

Optimal consumption-investment with coupled constraints on consumption and investment strategies in a regime switching market with random coefficients

Ying Hu ^{*} Xiaomin Shi[†] Zuo Quan Xu[‡]

January 30, 2024

This paper studies finite-time optimal consumption-investment problems with power, logarithmic and exponential utilities, in a regime switching market with random coefficients and possibly subject to non-convex constraints. Compared to the existing models, one distinguish feature of our model is that the trading constraints put on the consumption and investment strategies are coupled together in the cases of power and logarithmic utilities, leading to new technical challenges. We provide explicit optimal consumption-investment strategies and optimal values for these consumption-investment problems, which are expressed in terms of the solutions to some multi-dimensional diagonally quadratic backward stochastic differential equation (BSDE) and linear BSDE with unbound coefficients. Some of these BSDEs are new in the literature and solving them is one of the main theoretical contributions of this paper. We accomplish the latter by applying the truncation, approximation technique to get some a priori uniformly lower and upper bounds for their solutions.

Key words. Optimal consumption-investment, regime switching, random coefficients, multi-dimensional quadratic backward stochastic differential equation.

Mathematics Subject Classification (2020) 91B16 93E20 60H30 91G10

1 Introduction

In a consumption-investment problem in a financial market, one aims at maximizing his/her utility of consumption and wealth by choosing the best consumption and investment strategies. Following the pioneering work of Merton [29], a large volume of works has been done on the model. In particular, various constraints such as bankruptcy prohibition, subsistence consumption requirement, wealth-dependent investment and consumption constraints are introduced into the model; see, e.g.,

^{*}Univ Rennes, CNRS, IRMAR-UMR 6625, F-35000 Rennes, France. Partially supported by Lebesgue Center of Mathematics “Investissements d’avenir” program-ANR-11-LABX-0020-01. Email: ying.hu@univ-rennes1.fr

[†]School of Statistics and Mathematics, Shandong University of Finance and Economics, Jinan 250100, China. Partially supported by NSFC (No. 11801315). Email: shixm@mail.sdu.edu.cn

[‡]Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong. Partially supported by NSFC (No.11971409), The Hong Kong RGC (GRF No.15202421 and No.15204622). Email: maxu@polyu.edu.hk

Guan, Xu and Yi [11], Sethi [34], Xu and Yi [40], Zariphopoulou [43]. All these papers assume Markovian markets. The model has also been considered in non-Markovian markets with general random coefficients; see, Cox and Huang [6], Karatzas, Lehoczky and Shreve [20], Matoussi and Xing [28], Xing [39]. The method used in [6] and [20] is known as martingale method (or convex duality) nowadays. Please refer to the seminar monograph Karatzas and Shreve [21] for a systematic account on this method for more advanced (incomplete/constrained) markets.

On the other hand, Markov chain is often adopted to reflect, at the macroeconomic level, the market status (such as bull and bear). If such as a Markov chain is introduced to a market model, then the market becomes the famous regime switching market; see Hamilton [12]. Utility maximization and mean-variance models in regime switching markets have been studied by many researchers; see, e.g., [2, 26, 35, 36, 38, 41, 42, 44, 45]. In these works, the market parameters are assumed to be deterministic functions of time for each given regime. In this case, the Hamilton-Jacobi-Bellman (HJB, for short) equations for their problems can be reduced to some ordinary or partial differential equations (ODE or PDE), whose solvability is easier to establish. In practice, however, even in a bull market, the market parameters, such as the interest rate, stock appreciation rates and volatilities are affected by numerous uncertainty factors such as politics, economic, legal, military, corporate governance. Thus, it is too restrictive to set market parameters as deterministic function of time even if the market status is known. It is necessary to allow the market parameters to depend on not only the Markov chain but also other random resources. With this in mind, the authors consider stochastic linear quadratic control and mean-variance problems in [15–17]. Along this framework, this paper generalizes the consumption-investment model of Cheridito and Hu [4] to a regime switching market with random coefficients. But some new mathematical challenges will appear, compared to our existing works.

Precisely, this paper studies optimal consumption-investment problems with power, logarithmic and exponential utilities in a regime switching market with parameters depending on both a Markov chain and a Brownian motion. To the best of our knowledge, concerning utility maximization problems with possibly non-convex constraints and random coefficients, the most powerful (and possibly the unique) tool is still the backward stochastic differential equation (BSDE) theory introduced by Pardoux and Peng [31]. The convex duality method mainly solves problems with convex constraints, and PDE method mainly deals with Markovian models¹. The goal of this paper is to solve the classical power, logarithmic, and exponential utilities maximization problems in the most general setting (with possible non-convex constraints, random parameters and regime switching). We believe there are no well-established methods other BSDE theory that can achieve this goal. Also, our method can take the coupled constraints on consumption and investment strategies into account for power and logarithmic utilities. The coupled constraint leads to some new type of multi-dimensional quadratic BSDEs that brings fundamental mathematical challenges and calls for new methods.

Since its inception in Pardoux and Peng [31], the BSDE theory has become a powerful tool in dealing with modern financial engineering problems including portfolio selection and asset pricing problems. For instance, using quadratic BSDE theory, Rouge and El Karoui [33] characterize the price equation for exponential utility maximization problems. With trading constraints in mind,

¹Of course, we can invoke the delicate stochastic HJB equation method in [32] to study the value function and the optimal consumption and investment strategies, but one still needs to solve the corresponding stochastic HJB equations in terms of BSDE theory.

Hu, Imkeller and Müller [13] apply BSDE theory to give solutions to exponential/power/logarithmic utility maximization of terminal wealth. Later, Cheridito and Hu [4] extend the model to incorporate intermediate consumptions. Using quadratic BSDEs driven by random measures, Becherer [3] and Morlais [30] solve exponential utility optimization and indifference valuation problems with jumps. Using quadratic BSDEs with infinite activity jumps, Laeven and Stadye [25] investigate robust portfolio choice and indifference valuation problems which take ambiguity and time-consistent ambiguity-averse preferences into account. Because of regime switching, the BSDE in our problems are all multi-dimensional, making them harder to study compared to the scalar case.

In the existing studies of consumption-investment problems, one often assumes that the consumption and investment strategies are subject to decoupled (separated) and **mainly convex constraints**; see, e.g., [4]. In practice, however, the constraints on them may be coupled together and **non-convex**. For instance, if there is no individual constraint on the consumption or investment strategies, but borrowing money is prohibited, then the constraints on the consumption and investment strategies are indeed coupled together. To the best of our knowledge, this important coupled case is still lack of study in the literature, which we believe is due to its complexity. Different from [4], this paper considers general coupled constraints on consumption and investment strategies. The coupled constraints bring new mathematical challenges to our analysis for the related multi-dimensional BSDEs.

In the end, we provide explicit optimal consumption and investment strategies, which are expressed in terms of the solutions to some multi-dimensional BSDEs. Due to the presence of the regime switching, the BSDEs in our model are actually diagonally quadratic (after proper truncation) coupled through the generator of the Markov chain. It is important to notice that there are simple examples of multi-dimensional quadratic BSDEs which are not solvable (see, e.g., Frei and Dos Reis [10, P.164]). This means multi-dimensional quadratic BSDEs are essentially different from and harder to study than the linear growth or one-dimensional counterparts. Given that the terminal value is small enough, Tevzadze [37] derives a general existence and uniqueness result for multi-dimensional quadratic BSDEs. Kramkov and Pulido [24] obtain a similar result under another type of smallness assumption. Cheridito and Nam [5], Hu and Tang [19] and Luo [27] establish existence and uniqueness results of local and global solutions for multi-dimensional BSDEs with projectable quadratic generators and sub-quadratic generators, with diagonally quadratic generators and with triangularly quadratic generators respectively.

To ensure the solvability of multi-dimensional BSDEs, the quadratic growth and local Lipschitz conditions are usually required in the literature; see Fan, Hu and Tang [9] and reference therein. Unfortunately, due to the presence of the consumption term as well as the coupled constraints on consumption and investment strategies, the generators in our multi-dimensional BSDEs do not fit the usual quadratic growth or local Lipschitz condition in [9]. Therefore, the solvability of our BSDEs cannot be covered by the results of [9]. On the other hand, the inherent structures of the BSDEs in hand are different and more involved than those studied in our previous paper [15], where only convex constraints are considered, therefore they have to be treated differently. Multi-dimensional quadratic BSDE theory is still on its early state of development, and we believe that our new examples will contribute to the development of this theory.

Solving some new multi-dimensional BSDEs constitutes the major theoretical contribution of this paper. In these BSDEs, the growth speeds or local Lipschitz condition of their generators are not clear because of the coupled constraints on consumption and investment strategies. We

establish their solvability by pure BSDE techniques, so the method may be applied to tackle other BSDE problems. Our key idea is first to do some approximation and truncation of the generators so that the existing BSDE theory with Lipschitz continuous generators can be applied. Then we establish uniformly lower and upper bounds for the approximated solutions, which means that the growth rates of the generators are no more important. Eventually, we take limit to obtain the desired solutions.

The rest part of this paper is organized as follows. We first introduce some notations and recall some facts about bounded mean oscillation (BMO) martingales that will be used frequently in the subsequent analysis. In Section 2, we introduce the financial market and formulate the consumption-investment problem in a regime switching market with random coefficients. In Sections 3, 4, 5, we solve the problem for the power, logarithmic and exponential utilities, respectively. For each utility, we first specify the consumption-investment constraint and define the admissible strategies. We then accomplish the solvability of some related multi-dimensional BSDEs. With the aid of their solutions, we finally provide the optimal consumption and investment strategies and the optimal values of the problems. We conclude the paper in Section 6.

Notation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space on which are defined a standard n -dimensional Brownian motion $W_t = (W_{1,t}, \dots, W_{n,t})'$ and a continuous-time stationary Markov chain α_t valued in a finite state space $\mathcal{M} = \{1, 2, \dots, \ell\}$ with $\ell \geq 1$. We assume $\{W_t\}_{t \geq 0}$ and $\{\alpha_t\}_{t \geq 0}$ are independent processes. The generator of the Markov chain α_t is given by $Q = (q^{ij})_{\ell \times \ell}$ with $q^{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{\ell} q^{ij} = 0$ for every $i \in \mathcal{M}$. Define the filtrations $\mathcal{F}_t = \sigma\{W_s, \alpha_s : 0 \leq s \leq t\} \vee \mathcal{N}$ and $\mathcal{F}_t^W = \sigma\{W_s : 0 \leq s \leq t\} \vee \mathcal{N}$, where \mathcal{N} is the totality of all the \mathbb{P} -null sets of \mathcal{F} .

Throughout this paper, we denote by \mathbb{R}^n the set of n -dimensional column vectors, by \mathbb{R}_+^n the set of vectors in \mathbb{R}^n whose components are nonnegative, by $\mathbb{R}^{m \times n}$ the set of $m \times n$ real matrices, and by \mathbb{S}^n the set of symmetric $n \times n$ real matrices. For $x \in \mathbb{R}$, we define $x^+ := \max\{x, 0\}$, and $x^- := \max\{-x, 0\}$. For any matrix $M = (m_{ij}) \in \mathbb{R}^{m \times n}$, we denote its transpose by M' , and its norm by $|M| = \sqrt{\sum_{ij} m_{ij}^2}$. If $M \in \mathbb{S}^n$ is positive definite (resp. positive semidefinite), we then write $M > 0$ (resp. $M \geq 0$). We write $A > B$ (resp. $A \geq B$) if $A, B \in \mathbb{S}^n$ and $A - B > 0$ (resp. $A \geq B$). For any two $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted vector-valued stochastic processes ϕ and φ , we write $\phi \geq \varphi$ (resp. $\phi > \varphi$) if it holds that

$$\phi_t^i \geq \varphi_t^i \text{ (resp. } \phi_t^i > \varphi_t^i), \quad dt \otimes d\mathbb{P} - \text{a.s. for every component } \phi_t^i \text{ and } \varphi_t^i \text{ of } \phi \text{ and } \varphi.$$

We use the following spaces throughout the paper:

$$\begin{aligned} L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}) &= \left\{ \xi : \Omega \rightarrow \mathbb{R} \mid \xi \text{ is } \mathcal{F}_T\text{-measurable, and } \mathbb{E}(|\xi|^2) < \infty \right\}, \\ L_{\mathcal{F}_T}^\infty(\Omega; \mathbb{R}) &= \left\{ \xi : \Omega \rightarrow \mathbb{R} \mid \xi \text{ is } \mathcal{F}_T\text{-measurable, and essentially bounded} \right\}, \\ L_{\mathcal{F}}^2(0, T; \mathbb{R}) &= \left\{ \phi : [0, T] \times \Omega \rightarrow \mathbb{R} \mid \phi \text{ is an } \{\mathcal{F}_t\}_{t \geq 0}\text{-predictable process with} \right. \\ &\quad \left. \mathbb{E} \int_0^T |\phi_t|^2 dt < \infty \right\}, \\ L_{\mathcal{F}}^1(0, T; \mathbb{R}) &= \left\{ \phi : [0, T] \times \Omega \rightarrow \mathbb{R} \mid \phi \text{ is an } \{\mathcal{F}_t\}_{t \geq 0}\text{-predictable process with} \right. \end{aligned}$$

$$L_{\mathcal{F}}^{\infty}(0, T; \mathbb{R}) = \left\{ \phi : [0, T] \times \Omega \rightarrow \mathbb{R} \mid \begin{array}{l} \mathbb{E} \int_0^T |\phi_t| dt < \infty, \\ \phi \text{ is an } \{\mathcal{F}_t\}_{t \geq 0}\text{-predictable essentially} \\ \text{bounded process} \end{array} \right\}.$$

These definitions are generalized in the obvious way to the cases that \mathcal{F} is replaced by \mathcal{F}^W and \mathbb{R} by \mathbb{R}^n , $\mathbb{R}^{n \times m}$ or \mathbb{S}^m .

In our analysis, some arguments (in particular in integrals) such s , t , ω , α_t , as well as some terms including “almost surely” (a.s.) and “almost everywhere” (a.e.) may be suppressed for simplicity in some circumstances if no confusion would occur.

BMO martingales

We recall some facts about BMO martingales (see Kazamaki [22]). They will play a key role in our subsequent analysis.

For a process $\Lambda \in L_{\mathcal{F}^W}^2(0, T; \mathbb{R}^n)$, the process $\int_0^\cdot \Lambda' dW$ is called a BMO martingale (on $[0, T]$) if there exists a constant $K > 0$ such that

$$\mathbb{E} \left[\int_{\tau}^T |\Lambda|^2 ds \mid \mathcal{F}_{\tau}^W \right] \leq K$$

holds for all $\{\mathcal{F}_t^W\}_{t \geq 0}$ -stopping times $\tau \leq T$. The set of BMO martingales is defined as

$$L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n) = \left\{ \Lambda \in L_{\mathcal{F}^W}^2(0, T; \mathbb{R}^n) \mid \int_0^\cdot \Lambda' dW \text{ is a BMO martingale on } [0, T] \right\}.$$

The following two important properties of BMO martingales will be used in our below arguments without claim.

- If $\int_0^\cdot \Lambda' dW$ is a BMO martingale, then Doléans-Dade stochastic exponential

$$\mathcal{E} \left(\int_0^\cdot \Lambda' dW \right)$$

is a uniformly integrable martingale.

- If $\int_0^\cdot \Lambda' dW$ and $\int_0^\cdot Z' dW$ are both BMO martingales, then $\widetilde{W} := W - \int_0^\cdot Z ds$ is a standard n -dimensional Brownian motion, and $\int_0^\cdot \Lambda' d\widetilde{W}$ is a BMO martingale under the probability measure $\widetilde{\mathbb{P}}$ defined by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_T^W} = \mathcal{E} \left(\int_0^T Z' dW \right).$$

2 Problem formulation

2.1 Financial market

Consider a financial market consisting of a risk-free asset (called the money market instrument or bond) whose price is S_0 and m risky securities (called the stocks) whose prices are S_1, \dots, S_m .

Assume $m \leq n$, i.e., the number of risky securities is no more than that of the market noises (namely the Brownian motion). Note the market is incomplete if $m < n$. These asset prices are driven by SDEs:

$$\begin{cases} dS_{0,t} = r_t^{\alpha_t} S_{0,t} dt, \\ S_{0,0} = s_0, \end{cases}$$

and

$$\begin{cases} dS_{k,t} = S_{k,t} \left(\mu_{k,t}^{\alpha_t} dt + \sum_{j=1}^n \sigma_{kj,t}^{\alpha_t} dW_{j,t} \right), \\ S_{k,0} = s_k, \end{cases}$$

where r_t^i is the interest rate process and $\mu_{k,t}^i$ and $\sigma_{k,t}^i := (\sigma_{k1,t}^i, \dots, \sigma_{kn,t}^i)$ are the appreciation rate process and volatility rate process of the k th risky security corresponding to a market regime $\alpha_t = i$, for every $k = 1, \dots, m$ and $i \in \mathcal{M}$. Recall that α_t follows a continuous-time stationary Markov chain valued in a finite state space $\mathcal{M} = \{1, 2, \dots, \ell\}$ with $\ell \geq 1$. If $\ell = 1$, then there is no regime switching and the market becomes the classical Black-Scholes market. In our below argument we assume $\ell > 1$, although all the results remain true if $\ell = 1$.

Define the appreciate vector

$$\mu_t^i = (\mu_{1,t}^i, \dots, \mu_{m,t}^i)',$$

and volatility matrix

$$\sigma_t^i = \begin{pmatrix} \sigma_{1,t}^i \\ \vdots \\ \sigma_{m,t}^i \end{pmatrix} \equiv (\sigma_{kj,t}^i)_{m \times n}, \text{ for each } i \in \mathcal{M}.$$

2.2 Optimal investment-consumption problem

Consider a representative small investor, whose actions cannot affect the asset prices. He will decide at every time $t \in [0, T]$ the *proportion* $\pi_{j,t}$ of his wealth to invest in the j th risky asset, $j = 1, \dots, m$, as well as the *proportion* c_t of his wealth to consume. The vector process $\pi_t := (\pi_{1,t}, \dots, \pi_{m,t})'$ is called a portfolio of the investor. The pair (c, π) is called a consumption-investment strategy. The investor's self-financing wealth process X corresponding to a consumption-investment strategy (c, π) is a strong solution of the SDE (see, e.g., [21]):

$$\begin{cases} dX_t = X_t [r_t^{\alpha_t} + \pi_t' b_t^{\alpha_t} - c_t] dt + X_t \pi_t' \sigma_t^{\alpha_t} dW_t, \\ X_0 = x > 0, \alpha_0 = i_0 \in \mathcal{M}, \end{cases} \quad (2.1)$$

where $b_t^i := \mu_t^i - r_t^i \mathbf{1}_m$ for all $i \in \mathcal{M}$ and $\mathbf{1}_m$ is the m -dimensional vector with all entries being one. By Itô's lemma, one can easily show that the process X is always positive, hence the investor would never be bankrupt.

The investor's problem is to maximize his cost functional

$$J(x, i_0; \pi, c) := \mathbb{E} \left[\int_0^T e^{-\int_0^t \rho_u^{\alpha u} du} U(c_t X_t) dt + e^{-\int_0^T \rho_u^{\alpha u} du} U(X_T) \right], \quad \text{s.t. } (c, \pi) \in \mathcal{U}, \quad (2.2)$$

and to determine the value function

$$V(x, i_0) := \sup_{(c, \pi) \in \mathcal{U}} J(x, i_0; \pi, c),$$

where $\rho^i \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R})$, $i \in \mathcal{M}$, are the discount factor processes, U is the utility function of the investor, and \mathcal{U} is the set of admissible consumption-investment strategies. The utility U and admissible set \mathcal{U} will be defined in the sequel case by case.

Our model distinguishes from the existing models in the following aspects. Usually, the discount factor ρ is assumed to be deterministic and market regime independent in the literature; by contrast, it is allowed to be noise and regime dependent (thus stochastic) in our model. Our model not only generalizes the existing ones, but also fits the reality better sometimes. For instance, many investors use relative discount factors that depend on the interest rate or some stock index, thus become noise and regime dependent.

Another distinguish feature of our model is that the trading constraints put on the consumption and investment strategies. In the existing literature, their constraints are separately considered and convex. In our model, they are subject to fairly general coupled constraints so that may be non-convex. This is not only a more general case but also of great practical importance. For instance, it happens if one cannot borrow too much money (which is generally true in practice) in the market so that the amount of consumption and investment together cannot beyond what the investor can use. The study of this kind of coupled situations is rare in the literature, due to, we believe, the technical challenges. We will solve the problem completely for power and logarithmic utilities, but cannot do it perfectly for exponential utilities. Indeed, we suspect that the value functions are not of the exponential form any more when the constraints are coupled together in the latter case.

Let Θ denote the set of constraints on the consumption-investment strategies, which is assumed to be a given closed nonempty set in $\mathbb{R}_+ \times \mathbb{R}^m$.

Example 2.1 *Here are some important and interesting examples for the constraint set Θ in financial practice.*

- *If the investor is not allowed to borrow money, then the total consumption and investment cannot be beyond 100%, that is,*

$$\Theta = \left\{ (c, \pi) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \sum_{j=1}^m \pi_j + c \leq 1 \right\}.$$

Clearly, the constraints on the consumption and investment strategies are coupled together in this case. It will bring some mathematical challenges to our analysis for the related multi-dimensional BSDEs below.

- *If shorting is prohibited in the stock market, then*

$$\Theta = \left\{ (c, \pi) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \pi_j \geq 0, j = 1, \dots, m \right\}.$$

In this case, Θ is cone and the constraints on the consumption and investment strategies are decoupled.

- In practice, some stocks may be restricted to trade; for instance

$$\Theta = \left\{ (c, \pi) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \pi_j \in [d_j, e_j], j = 1, \dots, m \right\}.$$

Or some constraint may be put on the consumption strategies such as

$$\Theta = \left\{ (c, \pi) \in \mathbb{R}_+ \times \mathbb{R}^m \mid c \in [0, c_0] \right\}.$$

Or the above constraints are coupled together so that

$$\Theta = \left\{ (c, \pi) \in \mathbb{R}_+ \times \mathbb{R}^m \mid c \in [0, c_0], \pi_j \in [d_j, e_j], j = 1, \dots, m \right\}.$$

In these cases, the constraints on the consumption and investment strategies are also decoupled.

We put the following standard assumption on the market parameters.

Assumption 1 For all $i \in \mathcal{M}$,

$$\begin{cases} r^i \in L_{\mathbb{F}^W}^\infty(0, T; \mathbb{R}), \mu^i \in L_{\mathbb{F}^W}^\infty(0, T; \mathbb{R}^m), \\ \sigma^i \in L_{\mathbb{F}^W}^\infty(0, T; \mathbb{R}^{m \times n}), \rho^i \in L_{\mathbb{F}^W}^\infty(0, T; \mathbb{R}). \end{cases}$$

Also, there exists a constant $\delta > 0$ such that $\sigma_t^i (\sigma_t^i)' \geq \delta \mathbf{1}_m$ for all $i \in \mathcal{M}$ and $t \in [0, T]$ a.s., where $\mathbf{1}_m$ denotes the m -dimensional identity matrix.

Due to different features of the power, logarithmic and exponential utilities, they must be dealt separately. We study them in the subsequent Sections 3, 4 and 5, respectively.

3 Power utility

In this section, we assume that the investor's utility function is

$$U(x) = \frac{1}{\gamma} x^\gamma, \quad x > 0, \quad \gamma \in (-\infty, 0) \cup (0, 1)$$

and the admissible consumption-investment set is defined as

$$\mathcal{U} = \left\{ (c, \pi) \mid \int_0^T (|\pi|^2 + c) dt < \infty, a.s., (c, \pi) \in \Theta, dt \otimes d\mathbb{P} - a.s. \right\},$$

if $\gamma \in (0, 1)$; and

$$\begin{aligned} \mathcal{U} = \left\{ (c, \pi) \mid \int_0^T (|\pi|^2 + c) dt < \infty, a.s., (c, \pi) \in \Theta, dt \otimes d\mathbb{P} - a.s., \right. \\ \left. (X_t^\gamma)_{0 \leq t \leq T} \text{ belongs to class (D) and } \mathbb{E} \left[\int_0^T (cX)^\gamma dt \right] < \infty \right\}, \end{aligned}$$

if $\gamma < 0$.²

²We say that a process $(X_t^\gamma)_{0 \leq t \leq T}$ belongs to class (D) if $\{X_t^\gamma : \tau \text{ stopping times valued in } [0, T]\}$ is a family of uniformly integrable random variables.

Assumption 2 If $\gamma \in (0, 1)$, then no consumption or investment is always permitted, namely $(0, \mathbf{0}_m) \in \Theta$, where $\mathbf{0}_m$ is the m -dimensional vector with all entries being 0. If $\gamma < 0$, then there exists a constant $\varepsilon > 0$ such that $(\varepsilon, \mathbf{0}_m) \in \Theta$.

Due to homogeneity, we conjecture that its optimal value function of problem (2.2) has the following decomposition

$$V(x, i) = \frac{1}{\gamma} x^\gamma P_0^i$$

for some stochastic process P^i , which is governed by some BSDE. Indeed, $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ satisfies the following ℓ -dimensional BSDE:

$$\begin{cases} dP^i = -[f^i(t, P^i, \Lambda^i) - (\rho^i - \gamma r^i)P^i + \sum_{j=1}^{\ell} q^{ij} P^j] dt + (\Lambda^i)' dW, \\ P_T^i = 1, \\ P^i > 0 \text{ for all } i \in \mathcal{M}, \end{cases} \quad (3.1)$$

where, for any $(t, P, \Lambda) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}^n$,

$$f^i(t, P, \Lambda) = \gamma \sup_{(c, \pi) \in \Theta} \left[-\frac{1-\gamma}{2} |\pi' \sigma_t^i|^2 P + \pi' (P b_t^i + \sigma_t^i \Lambda) + \frac{c^\gamma}{\gamma} - P c \right].$$

In particular, if there is no consumption-investment constraint, namely $\Theta = \mathbb{R}_+ \times \mathbb{R}^m$, then

$$f^i(t, P, \Lambda) = \frac{1}{2} \frac{\gamma}{1-\gamma} \frac{1}{P} (P b^i + \sigma^i \Lambda)' (\sigma^i (\sigma^i)')^{-1} (P b^i + \sigma^i \Lambda) + (1-\gamma) P^{-\frac{\gamma}{1-\gamma}}.$$

We will first show the BSDE (3.1) has a solution, and then prove our conjecture.

Note that the coupling phenomenon of P and Λ in the first term, the range of $-\frac{\gamma}{1-\gamma}$ is $(-\infty, 0) \cup (0, \infty)$ for $\gamma \in (-\infty, 0) \cup (0, 1)$, the generators f^i , $i \in \mathcal{M}$, violate the quadratic growth and local Lipschitz conditions in (P, Λ) , which conditions are often required to ensure the solvability of BSDEs in the literature; see [9] and reference therein. Therefore, the solvability of (3.1) cannot be covered by the existing literature such as [9] and [15]. To solve such kind of BSDEs is one of the major contributions of this paper.

Definition 3.1 A vector process $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ is called a solution to the ℓ -dimensional BSDE (3.1), if it satisfies (3.1), and $(P^i, \Lambda^i) \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$. A solution $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ to (3.1) is called uniformly positive if $P^i \geq \delta$ for all $i \in \mathcal{M}$ with some deterministic constant $\delta > 0$.

The following comparison theorem for multi-dimensional BSDE firstly appeared in [18] (one can find a concise version in [14, Lemma 2.2] or [15, Lemma 3.4]). We shall use it frequently in our subsequent study of BSDEs.

Lemma 3.2 Suppose $(Y^i, Z^i)_{i \in \mathcal{M}}$ and $(\bar{Y}^i, \bar{Z}^i)_{i \in \mathcal{M}}$ satisfy the following two ℓ -dimensional BSDEs, respectively:

$$Y_t^i = \xi^i + \int_t^T g^i(s, Y^1, \dots, Y^\ell, Z^i) ds - \int_t^T (Z^i)' dW$$

and

$$\bar{Y}_t^i = \bar{\xi}^i + \int_t^T \bar{g}^i(s, \bar{Y}^1, \dots, \bar{Y}^\ell, \bar{Z}^i) ds - \int_t^T (\bar{Z}^i)' dW$$

for all $i \in \mathcal{M}$. Also suppose that, for all $i \in \mathcal{M}$ and $s \in [0, T]$, we have

1. $\xi^i, \bar{\xi}^i \in L^2_{\mathcal{F}^W}(\Omega; \mathbb{R})$, and $\xi^i \leq \bar{\xi}^i$;
2. there exists a constant $K > 0$ such that

$$|g^i(s, y^1, \dots, y^\ell, z) - g^i(s, \bar{y}^1, \dots, \bar{y}^\ell, \bar{z})| \leq K \left(\sum_{i=1}^{\ell} |y^i - \bar{y}^i| + |z - \bar{z}| \right),$$

for any $(y^1, \dots, y^\ell), (\bar{y}^1, \dots, \bar{y}^\ell) \in \mathbb{R}^\ell$, and $z, \bar{z} \in \mathbb{R}^n$;

3. $g^i(s, y^1, \dots, y^\ell, z)$ is nondecreasing in every y^j , $j \neq i \in \mathcal{M}$; and
4. $g^i(s, \bar{Y}_s^1, \dots, \bar{Y}_s^\ell, \bar{Z}_s^i) \leq \bar{g}^i(s, \bar{Y}_s^1, \dots, \bar{Y}_s^\ell, \bar{Z}_s^i)$.

Then $Y^i \leq \bar{Y}^i$ for all $i \in \mathcal{M}$.

In the above lemma, if conditions 2 and 3 hold for \bar{g}^i instead of g^i , and condition 4 is replaced by

$$g^i(s, Y_s^1, \dots, Y_s^\ell, Z_s^i) \leq \bar{g}^i(s, Y_s^1, \dots, Y_s^\ell, Z_s^i),$$

then the conclusion still holds.

We now focus on the solvability of the BSDE (3.1). The cases $\gamma \in (0, 1)$ and $\gamma < 0$ will be treated separately. We start with the former one.

Theorem 3.3 *Suppose $\gamma \in (0, 1)$ and Assumptions 1 and 2 hold. Then there is a unique uniformly positive solution $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ to the BSDE (3.1).*

Proof: To find a solution to the BSDE (3.1) that violate both the quadratic growth and local Lipschitz conditions, our key idea is first to do some approximation and truncation of the generators so that the existing results for BSDEs with Lipschitz continuous generators can be applied.

To this end, we first notice the coefficients are bounded, so there exists a constant $a > 1$ such that

$$-\rho^i + \gamma r^i \geq -a, \tag{3.2}$$

and

$$\frac{\gamma}{2(1-\gamma)} (b^i)' (\sigma^i (\sigma^i)')^{-1} b^i - \rho^i + \gamma r^i \leq a(1-\gamma), \tag{3.3}$$

hold simultaneously for all $i \in \mathcal{M}$. Define two constants $a_1 = e^{-aT}$ and $a_2 = (e^{aT} + \frac{1}{a}(e^{aT} - 1))^{1-\gamma}$.

Let $g : \mathbb{R} \rightarrow [0, 1]$ be a smooth truncation function satisfying $g(x) = 0$ for $x \in (-\infty, \frac{1}{2}a_1]$, and $g(x) = 1$ for $x \in [a_1, +\infty)$. For each $k \geq 1$, $(t, P, \Lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$, $i \in \mathcal{M}$, define

$$f^{k,i}(t, P, \Lambda) = \inf_{\tilde{P} \in \mathbb{R}, \tilde{\Lambda} \in \mathbb{R}^n} \left[f^i(t, \tilde{P}, \tilde{\Lambda}) g(\tilde{P}) + k|P - \tilde{P}| + k|\Lambda - \tilde{\Lambda}| \right].$$

Then $f^{k,i}(t, P, \Lambda)$ is uniformly Lipschitz continuous in (P, Λ) , and increasingly approaches to $f^i(t, P, \Lambda)g(P)$ as k goes to infinity. According to Assumption 2, we have $f^i \geq 0$, hence $f^{k,i} \geq 0$.

The following ℓ -dimensional BSDE

$$\begin{cases} dP^{k,i} = -\left[f^{k,i}(t, P^{k,i}, \Lambda^{k,i}) - (\rho^i - \gamma r^i)P^{k,i} + \sum_{j=1}^{\ell} q^{ij}P^{k,j} \right] dt + (\Lambda^{k,i})' dW, \\ P_T^{k,i} = 1 \text{ for all } i \in \mathcal{M}, \end{cases}$$

has a Lipschitz generator³, so it admits a unique solution, denoted by $(P^{k,i}, \Lambda^{k,i})_{i \in \mathcal{M}}$.

It is direct to verify that

$$(\underline{P}_t^i, \underline{\Lambda}_t^i) = \left(e^{-a(T-t)}, 0 \right), \quad i \in \mathcal{M}$$

is the unique solution to the following linear BSDE system:

$$\begin{cases} d\underline{P}^i = -\left[-a\underline{P}^i + \sum_{j=1}^{\ell} q^{ij}\underline{P}^j \right] dt + (\underline{\Lambda}^i)' dW, \\ \underline{P}_T^i = 1 \text{ for all } i \in \mathcal{M}. \end{cases}$$

Notice that $f^{k,i}(t, P, \Lambda) \geq 0$ and recall (3.2), so

$$f^{k,i}(t, \underline{P}^i, \underline{\Lambda}^i) - (\rho^i - \gamma r^i)\underline{P}^i + \sum_{j=1}^{\ell} q^{ij}\underline{P}^j \geq -(\rho^i - \gamma r^i)\underline{P}^i + \sum_{j=1}^{\ell} q^{ij}\underline{P}^j \geq -a\underline{P}^i + \sum_{j=1}^{\ell} q^{ij}\underline{P}^j.$$

By Lemma 3.2, we have

$$a_1 \leq e^{-a(T-t)} = \underline{P}_t^i \leq P_t^{k,i} \text{ for all } i \in \mathcal{M}.$$

On the other hand, direct calculation shows that

$$(\bar{P}_t^i, \bar{\Lambda}_t^i) = \left(\left(e^{a(T-t)} + \frac{1}{a}(e^{a(T-t)} - 1) \right)^{1-\gamma}, 0 \right) \text{ for all } i \in \mathcal{M},$$

is a solution to the following BSDE system:

$$\begin{cases} d\bar{P}^i = -\left[a(1-\gamma)\bar{P}^i + (1-\gamma)(\bar{P}^i)^{-\frac{\gamma}{1-\gamma}} + \sum_{j=1}^{\ell} q^{ij}\bar{P}^j \right] dt + (\bar{\Lambda}^i)' dW, \\ \bar{P}_T^i = 1 \text{ for all } i \in \mathcal{M}. \end{cases}$$

Notice, for $P > 0$,

$$\begin{aligned} f^{k,i}(t, P, \Lambda) &\leq f^i(t, P, \Lambda)g(P) \\ &\leq \gamma \sup_{(c, \pi) \in \mathbb{R}^n \times \mathbb{R}_+} \left[-\frac{1-\gamma}{2} |\pi' \sigma^i|^2 P + \pi'(Pb^i + \sigma^i \Lambda) + \frac{c^\gamma}{\gamma} - Pc \right] g(P) \\ &= \frac{1}{2} \frac{\gamma}{1-\gamma} \frac{1}{P} (Pb^i + \sigma^i \Lambda)' (\sigma^i (\sigma^i)')^{-1} (Pb^i + \sigma^i \Lambda) g(P) + (1-\gamma) P^{-\frac{\gamma}{1-\gamma}} g(P) \\ &\leq \frac{\gamma}{2(1-\gamma)} (b^i)' (\sigma^i (\sigma^i)')^{-1} b^i P + \frac{\gamma}{1-\gamma} (b^i)' (\sigma^i (\sigma^i)')^{-1} \sigma^i \Lambda \\ &\quad + \frac{\gamma}{2(1-\gamma)} \frac{1}{P} (\sigma^i \Lambda)' (\sigma^i (\sigma^i)')^{-1} \sigma^i \Lambda + (1-\gamma) P^{-\frac{\gamma}{1-\gamma}}. \end{aligned} \tag{3.4}$$

³For a BSDE $Y_t = \xi + \int_t^T f(s, Y, Z) ds - \int_t^T Z' dW$, f is called the generator and ξ is called the terminal value.

Hence, by (3.3),

$$\begin{aligned}
& f^{k,i}(t, \bar{P}^i, \bar{\Lambda}^i) - (\rho^i - \gamma r^i) \bar{P}^i + \sum_{j=1}^{\ell} q^{ij} \bar{P}^j \\
& \leq \frac{\gamma}{2(1-\gamma)} (b^i)' (\sigma^i (\sigma^i)')^{-1} b^i \bar{P}^i + (1-\gamma) (\bar{P}^i)^{-\frac{\gamma}{1-\gamma}} - (\rho^i - \gamma r^i) \bar{P}^i + \sum_{j=1}^{\ell} q^{ij} \bar{P}^j \\
& \leq a(1-\gamma) \bar{P}^i + (1-\gamma) (\bar{P}^i)^{-\frac{\gamma}{1-\gamma}} + \sum_{j=1}^{\ell} q^{ij} \bar{P}^j.
\end{aligned}$$

Then by Lemma 3.2 again, we have

$$P_t^{k,i} \leq \bar{P}_t^i \leq \left(e^{aT} + \frac{1}{a} (e^{aT} - 1) \right)^{1-\gamma} = a_2,$$

and $P^{k,i}$ is increasing in k , for each $i \in \mathcal{M}$.

Let $P_t^i = \lim_{k \rightarrow \infty} P_t^{k,i}$, $i \in \mathcal{M}$. Since a_1 and a_2 are independent of k , it follows that $a_1 \leq P^i \leq a_2$. Recalling $f^{k,i} \geq 0$, (3.4) and the role of the truncation function g , we can regard $(P^{k,i}, \Lambda^{k,i})$ as the solution of a scalar-valued quadratic BSDE for each $i \in \mathcal{M}$. Thus by [23, Proposition 2.4], there exists a process $\Lambda \in L^2_{\mathcal{F}^W}(0, T; \mathbb{R}^{n \times \ell})$ such that (P, Λ) satisfies the BSDE (3.1).

Applying Itô's formula to the process $(P_t^i - a_2)^2$, we obtain, for any stopping times $\tau \leq T$,

$$\begin{aligned}
\mathbb{E} \left[\int_{\tau}^T |\Lambda^i|^2 ds \middle| \mathcal{F}_{\tau}^W \right] &= (1 - a_2)^2 - (P_{\tau}^i - a_2)^2 \\
&+ 2\mathbb{E} \left[\int_{\tau}^T (P^i - a_2) \left[f^i(t, P^i, \Lambda^i) - (\rho^i - \gamma r^i) P^i + \sum_{j=1}^{\ell} q^{ij} P^j \right] ds \middle| \mathcal{F}_{\tau}^W \right] \\
&\leq (1 - a_2)^2 - (P_{\tau}^i - a_2)^2 \\
&+ 2\mathbb{E} \left[\int_{\tau}^T (P^i - a_2) \left[-(\rho^i - \gamma r^i) P^i + \sum_{j=1}^{\ell} q^{ij} P^j \right] ds \middle| \mathcal{F}_{\tau}^W \right] \\
&\leq (1 - a_2)^2 + 2\mathbb{E} \left[\int_0^T \left| (P^i - a_2) \left[-(\rho^i - \gamma r^i) P^i + \sum_{j=1}^{\ell} q^{ij} P^j \right] \right| ds \middle| \mathcal{F}_{\tau}^W \right] \\
&\leq (1 - a_2)^2 + 2T \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} \left| (P^i - a_2) \left[-(\rho^i - \gamma r^i) P^i + \sum_{j=1}^{\ell} q^{ij} P^j \right] \right|
\end{aligned}$$

where we used the fact that $P^i \leq a_2$ and $f^i \geq 0$ to get the first inequality. Because $a_1 \leq P^i \leq a_2$ and Assumption 1, we see that the right hand side is a finite constant. Hence $\Lambda^i \in L^2_{\mathcal{F}^W, \text{BMO}}(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$. We have now established the existence of the solution.

Next, let us prove the uniqueness. Suppose $(P^i, \Lambda^i)_{i \in \mathcal{M}}, (\tilde{P}^i, \tilde{\Lambda}^i)_{i \in \mathcal{M}}$ are two uniformly positive solutions of (3.1). For every $i \in \mathcal{M}$, define processes

$$(Y^i, Z^i) = \left(\ln P^i, \frac{\Lambda^i}{P^i} \right), (\tilde{Y}^i, \tilde{Z}^i) = \left(\ln \tilde{P}^i, \frac{\tilde{\Lambda}^i}{\tilde{P}^i} \right). \quad (3.5)$$

Then $(Y^i, Z^i), (\tilde{Y}^i, \tilde{Z}^i) \in L_{\mathcal{F}W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$. Furthermore, by Itô's formula, $(Y^i, Z^i)_{i \in \mathcal{M}}$ satisfies the following ℓ -dimensional BSDE:

$$\begin{cases} dY^i = - \left[F^i(t, Y^i, Z^i) + \frac{1}{2} |Z^i|^2 - \rho^i + \gamma r^i + \sum_{j=1}^{\ell} q^{ij} e^{Y^j - Y^i} \right] dt + (Z^i)' dW, \\ Y_T^i = 0 \text{ for all } i \in \mathcal{M}, \end{cases} \quad (3.6)$$

where for any $(Y, Z) \in \mathbb{R} \times \mathbb{R}^n$,

$$F^i(t, Y, Z) = \gamma \sup_{(c, \pi) \in \Theta} \left[-\frac{1-\gamma}{2} |\pi' \sigma_t^i|^2 + \pi'(b_t^i + \sigma_t^i Z) + \frac{c^\gamma}{\gamma} e^{-Y} - c \right].$$

The coupled constraint makes it hard to verify the local Lipschitz condition for the above generators, which is often needed to solve BSDEs. To the best of our knowledge, such kind of BSDEs has not been studied in the literature. See our further discussion on the new BSDE (3.6) after the proof in Remark 3.4.

Also, $(\tilde{Y}^i, \tilde{Z}^i)_{i \in \mathcal{M}}$ satisfies a similar BSDE.

Because $e^{-Y^i} = \frac{1}{P^i} \leq \frac{1}{a_1}$ and use Assumption 1, for any $(c, \pi) \in \Theta$,

$$\begin{aligned} & -\frac{1-\gamma}{2} |\pi' \sigma^i|^2 + \pi'(b^i + \sigma^i Z^i) + \frac{c^\gamma}{\gamma} e^{-Y^i} - c \\ & \leq -\delta \frac{1-\gamma}{2} |\pi|^2 + K |\pi| (1 + |Z|) + \sup_{c \in \mathbb{R}_+} \left(\frac{c^\gamma}{\gamma} \frac{1}{a_1} - c \right) \\ & = -\delta \frac{1-\gamma}{2} |\pi|^2 + K |\pi| (1 + |Z|) + \frac{1-\gamma}{\gamma} a_1^{\frac{1}{1-\gamma}} \\ & < 0, \end{aligned}$$

if $|\pi| > K(1 + |Z^i|)$ for sufficient large K .⁴ Since $(0, \mathbf{0}_{m+1}) \in \Theta$, $F^i(t, Y, Z) \geq 0$. Therefore

$$F^i(t, Y^i, Z^i) = \gamma \sup_{\substack{(c, \pi) \in \Theta \\ |\pi| \leq K(1 + |Z^i|)}} \left[-\frac{1-\gamma}{2} |\pi' \sigma_t^i|^2 + \pi'(b_t^i + \sigma_t^i Z^i) + \frac{c^\gamma}{\gamma} e^{-Y^i} - c \right]. \quad (3.7)$$

Note this still holds if $K(1 + |Z^i|)$ is replaced by any bigger quantity. A similar expression holds for $F^i(t, \tilde{Y}^i, \tilde{Z}^i)$.

Set $\bar{Y}^i = Y^i - \tilde{Y}^i$, $\bar{Z}^i = Z^i - \tilde{Z}^i$, for all $i \in \mathcal{M}$. From (3.7), we have

$$\left| F^i(t, Y^i, Z^i) - F^i(t, \tilde{Y}^i, \tilde{Z}^i) \right| \leq \gamma \sup_{\substack{(c, \pi) \in \Theta \\ |\pi| \leq K(1 + |Z^i| + |\tilde{Z}^i|)}} |\pi' \sigma^i \bar{Z}^i| \leq K(1 + |Z^i| + |\tilde{Z}^i|) |\bar{Z}^i|.$$

We can define β^i in an obvious way such that

$$F^i(t, Y^i, Z^i) - F^i(t, \tilde{Y}^i, \tilde{Z}^i) = (\beta^i)' \bar{Z}^i,$$

and

$$|\beta^i| \leq K(1 + |Z^i| + |\tilde{Z}^i|).$$

⁴ Hereafter, we shall use K to represent a generic positive constant independent of i , m , n and t , which can be different from line to line.

Since $Z^i, \tilde{Z}^i \in L^2_{\mathcal{F}^W, \text{BMO}}(0, T; \mathbb{R}^n)$, we have $\beta^i \in L^2_{\mathcal{F}^W, \text{BMO}}(0, T; \mathbb{R}^n)$.

Now applying Itô's formula to $(\bar{Y}^i)^2$, we deduce that

$$\begin{aligned} (\bar{Y}_t^i)^2 &= \int_t^T \left\{ 2\bar{Y}^i \left[F^i(t, Y^i, Z^i) - F^i(\tilde{Y}^i, \tilde{Z}^i) + \frac{1}{2}(|Z^i|^2 - |\tilde{Z}^i|^2) \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^{\ell} q^{ij} \left(e^{Y^j - Y^i} - e^{\tilde{Y}^j - \tilde{Y}^i} \right) \right] - |\bar{Z}^i|^2 \right\} ds - \int_t^T 2\bar{Y}^i (\bar{Z}^i)' dW. \end{aligned} \quad (3.8)$$

Notice that the map

$$Y \mapsto F^i(Y, Z), \quad Y \in \mathbb{R}_+,$$

is non-increasing for each fixed $Z \in \mathbb{R}^n$ and $i \in \mathcal{M}$. Therefore,

$$\begin{aligned} &\bar{Y}^i [F^i(t, Y^i, Z^i) - F^i(t, \tilde{Y}^i, \tilde{Z}^i)] \\ &= \bar{Y}^i [F^i(t, Y^i, Z^i) - F^i(t, Y^i, \tilde{Z}^i) + F^i(t, Y^i, \tilde{Z}^i) - F^i(t, \tilde{Y}^i, \tilde{Z}^i)] \\ &\leq \bar{Y}^i [F^i(t, Y^i, Z^i) - F^i(t, Y^i, \tilde{Z}^i)] \\ &= \bar{Y}^i (\beta^i)' \bar{Z}^i. \end{aligned}$$

For each fixed $i \in \mathcal{M}$, let us introduce the process

$$N_t^i = \mathcal{E} \left(\int_0^t (\beta^i + \frac{1}{2}(Z^i + \tilde{Z}^i))' dW \right).$$

Then N_t^i is a uniformly integrable martingale. Notice $Y^i, i \in \mathcal{M}$, are bounded processes, from (3.8),

$$\begin{aligned} (\bar{Y}_t^i)^2 &\leq \int_t^T \left\{ 2\bar{Y}^i \left[(\beta^i)' \bar{Z}^i + \frac{1}{2}(Z^i + \tilde{Z}^i)' \bar{Z}^i + K \sum_{j \neq i}^{\ell} \bar{Y}^j \right] \right\} ds - \int_t^T 2\bar{Y}^i (\bar{Z}^i)' dW \\ &= \int_t^T 2K\bar{Y}^i \sum_{j \neq i}^{\ell} \bar{Y}^j ds - \int_t^T 2\bar{Y}^i (\bar{Z}^i)' d\tilde{W}^i \end{aligned}$$

where

$$\tilde{W}_t^i := W_t - \int_0^t \left(\beta^i + \frac{1}{2}(Z^i + \tilde{Z}^i) \right) ds,$$

is a Brownian motion under the probability $\tilde{\mathbb{P}}^i$ defined by

$$\frac{d\tilde{\mathbb{P}}^i}{d\mathbb{P}} \Big|_{\mathcal{F}_T^W} = N_T^i.$$

Taking expectation $\tilde{\mathbb{E}}^i$ w.r.t. the probability measure $\tilde{\mathbb{P}}^i$,

$$(\bar{Y}_t^i)^2 \leq \tilde{\mathbb{E}}^i \left[\int_t^T 2K\bar{Y}^i \sum_{j \neq i}^{\ell} \bar{Y}^j ds \mid \mathcal{F}_t^W \right] \leq K\tilde{\mathbb{E}}^i \left[\int_t^T \sum_{j=1}^{\ell} (\bar{Y}^j)^2 ds \mid \mathcal{F}_t^W \right] \leq K \int_t^T \sum_{j=1}^{\ell} E^j ds,$$

by the inequality of arithmetic and geometric means (AM-GM inequality), where

$$E_t^i = \text{ess sup}_{\omega \in \Omega} (\bar{Y}_t^i)^2.$$

Taking essential supreme on both sides of the above inequality yields

$$E_t^i \leq K \int_t^T \sum_{j=1}^{\ell} E^j ds.$$

Thus

$$0 \leq \sum_{j=1}^{\ell} E_t^j \leq K\ell \int_t^T \sum_{j=1}^{\ell} E^j ds.$$

We infer from Gronwall's inequality that $\sum_{j=1}^{\ell} E^j = 0$, so $\bar{Y}^i = 0$ for all $i \in \mathcal{M}$. This completes the proof of the uniqueness. \square

Remark 3.4 *If the constraints on consumption and investment are decoupled, i.e. $\Theta = \mathcal{C} \times \Pi$, then one can verify that*

$$F^i(t, Y, Z) = \gamma \sup_{c \in \mathcal{C}} \left[\frac{c^\gamma}{\gamma} e^{-Y} - c \right] + \gamma \sup_{\pi \in \Pi} \left[-\frac{1-\gamma}{2} |\pi' \sigma_t^i|^2 + \pi'(b_t^i + \sigma_t^i Z) \right], \quad i \in \mathcal{M},$$

satisfy the local Lipschitz condition [9, Assumption (H2)]. Although the term $\sum_{j=1}^{\ell} \ell q^{ij} e^{Y^j - Y^i}$ in (3.6) violates [9, Assumption (H4)], the BSDE (3.6) can still be solved by [9, Theorem 2.4] with some suitable truncations on Y_j , $j \in \mathcal{M}$.

By contrast, if the constraints on consumption and investment are coupled together, then it is difficult to verify [9, Assumption (H2)] for the above generators, so we could not use [9, Theorem 2.4, Theorem 2.5] to solve our BSDE (3.6).

Now we turn to the case $\gamma < 0$.

Theorem 3.5 *Suppose $\gamma < 0$ and Assumption 1 and 2 hold. Then there is a unique uniformly positive solution $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ to the BSDE (3.1).*

Proof: The proof is similar to the procedure of Theorem 3.3 with different uniformly lower and upper bounds. Here, we only present how to find these bounds, the remainder details are left to the interested readers.

Due to the boundedness of the coefficients, there exists a constant $a > 0$ such that

$$\frac{\gamma}{2(1-\gamma)} (b^i)' (\sigma^i (\sigma^i)')^{-1} b^i - \rho^i + \gamma r^i \geq -a(1-\gamma), \quad (3.9)$$

and

$$-\rho^i + \gamma r^i - \gamma \varepsilon \leq a, \quad (3.10)$$

hold simultaneously for all $i \in \mathcal{M}$, where $\varepsilon > 0$ is given in Assumption 2. Define two constants $a_1 = e^{-a(1-\gamma)T}$ and $a_2 = e^{aT} + \frac{\varepsilon \gamma}{a} (e^{aT} - 1)$.

Let $g : \mathbb{R} \rightarrow [0, 1]$ be a smooth truncation function satisfying $g(x) = 0$ for $x \in (-\infty, \frac{1}{2}a_1]$, and $g(x) = 1$ for $x \in [a_1, +\infty)$. Notice that $|xg(x)| \leq |x|$ for all $x \in \mathbb{R}$.

For each positive integer $k \geq -\gamma \varepsilon$, $(t, P, \Lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$, $i \in \mathcal{M}$, define

$$f^{k,i}(t, P, \Lambda) = \sup_{\tilde{P} \in \mathbb{R}, \tilde{\Lambda} \in \mathbb{R}^n} \left[f^i(t, \tilde{P}, \tilde{\Lambda}) g(\tilde{P}) - k|P - \tilde{P}| - k|\Lambda - \tilde{\Lambda}| \right].$$

And consider the following ℓ -dimensional BSDE:

$$\begin{cases} dP^{k,i} = - \left[f^{k,i}(t, P^{k,i}, \Lambda^{k,i}) - (\rho^i - \gamma r^i) P^{k,i} g(P^{k,i}) + \sum_{j=1}^{\ell} q^{ij} P^{k,j} \right] dt + (\Lambda^{k,i})^\top dW, \\ P_T^{k,i} = 1 \text{ for all } i \in \mathcal{M}. \end{cases}$$

Since its generator is Lipschitz continuous, it admits a unique solution, denoted by $(P^{k,i}, \Lambda^{k,i})_{i \in \mathcal{M}}$.

It is easy to verify that

$$(\underline{P}_t^i, \underline{\Lambda}_t^i) = \left(\left(e^{-a(T-t)} + \frac{1}{a}(1 - e^{-a(T-t)}) \right)^{1-\gamma}, 0 \right) \text{ for all } i \in \mathcal{M}, \quad (3.11)$$

satisfies the following BSDE:

$$\begin{cases} d\underline{P}^i = - \left[-a(1-\gamma)\underline{P}^i g(\underline{P}^i) + (1-\gamma)(\underline{P}^i)^{-\frac{\gamma}{1-\gamma}} g(\underline{P}^i) + \sum_{j=1}^{\ell} q^{ij} \underline{P}^j \right] dt + (\underline{\Lambda}^i)' dW, \\ \underline{P}_T^i = 1 \text{ for all } i \in \mathcal{M}. \end{cases}$$

Since $\gamma < 0$, we have the following estimate for $f^{k,i}(t, P, \Lambda)$ with $P > 0$:

$$\begin{aligned} f^{k,i}(t, P, \Lambda) &\geq f^i(t, P, \Lambda)g(P) \\ &= \gamma \sup_{(c,\pi) \in \Theta} \left[-\frac{1-\gamma}{2} |\pi' \sigma^i|^2 P + \pi'(Pb^i + \sigma^i \Lambda) + \frac{c^\gamma}{\gamma} - Pc \right] g(P) \\ &\geq \gamma \sup_{(c,\pi) \in \mathbb{R}_+ \times \mathbb{R}^m} \left[-\frac{1-\gamma}{2} |\pi' \sigma^i|^2 P + \pi'(Pb^i + \sigma^i \Lambda) + \frac{c^\gamma}{\gamma} - Pc \right] g(P) \\ &= \frac{\gamma}{2(1-\gamma)} \frac{1}{P} (Pb^i + \sigma^i \Lambda)' (\sigma^i (\sigma^i)')^{-1} (Pb^i + \sigma^i \Lambda) g(P) + (1-\gamma) P^{-\frac{\gamma}{1-\gamma}} g(P). \end{aligned}$$

Hence, we deduce

$$\begin{aligned} &f^{k,i}(t, \underline{P}^i, \underline{\Lambda}^i) - (\rho^i - \gamma r^i) \underline{P}^i g(\underline{P}^i) + \sum_{j=1}^{\ell} q^{ij} \underline{P}^j \\ &\geq \frac{\gamma}{2(1-\gamma)} (b^i)' (\sigma^i (\sigma^i)')^{-1} b^i \underline{P}^i g(\underline{P}^i) + (1-\gamma) \underline{P}^{-\frac{\gamma}{1-\gamma}} g(\underline{P}^i) - (\rho^i - \gamma r^i) \underline{P}^i g(\underline{P}^i) + \sum_{j=1}^{\ell} q^{ij} \underline{P}^j \\ &\geq -a(1-\gamma) \underline{P}^i g(\underline{P}^i) + (1-\gamma) \underline{P}^{-\frac{\gamma}{1-\gamma}} g(\underline{P}^i) + \sum_{j=1}^{\ell} q^{ij} \underline{P}^j. \end{aligned}$$

By Lemma 3.2, we have

$$a_1 = e^{-a(1-\gamma)T} \leq \underline{P}_t^i \leq P_t^{k,i} \text{ for all } i \in \mathcal{M}.$$

To give an upper bound, we notice that

$$(\bar{P}_t^i, \bar{\Lambda}_t^i) = \left(e^{a(T-t)} + \frac{\varepsilon^\gamma}{a} (e^{a(T-t)} - 1), 0 \right) \text{ for all } i \in \mathcal{M}, \quad (3.12)$$

satisfies the following BSDE:

$$\begin{cases} d\bar{P}^i = - \left[a\bar{P}^i + \varepsilon^\gamma + \sum_{j=1}^{\ell} q^{ij} \bar{P}^j \right] dt + (\bar{\Lambda}^i)' dW, \\ \bar{P}_T^i = 1 \text{ for all } i \in \mathcal{M}. \end{cases}$$

Also, $(\varepsilon, \mathbf{0}_m) \in \Theta$ under Assumption 2, so, for $k \geq -\gamma\varepsilon$,

$$\begin{aligned} f^{k,i}(t, P, \Lambda) &\leq \sup_{\tilde{P} \in \mathbb{R}, \tilde{\Lambda} \in \mathbb{R}^n} \left[(\varepsilon^\gamma - \gamma\tilde{P}\varepsilon)g(\tilde{P}) - k|P - \tilde{P}| - k|\Lambda - \tilde{\Lambda}| \right] \\ &\leq \sup_{\tilde{P} \in \mathbb{R}} \left[(\varepsilon^\gamma - \gamma\tilde{P}\varepsilon)g(\tilde{P}) + \gamma\varepsilon|P - \tilde{P}| \right] \\ &\leq \sup_{\tilde{P} \in \mathbb{R}} \left[(\varepsilon^\gamma - \gamma\tilde{P}\varepsilon)g(\tilde{P}) + \gamma\varepsilon(|\tilde{P}| - |P|) \right] \\ &\leq \varepsilon^\gamma - \gamma\varepsilon|P|, \end{aligned}$$

where the third and fourth inequalities are due to that $\gamma < 0$, $g(x) \leq 1$, $|xg(x)| \leq |x|$ for all $x \in \mathbb{R}$. Hence, using (3.10) and $\bar{P}^i > 0$, we get

$$\begin{aligned} f^{k,i}(t, \bar{P}^i, \bar{\Lambda}^i) - (\rho^i - \gamma r^i)\bar{P}^i g(\bar{P}^i) + \sum_{j=1}^{\ell} q^{ij} \bar{P}^j &\leq \varepsilon^\gamma - \gamma\varepsilon|\bar{P}^i| - (\rho^i - \gamma r^i)\bar{P}^i g(\bar{P}^i) + \sum_{j=1}^{\ell} q^{ij} \bar{P}^j \\ &\leq a\bar{P}^i + \varepsilon^\gamma + \sum_{j=1}^{\ell} q^{ij} \bar{P}^j. \end{aligned}$$

By Lemma 3.2 again, we obtain the upper bound

$$P_t^{k,i} \leq \bar{P}_t^i \leq e^{aT} + \frac{\varepsilon^\gamma}{a}(e^{aT} - 1) = a_2, \quad t \in [0, T] \quad \text{for all } i \in \mathcal{M}.$$

This completes the proof. \square

Based on the above two theorems, we can provide a complete answer to problem (2.2).

Theorem 3.6 *Suppose $\gamma \in (-\infty, 0) \cup (0, 1)$ and Assumptions 1 and 2 hold. Let $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ be the unique solution to (3.1). Then the value function of the optimization problem (2.2) is given by*

$$V(x, i_0) = \frac{1}{\gamma} x^\gamma P_0^{i_0},$$

and $(\hat{c}, \hat{\pi}) \in \mathcal{U}$ is an optimal investment-consumption pair if and only if

$$(\hat{c}_t, \hat{\pi}_t) \in \operatorname{argmax}_{(c, \pi) \in \Theta} \left[-\frac{1-\gamma}{2} |\pi' \sigma_t^{\alpha_t}|^2 P_t^{\alpha_t} + \pi'(P_t^{\alpha_t} b_t^{\alpha_t} + \sigma_t^{\alpha_t} \Lambda_t^{\alpha_t}) + \frac{c^\gamma}{\gamma} - P_t^{\alpha_t} c \right].$$

In particular, if $\Theta = \mathbb{R}_+ \times \mathbb{R}^m$, then the optimal investment-consumption pair is uniquely given by

$$\hat{\pi}_t = \frac{1}{1-\gamma} (\sigma_t^{\alpha_t} (\sigma_t^{\alpha_t})')^{-1} \left(b_t^{\alpha_t} + \sigma_t^{\alpha_t} \frac{\Lambda_t^{\alpha_t}}{P_t^{\alpha_t}} \right), \quad \hat{c}_t = (P_t^{\alpha_t})^{\frac{1}{\gamma-1}}.$$

Remark 3.7 *Notice*

$$\lim_{K \rightarrow +\infty} \sup_{|\pi|+c > K} \left[-\frac{1-\gamma}{2} |\pi' \sigma_t^{\alpha_t}|^2 P_t^{\alpha_t} + \pi'(P_t^{\alpha_t} b_t^{\alpha_t} + \sigma_t^{\alpha_t} \Lambda_t^{\alpha_t}) + \frac{c^\gamma}{\gamma} - P_t^{\alpha_t} c \right] = -\infty,$$

so $(\hat{c}_t, \hat{\pi}_t)$ is well-defined. Let $G : \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^m \rightarrow \mathbb{R}$ be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable map, where \mathcal{P} is the σ -field of predictable sets of $\Omega \times [0, T]$, and $\mathcal{B}(\mathbb{R}^m)$ (resp. $\mathcal{B}(\mathbb{R}_+)$) is the Borelian σ -algebra on \mathbb{R}^m (resp. \mathbb{R}_+). Let

$$\operatorname{argmax}_{(c, \pi) \in \Theta} \left[G(\omega, t, \pi, c) \right],$$

denote the set of all predictable processes valued in Θ which attain the essential supremum of G w.r.t $(c, \pi) \in \Theta$. From the measurable selection theorem (see e.g. [13, Lemma 11], [1, Corollary 18.14] or [8, Proposition 2.4]), the set is not empty. If Θ is further convex, then $\operatorname{argmax}_{(c, \pi) \in \Theta} [G(t, \omega, \pi, c)]$ degenerates to a singleton set.

Proof: In this proof, the arguments (s, α_s) and (u, α_u) in the integrals will be omitted for notation simplicity, e.g., P means $P_s^{\alpha_s}$, ρ means $\rho_u^{\alpha_u}$. For any $(c, \pi) \in \mathcal{U}$, applying Itô's formula to $\frac{1}{\gamma} e^{-\int_0^t \rho \, du} X_t^\gamma P_t^{\alpha_t}$, (here we use Itô's formula for Markovian chain; please refer to [14, Lemma 4.3]), we have

$$\begin{aligned} & \frac{1}{\gamma} e^{-\int_0^t \rho \, du} X_t^\gamma P_t^{\alpha_t} + \int_0^t \frac{1}{\gamma} e^{-\int_0^s \rho \, du} X^\gamma c^\gamma \, ds \\ &= \frac{1}{\gamma} x^\gamma P_0^{i_0} + \int_0^t e^{-\int_0^s \rho \, du} X^\gamma \left[\frac{1}{\gamma} c^\gamma - cP - \frac{1-\gamma}{2} P |\pi' \sigma|^2 + \pi'(Pb + \sigma\Lambda) - \frac{f}{\gamma} \right] \, ds \\ & \quad + \int_0^t \frac{1}{\gamma} e^{-\int_0^s \rho \, du} X^\gamma (\Lambda' + \gamma P \pi' \sigma) \, dW \\ & \quad + \int_0^t \frac{1}{\gamma} e^{-\int_0^s \rho \, du} X^\gamma \sum_{j, j' \in \mathcal{M}} (P^j - P^{j'}) I_{\{\alpha_s = j'\}} d\tilde{N}^{j'j}, \end{aligned}$$

where $(N^{j'j})_{j, j' \in \mathcal{M}}$ are independent Poisson processes each with intensity $q^{j'j}$, and $\tilde{N}_t^{j'j} = N_t^{j'j} - q^{j'j}t$, $t \geq 0$ are the corresponding compensated Poisson martingales under the filtration \mathcal{F} .

For any $(c, \pi) \in \mathcal{U}$, the wealth process X is continuous and strictly positive, hence bounded on $[0, T]$ away from 0. Therefore, the stochastic integrals in the last equation are local martingales. Hence, there exists an increasing sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ satisfying $\lim_{n \rightarrow \infty} \tau_n = T$, a.s. such that

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{\gamma} e^{-\int_0^{\tau_n} \rho \, du} X_{\tau_n}^\gamma P_{\tau_n}^{\alpha_{\tau_n}} + \int_0^{\tau_n} \frac{1}{\gamma} e^{-\int_0^s \rho \, du} X^\gamma c^\gamma \, ds \right] \\ &= \frac{1}{\gamma} x^\gamma P_0^{i_0} + \mathbb{E} \int_0^{\tau_n} e^{-\int_0^s \rho \, du} X^\gamma \left[-\frac{1-\gamma}{2} P |\pi' \sigma|^2 + \pi'(Pb^i + \sigma\Lambda) + \frac{1}{\gamma} c^\gamma - cP - \frac{f}{\gamma} \right] \, ds \\ &\leq \frac{1}{\gamma} x^\gamma P_0^{i_0}, \end{aligned}$$

thanks to the definition of f .

Case $\gamma \in (0, 1)$: In this case, $\frac{1}{\gamma} e^{-\int_0^{\tau_n} \rho \, du} X_{\tau_n}^\gamma P_{\tau_n}^{\alpha_{\tau_n}} + \int_0^{\tau_n} \frac{1}{\gamma} e^{-\int_0^s \rho \, du} X^\gamma c^\gamma \, ds$ is bounded from below by 0. Passing to limit and applying Fatou's lemma in the above inequality yields

$$\mathbb{E} \left[\frac{1}{\gamma} e^{-\int_0^T \rho \, du} X_T^\gamma + \int_0^T \frac{1}{\gamma} e^{-\int_0^s \rho \, du} X^\gamma c^\gamma \, ds \right] \leq \frac{1}{\gamma} x^\gamma P_0^{i_0}, \quad (3.13)$$

by virtue of the terminal condition $P_T^{\alpha_T} = 1$. And the inequality (3.13) becomes an equality for some $(\hat{c}, \hat{\pi})$ if and only if

$$(\hat{c}, \hat{\pi}) \in \operatorname{argmax}_{(c, \pi) \in \Theta} \left[-\frac{1-\gamma}{2} |\pi' \sigma|^2 P + \pi'(Pb + \sigma\Lambda) + \frac{c^\gamma}{\gamma} - Pc \right].$$

It remains to show that $(\hat{c}, \hat{\pi}) \in \mathcal{U}$. Recall that the unique solution of (3.1), $(P^i, \Lambda^i) \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$. From (3.7),

$$|\hat{\pi}| \leq K(1 + |Z|) \leq K(1 + \sum_{i \in \mathcal{M}} |Z^i|),$$

hence $\hat{\pi} \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$. Because P^i , $i \in \mathcal{M}$, and the coefficients are bounded, there exists some constant $K_1 > 0$ such that

$$\begin{aligned} & -\frac{1-\gamma}{2}|\pi'\sigma|^2P + \pi'(Pb + \sigma\Lambda) + \frac{c^\gamma}{\gamma} - Pc \\ & \leq \sup_{\pi \in \mathbb{R}^m} \left[-\frac{1-\gamma}{2}|\pi'\sigma|^2P + \pi'(Pb + \sigma\Lambda) \right] + \frac{c^\gamma}{\gamma} - Pc \\ & = \frac{1}{2} \frac{\gamma}{1-\gamma} \frac{1}{P} (Pb + \sigma\Lambda)'(\sigma(\sigma)')^{-1}(Pb + \sigma\Lambda) + \frac{c^\gamma}{\gamma} - Pc \\ & \leq K_1(1 + |\Lambda|^2) + \frac{c^\gamma}{\gamma} - K_1c. \end{aligned}$$

The right hand side is negative if $c > K(1 + |\Lambda|^2)$ for sufficient large $K > 0$, so

$$\hat{c} \leq K(1 + |\Lambda|^2) \leq K(1 + \sum_{i \in \mathcal{M}} |\Lambda^i|^2),$$

which implies $(\hat{c}, \hat{\pi}) \in \mathcal{U}$.

Case $\gamma < 0$: For any admissible $(c, \pi) \in \mathcal{U}$, $(X^\gamma)_{0 \leq t \leq T}$ belongs to class (D) and $\mathbb{E}[\int_0^T (cX)^\gamma ds] < \infty$, so by the dominated convergence theorem, we also have (3.13). By a similar argument for the previous case $\gamma \in (0, 1)$, we can prove $\hat{\pi} \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ and $\int_0^T \hat{c} ds < \infty$, a.s. It remains to prove $(\hat{X}_t^\gamma)_{0 \leq t \leq T}$ belongs to class (D) and $\mathbb{E}[\int_0^T (\hat{c}\hat{X})^\gamma ds] < \infty$.

Since ρ and P are bounded, and

$$\mathbb{E} \left[e^{-\int_0^{\tau_n} \rho du} \hat{X}_{\tau_n}^\gamma P_{\tau_n}^{\alpha_{\tau_n}} + \int_0^{\tau_n} e^{-\int_0^s \rho du} \hat{X}^\gamma \hat{c}^\gamma ds \right] = x^\gamma P_0^i,$$

Fatou's Lemma yields

$$\mathbb{E} \left[\hat{X}_T^\gamma + \int_0^T (\hat{c}\hat{X})^\gamma ds \right] < \infty. \quad (3.14)$$

Denote $\theta^i := (\sigma^i)'(\sigma^i(\sigma^i)')^{-1}$ for all $i \in \mathcal{M}$. By Girsanov's theorem,

$$W_t^{\mathbb{Q}} := W_t + \int_0^t \theta ds$$

is a Brownian motion under the measure \mathbb{Q} defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E} \left(- \int_0^T \theta dW \right).$$

Under the strategy $(\hat{c}, \hat{\pi})$, we obtain from (2.1) that

$$\hat{X}_t = x e^{\int_0^t (r-\hat{c}) ds} \mathcal{E} \left(\int_0^t \hat{\pi}' \sigma dW^{\mathbb{Q}} \right).$$

Denote

$$J_t := e^{-\int_0^t \hat{c} ds} \mathcal{E} \left(\int_0^t \hat{\pi}' \sigma dW^{\mathbb{Q}} \right).$$

Thus it is sufficient to prove $(J_t^\gamma)_{0 \leq t \leq T}$ belongs to class (D) since r is bounded.

It can be seen from (3.14) that $\mathbb{E}[J_T^\gamma] < \infty$. One obtains for every stopping time $\tau \leq T$,

$$\begin{aligned} J_\tau^{\frac{\gamma}{2}} &= \left\{ \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^\tau \hat{c} ds} \mathcal{E} \left(\int_0^\tau \hat{\pi}' \sigma dW^{\mathbb{Q}} \right) \mid \mathcal{F}_\tau \right] \right\}^{\frac{\gamma}{2}} \\ &\leq \left\{ \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^\tau \hat{c} ds} \mathcal{E} \left(\int_0^\tau \hat{\pi}' \sigma dW^{\mathbb{Q}} \right) \mid \mathcal{F}_\tau \right] \right\}^{\frac{\gamma}{2}} \\ &= \left\{ \mathbb{E}^{\mathbb{Q}} \left[J_T \mid \mathcal{F}_\tau \right] \right\}^{\frac{\gamma}{2}} \leq \mathbb{E}^{\mathbb{Q}} \left[J_T^{\frac{\gamma}{2}} \mid \mathcal{F}_\tau \right], \end{aligned} \quad (3.15)$$

where we used that $\hat{c} \geq 0$ and $\gamma < 0$ in the first inequality, and Jensen's inequality in the second one. By the Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[J_T^{\frac{\gamma}{2}} \mid \mathcal{F}_\tau \right] &= \mathbb{E} \left[J_T^{\frac{\gamma}{2}} \mathcal{E} \left(- \int_\tau^T \theta dW \right) \mid \mathcal{F}_\tau \right] \\ &\leq \left\{ \mathbb{E} \left[J_T^\gamma \mid \mathcal{F}_\tau \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[\mathcal{E} \left(- \int_\tau^T \theta dW \right)^2 \mid \mathcal{F}_\tau \right] \right\}^{\frac{1}{2}} \leq K \left\{ \mathbb{E} \left[J_T^\gamma \mid \mathcal{F}_\tau \right] \right\}^{\frac{1}{2}}, \end{aligned} \quad (3.16)$$

where the last inequality is due to the boundedness of θ . Combining $\mathbb{E}[J_T^\gamma] < \infty$, (3.15) and (3.16), we conclude that $(J_t^\gamma)_{0 \leq t \leq T}$ belongs to class (D). \square

4 Logarithmic utility

Let us now turn to the logarithmic utility $U(x) = \ln x$, $x > 0$. The admissible set is now defined as

$$\begin{aligned} \mathcal{U} = \left\{ (c, \pi) \mid \pi \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^m), c \in L_{\mathcal{F}}^1(0, T; \mathbb{R}_+), (c, \pi) \in \Theta, dt \otimes d\mathbb{P} - \text{a.s.}, \right. \\ \left. (\ln X_t)_{0 \leq t \leq T} \text{ and } \left(\int_0^t \ln(cX) ds \right)_{0 \leq t \leq T} \text{ belong to class (D)} \right\}. \end{aligned}$$

Clearly \mathcal{U} would be empty if c could not take positive values. Hence, it is reasonable to put the following assumption.

Assumption 3 *There exists a constant $\varepsilon > 0$ such that $(\varepsilon, \mathbf{0}_m) \in \Theta$.*

Economically speaking, it means that consuming at a constant rate ε and investing nothing in the stocks is an admissible strategy.

Since the utility is logarithmic, we conjecture that the optimal value function of the problem (2.2) has the following decomposition

$$V(x, i) = h_0^i \ln x + P_0^i$$

for some stochastic processes h^i and P^i , which are governed by two coupled BSDEs.

The first one is a simple ℓ -dimensional linear BSDE system with bounded coefficients for $(h^i, \eta^i)_{i \in \mathcal{M}}$:

$$\begin{cases} dh^i = -(1 - \rho^i h^i + \sum_{j=1}^{\ell} q^{ij} h^j) dt + (\eta^i)' dW, \\ h_T^i = 1 \text{ for all } i \in \mathcal{M}. \end{cases} \quad (4.1)$$

Its unique solution $(h^i, \eta^i)_{i \in \mathcal{M}}$ satisfies $(h^i, \eta^i) \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$.

We claim that h^i is uniformly positive. Indeed, under Assumption 1, there exists a constant $k > 0$ such that $-\rho^i \geq -k$ for all $i \in \mathcal{M}$. Notice that

$$(\bar{h}_t, \bar{\eta}_t) = \left(e^{-k(T-t)} + \frac{1 - e^{-k(T-t)}}{k}, 0 \right)$$

is a solution to the following BSDE:

$$\begin{cases} d\bar{h} = -(1 - k\bar{h}) dt + \bar{\eta}' dW, \\ \bar{h}_T = 1, \end{cases}$$

From Lemma 3.2, we have $h_t^i \geq \bar{h}_t = e^{-k(T-t)} + \frac{1 - e^{-k(T-t)}}{k} \geq e^{-kT}$.

The second BSDE system is also linear but has unbounded coefficients:

$$\begin{cases} dP^i = - \left[f^i - \rho^i P^i + r^i h^i + \sum_{j=1}^{\ell} q^{ij} P^j \right] dt + (\Lambda^i)' dW \\ P_T^i = 0 \text{ for all } i \in \mathcal{M}, \end{cases} \quad (4.2)$$

where

$$f^i := \sup_{(c, \pi) \in \Theta} \left[-\frac{1}{2} h^i |\pi' \sigma^i|^2 + \pi' (h^i b^i + \sigma^i \eta^i) + \ln c - h^i c \right].$$

As $(\varepsilon, \mathbf{0}_m) \in \Theta$, and $h^i \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R})$, we have

$$f^i \geq \ln \varepsilon - h^i \varepsilon. \quad (4.3)$$

On the other hand,

$$\begin{aligned} f^i &\leq \sup_{(c, \pi) \in \mathbb{R}_+ \times \mathbb{R}^m} \left[-\frac{1}{2} h^i |\pi' \sigma^i|^2 + \pi' (h^i b^i + \sigma^i \eta^i) + \ln c - h^i c \right] \\ &= \frac{1}{2h^i} (h^i b^i + \sigma^i \eta^i)' (\sigma^i (\sigma^i)')^{-1} (h^i b^i + \sigma^i \eta^i) - 1 - \ln h^i. \end{aligned}$$

Thus by the boundedness of h and the coefficients, there exists some constant $K > 0$ such that

$$|f^i| \leq K(1 + |\eta^i|^2) \text{ for all } i \in \mathcal{M}.$$

By [17, Theorem 3.6], the system of (4.2) admits a unique solution $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ such that $(P^i, \Lambda^i) \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$.

Set $(Y^i, Z^i) = (-\frac{P^i}{h^i}, \frac{P^i}{(h^i)^2} \eta^i - \frac{1}{h^i} \Lambda^i)$ for all $i \in \mathcal{M}$, then $(Y^i, Z^i) \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ and $(Y^i, Z^i)_{i \in \mathcal{M}}$ is the unique solution to

$$\begin{cases} dY^i = - \left\{ \inf_{(c, \pi) \in \Theta} \left[\frac{1}{2} |\pi' \sigma^i|^2 - \pi' (b^i + \frac{1}{h^i} \sigma^i \eta^i) - \frac{\ln c}{h^i} + c \right] - \frac{1}{h^i} (1 + \sum_{j=1}^{\ell} q^{ij} h^j) Y^i \right. \\ \quad \left. + \frac{1}{h^i} (\eta^i)' Z^i - r^i + \frac{1}{h^i} \sum_{j=1}^{\ell} q^{ij} h^j Y^j \right\} dt + (Z^i)' dW, \\ Y_T^i = 0 \text{ for all } i \in \mathcal{M}. \end{cases} \quad (4.4)$$

If there is no regime switching, ρ is a constant and the constraints on consumption and investment are decoupled, then $\eta^i \equiv 0$ and (4.4) degenerates to (4.6) in [4].

Theorem 4.1 *Suppose Assumptions 1 and 3 hold. Let $(h^i, \eta^i)_{i \in \mathcal{M}}$ and $(Y^i, Z^i)_{i \in \mathcal{M}}$ be the unique solutions to (4.1) and (4.4), respectively. Then the value function of the optimization problem (2.2) is given by*

$$V(x, i_0) = h_0^{i_0} \ln x + P_0^{i_0},$$

and $(\hat{c}, \hat{\pi}) \in \mathcal{U}$ is an optimal consumption-investment pair if and only if

$$(\hat{c}_t, \hat{\pi}_t) \in \operatorname{argmax}_{(c, \pi) \in \Theta} \left[-\frac{1}{2} h_t^{\alpha_t} |\pi' \sigma_t^{\alpha_t}|^2 + \pi' (h_t^{\alpha_t} b_t^{\alpha_t} + \sigma_t^{\alpha_t} \eta_t^{\alpha_t}) + \ln c - h_t^{\alpha_t} c \right].$$

In particular, if $\Theta = \mathbb{R}_+ \times \mathbb{R}^m$, then the optimal consumption-investment pair is uniquely given by

$$\hat{\pi}_t = (\sigma^{\alpha_t} (\sigma^{\alpha_t})')^{-1} \left(b^{\alpha_t} + \frac{1}{h^{\alpha_t}} \sigma^{\alpha_t} \eta^{\alpha_t} \right), \quad \hat{c}_t = \frac{1}{h_t^{\alpha_t}}.$$

Proof: Same as before, we will omit the arguments (s, α_s) and (u, α_u) in the integrals for notation simplicity. For any $(c, \pi) \in \mathcal{U}$, applying Itô's formula to $e^{-\int_0^t \rho \, du} (h_t^{\alpha_t} \ln X_t + P_t^{\alpha_t})$, we have

$$\begin{aligned} & e^{-\int_0^t \rho \, du} (h_t^{\alpha_t} \ln X_t + P_t^{\alpha_t}) + \int_0^t e^{-\int_0^s \rho \, du} \ln(cX) \, ds \\ &= h_0^{i_0} \ln x + P_0^{i_0} + \int_0^t e^{-\int_0^s \rho \, du} \left[\ln c - ch - \frac{1}{2} |\pi' \sigma|^2 + \pi' (h\eta + \sigma\eta) - f \right] \, ds \\ & \quad + \int_0^t e^{-\int_0^s \rho \, du} \left[h\sigma' \pi + \eta \ln X + \Lambda \right]' \, dW \\ & \quad + \int_0^t e^{-\int_0^s \rho \, du} \sum_{j, j' \in \mathcal{M}} [(h^j \ln X + P^j) - (h^{j'} \ln X + P^{j'})] I_{\{\alpha_s = j'\}} \, d\tilde{N}_s^{j'j}. \end{aligned}$$

Then by the definition of f^i , the process

$$e^{-\int_0^t \rho \, du} (h_t^{\alpha_t} \ln X_t + P_t^{\alpha_t}) + \int_0^t e^{-\int_0^s \rho \, du} \ln(cX) \, ds$$

is a local supermartingale. Hence, there exists a sequence of increasing stopping times $(\tau_n)_{n \in \mathbb{N}}$ satisfying $\lim_{n \rightarrow \infty} \tau_n = T$, a.s. such that

$$\mathbb{E} \left[e^{-\int_0^{\tau_n} \rho \, du} (h_{\tau_n}^{\alpha_{\tau_n}} \ln X_{\tau_n} + P_{\tau_n}^{\alpha_{\tau_n}}) + \int_0^{\tau_n} e^{-\int_0^s \rho \, du} \ln(cX) \, ds \right] \leq h_0^{i_0} \ln x + P_0^{i_0}.$$

Recalling the admissibility of (c, π) , we deduce from the dominated convergence theorem that

$$\mathbb{E} \left[e^{-\int_0^T \rho \, du} (h_T^{\alpha_T} \ln X_T + P_T^{\alpha_T}) + \int_0^T e^{-\int_0^s \rho \, du} \ln(cX) \, ds \right] \leq h_0^{i_0} \ln x + P_0^{i_0}, \quad (4.5)$$

with the equality holds for some $(\hat{c}, \hat{\pi})$ if and only if

$$(\hat{c}, \hat{\pi}) \in \operatorname{argmax}_{(c, \pi) \in \Theta} \left[-\frac{1}{2} h |\pi' \sigma|^2 + \pi' (hb + \sigma\eta) + \ln c - hc \right].$$

It remains to prove $(\hat{c}, \hat{\pi}) \in \mathcal{U}$ for such a pair. ⁵

⁵ Compared with [4], a new, non-zero term $\eta \in L_{\mathcal{F}W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ emerges (In [4], ρ is a constant, hence $\eta = 0$). So we need conditions like class (D) to apply the dominated convergence theorem in (4.5). But this brings difficulty in proving the admissibility of $(\hat{c}, \hat{\pi})$.

From (4.3) and $h \in L^\infty(0, T; \mathbb{R})$, $f \geq -K_1$ for some $K_1 > 0$. On the other hand, Assumption 1 and $h \geq e^{-kT}$ imply

$$\begin{aligned} & -\frac{1}{2}h|\pi'\sigma|^2 + \pi'(hb + \sigma\eta) + \ln c - hc \\ & \leq -\frac{1}{2}h|\pi'\sigma|^2 + \pi'(hb + \sigma\eta) + \sup_{c \in \mathbb{R}_+} (\ln c - hc) \\ & = -\frac{1}{2}h|\pi'\sigma|^2 + \pi'(hb + \sigma\eta) - \ln h - 1 \\ & \leq -K_2|\pi|^2 + K_3|\pi|(1 + |\eta|) + K_4, \end{aligned}$$

for some positive constants K_2, K_3, K_4 . Thus

$$-\frac{1}{2}h|\pi'\sigma|^2 + \pi'(hb + \sigma\eta) + \ln c - hc < -K_1 \leq f,$$

if $|\pi| > K(1 + |\eta|)$ with K being sufficient large, which implies

$$|\hat{\pi}| \leq K(1 + |\eta|) \leq K\left(1 + \sum_{i \in \mathcal{M}} |\eta^i|\right). \quad (4.6)$$

Similarly, since h is uniformly positive, we have for some $K_5, K_6 > 0$,

$$\begin{aligned} & -\frac{1}{2}h|\pi'\sigma|^2 + \pi'(hb + \sigma\eta) + \ln c - hc \\ & \leq \sup_{\pi \in \mathbb{R}^m} \left[-\frac{1}{2}h|\pi'\sigma|^2 + \pi'(hb + \sigma\eta) \right] + \ln c - K_5c \\ & = \frac{1}{2h}(hb + \sigma\eta)'(\sigma(\sigma)')^{-1}(hb + \sigma\eta) + \ln c - K_5c \\ & \leq K_6(1 + |\eta|^2) + \ln c - K_5c \\ & < -K_1 \\ & \leq f, \end{aligned}$$

if $c \geq K(1 + |\eta|^2)$ or $\ln c \leq -K(1 + |\eta|^2)$ for sufficient large K . Therefore,

$$e^{-K(1+|\eta|^2)} \leq \hat{c} \leq K(1 + |\eta|^2), \quad (4.7)$$

and consequently,

$$|\ln \hat{c}| \leq K(1 + |\eta|^2) \leq K\left(1 + \sum_{i \in \mathcal{M}} |\eta^i|^2\right). \quad (4.8)$$

From (2.1),

$$\ln \hat{X}_t = \ln x + \int_0^t \left(r + \hat{\pi}'b - \hat{c} - \frac{1}{2}|\hat{\pi}'\sigma|^2 \right) ds + \int_0^t \hat{\pi}'\sigma dW.$$

It follows from the boundedness of coefficients and the AM-GM inequality that

$$\sup_{t \leq T} |\ln \hat{X}_t| \leq K + K \int_0^T (|\hat{c}| + |\hat{\pi}|^2) ds + K \sup_{t \leq T} \left| \int_0^t \hat{\pi}'\sigma dW \right|.$$

Using the AM-GM inequality, the Burkholder-Davis-Gundy inequality, (4.6) and (4.7), we have

$$\begin{aligned}
\mathbb{E} \left[\sup_{\tau \leq T} (\ln \hat{X}_\tau)^2 \right] &\leq K + K \mathbb{E} \left[\left(\int_0^T (|\hat{c}| + |\hat{\pi}|^2) ds \right)^2 + \sup_{t \leq T} \left| \int_0^t \hat{\pi}' \sigma dW \right|^2 \right] \\
&\leq K + K \mathbb{E} \left[\left(\int_0^T \left(1 + \sum_{i \in \mathcal{M}} |\eta^i|^2 \right) ds \right)^2 + \int_0^T |\hat{\pi}' \sigma|^2 ds \right] \\
&\leq K + K \mathbb{E} \left[\left(\int_0^T \left(1 + \sum_{i \in \mathcal{M}} |\eta^i|^2 \right) ds \right)^2 + \int_0^T \left(1 + \sum_{i \in \mathcal{M}} |\eta^i|^2 \right) ds \right] \\
&\leq K,
\end{aligned}$$

where the last inequality is due to the following fact (see [22, Page 26]) that, for any $j > 0$,

$$\mathbb{E} \left[\left(\int_0^T |\phi|^2 ds \right)^j \right] < \infty,$$

if $\phi \in L^2_{\mathcal{F}W}, \text{BMO}(0, T; \mathbb{R}^n)$. This shows that $(\ln \hat{X}_t)_{0 \leq t \leq T}$ belongs to class (D). By the same argument and using (4.8), we can prove that $\left(\int_0^t \ln(\hat{c}\hat{X}) ds \right)_{0 \leq t \leq T}$ belongs to class (D) as well. \square

5 Exponential utility

In this case, we do not consider the proportional consumption-investment strategies. Instead, let $\pi_{j,t}$ be the *amount* invested in the j th risky asset, $j = 1, \dots, m$, and let c_t be the *amount* to consume, at time t . Then the investor's self-financing wealth process X corresponding to a consumption-investment strategy (c, π) is the unique strong solution of the SDE:

$$\begin{cases} dX_t = (r_t X_t + \pi'_t b_t^{\alpha_t} - c_t) dt + \pi'_t \sigma_t^{\alpha_t} dW_t, \\ X_0 = x, \alpha_0 = i_0. \end{cases} \quad (5.1)$$

Let Π be a given closed nonempty set in \mathbb{R}^m with $\mathbf{0}_m \in \Pi$ to represent the constraint set for portfolios. We do not put constraint on consumptions, so $-c_t$ can be interpreted as an income if $c_t < 0$. The admissible consumption-investment set is defined as

$$\begin{aligned}
\mathcal{U} = \left\{ (c, \pi) \mid \int_0^T (|\pi|^2 + |c|) dt < \infty, (c, \pi) \in \mathbb{R} \times \Pi, dt \otimes d\mathbb{P} - \text{a.s.}, \right. \\
\left. \text{and } (e^{-\beta h_t X_t^{\pi, c}})_{0 \leq t \leq T} \text{ belongs to class (D)} \right\},
\end{aligned}$$

where the process h_t will be specified case by case in the following subsections.

The investor's problem is to maximize

$$J(x, i_0; \pi, c) := \mathbb{E} \left[\int_0^T e^{-\int_0^t \rho_u^{\alpha_u} du} U(c_t) dt + e^{-\int_0^T \rho_u^{\alpha_u} du} U(X_T) \right], \quad \text{s.t. } (c, \pi) \in \mathcal{U}, \quad (5.2)$$

and determine the value function

$$V(x, i_0) := \sup_{(c, \pi) \in \mathcal{U}} J(x, i_0; \pi, c),$$

where the utility is exponential and given by

$$U(x) = -\exp(-\beta x), \quad x \in \mathbb{R}, \quad \beta > 0.$$

It turns out that whether the interest rate is deterministic or random is crucial, we have to deal with these two cases separately.

5.1 Case I: Deterministic interest rate

Assumption 4 *The process r is a deterministic bounded measurable function of t . For all $i \in \mathcal{M}$,*

$$\mu^i \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}^m), \quad \sigma^i \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}^{m \times n}), \quad \rho^i \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}),$$

and $\sigma^i(\sigma^i)' \geq \delta \mathbf{1}_m$ with some constant $\delta > 0$.

In this case, we conjecture that the optimal value function of problem (2.2) has the following decomposition

$$V(x, i) = -e^{-\beta(h_0 x + Y_0^i)}$$

for some stochastic processes h^i and Y^i , which are governed by two coupled BSDEs.

Under Assumption 4, the first one reduces to an ODE:

$$\begin{cases} dh = -h(r - h) dt, \\ h_T = 1. \end{cases}$$

It admits a unique explicit solution

$$h_t = \left(e^{-\int_t^T r du} + \int_t^T e^{-\int_t^s r du} ds \right)^{-1} > 0, \quad (5.3)$$

Clearly, h is bounded and uniformly positive.

The second BSDE system for $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ is given as follows:

$$\begin{cases} dP^i = -\left\{ f^i(t, P^i, \Lambda^i) - hP^i \ln P^i - \rho^i P^i + h(1 - \ln h)P^i + \sum_{j=1}^{\ell} q^{ij} P^j \right\} dt + (\Lambda^i)' dW, \\ P_T^i = 1 \quad \text{for all } i \in \mathcal{M}, \end{cases} \quad (5.4)$$

where

$$f^i(t, P, \Lambda) := \beta h_t \inf_{\pi \in \Pi} \left[\frac{1}{2} \beta h_t P |\pi' \sigma_t^i|^2 - \pi'(P b_t^i + \sigma_t^i \Lambda) \right].$$

Thanks to $\mathbf{0}_m \in \Pi$, we have $f^i \leq 0$.

Theorem 5.1 *Under Assumption 4, there is a unique uniformly positive solution $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ to the BSDE (5.4) such that $(P^i, \Lambda^i) \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$.*

Proof: We will only present how to find the uniformly bounds. The remainder of the proof is left to the interested readers.

By the boundedness of the coefficients and h , there exist three positive constants a_1, a_2, a_3 such that, for all $i \in \mathcal{M}$,

$$-\rho^i + h(1 - \ln h) \leq a_1, \quad \frac{1}{2}(b^i)'(\sigma^i(\sigma^i)')^{-1}b^i - \rho^i + h(1 - \ln h) \geq -a_2, \quad h \geq a_3. \quad (5.5)$$

Define two constants $a = e^{a_1 T}$ and $\epsilon = e^{-\frac{a_2}{a_3}(1 - e^{-a_3 T})}$. Note $a > 1 > \epsilon > 0$.

Let $g : \mathbb{R} \rightarrow [0, 1]$ be a smooth truncation function satisfying $g(x) = 0$ for $x \in (-\infty, \epsilon/2] \cup [2a, \infty)$, and $g(x) = 1$ for $x \in [\epsilon, a]$.

For $k \geq 1$, $(t, P, \Lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$, $i \in \mathcal{M}$, define

$$f^{k,i}(t, P, \Lambda) = \sup_{\tilde{P} \in \mathbb{R}, \tilde{\Lambda} \in \mathbb{R}^n} \left[f^i(t, \tilde{P}, \tilde{\Lambda})g(\tilde{P}) - k|P - \tilde{P}| - k|\Lambda - \tilde{\Lambda}| \right].$$

Let $(P^{k,i}, \Lambda^{k,i})_{i \in \mathcal{M}}$ denote the unique solution to the following ℓ -dimensional BSDE system with a Lipschitz generator

$$\begin{cases} dP^{k,i} = - \left[f^{k,i}(t, P^{k,i}, \Lambda^{k,i}) - hP^{k,i}(\ln P^{k,i})g(P^{k,i}) - \rho^i P^{k,i}g(P^{k,i}) \right. \\ \quad \left. + h(1 - \ln h)P^{k,i}g(P^{k,i}) + \sum_{j=1}^{\ell} q^{ij}P^{k,j} \right] dt + (\Lambda^{k,i})^\top dW, \\ P_T^{k,i} = 1 \text{ for all } i \in \mathcal{M}. \end{cases}$$

It is direct to verify that

$$(\underline{P}_t^i, \underline{\Lambda}_t^i) = \left(e^{-\frac{a_2}{a_3}(1 - e^{-a_3(T-t)})}, 0 \right) \text{ for all } i \in \mathcal{M}, \quad (5.6)$$

is a solution to the following BSDE system:

$$\begin{cases} d\underline{P}^i = - \left[-a_2 \underline{P}^i g(\underline{P}^i) - a_3 \underline{P}^i (\ln \underline{P}^i) g(\underline{P}^i) + \sum_{j=1}^{\ell} q^{ij} \underline{P}^j \right] dt + (\underline{\Lambda}^i)' dW, \\ \underline{P}_T^i = 1 \text{ for all } i \in \mathcal{M}. \end{cases}$$

Noticing that $\underline{P}^i \in (0, 1)$ and, for $P > 0$,

$$\begin{aligned} f^{k,i}(t, P, \Lambda) &\geq f^i(t, P, \Lambda)g(P) \\ &= \beta h \inf_{\pi \in \Pi} \left[\frac{1}{2} \beta h P^i |\pi' \sigma^i|^2 - \pi'(P^i b^i + \sigma^i \Lambda^i) \right] g(P) \\ &\geq \beta h \inf_{\pi \in \mathbb{R}^m} \left[\frac{1}{2} \beta h P^i |\pi' \sigma^i|^2 - \pi'(P^i b^i + \sigma^i \Lambda^i) \right] g(P) \\ &= \frac{1}{2} (b^i)'(\sigma^i(\sigma^i)')^{-1} b^i P g(P) + (b^i)'(\sigma^i(\sigma^i)')^{-1} \sigma^i \Lambda g(P) \\ &\quad + \frac{1}{2P} \Lambda'(\sigma^i)'(\sigma^i(\sigma^i)')^{-1} \sigma^i \Lambda g(P), \end{aligned}$$

we deduce from (5.5) that

$$\begin{aligned} &f^{k,i}(t, \underline{P}^i, \underline{\Lambda}^i) - h \underline{P}^i (\ln \underline{P}^i) g(\underline{P}^i) - \rho^i \underline{P}^i g(\underline{P}^i) + h(1 - \ln h) \underline{P}^i g(\underline{P}^i) + \sum_{j=1}^{\ell} q^{ij} \underline{P}^j \\ &\geq \frac{1}{2} (b^i)'(\sigma^i(\sigma^i)')^{-1} b^i \underline{P}^i g(\underline{P}^i) - h \underline{P}^i (\ln \underline{P}^i) g(\underline{P}^i) - \rho^i \underline{P}^i g(\underline{P}^i) + h(1 - \ln h) \underline{P}^i g(\underline{P}^i) + \sum_{j=1}^{\ell} q^{ij} \underline{P}^j \\ &\geq -a_2 \underline{P}^i g(\underline{P}^i) - a_3 \underline{P}^i (\ln \underline{P}^i) g(\underline{P}^i) + \sum_{j=1}^{\ell} q^{ij} \underline{P}^j. \end{aligned}$$

By Lemma 3.2, we have

$$P_t^{k,i} \geq \underline{P}_t^i \geq e^{-\frac{a_2}{a_3}(1-e^{-a_3 T})} = \epsilon \text{ for all } i \in \mathcal{M}. \quad (5.7)$$

This gives a uniform lower bound.

To give a uniform upper bound, we notice

$$(\bar{P}_t^i, \bar{\Lambda}_t^i) = (e^{a_1(T-t)}, 0) \text{ for all } i \in \mathcal{M}, \quad (5.8)$$

is a solution to the following BSDE system:

$$\begin{cases} d\bar{P}^i = -\left[a_1 \bar{P}^i g(\bar{P}^i) + \sum_{j=1}^{\ell} q^{ij} \bar{P}^j \right] dt + (\bar{\Lambda}^i)' dW, \\ \bar{P}_T^i = 1 \text{ for all } i \in \mathcal{M}. \end{cases}$$

Since $f^i \leq 0$, we have $f^{k,i} \leq 0$. Notice that $\bar{P}_t^i \geq 1$, thus we have

$$\begin{aligned} & f^{k,i}(t, \bar{P}^i, \bar{\Lambda}^i) - h\bar{P}^i(\ln \bar{P}^i)g(\bar{P}^i) - \rho^i \bar{P}^i g(\bar{P}^i) + h(1 - \ln h)\bar{P}^i g(\bar{P}^i) + \sum_{j=1}^{\ell} q^{ij} \bar{P}^j \\ & \leq a_1 \bar{P}^i g(\bar{P}^i) + \sum_{j=1}^{\ell} q^{ij} \bar{P}^j. \end{aligned}$$

By Lemma 3.2, we have

$$P_t^{k,i} \leq \bar{P}_t^i \leq e^{a_1 T} = a \text{ for all } i \in \mathcal{M}. \quad (5.9)$$

This gives a uniform upper bound, completing the proof. \square

Set $Y^i = -\frac{1}{\beta} \ln P^i$, $Z^i = -\frac{\Lambda^i}{\beta P^i}$ for all $i \in \mathcal{M}$, then $(Y^i, Z^i) \in L_{\mathcal{F}^W}^{\infty}(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ and $(Y^i, Z^i)_{i \in \mathcal{M}}$ is the unique solution to

$$\begin{cases} dY^i = -\left[F^i(t, Z^i) - hY^i - \frac{\beta}{2}|Z^i|^2 + \frac{\rho^i}{\beta} - \frac{h}{\beta}(1 - \ln h) \right. \\ \quad \left. - \frac{1}{\beta} \sum_{j=1}^{\ell} q^{ij} \left(e^{-\beta(Y^j - Y^i)} - 1 \right) \right] dt + (Z^i)' dW, \\ Y_T^i = 0 \text{ for all } i \in \mathcal{M}, \end{cases} \quad (5.10)$$

where

$$F^i(t, z) := h_t \sup_{\pi \in \Pi} \left[-\frac{1}{2} \beta h_t |\pi' \sigma_t^i|^2 + \pi'(b_t^i - \beta \sigma_t^i z) \right], \quad (t, z) \in [0, T] \times \mathbb{R}^n.$$

Now we can present the solution to problem (5.2).

Theorem 5.2 *Suppose Assumption 4 holds. Let h be defined in (5.3) and $(Y^i, Z^i)_{i \in \mathcal{M}}$ be the unique solution to (5.10). Then the value function of the optimization problem (5.2) is given by*

$$V(x, i_0) = -e^{-\beta(h_0 x + Y_0^{i_0})},$$

and the optimal portfolio is

$$\hat{\pi}_t \in \operatorname{argmax}_{\pi \in \Pi} \left[-\frac{1}{2} \beta h_t |\pi' \sigma_t^{\alpha t}|^2 + \pi'(b_t^{\alpha t} - \beta \sigma_t^{\alpha t} Z_t^{\alpha t}) \right],$$

the optimal consumption strategy is

$$\hat{c}_t = h_t X_t + Y_t^{\alpha t} - \frac{1}{\beta} \ln h_t.$$

Proof: Same as before, we will omit the arguments (s, α_s) and (u, α_u) in the integrals for notation simplicity. For any $(c, \pi) \in \mathcal{U}$, applying Itô's formula to $-e^{-\int_0^t \rho \, du} e^{-\beta(h_t X_t + Y_t^{\alpha t})}$, we have

$$\begin{aligned} & -e^{-\int_0^t \rho \, du} e^{-\beta(h_t X_t + Y_t^{\alpha t})} - \int_0^t e^{-\int_0^s \rho \, du} e^{-\beta c} \, ds \\ &= -e^{-\beta(h_0 x + Y_0^{\alpha 0})} + \int_0^t \beta e^{-\int_0^s \rho \, du} e^{-\beta(hX+Y)} \left[-\frac{1}{\beta} e^{\beta(hX+Y)} e^{-\beta c} - hc \right. \\ & \quad \left. - \left(\frac{h}{\beta} (\ln h - 1) - h^2 X - hY \right) + h \left(-\frac{1}{2} \beta h |\pi' \sigma|^2 + \pi'(b - \beta \sigma z) \right) - F(s, Z) \right] \, ds \\ & \quad + \int_0^t \beta e^{-\int_0^s \rho \, du} e^{-\beta(hX+Y)} (h\pi' \sigma + Z') \, dW \\ & \quad + \int_0^t e^{-\int_0^s \rho \, du} e^{-\beta h X} \sum_{j, j' \in \mathcal{M}} (e^{-\beta Y^j} - e^{-\beta Y^{j'}}) I_{\{\alpha_s = j'\}} d\tilde{N}_s^{j'j}, \end{aligned}$$

where $(N^{j'j})_{j, j' \in \mathcal{M}}$ are independent Poisson processes each with intensity $q^{j'j}$, and $\tilde{N}_t^{j'j} = N_t^{j'j} - q^{j'j}t$, $t \geq 0$ are the corresponding compensated Poisson martingales under the filtration \mathcal{F} .

For any $(c, \pi) \in \mathcal{U}$, the wealth process X is continuous, hence bounded on $[0, T]$. Therefore, the stochastic integrals in the last equation are local martingales. Hence, there exists a sequence of increasing stopping times $(\tau_n)_{n \in \mathbb{N}}$ satisfying $\lim_{n \rightarrow \infty} \tau_n = T$, a.s. such that

$$\begin{aligned} & \mathbb{E} \left[-e^{-\int_0^{\tau_n} \rho \, du} e^{-\beta(h_{\tau_n} X_{\tau_n} + Y_{\tau_n}^{\alpha \tau_n})} - \int_0^{\tau_n} e^{-\int_0^s \rho \, du} e^{-\beta c} \, ds \right] \\ &= -e^{-\beta(h_0 x + Y_0^{\alpha 0})} + \mathbb{E} \left[\int_0^{\tau_n} \beta e^{-\int_0^s \rho \, du} e^{-\beta(hX+Y)} \left[-\frac{1}{\beta} e^{\beta(hX+Y)} e^{-\beta c} - hc \right. \right. \\ & \quad \left. \left. - \left(\frac{h}{\beta} (\ln h - 1) - h^2 X - hY \right) + h \left(-\frac{1}{2} \beta h |\pi' \sigma|^2 + \pi'(b - \beta \sigma z) \right) - F(s, Z) \right] \, ds \right]. \end{aligned}$$

Because $h, \beta > 0$ and

$$\sup_{c \in \mathbb{R}} \left[-\frac{1}{\beta} e^{\beta(hX+Y)} e^{-\beta c} - hc \right] = \frac{h}{\beta} (\ln h - 1) - h^2 X - hY,$$

we obtain from the definition of F^i that

$$\mathbb{E} \left[-e^{-\int_0^{\tau_n} \rho \, du} e^{-\beta(h_{\tau_n} X_{\tau_n} + Y_{\tau_n}^{\alpha \tau_n})} \right] - \mathbb{E} \left[\int_0^{\tau_n} e^{-\int_0^s \rho \, du} e^{-\beta c} \, ds \right] \leq -e^{-\beta(h_0 x + Y_0^{\alpha 0})},$$

for any $(c, \pi) \in \mathcal{U}$. As $n \rightarrow \infty$, the second expectation is convergent by the monotone convergence theorem; while the first expectation, because $(e^{-\beta h_t X_t^{\pi, c}})_{0 \leq t \leq T}$ belongs to class (D), Y^i , $i \in \mathcal{M}$, are bounded, is also convergent by the dominated convergence theorem. Hence

$$\mathbb{E} \left[-e^{-\int_0^T \rho \, du} e^{-\beta X_T} - \int_0^T e^{-\int_0^s \rho \, du} e^{-\beta c} \, ds \right] \leq -e^{-\beta(h_0 x + Y_0^{\alpha 0})}, \quad (5.11)$$

by virtue of the terminal conditions $h_T = 1$ and $Y_T = 0$. Furthermore, the inequality (5.11) becomes an equality for some $(\hat{c}, \hat{\pi})$ if and only if

$$\hat{c}_t = \operatorname{argmax}_{c \in \mathbb{R}} \left[-\frac{1}{\beta} e^{\beta(h_t X_t + Y_t^{\alpha t})} e^{-\beta c} - h_t c \right] = -\frac{1}{\beta} \ln h_t + h_t X_t + Y_t^{\alpha t},$$

and

$$\hat{\pi}_t \in \operatorname{argmax}_{\pi \in \Pi} \left[-\frac{1}{2} \beta h_t |\pi' \sigma_t^{\alpha t}|^2 + \pi'(b_t^{\alpha t} - \beta \sigma_t^{\alpha t} Z_t^{\alpha t}) \right].$$

The admissibility issue $(\hat{c}, \hat{\pi}) \in \mathcal{U}$ is covered by the proof of Theorem 5.5. \square

5.2 Case II: Random interest rate

Assumption 5 For all $i \in \mathcal{M}$,

$$\rho^i \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}).$$

Furthermore, the processes r , μ , σ are independent of the regime $i \in \mathcal{M}$ and such that

$$r \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}), \quad \mu \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}^m), \quad \sigma \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}^{m \times n}),$$

and $\sigma\sigma' \geq \delta \mathbf{1}_m$ with some constant $\delta > 0$.

Assumption 6 We have $m = n$ and $\Pi = \mathbb{R}^m$.

This assumption implies that we have neither the portfolio nor the consumption constraints. Under Assumptions 5 and 6, σ is a square invertible matrix.

Same as before, we need two BSDEs to express the optimal solution to (5.2). Now because the interest rate is stochastic, the first BSDE system for (h, η) is no more an ODE but a BSDE:

$$\begin{cases} dh = -[rh - h^2 - \eta'\sigma^{-1}b - \frac{1}{h}|\eta|^2] dt + \eta' dW, \\ h_T = 1. \end{cases} \quad (5.12)$$

and the second BSDE system for $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ is given by:

$$\begin{cases} dP^i = - \left[-rP^i - (\sigma^{-1}b)'\Lambda^i + \frac{1}{2\beta h^3} |h\sigma^{-1}b + \eta|^2 \right. \\ \quad \left. + \frac{\rho^i}{\beta h} - \frac{1}{\beta}(1 - \ln h) - \frac{1}{\beta h} \sum_{j=1}^{\ell} q^{ij} e^{-\beta h(P^j - P^i)} \right] dt + (\Lambda^i)' dW, \\ P_T^i = 0 \text{ for all } i \in \mathcal{M}. \end{cases} \quad (5.13)$$

Theorem 5.3 Under Assumptions 5 and 6, there is a unique solution (h, η) to the BSDE (5.12) such that $(h, \eta) \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ and $h \geq \delta$ for some positive constant δ .

Proof: Let $(p, q) \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ denote unique solution to the following linear BSDE:

$$\begin{cases} dp = -[1 - rp - q'\sigma^{-1}b] dt + q' dW, \\ p_T = 1. \end{cases} \quad (5.14)$$

Indeed, the process p admits an explicit representation:

$$p_t = \tilde{\mathbb{E}} \left[e^{-\int_t^T r du} + \int_t^T e^{-\int_t^s r du} ds \mid \mathcal{F}_t^W \right],$$

where $\tilde{\mathbb{E}}$ is the expectation with respect to the probability measure $\tilde{\mathbb{P}}$ defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E} \left(\int_0^T (\sigma^{-1}b)' dW \right).$$

It follows from Assumption 5 that $\delta_1 \leq p \leq \delta_2$ for some $0 < \delta_1 < \delta_2$. Therefore $(h, \eta) := (\frac{1}{p}, -\frac{q}{p^2})$ is well defined and $(h, \eta) \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ with $h \geq \frac{1}{\delta_2} > 0$. It can be directly

verified, using Itô's formula, that (h, η) is a solution of (5.12). As for uniqueness, observe that if (h, η) is a solution of (5.12) with $h \geq \delta$ for some $\delta > 0$, then $(p, q) = (\frac{1}{h}, -\frac{\eta}{h^2})$ is a solution to (5.14). Uniqueness follows from the fact that (5.14) has a unique solution. \square

Because $\eta \in L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ is not bounded, we cannot apply [9, Theorem 2.4] to solve (5.13).

Theorem 5.4 *Under Assumptions 5 and 6, there is a unique solution $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ to the BSDE (5.13) such that $(P^i, \Lambda^i) \in L_{\mathcal{F}^W}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$.*

Proof: We use the notations in the proof of Theorem 5.3. Under the probability measure $\tilde{\mathbb{P}}$,

$$\tilde{W}_t := W_t + \int_0^t \sigma^{-1} b \, ds,$$

is an n -dimensional Brownian motion. Notice that $\eta \in L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$, hence $\int_0^t \eta' \, d\tilde{W}$ is a BMO martingale under $\tilde{\mathbb{P}}$.

By boundedness, there exists a constant $k_1 > 0$ such that, for all $i \in \mathcal{M}$,

$$-r \leq k_1, \quad \frac{\rho^i}{\beta h} - \frac{1}{\beta}(1 - \ln h) \geq -k_1, \quad (5.15)$$

and

$$\tilde{\mathbb{E}} \left[\int_t^T e^{-\int_t^s r \, du} \left(\frac{1}{2\beta h^3} |h\sigma^{-1}b + \eta|^2 + \frac{\rho^i}{\beta h} - \frac{1}{\beta}(1 - \ln h) - \frac{1}{\beta h} q^{ii} \right) \, ds \mid \mathcal{F}_t^W \right] \leq k_1,$$

where the last estimate is due to $\eta \in L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$.

Denote $k = \max\{e^{k_1 T}, k_1\}$ and define a truncating function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\psi(y) := \max\{-k, \min\{y, k\}\}.$$

Consider the following truncated system of (5.13),

$$\begin{cases} dP^i = - \left[-rP^i - (\sigma^{-1}b)' \Lambda^i + \frac{1}{2\beta h^3} |h\sigma^{-1}b + \eta|^2 \right. \\ \quad \left. + \frac{\rho^i}{\beta h} - \frac{1}{\beta}(1 - \ln h) - \frac{1}{\beta h} \sum_{j=1}^\ell q^{ij} e^{-\beta h(\psi(P^j) - \psi(P^i))} \right] dt + (\Lambda^i)' dW, \\ P_T^i = 0 \text{ for all } i \in \mathcal{M}. \end{cases} \quad (5.16)$$

The driver is not Lipschitz continuous since $\eta \in L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ is unbounded. By [7, Theorem 9.3.5], there is a unique solution $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ to (5.16) such that $(P^i, \Lambda^i) \in L_{\mathcal{F}^W}^2(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^2(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$. If we can show $|P^i| \leq k$ and $\Lambda^i \in L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$, then $\psi(P^i) = P^i$ and consequently, $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ is a solution to (5.13).

We first show that $P^i \geq -k$. It is direct to verify that

$$(\underline{P}, \underline{\Lambda}) = (1 - e^{k_1(T-t)}, 0),$$

is a solution to the following BSDE:

$$\begin{cases} d\underline{P} = -(k_1 \underline{P} - k_1) dt + \underline{\Lambda}' dW \\ \underline{P}_T = 0. \end{cases} \quad (5.17)$$

Notice that $\underline{P} \leq 0$ and

$$\begin{aligned} k_1 \underline{P} - k_1 &\leq -r \underline{P} - (\sigma^{-1}b)' \underline{\Lambda} + \frac{1}{2\beta h^3} |h\sigma^{-1}b + \eta|^2 \\ &\quad + \frac{\rho^i}{\beta h} - \frac{1}{\beta} (1 - \ln h) - \frac{1}{\beta h} \sum_{j=1}^{\ell} q^{ij} e^{-\beta h(\psi(\underline{P}) - \psi(P))}, \end{aligned}$$

so by Lemma 3.2

$$-k \leq -e^{k_1 T} \leq \underline{P}_t \leq P_t^i.$$

We next show that $P^i \leq k$. From (5.16), we obtain

$$\begin{aligned} e^{-\int_0^t r \, du} P_t^i &= \int_t^T e^{-\int_0^s r \, du} \left[-(\sigma^{-1}b)' \Lambda^i + \frac{1}{2\beta h^3} |h\sigma^{-1}b + \eta|^2 \right. \\ &\quad \left. + \frac{\rho^i}{\beta h} - \frac{1}{\beta} (1 - \ln h) - \frac{1}{\beta h} \sum_{j=1}^{\ell} q^{ij} e^{-\beta h(\psi(P^j) - \psi(P^i))} \right] ds - \int_t^T (\Lambda^i)' dW \\ &= \int_t^T e^{-\int_0^s r \, du} \left[\frac{1}{2\beta h^3} |h\sigma^{-1}b + \eta|^2 + \frac{\rho^i}{\beta h} - \frac{1}{\beta} (1 - \ln h) \right. \\ &\quad \left. - \frac{1}{\beta h} \sum_{j=1}^{\ell} q^{ij} e^{-\beta h(\psi(P^j) - \psi(P^i))} \right] ds - \int_t^T (\Lambda^i)' d\widetilde{W}. \end{aligned}$$

Taking expectation $\widetilde{\mathbb{E}}$, and noting $q^{ij} \geq 0$ for $i \neq j$, we have

$$\begin{aligned} P_t^i &= e^{\int_0^t r \, du} \widetilde{\mathbb{E}} \left[\int_t^T e^{-\int_0^s r \, du} \left(\frac{1}{2\beta h^3} |h\sigma^{-1}b + \eta|^2 + \frac{\rho^i}{\beta h} - \frac{1}{\beta} (1 - \ln h) \right. \right. \\ &\quad \left. \left. - \frac{1}{\beta h} q^{ii} - \frac{1}{\beta h} \sum_{j \neq i}^{\ell} q^{ij} e^{-\beta h(\psi(P^j) - \psi(P^i))} \right) ds \middle| \mathcal{F}_t^W \right] \\ &\leq \widetilde{\mathbb{E}} \left[\int_t^T e^{-\int_0^s r \, du} \left(\frac{1}{2\beta h^3} |h\sigma^{-1}b + \eta|^2 + \frac{\rho^i}{\beta h} - \frac{1}{\beta} (1 - \ln h) - \frac{1}{\beta h} q^{ii} \right) ds \middle| \mathcal{F}_t^W \right] \\ &\leq k_1 \leq k. \end{aligned}$$

Now, we have proved $|P^i| \leq k$. Consequently, $(P^i, \Lambda^i)_{i \in \mathcal{M}}$ satisfies (5.13) as $\psi(P^i) = P^i$.

We now show $\Lambda^i \in L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$. To this end, applying Itô's formula to $(P_t^i)^2$, it then follows, for any stopping time $\tau \leq T$,

$$\begin{aligned} &(P_\tau^i)^2 + \mathbb{E} \left[\int_\tau^T |\Lambda^i|^2 ds \middle| \mathcal{F}_\tau^W \right] \\ &= \mathbb{E} \left[\int_\tau^T 2P^i \left(-rP^i - (\sigma^{-1}b)' \Lambda^i + \frac{1}{2\beta h^3} |h\sigma^{-1}b + \eta|^2 \right. \right. \\ &\quad \left. \left. + \frac{\rho^i}{\beta h} - \frac{1}{\beta} (1 - \ln h) - \frac{1}{\beta h} \sum_{j=1}^{\ell} q^{ij} e^{-\beta h(P^j - P^i)} \right) ds \middle| \mathcal{F}_\tau^W \right] \\ &\leq \mathbb{E} \left[\int_\tau^T \left(K + \frac{1}{2} |\Lambda^i|^2 + K |\eta|^2 \right) ds \middle| \mathcal{F}_\tau^W \right]. \end{aligned}$$

From which, we obtain

$$\mathbb{E} \left[\int_{\tau}^T |\Lambda^i|^2 ds \mid \mathcal{F}_{\tau}^W \right] \leq K + K \mathbb{E} \left[\int_{\tau}^T |\eta|^2 ds \mid \mathcal{F}_{\tau}^W \right].$$

Since $\eta \in L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$, the right hand side is uniformly bounded for all stopping times $\tau \leq T$, which yields $\Lambda^i \in L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$.

The uniqueness comes from [15, Lemma 3.4]. \square

Now set $(Y^i, Z^i) = (hP^i, h\Lambda^i + \frac{Y^i}{h}\eta)$ for all $i \in \mathcal{M}$. From Theorem 5.3 and Theorem 5.4, we immediately see that $(Y^i, Z^i) \in L_{\mathcal{F}^W}^{\infty}(0, T; \mathbb{R}) \times L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$ and $(Y^i, Z^i)_{i \in \mathcal{M}}$ is the unique solution to the following BSDE system:

$$\begin{cases} dY^i = - \left[-hY^i - (\sigma^{-1}b + \frac{\eta}{h})'Z^i + \frac{1}{2\beta h^2} |h\sigma^{-1}b + \eta|^2 \right. \\ \quad \left. + \frac{\rho^i}{\beta} - \frac{h}{\beta}(1 - \ln h) - \frac{1}{\beta} \sum_{j=1}^{\ell} q^{ij} e^{-\beta(Y^j - Y^i)} \right] dt + (Z^i)' dW, \\ Y_T^i = 0 \text{ for all } i \in \mathcal{M}. \end{cases} \quad (5.18)$$

Theorem 5.5 *Suppose Assumptions 5 and 6 hold. Let (h, η) and $(Y^i, Z^i)_{i \in \mathcal{M}}$ be the unique solutions to (5.12) and (5.18), respectively. Then the value function of the optimization problem (5.2) is given by*

$$V(x, i_0) = -e^{-\beta(h_0x + Y_0^{i_0})},$$

and the optimal portfolio is

$$\hat{\pi}_t = -\frac{1}{\beta h_t^2} (\sigma_t')^{-1} (\beta h_t \eta_t X_t + \beta h_t Z_t^{\alpha_t} - h_t \sigma_t^{-1} b_t - \eta_t), \quad (5.19)$$

the optimal consumption strategy is

$$\hat{c}_t = h_t X_t + Y_t^{\alpha_t} - \frac{1}{\beta} \ln h_t. \quad (5.20)$$

Proof: Same as before, we will omit the arguments (s, α_s) and (u, α_u) in the integrals for notation simplicity. For any $(c, \pi) \in \mathcal{U}$, applying Itô's formula to $-e^{-\int_0^t \rho du} e^{-\beta(h_t X_t + Y_t^{\alpha_t})}$, we obtain

$$\mathbb{E} \left[-e^{-\int_0^{\tau_n} \rho du} e^{-\beta(h_{\tau_n} X_{\tau_n} + Y_{\tau_n}^{\alpha_{\tau_n}})} \right] - \mathbb{E} \left[\int_0^{\tau_n} e^{-\int_0^s \rho du} e^{-\beta c} ds \right] \leq -e^{-\beta(h_0x + Y_0^{i_0})},$$

for a sequence of increasing localizing stopping times $(\tau_n)_{n \in \mathbb{N}}$ satisfying $\lim_{n \rightarrow \infty} \tau_n = T$. Same as the deterministic case, the two expectations are both convergence as $n \rightarrow \infty$ so that

$$\mathbb{E} \left[-e^{-\int_0^T \rho du} e^{-\beta X_T} \right] - \mathbb{E} \left[\int_0^T e^{-\int_0^s \rho du} e^{-\beta c} ds \right] \leq -e^{-\beta(h_0x + Y_0^{i_0})}, \quad (5.21)$$

thanks to the terminal conditions $h_T = 1$ and $Y_T = 0$. Moreover, the inequality (5.21) becomes an equality for some $(\hat{c}, \hat{\pi})$ if and only if both (5.19) and (5.20) hold.

Denote \hat{X} by the corresponding wealth process under the strategy $(\hat{c}, \hat{\pi})$. Then we have

$$\mathbb{E} \left[-e^{-\int_0^{\tau_n} \rho du} e^{-\beta(h_{\tau_n} \hat{X}_{\tau_n} + Y_{\tau_n}^{\alpha_{\tau_n}})} - \int_0^{\tau_n} e^{-\int_0^s \rho du} e^{-\beta \hat{c}} ds \right] = -e^{-\beta(h_0x + Y_0^{i_0})}.$$

Note that ρ , Y^i , $i \in \mathcal{M}$, are all bounded processes, so

$$\begin{aligned} \mathbb{E} \left[e^{-\beta h_{\tau_n} \hat{X}_{\tau_n}} + \int_0^{\tau_n} e^{-\beta \hat{c}} ds \right] &\leq K \mathbb{E} \left[e^{-\int_0^{\tau_n} \rho du} e^{-\beta(h_{\tau_n} \hat{X}_{\tau_n} + Y_{\tau_n}^{\alpha \tau_n})} + \int_0^{\tau_n} e^{-\int_0^s \rho du} e^{-\beta \hat{c}} ds \right] \\ &= K e^{-\beta(h_0 x + Y_0^{i_0})}. \end{aligned}$$

Applying Fatou's lemma to the above, we obtain

$$\mathbb{E} \left[e^{-\beta \hat{X}_T} + \int_0^T e^{-\beta \hat{c}} ds \right] \leq K. \quad (5.22)$$

Set $\theta = \frac{1}{h} \sigma^{-1}(hb + \sigma \eta)$, then $\theta \in L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ and $\mathcal{E}(\int_0^t \theta dW)$ is a uniformly integrable martingale. By Girsanov's theorem,

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \theta ds$$

is an n -dimensional Brownian motion under the measure \mathbb{Q} defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E} \left(- \int_0^T \theta dW \right).$$

We need to show that $(e^{-\beta h_t \hat{X}_t})_{0 \leq t \leq T}$ belongs to class (D). Compared with Section 4 or [4], a new term $\eta \in L_{\mathcal{F}^W}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ emerges (note r is deterministic in Section 4 or [4], hence $\eta = 0$). Applying Itô's formula to $\beta h_t \hat{X}_t$,

$$\begin{aligned} \beta h_t \hat{X}_t &= \beta h_0 x + \int_0^t \left[-\frac{1}{h^2} (\beta h Z - h \sigma^{-1} b - \eta)' \sigma^{-1} (hb + \sigma \eta) - h(\beta Y - \ln h) \right] ds \\ &\quad - \int_0^t \frac{1}{h} (\beta h Z - h \sigma^{-1} b - \eta)' dW \\ &= \beta h_0 x + \int_0^t \left[-h(\beta Y - \ln h) \right] ds + L_t, \end{aligned}$$

where

$$L_t = - \int_0^t \frac{1}{h} (\beta h Z - h \sigma^{-1} b - \eta)' dW^{\mathbb{Q}}.$$

Since $-\frac{1}{h} (\beta h Z - h \sigma^{-1} b - \eta) \in L_{\mathcal{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$, the process L_t is a BMO-martingale under \mathbb{Q} . Because h , Y^i , $i \in \mathcal{M}$, are bounded, there exists a positive constant K such that

$$\frac{1}{K} e^{-L_t} \leq e^{-\beta h_t \hat{X}_t} \leq K e^{-L_t} \text{ for all } t \in [0, T].$$

Hence our problem is reduced to showing that $(e^{-L_t})_{0 \leq t \leq T}$ belongs to class (D). By the reverse Hölder inequality, there exists $p > 1$ such that for all stopping times $\tau \leq T$,

$$\left(\mathbb{E} \left[\mathcal{E} \left(- \int_0^T \theta dW \right)^p \mid \mathcal{F}_\tau \right] \right)^{\frac{1}{p}} \leq K \mathcal{E} \left(- \int_0^\tau \theta dW \right). \quad (5.23)$$

Let q be the conjugate exponent of p , i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Since the function $x \mapsto e^{-\frac{1}{q}x}$ is convex, by Jensen's inequality, we have for any stopping time $\tau \leq T$,

$$e^{-\frac{1}{q}L_\tau} \leq \mathbb{E}^{\mathbb{Q}} \left[e^{-\frac{1}{q}L_T} \mid \mathcal{F}_\tau \right].$$

It thus follows from Hölder's inequality that

$$\begin{aligned}
e^{-L\tau} &\leq \left(\mathbb{E}^{\mathbb{Q}} \left[e^{-\frac{1}{q}L\tau} \mid \mathcal{F}_\tau \right] \right)^q \\
&= \left(\mathbb{E} \left[e^{-\frac{1}{q}L\tau} \mathcal{E} \left(- \int_0^\tau \theta \, dW \right) \mid \mathcal{F}_\tau \right] \right)^q \mathcal{E} \left(- \int_0^\tau \theta \, dW \right)^{-q} \\
&\leq \mathbb{E} \left[e^{-L\tau} \mid \mathcal{F}_\tau \right] \mathbb{E} \left[\mathcal{E} \left(- \int_0^\tau \theta \, dW \right)^p \mid \mathcal{F}_\tau \right]^{\frac{q}{p}} \mathcal{E} \left(- \int_0^\tau \theta \, dW \right)^{-q} \\
&\leq K \mathbb{E} \left[e^{-L\tau} \mid \mathcal{F}_\tau \right],
\end{aligned}$$

where the last inequality is due to (5.23). This shows that $(e^{-\beta h_t \hat{X}_t})_{0 \leq t \leq T}$ belongs to class (D) noting (5.22), hence the admissibility of $(\hat{c}, \hat{\pi})$ follows. \square

6 Concluding remarks

In this paper, we studied optimal consumption-investment problems in a market with regime switching and random coefficients with coupled constraints. Explicit solutions in term of the solutions of BSDE systems are given for the cases of power, logarithmic and exponential utilities. Some of these BSDE systems are new in the literature. Their solvability consists the main mathematical contributions and is interesting in its own from the point of view of BSDE theory. Extensions in other directions can be interesting as well; for instance: (1) The problem in infinite time horizon (lifetime). Actually the discount factor ρ was introduced in problems (2.2) and (5.2) for this possible extension. (2) The problem in an incomplete market, i.e. if $m < n$ in Subsection 5.2 or $r, \mu,$ or σ depends on the regime i .

References

- [1] Aliprantis C, Border K. Infinite dimensional analysis. New York: Springer, 2006.
- [2] Bauerle N, Rieder U. Portfolio optimization with Markov-modulated stock prices and interest rates. IEEE Trans. Automat. Control, 2004, 49(3): 442-447.
- [3] Becherer D. Bounded solutions to backward SDEs with jumps for utility optimization and indifference hedging. Ann. Appl. Probab., 2006, 16(4): 2027-2054.
- [4] Cheridito P, Hu Y. Optimal consumption and investment in incomplete markets with general constraints. Stoch. Dyn., 2011, 11(02n03): 283-299.
- [5] Cheridito P, Nam K. Multidimensional quadratic and sub-quadratic BSDEs with special structure. Stochastics, 2015, 87(5): 871-884.
- [6] Cox J C, Huang C. Optimal consumption and portfolio policies when asset prices follow a diffusion process. J. Econom. Theory, 1989, 49(1): 33-83.

- [7] Cvitanic J, and Zhang J. Contract theory in continuous-time models, Springer Science and Business Media, 2012.
- [8] Czichowsky C, Schweizer M. Cone-constrained continuous-time Markowitz problems. *Ann. Appl. Probab.*, 2013, 23(2): 764-810.
- [9] Fan S, Hu Y, Tang S. Multi-dimensional backward stochastic differential equations of diagonally quadratic generators: the general result. *J. Differential Equations*, 2023, 368:105-140.
- [10] Frei C, Dos Reis G. A financial market with interacting investors: does an equilibrium exist?. *Math. Financ. Econ.*, 2011, 4: 161-182.
- [11] Guan C, Xu Z Q, Yi F. A consumption-investment model with state-dependent lower bound constraint on consumption. *J. Math. Anal. Appl.*, 2022, 516(1): 126511.
- [12] Hamilton J. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, 1989, 57(2), 357-384.
- [13] Hu Y, Imkeller P, Muller M. Utility maximization in incomplete markets. *Ann. Appl. Probab.*, 2005, 15(3): 1691-1712.
- [14] Hu Y, Liang G, Tang S. Systems of ergodic BSDEs arising in regime switching forward performance processes. *SIAM J. Control Optim.*, 2020, 58(4): 2503-2534.
- [15] Hu Y, Shi X, Xu Z Q. Constrained stochastic LQ control with regime switching and application to portfolio selection. *Ann. Appl. Probab.*, 2022, 32 (1):426-460.
- [16] Hu Y, Shi X, Xu Z Q. Constrained stochastic LQ control on infinite time horizon with regime switching, *ESAIM Control Optim. Calc. Var.*, 28 (2022), 5.
- [17] Hu Y, Shi X, Xu Z Q. Non-homogeneous stochastic LQ control with regime switching and random coefficients. *arXiv:2201.01433v6*.
- [18] Hu Y, Peng S. On the comparison theorem for multidimensional BSDEs, *C. R. Acad. Sci. Paris, Ser. I*, 343 (2006), 135-140.
- [19] Hu Y, Tang S. Multi-dimensional backward stochastic differential equations of diagonally quadratic generators. *Stochastic Process. Appl.*, 2016, 126(4): 1066-1086.
- [20] Karatzas I, Lehoczky J, Shreve S. Optimal portfolio and consumption decisions for a “small investor” on a finite horizon, *SIAM J. Control Optim.*, 25(1987), 1557-1586.
- [21] Karatzas I, Shreve S. *Methods of mathematical finance*. Springer Science and Business Media, 1998.
- [22] Kazamaki N. *Continuous exponential martingales and BMO*. Springer, 2006.
- [23] Kobylanski M. Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.*, 28 (2000), 558-602.

- [24] Kramkov D, Pulido S. A system of quadratic BSDEs arising in a price impact model. *Ann. Appl. Probab.*, 2016, 26(2): 794-817.
- [25] Laeven R J A, Stajje M. Robust portfolio choice and indifference valuation. *Math. Oper. Res.*, 2014, 39(4): 1109-1141.
- [26] Liu R H. A finite-horizon optimal investment and consumption problem using regime-switching models. *Int. J. Theor. Appl. Finance*, 2014, 17(04): 1450027.
- [27] Luo P. A type of globally solvable BSDEs with triangularly quadratic generators. *Electron. J. Probab.*, 2020, 25: 1-23 (2020)
- [28] Matoussi A, Xing H. Convex duality for Epstein-Zin stochastic differential utility. *Math. Finance*, 2018, 28(4), 991-1019.
- [29] Merton R C. Lifetime portfolio selection under uncertainty: The continuous-time case. *Rev. Econ. Stat.*, 1969: 247-257.
- [30] Morlais M A. A new existence result for quadratic BSDEs with jumps with application to the utility maximization problem. *Stochastic Process. Appl.*, 2010, 120(10): 1966-1995.
- [31] Pardoux E, Peng S. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 1990, 14(1): 55-61.
- [32] Peng S. Stochastic Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.*, 1992, 30(2): 284-304.
- [33] Rouge R, El Karoui N. Pricing via utility maximization and entropy. *Math. Finance*, 2000, 10(2): 259-276.
- [34] Sethi S P. Optimal consumption and investment with bankruptcy. Springer Science & Business Media, 1996.
- [35] Shin Y H. Optimal consumption and investment problem with Regime-Switching and CARA Utility. *J. Chungcheong Math. Soc.*, 2013, 26(1): 85-85.
- [36] Sotomayor L R. Cadenillas A. Explicit solutions of consumption-investment problems in financial markets with regime switching. *Math. Finance*, 2009, 19(2): 251-279.
- [37] Tevzadze R. Solvability of backward stochastic differential equations with quadratic growth. *Stochastic processes and their Applications*, 2008, 118(3): 503-515.
- [38] Wei J, Shen Y, Zhao Q. Portfolio selection with regime-switching and state-dependent preferences. *J. Comput. Appl. Math.*, 2020, 365: 112361.
- [39] Xing H. Consumption-investment optimization with Epstein-Zin utility in incomplete markets. *Finance Stoch.*, 2017, 21, 227-262.
- [40] Xu Z Q, Yi F. An optimal consumption-investment model with constraint on consumption. *Math. Control Relat. Fields*, 2016, 6(3): 517-534.

- [41] Yin G, Zhou X Y Markowitz's mean-variance portfolio selection with regime switching: from discrete-time models to their continuous-time limits. *IEEE Trans. Automat. Contr.*, 2004, 49(3), 349-360.
- [42] Zariphopoulou T. Investment-consumption models with transaction fees and Markov-chain parameters. *SIAM J. Control Optim.*, 1992, 30(3): 613-636.
- [43] Zariphopoulou T. Consumption-investment model with constraints. *SIAM J. Control Optim.*, 1994, 32(1): 59-85.
- [44] Zhang Q, Yin G. Nearly-optimal asset allocation in hybrid stock investment models. *J. Optim. Theory Appl.*, 2004, 121(2): 419-444.
- [45] Zhou X Y, Yin G. Markowitz's mean-variance portfolio selection with regime switching: A continuous-time model. *SIAM J. Control Optim.*, 2003, 42, 1466-1482.