Constrained monotone mean–variance investment-reinsurance under the Cramér–Lundberg model with random coefficients

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Abstract

This paper studies an optimal investment-reinsurance problem for an insurer (she) under the Cramér–Lundberg model with monotone mean–variance (MMV) criterion. At any time, the insurer can purchase reinsurance (or acquire new business) and invest in a security market consisting of a risk-free asset and multiple risky assets whose excess return rate and volatility rate are allowed to be random. The trading strategy is subject to a general convex cone constraint, encompassing no-shorting constraint as a special case. The optimal investment-reinsurance strategy and optimal value for the MMV problem are deduced by solving certain backward stochastic differential equations with jumps. In the literature, it is known that models with MMV criterion and mean–variance criterion lead to the same optimal strategy and optimal value when the wealth process is continuous. Our result shows that the conclusion remains true even if the wealth process has compensated Poisson jumps and the market coefficients are random.

Keywords: Monotone mean-variance, the Cramér-Lundberg model, cone constraints, BSDE with jumps, random coefficients

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1   Introduction

In order to tame the drawback of non-monotonicity of the celebrated mean–variance (MV) portfolio selection theory, Maccheroni et al. [15] propose and solve, in a single-period setting, the monotone mean–variance (MMV) model in which the objective functional is the best approximation of the MV functional among those which are monotone. Trybuła and Zawisza [21] consider the MMV problem in a stochastic factor model using the Hamilton–Jacobi-Bellman-Isaacs (HJBI) equations approach. With conic convex constraints on the portfolios, Shen and Zou [18], Hu, Shi and Xu [7] solve the MMV problem with deterministic and random coefficients by means of the HJBI equation

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approach and backward stochastic differential equation (BSDE) approach. All the obtained optimal investment strategy and the optimal value in [7, 18, 21] coincide with that for the classical MV problem.

In a rather different way, when the underlying asset prices are continuous, or even with general trading constraints, Strub and Li [20], Du and Strub [4] prove directly that the terminal wealth levels corresponding to the optimal portfolio strategies for both the MMV and the MV problems will drop in the domain of the monotonicity of the classical MV functional. This, with earlier results in [15], leads to the conclusion that the optimal strategies for the MMV and the MV problems always coincide when the underlying asset prices are continuous. Hence further research on the MMV problems should focus on models where asset prices are not continuous as claimed in [4, Conlusions].

Li, Liang, Pang [13] solve, in a jump diffusion factor model, hence with discontinuous wealth dynamics, the MMV problem by HJBI equation method. But the results in [13] seem not correct since the Doléans-Dade stochastic exponential is not required to be positive. In the Cramér-Lundberg risk model, a particular jump diffusion scenario, Li, Guo and Tian [11], Li, Liang, Pang [12] study the optimal investment-reinsurance problems under the criteria of MMV and MV respectively and indicate that solutions to these two problems coincide as well. In their model, all the coefficients are deterministic, so the problem is reduced to solve a partial differential equation.

In practice, however, the market parameters, such as the interest rate, stock appreciation and volatility rates are affected by uncertainties caused by the Brownian motion and Poisson random measure. Thus, it is too restrictive to set market parameters as deterministic functions of t. On the other hand, in real financial market, especially for an insure, short-selling some or all the stocks are prohibited. Thus we aim to generalize the models in [11, 12] to allowing random coefficients (both excess return rates and volatility rate) and convex cone trading constraints. We follow the same idea of our previous work [8] without jumps: firstly guess an optimal portfolio candidate, via a heuristic argument, by means of specific BSDEs, and then prove a verification theorem rigorously. Along this line, we eventually provide a semi-closed-form solution for the constrained MMV investment-reinsurance problem in terms of some BSDE with jumps. Unlike models without jumps, ψ in our model is required to be no less than $-1 + \epsilon$ to guarantee that $\Lambda^{\eta,\psi}$ is a strictly positive martingale. Properties of the corresponding BSDE help to fit exactly this subtle point.

It is worth pointing out that, in our another preprint [19], we study the optimal investment-reinsurance problems under the MV criterion, with the same random coefficients and cone constraints setting. Explicit MV optimal investment-reinsurance strategies in terms of a partially coupled BSDEs with jumps (i.e. (5.6), (5.7)) are provided, and the existence and uniqueness of solutions to the corresponding BSDEs are established, which are the main theoretical contributions of [19]. Then we compare carefully the optimal strategies for MMV problem and MV problems obtained in the current paper and [19] respectively, and reach the same conclusion as all the existing models, that is, the MMV and MV criteria lead to the same optimal strategy.

The rest part of this paper is organized as follows. In Section 2, we present the financial market and formulate the constrained MMV problem with random coefficients. In Section 3, we derive heuristically the optimal candidate. Section 4 provides a rigorous verification for the optimal investment-reinsurance strategy and optimal value. In Section 5, we solve the constrained MV problem with random coefficients and make a comparison with the MMV problem. Some concluding remarks are given in Section 6.

2   Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space on which are defined a standard n-dimensional Brownian motion W. We denote by $\mathbb{R}^m$ the set of m-dimensional column vectors, by $\mathbb{R}^m_+$ the set of vectors in $\mathbb{R}^m$ whose components are nonnegative, by $\mathbb{R}^{m \times n}$ the set of $m \times n$ real matrices, and by $\mathbb{S}^n$ the set of symmetric $n \times n$ real matrices. For $M = (m_{ij}) \in \mathbb{R}^{m \times n}$, we denote its transpose by M', and its norm by $|M| = \sqrt{\sum_{ij} m_{ij}^2}$. If $M \in \mathbb{S}^n$ is positive definite (resp. positive semidefinite), we write $M > $ (resp. $\geqslant$) 0. We write $A > $ (resp. $\geqslant$) B if $A, B \in \mathbb{S}^n$ and $A - B > $ (resp. $\geqslant$) 0. As usual, we write $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$ for $x \in \mathbb{R}$.

2.1   The insurance and financial model

Let $\{N_t\}$ be a homogeneous Poisson process with intensity $\lambda > 0$ which counts the number of claims. Let Y_i denote the payment amount of the ith claim, $i = 1, 2, \ldots$, and assume they are bounded independent and identically distributed nonnegative random variables with a common probability distribution function $\nu : \mathbb{R}_+ \to [0, 1]$. In insurance practice, an insurance claim shall never exceed the value of the insured asset, which is usually upper bounded by some constant (such as the value of a new replacement of the insured car or house). Even if the insured asset (such as human life) is invaluable, there is usually a maximum payment amount in practice. Therefore, it is reasonable to assume the claims Y_i are bounded. Then the first and second moments of the claims exist, denoted by

$$b_Y = \int_{\mathbb{R}_+} y \nu(\mathrm{d}y) > 0, \quad \sigma_Y^2 = \int_{\mathbb{R}_+} y^2 \nu(\mathrm{d}y) > 0. \tag{2.1}$$

In the classical Cramér-Lundberg model, the surplus R_t dynamic without reinsurance or investment follows

$$R_t = R_0 + pt - \sum_{i=1}^{N_t} Y_i, \tag{2.2}$$

where p is the premium rate which is assumed to be calculated according to the expected value principle, i.e. $p = (1 + \eta)\lambda b_Y > 0$ and $\eta > 0$ is the relative safety loading of the insurer. As in [17], we use a Poisson random measure γ on $[0, T] \times \Omega \times \mathbb{R}_+$ to represent the compound Poisson process $\sum_{i=1}^{N_t} Y_i$ as[1]

$$\int_0^t \int_{\mathbb{R}_+} y \gamma(\mathrm{d}s, \mathrm{d}y) = \sum_{i=1}^{N_t} Y_i.$$

Assume that $N, Y_i, i = 1, 2, \ldots$ are independent, then

$$\mathbb{E}\Big[\sum_{i=1}^{N_t} Y_i\Big] = \mathbb{E}[N_t]\mathbb{E}[Y_1] = \lambda t \int_{\mathbb{R}_+} y \nu(\mathrm{d}y).$$

The compensated Poisson random measure is denoted by

$$\tilde{\gamma}(\mathrm{d}t, \mathrm{d}y) = \gamma(\mathrm{d}t, \mathrm{d}y) - \lambda \nu(\mathrm{d}y)\,\mathrm{d}t.$$

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[1]This method was also used to study a more general process in Elliott, Siu, Yang [5].

We suppose that the Brownian motion W and the Poisson random measure $\gamma(\mathrm{d}t, \mathrm{d}y)$ are independent processes under the probability measure $\mathbb{P}$. Define the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geqslant 0\}$ as the augmented natural filtration generated by W and γ.

Let $T > 0$ be a constant that will stand for the investment horizon throughout the paper. Let $\mathcal{P}$ be the $\mathbb{F}$-predictable σ-field on $[0, T] \times \Omega$, $\mathcal{B}(\mathbb{R}_+)$ the Borel σ-algebra of $\mathbb{R}_+$.

We now introduce several spaces. Let $L^2_{\mathbb{F}}(0, T; \mathbb{R})$ be the set of $\mathcal{P}$-measurable functions $\varphi : [0, T] \times \Omega \to \mathbb{R}$ such that $\mathbb{E} \int_0^T |\varphi|^2 \, \mathrm{d}t < \infty$. Let $L^{2,\nu}$ be the set of $\mathcal{B}(\mathbb{R}_+)$-measurable functions $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that $\int_{\mathbb{R}_+} \varphi(y)^2 \nu(\mathrm{d}y) < \infty$. Let $L^{2,\nu}_{\mathcal{P}}(0, T; \mathbb{R})$ be the set of $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions $\varphi : [0, T] \times \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that $\mathbb{E} \int_0^T \int_{\mathbb{R}_+} |\phi|^2 \lambda \nu(\mathrm{d}y) \, \mathrm{d}t < \infty$. Let $L^{\infty,\nu}_{\mathcal{P}}(0, T; \mathbb{R})$ be the set of functions $\varphi \in L^{2,\nu}_{\mathcal{P}}(0, T; \mathbb{R})$ which are essentially bounded w.r.t. $\mathrm{d}t \otimes \mathrm{d}\mathbb{P} \otimes \nu(\mathrm{d}y)$. Let $S^\infty_{\mathbb{F}}(0, T; \mathbb{R})$ be the set of $\mathbb{F}$-adapted functions $\varphi : [0, T] \times \Omega \to \mathbb{R}$ which are càd-làg and essentially bounded w.r.t. $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$. The above definitions are generalized in the obvious way to the cases that $\mathbb{R}$ is replaced by $\mathbb{R}^n$, $\mathbb{R}^{n \times m}$ or $\mathbb{S}^n$. In our argument, s, t, ω, "almost surely" and "almost everywhere", will be suppressed for simplicity in many circumstances, when no confusion occurs.

We consider an insurer (she) who is allowed to purchase reinsurances or acquire new businesses to control its exposure to the insurance risk. Let q_t be the value of risk exposure, which represents the insurer's retention level of insurance risk at time t. When $q_t \in [0, 1]$, it corresponds to a proportional reinsurance cover; in this case, the insurer only need pays $100q_t\%$ of each claim, and the reinsurer pays the rest; as a remedy, the insurer diverts part of the premium to the reinsurer at the rate of $(1 - q_t)(1 + \eta_r)\lambda b_Y$, where $\eta_r \geqslant \eta$ is the reinsurer's relative security loading. When $q_t > 1$, it corresponds to acquiring new business. The process q_t is called a reinsurance strategy for convenience. By adopting a reinsurance strategy q_t, the insurer's surplus process R^q follows

$$\mathrm{d}R^q_t = (1 + \eta)\lambda b_Y \, \mathrm{d}t - (1 - q_t)(1 + \eta_r)\lambda b_Y \, \mathrm{d}t - q_t \int_{\mathbb{R}_+} y\gamma(\mathrm{d}t, \mathrm{d}y)$$

$$= (\eta_r q_t + \eta - \eta_r)\lambda b_Y \, \mathrm{d}t - q_t \int_{\mathbb{R}_+} y\tilde{\gamma}(\mathrm{d}t, \mathrm{d}y).$$

The insurer is allowed to invest her wealth in a financial market consisting of a risk-free asset (the money market instrument or bond) whose price is S_0 and m risky securities (the stocks) whose prices are $S_1, \ldots, S_m$. And assume $m \leqslant n$, i.e. the number of risky securities is no more than the dimension of the Brownian motion (part of the source of market randomness). In this case, there always have randomness that cannot be perfectly hedged. Hence this is an incomplete market. The asset prices S_k, $k = 0, 1, \ldots, m$, are driven by stochastic differential equations (SDEs):

$$\begin{cases} \mathrm{d}S_{0,t} = r_t S_{0,t} \, \mathrm{d}t, \\ S_{0,0} = s_0, \end{cases}$$

and

$$\begin{cases} \mathrm{d}S_{k,t} = S_{k,t}\left((\mu_{k,t} + r_t) \, \mathrm{d}t + \sum_{j=1}^n \sigma_{kj,t} \, \mathrm{d}W_{j,t}\right), \\ S_{k,0} = s_k, \end{cases}$$

where, for every $k = 1, \ldots, m$, r is the interest rate process, μ_k and $\sigma_k := (\sigma_{k1}, \ldots, \sigma_{kn})$ are the mean excess return rate process and volatility rate process of the kth risky security. Define the

mean excess return vector $\mu = (\mu_1, \ldots, \mu_m)^\top$, and the volatility matrix $\sigma = (\sigma_{kj})_{m \times n}$. Unlike most existing models (e.g. Li, Liang, Pang [13]), we allow them to be random, non-anticipative with respect to the filtration $\mathbb{F}$. Precisely, we assume $\mu \in L_{\mathbb{F}}^\infty(0, T; \mathbb{R}^m)$ and $\sigma \in L_{\mathbb{F}}^\infty(0, T; \mathbb{R}^{m \times n})$. Furthermore, there exists a constant $\delta > 0$ such that $\sigma\sigma' \geqslant \delta 1_m$ for a.e. $t \in [0, T]$, where 1_m denotes the m-dimensional identity matrix. Same as most of existing models, we assume the interest rate r is a bounded deterministic measurable function of t. It is a long-standing problem how to solve cone constrained MV problem when the interest rate r is random. Actually, with random interest rate r, it is very difficult to construct one (or even more are required) auxiliary adapted process to recover the homogeneity, a critical property used in [7, 9] for studying cone constrained stochastic control problems. In the cone constrained MMV problem [8], the main difficulty in carrying out the construction of $R^{(\eta, \psi, \pi, q)}$ could not be avoided in this paper if r is random.

We assume the insurer is a small investor so that her actions cannot affect the asset prices. She can decide at every time $t \in [0, T]$ the amount $\pi_{j,t}$ of her wealth to invest in the jth risky asset, $j = 1, \ldots, m$. The vector process $\pi := (\pi_1, \ldots, \pi_m)'$ is called a portfolio of the investor. Then the investor's self-financing wealth process X corresponding to an (investment-reinsurance) strategy (π, q) satisfies the following SDE:

$$\begin{cases} dX_t = [r_t X_{t-} + \pi_t' \sigma_t \phi_t + b q_t + a]\, dt + \pi_t' \sigma_t\, dW_t - q_t \int_{\mathbb{R}_+} y\tilde{\gamma}(dt, dy), \\ X_0 = x, \end{cases} \tag{2.3}$$

where

$$\phi := \sigma'(\sigma\sigma)^{-1}\mu, \quad b := \lambda b_Y \eta_r > 0, \quad a := \lambda b_Y(\eta - \eta_r).$$

We may write $X^{\pi, q}$ instead of X to emphasis the dependent of X on the strategy (π, q) in (2.3).

Let Π be a given closed convex cone in $\mathbb{R}^m$, i.e., Π is closed, convex, and if $u \in \Pi$, then $cu \in \Pi$, for all $c \geqslant 0$. It is the constraint set for investment. Note that both $\Pi = \mathbb{R}_+^m$ and $\Pi = \mathbb{R}_+^{m_0} \times \mathbb{R}^{m-m_0}$ ($m_0 < m$) are exactly closed convex cones. And the former means no-shorting constraint; while the later means that shorting-selling the first m_0 stocks are prohibited. The class of admissible investment-reinsurance strategies is defined as

$$\mathcal{U} := \left\{ (\pi, q) \in L_{\mathbb{F}}^2(0, T; \mathbb{R}^{m+1}) \,\Big|\, \pi \in \Pi,\ q \geqslant 0 \right\}.$$

For any admissible strategy $(\pi, q) \in \mathcal{U}$, the wealth process (2.3) admits a unique strong solution $X^{\pi, q}$.

2.2 A family of probability measures

For any $(\eta, \psi) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{P}}^{2,\nu}(0, T; \mathbb{R})$, the process

$$M_t := \int_0^t \eta_s'\, dW_s + \int_0^t \int_{\mathbb{R}_+} \psi_s(y)\tilde{\gamma}(ds, dy)$$

is a square integrable martingale. According to [? , Theorem 10.9], the above process M_t is a BMO martingale if and only if there exists a constant $c > 0$ such that

$$\mathbb{E}\Big[\int_\tau^T \Big(|\eta_s|^2 + \int_{\mathbb{R}_+} |\psi_s(y)|^2 \nu(dy)\Big) ds\Big|\mathcal{F}_\tau\Big] \leqslant c, \quad |M_\tau - M_{\tau-}|^2 \leqslant c, \tag{2.4}$$

for any $\mathbb{F}$-stopping times $\tau \leqslant T$. The following result concerning the Doléans-Dade stochastic exponential of a BMO martingale, can be found in [10].

5

Lemma 2.1 (Kazamaki's Criterion) *Let M be a BMO martingale such that there exists $\varepsilon > 0$ with $M_t - M_{t-} \geqslant -1 + \varepsilon$, for all $t \in [0,T]$, $\mathbb{P}$-a.s., then $\mathcal{E}(M)$ is a strictly positive martingale.*

Denote

$$\mathcal{A} := \bigcup_{\epsilon > 0} \Big\{ (\eta, \psi) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \times L^{\infty,\nu}_{\mathcal{P}}(0,T;\mathbb{R})$$

$$\Big| (\eta, \psi) \text{ satisfies } (2.4), \ \psi \geqslant -1 + \epsilon, \ \mathbb{E}[(\Lambda^{\eta,\psi}_T)^2] < \infty \Big\}. \tag{2.5}$$

Then for any processes $(\eta, \psi) \in \mathcal{A}$, the Kazamaki's criterion clearly holds. Therefore, we can define a probability $\mathbb{P}^{\eta,\psi}$ through

$$\frac{d\mathbb{P}^{\eta,\psi}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \Lambda^{\eta,\psi}_t, \ t \in [0,T],$$

where the Doléans-Dade stochastic exponential

$$\Lambda^{\eta,\psi} := \mathcal{E}\left(\int_0^t \eta'_s \, dW_s + \int_0^t \int_{\mathbb{R}_+} \psi_s(y) \tilde{\gamma}(ds, dy) \right)$$

is a strictly positive martingale on $[0,T]$.

By Girsanov's theorem,

$$W^\eta_t := W_t - \int_0^t \eta_s \, ds, \ t \in [0,T]$$

is a Brownian motion, and

$$\tilde{\gamma}^\psi(dt, dy) := \tilde{\gamma}(dt, dy) - \lambda \psi_t(y) \nu(dy), \ t \in [0,T]$$

is a compensated Poisson random measure under $\mathbb{P}^{\eta,\psi}$.

2.3 The monotone mean–variance problem

According to [15, Page 489], the MMV functional, defined by

$$\inf_{(\eta,\psi) \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^{\eta,\psi}} \left[X^{\pi,q}_T + \frac{1}{2\theta}(\Lambda^{\eta,\psi}_T - 1) \right]$$

is the minimal monotone functional that dominates the MV functional

$$\mathbb{E}^{\mathbb{P}}(X^{\pi,q}_T) - \frac{\theta}{2} Var^{\mathbb{P}}(X^{\pi,q}_T),$$

where θ is a given positive constant measuring the risk aversion of the insurer.

In this paper, we consider the following MMV problem:

$$\sup_{(\pi,q) \in \mathcal{U}} \inf_{(\eta,\psi) \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^{\eta,\psi}} \left[X^{\pi,q}_T + \frac{1}{2\theta}(\Lambda^{\eta,\psi}_T - 1) \right], \tag{2.6}$$

The last condition in (2.5) ensures that the expectation in (2.6) is finite.

Without loss of generality, we may assume $a = 0$ in (2.3). Indeed, by setting $\widetilde{X}_t = X_t + a \int_t^T e^{-\int_t^s r_\alpha \, d\alpha} \, ds$, the optimization problem in (2.6) will not change and all the results in this paper remain true with X_t replaced by $\widetilde{X}_t$.

3 Optimal candidate: a heuristic derivation

In order to solve the problem (2.6), we hope, via a heuristic argument, to find a family of stochastic process $R^{(\eta,\psi,\pi,q)}$ and a quaternion $(\hat{\eta}, \hat{\psi}, \hat{\pi}, \hat{q})$ with the following properties:

1. $R_T^{(\eta,\psi,\pi,q)} = X_T^{\pi,q} + \frac{1}{2\theta}(\Lambda_T^{\eta,\psi} - 1)$ for all $(\eta, \psi, \pi, q) \in \mathcal{A} \times \mathcal{U}$.

2. $R_0^{(\eta,\psi,\pi,q)} = R_0$ is constant for all $(\eta, \psi, \pi, q) \in \mathcal{A} \times \mathcal{U}$.

3. $\mathbb{E}^{\mathbb{P}^{\hat{\eta},\hat{\psi}}}\left[X_T^{\pi,q} + \frac{1}{2\theta}(\Lambda_T^{\hat{\eta},\hat{\psi}} - 1)\right] \leqslant R_0$ for all $(\pi, q) \in \mathcal{U}$.

4. $\mathbb{E}^{\mathbb{P}^{\eta,\psi}}\left[X_T^{\hat{\pi},\hat{q}} + \frac{1}{2\theta}(\Lambda_T^{\eta,\psi} - 1)\right] \geqslant R_0$ for all $(\eta, \psi) \in \mathcal{A}$.

If this is done, we will then rigorously show that $(\hat{\eta}, \hat{\psi}, \hat{\pi}, \hat{q})$ is an optimal solution for the problem (2.6) and R_0 is its optimal value. Clearly, the third and fourth properties ensure that $(\hat{\eta}, \hat{\psi}, \hat{\pi}, \hat{q})$ is a saddle point for (2.6) with the optimal value

$$\mathbb{E}^{\mathbb{P}^{\hat{\eta},\hat{\psi}}}\left[X_T^{\hat{\pi},\hat{q}} + \frac{1}{2\theta}(\Lambda_T^{\hat{\eta},\hat{\psi}} - 1)\right] = R_0.$$

We consider the following family:

$$R_t^{(\eta,\psi,\pi,q)} = X_t^{\pi,q} h_t + \frac{1}{2\theta}(\Lambda_t^{\eta,\psi} Y_t - 1),$$

where (Y, Z, V) and (h, L, Φ) satisfy the following BSDEs with jumps, respectively, (we shall often suppress the argument t in BSDEs for notation simplicity)

$$\begin{cases} dY_t = -f\, dt + Z'\, dW + \int_{\mathbb{R}_+} V(y)\tilde{\gamma}(dt, dy), \\ Y_T = 1, \quad Y > 0, \quad Y + V > 0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} dh_t = -g\, dt + L'\, dW + \int_{\mathbb{R}_+} \Phi(y)\tilde{\gamma}(dt, dy), \\ h_T = 1. \end{cases} \tag{3.2}$$

Our aim reduces to finding proper drivers f and g, which are independent of (η, ψ, π, q), such that the desired properties hold.

Notice the first property is already satisfied by the choice of terminal conditions $Y_T = h_T = 1$ in the above BSDEs with jumps. Because Y and h are independent of (η, ψ, π, q), it follows

$$R_0^{(\eta,\psi,\pi,q)} = X_0^{\pi,q} h_0 + \frac{1}{2\theta}(\Lambda_0^{\eta,\psi} Y_0 - 1) = x h_0 + \frac{1}{2\theta}(Y_0 - 1) := R_0$$

is a constant. So the second property follows.

Now rewrite the BSDEs (3.1) and (3.2) in terms of W^η and $\tilde{\gamma}^\psi$ as

$$\begin{cases} dY_t = \left(-f + Z'\eta + \int_{\mathbb{R}_+} V(y)\psi(y)\lambda\nu(dy)\right) dt + Z'\, dW^\eta + \int_{\mathbb{R}_+} V(y)\tilde{\gamma}^\psi(dt, dy), \\ Y_T = 1, \quad Y > 0, \quad Y + V > 0, \end{cases}$$

and

$$\begin{cases} dh_t = \left(-g + L'\eta + \int_{\mathbb{R}_+} \Phi(y)\psi(y)\lambda\nu(dy)\right) dt + L'\, dW^\eta + \int_{\mathbb{R}_+} \Phi\tilde{\gamma}^\psi(dt, dy), \\ h_T = 1. \end{cases} \tag{3.3}$$

Applying Itô's formula we obtain

$$
\begin{aligned}
\mathrm{d}R_t^{(\eta,\psi,\pi,q)} = \Big[& \frac{\Lambda}{2\theta}\Big(Y|\eta|^2 + 2\eta'Z + \frac{2\theta}{\Lambda}h\eta'\sigma'\pi\Big) \\
& + \frac{\Lambda}{2\theta}\int_{\mathbb{R}_+}\Big((Y+V)\psi^2 + 2V\psi - \frac{2\theta}{\Lambda}hqy\psi\Big)\lambda\nu(\mathrm{d}y) \\
& + \Big(rh - g + L'\eta + \int_{\mathbb{R}_+}\Phi\psi\lambda\nu(\mathrm{d}y)\Big)X + \pi'(h\sigma\phi + \sigma L) \\
& + q\Big(hb - \int_{\mathbb{R}_+}y\Phi(1+\psi)\lambda\nu(\mathrm{d}y)\Big) - \frac{\Lambda}{2\theta}f\Big]\mathrm{d}t \\
& + (\cdots)\,\mathrm{d}W^\eta + \int_{\mathbb{R}_+}(\cdots)\tilde{\gamma}^\psi(\mathrm{d}t,\mathrm{d}y).
\end{aligned}
\tag{3.4}
$$

Since r is deterministic function of t, we see

$$
g = rh, \ \ h_t = e^{\int_t^T r_s\,\mathrm{d}s}, \ \ L_t = 0, \ \ \Phi_t(y) = 0,
\tag{3.5}
$$

fulfills (3.3). From now on, we fix this choice, under which (3.4) becomes

$$
\begin{aligned}
\mathrm{d}R_t^{(\eta,\psi,\pi,q)} = \Big\{ & \frac{\Lambda Y}{2\theta}\Big|\eta + \frac{1}{Y}\Big(Z + \frac{\theta h}{\Lambda}\sigma'\pi\Big)\Big|^2 - \frac{\Lambda Y}{2\theta}\frac{1}{Y^2}\Big|\Big(Z + \frac{\theta h}{\Lambda}\sigma'\pi\Big)\Big|^2 \\
& + \frac{\Lambda}{2\theta}\int_{\mathbb{R}_+}\Big[(Y+V)\Big|\psi + \frac{1}{Y+V}\Big(V - \frac{\theta}{\Lambda}hqy\Big)\Big|^2 \\
& - \frac{1}{Y+V}\Big|V - \frac{\theta}{\Lambda}hqy\Big|^2\Big]\lambda\nu(\mathrm{d}y) + \pi'h\sigma\phi + qhb - \frac{\Lambda}{2\theta}f\Big\}\mathrm{d}t \\
& + (\cdots)\,\mathrm{d}W^\eta + \int_{\mathbb{R}_+}(\cdots)\tilde{\gamma}^\psi(\mathrm{d}t,\mathrm{d}y).
\end{aligned}
$$

Integrating from 0 to T, and taking expectation $\mathbb{E}^{\mathbb{P}^{\eta,\psi}}$, we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}^{\eta,\psi}}[R_T^{(\eta,\psi,\pi,q)}] = R_0 + \mathbb{E}^{\mathbb{P}^{\eta,\psi}}\int_0^T \frac{\Lambda}{2\theta}\Big\{ & Y\Big|\eta + \frac{1}{Y}\Big(Z + \frac{\theta}{\Lambda}h\sigma'\pi\Big)\Big|^2 \\
& + \int_{\mathbb{R}_+}(Y+V)\Big|\psi + \frac{1}{Y+V}\Big(V - \frac{\theta}{\Lambda}hqy\Big)\Big|^2\lambda\nu(\mathrm{d}y) \\
& - \Big[f + \frac{1}{Y}\Big|Z + \frac{\theta}{\Lambda}h\sigma'\pi\Big|^2 + \int_{\mathbb{R}_+}\frac{1}{Y+V}\Big|V - \frac{\theta}{\Lambda}hqy\Big|^2\lambda\nu(\mathrm{d}y) \\
& - \frac{2\theta}{\Lambda}h\pi'\sigma\phi - \frac{2\theta}{\Lambda}hqb\Big]\Big\}\mathrm{d}s.
\end{aligned}
\tag{3.6}
$$

To fulfill the third and fourth properties, since $\theta, \Lambda > 0$, it suffices to find $f, \hat{\eta}, \hat{\psi}, \hat{\pi}$ and $\hat{q}$ such that

$$
\begin{aligned}
& Y\Big|\hat{\eta} + \frac{1}{Y}\Big(Z + \frac{\theta}{\Lambda}h\sigma'\pi\Big)\Big|^2 + \int_{\mathbb{R}_+}(Y+V)\Big|\hat{\psi} + \frac{1}{Y+V}\Big(V - \frac{\theta}{\Lambda}hqy\Big)\Big|^2\lambda\nu(\mathrm{d}y) \\
& - \Big[f + \frac{1}{Y}\Big|Z + \frac{\theta}{\Lambda}h\sigma'\pi\Big|^2 + \int_{\mathbb{R}_+}\frac{1}{Y+V}\Big|V - \frac{\theta}{\Lambda}hqy\Big|^2\lambda\nu(\mathrm{d}y) - \frac{2\theta}{\Lambda}h\pi'\sigma\phi - \frac{2\theta}{\Lambda}hqb\Big] \leqslant 0
\end{aligned}
\tag{3.7}
$$

for all π, q;

$$
Y\Big|\eta + \frac{1}{Y}\Big(Z + \frac{\theta}{\Lambda}h\sigma'\hat{\pi}\Big)\Big|^2 + \int_{\mathbb{R}_+}(Y+V)\Big|\psi + \frac{1}{Y+V}\Big(V - \frac{\theta}{\Lambda}h\hat{q}y\Big)\Big|^2\lambda\nu(\mathrm{d}y)
$$
$$
- \Big[f + \frac{1}{Y}\Big|Z + \frac{\theta}{\Lambda}h\sigma'\hat{\pi}\Big|^2 + \int_{\mathbb{R}_+}\frac{1}{Y+V}\Big|V - \frac{\theta}{\Lambda}h\hat{q}y\Big|^2\lambda\nu(\mathrm{d}y) - \frac{2\theta}{\Lambda}h\hat{\pi}'\sigma\phi - \frac{2\theta}{\Lambda}h\hat{q}b\Big] \geqslant 0 \qquad (3.8)
$$

for all η, ψ. The above two estimates clearly imply

$$
Y\Big|\hat{\eta} + \frac{1}{Y}\Big(Z + \frac{\theta}{\Lambda}h\sigma'\hat{\pi}\Big)\Big|^2 + \int_{\mathbb{R}_+}(Y+V)\Big|\hat{\psi} + \frac{1}{Y+V}\Big(V - \frac{\theta}{\Lambda}h\hat{q}y\Big)\Big|^2\lambda\nu(\mathrm{d}y)
$$
$$
- \Big[f + \frac{1}{Y}\Big|Z + \frac{\theta}{\Lambda}h\sigma'\hat{\pi}\Big|^2 + \int_{\mathbb{R}_+}\frac{1}{Y+V}\Big|V - \frac{\theta}{\Lambda}h\hat{q}y\Big|^2\lambda\nu(\mathrm{d}y) - \frac{2\theta}{\Lambda}h\hat{\pi}'\sigma\phi - \frac{2\theta}{\Lambda}h\hat{q}b\Big] = 0,
$$

which motives us to take

$$
\hat{\eta} = -\frac{1}{Y}\Big(Z + \frac{\theta}{\Lambda}h\sigma'\hat{\pi}\Big), \ \hat{\psi} = -\frac{1}{Y+V}\Big(V - \frac{\theta}{\Lambda}h\hat{q}y\Big)
$$

and

$$
f = -\frac{1}{Y}\Big|Z + \frac{\theta}{\Lambda}h\sigma'\hat{\pi}\Big|^2 + \frac{2\theta}{\Lambda}h\hat{\pi}'\sigma\phi + \frac{2\theta}{\Lambda}h\hat{q}b - \int_{\mathbb{R}_+}\frac{1}{Y+V}\Big|V - \frac{\theta}{\Lambda}h\hat{q}y\Big|^2\lambda\nu(\mathrm{d}y). \qquad (3.9)
$$

To find proper $\hat{\pi}$ and $\hat{q}$, substituting these expressions into (3.7), we get an equivalent condition of (3.7):

$$
Y\Big| -\frac{1}{Y}\Big(Z + \frac{\theta}{\Lambda}h\sigma'\hat{\pi}\Big) + \frac{1}{Y}\Big(Z + \frac{\theta}{\Lambda}h\sigma'\pi\Big)\Big|^2
$$
$$
+ \int_{\mathbb{R}_+}(Y+V)\Big| -\frac{1}{Y+V}\Big(V - \frac{\theta}{\Lambda}h\hat{q}y\Big) + \frac{1}{Y+V}\Big(V - \frac{\theta}{\Lambda}hqy\Big)\Big|^2\lambda\nu(\mathrm{d}y)
$$
$$
\leqslant -\frac{1}{Y}\Big|Z + \frac{\theta}{\Lambda}h\sigma'\hat{\pi}\Big|^2 + \frac{1}{Y}\Big|Z + \frac{\theta}{\Lambda}h\sigma'\pi\Big|^2
$$
$$
- \int_{\mathbb{R}_+}\frac{1}{Y+V}\Big|V - \frac{\theta}{\Lambda}h\hat{q}y\Big|^2\lambda\nu(\mathrm{d}y) + \int_{\mathbb{R}_+}\frac{1}{Y+V}\Big|V - \frac{\theta}{\Lambda}hqy\Big|^2\lambda\nu(\mathrm{d}y)
$$
$$
+ \frac{2\theta}{\Lambda}h\hat{\pi}'\sigma\phi - \frac{2\theta}{\Lambda}h\pi'\sigma\phi + \frac{2\theta}{\Lambda}h\hat{q}b - \frac{2\theta}{\Lambda}hqb. \qquad (3.10)
$$

Write $u = \frac{\theta}{\Lambda}h\sigma'\pi$, $\xi = \frac{\theta}{\Lambda}h\sigma'\hat{\pi}$, $\alpha = \frac{\theta}{\Lambda}hq$ and $\rho = \frac{\theta}{\Lambda}h\hat{q}$, then the above becomes

$$
(Y\phi - Z - \xi)'(u - \xi) + Y(\alpha - \rho)\Big[\int_{\mathbb{R}_+}\frac{V}{Y+V}y\lambda\nu(\mathrm{d}y) + b - \rho\int_{\mathbb{R}_+}\frac{1}{Y+V}y^2\lambda\nu(\mathrm{d}y)\Big] \leqslant 0. \qquad (3.11)
$$

Because $\theta, \Lambda, h > 0$, we have $q \in \mathbb{R}^+$ if and only if so is α. Because Π is a cone and $\pi \in \Pi$, we see $u \in \sigma'\Pi$, where, for every (t, ω), the set $\sigma_t(\omega)'\Pi$ is a closed convex cone, defined as $\sigma_t(\omega)'\Pi = \{\sigma_t(\omega)'\pi \mid \pi \in \Pi\}$.

Now we conclude (3.11) holds for all $u \in \sigma'\Pi$, $\alpha \in \mathbb{R}^+$ if

$$
(Y\phi - Z - \xi)'(u - \xi) \leqslant 0, \qquad (3.12)
$$

holds for all $u \in \sigma'\Pi$, and

$$
\Big[\frac{\int_{\mathbb{R}_+}\frac{V}{Y+V}y\lambda\nu(\mathrm{d}y) + b}{\int_{\mathbb{R}_+}\frac{1}{Y+V}y^2\lambda\nu(\mathrm{d}y)} - \rho\Big](\alpha - \rho) \leqslant 0, \qquad (3.13)
$$

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holds for all $\alpha \in \mathbb{R}^+$.

The following result is a consequence of the second projection theorem in [1, Theorem 9.8].

Lemma 3.1 *Suppose $\xi \in \sigma'\Pi$, $\rho \geqslant 0$. Then the inequality (3.12) holds for all $u \in \sigma'\Pi$ if and only if*

$$\xi = \mathrm{Proj}_{\sigma'\Pi}(\phi Y - Z), \tag{3.14}$$

and the inequality (3.13) holds for all $\alpha \in \mathbb{R}_+$ if and only if

$$\rho = \frac{\left(\int_{\mathbb{R}_+} \frac{V}{Y+V} y \lambda \nu(\mathrm{d}y) + b \right)^+}{\int_{\mathbb{R}_+} \frac{1}{Y+V} y^2 \lambda \nu(\mathrm{d}y)}. \tag{3.15}$$

Here, $\mathrm{Proj}_C(y)$ *denotes the projection of y to a closed convex set C, which is uniquely determined by*

$$|y - \mathrm{Proj}_C(y)| = \min_{c \in C} |y - c|.$$

Taking the above expressions into (3.9) yields

$$
\begin{aligned}
f &= -\frac{1}{Y} \left| Z + \frac{\theta}{\Lambda} h \sigma' \hat{\pi} \right|^2 - \int_{\mathbb{R}_+} \frac{1}{Y+V} \left| V - \frac{\theta}{\Lambda} h \hat{q} y \right|^2 \lambda \nu(\mathrm{d}y) + \frac{2\theta}{\Lambda} h \hat{\pi}' \sigma \phi + \frac{2\theta}{\Lambda} h \hat{q} b \\
&= -\frac{1}{Y} |Z + \xi|^2 + 2\phi'\xi - \int_{\mathbb{R}_+} \frac{1}{Y+V} |V - \rho y|^2 \lambda \nu(\mathrm{d}y) + 2b\rho.
\end{aligned}
$$

Overall, we conjecture that

$$f = -\frac{1}{Y} |Z + \xi|^2 + 2\phi'\xi - \int_{\mathbb{R}_+} \frac{1}{Y+V} |V - \rho y|^2 \lambda \nu(\mathrm{d}y) + 2b\rho, \tag{3.16}$$

$$\hat{\eta} = -\frac{1}{Y} \left(Z + \mathrm{Proj}_{\sigma'\Pi}(Y\phi - Z) \right), \quad \hat{\psi} = -\frac{1}{Y+V}(V - \rho y), \tag{3.17}$$

$$\hat{\pi} = \frac{\Lambda^{\hat{\eta},\hat{\psi}}}{h\theta} (\sigma\sigma')^{-1} \sigma \mathrm{Proj}_{\sigma'\Pi}(Y\phi - Z), \quad \hat{q} = \frac{\Lambda^{\hat{\eta},\hat{\psi}}}{h\theta} \rho, \tag{3.18}$$

where ξ and ρ are given in (3.14) and (3.15) respectively. Notice that

$$
\begin{aligned}
-\frac{1}{Y} |Z + \xi|^2 + 2\phi'\xi &= -\frac{1}{Y}|\xi|^2 + \frac{2}{Y}(\phi Y - Z)'\xi - \frac{1}{Y}|Z|^2 \\
&= -\frac{1}{Y}|\xi - (\phi Y - Z)|^2 + \frac{1}{Y}|\phi Y - Z|^2 - \frac{1}{Y}|Z|^2 \\
&= -\frac{1}{Y} \inf_{\pi \in \Pi} |\sigma'\pi - (\phi Y - Z)|^2 + \frac{1}{Y}|\phi Y - Z|^2 - \frac{1}{Y}|Z|^2 \\
&= -\frac{1}{Y} \inf_{\pi \in \Pi} \left[\pi'\sigma\sigma'\pi - 2\pi'\sigma(\phi Y - Z) \right] - \frac{1}{Y}|Z|^2, \tag{3.19}
\end{aligned}
$$

and

$$\int_{\mathbb{R}_+} \frac{1}{Y+V} \left| V - \rho y \right|^2 \lambda \nu(\mathrm{d}y) + 2b\rho = \frac{\left[\left(\int_{\mathbb{R}_+} \frac{V}{Y+V} y \lambda \nu(\mathrm{d}y) + b \right)^+ \right]^2}{\int_{\mathbb{R}_+} \frac{1}{Y+V} y^2 \lambda \nu(\mathrm{d}y)} - \int_{\mathbb{R}_+} \frac{V^2}{Y+V} \lambda \nu(\mathrm{d}y). \tag{3.20}$$

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Now we get the desired BSDE (3.1):

$$
\begin{cases}
\mathrm{d}Y = \Bigg\{ \ \frac{1}{Y} \inf_{\pi \in \Pi} \Big[\pi' \sigma \sigma' \pi - 2\pi' \sigma (\phi Y - Z) \Big] + \frac{1}{Y} |Z|^2 \\
\qquad\quad - \frac{\Big[\Big(\int_{\mathbb{R}_+} \frac{V}{Y+V} y \lambda \nu(\mathrm{d}y) + b \Big)^+ \Big]^2}{\int_{\mathbb{R}_+} \frac{1}{Y+V} y^2 \lambda \nu(\mathrm{d}y)} + \int_{\mathbb{R}_+} \frac{V^2}{Y+V} \lambda \nu(\mathrm{d}y) \ \Bigg\} \, \mathrm{d}t \\
\qquad\quad + Z' \, \mathrm{d}W + \int_{\mathbb{R}_+} V(y) \tilde{\gamma}(\mathrm{d}t, \mathrm{d}y), \\
Y_T = 1, \quad Y > 0, \quad Y + V > 0,
\end{cases}
\tag{3.21}
$$

In the next section, we will solve this BSDE and use its solution to derive a solution to the problem (2.6).

4 Solutions to the BSDE (3.21) and the problem (2.6)

Definition 4.1 *A triple (Y, Z, V) is called a solution to the BSDE (3.21) if it satisfies all the equalities and inequalities in (3.21) and $(Y, Z, V) \in S_{\mathbb{F}}^\infty(0, T; \mathbb{R}) \times L_{\mathbb{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{P}}^{\infty, \nu}(0, T; \mathbb{R})$. The solution is called uniformly positive if both $Y \geqslant \delta$ and $Y + V \geqslant \delta$ a.s. with some deterministic constant $\delta > 0$.*

Lemma 4.2 *There exists a unique uniformly positive solution (Y, Z, V) to the BSDE (3.21).*

Proof. Consider (the arguments t and ω are suppressed)

$$
\begin{cases}
\mathrm{d}P_t = - \Bigg\{ \ \inf_{\pi \in \Pi} \Big[P \pi' \sigma \sigma' \pi - 2\pi' (P \sigma \phi + \sigma \Delta) \Big] - \frac{\Big[\Big(Pb - \int_{\mathbb{R}_+} \Gamma(y) y \lambda \nu(\mathrm{d}y) \Big)^+ \Big]^2}{\int_{\mathbb{R}_+} (P + \Gamma(y)) y^2 \lambda \nu(\mathrm{d}y)} \ \Bigg\} \, \mathrm{d}t \\
\qquad\quad + \Delta' \, \mathrm{d}W + \int_{\mathbb{R}_+} \Gamma(y) \tilde{\gamma}(\mathrm{d}t, \mathrm{d}y), \\
P_T = 1, \quad P > 0, \quad P + \Gamma > 0.
\end{cases}
\tag{4.1}
$$

From [19, Theorem 3.3], the above BSDE (4.1) admits a unique solution $(P, \Delta, \Gamma) \in S_{\mathbb{F}}^\infty(0, T; \mathbb{R}) \times L_{\mathbb{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{P}}^{\infty, \nu}(0, T; \mathbb{R})$ such that $c_1 \leqslant P$, $P + \Gamma \leqslant c_2$ with some deterministic constants $c_2 > c_1 > 0$. Then

$$
(Y, Z, V) := \Big(\frac{1}{P}, -\frac{\Delta}{P^2}, \frac{1}{P + \Gamma} - \frac{1}{P} \Big),
\tag{4.2}
$$

is well-defined. It can be directly verified, using Itô's formula, that (Y, Z, V) is a solution to (3.21). The above change $(P, \Delta, \Gamma) \to (Y, Z, V)$ is invertible, so the uniqueness of uniformly positive solution to (3.21) follows from the fact that (4.1) admits a unique solution. ∎

The following result provides a complete answer to the problem (2.6).

Theorem 4.3 *Let (Y, Z, V) be the unique uniformly positive solution to (3.21). Let $h, \hat{\eta}, \hat{\psi}, \hat{\pi}, \rho, \hat{q}$, be defined in (3.5), (3.17), (3.15), (3.18), respectively. Then $(\hat{\pi}, \hat{q}, \hat{\eta}, \hat{\psi})$ is a saddle point for the constrained MMV problem (2.6) with the optimal value*

$$
\sup_{(\pi, q) \in \mathcal{U}} \inf_{(\eta, \psi) \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^\eta} \Big[X_T + \frac{1}{2\theta} (\Lambda_T^\eta - 1) \Big] = x h_0 + \frac{1}{2\theta} (Y_0 - 1).
\tag{4.3}
$$

Proof. To prove the theorem, it suffices to prove the following three claims.

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1. $(\hat{\pi}, \hat{q}) \in \mathcal{U}$, $(\hat{\eta}, \hat{\psi}) \in \mathcal{A}$;

2. it holds for all $(\pi, q) \in \mathcal{U}$ that

$$\mathbb{E}^{\mathbb{P}^{\hat{\eta}, \hat{\psi}}} \Big[X_T^{\pi, q} + \frac{1}{2\theta} (\Lambda_T^{\hat{\eta}, \hat{\psi}} - 1) \Big] \leqslant x h_0 + \frac{1}{2\theta} (Y_0 - 1); \tag{4.4}$$

3. it holds for all $(\eta, \psi) \in \mathcal{A}$ that

$$\mathbb{E}^{\mathbb{P}^{\eta, \psi}} \Big[X_T^{\hat{\pi}, \hat{q}} + \frac{1}{2\theta} (\Lambda_T^{\eta, \psi} - 1) \Big] \geqslant x h_0 + \frac{1}{2\theta} (Y_0 - 1). \tag{4.5}$$

Now we prove these claims one by one.

Claim 1. We have $(\hat{\pi}, \hat{q}) \in \mathcal{U}$ and $(\hat{\eta}, \hat{\psi}) \in \mathcal{A}$.

Let $\xi = \mathrm{Proj}_{\sigma' \Gamma}(\phi Y - Z)$. Then according to Lemma 3.1, (3.12) holds. By taking u as 2ξ and 0 in (3.12) respectively, we immediately have

$$|\xi|^2 - \xi'(\phi Y - Z) = -\xi'(\phi Y - Z - \xi) = 0. \tag{4.6}$$

Using this and applying Itô's formula to $\theta h_t X_t^{\hat{\pi}, \hat{q}} + Y_t \Lambda_t^{\hat{\eta}, \hat{\psi}}$, one can obtain

$$\mathrm{d}\big(\theta h_t X_t^{\hat{\pi}, \hat{q}} + Y_t \Lambda_t^{\hat{\eta}, \hat{\psi}}\big) = 0$$

so that

$$\theta h_t X_t^{\hat{\pi}, \hat{q}} + Y_t \Lambda_t^{\hat{\eta}, \hat{\psi}} \equiv \theta h_0 x + Y_0, \ t \in [0, T]. \tag{4.7}$$

Since $\xi \in \sigma'\Pi$, there exists $\beta \in \Pi$, such that $\xi = \sigma'\beta$. Let ρ be defined by (3.15). From (3.18), we get

$$\hat{\pi} = \frac{\Lambda^{\hat{\eta}, \hat{\psi}}}{h\theta} (\sigma\sigma')^{-1} \sigma \mathrm{Proj}_{\sigma'\Gamma}(Y\phi - Z) = \frac{\Lambda^{\hat{\eta}, \hat{\psi}}}{h\theta} \beta.$$

Because Π is cone, Λ, h, $\theta > 0$ and $\beta \in \Pi$, we see $\hat{\pi} \in \Pi$. By (4.7),

$$\sigma'_t \hat{\pi}_t = \frac{\Lambda_t^{\hat{\eta}, \hat{\psi}}}{h_t \theta} \sigma'_t \beta_t = \frac{\Lambda_t^{\hat{\eta}, \hat{\psi}}}{h_t \theta} \xi_t = \frac{\tilde{a} - h_t X_t^{\hat{\pi}, \hat{q}}}{h_t Y_t} \xi_t, \tag{4.8}$$

where $\tilde{a} := h_0 x + Y_0/\theta$ is a constant. Similarly, $\hat{q}$ can be rewritten as

$$\hat{q}_t = \frac{\tilde{a} - h_t X_t^{\hat{\pi}, \hat{q}}}{h_t Y_t} \rho_t. \tag{4.9}$$

Substituting (4.8) and (4.9) into (2.3), and recalling (3.5), we have

$$\mathrm{d}\big(h X^{\hat{\pi}, \hat{q}} - \tilde{a}\big) = -\frac{1}{Y}(h X^{\hat{\pi}, \hat{q}} - \tilde{a})\Big[(\xi'\phi + \rho b)\, \mathrm{d}t + \xi'\, \mathrm{d}W - \rho \int_{\mathbb{R}_+} y \tilde{\gamma}(\mathrm{d}t, \mathrm{d}y)\Big].$$

Applying Itô's formula to $\frac{1}{Y}(h X^{\hat{\pi}, \hat{q}} - \tilde{a})^2$ and using (3.16) and f(3.21), we have

$$\begin{aligned}
\mathrm{d}\Big[\frac{1}{Y}(h X^{\hat{\pi}, \hat{q}} - \tilde{a})^2\Big] = \Big\{ &\frac{|\xi|^2}{Y} - 2(\xi'\phi + \rho b) - \frac{|Z + \xi|^2}{Y} + 2\phi'\xi + 2b\rho + \frac{|Z|^2}{Y} + \frac{2}{Y} Z'\xi \\
&+ \frac{1}{Y} \int_{\mathbb{R}_+} \Big[\rho^2 y^2 - \frac{Y}{Y + V}(V - \rho y)^2 + Y^3\Big(\frac{1}{Y + V} - \frac{1}{Y} + \frac{V}{Y^2}\Big) \\
&+ Y^3\Big(\frac{\rho^2 y^2}{Y^2} + 2\frac{\rho}{Y} y\Big)\Big(\frac{1}{Y + V} - \frac{1}{Y}\Big)\Big] \lambda \nu(\mathrm{d}y) \Big\} \frac{1}{Y^2}(h X^{\hat{\pi}, \hat{q}} - \tilde{a})^2\, \mathrm{d}t \\
&+ (\cdots)\, \mathrm{d}W + (\cdots) \tilde{\gamma}(\mathrm{d}t, \mathrm{d}y).
\end{aligned}$$

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Using (4.6), a direct calculation shows that the term in $\{\cdots\}$ before dt equals 0. Therefore, $\frac{1}{Y}(hX^{\hat{\pi},\hat{q}} - \tilde{a})^2$ is a local martingale. From now on, let $\tau_n, n = 1, 2, \ldots$, be a localizing sequence of stopping times for the local martingales given by some stochastic integrals with respect to the Brownian motion and the compensated Poisson random measure which may vary in different cases. Then

$$\mathbb{E}\left[\frac{1}{Y_{\iota \wedge \tau_n}}(h_{\iota \wedge \tau_n}X^{\hat{\pi},\hat{q}}_{\iota \wedge \tau_n} - \tilde{a})^2\right] = \frac{1}{Y_0}(h_0 x - \tilde{a})^2 = \frac{Y_0}{\theta^2},$$

for any stopping time $\iota \leqslant T$. Sending $n \to \infty$, it follows from Fatou's Lemma that

$$\mathbb{E}\left[\frac{1}{Y_\iota}(h_\iota X^{\hat{\pi},\hat{q}}_\iota - \tilde{a})^2\right] \leqslant \frac{Y_0}{\theta^2}.$$

Since $c_1 \leqslant Y, h \leqslant c_2$ for some constants $c_2 > c_1 > 0$, we get for some constant c_3

$$\mathbb{E}\left[(X^{\hat{\pi},\hat{q}}_\iota)^2\right] \leqslant c_3, \tag{4.10}$$

holds for any stopping time $\iota \leqslant T$. Now it is standard to prove $(\hat{\pi}, \hat{q}) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+1})$, see e.g. [19, Lemma 3.8]. This proves $(\hat{\pi}, \hat{q}) \in \mathcal{U}$.

From 4.2, there exist two positive constants $c_1 \leqslant c_2$ such that $c_1 \leqslant Y, Y + V, h \leqslant c_2$. Recalling the definition of $\hat{\psi}$ in (3.17),

$$\hat{\psi}_t(y) = -1 + \frac{Y_t}{Y_t + V_t(y)} + \rho y \geqslant -1 + \frac{c_1}{c_2}.$$

As we assumed the size of all claims is uniformly bounded, so ν is compactly supported, which implies that $\hat{\psi} \in L^{\infty,\nu}_{\mathcal{P}}(0, T; \mathbb{R})$.

From the proof of [19, Theorem 3.2, Theorem 3.3], Δ, the second component of the unique solution to (4.1), actually satisfies (2.4). Moreover, from the fact that P is uniformly positive and bounded, and the relationship (4.2), we have

$$|\hat{\eta}| = \left| -\frac{1}{Y}\left(Z + \text{Proj}_{\sigma'\Pi}(Y\phi - Z)\right)\right| \leqslant c_4|Z| \leqslant c_5|\Delta|,$$

for some positive constants c_4, c_5. Therefore $\hat{\eta}$ also satisfies (2.4).

On the other hand, by virtue of (4.7) and (4.10), we immediately get that there exists some positive constant c_6,

$$\mathbb{E}[(\Lambda^{\hat{\eta},\hat{\psi}}_\iota)^2] < c_6, \tag{4.11}$$

for any stopping time $\iota \leqslant T$. Hence the last requirement in (2.5) is also satisfied. This proves $(\hat{\eta}, \hat{\psi}) \in \mathcal{A}$, completing the proof of the first claim.

Claim 2. The inequality (4.4) holds for all $(\pi, q) \in \mathcal{U}$.

Write

$$R^{(\eta,\psi,\pi,q)}_t = X^{\pi,q}_t h_t + \frac{1}{2\theta}(\Lambda^{\eta,\psi}_t Y_t - 1). \tag{4.12}$$

For any $(\pi, q) \in \mathcal{U}$, applying Itô's formula to $R^{(\hat{\eta},\hat{\psi},\pi,q)}_t$, and taking expectation with respect to the

probability measure $\mathbb{P}^{\hat{\eta},\hat{\psi}}$, we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}^{\hat{\eta},\hat{\psi}}}[R^{(\hat{\eta},\hat{\psi},\pi,q)}_{T\wedge\tau_n}] = {} & xh_0 + \frac{1}{2\theta}(Y_0 - 1) + \mathbb{E}^{\mathbb{P}^{\hat{\eta},\hat{\psi}}} \int_0^{T\wedge\tau_n} \Big\{ h\pi'(\sigma\phi + \sigma\hat{\eta}) \\
& + hq\Big(b - \int_{\mathbb{R}_+} y\hat{\psi}\lambda\nu(\mathrm{d}y)\Big) + \frac{\Lambda}{2\theta}\Big[\frac{1}{Y}|Z + \xi|^2 - 2\phi'\xi \\
& - 2b\rho + \hat{\eta}'Z + \int_{\mathbb{R}_+}\Big(\hat{\psi}V + \frac{(V - \rho y)^2}{Y + V}\Big)\lambda\nu(\mathrm{d}y) \\
& + (Z + Y\hat{\eta})'\hat{\eta} + \int_{\mathbb{R}_+}(V + Y\hat{\psi} + V\hat{\psi})\hat{\psi}\lambda\nu(\mathrm{d}y)\Big]\Big\}\,\mathrm{d}t,
\end{aligned}
\tag{4.13}
$$

recalling the stopping times $\tau_n, n = 1, 2, \ldots$, are a localizing sequence of the local martingales given by some stochastic integrals with respect to the Brownian motion and the compensated Poisson random measure. On one hand, for any $\pi \in \Pi$, according to (3.12), 3.1 and (4.6), we get

$$
h\pi'(\sigma\phi + \sigma\hat{\eta}) = \frac{h}{Y}\pi'\sigma(\phi Y - Z - \xi) \leqslant \frac{h}{Y}\xi'(\phi Y - Z - \xi) = 0.
\tag{4.14}
$$

A similar consideration shows that

$$
hq\Big(b - \int_{\mathbb{R}_+} y\hat{\psi}\lambda\nu(\mathrm{d}y)\Big) \leqslant 0.
$$

On the other hand, using (3.14), (3.15), (3.17) and (4.6), it is not hard to show the term in $[\cdots]$ in (4.13) equals 0.

Combining above, we arrive at

$$
\mathbb{E}^{\mathbb{P}^{\hat{\eta},\hat{\psi}}}[R^{(\hat{\eta},\hat{\psi},\pi,q)}_{T\wedge\tau_n}] = \mathbb{E}[\Lambda^{\hat{\eta},\hat{\psi}}_{T\wedge\tau_n}R^{(\hat{\eta},\hat{\psi},\pi,q)}_{T\wedge\tau_n}] \leqslant xh_0 + \frac{1}{2\theta}(Y_0 - 1), \ \forall(\pi,q) \in \mathcal{U}.
\tag{4.15}
$$

For any $(\pi, q) \in \mathcal{U}$, it is standard to verify

$$
\mathbb{E}\Big[\sup_{t\in[0,T]}(X^{\pi,q}_t)^2\Big] < \infty.
$$

Since h and Y are bounded, $Y \geqslant c > 0$, we get from (4.7) and (4.12) that

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[0,T]}|\Lambda^{\hat{\eta},\hat{\psi}}_t R^{\hat{\eta},\hat{\psi},\pi,q}_t|\Big] & \leqslant c\mathbb{E}\Big[\sup_{t\in[0,T]}(|X^{\hat{\pi},\hat{q}}_t| + 1)(|X^{\hat{\pi},\hat{q}}_t| + |X^{\pi,q}_t| + 1)\Big] \\
& \leqslant c\mathbb{E}\Big[\sup_{t\in[0,T]}|X^{\hat{\pi},\hat{q}}_t|^2\Big] + c\mathbb{E}\Big[\sup_{t\in[0,T]}|X^{\pi,q}_t|^2\Big] + c < \infty,
\end{aligned}
$$

where $c > 0$ is a constant that may vary from line to line.

Sending $n \to \infty$ in (4.15) and using the dominated convergence theorem, we get (4.4).

Claim 3. The inequality (4.5) holds for all $(\eta, \psi) \in \mathcal{A}$.

For any $(\eta, \psi) \in \mathcal{A}$, we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}^{\eta,\psi}}[R^{(\eta,\psi,\hat{\pi},\hat{q})}_{T\wedge\tau_n}] = {} & xh_0 + \frac{1}{2\theta}(Y_0 - 1) + \mathbb{E}^{\mathbb{P}^{\eta,\psi}} \int_0^{T\wedge\tau_n} \frac{\Lambda^{\eta,\psi}}{2\theta}\Big[Y|\eta|^2 + 2\eta'Z \\
& + \frac{2\theta}{\Lambda^{\eta,\psi}}h\hat{\pi}'\sigma\eta + \frac{2\theta}{\Lambda^{\eta,\psi}}h\hat{\pi}'\sigma\phi + \frac{2\theta}{\Lambda^{\eta,\psi}}h\hat{q}b - f \\
& + \int_{\mathbb{R}_+}\Big((Y + V)\psi^2 + 2\psi V - \frac{2\theta}{\Lambda^{\eta,\psi}}h\hat{q}y\psi\Big)\lambda\nu(\mathrm{d}y)\Big]\,\mathrm{d}t.
\end{aligned}
\tag{4.16}
$$

Completing the squares with respect to η and ψ respectively, the drift term in (4.16) is no less than

$$\frac{\Lambda^{\eta,\psi}}{2\theta}\left[-\frac{1}{Y}\left|Z+\frac{\theta}{\Lambda^{\eta,\psi}}h\sigma'\hat{\pi}\right|^2+\frac{2\theta}{\Lambda^{\eta,\psi}}h\hat{\pi}'\sigma\phi-\frac{1}{Y+V}\left(V-\frac{\theta}{\Lambda^{\eta,\psi}}h\hat{q}y\right)^2+\frac{2\theta}{\Lambda^{\eta,\psi}}h\hat{q}b-f\right]. \quad (4.17)$$

Please note that $(\hat{\pi},\hat{q})$ is a feedback of Λ, so under (η,ψ), the optimal $(\hat{\pi},\hat{q})$ in (3.18) should take

$$\hat{\pi}=\frac{\Lambda^{\eta,\psi}}{h\theta}(\sigma\sigma')^{-1}\sigma\mathrm{Proj}_{\sigma'\Gamma}(Y\phi-Z),\ \ \hat{q}=\frac{\Lambda^{\eta,\psi}}{h\theta}\rho.$$

Recall (4.8), (3.14) and (3.16), so (4.17) is equal to

$$\frac{\Lambda^{\eta,\psi}}{2\theta}\left[-\frac{1}{Y}|Z+\xi|^2+2\phi'\xi-\frac{1}{Y+V}(V-\rho y)^2+2\rho b-f\right]=0. \quad (4.18)$$

Hence from (4.16), (4.17) and (4.18), we have

$$\mathbb{E}^{\mathbb{P}^{\eta,\psi}}[R_{T\wedge\tau_n}^{(\eta,\psi,\hat{\pi},\hat{q})}]\geqslant xh_0+\frac{1}{2\theta}(Y_0-1).$$

Similar to the previous argument, we can prove

$$\mathbb{E}[\sup_{t\in[0,T]}|\Lambda_t^{\eta,\psi}R_t^{(\eta,\psi,\hat{\pi},\hat{q})}|]<\infty.$$

By sending $n\to\infty$ in above, we get (4.5). This complete the proof. ∎

5 Link to the classical MV problem

The classical MV problem (see, e.g. [22]) is

$$\sup_{(\pi,q)\in\mathcal{U}}\left[\mathbb{E}(X_T)-\frac{\theta}{2}\mathrm{Var}(X_T)\right]. \quad (5.1)$$

We will firstly solve the problem (5.1) and then draw a comparison with the MMV problem (2.6).

For any $z\in\mathbb{R}$, let us consider

$$F(z):=\inf_{(\pi,q)\in\mathcal{U}^z}\mathbb{E}\left[(X_T-z)^2\right]=\inf_{(\pi,q)\in\mathcal{U}^z}\left[\mathbb{E}(X_T^2)-z^2\right], \quad (5.2)$$

where

$$\mathcal{U}^z:=\left\{(\pi,q)\in\mathcal{U}\ \middle|\ \mathbb{E}(X_T^{\pi,q})=z\right\},$$

with the convention that $\inf\emptyset=+\infty$. By definition, $F(z)\geqslant 0$.

As illustrated in [21, page 12], the connection between problems (5.1) and (5.2) is given as follows:

$$\sup_{(\pi,q)\in\mathcal{U}}\left[\mathbb{E}(X_T)-\frac{\theta}{2}\mathrm{Var}(X_T)\right]=\sup_{z\in\mathbb{R}}\sup_{(\pi,q)\in\mathcal{U}^z}\left[z-\frac{\theta}{2}\left(E(X_T^2)-z^2\right)\right]$$

$$=\sup_{z\in\mathbb{R}}\left[z-\frac{\theta}{2}\inf_{(\pi,q)\in\mathcal{U}^z}\left(E(X_T^2)-z^2\right)\right]=\sup_{z\in\mathbb{R}}\left[z-\frac{\theta}{2}F(z)\right]. \quad (5.3)$$

In order to solve (5.2), we introduce a Lagrange multiplier $\zeta \in \mathbb{R}$ and consider

$$J(z,\zeta) := \inf_{(\pi,q)\in\mathcal{U}} \mathbb{E}\Big[X_T^2 - z^2 - 2\zeta(X_T - z)\Big] = \inf_{(\pi,q)\in\mathcal{U}}\Big[\mathbb{E}(X_T - \zeta)^2 - (z - \zeta)^2\Big]. \tag{5.4}$$

By the Lagrange duality theorem (see Luenberger [14]),

$$F(z) = \sup_{\zeta\in\mathbb{R}} J(z,\zeta), \ z \in \mathbb{R}. \tag{5.5}$$

According to [19], the solution for problem (5.4) depends on the following two coupled BSDEs:

$$\begin{cases} \mathrm{d}P_1 = -\Big[2rP_1 + F_1^*(t,P_1,\Delta_1) + G_1^*(P_1,\Gamma_1,P_2,\Gamma_2)\Big]\mathrm{d}t + \Delta_1'\,\mathrm{d}W + \int_{\mathbb{R}_+}\Gamma_1(y)\tilde{\gamma}(\mathrm{d}t,\mathrm{d}y), \\ P_{1,T} = 1, \ P_1 > 0, \ P_1 + \Gamma_1 > 0, \ P_2 + \Gamma_2 > 0, \end{cases} \tag{5.6}$$

and

$$\begin{cases} \mathrm{d}P_2 = -\Big[2rP_2 + F_2^*(t,P_2,\Delta_2) + G_2^*(P_2,\Gamma_2)\Big]\mathrm{d}t + \Delta_2'\,\mathrm{d}W + \int_{\mathbb{R}_+}\Gamma_2(y)\tilde{\gamma}(\mathrm{d}t,\mathrm{d}y), \\ P_{2,T} = 1, \ P_2 > 0, \ P_2 + \Gamma_2 > 0, \end{cases} \tag{5.7}$$

where, for $(t,\omega,v,u,P_1,\Delta_1,\Gamma_1,P_2,\Delta_2,\Gamma_2) \in [0,T]\times\Omega\times\Pi\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^m\times L^{2,\nu}\times\mathbb{R}_+\times\mathbb{R}^m\times L^{2,\nu}$, we define

$$F_1(t,\omega,v,P_1,\Delta_1) := P_1|\sigma_t'v|^2 + 2v'(P_1\mu_t + \sigma_t\Delta_1),$$
$$F_2(t,\omega,v,P_2,\Delta_2) := P_2|\sigma_t'v|^2 - 2v'(P_2\mu_t + \sigma_t\Delta_2),$$
$$F_1^*(t,\omega,P_1,\Delta_1) := \inf_{v\in\Pi} F_1(t,\omega,v,P_1,\Delta_1),$$
$$F_2^*(t,\omega,P_2,\Delta_2) := \inf_{v\in\Pi} F_2(t,\omega,v,P_2,\Delta_2),$$

and

$$G_1^*(P_1,\Gamma_1,P_2,\Gamma_2) := \inf_{u\geqslant 0} G_1(u,P_1,\Gamma_1,P_2,\Gamma_2),$$

$$G_2^*(P_2,\Gamma_2) := -\frac{\left[\left(P_2 b - \int_{\mathbb{R}_+}\Gamma_2(y)y\lambda\nu(\mathrm{d}y)\right)^+\right]^2}{\int_{\mathbb{R}_+}(P_2 + \Gamma_2(y))y^2\lambda\nu(\mathrm{d}y)},$$

and

$$G_1(u,P_1,\Gamma_1,P_2,\Gamma_2) := \int_{\mathbb{R}_+}\Big[(P_1 + \Gamma_1(y))\big[[(1 - uy)^+]^2 - 1\big]$$
$$+ (P_2 + \Gamma_2(y))[(1 - uy)^-]^2\Big]\lambda\nu(\mathrm{d}y) + 2uP_1(b + \lambda b_Y).$$

The definitions of solutions to (5.6) and (5.7) agree with the one in 4.1. Recall that $h_t = e^{\int_t^T r_s\,\mathrm{d}s}$ is deterministic and bounded. From [19, Theorem 3.2, Theorem 3.3 and Lemma 4.1], there exists a unique uniformly positive solution $(P_1,\Delta_1,\Gamma_1,P_2,\Delta_2,\Gamma_2)$ to the coupled BSDEs (5.6) and (5.7); moreover, one has $P_{1,0} \leqslant h_0^2$ and $P_{2,0} < h_0^2$.

The following result for the problem (5.4) comes from [19, Theorem 3.7]. We present it here in terms of our notation.

Lemma 5.1 *Let $(P_1, \Delta_1, \Gamma_1)$ and $(P_2, \Delta_2, \Gamma_2)$ be the unique uniformly positive solutions to (5.6) and (5.7), respectively. Let*

$$\xi_{1,t} := (\sigma_t \sigma_t')^{-1} \sigma_t Proj_{\sigma'\Gamma}\Big(-\phi_t - \frac{\Delta_{1,t}}{P_{1,t}}\Big),$$

$$\xi_{2,t} := (\sigma_t \sigma_t')^{-1} \sigma_t Proj_{\sigma'\Gamma}\Big(\phi_t + \frac{\Delta_{2,t}}{P_{2,t}}\Big),$$

$$\rho_{1,t} := \underset{u \geqslant 0}{\operatorname{argmin}}\, G_1(u, P_{1,t}, \Gamma_{1,t}, P_{2,t}, \Gamma_{2,t}),$$

$$\rho_{2,t} := \frac{\Big(P_{2,t}b - \int_{\mathbb{R}_+} \Gamma_{2,t}(y) y \lambda \nu(\mathrm{d}y)\Big)^+}{\int_{\mathbb{R}_+}(P_{2,t} + \Gamma_{2,t}(y)) y^2 \lambda \nu(\mathrm{d}y)}.$$

Then the pair of feedback investment-reinsurance strategies

$$\pi^\zeta(t, X) = \Big(X_t - \frac{\zeta}{h_t}\Big)^+ \xi_{1,t} + \Big(X_t - \frac{\zeta}{h_t}\Big)^- \xi_{2,t}, \tag{5.8}$$

$$q^\zeta(t, X) = \Big(X_t - \frac{\zeta}{h_t}\Big)^+ \rho_{1,t} + \Big(X_t - \frac{\zeta}{h_t}\Big)^- \rho_{2,t} \tag{5.9}$$

is optimal for the problem (5.4) with the optimal value

$$J(z, \zeta) = P_{1,0}\Big[\Big(x - \frac{\zeta}{h_0}\Big)^+\Big]^2 + P_{2,0}\Big[\Big(x - \frac{\zeta}{h_0}\Big)^-\Big]^2 - (\zeta - z)^2.$$

To solve (5.5), we do a tedious calculation and obtain

$$F(z) = \begin{cases} \frac{P_{2,0}(z - xh_0)^2}{h_0^2 - P_{2,0}}, & \text{if } z > xh_0 \text{ and } P_{2,0} < h_0^2; \\ +\infty, & \text{if } z < xh_0 \text{ and } P_{1,0} = h_0^2; \\ \frac{P_{1,0}(z - xh_0)^2}{h_0^2 - P_{1,0}}, & \text{if } z < xh_0 \text{ and } P_{1,0} < h_0^2; \\ 0, & \text{if } z = xh_0, \end{cases}$$

with the argument maximum

$$\hat{\zeta}(z) = \begin{cases} \frac{h_0^2 z - x P_{2,0} h_0}{h_0^2 - P_{2,0}}, & \text{if } z > xh_0 \text{ and } P_{2,0} < h_0^2; \\ -\infty, & \text{if } z < xh_0 \text{ and } P_{1,0} = h_0^2; \\ \frac{h_0^2 z - x P_{1,0} h_0}{h_0^2 - P_{1,0}}, & \text{if } z < xh_0 \text{ and } P_{1,0} < h_0^2; \\ xh_0, & \text{if } z = xh_0. \end{cases} \tag{5.10}$$

Accordingly,

$$\sup_{z \in \mathbb{R}}\Big[z - \frac{\theta}{2} F(z)\Big] = xh_0 + \frac{1}{2\theta}\Big(\frac{h_0^2}{P_{2,0}} - 1\Big), \tag{5.11}$$

with the argument maximum

$$\hat{z} = xh_0 + \frac{1}{\theta}\Big(\frac{h_0^2}{P_{2,0}} - 1\Big) > xh_0. \tag{5.12}$$

Substituting (5.12) into (5.10), we obtain

$$\hat{\zeta}(\hat{z}) = xh_0 + \frac{1}{\theta}\frac{h_0^2}{P_{2,0}}.$$

The above analysis leads to the following results for the problem (5.1).

Theorem 5.2 *Use the notations in 5.1. Define a constant*

$$\hat{\zeta} = xh_0 + \frac{1}{\theta}\frac{h_0^2}{P_{2,0}}, \tag{5.13}$$

and a pair of feedback strategies

$$\pi^{\hat{\zeta}}(t,X) = -\Big(X - \frac{\hat{\zeta}}{h_t}\Big)\xi_{2,t}, \quad q^{\hat{\zeta}} = -\Big(X - \frac{\zeta}{h_t}\Big)\rho_{2,t}. \tag{5.14}$$

Then $(\pi^{\hat{\zeta}}, q^{\hat{\zeta}})$ is the optimal feedback investment-reinsurance strategy for the problem (5.1) with the optimal value

$$\sup_{(\pi,q)\in\mathcal{U}}\Big[\mathbb{E}(X_T) - \frac{\theta}{2}Var(X_T)\Big] = xh_0 + \frac{1}{2\theta}\Big(\frac{h_0^2}{P_{2,0}} - 1\Big). \tag{5.15}$$

Proof. Notice $\hat{\zeta} = \hat{\zeta}(\hat{z})$; so, by (5.3) and (5.5),

$$\sup_{(\pi,q)\in\mathcal{U}}\Big[\mathbb{E}(X_T) - \frac{\theta}{2}Var(X_T)\Big] = \sup_{z\in\mathbb{R}}\Big[z - \frac{\theta}{2}F(z)\,\mathrm{d}p)\Big]$$

$$= \hat{z} - \frac{\theta}{2}F(\hat{z}) = \hat{z} - \frac{\theta}{2}J(\hat{z}, \hat{\zeta}(\hat{z})) = \hat{z} - \frac{\theta}{2}J(\hat{z}, \hat{\zeta}).$$

Therefore, the optimal portfolio for the problem (5.4) with $(z,\zeta) = (\hat{z}, \hat{\zeta})$ is also optimal to (5.1).

We now show that the strategy (5.14) is optimal to the problem (5.4) with $(z,\zeta) = (\hat{z}, \hat{\zeta})$. Taking (5.14) to the wealth process (2.3), we get

$$\begin{cases} \mathrm{d}\Big(X_t^{\pi^{\hat{\zeta}}} - \frac{\hat{\zeta}}{h_t}\Big) = \Big(X_t^{\pi^{\hat{\zeta}}} - \frac{\hat{\zeta}}{h_t}\Big)\Big((r - \xi_2'\mu - b\rho_2)\,\mathrm{d}t - \xi_2\sigma'\,\mathrm{d}W + \int_{\mathbb{R}_+}\rho_2 y\tilde{\gamma}(\mathrm{d}t,\mathrm{d}y) \Big), \\ X_0^{\pi^{\hat{\zeta}}} - \frac{\hat{\zeta}}{h_0} = -\frac{h_0}{\theta P_{2,0}}, \end{cases}$$

Since $-\frac{h_0}{\theta P_{2,0}} \leqslant 0$ and $\rho_2 \geqslant 0$, it follows that $X_t^{\pi^{\hat{\zeta}}} \leqslant \frac{\hat{\zeta}}{h_t}$. Accordingly, we can rewrite (5.14) as

$$\pi^{\hat{\zeta}}(t,X) = \Big(X_t - \frac{\hat{\zeta}}{h_t}\Big)^{+}\xi_{1,t} + \Big(X_t - \frac{\hat{\zeta}}{h_t}\Big)^{-}\xi_{2,t},$$

and

$$q^{\hat{\zeta}}(t,X) = \Big(X_t - \frac{\hat{\zeta}}{h_t}\Big)^{+}\rho_{1,t} + \Big(X_t - \frac{\hat{\zeta}}{h_t}\Big)^{-}\rho_{2,t},$$

which is just (5.8), the optimal strategy for the problem (5.4) with $(z,\zeta) = (\hat{z}, \hat{\zeta})$.

Finally, (5.15) comes from (5.3) and (5.11) evidently. ∎

In the end, we have the following connection between the problems (2.6) and (5.1).

Theorem 5.3 *The problems (2.6) and (5.1) share the same optimal value $xh_0 + \frac{1}{2\theta}(Y_0 - 1)$ and the same optimal feedback portfolio $\hat{\pi} = \pi^{\hat{\zeta}}$, where $\hat{\pi}$ and $\pi^{\hat{\zeta}}$ are defined in (3.18) and (5.14), respectively.*

Proof. Recall that (Y, Z, V) is the unique uniformly positive solution to (3.21), and $h_t = e^{\int_t^T r_s\,\mathrm{d}s}$. It can be directly verified, by Itô's formula, that

$$(Y, Z, V) = \Big(\frac{h^2}{P_2}, -\frac{h^2}{P_2^2}\Delta_2, \frac{h^2}{P_2 + \Gamma_2} - \frac{h^2}{P_2}\Big), \tag{5.16}$$

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where $(P_2, \Lambda_2, \Lambda_2)$ is the unique uniformly positive solution to (5.7). Comparing (4.3) and (5.15), the first claim follows.

By (4.8), (4.9), and (5.16), we have

$$\hat{\pi} = \frac{xh_0 + \frac{Y_0}{\theta} - hX^{\hat{\pi},\hat{q}}}{hY}(\sigma\sigma')^{-1}\sigma\mathrm{Proj}_{\sigma'\Gamma}(\phi Y - Z)$$

$$= \frac{\hat{\zeta} - hX^{\hat{\pi},\hat{q}}}{h}(\sigma\sigma')^{-1}\sigma\mathrm{Proj}_{\sigma'\Gamma}\Big(\phi + \frac{\Delta_2}{P_2}\Big) = -\Big(X^{\hat{\pi},\hat{q}} - \frac{\hat{\zeta}}{h}\Big)\xi_2,$$

and

$$\hat{q} = \frac{xh_0 + \frac{Y_0}{\theta} - hX^{\hat{\pi},\hat{q}}}{hY}\frac{\Big(\int_{\mathbb{R}_+}\frac{V}{Y+V}y\lambda\nu(\mathrm{d}y) + b\Big)^+}{\int_{\mathbb{R}_+}\frac{1}{Y+V}y^2\lambda\nu(\mathrm{d}y)}$$

$$= \frac{xh_0 + \frac{h_0^2}{\theta P_{2,0}} - hX^{\hat{\pi},\hat{q}}}{h^2}P_2\frac{-\Big(\int_{\mathbb{R}_+}\frac{\Gamma_2}{P_2}y\lambda\nu(\mathrm{d}y) + b\Big)^+}{\int_{\mathbb{R}_+}(P_2 + \Gamma_2)y^2\lambda\nu(\mathrm{d}y)} = -\Big(X^{\hat{\pi},\hat{q}} - \frac{\hat{\zeta}}{h}\Big)\rho_2.$$

which is exactly the feedback optimal strategy (5.14). $\blacksquare$

6 Concluding remarks

In this paper, we studied the MMV optimal investment-reinsurance problem under the Cramér-Lundberg model with random coefficients and strategy constraints. Semi-closed-form solutions, the optimal probability measure and optimal investment-reinsurance in terms of the solutions to some BSDEs with jumps are provided. Eventually, we conclude that the solution coincides with the corresponding optimal investment-reinsurance problem under the classical MV criterion.

In the definition (2.5), the condition $\psi \geqslant -1 + \epsilon$ ensures that the Doléans-Dade stochastic exponential $\Lambda^{\eta,\psi}$ is strictly positive. The properties of the solution (Y, Z, V) to BSDE with jumps (3.21) help to ensure that the optimal $\hat{\psi}$ defined in (3.17) fits exactly the subtle point $\hat{\psi} \geqslant -1 + \epsilon$. It is critical to notice that our model only considers down side jumps since they are coming from claims. It is interesting to consider general MMV problems with two-side Poisson jumps, in which case the optimal $\hat{\psi}$ may not be easy to verify that it satisfies $\hat{\psi} \geqslant -1 + \epsilon$.

References

[1] Beck A. Introduction to nonlinear optimization: Theory, algorithms, and applications with MATLAB. Society for Industrial and Applied Mathematics, 2014.

[2] Černý A. Semimartingale theory of monotone mean-variance portfolio allocation. Math. Finance, 2020, 30(3): 1168-1178.

[3] Černý A, Maccheroni F, Marinacci M, Rustichini. On the computation of optimal monotone mean-variance portfolios via truncated quadratic utility. J. Math. Econom., 2012, 48(6): 386-395.

[4] Du J, Strub MS. Monotone and classical mean-variance preferences coincide when asset prices are continuous, SSRN preprint, 2023, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4359422.

[5] Elliott R, Siu T K, Yang H. Filtering a Markov modulated random measure. IEEE Trans. Automat. Control, 2009, 55(1): 74-88.

[6] He S, Wang J, Yan J. Semimartingale theory and stochastic calculus. Science press and CRC press, 1992.

[7] Hu Y, Shi X, Xu Z Q. Constrained stochastic LQ control with regime switching and application to portfolio selection. Ann. Appl. Probab., 2022, 32 (1):426-460.

[8] Hu Y, Shi X, Xu Z Q. Constrained monotone mean-variance problem with random coefficients. SIAM J. Financial Math., 2023, 14(3): 838 - 854.

[9] Hu Y, Zhou X. Constrained stochastic LQ control with random coefficients, and application to portfolio selection. SIAM J. Control Optim., 2005, 44(2): 444-466.

[10] Kazamaki N. Continuous exponential martingales and BMO. Springer, 2006.

[11] Li B, Guo J, Tian L. Optimal investment and reinsurance policies for the Cramér-Lundberg risk model under monotone mean-variance preference. Internat. J. Control, 2023: 1-15.

[12] Li Y, Liang Z, Pang S. Optimal reinsurance-investment strategy for a monotone mean-variance insurer in the cramér-lundberg model. arXiv preprint, 2022, https://arxiv.org/abs/2211.12168.

[13] Li Y, Liang Z, and Pang S. Comparison between mean-variance and monotone mean- variance preferences under jump diffusion and stochastic factor model. arXiv preprint, 2022, https://arxiv.org/abs/2211.14473.

[14] Luenberger D. G. Optimization by vector space methods, John Wiley and Sons, New York, 1997.

[15] Maccheroni F, Marinacci M, Rustichini A, Taboga M. Portfolio selection with monotone mean-variance preferences. Math. Finance, 2009, 19(3): 487-521.

[16] Markowitz H. Portfolio selection. J. Finance, 1952, 7: 77-91.

[17] Shen Y, Zeng Y. Optimal investment-reinsurance strategy for mean-variance insurers with square-root factor process. Insurance Math. Econom., 2015, 62: 118-137.

[18] Shen Y, Zou B. Cone-constrained monotone mean-variance portfolio selection under diffusion models. SIAM J. Financial Math., 2022, 13(4):SC99-SC112.

[19] Shi X, Xu Z. Constrained mean-variance investment-reinsurance under the Cramér-Lundber gmodel with random coefficients. working paper.

[20] Strub M S, Li D. A note on monotone mean-variance preferences for continuous processes. Oper. Res. Lett., 2020, 48(4): 397-400.

[21] Trybuła J, Zawisza D. Continuous-time portfolio choice under monotone mean-variance preferences-stochastic factor case. Math. Oper. Res., 2019, 44(3): 966-987.

[22] Zhou X Y, Li D. Continuous-time mean-variance portfolio selection: A stochastic LQ framework. Appl. Math. Optim., 2000, 42(1): 19-33.