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A relaxed condition for "perfect" cancellation of broadband noise in 3D enclosures

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This paper presents a relaxed condition for ''perfect'' cancellation of broadband noise in 3D enclosures. On the basis of a truncated modal model, it can be shown that the primary and secondary paths belong to a same subspace if a certain condition is satisfied. There exists a finite impulse response (FIR) filter transfer function vector for perfect cancellation of the primary paths. The analytical result is verified numerically with an active noise control (ANC) system in a 3D rectangular enclosure. The proposed ANC scheme is shown to fit well into the framework of an existing multichannel least-mean squares (LMS) algorithm for adaptive implementation. © *2000 Acoustical Society of America.* [S0001-4966(00)00806-7]

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INTRODUCTION

Active noise control (ANC) is a technique that uses actuators to synthesize destructive interference at selected locations in a sound field.[1] It is most effective in low-frequency ranges where the performance of passive noise control methods tends to deteriorate. This reason makes ANC an important alternative to passive noise control methods.

The objective of ANC is to cancel or minimize noise signals in n sensor locations. Its mathematical model consists of a primary path vector $P(z)$ and a secondary path matrix $\mathbf{G}(z)$. ''Perfect'' cancellation requires $\mathbf{G}(z)H(z)+P(z)=0$, where $H(z)$ denotes the ANC transfer function. An ideal solution may be $H_I(z)=-\mathbf{G}^{-1}(z)P(z)$ that is not necessarily applicable. Stability of $H_I(z)$ requires a minimum phase $\mathbf{G}(z)$ that may be true if the error sensors collocate with the actuators. In practice, error sensors are usually placed away from actuators to avoid the near-field effects. With few exceptions, $\mathbf{G}(z)$ is usually nonminimum phase and $H_1(s)$ is not stable. Some researchers[2] derive stable controllers to approximate $H_I(s)$ under proper criteria. Most others adjust the weights of an FIR filter $H(z)$ to minimize $\min_{H(z)}\|\mathbf{G}(z)H(z)+P(z)\|$ in the least-mean squares (LMS) sense.[3,4] It is suggested[5] that an IIR filter $H(z)$ will minimize $\min_{H(z)}\|\mathbf{G}(z)H(z)+P(z)\|$ with better effects. These are stable approximations of an unstable, albeit exact, solution $H_I(s)$, not perfect cancellation solutions.

In this study, perfect cancellation means a stable and exact solution of $\mathbf{G}(z)H(z)+P(z)=0$, which is achievable by an interesting idea.[6] The method, named ''MINT'' by its authors, depends on an important result in general control theory:[7] If the Smith form of an $n\times(n+1)$ polynomial matrix $\mathbf{G}(z)$ is equivalent to $[\mathbf{I}_n \quad O]$, then there exists a polynomial matrix $\mathbf{H}(z)$ with a finite degree such that

$$\mathbf{I}_n=\mathbf{G}(z)\mathbf{H}(z), \tag{1}$$

where $\mathbf{I}_n$ is an $n\times n$ identity and O an all-zero vector. An experimental verification of MINT creates a zone of quiet in a wild frequency range.[8] For ease of reference, the condition on the Smith form of $\mathbf{G}(z)$ is called the MINT assumption here.

This study uses a truncated modal model to analyze 3D reverberant sound fields. It reveals that the MINT assumption is not easy to satisfy for $n>2$ in a 3D reverberant field, though it is generally true for $n=1$. Perfect cancellation is still possible even if the MINT assumption is not valid. The reason is rather simple: $P(z)$ is not an arbitrary vector. It is subject to the same acoustical constraints as $\mathbf{G}(z)$. Under a relaxed condition, $\mathbf{G}(z)$ will consist of a sufficient number of basis to span a polynomial subspace that includes $P(z)$. A multichannel filtered-x algorithm[9] may be applied to implement an FIR filter $H(z)$ such that $P(z)+\mathbf{G}(z)H(z)=0$ for all z. Contrary to MINT, this approach may be called SCBN for ''subspace cancellation of broadband noise'' in 3D enclosures. The corresponding condition may be called the SCBN assumption as well.

I. MATHEMATICAL MODEL

Let $\mathbf{x}$ denote a 3D coordinate; then, a pressure signal, measured at $\mathbf{x}$, can be expressed as

$$y(t,\mathbf{x})=\sum_{i=1}^{\infty}\alpha_i(t)\phi_i(\mathbf{x}), \tag{2}$$

where ϕ_i is a spatial mode function and α_i the temporal mode magnitude. The Laplace transform of α_i is the response of a second-order subsystem

$$\alpha_i(s)=\frac{\kappa_{ij}}{s^2+2\xi_i\omega_i s+\omega_i^2}u_j(s), \tag{3}$$

where ω_i and ξ_i are, respectively, the ith resonance frequency and damping ratio; $u_j(s)$ is a source. The spatial source effect on the ith mode is evaluated by

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$$\kappa_{ij}=\int\int_V\int \phi_i(\mathbf{x})f_j(\mathbf{x})dv, \tag{4}$$

where $f_j(\mathbf{x})$ describes the spatial magnitude of the source. The integral reduces to $\kappa_{ij}=\phi_i(\mathbf{x}_j)$ if the source can be approximated by a point excitation at location $\mathbf{x}_j$.

A. Single-path model

Since ANC is most effective in low-frequency ranges, a measured signal is usually low-pass filtered by a sharp filter $F(s)$ to cut off the high-frequency components. Let $y_k(s)$ denote the pressure signal sensed at $\mathbf{x}_k$ and filtered by $F(s)$; one can write

$$\begin{aligned} y_k(s)&=\left(\varphi_{kj}+\sum_{i=1}^{m}\frac{\phi_i(\mathbf{x}_k)\phi_i(\mathbf{x}_j)}{s^2+2\xi_i\omega_i s+\omega_i^2}\right)u_j(s)\\ &=g_{kj}(s)u_j(s),\end{aligned}$$

where $g_{kj}(s)$ denotes the transfer function from source location $\mathbf{x}_j$ to sensor location $\mathbf{x}_k$. It is derived from Eqs. (2), (3), and (4), assuming a point excitation. The 0th mode of $g_{kj}(s)$, denoted by

$$\begin{aligned}\varphi_{kj}&=\sum_{i=m+1}^{\infty}\frac{\phi_i(\mathbf{x}_k)\phi_i(\mathbf{x}_j)}{\omega_i^2}\\ &\approx\sum_{i=m+1}^{\infty}\frac{\phi_i(\mathbf{x}_k)\phi_i(\mathbf{x}_j)}{s^2+2\xi_i\omega_i s+\omega_i^2}F(s),\end{aligned}$$

is due to high-order modes in the passband of the sharp low-pass filter.[10]

In control theory, cancellation of the 0th mode is a trivial problem. One may ignore φ_{kj} in $g_{kj}(s)$ without losing generality while keeping a better focus on the main problem. The path model is therefore simplified to

$$g_{kj}(s)=\sum_{i=1}^{m}\frac{\phi_i(\mathbf{x}_k)\phi_i(\mathbf{x}_j)}{s^2+2\xi_i\omega_i s+\omega_i^2}=\frac{g_{kj}^*(s)}{D(s)}, \tag{5}$$

where $D(s)$ is the system characteristic polynomial; $g_{kj}^*(s)$ is the numerator polynomial of $g_{kj}(s)$. Equation (5) implies

$$g_{kj}^*(s)=\sum_{i=1}^{m}\phi_i(\mathbf{x}_k)\phi_i(\mathbf{x}_j)\theta_i(s), \tag{6}$$

where

$$\theta_i(s)=\prod_{l\neq i}^{m}(s^2+2\xi_l\omega_l s+\omega_l^2)\quad\text{for}\ 1\leq i\leq m. \tag{7}$$

These m scalar polynomials form a polynomial diagonal matrix

$$\mathbf{\Theta}(s)=\begin{bmatrix}\theta_1(s)&&&\\&\theta_2(s)&&\\&&\ddots&\\&&&\theta_m(s)\end{bmatrix}=\sum_{l=0}^{2m-2}\mathbf{\Theta}_l s^l, \tag{8}$$

where the ith diagonal element of $\mathbf{\Theta}_l$ is the lth-order coefficient of $\{\theta_i(s)\}_{i=1}^m$.

The mode functions may be spatially sampled at a coordinate $\mathbf{x}=[xyz]$ to form a mode vector

$$\Phi^T(\mathbf{x})=[\phi_1(\mathbf{x})\ \ \phi_2(\mathbf{x})\cdots\phi_m(\mathbf{x})]. \tag{9}$$

Equations (8) and (9) allow one to express Eq. (6) in a concise form

$$g_{kj}^*(s)=\Phi^T(\mathbf{x}_k^s)\mathbf{\Theta}(s)\Phi(\mathbf{x}_j^a), \tag{10}$$

where superscripts "s" and "a" indicate that the mode vectors are sampled at sensor location $\mathbf{x}_k^s$ and actuator location $\mathbf{x}_j^a$, respectively.

B. Multiple-path model

Generally, an ANC system has n_s sensors and n_a actuators. Locations of actuators and sensors affect the transfer functions in terms of an actuation matrix

$$\begin{aligned}\mathbf{M}_a&=\begin{bmatrix}\phi_1(\mathbf{x}_1^a)&\phi_1(\mathbf{x}_2^a)&\cdots&\phi_1(\mathbf{x}_{n_a}^a)\\ \phi_2(\mathbf{x}_1^a)&\phi_2(\mathbf{x}_2^a)&\cdots&\phi_2(\mathbf{x}_{n_a}^a)\\ \vdots&\vdots&\vdots&\vdots\\ \phi_m(\mathbf{x}_1^a)&\phi_m(\mathbf{x}_2^a)&\cdots&\phi_m(\mathbf{x}_{n_a}^a)\end{bmatrix}\\ &=[\Phi(\mathbf{x}_1^a)\ \ \Phi(\mathbf{x}_2^a)\ \cdots\ \Phi(\mathbf{x}_{n_a}^a)],\end{aligned} \tag{11}$$

and a sensor matrix

$$\mathbf{M}_s^T=\begin{bmatrix}\phi_1(\mathbf{x}_1^s)&\phi_2(\mathbf{x}_1^s)&\cdots&\phi_m(\mathbf{x}_1^s)\\ \phi_1(\mathbf{x}_2^s)&\phi_2(\mathbf{x}_2^s)&\cdots&\phi_m(\mathbf{x}_2^s)\\ \vdots&\vdots&\vdots&\vdots\\ \phi_1(\mathbf{x}_{n_s}^s)&\phi_2(\mathbf{x}_{n_s}^s)&\cdots&\phi_m(\mathbf{x}_{n_s}^s)\end{bmatrix}=\begin{bmatrix}\Phi^T(\mathbf{x}_1^s)\\ \Phi^T(\mathbf{x}_2^s)\\ \cdots\\ \Phi^T(\mathbf{x}_{n_s}^s)\end{bmatrix}. \tag{12}$$

The system has $n_s\times n_a$ transfer functions representing paths from n_a actuators to n_s sensors. Similar to the derivation of Eqs. (5) and (10), one can express these transfer functions in a concise $n_s\times n_a$ matrix

$$\mathbf{G}(s)=\frac{1}{D(s)}\mathbf{M}_s^T\mathbf{\Theta}(s)\mathbf{M}_a. \tag{13}$$

The effect of the primary source can be modeled by $\kappa_p^T=[\kappa_{1p}\ \ \kappa_{2p}\cdots\ \kappa_{mp}]$. The ith element of κ_p is an integral similar to Eq. (4), with $f_j(\mathbf{x})$ replaced by $f_p(\mathbf{x})$. Since the primary source is not necessarily a point excitation, κ_p is not necessarily Φ sampled at a specific coordinate. Let $P(s)$ denote the primary path vector; then, it differs from Eq. (13) in terms κ_p. The sensor matrix $\mathbf{M}_s^T$ remains unchanged because the same set of sensors measure a sound field due to both primary source and secondary actuators. It then follows

$$P(s)=\frac{1}{D(s)}\mathbf{M}_s^T\mathbf{\Theta}(s)\kappa_p. \tag{14}$$

Combining Eqs. (13) and (14), one expresses the sensor signals as

$$Y(s)=\frac{1}{D(s)}\mathbf{M}_s^T\mathbf{\Theta}(s)[\kappa_p r(s)+\mathbf{M}_a U(s)],$$

where $r(s)$ is a scalar transfer function of the primary source and $U(s)$ is a vector transfer function of n_a secondary actuators. Cancellation errors are measured by the sensors and expressed in Laplace transform domain as an n_s-dimensional vector $Y(s)$.

C. Definition of perfect cancellation

Since the primary path vector and the secondary path matrix share the same denominator $D(s)$, only the numerators are of interest here. If the actuator signals are synthesized by $U(s)=H(s)r(s)$, then the problem becomes

$$\mathbf{M}_s^T\mathbf{\Theta}(s)[\kappa_p+\mathbf{M}_aH(s)]=0. \tag{15}$$

It defines the mathematical meaning of perfect cancellation of the primary paths.

The analysis is based on the modal theory that does not take pure propagation delays into explicit account. Instead, a modal model uses infinite modes to describe a reverberant field collectively. While mode truncation introduces inevitable errors, the theory is reasonably accurate at predicting path transfer functions in low-frequency ranges when mode density is sparse.

One may be concerned with the sharp low-pass filter as it introduces a phase shift to Eq. (5). This is not a problem when the filters are only applied to the error sensors. The filter transfer function $F(s)$ need not be included in the model since it is a common multiplier to all paths and $F(s)\neq 0$ in the frequency range of interest. Equation (5) uses $|F(s)|$ to justify mode truncation in the passband. The truncated model is inaccurate in the transition and stop bands that are not of interest here. The reference signal $r(s)$ does not pass the sharp low-pass filter and hence avoids phase distortion.

II. CONDITIONS FOR PERFECT CANCELLATION

This section shows the existence of a polynomial solution $H(s)$ for Eq. (15) under a relaxed condition. For better focus, the reference signal is assumed available. Otherwise, it will involve more problems to be addressed in Sec. IV.

A. System assumptions

If there are m properly placed actuators to ensure a square and full rank $\mathbf{M}_a$, then there exists a vector H such that $\kappa_p+\mathbf{M}_aH=0$ to achieve perfect cancellation. This, however, is not the case to be studied here. A more general problem is to design an ANC when the number of actuators is smaller than the number of modes. Hence, the following assumption:

Assumption 1: Rank $(\mathbf{M}_s^T)=n$ and rank $(\mathbf{M}_a)=n+r^*<m$ for an integer $r^*>0$.

Generally, $\kappa_p+\mathbf{M}_aH=0$ is not possible when assumption 1 holds. An ANC transfer function should be a polynomial vector $H(s)$ of degree n_0. Equation (15) becomes

$$P^*(s)=-\mathbf{G}^*(s)H(s), \tag{16}$$

where $P^*(s)=\mathbf{M}_s^T\mathbf{\Theta}(s)\kappa_p$ and $\mathbf{G}^*(s)=\mathbf{M}_s^T\mathbf{\Theta}(s)\mathbf{M}_a$ are numerator parts of transfer functions $P(s)$ and $\mathbf{G}(s)$, respectively.

B. Matrix form of polynomial products

To solve Eq. (16), a matrix form[8] of polynomial products is adopted here. It expresses a polynomial product $C(s)=A(s)B(s)$ as

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n_1+n_2} \end{bmatrix} = \underbrace{\begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & a_1 & \ddots & \\ a_{n_1} & \vdots & \ddots & a_0 \\ & a_{n_1} & \ddots & a_1 \\ & & \ddots & \vdots \\ & & & a_{n_1} \end{bmatrix}}_{n_2+1 \quad \text{columns}} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n_2} \end{bmatrix},$$

where

$$A(s)=\sum_{i=0}^{n_1} a_i s^i$$

and

$$B(s)=\sum_{i=0}^{n_2} b_i s^i.$$

Coefficients of $A(s)$ contribute to the above matrix that has a row size n_1+n_2+1 and a column size n_2+1. Coefficient vector of $B(s)$ multiplies to coefficient matrix of $A(s)$ to produce coefficient vector of $C(s)$.

Without excessive use of symbols, one may express the above matrix form as $C=\mathbf{A}B$ with C denoting coefficient vector of $C(s)$, $\mathbf{A}$ representing coefficient matrix of $A(s)$ and B coefficient vector of $B(s)$. By the same token, the matrix form of Eq. (16) is denoted as $P^*=-\mathbf{G}^*H$. Detailed expressions of P^* and $\mathbf{G}^*$ read, respectively,

$$P^*=\begin{bmatrix} P_0^* \\ P_1^* \\ \vdots \\ P_{2m-2}^* \\ O \\ \vdots \\ O \end{bmatrix}$$

and

$$\mathbf{G}^*=\underbrace{\begin{bmatrix} \mathbf{G}_0^* & & & \\ \mathbf{G}_1^* & \mathbf{G}_0^* & & \\ \vdots & \mathbf{G}_1^* & \ddots & \\ \mathbf{G}_{2m-2}^* & \vdots & \ddots & \mathbf{G}_0^* \\ & \mathbf{G}_{2m-2}^* & \ddots & \mathbf{G}_1^* \\ & & \ddots & \vdots \\ & & & \mathbf{G}_{2m-2}^* \end{bmatrix}}_{n_0+1 \quad \text{blocks}},$$

$$(17)$$

where $\{P_i^* = \mathbf{M}_s^T \boldsymbol{\Theta}_i \kappa_p\}_{i=0}^{2m-2}$, $\{G_i^* = M_s^T \Theta_i M_a\}_{i=0}^{2m-2}$ and $\boldsymbol{\Theta}_i$ is the ith-order coefficient of $\boldsymbol{\Theta}(s)$.

A subspace analysis is conducted in the Appendix to show that P^* belongs to the column space of $\mathbf{G}^*$ when assumption 1 is true and n_0 satisfies Eq. (A5).

C. MINT and SCBN solutions

Both MINT and SCBN try to solve a finite-degree polynomial $H(s)$ whose coefficient vector H satisfies $P^* = -\mathbf{G}^* H$. MINT depends on Eq. (1) to match an arbitrary transfer function. It requires a square and full rank $\mathbf{G}^*$. With n error sensors and $(n+1)$ actuators, $\mathbf{G}^*$ is square if $n_0 = (2m-2)n-1$. Its rank, however, depends on the Smith form of $\mathbf{G}^*(s)$ (Ref. 7)—a condition derived for general polynomial matrices without any specific application background. The Appendix reveals that $\mathbf{G}^*$ is not necessarily full rank in a 3D reverberant field. That means the MINT assumption is not necessarily true. Fortunately, P^* is not an arbitrary vector. It belongs to the column space of $\mathbf{G}^*$ under a relaxed condition. Physically, this is understandable since all sources are subject to the same wave equation under the same boundary conditions when placed in the same enclosure. Therefore, $P^* = -\mathbf{G}^* H$ has an exact solution even if $\mathbf{G}^*$ is neither square nor full rank.

D. A solution based on a discrete-time model

A continuous-time sound field becomes a hybrid system when subject to the interference of a discrete-time ANC. A well-accepted way[11] to obtain a discrete-time model for a hybrid system is mathematically given by

$$G(z)H(z) = (1-z^{-1})Z\left[\frac{G(s)}{s}\right]H(z),$$

where $G(s)$ and $H(z)$ denote, respectively, transfer functions of the continuous-time part and the discrete-time part of a hybrid system. The output of $H(z)$ excites $G(s)$ via a first-order hold circuit represented by $(1-z^{-1})/s$.

Comparing the two sides of the above equation, one can see $G(z) = (1-z^{-1})Z[G(s)/s]$. Applying this to Eq. (5), one obtains

$$\begin{aligned} g_{kj}(z) &= (1-z^{-1})\sum_{i=1}^{m} Z\left[\frac{\phi_i(\mathbf{x}_k^s)\phi_i(\mathbf{x}_j^a)}{s(s^2+2\xi_i\omega_i s+\omega_i^2)}\right] \\ &= \sum_{i=1}^{m} \frac{\phi_i(\mathbf{x}_k^s)\phi_i(\mathbf{x}_j^a)(\eta_i z^{-1}+\rho_i z^{-2})}{[1-2z^{-1}e^{-T\sigma_i}\cos(\beta_i T)+e^{-2T\sigma_i}z^{-2}]\omega_i^2} \\ &= \frac{g_{kj}^*(z)}{D(z)}, \end{aligned} \tag{18}$$

where $\sigma_i = \xi_i\omega_i$, $\beta_i = \omega_i\sqrt{1-\xi_i^2}$, $\eta_i = 1-e^{-T\sigma_i}\cos(\beta_i T) + e^{-T\sigma_i}(\sigma_i/\beta_i)\sin(\beta_i T)$, $\rho_i = e^{-2T\sigma_i} + e^{-T\sigma_i}(\sigma_i/\beta_i)\sin(\beta_i T) - e^{-T\sigma_i}\cos(\beta_i T)$ and T is the sampling interval. The characteristic polynomial is given by

$$D(z) = \prod_{l=1}^{m} [1-2z^{-1}e^{-T\sigma_l}\cos(\beta_l T)+e^{-2T\sigma_l}z^{-2}].$$

Similar to analog systems, the numerator of Eq. (18) has a concise form $g_{kj}^*(z) = \Phi^T(\mathbf{x}_k^s)\boldsymbol{\Theta}(z)\Phi(\mathbf{x}_j^a)$. It uses exactly the same $\Phi^T(\mathbf{x}_k^s)$ and $\Phi(\mathbf{x}_j^a)$ as Eq. (10), but a different $\boldsymbol{\Theta}(z)$. The ith diagonal element of $\Theta(z)$ is a scalar polynomial

$$\theta_i(z) = \frac{(\eta_i z^{-1}+\rho_i z^{-2})}{\omega_i^2}\prod_{l\neq i}[1-2z^{-1}e^{-T\sigma_l}\cos(\beta_l T) + e^{-2T\sigma_l}z^{-2}],$$

whose degree increases to $2m$ due to $(\eta_i z^{-1}+\rho_i z^{-2})/\omega_i^2$ in the above expression. The secondary path matrix and the primary path vector become, respectively,

$$\mathbf{G}(s) = \frac{1}{D(z)}\mathbf{M}_s^T\boldsymbol{\Theta}(z)\mathbf{M}_a$$

and

$$P(s) = \frac{1}{D(z)}\mathbf{M}_s^T\boldsymbol{\Theta}(z)\kappa_p. \tag{19}$$

The primary paths and secondary paths still share the same denominator $D(z)$. The numerators differ from their analog counterparts in terms of polynomial matrix $\boldsymbol{\Theta}(z)$.

Perfect cancellation means

$$\mathbf{M}_s^T\boldsymbol{\Theta}(z)[\kappa_p + \mathbf{M}_a H(z)] = 0, \tag{20}$$

which is very similar to Eq. (15), only with a different $\boldsymbol{\Theta}(z)$. Such a similarity implies a matrix form of Eq. (20) as $P^* = -\mathbf{G}^* H$, where

$$P^* = \begin{bmatrix} P_1^* \\ P_2^* \\ \vdots \\ P_{2m}^* \\ O \\ \vdots \\ O \end{bmatrix} \quad \text{and} \quad \mathbf{G}^* = \underbrace{\begin{bmatrix} \mathbf{G}_1^* & & & \\ \mathbf{G}_2^* & \mathbf{G}_1^* & & \\ \vdots & \mathbf{G}_2^* & \ddots & \\ \mathbf{G}_{2m}^* & \vdots & \ddots & \mathbf{G}_1^* \\ & \mathbf{G}_{2m}^* & \ddots & \mathbf{G}_2^* \\ & & \ddots & \vdots \\ & & & \mathbf{G}_{2m}^* \end{bmatrix}}_{n_0+1 \text{ blocks}}. \tag{21}$$

It can be shown, similar to the Appendix, that P^* belongs to the column space of $\mathbf{G}^*$ when assumption 1 is true and n_0 satisfies

$$n_0 = \left\lceil \frac{2m+\min(2n,m)-3n-r^*}{r^*} \right\rceil \tag{22}$$

which denotes the smallest positive integer larger than or equal to $[2m+\min(2n,m)-3n-r^*]/r^*$. The rank of $\mathbf{G}^*$ is bounded by

$$\operatorname{rank}(\mathbf{G}^*) \leq 2m + \min(2n,m) + n(n_0-2). \tag{23}$$

In the presence of 0th mode, Eqs. (22) and (23) become

$$\begin{aligned} n_0 &= \left\lceil \frac{2m+\min(3n,m)-3n-r^*}{r^*} \right\rceil \\ \operatorname{rank}(\mathbf{G}^*) &\leq 2m + \min(3n,m) + n(n_0-2). \end{aligned} \tag{24}$$

Under these conditions, $P^* = -\mathbf{G}^* H$ has an exact solution as coefficients of an FIR filter vector.

E. Adaptive implementation

Like all other adaptive ANC schemes, SCBN depends on a pre-estimated model $\mathbf{G}(z) = 1/[D(z)]\mathbf{G}^*(z)$ that includes system characteristic polynomial $D(z)$ and a numerator polynomial matrix $\mathbf{G}^*(z)$. The error sensors measure a signal vector $Y(z)$ to get $\epsilon(z) = D(z)Y(z)$. It can be shown that

$$\epsilon(z) = Y_p(z) + \mathbf{G}^*(z)H(z)r(z), \tag{25}$$

where the primary source contributes $Y_p(z) = P^*(z)r(z)$ and the ANC contributes $\mathbf{G}^*(z)H(z)r(z)$. The filtered residue $\epsilon(z)$ may be used by a proper adaptation law to adjust $H(z)$.

An excellent study[9] has been conducted to address the application of LMS to multichannel ANC systems, with a thorough convergence analysis. All left to do here is to show that Eq. (25) fits into the framework studied in Ref. 9.

Let $G_j^*(z)$ denote the jth column of matrix $\mathbf{G}^*(z)$; then, Eq. (25) may be expressed as

$$\begin{aligned}\epsilon(z) &= Y_p(z) + \sum_{j=1}^{n+r^*} G_j^*(z)h_j(z)r(z) \\ &= Y_p(z) + \sum_{j=1}^{n+r^*} h_j(z)\Psi_j(z) = Y_p(z) + \mathbf{\Psi}(z)H(z),\end{aligned} \tag{26}$$

where $\Psi_j(z) = G_j^*(z)r(z)$ represents the jth column of filtered signals and matrix

$$\mathbf{\Psi}(z) = [\Psi_1(z)\ \Psi_2(z)\ \cdots\ \Psi_{n+r^*}(z)]$$

contains the Z transforms of all filtered signals. A regressive form is now derived in Z-transform domain as Eq. (26). It has a time-domain expression of

$$\epsilon(t) = Y_p(t) + [\mathbf{\Psi}(t)\ \mathbf{\Psi}(t-1)\ \cdots\ \mathbf{\Psi}(t-n_0)]\begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_{n_0}\end{bmatrix},$$

where $Y_p(t)$ and $\mathbf{\Psi}(t)$ are the time-domain versions of $Y_p(z)$ and $\mathbf{\Psi}(z)$, respectively. Adaptive implementation of SCBN becomes straightforward thereafter.

III. NUMERICAL VERIFICATION

This section presents a numerical example to verify the theoretical results obtained by Sec. II. All path transfer functions are available analytically by the modal theory.

A. Model selection

The sound field is enclosed in a rectangle cavity of $260 \times 64 \times 60$ in.[3] with modes and resonant frequencies available from Ref. 10. The numerical example excludes the 0th mode and uses the next 14 modes. The ANC consists of three sensors and five actuators with rank $(\mathbf{M}_s^T) = 3$ and rank $(\mathbf{M}_a) = 5$, respectively. Figure 1 illustrates the sensor and actuator locations, where the sensors are represented by small spheres and the actuators are represented by larger ones. Three lines connect the primary source to the sensors to distinguish the primary paths.

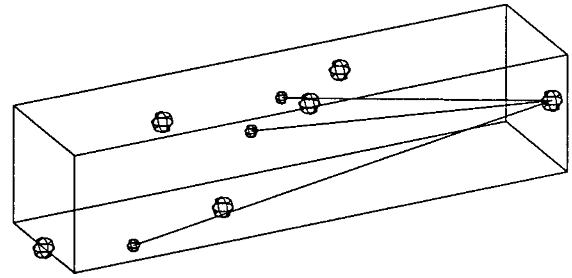

FIG. 1. Locations of sensors and actuators.

Since the SCBN condition is rank $(\mathbf{M}_s^T) <$ rank$(\mathbf{M}_a)$, one may worry about the accessibility of this criterion when the mode functions are not accessible. Usually, locations of error sensors may be specified by a quiet zone. The rank of $\mathbf{M}_s^T$ may not be changed even if the mode functions are accessible. However, one always knows rank $(\mathbf{M}_s^T) \leq n_s$ without the exact form of $\mathbf{M}_s^T$. The problem becomes how to make $n_s <$ rank$(\mathbf{M}_a)$. Generally, an ANC designer has the flexibility to determine the number and locations of actuators. It is not difficult to use a conservatively large number of actuators to meet the requirement, as long as the actuators do not collocate substantially.

In the present example, it is not difficult to get rank $(\mathbf{M}_a) = 5$ by placing actuators sufficiently away from each other. The actuator placement is disorganized as shown in Fig. 1. Available information of mode shapes could suggest a more organized actuator placement. This demonstrates the ease to satisfy assumption 1. In Fig. 1, the actuators are closer to the error sensors than the primary source. Therefore, the secondary fields reach the sensors ahead of the primary field. This is an important principle of actuator placement. It should not be difficult to implement in practice.

B. Cancellation results

Substituting $m = 14$, $n = 3$, and $r^* = 2$ into Eq. (22), the degree of $\mathbf{H}(z)$ is estimated to be $n_0 = 12$. It suggests that P^* is a 120×1 vector and $\mathbf{G}^*$ is a 120×65 matrix. The rank of $\mathbf{G}^*$ is estimated by Eq. (23) to be 64, which matches exactly what is returned by a MATLAB command ''rank $(\mathbf{G}^*)$.'' Both Eq. (22) and Eq. (23) prove very accurate when different numbers of modes are included in the model, with different numbers of sensors and actuators tested at different locations.

A MATLAB command ''$H = -\text{pinv}(\mathbf{G}^*)^* P^*$'' solves the coefficient vector of $H(z)$. Cancellation effect is evaluated by a residue transfer function vector $R(z) = P(z) + \mathbf{G}(z)H(z)$. The primary paths $P(z)$ are compared with $R(z)$ sensor by sensor in Figs. 2, 3, and 4, respectively. Residue paths obtained by a standard multichannel filtered-x algorithm are also compared in these figures. The two kinds of residues are labeled as ''standard residue'' and ''SCBN residue,'' respectively. The SCBN residues demonstrate reasonably good cancellation effects over a wild frequency

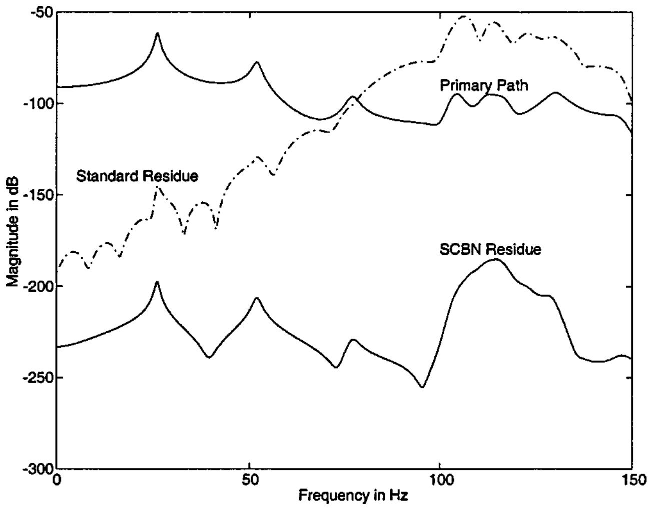

FIG. 2. Primary and residue paths to sensor 1.

range. The standard residues, however, attenuate noise in low-frequency range but enhance noise in high-frequency range.

In this example, $\mathbf{G}^*$ is a 120×65 matrix with rank 64—neither square nor full rank. Therefore, the MINT condition cannot be satisfied under the present situation. It is not analytically clear how the MINT condition can be met in a 3D reverberant sound field. On the contrary, SCBN manages to achieve perfect cancellation under a relaxed condition.

IV. RECOVERING THE REFERENCE SIGNAL

In many applications, the reference signal has to be measured by a reference sensor placed near the primary source. The sensor measures signals from the primary source and secondary actuators. It is important to estimate the paths from the actuators to the sensor and cancel the acoustic feedback by a software subtraction.[5,12] This point has been addressed[5,12] thoroughly and hence is not repeated here.

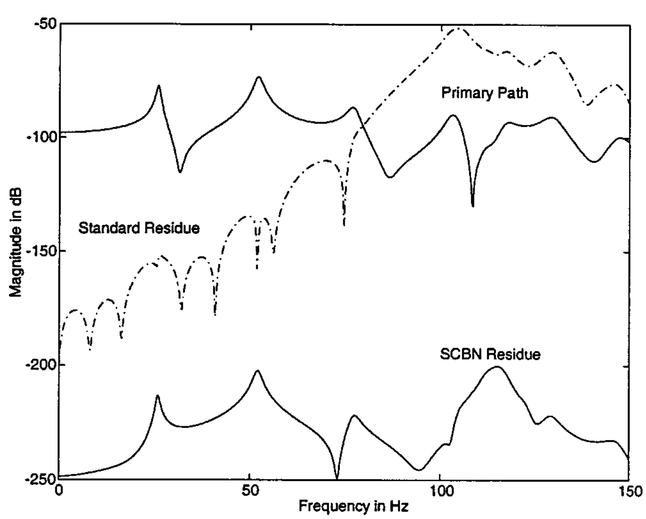

FIG. 3. Primary and residue paths to sensor 2.

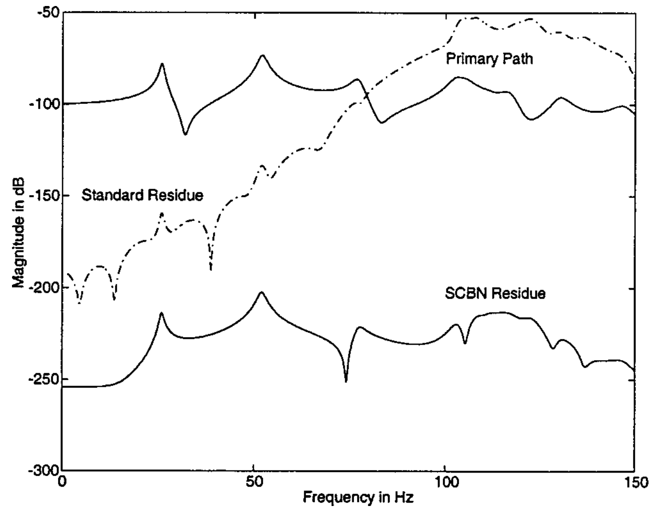

FIG. 4. Primary and residue paths to sensor 3.

A. Another problem

There is another problem with the reference signal that also affects the performance of an ANC system. For better focus, one may assume perfect cancellation of acoustic feedback by software subtraction. The sensed signal is then given by $y_s(z)=[N(z)]/[D(z)]r(z)$, where $N(z)$ is the numerator polynomial and $D(z)$ is the system characteristic polynomial. Perfect recovery of $r(z)$ calls for an inverse filtering $r(z)=[D(z)]/[N(z)]y_s(z)$ that may be unstable if $N(z)$ is nonminimum phase.

B. Suggested solution

A possible solution is to place two reference sensors at separate spots near the primary source. The sensed signals are expressed as

$$\begin{aligned} s_1(z)&=D(z)y_{s1}(z)=N_1(z)r(z)\\ s_2(z)&=D(z)y_{s2}(z)=N_2(z)r(z), \end{aligned} \tag{27}$$

where $N_1(z)$ and $N_2(z)$ are, respectively, numerators of the transfer functions from the primary source to the reference sensors. The reference signal can be recovered perfectly using FIR filters $F_1(z)$ and $F_2(z)$ via $r(z)=F_1(z)s_1(z)+F_2(z)s_2(z)$. It is equivalent to

$$1=N_1(z)F_1(z)+N_2(z)F_2(z), \tag{28}$$

after substituting Eq. (27). This is a special case of Eq. (1) with $n=1$. It is also known as the Bezout equation, solvable if $N_1(z)$ and $N_2(z)$ are relatively coprime.[13]

C. SCBN and the Bezout equation

It is interesting that SCBN also becomes the Bezout equation when $n=r^*=1$. In the presence of 0th mode, the degree of $F_1(z)$ and $F_2(z)$ is estimated by Eq. (24) to be $n_0=2m-1$. Let

$$N_1(z)=\sum_{i=0}^{2m} n_{i,1}z^{-i}$$

and

$$N_2(z)=\sum_{i=0}^{2m} n_{i,2}z^{-i}.$$

A matrix form of

$$\mathbf{G}^*(z)=[N_1(z)\ N_2(z)]=\sum_{i=0}^{2m}[n_{i,1}\ \ n_{i,2}]z^{-i}$$

can be constructed as

$$\mathbf{G}^*=\underbrace{\begin{bmatrix} \mathbf{G}_0^* & & & \\ \mathbf{G}_1^* & \mathbf{G}_0^* & & \\ \vdots & \mathbf{G}_1^* & \ddots & \\ \mathbf{G}_{2m}^* & \vdots & \ddots & \mathbf{G}_0^* \\ & \mathbf{G}_{2m}^* & \ddots & \mathbf{G}_1^* \\ & & \ddots & \vdots \\ & & & \mathbf{G}_{2m}^* \end{bmatrix}}_{2m\ \text{blocks}}$$

$$=\begin{bmatrix} n_{0,1} & n_{0,2} & & & & & \\ n_{1,1} & n_{1,2} & n_{0,1} & n_{0,2} & & & \\ \vdots & \vdots & n_{1,1} & n_{1,2} & \ddots & & \\ n_{2m,1} & n_{2m,2} & \vdots & \vdots & \ddots & n_{0,1} & n_{0,2} \\ & & n_{2m,1} & n_{2m,2} & \ddots & n_{1,1} & n_{1,2} \\ & & & & \ddots & \vdots & \vdots \\ & & & & & n_{2m,1} & n_{2m,2} \end{bmatrix}.$$

By a proper column permutation, this matrix is equivalent to

$$\mathbf{N}=\begin{bmatrix} n_{0,1} & & & & n_{0,2} & & & \\ n_{1,1} & n_{0,1} & & & n_{1,2} & n_{0,2} & & \\ \vdots & n_{1,1} & \ddots & & \vdots & n_{1,2} & \ddots & \\ n_{2m,1} & \vdots & \ddots & n_{0,1} & n_{2m,2} & \vdots & \ddots & n_{0,2} \\ & n_{2m,1} & \ddots & n_{1,1} & & n_{2m,2} & \ddots & n_{1,2} \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & n_{2m,1} & & & & n_{2m,2} \end{bmatrix}.$$

It turns out to be the Sylvester resultant matrix of Bezout equation Eq. (28). Let

$$F_1(z)=\sum_{i=0}^{2m-1} f_{i,1}z^{-i}$$

and

$$F_2(z)=\sum_{i=0}^{2m-1} f_{i,2}z^{-i},$$

then Eq. (28) can be expressed as

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} n_{0,1} & & & & n_{0,2} & & & \\ n_{1,1} & n_{0,1} & & & n_{1,2} & n_{0,2} & & \\ \vdots & n_{1,1} & \ddots & & \vdots & n_{1,2} & \ddots & \\ n_{2m,1} & \vdots & \ddots & n_{0,1} & n_{2m,2} & \vdots & \ddots & n_{0,2} \\ & n_{2m,1} & \ddots & n_{1,1} & & n_{2m,2} & \ddots & n_{1,2} \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & n_{2m,1} & & & & n_{2m,2} \end{bmatrix}\begin{bmatrix} f_{0,1} \\ f_{1,1} \\ \vdots \\ f_{2m-1,1} \\ f_{0,2} \\ f_{1,2} \\ \vdots \\ f_{2m-1,2} \end{bmatrix}$$

$$=\begin{bmatrix} n_{0,1} & n_{0,2} & & & & & \\ n_{1,1} & n_{1,2} & n_{0,1} & n_{0,2} & & & \\ \vdots & \vdots & n_{1,1} & n_{1,2} & \ddots & & \\ n_{2m,1} & n_{2m,2} & \vdots & \vdots & \ddots & n_{0,1} & n_{0,2} \\ & & n_{2m,1} & n_{2m,2} & \ddots & n_{1,1} & n_{1,2} \\ & & & & \ddots & \vdots & \vdots \\ & & & & & n_{2m,1} & n_{2m,2} \end{bmatrix}\begin{bmatrix} f_{0,1} \\ f_{0,2} \\ f_{1,1} \\ f_{1,2} \\ \vdots \\ f_{2m-1,1} \\ f_{2m-1,2} \end{bmatrix}. \tag{29}$$

The above matrix is nonsingular if $N_1(z)$ and $N_2(z)$ are relatively coprime (the MINT assumption). The product is able to match any $4m$-dimensional vectors that correspond to polynomials of degree less than or equal to $4m-1$.

The objective of SCBN is matching the right side of Eq. (29) to the numerator of a path transfer function in the same enclosure, but not an arbitrary polynomial. It does not necessarily need a full rank $\mathbf{G}^*$ as explained in Sec. II; hence, a relaxed condition. By coincident, $N_1(z)$ and $N_2(z)$ are found relatively coprime when the two sensors locate sufficiently away from each other. Therefore, both MINT and SCBN merge to the Bezout equation in a special case when $n=r^*=1$.

V. CONCLUSIONS

This study presents SCBN to relax the condition of MINT for perfect noise cancellation in n_s sensor locations. The SCBN assumption depends on an actuation matrix $\mathbf{M}_a$ and a sensor matrix $\mathbf{M}_s^T$. Perfect cancellation is possible if $\mathrm{rank}(\mathbf{M}_a)>\mathrm{rank}(\mathbf{M}_s^T)$.

The principle of SCBN is verified numerically with an exemplary ANC in a 3D rectangular enclosure. It is not difficult to satisfy the SCBN assumption by placing actuators sufficiently away from each other. In a special case when an ANC system consists of two actuators and a single error sensor, both MINT and SCBN are equivalent to a Bezout equation. The principle can be applied to recover the reference signal with two reference sensors.

In a sound field, all acoustical paths are subject to the same physical constraints. The primary paths are therefore acoustically correlated to the secondary paths to a certain degree. SCBN uses a truncated modal model to show complete correlation between the primary and secondary paths. The uncorrelated parts are due to the truncated modes. The method works well in a lightly damped sound field with sparse modes in the frequency range of interest. In highly damped enclosures, the direct field becomes dominant and a separate study is needed to reach a conclusion. A proper placement of actuators, however, will enable secondary fields to reach the sensors ahead of the primary field. Therefore, SCBN is likely to work well in that case.

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APPENDIX

Equation (17) allows one to express P^* as

$$P^*=\begin{bmatrix}\mathbf{B}\\ O\\ \vdots\\ O\end{bmatrix}k_p \quad \text{where} \quad \mathbf{B}=\begin{bmatrix}\mathbf{M}_s^T\mathbf{\Theta}_0\\ \mathbf{M}_s^T\mathbf{\Theta}_1\\ \vdots\\ \mathbf{M}_s^T\mathbf{\Theta}_{2m-2}\end{bmatrix}. \tag{A1}$$

Similarly, $\mathbf{G}^*$ can be expressed as a product of matrices $\mathbf{R}$ and $\mathbf{S}$ as follows:

$$\mathbf{G}^*=\underbrace{\begin{bmatrix}\mathbf{M}_s^T\mathbf{\Theta}_0 & & & \\ \mathbf{M}_s^T\mathbf{\Theta}_1 & \mathbf{M}_s^T\mathbf{\Theta}_0 & & \\ \vdots & \mathbf{M}_s^T\mathbf{\Theta}_1 & \ddots & \\ \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} & \vdots & \ddots & \mathbf{M}_s^T\mathbf{\Theta}_0\\ & \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} & \ddots & \mathbf{M}_s^T\mathbf{\Theta}_1\\ & & \ddots & \vdots\\ & & & \mathbf{M}_s^T\mathbf{\Theta}_{2m-2}\end{bmatrix}}_{\mathbf{R}\text{ with } n_0+1 \text{ shifted } \mathbf{B}\text{-blocks}}$$

$$\times\underbrace{\begin{bmatrix}\mathbf{M}_a & & & \\ & \mathbf{M}_a & & \\ & & \ddots & \\ & & & \mathbf{M}_a\end{bmatrix}}_{\mathbf{S}\text{ with } n_0+1 \text{ diagonal } \mathbf{M}_a\text{-blocks}}. \tag{A2}$$

A subspace analysis of P^* and $\mathbf{G}^*$ leads to a series of lemmas and properties:

Lemma 1: Let θ_i denote the coefficient vector of polynomial $\theta_i(s)$, then $\{\theta_i\}_{i=1}^m$ are linearly independent when the resonance frequencies $\{\omega_i\}_{i=1}^m$ are distinct.

Proof: Equation (7) suggests $\theta_i(s)=D(s)/(s^2+2\xi_i\omega_i s+\omega_i^2)$ for $1\leq i\leq m$. Let $p_i=-\xi_i\omega_i-j\omega_i\sqrt{1-\xi_i^2}$, then $1/(s^2+2\xi_i\omega_i s+\omega_i^2)=-1/(\omega_i\sqrt{1-\xi_i^2})\mathrm{Im}(1/(s-p_i))$. When the resonance frequencies are distinct, $\{p_i\}_{i=1}^m$ are distinct as well. As a result, Vandermonde matrix

$$\mathbf{V}=\begin{bmatrix}1 & p_1 & p_1^2 & \cdots & p_1^{m-1}\\ 1 & p_2 & p_2^2 & \cdots & p_2^{m-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & p_m & p_m^2 & \cdots & p_m^{m-1}\end{bmatrix}$$

is nonsingular. The ith row of the above matrix consists of the first m coefficients of polynomial

$$\frac{1}{(s-p_i)}=\sum_{k=0}^{\infty} p_i^k s^k.$$

Therefore, coefficient vectors of $\{1/(s-p_i)\}_{i=1}^m$ are linearly independent. The same applies to the coefficient vectors of $\{\theta_i(s)=-D(s)/(\omega_i\sqrt{1-\xi_i^2})\,\mathrm{Im}(1/(s-p_i))\}_{i=1}^m$. Q.E.D.

Lemma 2: $\mathrm{rank}(\mathbf{B})=m$ [matrix $\mathbf{B}$ is defined by Eq. (A1)].

Proof: Since $\mathbf{\Theta}_l$ is a diagonal matrix, $\mathbf{M}_s^T\mathbf{\Theta}_l$ has m columns, with the ith column of $\mathbf{M}_s^T$ multiplied by the ith diagonal element of $\mathbf{\Theta}_i$. Let $\mathbf{m}_i^s$ denote the ith column vector of $\mathbf{M}_s^T$, then the ith column of $\mathbf{B}$ forms a coefficient vector of a polynomial vector

$$\{\mathbf{m}_i^s\theta_i(s)\}_{i=1}^m.$$

Recalling from Eq. (12), one can see that

$$\mathbf{m}_i^{sT}=[\phi_i(\mathbf{x}_1^s)\ \phi_i(\mathbf{x}_2^s)\cdots\phi_i(\mathbf{x}_{n_s}^s)].$$

If $\|\mathbf{m}_i^s\|$ were zero, then all n_s sensors would locate at the node planes or node lines of the ith mode. That mode would not have been included in the truncated model in the first place. On the contrary, a mode is included in a truncated model only if it is measurable by at least one sensor. Therefore, $\|\mathbf{m}_i^s\| \neq 0$ for all $1 \leqslant i \leqslant m$; and the rank of $\mathbf{B}$ is the same as the rank of polynomials

$$\{\theta_i(s)\}_{i=1}^m.$$

Quoting from Lemma 1, one can see that rank($\mathbf{B}$)$=m$. Q.E.D.

A close look at Eq. (A2) indicates that $\mathbf{R}$ consists of n_0+1 blocks of $\mathbf{B}$. Counting from left to right, each column block of $\mathbf{B}$ is downshifted by the row size of $\mathbf{M}_s^T\mathbf{\Theta}_i$. Therefore, the row size of $\mathbf{R}$ is the row size of $\mathbf{B}$ plus n_0 times the row size of $\mathbf{M}_s^T\mathbf{\Theta}_i$, which is $n_s(2m-1+n_0)$. The column size of $\mathbf{R}$ is n_0+1 times the column size of $\mathbf{B}$; that is, $n_0 m + m$. Hence, $\mathbf{R}$ is an $[n_s(2m-1+n_0)]\times(n_0 m+m)$ matrix. Evidently, the rank of $\mathbf{R}$ depends on the integer value of n_0. When $n_0=1$, $\mathbf{R}$ consists of two blocks of $\mathbf{B}$. The second block is downshifted by n_s rows. These two blocks could be linearly independent, implying

Property 1: rank($\mathbf{R}$)$=\gamma \leqslant 2m$ when $n_0=1$.

When n_0 increases to 2, $\mathbf{R}$ contains one more block of $\mathbf{B}$ downshifted by n_s rows. Its rank, however, cannot reach $3m$. In order to prove this, one has to recall from Eq. (7) that $\theta_i(s)(s^2+2\xi_i\omega_i s+\omega_i^2)=D(s)$ for $1\leqslant i \leqslant m$. This is equivalent to a very useful equation

$$\mathbf{\Theta}(s)(s^2\mathbf{I}_m+\mathbf{\Xi}_s+\mathbf{\Omega})=D(s)\mathbf{I}_m, \tag{A3}$$

where $\mathbf{I}_m$ is an $m\times m$ identity

$$\mathbf{\Xi}(s)=\begin{bmatrix} 2\xi_1\omega_1 & & & \\ & 2\xi_2\omega_2 & & \\ & & \ddots & \\ & & & 2\xi_m\omega_m \end{bmatrix}$$

and

$$\mathbf{\Omega}(s)=\begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_m^2 \end{bmatrix}.$$

A matrix form is easily available for Eq. (A3) as

$$\begin{bmatrix} d_0\mathbf{I}_m \\ d_1\mathbf{I}_m \\ \vdots \\ d_{2m}\mathbf{I}_m \end{bmatrix}=\begin{bmatrix} \mathbf{\Theta}_0 & & \\ \mathbf{\Theta}_1 & \mathbf{\Theta}_0 & \\ \vdots & \mathbf{\Theta}_1 & \mathbf{\Theta}_0 \\ \mathbf{\Theta}_{2m-2} & \vdots & \mathbf{\Theta}_1 \\ & \mathbf{\Theta}_{2m-2} & \vdots \\ & & \mathbf{\Theta}_{2m-2} \end{bmatrix}\begin{bmatrix} \mathbf{\Omega} \\ \mathbf{\Xi} \\ \mathbf{I}_m \end{bmatrix} \tag{A4}$$

where $\{d_i\}_{i=0}^{2m}$ are the coefficients of system characteristic polynomial $D(s)$. This equation is useful in the proof of

Lemma 3: Rank($\mathbf{R}$)$=k_r=\gamma+n(n_0-1)\leqslant 2m+n(n_0-1)$ when $n_0\geqslant 2$.

Proof: To check rank($\mathbf{R}$) for $n_0\geqslant 1$, one may examine the product of the following two matrices:

$$\mathbf{R}|_{n_0=2}=\begin{bmatrix} \mathbf{M}_s^T\mathbf{\Theta}_0 & & \\ \mathbf{M}_s^T\mathbf{\Theta}_1 & \mathbf{M}_s^T\mathbf{\Theta}_0 & \\ \vdots & \mathbf{M}_s^T\mathbf{\Theta}_1 & \mathbf{M}_s^T\mathbf{\Theta}_0 \\ \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} & \vdots & \mathbf{M}_s^T\mathbf{\Theta}_1 \\ & \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} & \vdots \\ & & \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} \end{bmatrix}$$

and

$$\mathbf{T}=\begin{bmatrix} \mathbf{I}_m & & \mathbf{\Omega} \\ & \mathbf{I}_m & \mathbf{\Xi} \\ & & \mathbf{I}_m \end{bmatrix}.$$

Here, $\mathbf{R}|_{n_0=2}$ represents matrix $\mathbf{R}$ when $n_0=2$. It contains three blocks of $\mathbf{B}$. $\mathbf{T}$ is a triangle matrix with nonzero off-diagonal elements only in the last m columns. This implies rank($\mathbf{R}|_{n_0=2}$)$=$rank($\mathbf{R}|_{n_0=2}T$). Using Eq. (A4), one obtains

$$\mathbf{R}|_{n_0=2}\mathbf{T}=\begin{bmatrix} \mathbf{M}_s^T\mathbf{\Theta}_0 & & \mathbf{M}_s^T d_0 \\ \mathbf{M}_s^T\mathbf{\Theta}_1 & \mathbf{M}_s^T\mathbf{\Theta}_0 & \mathbf{M}_s^T d_1 \\ \vdots & \mathbf{M}_s^T\mathbf{\Theta}_1 & \mathbf{M}_s^T d_2 \\ \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} & \vdots & \mathbf{M}_s^T d_3 \\ & \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} & \vdots \\ & & \mathbf{M}_s^T d_{2m} \end{bmatrix}.$$

The next step is the use of a linear transformation

$$\mathbf{\Gamma}_0=\begin{bmatrix} \mathbf{I}_m & & -\mathbf{\Theta}_0^{-1}d_0 \\ & \mathbf{I}_m & \\ & & \mathbf{I}_m \end{bmatrix},$$

to get

$$\mathbf{R}|_{n_0=2}\mathbf{T}\mathbf{\Gamma}_0$$
$$=\begin{bmatrix} \mathbf{M}_s^T\mathbf{\Theta}_0 & & \\ \mathbf{M}_s^T\mathbf{\Theta}_1 & \mathbf{M}_s^T\mathbf{\Theta}_0 & \mathbf{M}_s^T(d_1\mathbf{I}_m-\mathbf{\Theta}_1\mathbf{\Theta}_0^{-1}d_0) \\ \vdots & \mathbf{M}_s^T\mathbf{\Theta}_1 & \mathbf{M}_s^T(d_2\mathbf{I}_m-\mathbf{\Theta}_2\mathbf{\Theta}_0^{-1}d_0) \\ \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} & \vdots & \mathbf{M}_s^T(d_3\mathbf{I}_m-\mathbf{\Theta}_3\mathbf{\Theta}_0^{1}d_0) \\ & \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} & \vdots \\ & & \mathbf{M}_s^T d_{2m} \end{bmatrix}.$$

It suggests the existence of a series of linear transformations $\mathbf{\Gamma}_1\cdots\mathbf{\Gamma}_{m-1}$, such that

$$\mathbf{R}|_{n_0=2}\mathbf{T}\Gamma_0\mathbf{\Gamma}_1\cdots\mathbf{\Gamma}_{m-1}$$

$$=\begin{bmatrix} \mathbf{M}_s^T\mathbf{\Theta}_0 & & \\ \mathbf{M}_s^T\mathbf{\Theta}_1 & \mathbf{M}_s^T\mathbf{\Theta}_0 & \\ \vdots & \mathbf{M}_s^T\mathbf{\Theta}_1 & \\ \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} & \vdots & \\ & \mathbf{M}_s^T\mathbf{\Theta}_{2m-2} & \\ & & \mathbf{M}_s^T d_{2m} \end{bmatrix}.$$

The above expression suggests rank$(\mathbf{R}|_{n_0=2})=\gamma+n$ by assumption 1 and Property 1. One can apply Eq. (A4) and construct matrices similar to $\mathbf{T}$ for consecutive increments of n_0. The result is similar: each increment of n_0 increases rank($\mathbf{R}$) by n. That proves the lemma. Q.E.D.

Property 2: P^* belongs to the column space of $\mathbf{R}$.

This is rather obvious because P^* is a linear combination of the first m columns of $\mathbf{R}$, as defined by Eq. (A1).

Property 3: Rank$(\mathbf{S})=(n+r^*)(n_0+1)$.

From Eq. (A2), one can see that $\mathbf{S}$ has n_0+1 diagonal blocks of $\mathbf{M}_a$. Hence, rank$(\mathbf{S})=(n_0+1)$rank$(\mathbf{M}_a)=(n+r^*)(n_0+1)$ by assumption 1.

Since $\mathbf{G}^*=\mathbf{RS}$, both rank($\mathbf{R}$) and rank($\mathbf{S}$) affect rank($\mathbf{G}^*$). According to Lemma 2 and Property 3, rank($\mathbf{R}$) $>$rank($\mathbf{S}$) for $n_0=0$. As n_0 increases, rank($\mathbf{R}$) increases by n for each increment of n_0 (Lemma 3); whereas rank($\mathbf{S}$) increases by $n+r^*$ for each increment of n_0 (Property 3). The role of integer $r^*>0$ now becomes clear: to make rank($\mathbf{S}$) grow faster than rank($\mathbf{R}$) does. There must exist an integer value of n_0 such that rank$(\mathbf{R})\leqslant$rank$(\mathbf{S})$ for the first time. It is solved by $n_0=\lceil(\gamma-2n-r^*)/r^*\rceil$ that is the smallest positive integer larger than or equal to $(\gamma-2n-r^*)/r^*$. Substituting $\gamma\leqslant 2m$ (Property 1), one obtains an estimate of n_0 as

$$n_0=\left\lceil\frac{2m-2n-r^*}{r^*}\right\rceil. \tag{A5}$$

Generally, any integer values of n_0, if larger than Eq. (A5), will cause rank$(\mathbf{R})\leqslant$rank$(\mathbf{S})$. Equation (A5) estimates the smallest integer to meet the requirement. It implies

Lemma 4: The column space of $\mathbf{G}^*$ is equivalent to the column space of $\mathbf{R}$ when assumption 1 is valid and n_0 satisfies Eq. (A5).

Combining Property 2 and Lemma 4, one concludes that P^* belongs to the column space of $\mathbf{G}^*$ when assumption 1 is true and n_0 satisfies Eq. (A5).

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