Tripartite entanglement measure under local operations and classical communication

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Multipartite entanglement is an indispensable resource in quantum communication and computation; however, it is a challenging task to faithfully quantify this global property of multipartite quantum systems. In this work we study the concurrence fill, which admits a geometric interpretation to measure genuine tripartite entanglement for the three-qubit system [Xie and Eberly, Phys. Rev. Lett. **127**, 040403 (2021)]. First, we use the well-known three-tangle and bipartite concurrence to reformulate this quantifier for all pure states. We then construct an explicit example to conclusively show that the concurrence fill can be increased under local operations and classical communication (LOCC) on average, implying it is not an entanglement monotone. Moreover, we give a simple proof of the LOCC monotonicity of the three-tangle and find that the bipartite concurrence and the squared concurrence can have distinct performances under the same LOCC. Finally, we propose a reliable monotone to quantify genuine tripartite entanglement, which can also be easily generalized to the multipartite system. Our results shed light on the study of genuine entanglement and also reveal the complex structure of multipartite systems.

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I. INTRODUCTION

Entanglement, having no classical counterpart, is of fundamental importance in quantum theory [1], and also an essential resource in various quantum information processing tasks, including cryptography [2,3], teleportation [4,5], dense coding [6,7], secret sharing [8,9], metrology [10,11], and computation [12]. Thereof, an important problem arises about how to quantitatively measure the degree of entanglement for the quantum system. Typically, one entanglement measure is defined as some non-negative function which maps any quantum state to a real number in the interval [0,1] and satisfies a set of reasonable assumptions [1,13,14], such as being zero for all nonentangled states and invariant under local unitary operations. Within the resource theory of entanglement [15,16], it is further required to be nonincreasing under local operations and classical communication (LOCC) on average, hence being an entanglement monotone of which entanglement never increases under the free LOCC operations. Correspondingly, numerous entanglement measures and monotones have been developed [17–22], especially for the low-dimensional bipartite system, such as concurrence [23,24] and logarithmic negativity [25].

It is challenging to find proper measures for multipartite entanglement due to the complicated partial separability structure of multipartite systems. Indeed, in order to detect genuine multipartite entanglement which is the key resource in multiparty information tasks, the reliable measure needs to meet extra conditions, like being zero for partial separable states and strictly positive for genuinely entangled states [26,27]. Thus, there have been some commonly used quantifiers, including α-entanglement entropy [27] and generalized concurrence [28,29], which cannot detect all genuinely entangled states, and few genuine multipartite entanglement monotones [30–35]. In particular, the concurrence fill, with a nice geometric interpretation as the square root of the concurrence triangle area, was recently proposed to measure the degree of genuine entanglement for the three-qubit system [36], together with an experimental test [37]. Although it conforms with almost all of the necessary conditions mentioned above, a fundamental problem still remains open about whether it is an entanglement monotone or, equivalently, it admits the LOCC monotonicity.

Here we first establish the close connections between the concurrence fill and the well-known three-tangle [38] and reduced bipartite concurrences [23,24,39,40], of some interest on their own. Then we solve the above open problem via an explicit example to show that it can be increased under LOCC on average, implying it is not an entanglement monotone. Furthermore, a simple proof of the LOCC monotonicity of three-tangle is presented, which completes the proof in [41]. It is also interesting to find that the bipartite concurrence and the squared concurrence can have distinct performances under the same LOCC. Finally, we conjecture that the area of the triangle with edges corresponding to the bipartite concurrences is an entanglement monotone and also propose a

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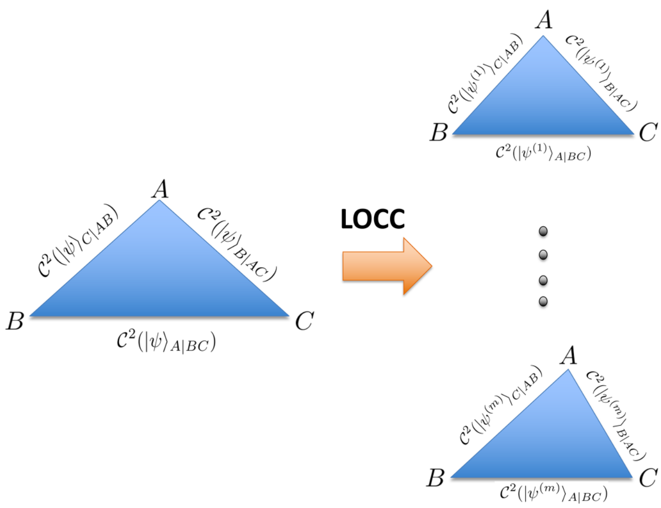

FIG. 1. For any pure three-qubit state $|\psi\rangle_{ABC}$, the concurrence triangle ΔABC is composed of three sides described by the three squared one-to-other concurrences $\mathcal{C}^2(|\psi\rangle_{i|jk})$. It leaves open in [36] that the square root of the area of this triangle, named the concurrence fill, is a genuine entanglement monotone or, equivalently, the quantity $\mathcal{F}$ as per (1) is nonincreasing under LOCC on average and thus satisfies the inequality (8). In this work we construct an explicit example to show that the concurrence fill can violate the inequality (8), thus implying it is not an entanglement monotone.

reliable monotone to quantify genuine tripartite entanglement, which can be easily generalized to the general multipartite systems.

The rest of this work is structured as follows. Section II introduces basic notation and useful relations about the concurrence fill for all pure three-qubit states. In Sec. III we address the fundamental issue about whether the concurrence fill is an entanglement monotone or not and also examine the LOCC monotonicity of three-tangle and bipartite concurrence. Section IV proposes a genuine multipartite entanglement monotone for the multipartite quantum system. A summary is given in Sec. V.

II. CONCURRENCE FILL

For an arbitrary pure three-qubit state $|\psi\rangle_{ABC}$ shared by three parties A, B, and C, we define the concurrence between the bipartition i and jk as $\mathcal{C}(|\psi\rangle_{i|jk}) = \sqrt{2[1-\mathrm{Tr}(\rho_i^2)]}$, with $\rho_i = \mathrm{Tr}_{jk}(|\psi\rangle_{ijk}\langle\psi|)$ and $i, j, k = A, B, C$. It follows from the relation $\mathcal{C}^2(|\psi\rangle_{i|jk}) \leqslant \mathcal{C}^2(|\psi\rangle_{j|ik}) + \mathcal{C}^2(|\psi\rangle_{k|ij})$ [42] that these three squared one-to-other concurrences can be geometrically interpreted as the lengths of three sides of a triangle, which is called the concurrence triangle ΔABC, as shown in Fig. 1. The concurrence fill is defined as the square root of the area of this concurrence triangle [36]

$$\mathcal{F}(|\psi\rangle_{ABC}) = \left(\frac{\mathcal{P}_{ABC}}{3}[\mathcal{P}_{ABC} - 2\mathcal{C}^2(|\psi\rangle_{A|BC})] [\mathcal{P}_{ABC} - 2\mathcal{C}^2(|\psi\rangle_{B|AC})][\mathcal{P}_{ABC} - 2\mathcal{C}^2(|\psi\rangle_{C|AB})]\right)^{1/4}, \quad (1)$$

with the perimeter

$$\mathcal{P}_{ABC} = \mathcal{C}^2(|\psi\rangle_{A|BC}) + \mathcal{C}^2(|\psi\rangle_{B|AC}) + \mathcal{C}^2(|\psi\rangle_{C|AB}). \quad (2)$$

It was shown in [36] that the concurrence fill (1) is useful to quantify genuine entanglement for three-qubit states as it satisfies almost all of the necessary conditions to be an entanglement measure. It was also noted that the perimeter $\mathcal{P}_{ABC}$ (2), known as global entanglement [43,44], is a feasible measure, but is not genuine in the sense that it can be nonzero for certain biseparable states.

Further, defining reduced states $\rho_{ij} = \mathrm{Tr}_k(|\psi\rangle_{ijk}\langle\psi|)$ and using the relations [38,45]

$$\mathcal{C}^2(|\psi\rangle_{i|jk}) = 2\big[1 - \mathrm{Tr}\big(\rho_i^2\big)\big] = \tau + \mathcal{C}^2(\rho_{ij}) + \mathcal{C}^2(\rho_{ik}), \quad (3)$$

we are able to derive

$$\mathcal{P}_{ABC} = 3\tau + 2[\mathcal{C}^2(\rho_{AB}) + \mathcal{C}^2(\rho_{AC}) + \mathcal{C}^2(\rho_{BC})] \quad (4)$$

and hence

$$\mathcal{P}_{ABC} - 2\mathcal{C}^2(|\psi\rangle_{i|jk}) = \tau + 2\mathcal{C}^2(\rho_{jk}). \quad (5)$$

Here τ is the three-tangle which quantifies the tripartite entanglement [38], and the concurrence of the reduced two-qubit state ρ_{ij} is given by $\max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$. with $\{\lambda_i\}$ the singular values of $X^\top(\sigma_y \otimes \sigma_y)X$ with $\rho_{ij} = XX^\dagger$ [46].

Substituting Eqs. (4) and (5) into (1) immediately leads to

$$\mathcal{F}(|\psi\rangle_{ABC}) = \left[\left(\tau + \frac{2[\mathcal{C}^2(\rho_{AB}) + \mathcal{C}^2(\rho_{AC}) + \mathcal{C}^2(\rho_{BC})]}{3}\right) \times [\tau + 2\mathcal{C}^2(\rho_{AB})][\tau + 2\mathcal{C}^2(\rho_{AC})][\tau + 2\mathcal{C}^2(\rho_{BC})]\right]^{1/4}. \quad (6)$$

This indicates that the concurrence fill can be fully determined by the well-known three-tangle and bipartite concurrence. By introducing the concurrence of assistance for a two-qubit state as $\mathcal{C}_a(\rho) = \mathrm{tr}[|X^\top(\sigma_y \otimes \sigma_y)X|] = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ [47] and using $\tau = \mathcal{C}_a^2(\rho_{ij}) - \mathcal{C}^2(\rho_{ij})$ [48], we can further obtain

$$\mathcal{F}(|\psi\rangle_{ABC}) = \left(\frac{\mathcal{P}_{ABC}}{3}\big[\mathcal{C}^2(\rho_{AB}) + \mathcal{C}_a^2(\rho_{AB})\big] \times \big[\mathcal{C}^2(\rho_{AC}) + \mathcal{C}_a^2(\rho_{AC})\big]\big[\mathcal{C}^2(\rho_{BC}) + \mathcal{C}_a^2(\rho_{BC})\big]\right)^{1/4}, \quad (7)$$

with

$$\mathcal{P}_{ABC} = \mathcal{C}^2(\rho_{AB}) + \mathcal{C}^2(\rho_{AC}) + \mathcal{C}^2(\rho_{BC}) + \mathcal{C}_a^2(\rho_{AB}) + \mathcal{C}_a^2(\rho_{AC}) + \mathcal{C}_a^2(\rho_{BC}).$$

This means that the tripartite entanglement quantifier can also be expressed via the reduced bipartite concurrences. Since both two bipartite concurrences and three-tangles are entanglement monotones, it is more likely that the concurrence fill shares many similarities with the known entanglement monotones. Thus, Eqs. (6) and (7) provide a useful way to investigating the property of $\mathcal{F}$, which will be discussed in the following sections.

III. CONCURRENCE FILL UNDER LOCC

Within the resource theory of entanglement, the LOCC is the most important class of free operations which cannot generate entanglement. Correspondingly, an important issue about the entanglement monotone is whether it is nonincreasing under LOCC on average. For the above concurrence fill $\mathcal{F}$, as LOCC do not increase the three lengths of the concurrence triangle ΔABC, it is natural to draw the conclusion that the corresponding triangle area is nonincreasing and hence $\mathcal{F}$ is an entanglement monotone, as conjectured in [36]. However, in this section we provide conclusive evidence to show that this is not the case.

Following from the arguments used to prove the LOCC monotonicity of a three-tangle [41], the whole class of LOCC can be restricted to the binary-outcome positive-operator-valued measures (POVMs) $\{X_1, X_2\}$ acting on the local party A, with $X_1^\dagger X_1 + X_2^\dagger X_2 = \mathbb{1}$. As a consequence, the problem about whether the concurrence fill is an entanglement monotone reduces to checking if it is nonincreasing under the above POVMs on average or, equivalently, the inequality

$$\mathcal{F}(|\psi\rangle_{ABC}) - [p_1\mathcal{F}(|\psi^{(1)}\rangle_{ABC}) + p_2\mathcal{F}(|\psi^{(2)}\rangle_{ABC})] \geqslant 0 \quad (8)$$

holds for all pure states and local measurements, where $|\psi^{(k)}\rangle_{ABC} = X_k \otimes \mathbb{1} \otimes \mathbb{1}|\psi\rangle_{ABC}/\sqrt{p_k}$, with $p_k = \langle\psi|_{ABC}X_k^\dagger X_k \otimes \mathbb{1} \otimes \mathbb{1}|\psi\rangle_{ABC}$ for $k = 1, 2$. If the above inequality is satisfied, then the quantifier $\mathcal{F}$ is an entanglement monotone. Otherwise, it is definitely not.

Noting further that $\mathcal{F}$ is invariant under the local unitary operations, any pure three-qubit state can be restricted to its standard form [49]

$$|\psi\rangle_{ABC}=l_0|000\rangle + l_1e^{i\phi}|100\rangle + l_2|101\rangle + l_3|110\rangle + l_4|111\rangle, \quad (9)$$

with $\sum_m l_m^2 = 1$ and $\phi \in [0, \pi]$, and the measurement operators can be parametrized as $X_i = D_iV$, where

$$D_1 = \begin{pmatrix} \sin\varphi_1 & 0 \\ 0 & \sin\varphi_2 \end{pmatrix}, \quad D_2 = \begin{pmatrix} \cos\varphi_1 & 0 \\ 0 & \cos\varphi_2 \end{pmatrix}, \quad (10)$$

and

$$V = \begin{pmatrix} \cos\psi_1 & -e^{i\psi_2}\sin\psi_1 \\ \sin\psi_1 & e^{i\psi_2}\cos\psi_1 \end{pmatrix}, \quad (11)$$

with $\varphi_i, \psi_i \in [-\pi, \pi]$. Then the numerical search leads us to find that if the state (9) is chosen as $l_0|000\rangle + 0.096|100\rangle + 0.238|101\rangle + 0.173|110\rangle$ and the measurements X_i given in Eqs. (10) and (11) are set with $\varphi_1 = 2\pi/5$, $\varphi_2 = \pi/5$, $\psi_1 = -\pi/2$, and $\psi_2 = -\pi/10$, then the inequality (8) is violated up to

$$\mathcal{F}(|\psi\rangle_{ABC}) - \sum_{k=1,2} p_k\mathcal{F}(|\psi^{(k)}\rangle_{ABC}) \approx -0.0086. \quad (12)$$

This immediately implies that the concurrence fill can be increased by some certain LOCC and hence is not a tripartite entanglement monotone.

We remark that two interesting results related to the concurrence fill can also be obtained. The first is a simple proof of the LOCC monotonicity of three-tangle τ in Eq. (6). Indeed, we are able to prove

$$\begin{aligned}
&\tau(|\psi\rangle_{ABC}) - [p_1\tau(|\psi^{(1)}\rangle_{ABC}) + p_2\tau(|\psi^{(2)}\rangle_{ABC})] \\
&= \frac{\tau(|\psi\rangle_{ABC})}{p_1p_2}[p_1p_2 - p_2|\det(X_1)|^2 - p_1|\det(X_2)|^2] \\
&= \frac{\tau(|\psi\rangle_{ABC})}{p_1p_2}[p_1p_2 - p_2|\det(D_1)|^2 - p_1|\det(D_2)|^2] \\
&= \frac{\tau(|\psi\rangle_{ABC})}{p_1(1-p_1)}(p_1 - \sin^2\varphi_1)(\sin^2\varphi_2 - p_1) \\
&\geqslant 0.
\end{aligned} \quad (13)$$

The first equality derives from the relation $\tau(|\psi^{(k)}\rangle_{ABC}) = \tau(|\psi\rangle_{ABC})|\det(X_k)|^2/p_k^2$ [41], the second from $|\det(X_k)| = |\det(D_k)|$, and the third from $\sin^2\varphi_k + \cos^2\varphi_k = 1$ and $p_1 + p_2 = 1$; the inequality follows from $\min\{\sin^2\varphi_1, \sin^2\varphi_2\} \leqslant p_1 = \mathrm{Tr}(X_1^\dagger X_1\rho_A) \leqslant \max\{\sin^2\varphi_1, \sin^2\varphi_2\}$ for an arbitrary local state ρ_A. This completes the proof that τ is an entanglement monotone because the LOCC monotonicity of $\tau^{1/2}$ is proven in [41].

It is also interesting to find that the bipartite concurrence $\mathcal{C}(\rho_{BC})$ and the squared $\mathcal{C}^2(\rho_{BC})$ in the concurrence fill (6) can have distinct performances under the same LOCCs. For example, when each local measurement operator $X_k = \left(\begin{smallmatrix} x_k & 0 \\ 0 & y_k \end{smallmatrix}\right)$ acts on the pure three-qubit state $|\psi\rangle_{ABC}$ (9), there are $l_0^{(k)} = x_kl_0/\sqrt{p_k}$ and $l_m^{(k)} = y_kl_m/\sqrt{p_k}$ for each $|\psi^{(k)}\rangle_{ABC}$. Noting that $\mathcal{C}^2(\rho_{BC}) = 4(l_2^2l_3^2 + l_1^2l_4^2 - 2l_1l_2l_3l_4\cos\phi)$ [45] and $p_1 = 1 - p_2 = x_1^2l_0^2 + y_1^2(1 - l_0^2)$, we then obtain

$$\begin{aligned}
&\mathcal{C}^2(\rho_{BC}) - \left[p_1\mathcal{C}^2\left(\rho_{BC}^{(1)}\right) + p_2\mathcal{C}^2\left(\rho_{BC}^{(2)}\right)\right] \\
&\quad = -\left(x_1^2 - y_1^2\right)^2 l_0^4\mathcal{C}^2(\rho_{BC})/p_1p_2 \leqslant 0,
\end{aligned} \quad (14)$$

while

$$\begin{aligned}
&\mathcal{C}(\rho_{BC}) - \left[p_1\mathcal{C}\left(\rho_{BC}^{(1)}\right) + p_2\mathcal{C}\left(\rho_{BC}^{(2)}\right)\right] \\
&\quad = \left(1 - y_1^2 - y_2^2\right)\mathcal{C}(\rho_{BC}) = 0.
\end{aligned} \quad (15)$$

It is evident that if $x_1 \neq y_1$ and $l_0, \mathcal{C}(\rho_{BC}) \neq 0$, then the squared concurrence is always strictly increased by the above POVMs while the concurrence remains unchanged. We clarify that Eq. (14) does not contradict the fact that the squared concurrence is an entanglement monotone for two-qubit states because entanglement can be increased or even generated via local measurements performed by a third party, if a multipartite entangled state is shared.

IV. RELIABLE MULTIPARTITE ENTANGLEMENT MONOTONE

We have shown via Eq. (12) that the concurrence fill is not an entanglement monotone, and the main reason lies in the fact that $\mathcal{G}(|\psi\rangle_{ABC}) \equiv \tau(|\psi\rangle_{ABC}) + 2\mathcal{C}^2(\rho_{BC})$ in the concurrence fill (6) can be increased by certain LOCC. This is based on the observation that if two non-negative quantifiers $\mathcal{E}_j$, $j = 1, 2$, are monotonic under LOCC, i.e., $\mathcal{E}_j(\rho) \geqslant \sum_k p_k\mathcal{E}(\rho^{(k)})$, then the square root of their product also obeys the monotonicity as

$$\sqrt{\mathcal{E}_1(\rho)\mathcal{E}_2(\rho)} \geqslant \sqrt{\left(\sum_k p_k\mathcal{E}_1(\rho^{(k)})\right)\left(\sum_k p_k\mathcal{E}_2(\rho^{(k)})\right)} \geqslant \sum_k p_k\sqrt{\mathcal{E}_1(\rho^{(k)})\mathcal{E}_2(\rho^{(k)})}, \quad (16)$$

where the second inequality follows directly from the Cauchy-Schwarz inequality. Hence, if $\mathcal{G}(|\psi\rangle_{ABC})$ admits the LOCC monotonicity, then using the relation (16) a proper number of times on four terms in the concurrence fill (6) leads to the monotonicity of $\mathcal{F}$. Specifically, one counterexample is given by $\mathcal{G}(|\psi\rangle_{ABC}) - p_1\mathcal{G}(|\psi^{(1)}\rangle_{ABC}) - p_2\mathcal{G}(|\psi^{(2)}\rangle_{ABC}) \approx -0.009$, with $|\psi_{ABC}\rangle = l_0|000\rangle + 0.095|100\rangle + 0.238|101\rangle + 0.086|110\rangle + 0.142|111\rangle$, and $\varphi_1 = \pi/10$, $\varphi_2 = 2\pi/5$, $\psi_1 = -3\pi/5$, and $\psi_2 = -\pi/2$ for the local POVM.

However, the numerical test strongly supports that if the triangle is formed by the bipartite concurrence, then the area of this new triangle

$$\mathcal{F}'(|\psi\rangle_{ABC}) = \left(\frac{\mathcal{P}'_{ABC}}{3}[\mathcal{P}'_{ABC} - 2\mathcal{C}(|\psi\rangle_{A|BC})] \times [\mathcal{P}'_{ABC} - 2\mathcal{C}(|\psi\rangle_{B|AC})][\mathcal{P}'_{ABC} - 2\mathcal{C}(|\psi\rangle_{C|AB})]\right)^{1/2}, \quad (17)$$

with $\mathcal{P}'_{ABC} = \mathcal{C}(|\psi\rangle_{A|BC}) + \mathcal{C}(|\psi\rangle_{B|AC}) + \mathcal{C}(|\psi\rangle_{C|AB})$, is always nonincreasing under LOCC. Here $\mathcal{F}'$ has a much more natural interpretation than the original concurrence fill (1) and we conjecture that it is a monotone for genuine entanglement.

We also propose using the square root of the product of three sides to measure tripartite entanglement

$$\mathcal{S}(|\psi\rangle_{ABC}) = \sqrt{\mathcal{C}(|\psi\rangle_{A|BC})\mathcal{C}(|\psi\rangle_{B|AC})\mathcal{C}(|\psi\rangle_{C|AB})} \quad (18)$$

for the pure case. It is easy to show that the above $\mathcal{S}$ is invariant under local unitary operations and permutations of the parties and is zero if and only if the state is biseparable, i.e., at least one bipartite concurrence is zero. Since the bipartite concurrence and squared concurrence are monotonic under LOCC, using the inequality (16) twice gives rise to a proof of the LOCC monotonicity of $\mathcal{S}$. We can also obtain that it achieves the maximal value 1 with the Greenberger-Horne-Zeilinger state $(|000\rangle + |111\rangle)/\sqrt{2}$, which is larger than that of the W state $(|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$. Consequently, the quantifier $\mathcal{S}$ as per (18) is a reliable monotone to measure genuine tripartite entanglement and is readily applied to the mixed states via the convex-roof extension $\mathcal{S}(\rho) = \min_{p_j,|\psi_j\rangle}\sum_j p_j\mathcal{S}(|\psi_j\rangle)$, where the minimization runs over all ensemble realizations $\{p_j, |\psi_j\rangle\}$ for ρ.

We note that the genuine entanglement monotone $\mathcal{S}$ (18) improves the one based on the geometric mean of bipartite concurrences introduced in [34] from the order $\frac{1}{3}$ to $\frac{1}{2}$. Moreover, we point out that it can be easily generalized to the multipartite system. For example, for any pure n-qubit state denoted by S, we can define

$$\mathcal{S}(|\psi\rangle_S) = \left(\prod_{S_1|S_2}\mathcal{C}(|\psi\rangle_{S_1|S_2})\right)^{1/2^{n-2}}, \quad (19)$$

where $S_1|S_2$ denotes any possible bipartition of the n-qubit S. Using the inequality (16) $2^{n-1} - 2$ times yields that it is a multipartite entanglement monotone.

V. CONCLUSION

We have studied the concurrence fill (1) which is introduced to measure genuine entanglement for the three-qubit system. We first established the close connections between the concurrence fill and the well-known entanglement monotones, including three-tangle and bipartite concurrences, as obtained in Eqs. (6) and (7). Then we constructed an explicit example (12) to conclusively show that the concurrence fill is not an entanglement monotone, thus answering the fundamental open question left in [36]. Moreover, we presented a simple and complete proof for the LOCC monotonicity of the three-tangle and also found that the bipartite concurrence can behave differently from the squared concurrences under the same LOCC. Finally, we proposed an entanglement monotone (16) for genuine tripartite entanglement, which is readily applied to the multipartite system.

There are many interesting questions left for future work. For example, is it possible to prove that the concurrence fill satisfies $\mathcal{F}(|\psi\rangle_{ABC}) \geqslant \mathcal{F}(\sum p_i|\psi^{(i)}\rangle_{ABC}\langle\psi^{(i)}|)$, thus admitting a less stringent monotonicity under LOCC? If so, then it could be still an entanglement measure. Otherwise, can we find the counterexample? Instead of the squared concurrence, is it possible to use other bipartite entanglement measures to form a similar triangle and hence to construct genuine multipartite entanglement measures or monotones which also have a nice geometric interpretation? It is also interesting to investigate the relevant problems about the genuine multipartite nonlocality [50] and steering [51].

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