A Meshless Collocation Method Based on Radial Basis Functions and Wavelets

S. L. Ho, Shiyou Yang, H. C. Wong, and Guangzheng Ni

***Abstract*—A meshless method based on collocation with radial basis functions (RBFs) and wavelets is proposed. It is shown that the proposed method takes full advantage of both RBFs and wavelets. The bridging scales are employed to preserve the mathematical properties of the entire bases in terms of consistency and linear independence. A numerical example that is used to validate the proposed method is reported.**

***Index Terms*—Collocation, meshless method, radial basis function, wavelet.**

I. Introduction

During the past decade, the application of meshless or mesh-free methods to solve partial differential equations (PDEs) has received considerable attention in virtually all engineering disciplines. Since the meshless method can eliminate the construction of a tedious and difficult mesh, which is required by the widely used finite element (FE) method, meshless methods are very attractive in solving problems involving large deformations or problems requiring repeatedly adaptive mesh updating. According to the approximation approaches used in the derivation of the discrete mathematical models, the meshless methods can be categorized mainly into two groups, namely, Galerkin integration-based methods and point collocation-based methods. For the former, the discritization is based on the approximation of the PDEs in a weak form, and for the latter, the discritization is set up by directly approximating the PDEs in a strong form. Since the shape functions of the Galerkin integration based methods do not have the delta function properties, some special techniques must be designed to enforce the boundary conditions. Moreover, integration "cells" are also necessary when computing the stiffness matrix. Hence, these algorithms are not truly meshless. Consequently, additional efforts are necessary in developing collocation-based meshless methods. The radial basis-based collocation method, among others, has been researched extensively with proven success in solving many PDE problems [1]–[4]. Numerical results have demonstrated that the RBF-based methods 1) are truly mesh-free and 2) are computationally accurate and simple in numerical implementation. However, the main drawback of globally supported RBFs is that the associated stiffness matrix is full. Due to its inefficiency to deal with a full matrix, there is an upper limit in the number of collocation points of globally supported RBF collocation methods. Moreover, when the spacing between collocation points is very small, the stiffness matrix will become very ill-conditioned, leading to serious numerical singularity and degradation in numerical accuracy. By using some compactly supported RBFs, one might avoid the need to manipulate the full coefficient matrix in the traditional globally supported RBFs. However, such an approach will have significant errors when one interpolates the derivatives on the boundary [1]. In order to address this problem and to take full advantage of both RBFs and wavelets, a meshless collocation method using RBF interpolation to enforce the boundary conditions is proposed. In order to retain the desirable mathematical properties such as the consistency and linear independence of the shape functions of the proposed method, the bridge scales are generalized and used in the proposed algorithm. An iterative solution procedure is also introduced to solve the discrete algebraic equation set. Computer simulations on a two-dimensional (2-D) problem are conducted, and the numerical results are presented to validate and demonstrate the advantages and shortcomings of the proposed method.

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II. Collocation Method Based on RBFs and Wavelets

A. Wavelet Approximation

For any function $u(x,y) \in \Omega$, its approximation using wavelets can be given as

$$u(x,y)=\sum_{i,j} c_{i,j}\phi_{i,j}^{J}(x,y) \tag{1}$$

where J is the resolution or scale parameter, and

$$\phi_{i,j}^{J}(x,y)=\phi_{i}^{J}(x)\phi_{j}^{J}(y) \tag{2}$$

where $\phi_i^J(z)=2^{J/2}\phi(2^J z-i)$ is the one dimensional (1-D) scale function of the wavelets, and it can be determined from the following two-scale relation:

$$\phi(x)=\sum_{k=0}^{L-1} p_k\phi(2x-k). \tag{3}$$

In the proposed algorithm, the Daubechies' scale function is used. Hence, L is an even integer. Due to the compact supporting properties of the wavelets, the stiffness matrix based on the collocation with wavelets is sparse and banded. However, the

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wavelet-based collocation method is rather inflexible in dealing with arbitrary solution domains.

B. Interpolation Using RBF

Although RBFs have originally been used in the interpolation of scattered multivariate data, recently, there has been an increasing interest to use them to solve PDEs since the meshless method based on collocation with RBFs is truly meshless, and the shape function is interpolant, rendering them simple in numerical implementation. However, one needs to manipulate a full stiffness matrix when using them to solve a practical problem. To make the best use of both wavelets and RBFs, a meshless collocation method based on a combination of wavelets and RBFs in which the RBF interpolation is used only to enforce the boundary conditions is proposed.

Since the RBF interpolation is only used for enforcing boundary conditions in the proposed method, the globally supported RBFs will be used in this paper because of their high interpolation accuracies. The interpolation of a function $u(x,y): D \to R$ on the basis of its values u_i at some scattered data points $X_i = (x_i, y_i) \in D (i = 1, 2, \ldots, N)$ in terms of some radial basis function H is

$$u(X) = \sum_{j=1}^{N} d_j H(\|X - X_j\|) \tag{4}$$

where $\|\cdot\|$ is the Euclidean norm.

In the numerical implementation, a transformation is used to consider the difference in the dimensional sizes of different coordinate directions, i.e.,

$$r = \|X\| = \sqrt{\left(\frac{x}{k_x}\right)^2 + \left(\frac{y}{k_y}\right)^2}. \tag{5}$$

C. Combined Interpolation of RBFs and Wavelets

For the proposed algorithm to work in a more general form, the entire domain of the problem is divided into three subregions (see Fig. 1): Ω_R is where only the RBF interpolation is present; Ω_w is where only the wavelets contribute to the approximation of the solution variable; and Ω_R^w is where both RBFs and wavelets have influences. In regions Ω_R and Ω_w, the interpolation of the solution variable is the standard form of (4) and (1), respectively. To develop a general interpolation formula in region Ω_R^w for the solution variable $u(X)$ using both RBF's and wavelets, one begins with

$$u(X) = \sum_{j=1}^{N} d_j H(\|X - X_j\|) + \sum c_{i,j} \phi_{i,j}^J(X). \tag{6}$$

To ensure that the required mathematical properties of the entire bases such as consistency and linear independence are retained, the bridging scale concept is used to modify the wavelets [5]. Thus, in region Ω_R^w, (6) becomes

$$u(X) = \sum_{j=1}^{N} d_j H(\|X - X_j\|) + \sum_{i,j} c_{i,j} \bar{\phi}_{i,j}^J(X) \tag{7}$$

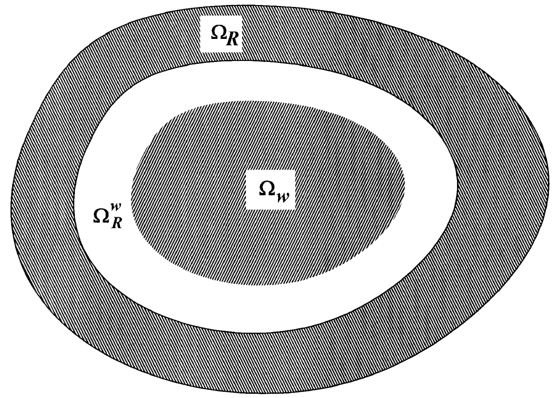

Fig. 1. Schematic diagram of the division of the solution domain.

where $\bar{\phi}_{i,j}^J(X)$ is the modified wavelet based on bridging scales and is defined as

$$\bar{\phi}_{j,k}^J(X) = \phi_{j,k}^J(X) - \sum_i H_i(\|X\|) \phi_{j,k}^J(X_i) \tag{8}$$

where $H_i(\|X\|) = H(\|X - X_i\|)$.

The objective to modify the wavelets as defined from (2) to (8) is to make the wavelet interpolation contain only the parts of the solution variable that are not included in the RBF interpolation, thereby ensuring a hierarchical decomposition of the solution variable.

D. Discrete Mathematical Model

Without a lost of generality, one considers the following 2-D Poisson's equation on the domain Ω bounded by boundary $\Gamma = \Gamma_D \cup \Gamma_N$:

$$\Omega : Lu = -f/\beta \tag{9}$$

$$\Gamma_D : u = u_0 \tag{10}$$

$$\Gamma_N : Du = q/\beta \tag{11}$$

where

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, D = \frac{\partial}{\partial n}.$$

As described previously, the RBF interpolation is used to impose only the boundary conditions in the proposed algorithm. In other words, the wavelet approximations have no contribution on boundary point collocations. Based on the collocation method, and substituting (7) into (9)–(11), one obtains the discrete equations as

$$\begin{bmatrix} L[H] & L[\bar{\phi}] \\ H & 0 \\ D[H] & 0 \end{bmatrix} \begin{Bmatrix} d \\ c \end{Bmatrix} = \begin{Bmatrix} -\frac{f}{\beta} \\ u_0 \\ \frac{q}{\beta} \end{Bmatrix}. \tag{12}$$

Obviously, even for a natural boundary condition of (11) when q is equal to zero, it should also be enforced explicitly in a collocation method.

In the wavelet approximation as formulated in (1), there is no intrinsic relation between the wavelet coefficients and collocation points. Thus, one needs to have some "virtual" collocation points only when deriving the discrete mathematical model of (12). Hence, the number of collocation points in the subregion

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where only wavelets have influence can be different from the number of the coefficient $c_{i,j}$. In such case, the least square approach is used to solve (12) since the number of the equation set is not equal to that of its degree of freedoms (DoFs). Furthermore, since the discretization of the boundary conditions precludes the discretizations of the derivatives at these boundary collocation points, the density of the collocation points in the boundaries should be higher than that of the inner regions in its neighborhood. In the numerical implementation, half of these boundary points are used for boundary condition collocations, whereas the other half of the boundary points are used for (partial) derivative collocations.

E. Decoupling and Solving of RBF and Wavelet Systems

Since the RBF interpolation is used only to enforce the boundary conditions, the DoF of the RBF system is very small in general when compared with that of the wavelet system. Moreover, the ratio between the values of quantities in submatrix $L[H]$ and those in submatrix $L[\bar{\phi}]$ of the stiffness matrix of (12) may be too large or too small, and this is a good recipe for poor matrix conditioning. Thus, if the discrete linear equation set of (12) is solved as a whole, some numerical technique are needed to guarantee good performances in the evaluation of the numerical solutions. To avoid this poor matrix conditioning problem and to consider the fact that the DoFs of the RBF system are far smaller in general than those of the wavelet system, the two matrix systems are decoupled and solved separately and iteratively in the proposed method. Mathematically, (12) is reformulated as

$$\begin{bmatrix} L(H) \\ H \\ D(H) \end{bmatrix} \{d\} = \begin{Bmatrix} -\frac{f}{\beta} \\ u_0 \\ \frac{q}{\beta} \end{Bmatrix} - \begin{bmatrix} L(\bar{\phi}) \\ 0 \\ 0 \end{bmatrix} \{c\} \qquad (13)$$

$$[L(\bar{\phi})]\{c\} = \left\{-\frac{f}{\beta}\right\} - [L(H)]\{d\}. \qquad (14)$$

Thus, the iterative solution procedure for the two matrix systems can be described as follows.

1) Equation set (13) is first solved by initializing the wavelet coefficients $\{c\}$ to zero.
2) Equation (14) is then solved with the values of the newly solved $\{d\}$ as known variables.
3) Equation (13) is solved again using $\{c\}$ of step (2).
4) The solutions of $\{c\}$ and $\{d\}$ between two successive iterations are compared. If the error is within a prescribed value, the iterative process is stopped; otherwise, go to step (2) for the next iteration cycle.

III. Numerical Example

The computation of the end fields of a practical power transformer with complicated geometries, as shown in Fig. 2, is selected as the numerical example to validate and to demonstrate the advantages and shortcomings of the proposed method. The governing equations are

$$\varepsilon\frac{\partial^2\varphi}{\partial x^2} + \varepsilon\frac{\partial^2\varphi}{\partial y^2} = 0$$

$$\varphi|_{\Gamma_1} = 1, \varphi|_{\Gamma_3} = 0, \left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma_2} = 0. \qquad (15)$$

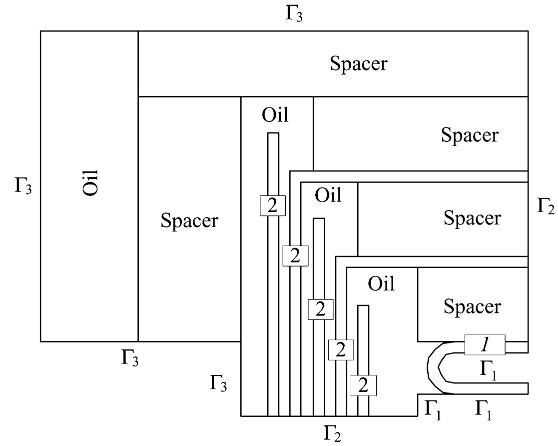

Fig. 2. Schematic diagram of the end region of the studied transformer (1-Electrostatic ring, 2-Insulation cylinder).

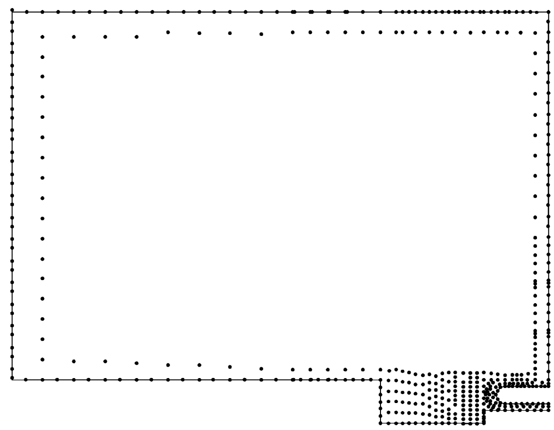

Fig. 3. Node arrangement of the proposed method.

Three different methods, i.e., the proposed one, the element-free Galerkin (EFG) method [6], and the FE method are used to study this problem. For the convenience of performance comparisons, the same node distribution with a total number of 1236 nodes is used for all three methods. In the proposed method, the Daubechies' scale function with $L = 8$ is used. The RBF being used is $h(r) = (r^2 + 0.1)^{-0.04}$.

In the numerical implementation of the proposed method, the solution domain is divided into two different subregions where i) Ω_R^w are very thin layers near the boundaries where both RBFs and wavelets have common influences, and ii) Ω_w is the residual of the solution domain where only the wavelets are contributing to the interpolation. As explained previously, since the coefficients of the wavelet approximation have no intrinsic relations with the collocation points, there is no need to generate the concrete collocation points and one only needs "virtual" collocation points in Ω_w. Thus, a very simple node arrangement, as depicted in Fig. 3, is needed for the proposed algorithm. It should be pointed out that in order to guarantee a high interpolation accuracy in the boundaries, the densities of the nodes on these boundaries are generally twice as high as those in its neighborhood inner regions, as shown in Fig. 3. For the EFG method, the Lagrange multiplier method is used to enforce the essential boundary conditions. Thus, a total of 102 additional DoFs are required for this method. To compare the accuracy of the

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TABLE I
PERFORMANCE COMPARISON OF DIFFERENT METHODS.

	No FE nodes	No wavelet Bases	Total DoFs	Relative Error	CPU (s)
Proposed	1236§	1044	1044+210©	1.14×10^{-2}	2.9
FEM	1236	/	1236	0.	3♦
EFG	1236	/	1236+102[a]	2.76×10^{-3}	4.2

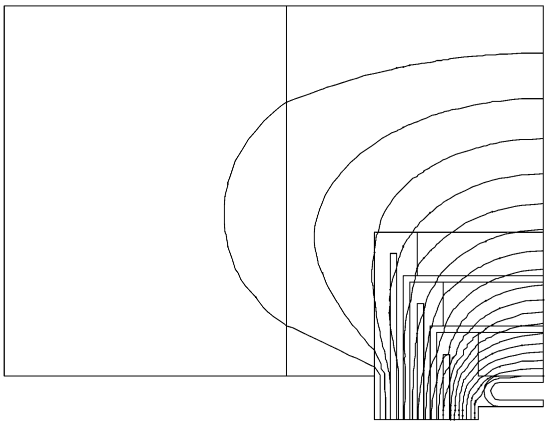

Fig. 4. Computed equipotential contours of the end field of the power transformer.

numerical solutions obtained by using different methods in this paper, a discrete L_2 norm of relative error is defined below. The values of the solution variable ϕ, as computed by using the FE method, which has now become the standard method for dealing with PDEs, are selected as the base values. Moreover

$$\text{Error} = \sqrt{\frac{\sum_{i=1}^{N}\left(\frac{\phi_i^{\text{fem}}-\phi_i^{\text{meshless}}}{\phi_i^{\text{fem}}+10^{-12}}\right)^2}{N}} \tag{16}$$

where N is the nodal number.

The corresponding performance comparison results of different methods on a practical power transformer are given in Table I. The equipotential lines of the end fields computed by using the proposed algorithm are shown in Fig. 4. From these numerical results one can see the following.

1) Unlike FE and Galerkin integration-based meshless methods, the numerical integration is not required for the proposed algorithm in assembling the stiffness matrix. This feature makes the proposed method the most efficient one among the three methods, although the number of DoFs of the proposed method is slightly higher than those of the other two methods.
2) Compared with the traditional FE method, the proposed algorithm is truly "meshless," even in terms of integration cells in numerical implementations.
3) Compared with the Galerkin integration-based meshless methods, the most salient characteristics of the proposed method are i) no special technique is required to deal with the essential boundary conditions, and ii) no matrix inversion is needed in the development of the shape functions, rendering it computationally efficient.
4) Compared with conventional globally supported RBF-based collocation methods, the stiffness matrix of the proposed algorithm is banded and sparse.
5) Compared with Galerkin integration-based meshless and FE methods, the quality of the solution of the proposed algorithm is, however, relatively low. In other words, to obtain the same level of accuracy of the numerical solutions, more collocation points will be required with the proposed method. In addition, the natural boundary conditions have to be enforced explicitly in the proposed algorithm.

IV. CONCLUSION

As part of our efforts toward designing an efficient and robust meshless method for solving 3-D boundary value problems, a mesh-free method based on collocation with RBFs and wavelets is proposed in this paper. Numerical results on a typical 2-D engineering problem are reported. The primary numerical results demonstrate that there are good potentials in using proposed algorithm to solve 3-D electromagnetic problems, and an extension of the proposed algorithm to 3-D problems is being investigated by the authors. Besides, the related techniques such as the incorporation of the symmetric RBF collocation formulation, the multilevel scheme for adjusting the scale of the radial functions, and that for updating the scale of wavelets are also being studied in order to develop a simple and efficient solver of boundary value problems in practical engineering studies.

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