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Analytical solutions to the Navier–Stokes equations with density-dependent viscosity and with pressure

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This article is the continued version of the analytical solutions for the pressureless Navier–Stokes equations with density-dependent viscosity [Yuen, M. W., "Analytical solutions to the Navier-Stokes equations," J. Math. Phys. **49**, 113102 (2008)]. We are able to extend the similar solution structure to the case with pressure under some restriction to the constants γ and θ. *© 2009 American Institute of Physics.* [DOI: 10.1063/1.3197860]

I. INTRODUCTION

The Navier–Stokes equations can be formulated in the following form:

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0,$$

$$(\rho \vec{u})_t + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla P = \mathrm{vis}(\rho, \vec{u}). \tag{1}$$

As usual, $\rho = \rho(x,t)$ and $\vec{u}(x,t)$ are the density and the velocity, respectively. $P = P(\rho)$ is the pressure function. We use a γ-law on the pressure, i.e.,

$$P(\rho) = K\rho^{\gamma}. \tag{2}$$

When the constant $K=0$, we call the system is pressureless; when $K>0$, we call that it is with pressure. The constant $\gamma = c_P/c_v \geq 1$, where c_p and c_v are the specific heats per unit mass under constant pressure and constant volume, respectively, is the ratio of the specific heats. γ is the adiabatic exponent in (2). In particular, the fluid is called isothermal if $\gamma=1$. It can be used for constructing models with nondegenerate isothermal fluid. In addition, $\mathrm{vis}(\rho, \vec{u})$ is the viscosity function. When $\mathrm{vis}(\rho, \vec{u}) = 0$, the system (1) becomes the Euler equations. For the detailed study of the Euler and Navier–Stokes equations, see Refs. 1 and 4. Here we consider the density-dependent viscosity function as follows:

$$\mathrm{vis}(\rho, \vec{u}) \doteq \nabla(\mu(\rho) \nabla \cdot \vec{u}). \tag{3}$$

where $\mu(\rho)$ is a density-dependent viscosity function, which is usually written as $\mu(\rho) \doteq \kappa \rho^{\theta}$ with the constants κ, $\theta > 0$. For the study of this kind of the above system, the readers may refer to Refs. 3, 8, 7, 9, 10, and 12.

The solutions to the Navier–Stokes equations (1) in radial symmetry,

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$$\rho(t,x)=\rho(t,r) \quad \text{and} \quad \vec{u}=\frac{\vec{x}}{r}V(t,r):=\frac{\vec{x}}{r}V, \tag{4}$$

can be expressed by

$$\rho_t+V\rho_r+\rho V_r+\frac{N-1}{r}\rho V=0, \tag{5a}$$

$$\rho(V_t+VV_r)+(K\rho^\gamma)_r=(\kappa\rho^\theta)_r\left(\frac{N-1}{r}V+V_r\right)+\kappa\rho^\theta\left(V_{rr}+\frac{N-1}{r}V_r-\frac{N-1}{r^2}V\right), \tag{5b}$$

with $r=(\Sigma_{i=1}^N x_i^2)^{1/2}$.

Recently, Yuen's[13] results showed that there exists a family of the analytical solutions for the pressureless Navier–Stokes equations with density-dependent viscosity (5).

For $\theta=1$,

$$\rho(t,r)=\frac{e^{y(r/a(t))}}{a(t)^N}, \quad V(t,r)=\frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t)=\frac{\lambda\dot{a}(t)}{a(t)^2}, \quad a(0)=a_0>0, \quad \dot{a}(0)=a_1,$$

$$y(x)=\frac{\lambda}{2N\kappa}x^2+\alpha. \tag{6}$$

For $\theta\neq 1$,

$$\rho(t,r)=\begin{cases}\dfrac{y(r/a(t))}{a(t)^N} & \text{for } y\left(\dfrac{r}{a(t)}\right)\geq 0,\\[2ex] 0 & \text{for } y\left(\dfrac{r}{a(t)}\right)<0\end{cases}, \quad V(t,r)=\frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t)=\frac{-\lambda\dot{a}(t)}{a(t)^{N\theta-N+2}}, \quad a(0)=a_0>0, \quad \dot{a}(0)=a_1,$$

$$y(x)=\sqrt[\theta-1]{\frac{1}{2}(\theta-1)\frac{-\lambda}{N\kappa\theta}x^2+\alpha^{\theta-1}}, \tag{7}$$

where $\alpha>0$.

In this article, we extend the results form the study of the analytical solutions in the Navier-Stokes equations without pressure[13] to the case with pressure. The techniques of separation method of self-similar solutions were also found to treat other similar systems in Refs. 2, 5, 6, and 11–13.

Our main result is the following theorem.

Theorem 1: *For the N-dimensional Navier–Stokes equations in radial symmetry (5), there exists a family of solutions, those are the following.*

For $\theta=\gamma=1$,

$$\rho(t,r)=\frac{Ae^{B(r/a(t))^2+C}}{a(t)^N}, \quad V(t,r)=\frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t) - \frac{2BK}{a(t)} + \frac{BN\kappa\dot{a}(t)}{a(t)^2} = 0, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \tag{8}$$

where $A \geq 0$, B, *and* C *are constants.*

For $\theta = \gamma > 1$,

$$\rho(t,r) = \begin{cases} \dfrac{y\left(\dfrac{r}{a(t)}\right)}{a(t)^N} & \text{for } y\left(\dfrac{r}{a(t)}\right) \geq 0 \\ 0 & \text{for } y\left(\dfrac{r}{a(t)}\right) < 0 \end{cases}, \quad V(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

$$\frac{\ddot{a}(t)}{a(t)^N} + \frac{K\gamma}{a(t)^{\theta N+1}} - \frac{N\kappa\theta\dot{a}(t)}{a(t)^{\theta N+2}} = 0, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$y(z) = \sqrt[\theta-1]{\tfrac{1}{2}(\theta-1)z^2 + \alpha^{\theta-1}}, \tag{9}$$

where a_0, a_1, *and* $\alpha > 0$ *are constants.*

For $\gamma/2 + 1/2 - 1/N = \theta \geq 1 - 1/N$,

$$\rho(t,r) = \begin{cases} \dfrac{y\left(\dfrac{r}{a(t)}\right)}{a(t)^N} & \text{for } y\left(\dfrac{r}{a(t)}\right) \geq 0 \\ 0 & \text{for } y\left(\dfrac{r}{a(t)}\right) < 0 \end{cases}, \quad V(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

$$a(t) = \sigma(mt+n)^s, \quad 0 < s = \frac{2}{\gamma N - N + 2} \leq 1,$$

$$\left[\frac{K\gamma}{s\sigma^{\gamma N+1}}y(z)^{\gamma-2} - \frac{mN\kappa\theta}{\sigma^{\theta N+1}}y(z)^{\theta-2}\right]\dot{y}(z) = \frac{(1-s)m^2}{\sigma^{N-1}}z, \quad y(0) = \alpha > 0, \tag{10}$$

where m, $n > 0$, $\sigma > 0$, *and* α *are constants.*

II. SEPARATION METHOD OF SELF-SIMILAR SOLUTIONS

Before we present the proof of Theorem 1, Lemma 3 and Lemma 12 of Ref. 13 are quoted here.

Lemma 2: (Lemma 3 of Ref. 13) *For the equation of conservation of mass in radial symmetry,*

$$\rho_t + V\rho_r + \rho V_r + \frac{N-1}{r}\rho V = 0, \tag{11}$$

there exist solutions,

$$\rho(t,r) = \frac{f(r/a(t))}{a(t)^N}, \quad V(t,r) = \frac{\dot{a}(t)}{a(t)}r, \tag{12}$$

with the form $f \geq 0 \in C^1$ *and* $a(t) > 0 \in C^1$.

Lemma 3: (Lemma 12 of Ref. 13) *For the ordinary differential equation,*

$$\dot{y}(z)y(z)^n - \xi x = 0,$$

$$y(0) = \alpha > 0, \quad n \neq -1, \tag{13}$$

where ξ and n are constants, we have the solution

$$y(z) = \sqrt[n+1]{\frac{1}{2}(n+1)\xi z^2 + \alpha^{n+1}}, \tag{14}$$

where the constant $\alpha > 0$.

At this stage, we can show the proof of Theorem 1.

Proof: Our solutions (8)–(10) fit the mass equation (5a) by Lemma 2. Next, for Eq. (5b), we plug our solutions to check that.

For $\theta = \gamma = 1$, we get

$$\rho(V_t + V \cdot V_r) + K(\rho)_r - (\kappa\rho)_r\left(\frac{N-1}{r}V + V_r\right) - \kappa\rho_r\left(V_{rr} + \frac{N-1}{r}V_r - \frac{N-1}{r^2}V\right), \tag{15}$$

$$= \frac{Ae^{B(r/a(t))^2+C}}{a(t)^3}\frac{\ddot{a}(t)}{a(t)}r + K\frac{Ae^{B(r/a(t))^2+C}}{a(t)^4}B\left[\frac{-2r}{a(t)}\right] - \frac{AN\kappa e^{B(r/a(t))^2+C}}{a(t)^4}B\left[\frac{-2r}{a(t)}\right]\frac{\dot{a}(t)}{a(t)}, \tag{16}$$

$$= \frac{Ae^{B(r/a(t))^2+C}r}{a(t)^4}\left[\ddot{a}(t) - \frac{2BK}{a(t)} + \frac{BN\kappa\dot{a}(t)}{a(t)^2}\right] \tag{17}$$

$$= 0, \tag{18}$$

where the function $a(t)$ is required to be

$$\ddot{a}(t) - \frac{2BK}{a(t)} + \frac{BN\kappa\dot{a}(t)}{a(t)^2} = 0, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \tag{19}$$

where a_0 and a_1 are constants.

For $\theta = \gamma > 1$, we have

$$\rho(V_t + V \cdot V_r) + K(\rho^\gamma)_r - (\kappa\rho^\theta)_r\left(\frac{N-1}{r}V + V_r\right) - \kappa\rho_r^\theta\left(V_{rr} + \frac{N-1}{r}V_r - \frac{N-1}{r^2}V\right) \tag{20}$$

$$= \frac{y\left(\frac{r}{a(t)}\right)}{a(t)^N}\frac{\ddot{a}(t)}{a(t)}r + \frac{K\theta y\left(\frac{r}{a(t)}\right)^{\theta-1}\dot{y}\left(\frac{r}{a(t)}\right)}{a(t)^{\gamma N+1}} - \frac{N\kappa\theta y\left(\frac{r}{a(t)}\right)^{\theta-1}\dot{y}\left(\frac{r}{a(t)}\right)}{a(t)^{\theta N+2}}\dot{a}(t). \tag{21}$$

By defining $z := r/a(t)$ and requiring

$$y(z)z - y(z)^{\theta-1}\dot{y}(z) = 0, \tag{22}$$

$$z - y(z)^{\theta-2}\dot{y}(z) = 0, \tag{23}$$

(21) becomes

$$= y(z)z\left[\frac{\ddot{a}(t)}{a(t)^N} + \frac{K\theta}{a(t)^{\theta N+1}} - \frac{N\kappa\theta\dot{a}(t)}{a(t)^{\theta N+2}}\right] = 0, \tag{24}$$

where the function $a(t)$ is required to be

$$\frac{\ddot{a}(t)}{a(t)^N}+\frac{K\gamma}{a(t)^{\theta N+1}}-\frac{N\kappa\theta\dot{a}(t)}{a(t)^{\theta N+2}}=0. \tag{25}$$

Therefore, Eq. (5b) is satisfied.

With $n:=\theta-2$ and $\xi:=1$, in Lemma 3, (23) can be solved by

$$y(z)=\sqrt[\theta-1]{\tfrac{1}{2}(\theta-1)z^2+\alpha^{\theta-1}}, \tag{26}$$

where $\alpha>0$ is a constant.

For the case of $\gamma/2+1/2-1/N=\theta\geq 1-1/N$, we have

$$\rho(V_t+V\cdot V_r)+K(\rho^\gamma)_r-(\kappa\rho^\gamma)_r\left(\frac{N-1}{r}V+V_r\right)-\kappa\rho_r^\theta\left(V_{rr}+\frac{N-1}{r}V_r-\frac{N-1}{r^2}V\right) \tag{27}$$

$$=\frac{y\left(\frac{r}{a(t)}\right)}{a(t)^N}\frac{\ddot{a}(t)}{a(t)}r+\frac{K\theta y\left(\frac{r}{a(t)}\right)^{\gamma-1}\dot{y}\left(\frac{r}{a(t)}\right)}{a(t)^{\gamma N+1}}-\frac{N\kappa\theta y\left(\frac{r}{a(t)}\right)^{\theta-1}\dot{y}\left(\frac{r}{a(t)}\right)}{a(t)^{\theta N+2}}\dot{a}(t). \tag{28}$$

By letting $a(t)=\sigma(mt+n)^s$, we have

$$=y\left(\frac{r}{a(t)}\right)\frac{s(s-1)(mt+n)^{s-2}}{\sigma^N(mt+n)^{sN}}\frac{m^2\sigma r}{a(t)}+\frac{K\theta y\left(\frac{r}{a(t)}\right)^{\gamma-1}\dot{y}\left(\frac{r}{a(t)}\right)}{\sigma^{\gamma N+1}(mt+n)^{s(\gamma N+1)}}-\frac{sm\sigma N\kappa\theta y\left(\frac{r}{a(t)}\right)^{\theta-1}\dot{y}\left(\frac{r}{a(t)}\right)}{\sigma^{\theta N+2}(mt+n)^{s(\theta N+2)}}(mt+n)^{s-1} \tag{29}$$

$$=y\left(\frac{r}{a(t)}\right)\frac{s(s-1)m^2 r}{\sigma^{N-1}a(t)}\frac{1}{(mt+n)^{sN-(s-2)}}+\frac{K\theta y\left(\frac{r}{a(t)}\right)^{\gamma-1}\dot{y}\left(\frac{r}{a(t)}\right)}{\sigma^{\gamma N+1}(mt+n)^{s(\gamma N+1)}}-\frac{smN\kappa\theta y\left(\frac{r}{a(t)}\right)^{\theta-1}\dot{y}\left(\frac{r}{a(t)}\right)}{\sigma^{\theta N+1}(mt+n)^{s(\theta N+2)-(s-1)}} \tag{30}$$

$$=y\left(\frac{r}{a(t)}\right)\frac{s(s-1)m^2 r}{\sigma^{N-1}a(t)}\frac{1}{(mt+n)^{sN-s+2}}+\frac{K\theta y\left(\frac{r}{a(t)}\right)^{\gamma-1}\dot{y}\left(\frac{r}{a(t)}\right)}{\sigma^{\gamma N+1}(mt+n)^{s(\gamma N+1)}}-\frac{smN\kappa\theta y\left(\frac{r}{a(t)}\right)^{\theta-1}\dot{y}\left(\frac{r}{a(t)}\right)}{\sigma^{\theta N+1}(mt+n)^{\theta N+s+1}}. \tag{31}$$

Here we require that

$$sN-s+2=s(\gamma N+1),$$

$$s(\gamma N+1)=s(\theta N+1)+1. \tag{32}$$

That is,

$$0<s=\frac{1}{(\gamma-\theta)N}=\frac{2}{\gamma N-N+2}\leq 1. \tag{33}$$

In the solutions (10), it fits to the conditions (33) by setting $\gamma/2+1/2-1/N=\theta\geq 1-1/N>0$ and $s=2/(\gamma N-N+2)$. Additionally by defining $z:=r/a(t)$, Eq. (31) becomes

$$= \frac{y(z)s}{(mt+n)^{Ns-s+2}}\left[\frac{(s-1)m^2}{\sigma^{N-1}}z + \frac{K\gamma}{s\sigma^{\gamma N+1}}y(z)^{\gamma-2}\dot{y}(z) - \frac{mN\kappa\theta}{\sigma^{\theta N+1}}y(z)^{\theta-2}\dot{y}(z)\right]. \tag{34}$$

Here we require that

$$\left[\frac{K\gamma}{s\sigma^{\gamma N+1}}y(z)^{\gamma-2} - \frac{mN\kappa\theta}{\sigma^{\theta N+1}}y(z)^{\theta-2}\right]\dot{y}(z) = \frac{(1-s)m^2}{\sigma^{N-1}}z. \tag{35}$$

The proof is completed. □

In the corollary can be followed immediately.

Corollary 4: For $m<0$, *the solutions (10) blowup at the finite time* $T=-m/n$.

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