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# On analysis of exponentially decaying pulse signals using stochastic volatility model. Part II: Student-*t* distribution (L)

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The authors have recently demonstrated how the stochastic volatility model incorporating the exponential power distribution can be used to retrieve the instant of initiation of an exponentially decaying pulse and its decay constants in the presence of background noises. In the present study, the Student-*t* distribution, which can be expressed in a two-stage scale mixtures representation, is adopted in the stochastic volatility model. It is found that the corresponding performance is comparable to that for the case of the exponential power distribution when the signal-to-noise ratio is larger than or equal to 3 dB. The performance deteriorates quickly when the signal-to-noise ratio drops below 0 dB. © *2006 Acoustical Society of America.* [DOI: 10.1121/1.2266455]

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## I. INTRODUCTION

Pulses and their decays are important in the studies of acoustics and building services engineering. In a diffused system, such decay is an exponential function of time and the rate of the decay, which is commonly described by the decay constant, contains information on the system damping and thus pulse decay analysis has attracted great attention in the past few decades (for instance, Heckl[1] and Tang[2]).

The recent study of the authors[3] illustrates that the stochastic volatility (SV) model incorporating the exponential power distribution (EP) is able to retrieve the instant of the pulse initiation and the decay constant within engineering tolerance even when the background noise level is comparable to that of the pulse. Its performance is much better than that of the conventional short-time Fourier transform when there is a small fluctuation in the frequency of the decaying pulse. Details on the properties of the EP distribution and its application in the Bayesian analysis can be found, for instance, in Choy and Walker.[4]

The Student-*t* distribution is a conventional distribution form and has been used in many applications (for instance, Lange *et al.*,[5] and Chan[7]). It can be expressed in a two-stage scale mixtures representation and can in principle be an alternative to the EP distribution.[6] In the present study, the performance of the SV model incorporating the Student-*t* distribution in analyzing decaying pulses is investigated. The results supplement those of the previous study of the authors.[3]

## II. TWO-STAGE SCALE MIXTURES REPRESENTATION

A standard random variable $X$ having the normal scale mixtures representation can be expressed in the form of $X = Z \times \lambda$ where $Z$ is the standard normal random variate and $\lambda$ is a positive random variate known as the mixing variable having a probability density function $g$, which can be either continuous or discrete.

Let $\theta$ and $\sigma$ be the location and scale parameter of a scale mixture distribution, respectively. The probability density function of $X$ takes the mixture form

$$f(x) = \int_{\Re^+} N(x|\theta, \lambda\sigma^2) g(\lambda) d\lambda, \tag{1}$$

where $N(\cdot|\cdot)$ denotes the normal density defined on $\Re^+ = (0, \infty)$. In the Bayesian framework, the mixture density in Eq. (1) can be expressed into a two-stage hierarchy of the form

$$X|\theta, \sigma^2, \lambda \sim N(\theta, \lambda\sigma^2), \quad \lambda \sim g(\lambda). \tag{2}$$

The Student-*t* distribution with degrees of freedom $\alpha$ corresponds to an inverse gamma mixing distribution

$$\lambda \sim g(\lambda) = G_{\text{inv}}(0.5\alpha, 0.5\alpha), \tag{3}$$

where $G_{\text{inv}}(a, b)$ is the inverse gamma distribution with density ($a > 0$ and $b > 0$)

$$g(\lambda) = b^a e^{-b/\lambda} \lambda^{-(a+1)} / \Gamma(a). \tag{4}$$

To facilitate an efficient computation for the SV models, use is made of the class of scale mixtures of uniform representation for the normal density. Since $X$ is a normal random variable with mean $\theta$ and variance $\sigma^2$, its density function can be rewritten into

$$N(x|\theta, \sigma^2) = \int_{\theta-\sigma\sqrt{u}}^{\theta+\sigma\sqrt{u}} \frac{1}{2\sigma\sqrt{u}} G(u|1.5, 0.5) du, \tag{5}$$

where $G(u|a, b)$ is the gamma density function with parameter $a$ and $b$.[8] The Student-*t* distribution with a degree of freedom $\alpha$ can be expressed into the following hierarchy:

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TABLE I. Full conditionals in the Gibbs sampling.

| Conditional | Remarks | Distribution |
|---|---|---|
| $h_t \mid \bar{h}_{-t}, \bar{\lambda}, \bar{u}, \sigma^2, \phi, \bar{y} \sim N(a_t, b_t\sigma^2)$ | $h_t > \ln y_t^2 - 2\ln\beta\ln\lambda_t - \ln u_t$, $a_t = \begin{cases} \phi h_{t+1} - 0.5\sigma^2 & t=1 \\ \dfrac{\phi(h_{t-1}+h_{t+1}) - 0.5\sigma^2}{1+\phi^2} & 2 \le t \le n=1, \\ \phi h_{t-1} - 0.5\sigma^2 & t=n \end{cases}$ $b_t = \begin{cases} 1 & t=1,n \\ (1+\phi^2)^{-1} & 2 \le t \le n-1 \end{cases}$ | Truncated normal |
| $\sigma^2 \mid \bar{h}, \bar{\lambda}, \bar{u}, \phi, \bar{y} \sim G_{\text{inv}}\left(a_\sigma + \dfrac{n}{2}, b_\sigma + \dfrac{1}{2}\left[(1-\phi^2)h_1^2 + \sum_{t=2}^{n}(h_t - \phi h_{t-1})^2\right]\right)$ | N.A. | Inverse gamma |
| $\lambda_t \mid \bar{h}, \bar{\lambda}_{-t}, \bar{u}, \phi, \sigma^2, \bar{y} \sim G_{\text{inv}}\left(\dfrac{\alpha+1}{2}, \dfrac{\alpha}{2}\right)$ | $\lambda_t > \dfrac{y_t^2}{\beta^2 H_t u_t}$ | Truncated inverse gamma |
| $u_t \mid \bar{h}, \bar{\lambda}, \bar{u}_{-t}, \phi, \sigma^2, \bar{y} \sim \exp(0.5)$ | $u_t > \dfrac{y_t^2}{\beta^2 H_t \lambda_t}$ | Truncated exponential |
| $\phi \mid \bar{h}, \bar{\lambda}, \bar{u}, \sigma^2, \bar{y} \sim N\left(\phi \mid \dfrac{\sum_{t=2}^{n} h_{t-1}h_t}{\sum_{t=2}^{n} h_t^2}, \dfrac{\sigma^2}{\sum_{t=2}^{n} h_t^2}\right)(1+\phi)^{a_\phi - 1/2}(1-\phi)^{b_\phi - 1/2}$ | $\lvert\phi\rvert \le 1$ | Product of normal and shifted beta |



$$X \mid \theta, \sigma^2, \lambda, u \sim U(\theta - \sigma\sqrt{\lambda u}, \theta + \sigma\sqrt{\lambda u})$$

$$,\lambda \sim G_{inv}(0.5\alpha, 0.5\alpha), \quad \text{and} \quad u \sim G(1.5, 0.5), \tag{6}$$

where $U(a,b)$ is a uniform distribution defined on the interval $(a,b)$.

## III. BAYESIAN STUDENT-*T* SV MODEL AND GIBBS SAMPLING

Let $H_t$ and $h_t$ be the volatilities, and log-volatilities, respectively. In the SV model, the signal data, $y_t$ (where $t = 1, 2, \ldots, n$), is defined as

$$y_t = \beta\sqrt{H_t}\varepsilon_t \quad \text{and} \quad h_t = \begin{cases} \sigma\eta_1/\sqrt{1-\phi^2} & t=1 \\ \phi h_{t-1} + \sigma\eta_t & t>1 \end{cases}, \tag{7}$$

where $\{\varepsilon_t\}$ and $\{\eta_t\}$ are independent standard Gaussian processes. $\beta$ is a constant representing the model instantaneous volatility, $\sigma$ the variance of the log volatilities, and $\phi \in (-1,1)$ the persistence of the volatility.

In the present study, the signal data set is modeled by a Student-*t* distribution while the log volatility is assumed to follow a normal distribution

$$y_t \mid h_t \sim t_\alpha(0, \beta^2 H_t) \tag{8}$$

and

$$h_t \mid h_{t-1}, \phi, \sigma^2 \sim N(\phi h_{t-1}, \sigma^2), \tag{9}$$

while the marginal distribution is $h_t \mid \phi, \sigma^2 \sim N[0, \sigma^2/(1-\phi^2)]$.

To complete the Bayesian framework, the shifted beta and inverse gamma distributions are assigned to be the independent priors for $\phi$ and $\sigma^2$, respectively

$$\phi \sim 2Be(a_\phi, b_\phi) - 1 \quad \text{and} \quad \sigma^2 \sim G_{\text{inv}}(a_\sigma, b_\sigma). \tag{10}$$

The SV model can then be rewritten hierarchically as

$$y_t \mid h_t, \lambda_t, u_t \sim U(-\beta H_t^{1/2}\lambda_t^{1/2}u_t^{1/2}, \beta H_t^{1/2}\lambda_t^{1/2}u_t^{1/2}) \quad \text{and}$$

$$h_t \mid h_{t-1}, \phi, \sigma^2 \sim N(\phi h_{t-1}, \sigma^2) \tag{11}$$

with $\lambda_t \sim G_{inv}(0.5\alpha, 0.5\alpha)$, $u_t \sim G(1.5, 0.5)$, $\phi \sim 2Be(a_\phi, b_\alpha) - 1$, and $\sigma^2 \sim G_{inv}(a_\sigma, b_\sigma)$. It is implemented using the Gibbs sampling approach with the variables $\bar{y} = (y_1, \ldots, y_n)$, $\bar{h} = (h_1, \ldots, h_n)$, $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $\bar{u} = (u_1, \ldots, u_n)$, $\bar{h}_{-t} = (h_1, \ldots, h_{t-1}, h_{t+1}, \ldots, h_n)$, $\bar{\lambda}_{-t} = (\lambda_1, \ldots, \lambda_{t-1}, \lambda_{t+1}, \ldots, \lambda_n)$, and $\bar{u}_{-t} = (u_1, \ldots, u_{t-1}, u_{t+1}, \ldots, u_n)$.

With arbitrarily chosen starting values for $\bar{h}, \bar{\lambda}, \bar{u}, \sigma^2$, and $\phi$, the Gibbs sampler iteratively sample random variates from a system of full conditional distributions and the resulting simulations are used to mimic a random sample from the

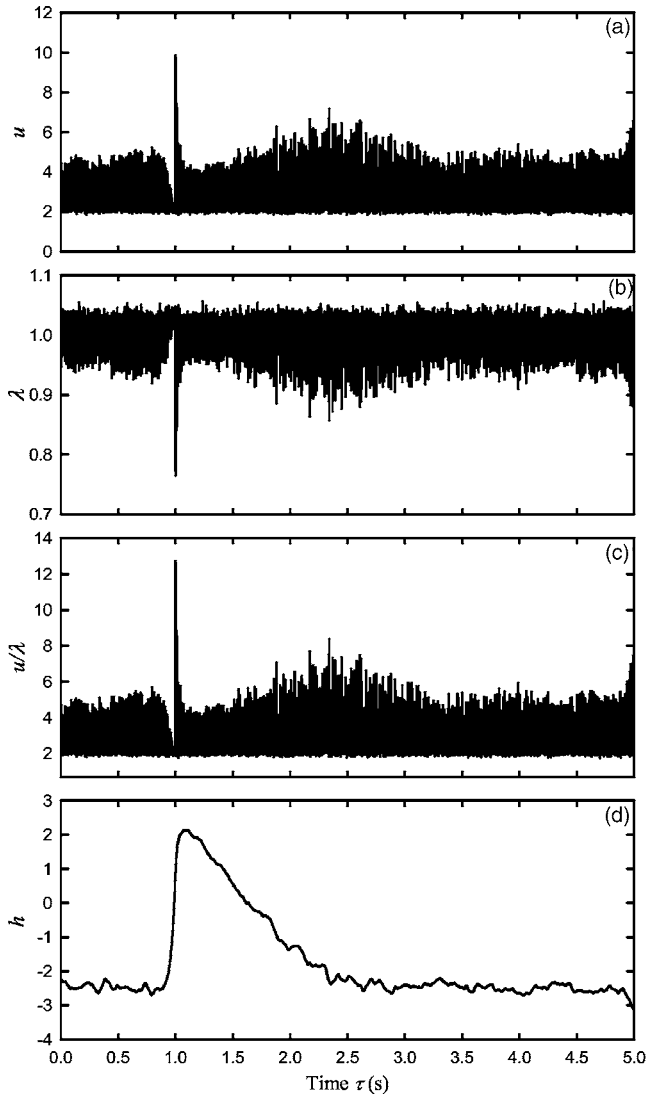

FIG. 1. Time variations of the mixing parameters for S/N=10 dB and $\alpha$ =30. (a) $u$; (b) λ; (c) $u/\lambda$; (d) $h$.

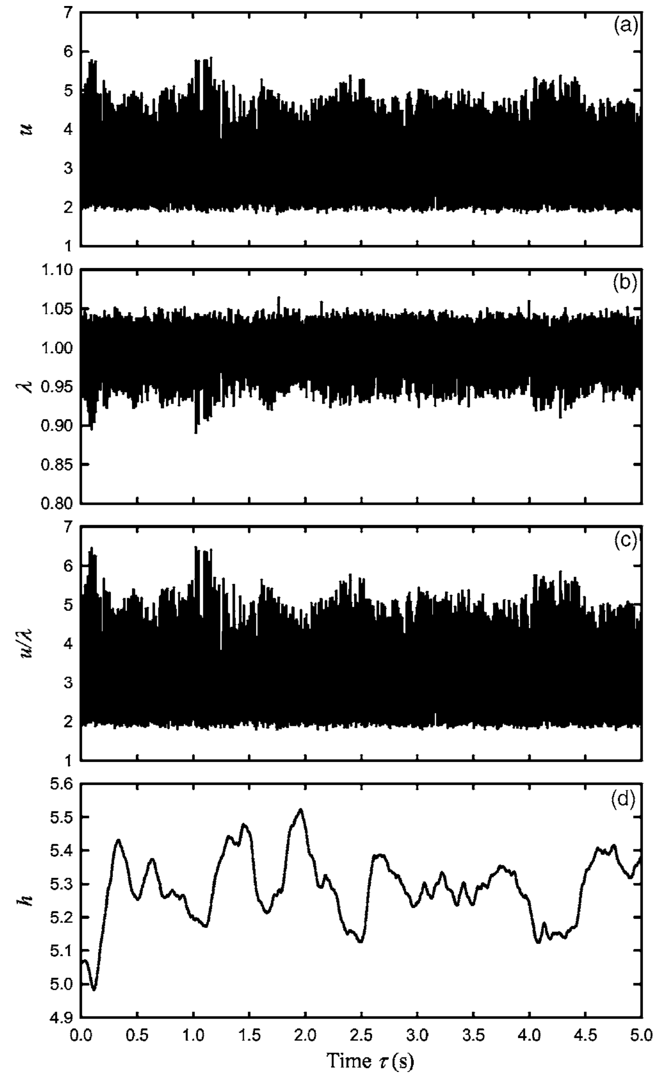

FIG. 2. Time variations of the mixing parameters for S/N=−3 dB and $\alpha$ =30. (a) $u$; (b) λ; (c) $u/\lambda$; (d) $h$.



targeted joint posterior distribution. The system of full conditionals is given in Table I. In the present study, $\beta$ is fixed at unity.

## IV. NUMERICAL EXAMPLES

The artificial signals in Chan *et al.*[3] are adopted again in the present study. They consist of an exponentially decaying harmonic wave $s$ and Gaussian white noises $\upsilon$ of various levels

$$y(\tau) = e^{-\eta(\tau-\tau_o)}\cos[2\pi f(\tau-\tau_o)]H(\tau-\tau_o) + \upsilon(\tau), \qquad (12)$$

where $H$ denotes the Heaviside step function, $f$ the frequency of the wave, $\eta$ the decay constant, $\tau_o$ the instant of pulse initiation, and $\tau$ the time in second. The signal-to-noise ratio (S/N) in decibels is defined as

$$\mathrm{S/N} = 10\log_{10}(|s|_{\max}/|\upsilon|_{\max}). \qquad (13)$$

Without loss of generality, $\eta$ is fixed at 2 and $f$ is normally set at 50 Hz in the present study.

The parameter $\alpha$ in the SV model determines the shapes of the Student-$t$ distributions and thus has a significant impact on the modeling of the decaying signal by the SV model. However, since the shape of the Student-$t$ distribution does not vary much for $\alpha > 30$ and the computation with large $\alpha$ is in general impractical as it is very demanding on computing resources, the largest $\alpha$ included in the present investigation is 30. The incorporation of the Student-$t$ distribution into the SV model gives rise to two mixing parameters, namely, $u$ and λ, and a log-volatility $h$ as shown in the previous section. The two mixing parameters illustrate fluctuation while the latter follows the envelope of the signal magnitude.

Figure 1 illustrates the time variations of $u, \lambda, u/\lambda$, and $h$ for S/N=10 dB and $\alpha$=30. One can observe that the variation patterns of $u$ and λ are opposite. The initiation of the pulse results in a prominent upward and downward spike in $u$ and λ, respectively. The parameter $u/\lambda$ shows a more prominent spike at the instant of pulse initiation. The present $u$ is basically the same as those obtained using the EP distribution with a kurtosis of 0.75.[3] The variation pattern of $h$ is also very close to that shown in Chan *et al.*[3] Further increas-

TABLE II. Estimated $\tau_o$ and $\eta$.[a]

| S/N (dB) | $\tau_o$ (s) | $\eta(\mathrm{s}^{-1})$ |
|---|---|---|
| $+\infty$ | 1.000(1.000) | 1.95(2.00) |
| 10 | 1.000(1.000) | 2.30 (2.05) |
| 3 | 1.000(1.000) | 2.20(2.16) |
| 0 | 1.010(1.001) | 2.60 (2.42) |
| −3 | 1.022(1.002) | —(2.36) |

[a]Numbers in parenthesis are those obtained with the EP distribution (Ref. 3).

ing $\alpha$ may result in a slightly better performance, but the very computer resources demanding calculation makes it impractical.

The performance of the SV model deteriorates when the background noise level increases. For $\mathrm{S/N}=-3$ dB with $\alpha=30, u, \lambda$, and $u/\lambda$ are unable to indicate without ambiguity the instant of the pulse initiation [Figs. 2(a) to 2(c)]. The log-volatility $h$ does not even suggest the presence of a decaying pulse [Fig. 2(d)]. However, the pulse initiation and its eventual decay are still observable with the EP distribution at this signal-to-noise ratio level.[3]

Results for $\alpha < 30$ are not presented as they are all worse than those shown in Figs. 1 and 2. Table II summarizes the performance of the present SV model and has it compared with that of the previous study of the authors.[3] One can find that the use of the EP distribution in the SV model results in a better analysis when the S/N drops below 3 dB. The somewhat worse performance of the Student-*t* distribution overall may be due to the relatively longer tail of the Student-*t* distribution compared to those of the EP family with small kurtosis even when the degree of freedom is large.

## V. CONCLUSIONS

A stochastic volatility model which incorporates the Student-*t* distribution is used in the present study to retrieve the instant of initiation and the decay constant of an exponentially decaying signal in the presence of random background noises. It is found that the performance of the Student-*t* distribution is comparable to that of the EP distribution for a signal-to-noise ratio higher than or equal to 3 dB. It deteriorates quickly when this ratio falls below 0 dB.


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