L^4} \\
&\leq C\|u(t_n)\|_{L^2}^{\frac{1}{2}}\|u(t_n)\|_{H^1}^{\frac{1}{2}}h\|u(t_{n-1})\|_{H^1}\|\nabla\phi_h^n\|_{L^2}^{\frac{1}{2}}\|A_h\phi_h^n\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

Since $\|u(t_n)\|_{L^2} \leq C$, the three estimates above imply that

$$
\begin{aligned}
|(\mathcal{F}_h^n, \phi_h^n)| &\leq Ch\|u(t_n)\|_{H^1}\|A_h\phi_h^n\|_{L^2} \\
&\quad + Ch\left(\|u(t_n)\|_{H^1}^{\frac{3}{2}} + \|u(t_{n-1})\|_{H^1}^{\frac{3}{2}}\right)\|\nabla\phi_h^n\|_{L^2}^{\frac{1}{2}}\|A_h\phi_h^n\|_{L^2}^{\frac{1}{2}} \\
&\leq \epsilon\|A_h\phi_h^n\|_{L^2}^2 + C\epsilon^{-1}h^2\left(\|u(t_n)\|_{H^1}^2 + \|u(t_{n-1})\|_{H^1}^2\right) \\
&\quad + \epsilon(\|u(t_n)\|_{H^1} + \|u(t_{n-1})\|_{H^1})\|\nabla\phi_h^n\|_{L^2}\|A_h\phi_h^n\|_{L^2} \quad \text{for } n \geq 2, \quad (3.30)
\end{aligned}
$$

where ϵ can be an arbitrary positive constant (arising from using Young's inequality). In the case $n = 1$, Lemma 3.3 implies that

$$\tau_1\big|\big(\mathcal{F}_h^1, \phi_h^1\big)\big| \leq C\tau_1^{\frac{1}{2}}\|\nabla\phi_h^1\|_{L^2} \leq C\epsilon^{-1}\tau_1 + \epsilon\|\nabla\phi_h^1\|_{L^2}^2 \leq C\epsilon^{-1}t_m^{-1}\tau_m^2 + \epsilon\|\nabla\phi_h^1\|_{L^2}^2, \tag{3.31}$$

where the last inequality is due to the stepsize choice in (2.10), which implies that

$$\tau_1 \leq C\tau^{\frac{1}{1-\alpha}} \leq C\tau^{\frac{1}{1-\alpha}-2}t_m^{1-2\alpha}\big(t_m^{2\alpha-1}\tau^2\big) \leq C\tau^{\frac{1}{1-\alpha}-2}t_m^{1-2\alpha}\big(t_m^{-1}\tau_m^2\big) \leq C\tau^{\left(\frac{1}{1-\alpha}-2\right)\alpha}t_m^{-1}\tau_m^2$$
$$\leq Ct_m^{-1}\tau_m^2 \quad \text{for } \alpha \in \big(\tfrac{1}{2}, 1\big). \tag{3.32}$$

By summing $\tau_n|(\mathcal{F}_h^n, \phi_h^n)|$ for $n = 1, \ldots, m$, and using the estimates in (3.30) and (3.31), we obtain

$$
\begin{aligned}
\sum_{n=1}^{m} \tau_n|(\mathcal{F}_h^n, \phi_h^n)| &= \tau_1\big|\big(\mathcal{F}_h^1, \phi_h^1\big)\big| + \sum_{n=2}^{m} \tau_n|(\mathcal{F}_h^n, \phi_h^n)| \\
&\leq \tau_1\big|\big(\mathcal{F}_h^1, \phi_h^1\big)\big| + \epsilon\sum_{n=2}^{m} \tau_n\|A_h\phi_h^n\|_{L^2}^2 + C\epsilon^{-1}h^2\sum_{n=2}^{m} \tau_n\Big(\|u(t_{n-1})\|_{H^1}^2 + \|u(t_n)\|_{H^1}^2\Big) \\
&\quad + \epsilon\sum_{n=2}^{m} \tau_n(\|u(t_n)\|_{H^1} + \|u(t_{n-1})\|_{H^1})\|\nabla\phi_h^n\|_{L^2}\|A_h\phi_h^n\|_{L^2} \\
&\leq C\epsilon^{-1}t_m^{-1}\tau_m^2 + \epsilon\big\|\nabla\phi_h^1\big\|_{L^2}^2 + \epsilon\sum_{n=1}^{m} \tau_n\|A_h\phi_h^n\|_{L^2}^2 + C\epsilon^{-1}h^2\sum_{n=1}^{m} \tau_n\|u(t_n)\|_{H^1}^2 \\
&\quad + \epsilon\left(\sum_{n=1}^{m} \tau_n\|u(t_n)\|_{H^1}^2\right)^{\frac{1}{2}} \max_{1\leq n\leq m}\|\phi_h^n\|_{H^1}\left(\sum_{n=1}^{m} \tau_n\|A_h\phi_h^n\|_{L^2}^2\right)^{\frac{1}{2}} \\
&\leq C\epsilon^{-1}t_m^{-1}\tau_m^2 + C\epsilon\sum_{n=1}^{m} \tau_n\|e_h^n\|_{L^2}^2 + C\epsilon^{-1}h^2\sum_{n=1}^{m} \tau_n\|u(t_n)\|_{H^1}^2 \\
&\quad + C\epsilon\left(\sum_{n=1}^{m} \tau_n\|u(t_n)\|_{H^1}^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{m} \tau_n\|e_h^n\|_{L^2}^2\right) \quad \text{(here (3.28) is used).} \tag{3.33}
\end{aligned}
$$

By using (3.23) we have

$$
\begin{aligned}
\sum_{n=1}^{m} \tau_n|(\mathcal{E}_h^n, \phi_h^n)| &\leq C\left(\sum_{n=1}^{m} \tau_n^2 t_n^{-\frac{3}{2}}\right)\max_{1\leq n\leq m}\|\phi_h^n\|_{H^1} \\
&\leq C\left(\sum_{n=1}^{m} \tau_n\tau t_n^{\alpha-\frac{3}{2}}\right)\left(\sum_{n=1}^{m} \tau_n\|e_h^n\|_{L^2}^2\right)^{\frac{1}{2}} \\
&\leq Ct_m^{\alpha-\frac{1}{2}}\tau\left(\sum_{n=1}^{m} \tau_n\|e_h^n\|_{L^2}^2\right)^{\frac{1}{2}} \quad \text{since } \alpha > \tfrac{1}{2} \\
&\leq Ct_m^{-\frac{1}{2}}\tau_m\left(\sum_{n=1}^{m} \tau_n\|e_h^n\|_{L^2}^2\right)^{\frac{1}{2}} \quad \text{here we used (2.10)} \\
&\leq C\epsilon^{-1}t_m^{-1}\tau_m^2 + \epsilon\sum_{n=1}^{m} \tau_n\|e_h^n\|_{L^2}^2. \tag{3.34}
\end{aligned}
$$

Note that

$$\sum_{n=1}^{m} \tau_n \|\hat{u}_h^n\|_{L^4}^4 \le C \sum_{n=1}^{m} \tau_n \|\hat{u}_h^n\|_{L^2}^2 \|\hat{u}_h^n\|_{H^1}^2 \le C \sum_{n=1}^{m} \tau_n \|\hat{u}_h^n\|_{H^1}^2 \le C \sum_{n=1}^{m} \tau_n \|u(t_n)\|_{H^1}^2, \quad (3.35)$$

$$\sum_{n=1}^{m} \tau_n \|u_h^{n-1}\|_{L^4}^4 \le C \sum_{n=1}^{m} \tau_n \|u_h^{n-1}\|_{L^2}^2 \|u_h^{n-1}\|_{H^1}^2 \le C \sum_{n=1}^{m} \tau_n \|u_h^{n-1}\|_{H^1}^2, \quad (3.36)$$

$$\sum_{n=1}^{m} \tau_n \|\nabla\phi_h^n\|_{L^4}^4 \le C \sum_{n=1}^{m} \tau_n \|\phi_h^n\|_{H^1}^2 \|A_h\phi_h^n\|_{L^2}^2 \le C \max_{1 \le n \le m} \|\phi_h^n\|_{H^1}^2 \sum_{n=1}^{m} \tau_n \|A_h\phi_h^n\|_{L^2}^2$$

$$\le C \left(\sum_{n=1}^{m} \tau_n \|e_h^n\|_{L^2}^2 \right)^2, \quad (3.37)$$

which are consequences of (3.1) and Lemma 3.1. Substituting (3.33)–(3.34) into (3.29) and using the three estimates above, we obtain

$$\sum_{n=1}^{m} \tau_n \|e_h^n\|_{L^2}^2 \le C \left(\sum_{n=1}^{m} \tau_n \|\hat{u}_h^n\|_{L^4}^4 \right)^{\frac{1}{4}} \left(\sum_{n=1}^{m} \tau_n \|e_h^{n-1}\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{m} \tau_n \|\nabla\phi_h^n\|_{L^4}^4 \right)^{\frac{1}{4}}$$

$$+ C \left(\sum_{n=2}^{m} \tau_n \|u_h^{n-1}\|_{L^4}^4 \right)^{\frac{1}{4}} \left(\sum_{n=2}^{m} \tau_n \|e_h^n\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_{n=2}^{m} \tau_n \|\nabla\phi_h^n\|_{L^4}^4 \right)^{\frac{1}{4}}$$

$$+ \tau_1 \left| \left(e_h^1, u_h^0 \cdot \nabla\phi_h^1 \right) \right| + \sum_{n=1}^{m} \tau_n |(\mathcal{E}_h^n, \phi_h^n)| + \sum_{n=1}^{m} \tau_n |(\mathcal{F}_h^n, \phi_h^n)|$$

$$\le C \left(\sum_{n=1}^{m} \tau_n \|\nabla u(t_n)\|_{L^2}^2 + \sum_{n=2}^{m} \tau_n \|\nabla u_h^{n-1}\|_{L^2}^2 \right)^{\frac{1}{4}} \left(\sum_{n=1}^{m} \tau_n \|e_h^n\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{m} \tau_n \|\nabla\phi_h^n\|_{L^4}^4 \right)^{\frac{1}{4}}$$

$$+ \tau_1 \left| \left(e_h^1, u_h^0 \cdot \nabla\phi_h^1 \right) \right| + \sum_{n=1}^{m} \tau_n |(\mathcal{E}_h^n, \phi_h^n)| + \sum_{n=1}^{m} \tau_n |(\mathcal{F}_h^n, \phi_h^n)|$$

$$\le C \left(\sum_{n=1}^{m} \tau_n \|u(t_n)\|_{H^1}^2 + \sum_{n=2}^{m} \tau_n \|u_h^{n-1}\|_{H^1}^2 \right)^{\frac{1}{4}} \left(\sum_{n=1}^{m} \tau_n \|e_h^n\|_{L^2}^2 \right) \quad \text{(here (3.37) is used)}$$

$$+ \tau_1 \left| \left(e_h^1, u_h^0 \cdot \nabla\phi_h^1 \right) \right| + C\epsilon^{-1} t_m^{-1} \tau_m^2 + C\epsilon \sum_{n=1}^{m} \tau_n \|e_h^n\|_{L^2}^2 + C\epsilon^{-1} h^2 \sum_{n=1}^{m} \tau_n \|u(t_n)\|_{H^1}^2$$

$$+ C\epsilon \left(\sum_{n=1}^{m} \tau_n \|u(t_n)\|_{H^1}^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{m} \tau_n \|e_h^n\|_{L^2}^2 \right). \quad (3.38)$$

The remaining term $\tau_1 \left| \left(e_h^1, u_h^0 \cdot \nabla\phi_h^1 \right) \right|$ in (3.38) can be estimated by using the basic energy estimate:

$$\left\| e_h^1 \right\|_{L^2}^2 + \tau_1 \left\| \nabla e_h^1 \right\|_{L^2}^2 \le C, \quad (3.39)$$

which is a combination of (3.9) and the regularity estimate (see Lem. 3.2)

$$\|u(t_1)\|_{L^2}^2 + \tau_1 \|\nabla u(t_1)\|_{L^2}^2 \le C$$

through the triangle inequality. By using (3.39) we have

$$\tau_1 \left| \left(e_h^1, u_h^0 \cdot \nabla\phi_h^1 \right) \right| \le C\tau_1 \left\| e_h^1 \right\|_{L^4} \|u_h^0\|_{L^2} \|\nabla\phi_h^1\|_{L^4}$$

$$\leq C\tau_1\|e_h^1\|_{L^2}^{\frac{1}{2}}\|\nabla e_h^1\|_{L^2}^{\frac{1}{2}}\|u_h^0\|_{L^2}\|\nabla\phi_h^1\|_{L^2}^{\frac{1}{2}}\|A_h\phi_h^1\|_{L^2}^{\frac{1}{2}}$$

$$\leq C\tau_1^{\frac{3}{4}}\|e_h^1\|_{L^2}^{\frac{1}{2}}\|\nabla\phi_h^1\|_{L^2}^{\frac{1}{2}}\|A_h\phi_h^1\|_{L^2}^{\frac{1}{2}} \quad \text{(using (3.39) and } \|u_h^0\|_{L^2}\leq C)$$

$$\leq C\epsilon^{-1}\tau_1\|e_h^1\|_{L^2} + \epsilon\tau_1^{\frac{1}{2}}\|\phi_h^1\|_{H^1}\|A_h\phi_h^1\|_{L^2}$$

$$\leq C\epsilon^{-3}\tau_1 + \epsilon\sum_{n=1}^m\tau_n\|e_h^n\|_{L^2}^2 + \frac{1}{2}\epsilon\big(\|\phi_h^1\|_{H^1}^2 + \tau_1\|A_h\phi_h^1\|_{L^2}^2\big)$$

$$\leq C\epsilon^{-3}t_m^{-1}\tau_m^2 + C\epsilon\sum_{n=1}^m\tau_n\|e_h^n\|_{L^2}^2, \tag{3.40}$$

where we have used (3.28) and (3.32) in deriving the last inequality. Substituting the last inequality into (3.38) and choosing a sufficiently small constant ϵ (so that the term $C\epsilon\sum_{n=1}^m\tau_n\|e_h^n\|_{L^2}^2$ can be absorbed by the left-hand side), we obtain that

$$\sum_{n=1}^m\tau_n\|e_h^n\|_{L^2}^2 \leq C\left[\left(\sum_{n=1}^m\tau_n\big(\|\nabla u(t_n)\|_{L^2}^2 + \|\nabla u_h^n\|_{L^2}^2\big)\right)^{\frac{1}{4}} + \left(\sum_{n=1}^m\tau_n\|u(t_n)\|_{H^1}^2\right)^{\frac{1}{2}}\right]\left(\sum_{n=1}^m\tau_n\|e_h^n\|_{L^2}^2\right)$$

$$+ Ct_m^{-1}\tau_m^2 + Ch^2\sum_{n=1}^m\tau_n\|u(t_n)\|_{H^1}^2. \tag{3.41}$$

The first and last terms on the right-hand side of (3.41) can be dealt with using the following lemma.

Lemma 3.4. *For any given $u^0\in\dot{L}^2$ the following result holds:*

$$\sum_{n=1}^N\tau_n\big(\|\nabla u(t_n)\|_{L^2}^2 + \|\nabla u_h^n\|_{L^2}^2\big) \leq C.$$

Furthermore, for any $\varepsilon > 0$ there exists positive constants T_ε, h_ε and τ_ε (depending on u^0, but independent of τ and h) such that for $h \leq h_\varepsilon$ and $\tau \leq \tau_\varepsilon$ the following result holds:

$$\sum_{n=1}^m\tau_n(\|\nabla u(t_n)\|_{L^2}^2 + \|\nabla u_h^n\|_{L^2}^2) \leq \varepsilon \quad \forall t_m\in(0,T_\varepsilon].$$

The proof of Lemma 3.4 is deferred to Section 3.5.

By using Lemma 3.4, there exist constants T_*, h_* and τ_* such that for $h \leq h_*$, $\tau \leq \tau_*$ and $t_m \leq T_*$ the quantity

$$\sum_{n=1}^m\tau_n\big(\|\nabla u(t_n)\|_{L^2}^2 + \|\nabla u_h^n\|_{L^2}^2\big)$$

is sufficiently small so that the first term on the right-hand side of (3.41) can be absorbed by the left hand side, and the last term on the right-hand side of (3.41) is bounded by Ch^2. Then we obtain

$$\sum_{n=1}^m\tau_n\|e_h^n\|_{L^2}^2 \leq C\big(t_m^{-1}\tau_m^2 + h^2\big) \quad \text{for } t_m\in(0,T_*]. \tag{3.42}$$

Hence, if we consider the problem in the time interval $[0,T_*]$, we obtain an error estimate (3.42) in the discrete $L^2(0,T_*;L^2)$ norm.

The error bound in (3.42) can be furthermore improved to an L^2 norm at a fixed time. To this end, we denote by $\chi(t)$ the nonnegative smooth cut-off function such that

$$\chi(t) = \begin{cases} 0 & \text{for } t \in (0, t_m/4], \\ 1 & \text{for } t \in [t_m/2, \infty), \end{cases} \quad \text{and} \quad \left|\partial_t^k \chi(t)\right| \le C t_m^{-k} \text{ for } k = 0, 1, 2, \ldots \tag{3.43}$$

which satisfies $|\partial_t \chi(t)| \le C t_m^{-1}$. Then, testing (3.26) by $\chi(t_n) e_h^n$, we obtain

$$\frac{\chi(t_n)\|e_h^n\|_{L^2}^2 - \chi(t_{n-1})\|e_h^{n-1}\|_{L^2}^2 + \chi(t_n)\|e_h^n - e_h^{n-1}\|_{L^2}^2}{2\tau_n} + \chi(t_n)\|\nabla e_h^n\|_{L^2}^2$$

$$= \chi(t_n)\big(\hat{u}_h^n, e_h^{n-1} \cdot \nabla e_h^n\big) + \chi(t_n)(\mathcal{E}^n, e_h^n) + \chi(t_n)(\mathcal{F}_h^n, e_h^n) + \frac{(\chi(t_n) - \chi(t_{n-1}))\|e_h^{n-1}\|_{L^2}^2}{2\tau_n}$$

$$\le \chi(t_n)\|\hat{u}_h^n\|_{L^4}\|e_h^{n-1}\|_{L^4}\|\nabla e_h^n\|_{L^2}$$
$$\quad + C\chi(t_n)\Big(\tau_n t_n^{-\frac{3}{2}} + h t_n^{-1}\Big)\|\nabla e_h^n\|_{L^2} + \max_{t \in [t_{n-1}, t_n]} |\partial_t \chi(t)|\|e_h^{n-1}\|_{L^2}^2 \quad (\text{Lem. 3.3 is used})$$

$$\le \chi(t_n)\|\hat{u}_h^n\|_{L^2}^{\frac{1}{2}}\|\hat{u}_h^n\|_{H^1}^{\frac{1}{2}}\|e_h^{n-1}\|_{L^2}^{\frac{1}{2}}\|\nabla e_h^{n-1}\|_{L^2}^{\frac{1}{2}}\|\nabla e_h^n\|_{L^2}$$
$$\quad + C\chi(t_n)\Big(\tau_n t_n^{-\frac{3}{2}} + h t_n^{-1}\Big)\|\nabla e_h^n\|_{L^2} + C t_m^{-1}\|e_h^{n-1}\|_{L^2}^2$$

$$\le C\chi(t_n)\|\hat{u}_h^n\|_{H^1}^2\|e_h^{n-1}\|_{L^2}^2 + C\chi(t_n)\big(\tau_n^2 t_n^{-3} + h^2 t_n^{-2}\big) + C t_m^{-1}\|e_h^{n-1}\|_{L^2}^2$$
$$\quad + \frac{1}{4}\chi(t_n)\big(\|\nabla e_h^{n-1}\|_{L^2}^2 + \|\nabla e_h^n\|_{L^2}^2\big)$$

$$\le C\|u(t_n)\|_{H^1}^2\chi(t_n)\|e_h^{n-1}\|_{L^2}^2 + C\chi(t_n)\big(\tau_n^2 t_n^{-3} + h^2 t_n^{-2}\big) + C t_m^{-1}\|e_h^{n-1}\|_{L^2}^2$$
$$\quad + \frac{1}{4}\chi(t_n)\big(\|\nabla e_h^{n-1}\|_{L^2}^2 + \|\nabla e_h^n\|_{L^2}^2\big), \tag{3.44}$$

where in the last inequality we have used $\|\hat{u}_h^n\|_{H^1} = \|P_{X_h} u(t_n)\|_{H^1} \le C\|u(t_n)\|_{H^1}$ as a result of (3.5). Absorbing the last term of (3.44) by its left-hand side and applying the discrete version of Gronwall's inequality (*cf.* [2], Lem. 5.1), we obtain

$$\max_{1 \le n \le m} \chi(t_n)\|e_h^n\|_{L^2}^2 \le \exp\left(\sum_{n=1}^m \tau_n \|u(t_n)\|_{H^1}^2\right)\left(C\sum_{n=1}^m \chi(t_n)\big(\tau_n^3 t_n^{-3} + \tau_n h^2 t_n^{-2}\big) + \sum_{n=1}^m \tau_n C t_m^{-1}\|e_h^{n-1}\|_{L^2}^2\right)$$

$$\le \exp\left(\sum_{n=1}^m \tau_n \|u(t_n)\|_{H^1}^2\right)\left(C\big(\tau_m^2 t_m^{-3} + h^2 t_m^{-2}\big)\sum_{n=1}^m \tau_n \mathbf{1}_{t_n \ge \frac{t_m}{4}} + C t_m^{-1}\sum_{n=1}^m \tau_n \|e_h^{n-1}\|_{L^2}^2\right)$$

$$\le C\tau_m^2 t_m^{-2} + C h^2 t_m^{-1}, \tag{3.45}$$

where the last inequality uses Lemma 3.4 and (3.42). Since this inequality holds for all $m \ge 1$ such that $t_m \in (0, T_*]$, it follows that

$$\|e_h^n\|_{L^2} \le C\big(t_n^{-2}\tau_n^2 + t_n^{-1}h^2\big) \quad \text{for } t_n \in (0, T_*]. \tag{3.46}$$

This proves the desired error bound in a time interval $(0, T_*]$, where T_* is a sufficiently small constant (depending on u^0 but independent of τ and h). In the next subsection, we extend the error estimate to the whole time interval $[0, T]$.

3.4. Error analysis in $[0, T]$

Let k be the maximal integer such that $t_k \in (0, T_*]$. When $\tau \le T_*/4$ there holds $t_k \ge T_*/2$ and therefore $t_k^{-1} \le C$. In this case, equation (3.46) implies

$$\left\|e_h^k\right\|_{L^2} \le C(\tau_k + h), \tag{3.47}$$

and Lemma 3.2 implies that

$$\|\partial_t^m u\|_{L^2} + \|\partial_t^m u\|_{H^1} + \|\partial_t^m u\|_{H^2} \le C_m, \qquad \forall t \in [t_k, T], \ m = 0, 1, \ldots \tag{3.48}$$

Since $t_k^{-1} \le C$, the estimates in Lemma 3.3 reduce to

$$|(\mathcal{E}^n, v_h)| + |(\mathcal{F}_h^n, v_h)| \le C(\tau_n + h)\|\nabla v_h\|_{L^2} \quad \text{for } k+1 \le n \le N, \quad v_h \in X_h.$$

Then, testing the error equation (3.26) by e_h^n, we obtain

$$
\begin{aligned}
\frac{\|e_h^n\|_{L^2}^2 - \|e_h^{n-1}\|_{L^2}^2 + \|e_h^n - e_h^{n-1}\|_{L^2}^2}{2\tau_n} &+ \|\nabla e_h^n\|_{L^2}^2 \\
&= \left(\hat{u}_h^n, e_h^{n-1} \cdot \nabla e_h^n\right) + (\mathcal{E}^n, e_h^n) + (\mathcal{F}_h^n, e_h^n) \\
&\le \|\hat{u}_h^n\|_{L^4}\|e_h^{n-1}\|_{L^4}\|\nabla e_h^n\|_{L^2} + C(\tau_n + h)\|\nabla e_h^n\|_{L^2} \\
&\le C\|\hat{u}_h^n\|_{L^2}^{\frac{1}{2}}\|\hat{u}_h^n\|_{H^1}^{\frac{1}{2}}\|e_h^{n-1}\|_{L^2}^{\frac{1}{2}}\|\nabla e_h^{n-1}\|_{L^2}^{\frac{1}{2}}\|\nabla e_h^n\|_{L^2} + C(\tau_n + h)\|\nabla e_h^n\|_{L^2} \\
&\le C\|\hat{u}_h^n\|_{H^1}^2\|e_h^{n-1}\|_{L^2}^2 + C\left(\tau_n^2 + h^2\right) + \frac{1}{4}\left(\|\nabla e_h^{n-1}\|_{L^2}^2 + \|\nabla e_h^n\|_{L^2}^2\right) \\
&\le C\|e_h^{n-1}\|_{L^2}^2 + C\left(\tau_n^2 + h^2\right) + \frac{1}{4}\left(\|\nabla e_h^{n-1}\|_{L^2}^2 + \|\nabla e_h^n\|_{L^2}^2\right) \qquad \text{for } n \ge k+1,
\end{aligned}
$$

where we have used the regularity estimate

$$\|\hat{u}_h^n\|_{H^1} = \|P_{X_h} u(t_n)\|_{H^1} \le \|u(t_n)\|_{H^1} \le C \quad \text{as a result of (3.5) and (3.48).}$$

By applying the discrete version of Gronwall's inequality, we obtain

$$\max_{k+1 \le n \le N} \|e_h^n\|_{L^2} \le C\|e_h^k\|_{L^2} + C(\tau + h) \tag{3.49}$$

which together with (3.47) yields the desired result of Theorem 2.1. $\qquad\square$

In the proof of Theorem 2.1 we have used the key technical Lemma 3.4, which is proved in the next subsection.

3.5. Proof of Lemma 3.4

Lemma 3.4 is a combination of (3.9) and the following two lemmas (Lems. 3.5 and 3.6).

Lemma 3.5. *Let $u^0 \in \dot{L}^2$ be given. Then for any $\varepsilon > 0$ there exist positive constants T_ε, h_ε and τ_ε such that for $h \le h_\varepsilon$ and $\tau \le \tau_\varepsilon$ there holds*

$$\sum_{n=1}^m \tau_n \|\nabla u_h^n\|_{L^2}^2 \le \varepsilon \qquad \forall t_m \in (0, T_\varepsilon]. \tag{3.50}$$

The constants T_ε, h_ε and τ_ε may depend on u^0 but are independent of τ and h.

Proof. Let $u_{\tau,h}(t)$ be a piecewise linear function in time, defined by

$$u_{\tau,h}(t) = \frac{t_n - t}{\tau_n} u_h^{n-1} + \frac{t - t_{n-1}}{\tau_n} u_h^n \quad \text{for } t \in (t_{n-1}, t_n].$$

We claim that

$$u_{\tau,h} \text{ converges to the unique weak solution } u \text{ weakly in } L^2(0, T; \dot{H}^1); \tag{3.51}$$

$$u_{\tau,h} \text{ converges to } u \text{ strongly in } C([T_1, T]; \dot{L}^2) \text{ for any fixed } T_1 \in (0, T); \tag{3.52}$$

$$u_{\tau,h} \text{ converges to } u \text{ strongly in } C\Big([0,T];\dot{H}^{-1}\Big). \tag{3.53}$$

The proof of (3.51)–(3.53) is presented in Appendix B.

In addition to (3.51)–(3.53), we claim that the following result holds:

$$\max_{t\in[0,T]}\Big|\|u_{\tau,h}(\cdot,t)\|_{L^2} - \|u(\cdot,t)\|_{L^2}\Big| \to 0 \text{ as } \tau,h \to 0. \tag{3.54}$$

We prove (3.54) by using the method of contradiction. If (3.54) does not hold then (3.52) implies that there exists a sequence $t_j \to 0$ and $\tau_j, h_j \to 0$ such that

$$\Big|\|u_{\tau_j,h_j}(\cdot,t_j)\|_{L^2} - \|u(\cdot,t_j)\|_{L^2}\Big| \geq \delta \text{ for } j \geq 1.$$

Since $u \in C([0,T];L^2)$ it follows that $\|u(\cdot,t_j)\|_{L^2} \to \|u^0\|_{L^2}$ and therefore

$$\Big|\|u_{\tau_j,h_j}(\cdot,t_j)\|_{L^2} - \|u^0\|_{L^2}\Big| \geq \frac{\delta}{2} \text{ for sufficiently large } j.$$

The energy inequality (3.9) implies that $\|u_{\tau_j,h_j}(\cdot,t_j)\|_{L^2} \leq \|u^0_{h_j}\|_{L^2} = \|P_{X_{h_j}}u^0\|_{L^2} \leq \|u^0\|_{L^2}$, and therefore

$$\|u_{\tau_j,h_j}(\cdot,t_j)\|_{L^2} - \|u^0\|_{L^2} \leq -\frac{\delta}{2} \text{ for sufficiently large } j. \tag{3.55}$$

Since $u_{\tau_j,h_j}(\cdot,t_j)$ converges to u^0 in $\dot{H}^{-1}$ (as a result of (3.53)), and $\|u_{\tau_j,h_j}(\cdot,t_j)\|_{L^2}$ is uniformly bounded as $j \to \infty$, it follows that $u_{\tau_j,h_j}(\cdot,t_j)$ also converges to u^0 weakly in L^2 and therefore

$$\|u^0\|_{L^2} \leq \liminf_{j\to\infty} \|u_{\tau_j,h_j}(\cdot,t_j)\|_{L^2}. \tag{3.56}$$

Substituting this into (3.55) yields that

$$\|u^0\|_{L^2} \leq \|u^0\|_{L^2} - \frac{\delta}{2}.$$

The contradiction implies that (3.54) holds.

We use the standard energy equality for the numerical solution:

$$\frac{\|u^n_h\|^2_{L^2} - \|u^{n-1}_h\|^2_{L^2}}{2\tau_n} + \frac{\tau_n}{2}\left\|\frac{u^n_h - u^{n-1}_h}{\tau_n}\right\|^2_{L^2} + \|\nabla u^n_h\|^2_{L^2} = 0, \tag{3.57}$$

which can be obtained through testing (2.12) by u^n_h. By summing up (3.57) for $n = 1, \ldots, m$, we have

$$
\begin{aligned}
\sum_{n=1}^m \tau_n \|\nabla u^n_h\|^2_{L^2} &\leq \frac{1}{2}\|u^0_h\|^2_{L^2} - \frac{1}{2}\|u^m_h\|^2_{L^2} \\
&= \frac{1}{2}\|u^0_h\|^2_{L^2} - \frac{1}{2}\|u(t_m)\|^2_{L^2} + \frac{1}{2}\Big(\|u(t_m)\|^2_{L^2} - \|u^m_h\|^2_{L^2}\Big) \\
&\leq \frac{1}{2}\|u^0\|^2_{L^2} - \frac{1}{2}\|u(t_m)\|^2_{L^2} + \frac{1}{2}\Big(\|u(t_m)\|^2_{L^2} - \|u^m_h\|^2_{L^2}\Big) \\
&= \int_0^{t_m} \|\nabla u(t)\|^2_{L^2}\,\mathrm{d}t + \frac{1}{2}\Big(\|u(t_m)\|^2_{L^2} - \|u^m_h\|^2_{L^2}\Big).
\end{aligned} \tag{3.58}
$$

First, equation (3.54) implies that there exist constants τ_ε and h_ε such that when $\tau \leq \tau_\varepsilon$ and $h \leq h_\varepsilon$ the following inequality holds:

$$\|u(t_m)\|^2_{L^2} - \|u^m_h\|^2_{L^2} < \varepsilon.$$

Second, $u \in L^2(0, T; H^1(\Omega)^2)$ implies that there exists a constant T_ε such that

$$\int_0^{T_\varepsilon} \|\nabla u(t)\|_{L^2}^2 \, dt < \frac{\varepsilon}{2}.$$

As a result, we have

$$\sum_{n=1}^m \tau_n \|\nabla u_h^n\|_{L^2}^2 \le \int_0^{t_m} \|\nabla u(t)\|_{L^2}^2 \, dt + \frac{1}{2}\big(\|u(t_m)\|_{L^2}^2 - \|u_h^m\|_{L^2}^2\big) < \varepsilon. \qquad (3.59)$$

This proves the desired result of Lemma 3.5. $\qquad \square$

Lemma 3.6. *Let $u^0 \in \dot{L}^2$ be given. Then $\sum_{n=1}^N \tau_n \|\nabla u(t_n)\|_{L^2}^2 \le C$. Moreover, for any $\varepsilon > 0$ there exist positive constants T_ε, h_ε and τ_ε such that for $h \le h_\varepsilon$ and $\tau \le \tau_\varepsilon$ there holds*

$$\sum_{n=1}^m \tau_n \|\nabla u(t_n)\|_{L^2}^2 \le \varepsilon \qquad \forall \, t_m \in (0, T_\varepsilon].$$

The constants T_ε, h_ε and τ_ε may depend on u^0 but are independent of τ and h.

Proof. By using the triangle inequality we have

$$\sum_{n=1}^m \tau_n \|\nabla u(t_n)\|_{L^2}^2 \le C \sum_{n=1}^m \tau_n \left\| \frac{2}{\tau_n} \int_{t_{n-\frac{1}{2}}}^{t_n} \nabla(u(t_n) - u(t)) \, dt \right\|_{L^2}^2 + C \sum_{n=1}^m \tau_n \left\| \frac{2}{\tau_n} \int_{t_{n-\frac{1}{2}}}^{t_n} \nabla u(t) \, dt \right\|_{L^2}^2$$

$$\le C \sum_{n=1}^m \tau_n \left\| \frac{2}{\tau_n} \int_{t_{n-\frac{1}{2}}}^{t_n} \int_t^{t_n} \nabla \partial_t u(s) \, ds \, dt \right\|_{L^2}^2 + C \sum_{n=1}^m \int_{t_{n-\frac{1}{2}}}^{t_n} \|\nabla u(t)\|_{L^2}^2 \, dt$$

$$\le C \sum_{n=1}^m \tau_n^2 \int_{t_{n-\frac{1}{2}}}^{t_n} \|\nabla \partial_t u(t)\|_{L^2}^2 \, dt + C \int_0^{t_m} \|\nabla u(t)\|_{L^2}^2 \, dt$$

$$\le C \|t \, \partial_t u(t)\|_{L^2(0,t_m;H^1)}^2 + C \|u\|_{L^2(0,t_m;H^1)}^2 \qquad (3.60)$$

where we have used $\tau_n \le 2t$ for $t \in [t_{n-\frac{1}{2}}, t_n]$. From (3.18) we see that $u \in L^2(0, T; H^1(\Omega)^2)$, which implies that

$$\|u\|_{L^2(0,t_m;H^1)}^2 < \varepsilon \quad \text{when } t_m \text{ is smaller than some constant } T_\varepsilon. \qquad (3.61)$$

In view of (3.60), it suffices to prove the following result

$$\|t \, \partial_t u\|_{L^2(0,s;H^1)}^2 \le C \|u\|_{L^2(0,s;H^1)}^2. \qquad (3.62)$$

Then substituting (3.61) and (3.62) into (3.60) yields the desired result of Lemma 3.6.

In order to prove (3.62), we differentiate (2.2) and consider the equation of $\partial_t u$, *i.e.,*

$$\partial_t^2 u - A \partial_t u = -P_X(\partial_t u \cdot \nabla u) - P_X(u \cdot \nabla \partial_t u) \quad \text{for } t \in (0, T]. \qquad (3.63)$$

Testing (3.63) by $t^2 \partial_t u$, we have

$$\frac{1}{2} t^2 \frac{d}{dt} \|\partial_t u\|_{L^2}^2 + t^2 \|\nabla \partial_t u\|_{L^2}^2 = -(\partial_t u \cdot \nabla u, t^2 \partial_t u)$$

$$= t^2(u, \partial_t u \cdot \nabla \partial_t u) \quad \text{(integration by parts)}$$

$$\leq Ct^2 \|u\|_{L^4}^2 \|\partial_t u\|_{L^4}^2 + \frac{1}{4} t^2 \|\nabla \partial_t u\|_{L^2}^2$$

$$\leq Ct^2 \|u\|_{L^2} \|u\|_{H^1} \|\partial_t u\|_{L^2} \|\nabla \partial_t u\|_{L^2} + \frac{1}{4} t^2 \|\nabla \partial_t u\|_{L^2}^2$$

$$\leq \frac{1}{2} C_* \|\nabla u\|_{L^2}^2 t^2 \|\partial_t u\|_{L^2}^2 + \frac{1}{2} t^2 \|\nabla \partial_t u\|_{L^2}^2,$$

where we have used $\|u\|_{L^2} \leq C$ in deriving the last inequality, as shown in (3.18). Since the second term on the right-hand side of the inequality above can be absorbed by its left-hand side, it follows from $t^2 \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_t u\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t}(t^2 \|\partial_t u\|_{L^2}^2) - 2t \|\partial_t u\|_{L^2}^2$ that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} t^2 \|\partial_t u\|_{L^2}^2 \right) + t^2 \|\nabla \partial_t u\|_{L^2}^2 \leq C_* \|\nabla u\|_{L^2}^2 t^2 \|\partial_t u\|_{L^2}^2 + 2C_* t \|\partial_t u\|_{L^2}^2. \tag{3.64}$$

Hence, it remains to estimate $\int_0^{t_m} t \|\partial_t u\|_{L^2}^2 \, \mathrm{d}t$ (the last term of the inequality above). To this end, we test (2.2) by $t\partial_t u$ and use (3.19). This yields that

$$\frac{1}{2} t \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla u\|_{L^2}^2 + t \|\partial_t u\|_{L^2}^2 = -(u \cdot \nabla u, t\partial_t u)$$

$$\leq Ct^2 \|u\|_{L^4}^2 \|\partial_t u\|_{L^4}^2 + \frac{1}{2} \|\nabla u\|_{L^2}^2$$

$$\leq Ct \|u\|_{L^2} \|\nabla u\|_{L^2} t \|\partial_t u\|_{L^2} \|\nabla \partial_t u\|_{L^2} + \frac{1}{2} \|\nabla u\|_{L^2}^2$$

$$\leq Ct \|\nabla u\|_{L^2} \|\nabla \partial_t u\|_{L^2} + \frac{1}{2} \|\nabla u\|_{L^2}^2 \quad \text{(here (3.19) is used)}$$

$$\leq C\sigma^{-1} \|\nabla u\|_{L^2}^2 + \sigma t^2 \|\nabla \partial_t u\|_{L^2}^2 + \frac{1}{2} \|\nabla u\|_{L^2}^2,$$

where $\sigma \in (0,1)$ is a constant arising from Young's inequality and therefore can be arbitrarily small. By using the identity $\frac{t}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla u\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{t}{2} \|\nabla u\|_{L^2}^2 \right) - \frac{1}{2} \|\nabla u\|_{L^2}^2$ we furthermore derive that

$$t \|\partial_t u\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{t}{2} \|\nabla u\|_{L^2}^2 \right) \leq (C\sigma^{-1} + 1) \|\nabla u\|_{L^2}^2 + \sigma t^2 \|\nabla \partial_t u\|_{L^2}^2. \tag{3.65}$$

Combining the two estimates above, *i.e.*, $2C_* \times$(3.65)+(3.64), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(t^2 \|\partial_t u\|_{L^2}^2 + C_* t \|\nabla u\|_{L^2}^2 \right) + t^2 \|\nabla \partial_t u\|_{L^2}^2$$

$$\leq C_* \|\nabla u\|_{L^2}^2 t^2 \|\partial_t u\|_{L^2}^2 + (2C_* C\sigma^{-1} + 2C_*) \|\nabla u\|_{L^2}^2 + 2C_* \sigma t^2 \|\nabla \partial_t u\|_{L^2}^2. \tag{3.66}$$

By choosing σ sufficiently small, the last term on the right-hand side can be absorbed by the left-hand side. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(t^2 \|\partial_t u\|_{L^2}^2 + C_* t \|\nabla u\|_{L^2}^2 \right) + t^2 \|\nabla \partial_t u\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 t^2 \|\partial_t u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2. \tag{3.67}$$

By applying Gronwall's inequality and using (3.18) we obtain

$$\max_{t \in [0,s]} \left(t^2 \|\partial_t u\|_{L^2}^2 + C_* t \|\nabla u\|_{L^2}^2 \right) + \int_0^s t^2 \|\nabla \partial_t u\|_{L^2}^2 \, \mathrm{d}t \leq \exp\left(\int_0^s C \|\nabla u\|_{L^2}^2 \mathrm{d}t \right) \int_0^s C \|\nabla u\|_{L^2}^2 \, \mathrm{d}t$$

$$\leq C \int_0^s \|\nabla u\|_{L^2}^2 \, \mathrm{d}t. \tag{3.68}$$

This proves (3.62) and completes the proof of Lemma 3.6. $\qquad\qquad\square$

Remark 3.7. In (3.29) we have used the following integration by parts:

$$-\sum_{n=1}^{m} \tau_n \left(e_h^{n-1} \cdot \nabla \hat{u}_h^n, \phi_h^n \right) = \sum_{n=1}^{m} \tau_n \left(\hat{u}_h^n, e_h^{n-1} \cdot \nabla \phi_h^n \right), \tag{3.69}$$

and then have estimated this term in (3.38) in terms of the discrete $L^4(0, T; W^{1,4})$ norm of ϕ_h^n. We need to put a sufficiently strong norm on ϕ_h^n, as strong as the $L^4(0, T; W^{1,4})$ norm, so that a sufficiently weak norm can be put on $\hat{u}_h^n e_h^{n-1}$, i.e., the discrete $L^4(0, T; L^4)$ norm on $\hat{u}_h^n$ and the discrete $L^2(0, T; L^2)$ norm on e_h^{n-1}. The integration by parts in (3.69) uses the divergence-free property of e_h^{n-1} and u_h^{n-1}. Without the divergence-free property, the extra term $\left(\nabla \cdot e_h^{n-1}, \hat{u}_h^n \phi_h^n \right)$ will appear, which contains $\nabla \cdot e_h^{n-1}$ and therefore cannot be estimated in terms of the $L^2(0, T; L^2)$ norm on e_h^{n-1}. This is the main technical reason that we consider divergence-free finite elements in the analysis of the NS equations with critical initial data.

4. NUMERICAL EXPERIMENTS

In this section we present numerical examples to support the theoretical result in Theorem 2.1. Both examples concern the incompressible NS problem

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = 0 & \text{in } \Omega \times (0, T], \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T], \\ u = 0 & \text{on } \partial\Omega \times (0, T], \\ u = u^0 & \text{at } \Omega \times \{0\}, \end{cases} \tag{4.1}$$

in the unit square $\Omega = (0, 1) \times (0, 1)$ with $T = 0.1$ and $\mu = 0.05$. The Scott–Vogelius (P_4, P_3^{-1}) finite elements are used for spatial discretization; see [36]. This finite element space has the required properties (P1)–(P2) mentioned in Section 2. All the computations are performed using the software package FEniCS (https://fenicsproject.org).

Example 4.1. Let $w = \sin(\pi x)^{\epsilon+0.5} \sin(\pi y)^{\epsilon+0.5}$ with $\epsilon = 0.01$, and consider the initial value

$$u^0 = \left(u_1^0(x, y), u_2^0(x, y) \right) := (w_y, -w_x),$$

which satisfies that

$$u^0 \in \dot{L}^2 \quad \text{but} \quad u^0 \notin H^\epsilon(\Omega)^2.$$

We solve problem (1.1) by the proposed method (2.11) and compare the numerical solutions with the reference solution given by sufficiently small stepsize and mesh size.

The time discretization errors $\left\| u_h^N - u_{h,\text{ref}}^N \right\|_{L^2(\Omega)}$ are presented in Table 1, where the reference solution $u_{h,\text{ref}}^N$ is chosen to be the numerical solution with maximal stepsize $\tau_{\text{ref}} = 1/1280$. We have used four sufficiently small spatial mesh sizes $h = 2^{-4-j}, j = 0, 1, 2, 3$, to investigate the influence of spatial discretization on the temporal discretization errors $\left\| u_h^N - u_{h,\text{ref}}^N \right\|_{L^2(\Omega)}$. From Table 1 we can see that the influence of spatial discretization is negligibly small in observing the first-order convergence in time, which is consistent with the result proved in Theorem 2.1.

The spatial discretization errors $\left\| u_h^N - u_{h,\text{ref}}^N \right\|_{L^2(\Omega)}$ are presented in Table 2, where the reference solution $u_{h,\text{ref}}^N$ is chosen to be the numerical solution with mesh size $h_{\text{ref}} = 1/128$. We have chosen several sufficiently small time stepsizes $\tau = 2^{-3-j}/10, j = 0, 1, 2, 3$ to investigate the influence of temporal discretization on the spatial discretization errors $\left\| u_h^N - u_{h,\text{ref}}^N \right\|_{L^2(\Omega)}$. From Table 2 we see that the influence of temporal discretization

TABLE 1. Example 4.1: Time discretization errors using variable stepsize with $\alpha = 0.55$.

τ	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
1/40	3.8127E–02	3.7780E–02	3.7664E–02	3.7624E–02
1/80	1.5696E–02	1.5545E–02	1.5493E–02	1.5475E–02
1/160	7.6949E–03	7.6225E–03	7.5968E–03	7.5879E–03
Convergence rate	$O(\tau^{1.03})$	$O(\tau^{1.03})$	$O(\tau^{1.03})$	$O(\tau^{1.03})$

TABLE 2. Example 4.1: Spatial discretization errors using variable stepsize with $\alpha = 0.55$.

h	$\tau = 1/80$	$\tau = 1/160$	$\tau = 1/320$	$\tau = 1/640$
1/8	5.0365E–03	4.7074E–03	4.5093E–03	4.4308E–03
1/16	1.6844E–03	1.5711E–03	1.5019E–03	1.4744E–03
1/32	5.4146E–04	5.0663E–04	4.8451E–04	4.7546E–04
Convergence rate	$O(h^{1.64})$	$O(h^{1.63})$	$O(h^{1.63})$	$O(h^{1.63})$

can be neglected compared with the spatial discretization errors, which are $O(h^{1.5})$ in the L^2 norm. This is half-order better than the result proved in Theorem 2.1. The rigorous proof of this sharper convergence rate for L^2 initial data still remains open.

Example 4.2. In the second example, we consider an initial value $u^0 = P_X w$ with

$$w = (w_1(x, y), w_2(x, y)) = (y^{\epsilon - 0.5}, x^{\epsilon - 0.5}) \quad \text{with } \epsilon = 0.01,$$

which is a function in $L^2(\Omega)^2$ but not in $H^\epsilon(\Omega)^2$. Since P_X is the L^2-orthogonal projection from $L^2(\Omega)^2$ onto $\dot{L}^2$, it follows that $u^0 \in \dot{L}^2$. But the analytical expression of u^0 is unknown. We solve problem (1.1) by the proposed method (2.11) with $u_h^0 = P_{X_h} u^0 = P_{X_h} w$, which can be computed from (2.7) with u^0 replaced by w therein. Then we compare the numerical solutions with a reference solution given by sufficiently small mesh size.

The temporal discretization errors $\left\| u_h^N - u_{h,\text{ref}}^N \right\|_{L^2(\Omega)}$ are presented in Table 3, where the reference solution $u_{h,\text{ref}}^N$ is chosen to be the numerical solution with maximal stepsize $\tau_\text{ref} = 1/1280$, and we have used several sufficiently small spatial mesh sizes $h = 2^{-4-j}, j = 0, 1, 2, 3$ to investigate the spatial discretization errors and to guarantees that the influence of spatial discretization error is negligibly small in observing the temporal convergence rates. From Table 3 we see that the temporal discretization errors are about $O(\tau)$, which is consistent with the result proved in Theorem 2.1.

The spatial discretization errors $\left\| u_h^N - u_{h,\text{ref}}^N \right\|_{L^2(\Omega)}$ are presented in Table 4, where the reference solution $u_{h,\text{ref}}^N$ is chosen to be the numerical solution with mesh size $h_\text{ref} = 1/128$. We have chosen several time stepsizes to investigate the influence of temporal discretization on the spatial discretization errors. From Table 4 we see that the influence of temporal discretization can be neglected compared with the spatial discretization errors, which are $O(h^{1.5})$ in the L^2 norm. This is better than the result proved in Theorem 2.1 (similarly as the results shown in the previous example).

TABLE 3. Example 4.2: Time discretization errors using variable stepsize with $\alpha = 0.55$.

τ	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
1/40	4.3461E–03	4.0663E–03	3.9460E–03	3.9014E–03
1/80	1.8754E–03	1.7471E–03	1.6919E–03	1.6711E–03
1/160	9.1079E–04	8.4811E–04	8.1938E–04	8.0861E–04
Convergence rate	$O(\tau^{1.04})$	$O(\tau^{1.04})$	$O(\tau^{1.05})$	$O(\tau^{1.05})$

TABLE 4. Example 4.2: Spatial discretization errors using variable stepsize with $\alpha = 0.55$.

h	$\tau = 1/80$	$\tau = 1/160$	$\tau = 1/320$	$\tau = 1/640$
1/8	1.0833E–03	8.6397E–04	7.4118E–04	6.8490E–04
1/16	4.1932E–04	3.1724E–04	2.5928E–04	2.3009E–04
1/32	1.4837E–04	1.1103E–04	8.9531E–05	7.8055E–05
Convergence rate	$O(h^{1.50})$	$O(h^{1.51})$	$O(h^{1.53})$	$O(h^{1.56})$

5. CONCLUSIONS

We have presented an error estimate for a fully discrete semi-implicit Euler finite element method for the NS equations with L^2 initial data based on the natural regularity of the solution with singularity at $t = 0$. The numerical solution is proved to be at least first-order convergent in both time and space without any CFL condition. The analysis makes use of the smoothing property of the NS equations under L^2 initial data and appropriate duality arguments to obtain a discrete $L^2(0, T_*; L^2)$ error bound for a sufficiently small constant T_* (which depends on the initial data u^0, but independent of τ and h). This is proved by utilizing Lemma 3.5, which says that the discrete $L^2(0, T_*; L^2)$ norm of the numerical solution is not only bounded but also small for sufficiently small T_*. The discrete error bound in $L^2(0, T_*; L^2)$ is furthermore improved to $L^2(0, T; L^2)$ (for a general $T > 0$) and a pointwise-in-time L^2 error bound away from $t = 0$. The extension of the analysis to the Taylor–Hood finite elements (which do not satisfy property (P2)) is also possible.

Several questions still remain open for the NS equations with nonsmooth initial data.

First, the numerical results show that 1.5th-order convergence is achieved in the space discretization. This is slightly better than the result proved in this article. The proof of this sharper convergence rate still remain open.

Second, the current numerical method and its error analysis requires variable stepsize and yields an error bound which holds only away from $t = 0$. The development of efficient numerical methods that may have some uniform temporal convergence up to $t = 0$ is still challenging. In view of the low-regularity integrators recently developed for dispersive equations [44–46] and semilinear parabolic equations [34] this is possible and worth to be considered (at least for semi-discretization in time).

Third, the development of numerical methods with higher-order convergence (*e.g.*, away from $t = 0$) for the NS equations with initial data in critical spaces is still challenging and worth to be studied.

Fourth, the error analysis of numerical methods for the three-dimensional NS equations with critical initial data in $H^{\frac{1}{2}}$ or L^3 still remains open.

APPENDIX A. PROOF OF LEMMA 3.2

We only need to prove (3.19) by assuming that the initial value is in $H^2(\Omega)^2$, provided that all the constant C_m below depends only on m and $\|u^0\|_{L^2}$ (independent of higher regularity of u^0). Then, for a nonsmooth initial value $u^0 \in \dot{L}^2$, we can choose a sequence of functions $u_n^0 \in \dot{H}_0^1 \cap H^2(\Omega)^2$, $n = 1, 2, \ldots$, converging to u^0 in $\dot{L}^2$. The solution u_n corresponding to the smooth initial value u_n^0 satisfies

$$\|\partial_t^m u_n(t)\|_{L^2} + t^{\frac{1}{2}}\|\partial_t^m u_n(t)\|_{H^1} + t\|\partial_t^m u_n(t)\|_{H^2} \leq C_m t^{-m} \quad \forall t > 0, \ m \geq 0, \tag{A.1}$$

with a constant C_m depending only on m and $\|u_n^0\|_{L^2}$ (thus independent of n). By a standard compactness argument and passing to the limit $n \to \infty$, one obtains that $u_n(t)$ converges to $u(t)$ for $t > 0$ and therefore (A.1) implies (3.19).

It remains to prove (3.19) for H^2 initial value (thus the solution is qualitatively smooth in time and H^2 in space; see [1], Rem. 3.7).

From (3.18) we immediately obtain

$$\|u\|_{L^\infty(0,\infty;L^2)} + \|u\|_{L^2(0,\infty;H^1)} \leq C. \tag{A.2}$$

To obtain higher-order estimates, we fix an arbitrary $s > 0$ and let $\chi(t)$ be a nonnegative smooth cut-off function of time (independent of x) satisfying that

$$\chi(t) = \begin{cases} 0 & \text{for } t \in (0, s/4), \\ 1 & \text{for } t \in [s/2, \infty), \end{cases} \quad \text{and} \quad |\partial_t^k \chi(t)| \leq Cs^{-k} \text{ for } k = 0, 1, 2, \ldots \tag{A.3}$$

Testing (1.1) by $\chi^2 \partial_t u$ yields

$$\begin{aligned}
\|\chi\partial_t u\|_{L^2}^2 &+ \frac{1}{2}\chi^2\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 \\
&= -(u \cdot \nabla u, \chi^2 \partial_t u) \\
&\leq C\epsilon^{-1}\|u \cdot \nabla(\chi u)\|_{L^2}^2 + \frac{\epsilon}{2}\|\chi\partial_t u\|_{L^2}^2 \\
&\leq C\epsilon^{-1}\|u\|_{L^4}^2\|\nabla(\chi u)\|_{L^4}^2 + \frac{\epsilon}{2}\|\chi\partial_t u\|_{L^2}^2 \\
&\leq C\epsilon^{-1}\|u\|_{L^2}\|u\|_{H^1}\|\nabla(\chi u)\|_{L^2}\|\Delta(\chi u)\|_{L^2} + \frac{\epsilon}{2}\|\chi\partial_t u\|_{L^2}^2 \text{ (here (3.1) and (3.2) are used)} \\
&\leq C\epsilon^{-3}\|u\|_{L^2}^2\|u\|_{H^1}^2\|\nabla(\chi u)\|_{L^2}^2 + \frac{\epsilon}{2}\|\Delta(\chi u)\|_{L^2}^2 + \frac{\epsilon}{2}\|\chi\partial_t u\|_{L^2}^2,
\end{aligned} \tag{A.4}$$

where ϵ is an arbitrary positive constant arising from using Young's inequality. Since

$$\chi^2\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\chi u)\|_{L^2}^2 - 2\chi\partial_t\chi\|\nabla u\|_{L^2}^2$$

and $|\partial_t\chi| \leq Cs^{-1}$, it follows from (A.4) that

$$\|\chi\partial_t u\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\chi u)\|_{L^2}^2 \leq C\epsilon^{-3}\|u\|_{L^2}^2\|u\|_{H^1}^2\|\nabla(\chi u)\|_{L^2}^2 + \frac{\epsilon}{2}\|\Delta(\chi u)\|_{L^2}^2 + \frac{\epsilon}{2}\|\chi\partial_t u\|_{L^2}^2 + Cs^{-1}\|\nabla u\|_{L^2}^2. \tag{A.5}$$

To estimate the term $\epsilon\|\Delta(\chi u)\|_{L^2}^2$ on the right-hand side of (A.5), we multiply (1.1) by χ and consider the resulting equation

$$\begin{cases} -\Delta(\chi u) + \nabla(\chi p) = -\chi\partial_t u - u \cdot \nabla(\chi u) & \text{in } \Omega, \\ \nabla \cdot (\chi u) = 0 & \text{in } \Omega, \\ \chi u = 0 & \text{on } \partial\Omega. \end{cases}$$

Through the standard H^2 estimate of the linear Stokes equation (*cf.* [41], Thm. 2) we obtain

$$
\begin{aligned}
\|\chi u\|_{H^2}^2 &\le C\| - \chi\partial_t u - u\cdot\nabla(\chi u)\|_{L^2}^2 \\
&\le C\|\chi\partial_t u\|_{L^2}^2 + C\|u\|_{L^4}^2\|\nabla(\chi u)\|_{L^4}^2 \\
&\le C\|\chi\partial_t u\|_{L^2}^2 + C\|u\|_{L^2}\|u\|_{H^1}\|\nabla(\chi u)\|_{L^2}\|\Delta(\chi u)\|_{L^2} \\
&\le C\|\chi\partial_t u\|_{L^2}^2 + C\|u\|_{L^2}^2\|u\|_{H^1}\|\nabla(\chi u)\|_{L^2}^2 + \frac{1}{2}\|\Delta(\chi u)\|_{L^2}^2,
\end{aligned}
\tag{A.6}
$$

where we have estimated the term $\|u\|_{L^4}^2\|\nabla(\chi u)\|_{L^4}^2$ similarly as in (A.4). The last term of (A.6) can be absorbed by the left-hand side. Then adding $\epsilon\times$(A.6) to (A.5) yields

$$
\begin{aligned}
&\|\chi\partial_t u\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\chi u)\|_{L^2}^2 + \epsilon\|\chi u\|_{H^2}^2 \\
&\qquad \le C\epsilon^{-3}\|u\|_{L^2}^2\|u\|_{H^1}^2\|\nabla(\chi u)\|_{L^2}^2 + \frac{\epsilon}{2}\|\Delta(\chi u)\|_{L^2}^2 + C\epsilon\|\chi\partial_t u\|_{L^2}^2 + Cs^{-1}\|\nabla u\|_{L^2}^2.
\end{aligned}
\tag{A.7}
$$

By choosing sufficiently small ϵ, the two terms involving $\|\Delta(\chi u)\|_{L^2}^2$ and $\|\chi\partial_t u\|_{L^2}^2$ can be absorbed by the left-hand side. Then, by using estimate $\|u\|_{L^\infty(0,\infty;L^2)} \le C$ from (A.2), we obtain

$$
\frac{1}{2}\|\chi\partial_t u\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\chi u)\|_{L^2}^2 + \frac{1}{2}\|\chi u\|_{H^2}^2 \le C\|u\|_{H^1}^2\|\nabla(\chi u)\|_{L^2}^2 + Cs^{-1}\|\nabla u\|_{L^2}^2.
\tag{A.8}
$$

Now, applying Gronwall's inequality, we have

$$
\begin{aligned}
\|\nabla(\chi u)\|_{L^\infty(0,s;L^2)}^2 + \|\chi\partial_t u\|_{L^2(0,s;L^2)}^2 + \|\chi u\|_{L^2(0,s;H^2)}^2 &\le \exp\Big(C\|u\|_{L^2(0,s;H^1)}^2\Big)Cs^{-1}\|\nabla u\|_{L^2(0,s;L^2)}^2 \\
&\le Cs^{-1},
\end{aligned}
\tag{A.9}
$$

where we have used the estimate $\|u\|_{L^2(0,\infty;H^1)} \le C$ from (A.2). Since $s > 0$ is arbitrary and $\chi(t) = 1$ for $t \ge s/2$, choosing $s = t$ in the inequality above yields that

$$
\|u\|_{L^\infty(t/2,t;H^1)} + \|\partial_t u\|_{L^2(t/2,t;L^2)} + \|u\|_{L^2(t/2,t;H^2)} \le Ct^{-\frac{1}{2}}, \quad \forall t > 0.
\tag{A.10}
$$

We consider the mathematical induction on m, assuming that

$$
\left\|u^{(j)}(t)\right\|_{L^2} + t^{\frac{1}{2}}\left(\left\|u^{(j)}(t)\right\|_{H^1} + \left\|\partial_t u^{(j)}(t)\right\|_{L^2(t/2,t;L^2)} + \left\|u^{(j)}(t)\right\|_{L^2(t/2,t;H^2)}\right) \le Ct^{-j},
\tag{A.11}
$$
$$
\forall t > 0, \; j = 0,\ldots,m-1,
$$

which holds for $m = 1$ in view of (A.2) and (A.10).

We denote $u^{(m)} = \partial_t^m u$ and differentiate (1.1) m times. This yields

$$
\begin{cases}
\partial_t u^{(m)} + \sum_{j=0}^{m}\binom{m}{j}u^{(j)}\cdot\nabla u^{(m-j)} - \Delta u^{(m)} + \nabla p^{(m)} = 0 & \text{in } \Omega \times (0,T], \\
\nabla\cdot u^{(m)} = 0 & \text{in } \Omega \times (0,T], \\
u^{(m)} = 0 & \text{on } \partial\Omega \times (0,T].
\end{cases}
\tag{A.12}
$$

Testing this equation by $u^{(m)}$ yields

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^{(m)}\right\|_{L^2}^2 + \left\|\nabla u^{(m)}\right\|_{L^2}^2 \le C\sum_{j=0}^{m}\left\|u^{(j)}\cdot\nabla u^{(m-j)}\right\|_{H^{-1}}\left\|u^{(m)}\right\|_{H_0^1}
$$

$$\leq C \sum_{j=0}^{m} \left\|u^{(j)}\right\|_{L^4}^2 \left\|u^{(m-j)}\right\|_{L^4}^2 + \frac{1}{4}\left\|\nabla u^{(m)}\right\|_{L^2}^2$$

$$\leq C \sum_{j=0}^{m} \left\|u^{(j)}\right\|_{L^2} \left\|\nabla u^{(j)}\right\|_{L^2} \left\|u^{(m-j)}\right\|_{L^2} \left\|\nabla u^{(m-j)}\right\|_{L^2} + \frac{1}{4}\left\|\nabla u^{(m)}\right\|_{L^2}^2, \quad (A.13)$$

where we have used the following fact to get the second to last inequality:

$$\left\|u^{(j)} \cdot \nabla u^{(m-j)}\right\|_{H^{-1}} = \sup_{v \in H_0^1, \|v\|_{H^1}=1} \left(u^{(j)} \cdot \nabla u^{(m-j)}, v\right)$$

$$= \sup_{v \in H_0^1, \|v\|_{H^1}=1} -\left(u^{(m-j)}, u^{(j)} \cdot \nabla v\right)$$

$$\leq \left\|u^{(j)}\right\|_{L^4} \left\|u^{(m-j)}\right\|_{L^4}.$$

Substituting (A.11) into (A.13) for $1 \leq j \leq m-1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|u^{(m)}\right\|_{L^2}^2 + \left\|\nabla u^{(m)}\right\|_{L^2}^2 \leq C\|u\|_{L^2}\|\nabla u\|_{L^2}\left\|u^{(m)}\right\|_{L^2}\left\|\nabla u^{(m)}\right\|_{L^2} + Ct^{-2m-1} + \frac{1}{4}\left\|\nabla u^{(m)}\right\|_{L^2}^2$$

$$\leq C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\left\|u^{(m)}\right\|_{L^2}^2 + Ct^{-2m-1} + \frac{1}{2}\left\|\nabla u^{(m)}\right\|_{L^2}^2.$$

Then, multiplying the inequality above by χ^2 and using $\|u(t)\|_{L^2} \leq C$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\chi u^{(m)}\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla\left(\chi u^{(m)}\right)\right\|_{L^2}^2 \leq C\|\nabla u\|_{L^2}^2\left\|\chi u^{(m)}\right\|_{L^2}^2 + C\chi(t)t^{-2m-1} + C|\partial_t\chi|\chi\left\|\partial_t u^{(m-1)}\right\|_{L^2}^2.$$

By using Gronwall's inequality, estimate (A.2) and property (A.3), we derive that

$$\left\|\chi u^{(m)}\right\|_{L^\infty(0,s;L^2)}^2 + \left\|\nabla\left(\chi u^{(m)}\right)\right\|_{L^2(0,s;L^2)}^2$$

$$\leq \exp\left(C\|\nabla u\|_{L^2(0,s;L^2)}^2\right)\int_0^s \left(\chi(t)^2 t^{-2m-1} + C|\partial_t\chi(t)|\chi(t)\left\|\partial_t u^{(m-1)}\right\|_{L^2}^2\right)dt$$

$$\leq \exp\left(C\|\nabla u\|_{L^2(0,s;L^2)}^2\right)\int_{\frac{s}{4}}^s \left(t^{-2m-1} + Cs^{-1}\left\|\partial_t u^{(m-1)}\right\|_{L^2}^2\right)dt$$

$$\leq Cs^{-2m}.$$

As a result, choosing $s = t$ in the inequality above, we have

$$\left\|u^{(m)}\right\|_{L^\infty(t/2,t;L^2)} + \left\|\nabla u^{(m)}\right\|_{L^2(t/2,t;L^2)} \leq Ct^{-m}, \quad \forall t > 0. \quad (A.14)$$

From (A.12) we know that

$$\begin{cases} -\Delta u^{(m)} + \nabla p^{(m)} = -\partial_t u^{(m)} - \sum_{j=0}^{m}\binom{m}{j}u^{(j)} \cdot \nabla u^{(m-j)}, & \text{in } \Omega, \\ \nabla \cdot u^{(m)} = 0 & \text{in } \Omega, \\ u^{(m)} = 0 & \text{on } \partial\Omega. \end{cases} \quad (A.15)$$

Through the standard H^2 estimate of linear Stokes equations (*cf.* [41], Thm. 2) we obtain

$$\left\|u^{(m)}\right\|_{H^2}^2 \le C\left\|\partial_t u^{(m)}\right\|_{L^2}^2 + C\sum_{j=0}^{m}\left\|u^{(j)}\cdot\nabla u^{(m-j)}\right\|_{L^2}^2. \tag{A.16}$$

Testing equation (A.12) by $\partial_t u^{(m)}$ gives

$$\left\|\partial_t u^{(m)}\right\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u^{(m)}\right\|_{L^2}^2 \le C\sum_{j=0}^{m}\left\|u^{(j)}\cdot\nabla u^{(m-j)}\right\|_{L^2}\left\|\partial_t u^{(m)}\right\|_{L^2}$$

$$\le C\epsilon^{-1}\sum_{j=0}^{m}\left\|u^{(j)}\cdot\nabla u^{(m-j)}\right\|_{L^2}^2 + \epsilon\left\|\partial_t u^{(m)}\right\|_{L^2}^2. \tag{A.17}$$

Summing up (A.17) and $\lambda\times$(A.16) yields

$$\left\|\partial_t u^{(m)}\right\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u^{(m)}\right\|_{L^2}^2 + \lambda\left\|\Delta u^{(m)}\right\|_{L^2}^2$$

$$\le C\epsilon^{-1}\sum_{j=0}^{m}\left\|u^{(j)}\cdot\nabla u^{(m-j)}\right\|_{L^2}^2 + (\epsilon + C\lambda)\left\|\partial_t u^{(m)}\right\|_{L^2}^2$$

$$\le C\epsilon^{-1}\sum_{j=0}^{m}\left\|u^{(j)}\right\|_{L^4}^2\left\|\nabla u^{(m-j)}\right\|_{L^4}^2 + (\epsilon + C\lambda)\left\|\partial_t u^{(m)}\right\|_{L^2}^2$$

$$\le C\epsilon^{-1}\sum_{j=0}^{m}\left\|u^{(j)}\right\|_{L^2}\left\|\nabla u^{(j)}\right\|_{L^2}\left\|\nabla u^{(m-j)}\right\|_{L^2}\left\|\Delta u^{(m-j)}\right\|_{L^2} + (\epsilon + C\lambda)\left\|\partial_t u^{(m)}\right\|_{L^2}^2,$$

where we have used (3.1) and (3.2) in deriving the last inequality. By choosing sufficiently small ϵ and λ, the term $(\epsilon + C\lambda)\left\|\partial_t u^{(m)}\right\|_{L^2}^2$ can be absorbed by the left-hand side. Then, substituting (A.11) and (A.14) into the inequality above, we obtain

$$\frac{1}{2}\left\|\partial_t u^{(m)}\right\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u^{(m)}\right\|_{L^2}^2 + \lambda\left\|\Delta u^{(m)}\right\|_{L^2}^2$$

$$\le C\|u\|_{L^2}\|\nabla u\|_{L^2}\left\|\nabla u^{(m)}\right\|_{L^2}\left\|\Delta u^{(m)}\right\|_{L^2} + C\sum_{j=1}^{m-1}t^{-j-m-1}\left\|\Delta u^{(m-j)}\right\|_{L^2}$$

$$+ C\left\|u^{(m)}\right\|_{L^2}\left\|\nabla u^{(m)}\right\|_{L^2}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}$$

$$\le C\lambda^{-1}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\left\|\nabla u^{(m)}\right\|_{L^2}^2 + \frac{\lambda}{2}\left\|\Delta u^{(m)}\right\|_{L^2}^2 + C\sum_{j=1}^{m-1}\left(t^{-2m-2} + t^{-2j}\left\|\Delta u^{(m-j)}\right\|_{L^2}^2\right)$$

$$+ C\|\nabla u\|_{L^2}^2\left\|\nabla u^{(m)}\right\|_{L^2}^2 + C\left\|u^{(m)}\right\|_{L^2}^2\|\Delta u\|_{L^2}^2$$

$$\le C\lambda^{-1}\|\nabla u\|_{L^2}^2\left\|\nabla u^{(m)}\right\|_{L^2}^2 + \frac{\lambda}{2}\left\|\Delta u^{(m)}\right\|_{L^2}^2 + C\sum_{j=1}^{m-1}\left(t^{-2m-2} + t^{-2j}\left\|\Delta u^{(m-j)}\right\|_{L^2}^2\right)$$

$$+ C\|\nabla u\|_{L^2}^2\left\|\nabla u^{(m)}\right\|_{L^2}^2 + Ct^{-2m}\|\Delta u\|_{L^2}^2.$$

After absorbing $\frac{\lambda}{2}\left\|\Delta u^{(m)}\right\|_{L^2}^2$ by the left-hand side, multiplying the last inequality by χ^2 yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\left(\chi u^{(m)}\right)\right\|_{L^2}^2 + \left\|\partial_t\left(\chi u^{(m)}\right)\right\|_{L^2}^2 + \lambda\left\|\Delta\left(\chi u^{(m)}\right)\right\|_{L^2}^2$$

$$\leq C\chi^2(t)\left[\sum_{j=1}^{m-1}\left(t^{-2m-2}+t^{-2j}\left\|\Delta u^{(m-j)}\right\|_{L^2}^2\right)+t^{-2m}\|\Delta u\|_{L^2}^2\right]$$
$$+C\|\nabla u\|_{L^2}^2\left\|\nabla\left(\chi u^{(m)}\right)\right\|_{L^2}^2+C|\partial_t\chi(t)|\chi(t)\left\|\nabla u^{(m)}\right\|_{L^2}^2+|\partial_t\chi|^2\left\|u^{(m)}\right\|_{L^2}^2.$$

Then we apply Gronwall's inequality in the time interval $[0,s]$ and using the estimate $|\partial_t\chi(t)|\leq Cs^{-1}$. This yields that

$$\left\|\nabla\left(\chi u^{(m)}\right)\right\|_{L^\infty(0,s;L^2)}^2+\left\|\partial_t\left(\chi u^{(m)}\right)\right\|_{L^2(0,s;L^2)}^2+\left\|\Delta\left(\chi u^{(m)}\right)\right\|_{L^2(0,s;L^2)}^2$$
$$\leq\exp\left(C\|\nabla u\|_{L^2(0,s;L^2)}^2\right)\int_{\frac{s}{4}}^s C\left[\sum_{j=1}^{m-1}\left(t^{-2m-2}+t^{-2j}\left\|\Delta u^{(m-j)}\right\|_{L^2}^2\right)+t^{-2m}\|\Delta u\|_{L^2}^2\right]dt$$
$$+\exp\left(C\|\nabla u\|_{L^2(0,s;L^2)}^2\right)\int_{\frac{s}{4}}^s Cs^{-1}\left\|\nabla u^{(m)}\right\|_{L^2}^2 dt$$
$$\leq Cs^{-2m-1},$$

where the last inequality uses (A.14). Since $s>0$ is arbitrary, choosing $s=t$ in the inequality above yields that

$$\left\|\nabla u^{(m)}\right\|_{L^\infty(t/2,t;L^2)}+\left\|\partial_t u^{(m)}\right\|_{L^2(t/2,t;L^2)}+\left\|\Delta u^{(m)}\right\|_{L^2(t/2,t;L^2)}\leq Ct^{-m-\frac{1}{2}}. \tag{A.18}$$

Combining (A.14) and (A.18), we have

$$\left\|u^{(m)}(t)\right\|_{L^2}+t^{\frac{1}{2}}\left(\left\|\partial_t u^{(m)}\right\|_{L^2(t/2,t;L^2)}+\|u^{(m)}(t)\|_{H^1}+\|u^{(m)}(t)\|_{L^2(t/2,t;H^2)}\right)\leq Ct^{-m},\quad\forall t>0. \tag{A.19}$$

This completes the mathematical induction on (A.11). Hence, equation (A.18) holds for all m.

By substituting estimates (A.10) and (A.19) into h(A.6) and considering $m=1$, we furthermore derive that

$$\|u(t)\|_{H^2}\leq C\|\partial_t u(t)\|_{L^2}+C\|u(t)\|_{L^2}\|u(t)\|_{H^1}^2\leq Ct^{-1},\quad\forall t>0. \tag{A.20}$$

From (A.16) we also obtain that

$$\left\|\Delta u^{(m)}\right\|_{L^2}\leq C\left\|u^{(m+1)}\right\|_{L^2}+C\sum_{j=0}^m\left\|u^{(j)}\cdot\nabla u^{(m-j)}\right\|_{L^2}$$
$$\leq Ct^{-m-1}+C\sum_{j=0}^m\left\|u^{(j)}\right\|_{L^4}\left\|\nabla u^{(m-j)}\right\|_{L^4}$$
$$\leq Ct^{-m-1}+C\sum_{j=0}^m\left\|u^{(j)}\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla u^{(j)}\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla u^{(m-j)}\right\|_{L^2}^{\frac{1}{2}}\left\|\Delta u^{(m-j)}\right\|_{L^2}^{\frac{1}{2}}$$
$$\leq Ct^{-m-1}+C\sum_{j=0}^m t^{-\frac{j+m+1}{2}}\left\|\Delta u^{(m-j)}\right\|_{L^2}^{\frac{1}{2}}$$
$$\leq Ct^{-m-1}+C\sum_{j=1}^m t^{-\frac{j+m+1}{2}}\left\|\Delta u^{(m-j)}\right\|_{L^2}^{\frac{1}{2}}+Ct^{-\frac{m+1}{2}}\left\|\Delta u^{(m)}\right\|_{L^2}^{\frac{1}{2}}.$$

Assuming that $\left\|\Delta u^{(j)}\right\|_{L^2}\leq Ct^{-j-1}$ for $j=0,\dots,m-1$ (which holds for $m=1$ in view of (A.20)), the last inequality furthermore implies that

$$\left\|\Delta u^{(m)}\right\|_{L^2}\leq Ct^{-m-1}. \tag{A.21}$$

By mathematical induction, equation (A.21) holds for all $m \geq 0$.

Combining (A.19) and (A.21), we obtain the desired result of Lemma 3.2. $\qquad \square$

Appendix B. Proof of (3.51)–(3.53)

From (2.12) we see that

$$
\begin{aligned}
\|A_h u_h^n\|_{L^2} &\leq \left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2} + \|u_h^{n-1}\|_{L^4} \|\nabla u_h^n\|_{L^4} \\
&\leq \left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2} + \|u_h^{n-1}\|_{L^2}^{\frac{1}{2}} \|u_h^{n-1}\|_{H^1}^{\frac{1}{2}} \|u_h^n\|_{H^1}^{\frac{1}{2}} \|A_h u_h^n\|_{L^2}^{\frac{1}{2}} \\
&\qquad \text{(here (3.1) and Lem. 3.1 are used)} \\
&\leq \left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2} + \frac{1}{2}\|u_h^{n-1}\|_{L^2} (\|u_h^{n-1}\|_{H^1}^2 + \|u_h^n\|_{H^1}^2) + \frac{1}{2}\|A_h u_h^n\|_{L^2}.
\end{aligned}
$$

As a result, we have

$$
\|A_h u_h^n\|_{L^2} \leq 2\left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2} + \|u_h^{n-1}\|_{L^2} \left(\|u_h^{n-1}\|_{H^1}^2 + \|u_h^n\|_{H^1}^2 \right). \tag{B.1}
$$

Testing (2.12) by $(u_h^n - u_h^{n-1})/\tau_n$ yields

$$
\begin{aligned}
&\left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2}^2 + \frac{\|\nabla u_h^n\|_{L^2}^2 - \|\nabla u_h^{n-1}\|_{L^2}^2}{2\tau_n} + \frac{\tau_n}{2}\left\| \frac{\nabla \left(u_h^n - u_h^{n-1} \right)}{\tau_n} \right\|_{L^2}^2 \\
&= -\left(u_h^{n-1} \cdot \nabla u_h^n, \frac{u_h^n - u_h^{n-1}}{\tau_n} \right) \\
&\leq \|u_h^{n-1}\|_{L^4} \|\nabla u_h^n\|_{L^4} \left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2} \\
&\leq \|u_h^{n-1}\|_{L^2}^{\frac{1}{2}} \|u_h^{n-1}\|_{H^1}^{\frac{1}{2}} \|u_h^n\|_{H^1}^{\frac{1}{2}} \|A_h u_h^n\|_{L^2}^{\frac{1}{2}} \left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2} \\
&\qquad \text{(again, Eq. (3.1) and Lem. 3.1 are used)} \\
&\leq C\|u_h^{n-1}\|_{L^2}^{\frac{1}{2}} \|u_h^{n-1}\|_{H^1}^{\frac{1}{2}} \|u_h^n\|_{H^1}^{\frac{1}{2}} \left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2}^{\frac{3}{2}} + C\|u_h^{n-1}\|_{L^2} \|u_h^{n-1}\|_{H^1}^{\frac{3}{2}} \|u_h^n\|_{H^1}^{\frac{1}{2}} \left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2} \\
&\quad + C\|u_h^{n-1}\|_{L^2} \|u_h^{n-1}\|_{H^1}^{\frac{1}{2}} \|u_h^n\|_{H^1}^{\frac{3}{2}} \left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2} \qquad \text{(here (B.1) is used)} \\
&\leq C\|u_h^{n-1}\|_{L^2}^2 \left(\|u_h^{n-1}\|_{H^1}^4 + \|u_h^n\|_{H^1}^4 \right) + \frac{1}{2}\left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2}^2.
\end{aligned}
$$

The last term can be absorbed by the left-hand side. Then, multiplying the result by the smooth cut-off function $\chi(t_n)$ in (3.43), and using the estimate $\max_{1 \leq n \leq N} \|u_h^n\|_{L^2} \leq C$, we obtain

$$
\begin{aligned}
&\chi(t_n)\left\| \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2}^2 + \frac{\chi(t_n)\|\nabla u_h^n\|_{L^2}^2 - \chi(t_{n-1})\|\nabla u_h^{n-1}\|_{L^2}^2}{\tau_n} \\
&\leq C\chi(t_n)\left(\|u_h^{n-1}\|_{H^1}^4 + \|u_h^n\|_{H^1}^4 \right) + \frac{\chi(t_n) - \chi(t_{n-1})}{\tau_n}\|\nabla u_h^{n-1}\|_{L^2}^2
\end{aligned}
$$

$$\leq C\big\|\nabla u_h^{n-1}\big\|_{L^2}^2\chi(t_n)\big\|\nabla u_h^{n-1}\big\|_{L^2}^2 + C\|\nabla u_h^n\|_{L^2}^2\chi(t_n)\|\nabla u_h^n\|_{L^2}^2 + Ct_m^{-1}\big\|\nabla u_h^{n-1}\big\|_{L^2}^2$$

for $n \geq 2$ and $m \geq 4$ (so $\chi(t_1) = 0$). By applying Gronwall's inequality, we obtain

$$\sum_{n=2}^N \tau_n \chi(t_n)\left\|\frac{u_h^n - u_h^{n-1}}{\tau_n}\right\|_{L^2}^2 + \max_{2 \leq n \leq N} \chi(t_n)\|\nabla u_h^n\|_{L^2}^2$$

$$\leq \exp\left(C\sum_{n=2}^N \tau_n\Big(\big\|\nabla u_h^{n-1}\big\|_{L^2}^2 + \|\nabla u_h^n\|_{L^2}^2\Big)\right)Ct_m^{-1}\sum_{n=2}^N \tau_n\big\|\nabla u_h^{n-1}\big\|_{L^2}^2$$

$$\leq Ct_m^{-1}, \quad \text{where we have used the basic energy estimate (3.9).}$$

Substituting this into (B.1), we also obtain

$$\sum_{n=2}^N \tau_n \chi(t_n)\|A_h u_h^n\|_{L^2}^2 \leq Ct_m^{-1}.$$

To summarize, the two estimates above imply that

$$\sum_{n=[\frac{m}{2}]+1}^m \tau_n\left(\|A_h u_h^n\|_{L^2}^2 + \left\|\frac{u_h^n - u_h^{n-1}}{\tau_n}\right\|_{L^2}^2\right) + \max_{[\frac{m}{2}]<n\leq m}\|\nabla u_h^n\|_{L^2}^2 \leq Ct_m^{-1} \quad \text{for } 4 \leq m \leq N. \tag{B.2}$$

Let $u_{\tau,h}(t)$ be a piecewise linear function in time, defined by

$$u_{\tau,h}(t) = \frac{t_n - t}{\tau_n}u_h^{n-1} + \frac{t - t_{n-1}}{\tau_n}u_h^n \quad \text{for } t \in (t_{n-1}, t_n].$$

Let $u_{\tau,h}^+(t)$ and $u_{\tau,h}^-(t)$ be piecewise constant functions in time, defined by

$$u_{\tau,h}^+(t) = u_h^n \quad \text{and} \quad u_{\tau,h}^-(t) = u_h^{n-1} \quad \text{for } t \in (t_{n-1}, t_n].$$

Then (3.9) and (B.2) imply that the following quantities remain bounded as $\tau, h \to 0$:

$$\|u_{\tau,h}\|_{L^\infty(0,T;L^2)} + \|u_{\tau,h}\|_{L^2(0,T;H^1)} + \big\|u_{\tau,h}^\pm\big\|_{L^\infty(0,T;L^2)} + \big\|u_{\tau,h}^\pm\big\|_{L^2(0,T;H^1)} \leq C, \tag{B.3}$$

$$\|\partial_t u_{\tau,h}\|_{L^2(T_1,T_2;L^2)} + \|A_h u_{\tau,h}\|_{L^2(T_1,T_2;L^2)} + \|u_{\tau,h}\|_{L^\infty(T_1,T_2;H^1)} \leq C, \tag{B.4}$$

$$\big\|A_h u_{\tau,h}^\pm\big\|_{L^2(T_1,T_2;L^2)} + \big\|u_{\tau,h}^\pm\big\|_{L^\infty(T_1,T_2;H^1)} \leq C, \tag{B.5}$$

for arbitrary fixed constants T_1 and T_2 such that $0 < T_1 < T_2 \leq T$. Since

$$L^\infty\big(0,T;L^2(\Omega)^2\big) \cap L^2\big(0,T;H^1(\Omega)^2\big) \hookrightarrow L^4\big(0,T;L^4(\Omega)^2\big),$$

from (2.12) we also derive that

$$\begin{aligned}
\|\partial_t u_{\tau,h}\|_{L^2(0,T;\dot{H}^{-1})} &\leq \big\|A_h u_{\tau,h}^+\big\|_{L^2(0,T;\dot{H}^{-1})} + C\Big\|P_{X_h}\big(u_{\tau,h}^- \cdot \nabla u_{\tau,h}^+\big)\Big\|_{L^2(0,T;\dot{H}^{-1})} \\
&\leq C\big\|u_{\tau,h}^+\big\|_{L^2(0,T;H^1)} + C\big\|u_{\tau,h}^- \cdot \nabla u_{\tau,h}^+\big\|_{L^2(0,T;H^{-1})} \\
&\leq C\big\|u_{\tau,h}^+\big\|_{L^2(0,T;H^1)} + C\big\|u_{\tau,h}^-\big\|_{L^4(0,T;L^4)}\big\|u_{\tau,h}^+\big\|_{L^4(0,T;L^4)} \\
&\leq C.
\end{aligned} \tag{B.6}$$

From (B.3) and (B.6) we see that $u_{\tau,h}$ is uniformly (with respect to τ and h) bounded in $L^\infty\big(0,T;L^2(\Omega)^2\big) \cap L^2\big(0,T;\dot{H}_0^1\big) \cap H^1\big(0,T;\dot{H}^{-1}\big) \hookrightarrow L^4\big(0,T;L^4(\Omega)^2\big)$, compactly embedded into $L^3\big(0,T;L^3(\Omega)^2\big)$ (see the Aubin–Lions–Simon theorem in [47], Thm. 7). In the meantime, $u_{\tau,h}$ is uniformly bounded in $H^1\big(T_1,T_2;L^2(\Omega)^2\big) \cap L^\infty\big(T_1,T_2;H^1(\Omega)^2\big)$, which is compactly embedded into $C\big([T_1,T_2];L^2(\Omega)^2\big)$ (*cf.* [47], Thm. 5) . As a result, for any sequence $(\tau_j,h_j) \to (0,0)$ there exists a subsequence, also denoted by (τ_j,h_j) for the simplicity of notation, such that

$$
\begin{aligned}
u_{\tau_j,h_j} &\to u \quad \text{weakly* in } L^\infty\big(0,T;L^2(\Omega)^2\big), \\
u_{\tau_j,h_j} &\to u \quad \text{weakly in } L^2\big(0,T;\dot{H}_0^1\big), \\
u_{\tau_j,h_j} &\to u \quad \text{strong in } L^3(0,T;L^3(\Omega)^2), \\
u_{\tau_j,h_j} &\to u \quad \text{weakly in } H^1\big(T_1,T_2;L^2(\Omega)^2\big) \quad \text{for arbitrary } 0<T_1<T_2\le T, \\
u_{\tau_j,h_j} &\to u \quad \text{weakly* in } L^\infty(T_1,T_2;H^1(\Omega)^2) \quad \text{for arbitrary } 0<T_1<T_2\le T, \\
u_{\tau_j,h_j} &\to u \quad \text{strongly in } C\big([T_1,T_2];L^2(\Omega)^2\big) \quad \text{for arbitrary } 0<T_1<T_2\le T,
\end{aligned}
\tag{B.7}
$$

for some function

$$
u \in L^\infty\big(0,T;L^2(\Omega)^2\big) \cap L^2\big(0,T;\dot{H}_0^1\big) \cap H^1\big(T_1,T_2;L^2(\Omega)^2\big) \cap L^\infty\big(T_1,T_2;H^1(\Omega)^2\big)
$$
$$
\hookrightarrow C^{\frac{1}{2}}\big([T_1,T_2];L^2(\Omega)^2\big).
$$

From (B.3) we see that the set of functions $\{u_{\tau,h}(\cdot,t) : t \in [0,T]\}$ is uniformly (with respect to τ and h) bounded in $\dot{L}^2$ and therefore precompact in $\dot{H}^{-1}$. From (B.6) we also know that $u_{\tau,h}$ is uniformly bounded in $H^1\big(0,T;\dot{H}^{-1}\big) \hookrightarrow C^{\frac{1}{2}}\big([0,T];\dot{H}^{-1}\big)$, which implies that the function $u_{\tau,h} : [0,T] \to \dot{H}^{-1}$ is equicontinuous with respect to $t \in [0,T]$. According to the Arzela–Ascoli theorem ([48], Chap. 7, Thm. 17), the functions $u_{\tau,h}$ are precompact in $C\big([0,T];\dot{H}^{-1}\big)$ and therefore a subsequence u_{τ_j,h_j} satisfies that

$$
u_{\tau_j,h_j} \text{ converges to } u \text{ in } C\big([0,T];\dot{H}^{-1}\big), \text{ with } u(0) = \lim_{j\to\infty} P_{X_{h_j}} u^0 = u^0 \text{ in } \dot{H}^{-1}.
\tag{B.8}
$$

It remains to prove that the limit function u is the unique weak solution of the NS equations (thus the limit function is independent of the choice of the subsequence $\tau_j, h_j \to 0$). This would imply that $u_{\tau,h}$ converges to u as $\tau, h \to 0$ in the sense of (B.7) and (B.8) without necessarily passing to a subsequence.

For $t \in (t_{n-1}, t_n] \subset [T_1, T_2]$ we have

$$
\big\| u_{\tau,h}^+ - u_{\tau,h} \big\|_{L^2} \le \big\| u_h^n - u_h^{n-1} \big\|_{L^2} \le \int_{t_{n-1}}^{t_n} \|\partial_t u_{\tau,h}\|_{L^2} \mathrm{d}t \le C\tau_n^{\frac{1}{2}} \|\partial_t u_{\tau,h}\|_{L^2(T_1,T_2;L^2)}.
$$

As a result of (B.4), we obtain

$$
\big\| u_{\tau,h}^+ - u_{\tau,h} \big\|_{L^\infty(T_1,T_2;L^2)} \to 0 \quad \text{as } \tau \to 0,
$$

and similarly,

$$
\| u_{\tau,h}^- - u_{\tau,h} \|_{L^\infty(T_1,T_2;L^2)} \to 0 \quad \text{as } \tau \to 0.
$$

Hence, there exists a subsequence, also denoted by (τ_j,h_j) for the simplicity of notation, such that

$$
u_{\tau_j,h_j}^\pm \to u \quad \text{strongly in } L^\infty(T_1,T_2;L^2(\Omega)^2).
$$

Note that

$$\int_0^T \left\| u_{\tau,h}^\pm - u_{\tau,h} \right\|_{L^2}^2 \mathrm{d}t \leq C \sum_{n=1}^N \tau_n \left\| u_h^n - u_h^{n-1} \right\|_{L^2}^2$$

$$\leq C\tau_1 \left\| u_h^n - u_h^{n-1} \right\|_{L^2}^2 + C \sum_{n=2}^N \tau_n \left\| u_h^n - u_h^{n-1} \right\|_{\dot{H}^{-1}} \left\| u_h^n - u_h^{n-1} \right\|_{\dot{H}_0^1}$$

$$\leq C\tau_1 + C \sum_{n=2}^N \tau_n^2 \left\| \partial_t u_{\tau,h}|_{[t_{n-1},t_n]} \right\|_{\dot{H}^{-1}} \left\| u_h^n - u_h^{n-1} \right\|_{H^1}$$

$$\leq C\tau_1 + C\tau \|\partial_t u_{\tau,h}\|_{L^2(0,T;\dot{H}^{-1})} \left(\sum_{n=2}^N \tau_n \left(\|u_h^n\|_{H^1}^2 + \|u_h^{n-1}\|_{H^1}^2 \right) \right)^{\frac{1}{2}}$$

$$\leq C\tau \quad \text{(here (3.9) and (B.6) are used)}$$

which immediately yields

$$\left\| u_{\tau,h}^\pm - u_{\tau,h} \right\|_{L^2(0,T;L^2)} \to 0 \quad \text{as } \tau \to 0.$$

Since $u_{\tau_j,h_j} \to u$ strongly in $L^2\left(0, T; L^2(\Omega)^2\right)$ as shown in (B.7), it follows that

$$u_{\tau_j,h_j}^\pm \to u \quad \text{strongly in } L^2\left(0, T; L^2(\Omega)^2\right).$$

From (B.3) and (B.7) we know that, by passing to a subsequence if necessary,

$$u_{\tau_j,h_j}^\pm \to u \quad \text{weakly* in } L^\infty\left(0, T; L^2(\Omega)^2\right),$$
$$u_{\tau_j,h_j}^+ \to u \quad \text{weakly in } L^2(0, T; H^1(\Omega)^2).$$

Now, testing (2.12) by $v_h \in C([0,T]; X_h)$ and integrating the result in time, we have

$$\int_0^T (\partial_t u_{\tau,h}, v_h) \, \mathrm{d}t + \int_0^T (\nabla u_{\tau,h}^+, \nabla v_h) \, \mathrm{d}t + \int_0^T (u_{\tau,h}^- \cdot \nabla u_{\tau,h}^+, v_h) \, \mathrm{d}t = 0.$$

For any given $v \in C([0,T]; \dot{H}_0^1 \cap H^2(\Omega)^2)$, we let $v_h = \Pi_h v$ (see (3.4)), which would converge to v strongly in $C([0,T]; \dot{H}^1)$ as $h \to 0$. Then the equation above implies that

$$\int_0^T (\partial_t u_{\tau,h}, v) \, \mathrm{d}t + \int_0^T (\nabla u_{\tau,h}^+, \nabla v) \, \mathrm{d}t + \int_0^T (u_{\tau,h}^- \cdot \nabla u_{\tau,h}^+, v) \, \mathrm{d}t$$

$$= \int_0^T (\partial_t u_{\tau,h}, v - v_h) \, \mathrm{d}t + \int_0^T (\nabla u_{\tau,h}^+, \nabla(v - v_h)) \, \mathrm{d}t + \int_0^T (u_{\tau,h}^- \cdot \nabla u_{\tau,h}^+, (v - v_h)) \, \mathrm{d}t$$

$$=: J_1^h(v) + J_2^h(v) + J_3^h(v), \tag{B.9}$$

where

$$\left| J_1^h(v) \right| = \left| \int_0^T (\partial_t u_{\tau,h}, v - v_h) \, \mathrm{d}t \right|$$

$$\leq C \|\partial_t u_{\tau,h}\|_{L^2(0,T;\dot{H}^{-1})} \|v - v_h\|_{L^2(0,T;H^1)} \to 0,$$

$$\left| J_2^h(v) \right| \leq C \left\| u_{\tau,h}^+ \right\|_{L^2(0,T;H^1)} \|v - v_h\|_{L^2(0,T;H^1)} \to 0,$$

$$\left| J_3^h(v) \right| = \left| \int_0^T \left(u_{\tau,h}^+, u_{\tau,h}^- \cdot \nabla(v - v_h) \right) \mathrm{d}t \right|$$

$$\leq C \left\| u_{\tau,h}^+ \right\|_{L^2(0,T;L^4)} \left\| u_{\tau,h}^- \right\|_{L^2(0,T;L^4)} \| \nabla(v - v_h) \|_{L^\infty(0,T;L^2)}$$

$$\leq C \left\| u_{\tau,h}^+ \right\|_{L^2(0,T;H^1)} \left\| u_{\tau,h}^- \right\|_{L^2(0,T;H^1)} \| v - v_h \|_{L^\infty(0,T;H^1)} \to 0.$$

Since $u_{\tau_j,h_j}, u_{\tau_j,h_j}^\pm \to u$ weakly in $L^2\left(0, T; \dot{H}_0^1\right)$ and $\partial_t u_{\tau_j,h_j} \to \partial_t u$ weakly in $L^2\left(0, T; \dot{H}^{-1}\right)$, it follows that

$$\int_0^T (\partial_t u_{\tau,h}, v) \, \mathrm{d}t + \int_0^T (\nabla u_{\tau,h}^+, \nabla v) \, \mathrm{d}t \to \int_0^T (\partial_t u, v) \, \mathrm{d}t + \int_0^T (\nabla u, \nabla v) \, \mathrm{d}t.$$

Since $u_{\tau_j,h_j}^- \to u$ strongly in $L^2\left(0, T; L^2(\Omega)^2\right)$ and $\nabla u_{\tau_j,h_j}^+ \to \nabla u$ weakly in $L^2\left(0, T; L^2(\Omega)^2\right)$, it follows that $u_{\tau_j,h_j}^- \cdot \nabla u_{\tau_j,h_j}^+$ convergence weakly in $L^1(0, T; L^1(\Omega)^2)$ and therefore

$$\int_0^T \left(u_{\tau,h}^- \cdot \nabla u_{\tau,h}^+, v \right) \mathrm{d}t \to \int_0^T (u \cdot \nabla u, v) \, \mathrm{d}t, \ \forall \, v \in C\left([0, T]; \dot{H}_0^1 \cap H^2(\Omega)^2\right) \hookrightarrow C\left([0, T]; L^\infty(\Omega)^2\right).$$

By using these results and passing to the limit $(\tau, h) = (\tau_j, h_j) \to (0, 0)$ in (B.9), we obtain that the limit function u satisfies the following weak form:

$$\int_0^T (\partial_t u, v) \, \mathrm{d}t + \int_0^T (\nabla u, \nabla v) \mathrm{d}t + \int_0^T (u \cdot \nabla u, v) \, \mathrm{d}t = 0 \quad \forall v \in C\left([0, T]; \dot{H}_0^1 \cap H^2(\Omega)^2\right). \tag{B.10}$$

Note that $\partial_t u \in L^2\left(0, T; \dot{H}^{-1}\right)$, $\nabla u \in L^2\left(0, T; L^2(\Omega)^2\right)$ and

$$u \cdot \nabla u \in L^{\frac{4}{3}}\left(0, T; L^{\frac{4}{3}}(\Omega)^2\right) \subset L^{\frac{4}{3}}\left(0, T; H^{-1}(\Omega)^2\right) \subset L^{\frac{4}{3}}\left(0, T; \dot{H}^{-1}\right).$$

Since $(P_X(u \cdot \nabla u), v) = (u \cdot \nabla u, v)$ for $v \in \dot{H}^1$, it follows that $P_X(u \cdot \nabla u) \in L^{\frac{4}{3}}\left(0, T; \dot{H}^{-1}\right)$. Therefore, $\partial_t u - Au + P_X(u \cdot \nabla u) \in L^{\frac{4}{3}}\left(0, T; \dot{H}^{-1}\right)$ and (B.10) implies that

$$\int_0^T (\partial_t u - Au + P_X(u \cdot \nabla u), v) \, \mathrm{d}t = 0 \quad \forall v \in C\left([0, T]; \dot{H}_0^1\right).$$

This implies that

$$\partial_t u - Au + P_X(u \cdot \nabla u) = 0 \quad \text{in } \dot{H}^{-1} \text{ a.e. } t \in (0, T]. \tag{B.11}$$

From (B.3) and (B.6) we conclude that, as the limit of u_{τ_j,h_j} when $j \to \infty$, the limit function u must satisfy

$$u \in L^\infty\left(0, T; \dot{L}^2\right) \cap L^2\left(0, T; \dot{H}_0^1\right) \cap H^1\left(0, T; \dot{H}^{-1}\right) \hookrightarrow C\left([0, T]; \dot{L}^2\right). \tag{B.12}$$

According to Problem 3.2 and Theorem 3.2 of Chapter 3 from [1], the equation (B.11) and the regularity result (B.12) imply that u must be the unique weak solution of the 2D NS equation.

Since every sequence u_{τ_j,h_j} contains a subsequence that converges to the unique weak solution u in the sense of (B.7) and (B.8), it follows that $u_{\tau,h}$ converges to u as $\tau, h \to 0$ (without passing to a subsequence). Then (B.7) and (B.8) imply the desired results in (3.51)–(3.53). $\qquad \square$

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