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Self-similar blowup solutions to the 2-component Camassa–Holm equations

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In this article, we study the self-similar solutions of the 2-component Camassa–Holm equations $\rho_t+u\rho_x+\rho u_x=0$, $m_t+2u_xm+um_x+\sigma\rho\rho_x=0$, with $m=u-\alpha^2u_{xx}$. By the separation method, we can obtain a class of blowup or global solutions for $\sigma=1$ or -1. In particular, for the integrable system with $\sigma=1$, we have the global solutions, $\rho(t,x)=f(\eta)/a(3t)^{1/3}$ for $\eta^2<\alpha^2/\xi$, $\rho(t,x)=0$ for $\eta^2\geq\alpha^2/\xi$, $u(t,x)=\dot{a}(3t)/a(3t)x$, $\ddot{a}(s)-\xi/3a(s)^{1/3}=0, a(0)=a_0>0, \dot{a}(0)=a_1$, $f(\eta)=\xi\sqrt{-1/\xi\eta^2+(\alpha/\xi)^2}$, where $\eta=\frac{x}{a(s)^{1/3}}$ with $s=3t$; $\xi>0$ and $\alpha\geq0$ are arbitrary constants. Our analytical solutions could provide concrete examples for testing the validation and stabilities of numerical methods for the systems. © *2010 American Institute of Physics.* [doi:10.1063/1.3490189]

I. INTRODUCTION

The 2-component Camassa–Holm equations can be expressed in the following form:

$$\begin{cases} \rho_t+u\rho_x+\rho u_x=0 \\ m_t+2u_xm+um_x+\sigma\rho\rho_x=0, \end{cases} \tag{1}$$

with

$$m=u-\alpha^2u_{xx}, \tag{2}$$

$x\in R$ and $u=u(x,t)\in R$ is the velocity of fluid and $\rho=\rho(t,x)\geq0$ is the density of fluid. The constant σ is equal to 1 or -1. If $\sigma=-1$, the gravity acceleration points upward.[2,3,13,16,15] If $\sigma=1$, there exist some papers regarding the corresponding models.[9,12,16,13] When $\rho\equiv0$, the system returns to the Camassa–Holm equation.[1] The searching of a model equation which can capture breaking waves and peaked traveling waves is a longstanding open problem.[24] Here, the Camassa–Holm equation satisfies the above conditions as a model equation. The searching of peaked traveling waves is motivated by the wish to discover waves replicating a characteristic for the wave of great height (waves of largest amplitude), that are exact traveling solutions of the shallow water equations, whether periodic or solitary.[5,23,8] The breaking waves can be understood by solutions which remain bounded but the slope at some point becomes unbounded in a finite time.[6] Meanwhile, there is an alternative derivation of the Camassa–Holm equation in Refs. 18 and 10.

With $\sigma=1$, system (1) is integrable.[17,1] It can be expressed as a compatibility condition of two linear systems (Lax pair),

$$\psi_{xx}=\left(\zeta^2\rho^2+\zeta m+\frac{1}{4}\right)\psi \tag{3}$$

and

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$$\psi_t = \left(\frac{1}{2\zeta} - u\right)\psi_x + \frac{1}{2}u_x\psi \tag{4}$$

with a spectral parameter ζ.

It is bi-Hamiltonian, the first Poisson bracket,

$$\{F_1,F_2\} = -\int \left[\frac{\delta F_1}{\delta m}(m\,\partial + \partial m)\frac{\delta F_2}{\delta m} + \frac{\delta F_1}{\delta m}\rho\,\partial\,\frac{\delta F_2}{\delta \rho} + \frac{\delta F_1}{\delta \rho}\,\partial\,\rho\frac{\delta F_2}{\delta m}\right]dx, \tag{5}$$

with the Hamiltonian

$$H = \frac{1}{2}\int (mu + \rho^2)dx \tag{6}$$

and the second Poisson bracket

$$\{F_1,F_2\} = -\int \left[\frac{\delta F_1}{\delta m}(\partial - \partial^3)\frac{\delta F_2}{\delta m} + \frac{\delta F_1}{\delta \rho}\,\partial\,\frac{\delta F_2}{\delta \rho}\right]dx, \tag{7}$$

with the Hamiltonian

$$H = \frac{1}{2}\int (u\rho^2 + u^3 + uu_x^2)dx. \tag{8}$$

There are two Casimirs,

$$Mass = \int \rho dx \tag{9}$$

and

$$\int m dx. \tag{10}$$

In this article, we adopt an alternative approach (method of separation) to study some self-similar solutions of 2-component Camassa–Holm equations (1). Indeed, we observe that the isentropic Euler, Euler–Poisson, Navier–Stokes, and Navier–Stokes–Poisson systems are written by

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0 \\ \rho[\,\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u})] + \nabla P = -\rho\,\nabla\,\Phi + vis(\rho,\vec{u}) \\ \Delta\Phi(t,x) = \alpha(N)\rho, \end{cases} \tag{11}$$

where $\alpha(N)$ is a constant related to the unit ball in R^N: $\alpha(1)=2$; $\alpha(2)=2\pi$ and for $N\geq 3$,

$$\alpha(N) = N(N-2)Vol(N) = N(N-2)\frac{\pi^{N/2}}{\Gamma(N/2+1)}, \tag{12}$$

where $Vol(N)$ is the volume of the unit ball in R^N and Γ is a gamma function. Moreover, as usual, $\rho=\rho(t,\vec{x})$ and $\vec{u}=\vec{u}(t,\vec{x})\in\mathbf{R}^N$ are the density and velocity, respectively. $P=P(\rho)=K\rho^{\gamma}$ is the pressure, the constant $K\geq 0$ and $\gamma\geq 1$. Moreover, $vis(\rho,\vec{u})$ is the viscosity function.

We may seek the radial solutions

$$\rho(t,\vec{x}) = \rho(t,r), \quad \vec{u} = \frac{\vec{x}}{r}V(t,r) =: \frac{\vec{x}}{r}V, \tag{13}$$

with $r=(\sum_{i=1}^N x_i^2)^{1/2}$.

By the standard computation, the Euler equations in radial symmetry can be written in the following form:

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{N-1}{r}\rho V = 0 \\ \rho(V_t + VV_r) + K\dfrac{\partial}{\partial r}\rho^{\gamma} = 0. \end{cases} \tag{14}$$

For the mass equation in radial symmetry, first equation of (14), we well know the solutions' structure (Lemma 3, Ref. 27),

$$\rho(t,r) = \frac{f\left(\dfrac{r}{a(t)}\right)}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(r)}{a(r)}r. \tag{15}$$

As the 2-component Camassa–Holm equations, (1), are very similar to Euler system (14), in some senses, we can apply the separation method[14,21,20,25,27] to systems (1). In fact, we can deduce the nonlinear partial differential equations into much simpler ordinary differential equations. In this way, we can contribute a new class of self-similar solutions in the following theorem.

Theorem 1: *We define the function $a(s)$ is the solution of the Emden equation,*

$$\begin{cases} \ddot{a}(s) - \dfrac{\xi}{3a(s)^{1/3}} = 0 \\ a(0) = a_0 \neq 0, \dot{a}(0) = a_1 \end{cases} \tag{16}$$

and

$$f(\eta) = \frac{\xi}{\sigma}\sqrt{-\frac{\sigma}{\xi}\eta^2 + \left(\frac{\sigma\alpha}{\xi}\right)^2}, \tag{17}$$

where $\eta = \frac{x}{a(s)^{1/3}}$ *with* $s=3t$; $\alpha \geq 0$, $\xi \neq 0$, *and* a_0 *and* a_1 *are arbitrary constants.*

For the 2-compenent Camassa–Holm equations (1), there exists a family of solutions,

(1) for $\sigma=-1$,

(a) with $\xi<0$ *and* $a_0>0$,

$$\rho(t,x) = \begin{cases} \dfrac{f(\eta)}{a(3t)^{1/3}}, & \text{for } \eta^2 < -\dfrac{\alpha^2}{\xi} \\ 0, & \text{for } \eta^2 \geq -\dfrac{\alpha^2}{\xi} \end{cases}, u(t,x) = \frac{\dot{a}(3t)}{a(3t)}x. \tag{18}$$

Solution (18) blows up in a finite time T .

(b) with $\xi>0$ *and* $a_0<0$,

$$\rho(t,x) = \frac{f(\eta)}{a(3t)^{1/3}}, \quad u(t,x) = \frac{\dot{a}(3t)}{a(3t)}x. \tag{19}$$

Solution (19) exists globally;

(2) for $\sigma=1$,

(a) with $\xi>0$ *and* $a_0>0$,

$$\rho(t,x)=\begin{cases}\dfrac{f(\eta)}{a(3t)^{1/3}}, & \text{for } \eta^2<\dfrac{\alpha^2}{\xi}\\ 0, & \text{for } \eta^2\geq\dfrac{\alpha^2}{\xi}\end{cases},u(t,x)=\frac{\dot{a}(3t)}{a(3t)}x. \tag{20}$$

Solution (20) globally exists.
(b) with $\xi<0$ and $a_0<0$,

$$\rho(t,x)=\frac{f(\eta)}{a(3t)^{1/3}},\quad u(t,x)=\frac{\dot{a}(3t)}{a(3t)}x. \tag{21}$$

Solution (21) blows up in a finite time T .

II. SEPARATION METHOD

First, we can design a nice functional structure for the mass equation.

Lemma 2: For the one-dimensional equation of mass $(4)_1$,

$$\rho_t+u\rho_x+\rho u_x=0, \tag{22}$$

there exist solutions

$$\rho(t,x)=\frac{f\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}},\quad u(t,x)=\frac{\dot{a}(3t)}{a(3t)}x, \tag{23}$$

with the form $f(\eta)\geq 0\in C^1$ with $\eta=\frac{x}{a(3t)^{1/3}}$ and $a(3t)>0\in C^1$

Proof: We just plug (23) into (22) to check

$$\rho_t+u\rho_x+\rho u_x \tag{24}$$

$$=\frac{\partial}{\partial t}\left(\frac{f\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}}\right)+\frac{\dot{a}(3t)}{a(3t)}x\frac{\partial}{\partial x}\left(\frac{f\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}}\right)+\frac{f\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}}\frac{\partial}{\partial x}\left(\frac{\dot{a}(3t)}{a(3t)}x\right) \tag{25}$$

$$=\frac{1}{a(3t)^{1/3+1}}\left(-\frac{1}{3}\right)\cdot\dot{a}(3t)\cdot 3\cdot f\left(\frac{x}{a(3t)^{1/3}}\right)+\frac{1}{a(3t)^{1/3}}\dot{f}\left(\frac{x}{a(3t)^{1/3}}\right)\frac{\partial}{\partial t}\left(\frac{x}{a(3t)^{1/3}}\right) \tag{26}$$

$$+\frac{\dot{a}(3t)x}{a(3t)}\frac{\dot{f}\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)}\frac{\partial}{\partial x}\left(\frac{x}{a(3t)^{1/3}}\right)+\frac{f\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}}\frac{\dot{a}(3t)}{a(3t)} \tag{27}$$

$$=-\frac{\dot{a}(3t)f\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3+1}}+\frac{1}{a(3t)^{1/3}}\dot{f}\left(\frac{x}{a(3t)^{1/3}}\right)\frac{x}{a(3t)^{1/3+1}}\frac{1}{3}\dot{a}(3t)3 \tag{28}$$

$$+\frac{\dot{a}(3t)x}{a(3t)}\frac{\dot{f}\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}}\frac{1}{a(3t)^{1/3}}+\frac{f\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}}\frac{\dot{a}(3t)}{a(3t)} \tag{29}$$

$$=0. \tag{30}$$

The proof is completed. ■

In Refs. 11 and 26, the qualitative properties of the Emden equation,

$$\begin{cases} \ddot{a}(s) - \dfrac{\xi}{a(s)^{N-1}} = 0 \\ a(0) = a_0 > 0, \dot{a}(0) = a_1, \end{cases} \tag{31}$$

where $N \geq 2$ were studied. Therefore, the similar local existence of Emden equation (16),

$$\begin{cases} \ddot{a}(s) - \dfrac{\xi}{a(s)^{1/3}} = 0 \\ a(0) = a_0 \neq 0, \dot{a}(0) = a_1, \end{cases} \tag{32}$$

can be proved by the standard fixed point theorem.[11,26] To additionally show the blowup property of the time function $a(s)$, the following lemmas are needed.

Lemma 3: For Emden equation (19),

$$\begin{cases} \ddot{a}(s) - \dfrac{\xi}{a(s)^{1/3}} = 0 \\ a(0) = a_0 > 0, \dot{a}(0) = a_1, \end{cases} \tag{33}$$

(1) if $\xi < 0$, there exists a finite time S, such that $\lim_{s \to S^-} a(s) = 0$ *;*
(2) if $\xi > 0$, the solution $a(s)$ globally exists, such that

$$\lim_{s \to +\infty} a(s) = +\infty. \tag{34}$$

Proof: (1) For Emden equation (33), we can multiply $\dot{a}(s)$ and then integrate it, as follows:

$$\frac{\dot{a}(s)^2}{2} - \frac{3}{2}\xi a(s)^{2/3} = \theta, \tag{35}$$

with the constant

$$\theta = \frac{a_1^2}{2} - \frac{3}{2}\xi a_0^{2/3} > 0 \tag{36}$$

for $\xi < 0$.

Equation (38) becomes

$$\dot{a}(s) = \pm\sqrt{2\theta + 3\xi a(s)^{2/3}}. \tag{37}$$

We can define the kinetic energy thus

$$F_{kin} := \frac{\dot{a}(s)^2}{2}, \tag{38}$$

and the potential energy thus

$$F_{pot} = -\frac{3}{2}\xi a(s)^{2/3}. \tag{39}$$

The total energy (Fig. 1) is conserved thus

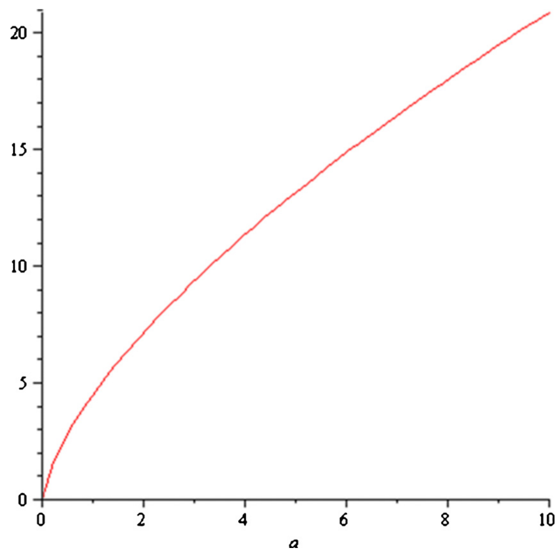

FIG. 1. (Color online) Curve for potential energy F_pot with $\xi=-3$.

$$\frac{d}{ds}(F_{kin}+F_{pot})=0. \tag{40}$$

By the classical energy method for conservative systems (in Sec. 4.3 of Ref. 19), the solution has a trajectory. The time for traveling the orbit $[0,a_{\sup}]$ can be estimated as

$$S=\int_0^S ds \le 2\int_{a_{\inf}}^{a_{\sup}} \frac{da(s)}{\sqrt{2\left(\theta+\frac{3\xi a(s)^{2/3}}{2}\right)}}=2\int_0^{(2\theta/3\xi)^{3/2}} \frac{da(s)}{\sqrt{2\left(\theta+\frac{3\xi a(s)^{2/3}}{2}\right)}}. \tag{41}$$

Then we let $G(s):=\sqrt{\frac{-3\xi}{2}}a(s)^{1/3}$ have

$$dG(s)=\sqrt{\frac{-\xi}{6}}\frac{da(s)}{a(s)^{2/3}} \tag{42}$$

and

$$da(s)=\sqrt{\frac{-6}{\xi}}a(s)^{2/3}dG(s)=\frac{2}{-3\xi}\sqrt{\frac{-6}{\xi}}G(s)^2dG(s). \tag{43}$$

Then, Eq. (44) becomes

$$2\int_0^{(2\theta/3\xi)^{3/2}} \frac{da(s)}{\sqrt{2\left(\theta+\frac{3\xi a(s)^{2/3}}{2}\right)}}=\frac{2\sqrt{2}}{-3\xi}\sqrt{\frac{-6}{\xi}}\int_0^{\sqrt{\theta}}\frac{G(s)^2dG(s)}{\sqrt{(\theta-G(s)^2)}}, \tag{44}$$

The integral is solvable exactly by the integration formula 17.11.3 on page 82 of Ref. 22,

$$\int_0^{\sqrt{\theta}}\frac{x^2dx}{\sqrt{(\theta-x^2)}}= \left[\frac{-x\sqrt{\theta-x^2}}{2}+\frac{\theta}{2}\sin^{-1}\left(\frac{x}{\sqrt{\theta}}\right)\right]\Bigg|_{x=0}^{x=\sqrt{\theta}}=\frac{\theta\pi}{4}. \tag{45}$$

Therefore, we showed that there exists a finite time S, such that $\lim_{s->S^-} a(s)=0$.

Intuitively we may see the particular phase diagram (Fig. 2) for the dynamical system for $\xi=-3$,

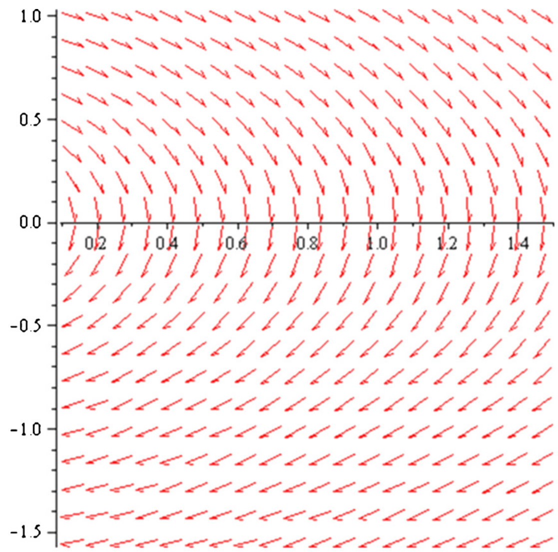

FIG. 2. (Color online) Phase diagram of the dynamical system.

$$\begin{cases} \dfrac{d}{ds}x(s) = y(s) \\ \dfrac{d}{ds}y(s) = \dfrac{-1}{x(s)^{1/3}} \\ x(0) = a_0 \neq 0, y(0) = a_1, \end{cases} \tag{46}$$

$$(x(s), y(s)). \tag{47}$$

(2) for the case of $\xi > 0$, the potential function (Fig. 3) is

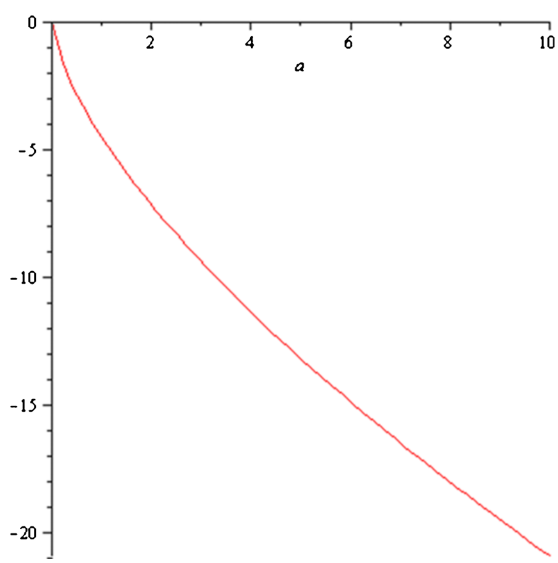

FIG. 3. (Color online) Curve for potential energy.

$$F_{pot} = -\frac{3}{2}\xi a(s)^{2/3}, \tag{48}$$

$$F_pot \quad \text{with} \quad \xi = 3. \tag{49}$$

As the infimum of $a(s)$ is

$$a_{\inf} = \left(\frac{-2\theta}{3\xi}\right)^{3/2} > 0, \tag{50}$$

the solution $a(s)$ is uniformly bounded below. The time for achieving the infimum is finite if $a_1 < 0$,

$$S_1 = \int_0^{S_1} ds = \int_{(-2\theta/3\xi)^{3/2}}^{a_0} \frac{da(s)}{\sqrt{2\left(\theta + \frac{3\xi a(s)^{2/3}}{2}\right)}} < +\infty. \tag{51}$$

Therefore, for any constant a_1, the solution $a(s)$ must increase, after some finite time. Moreover, the time for traveling the interval $[\inf a(s), +\infty)$ or $[a_1, +\infty)$ is infinite,

$$S_2 = \int_{(-2\theta/3\xi)^{3/2}}^{+\infty} \frac{da(s)}{\sqrt{2\left(\theta + \frac{3\xi a(s)^{2/3}}{2}\right)}} = +\infty. \tag{52}$$

Therefore, we showed that the solution $a(s)$ globally exists, such that

$$\lim_{s \to +\infty} a(s) = +\infty. \tag{53}$$

The proof is completed. ■

We may let $b(t) = -a(t)$, in Emden equation (16), for $a_0 < 0$,

$$\begin{cases} \ddot{a}(s) - \dfrac{\xi}{3a(s)^{1/3}} = 0 \\ a(0) = a_0 < 0, \dot{a}(0) = a_1, \end{cases} \tag{54}$$

have

$$\begin{cases} \ddot{b}(s) - \dfrac{\xi}{3b(s)^{1/3}} = 0 \\ b(0) = -a_0 > 0, \dot{b}(0) = -a_1. \end{cases} \tag{55}$$

Alternatively, the ordinary differential equation can be rewritten as the dynamical system,

$$\begin{cases} \dfrac{d}{ds}x(s) = y(s) \\ \dfrac{d}{ds}y(s) = \dfrac{\xi}{3x(s)^{1/3}} \\ x(0) = a_0 \neq 0, y(0) = a_1. \end{cases} \tag{56}$$

It is symmetric about the origin (0,0). Therefore, we can use the above lemma to drive the corresponding lemma for $a_0 < 0$.

Lemma 4: For Emden equation (16),

$$\begin{cases} \ddot{a}(s) - \dfrac{\xi}{a(s)^{1/3}} = 0 \\ a(0) = a_0 < 0, \dot{a}(0) = a_1, \end{cases} \tag{57}$$

(1) if $\xi < 0$*, there exists a finite time S, such that* $\lim_{s \to S^-} a(s) = 0$ *;*
(2) if $\xi > 0$*, the solution* $a(t)$ *globally exists, such that*

$$\lim_{s \to +\infty} a(s) = -\infty. \tag{58}$$

After obtaining the nice structure of solutions (23), we simply use the techniques of separation of variables,[14,21,20,25,27] to prove the theorem:

Proof of Theorem 1: From Lemma 2, it is clear to see our functions (18)–(21) fit well into the mass equation, first equation of (1), except for two boundary points.

The second equation of 2-component Camassa–Holm equations, second equation of (1), becomes

$$m_t + 2u_x m + u m_x + \sigma \rho \rho_x \tag{59}$$

$$= (u - u_{xx})_t + 2u_x(u - u_{xx}) + u(u - u_{xx})_x + \sigma \rho \rho_x. \tag{60}$$

As our velocity u, in solutions (18)–(21), is a linear flow,

$$u = \frac{\dot{a}(3t)}{a(3t)} x, \tag{61}$$

we have

$$u_{xx} = 0. \tag{62}$$

Equation (61) becomes

$$= u_t + 3u_x u + \sigma \rho \rho_x \tag{63}$$

$$= \frac{\partial}{\partial t}\left(\frac{\dot{a}(3t)}{a(3t)}\right) x + 3\left(\frac{\dot{a}(3t)}{a(3t)}\right) x \frac{\dot{a}(3t)}{a(3t)} + \sigma \frac{f\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}} \left(\frac{f\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}}\right)_x \tag{64}$$

$$= \left(3\frac{\ddot{a}(3t)}{a(3t)} - 3\frac{\dot{a}(3t)^2}{a(3t)^2}\right) x + 3\left(\frac{\dot{a}(3t)}{a(3t)}\right)\frac{\dot{a}(3t)}{a(3t)} x + \sigma \frac{f\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}} \frac{\dot{f}\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)^{1/3}} \frac{1}{a(3t)^{1/3}} \tag{65}$$

$$= 3\frac{\ddot{a}(3t)}{a(3t)} x + \sigma \frac{f\left(\dfrac{x}{a(3t)^{1/3}}\right)\dot{f}\left(\dfrac{x}{a(3t)^{1/3}}\right)}{a(3t)} \tag{66}$$

$$= \frac{\sigma}{a(3t)}\left(\frac{\xi}{\sigma}\eta + f(\eta)\dot{f}(\eta)\right), \tag{67}$$

with the Emden equation

$$\begin{cases} \ddot{a}(s) - \dfrac{\xi}{3a(s)^{1/3}} = 0 \\ a(0) = a_0 \neq 0, \dot{a}(0) = a_1, \end{cases} \tag{68}$$

by defining the variables $s := 3t$ and $\eta := x/a(s)^{1/3}$.

Now, we can separate the partial differential equations into two ordinary differential equations. Then, we only need to solve for $\frac{\xi}{\sigma} < 0$,

$$\begin{cases} \dfrac{\xi}{\sigma}\eta + f(\eta)\dot{f}(\eta) = 0 \\ f(0) = -\alpha \leq 0; \end{cases} \tag{69}$$

or for $\frac{\xi}{\sigma} > 0$,

$$\begin{cases} \dfrac{\xi}{\sigma}\eta + f(\eta)\dot{f}(\eta) = 0 \\ f(0) = \alpha \geq 0; \end{cases} \tag{70}$$

Ordinary differential equations (67) or (68) can be solved exactly as

$$f(\eta) = \frac{\xi}{\sigma}\sqrt{-\frac{\sigma}{\xi}\eta^2 + \left(\frac{\sigma\alpha}{\xi}\right)^2}. \tag{71}$$

In fact, we have the self-similar solutions in details,

(1) for $\sigma = -1$,

(a) with $\xi < 0$ and $a_0 > 0$,

$$\begin{cases} \rho(t,x) = \begin{cases} \dfrac{f(\eta)}{a(3t)^{1/3}} & \text{for } \eta^2 < -\dfrac{\alpha^2}{\xi} \\ 0 & \text{for } \eta^2 \geq -\dfrac{\alpha^2}{\xi} \end{cases}, u(t,x) = \dfrac{\dot{a}(3t)}{a(3t)}x \\ \ddot{a}(s) - \dfrac{\xi}{3a(s)^{1/3}} = 0, a(0) = a_0 > 0, \dot{a}(0) = a_1 \\ f(\eta) = -\xi\sqrt{\dfrac{1}{\xi}\eta^2 + \left(\dfrac{\alpha}{\xi}\right)^2}, \end{cases} \tag{72}$$

(b) with $\xi > 0$ and $a_0 < 0$,

$$\begin{cases} \rho(t,x) = \dfrac{f(\eta)}{a(3t)^{1/3}}, u(t,x) = \dfrac{\dot{a}(3t)}{a(3t)}x \\ \ddot{a}(s) - \dfrac{\xi}{3a(s)^{1/3}} = 0, a(0) = a_0 < 0, \dot{a}(0) = a_1 \\ f(\eta) = -\xi\sqrt{\dfrac{1}{\xi}\eta^2 + \left(\dfrac{\alpha}{\xi}\right)^2}. \end{cases} \tag{73}$$

(2) for $\sigma = 1$,

(a) with $\xi > 0$ and $a_0 > 0$,

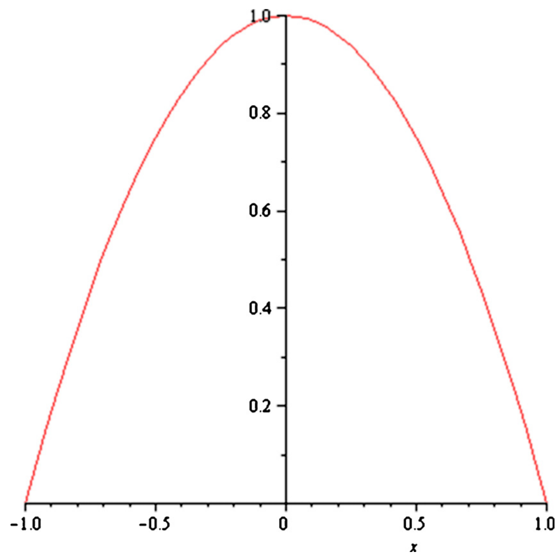

FIG. 4. (Color online) $\rho(0,x)=\sqrt{-x^2+1}$.

$$\begin{cases} \rho(t,x)=\begin{cases}\dfrac{f(\eta)}{a(3t)^{1/3}} & \text{for } \eta^2<\dfrac{\alpha^2}{\xi} \\ 0 & \text{for } \eta^2\geq\dfrac{\alpha^2}{\xi}\end{cases}, u(t,x)=\dfrac{\dot{a}(3t)}{a(3t)}x \\ \ddot{a}(s)-\dfrac{\xi}{3a(s)^{1/3}}=0, a(0)=a_0>0, \dot{a}(0)=a_1 \\ f(\eta)=\xi\sqrt{-\dfrac{1}{\xi}\eta^2+\left(\dfrac{\alpha}{\xi}\right)^2}. \end{cases} \tag{74}$$

(b) with $\xi<0$ and $a_0<0$,

$$\begin{cases} \rho(t,x)=\dfrac{f(\eta)}{a(3t)^{1/3}}, \quad u(t,x)=\dfrac{\dot{a}(3t)}{a(3t)}x \\ \ddot{a}(s)-\dfrac{\xi}{3a(s)^{1/3}}=0, \quad a(0)=a_0<0, \quad \dot{a}(0)=a_1 \\ f(\eta)=\xi\sqrt{-\dfrac{1}{\xi}\eta^2+\left(\dfrac{\alpha}{\xi}\right)^2}. \end{cases} \tag{75}$$

With the assistance of Lemmas 3 and 4, the blowup or global existence of the solutions can be determined under the prescribed conditions in the theorem.

For the graphical illustration of solution (74), for the integrable system with $\sigma=1$, $\xi=1$, $\alpha=1$, and $a_0=1$, the initial shape of the self-similar solution can be found (Fig. 4),

$$\rho(0,x)=\sqrt{-x^2+1}. \tag{76}$$

For solution (75) with $\sigma=1$, $\xi=-1$, $\alpha=1$, and $a_0=-1$, the corresponding graph is (Fig. 5)

$$\rho(0,x)=\sqrt{x^2+1}. \tag{77}$$

The proof is completed. ■

Remark 5: For $\xi<0$, blowup solutions (18) and (21) collapse at the origin

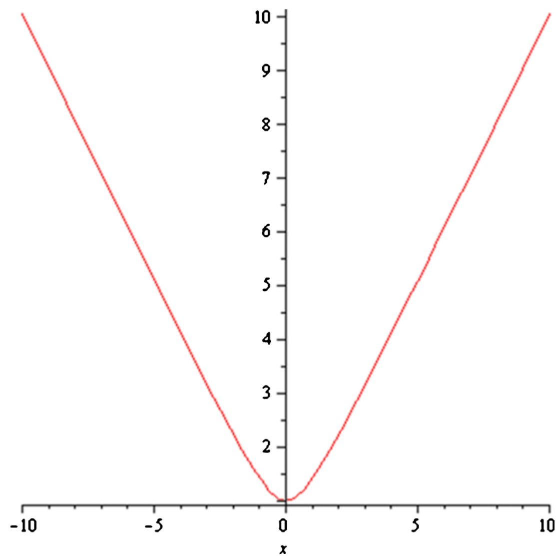

FIG. 5. (Color online) $\rho(0,x)=\sqrt{x^2+1}$.

$$\lim_{t\to T^-} \rho(t,0) = +\infty, \tag{78}$$

with a finite time T *.*

For $\xi > 0$ *, the global behavior of solutions (19) and (20) at the origin is*

$$\lim_{t\to +\infty} \rho(t,0) = 0. \tag{79}$$

Remark 6: Solutions (18) and (20) are only C^0 *functions, as the function* $f(\eta)$ *is discontinuous at the two boundary points, for* $\alpha > 0$ *,*

$$\lim_{\eta^2 \to \left|\frac{\alpha^2}{\xi}\right|} \dot{f}(\eta) \neq 0. \tag{80}$$

Remark 7: Our analytical solutions could provide concrete examples for testing the validation and stabilities of numerical methods for the systems. Additionally, our special solutions can shed some light on understanding of evolutionary pattern of the systems.

Remark 8: For the integrable system with $\sigma=1$ *, we may calculate the mass of*

(1) the blowup solution (21) [or (73)], $\xi < 0$ *and* $a_0 < 0$ *,*

$$Mass = \int_{-\infty}^{+\infty} \rho(0,x)dx = \frac{1}{\xi a_0^{1/3}} \int_{-\infty}^{+\infty} \sqrt{-\xi\left(\frac{x}{a_0^{1/3}}\right)^2 + (\xi\alpha)^2}dx = +\infty, \tag{81}$$

(2) Global solution (20) [or (74)], $\xi > 0$ *and* $a_0 > 0$ *,*

$$Mass = \int_{-\infty}^{+\infty} \rho(0,x)dx \tag{82}$$

$$= 2\int_0^{a_0^{1/3}\alpha\sqrt{1/\xi}} \frac{f\left(\frac{x}{a_0^{1/3}}\right)}{a_0^{1/3}}dx \tag{83}$$

$$=\frac{2\xi}{a_0^{1/3}}\int_0^{a_o^{1/3}\alpha\sqrt{1/\xi}}\sqrt{-\frac{1}{\xi}\left(\frac{x}{a_0^{1/3}}\right)^2+\left(\frac{\alpha}{\xi}\right)^2}dx \tag{84}$$

$$=2\xi\int_0^{\alpha\sqrt{1/\xi}}\sqrt{-\frac{1}{\xi}s^2+\left(\frac{\alpha}{\xi}\right)^2}ds \tag{85}$$

$$=\frac{2\xi}{\sqrt{\xi}}\int_0^{\alpha\sqrt{1/\xi}}\sqrt{\frac{\alpha^2}{\xi}-s^2ds} \tag{86}$$

$$=\frac{\alpha^2\pi}{2\sqrt{\xi}}. \tag{87}$$

III. BLOWUP RATES

We are interested in how fast the blowup solutions tend to the infinite as the time tends to the critical time T. The blowup rate of constructed solutions (18) and (21) at the origin is estimated in the following theorem.

Theorem 9: *The blowup rate of solutions (18) and (23) for* $\alpha > 0$ *is*

$$\lim_{s\to S^-}\rho(s,0)(S-s)^{1/3}\geq O(1). \tag{88}$$

Proof: We only need to study the blowup rate of Emden equation (16),

$$\begin{cases}\ddot{a}(s)-\dfrac{\xi}{a(s)^{1/3}}=0\\ a(0)=a_0>0,\dot{a}(0)=a_1,\end{cases} \tag{89}$$

with $\xi < 0$.

For the total energy $\theta > 0$, with the assistance of Eq. (37),

$$S=s+\int_s^S d\eta=s+\int_{a(s)}^0\frac{d\eta}{da}da=s-\int_{a(s)}^0\frac{1}{\sqrt{2\theta+3\xi a(\eta)^{2/3}}}da\geq s+\int_0^{a(s)}\frac{da(\eta)}{\sqrt{2\theta}}. \tag{90}$$

That is,

$$(S-s)^{1/3}\geq O(1)a(s)^{1/3}. \tag{91}$$

Therefore, we may estimate the blowup rates at the origin of the density function ρ,

$$\lim_{s\to S^-}\rho(s,0)(S-s)^{1/3}=\lim_{s\to S^-}\frac{\alpha}{a(s)^{1/3}}(S-s)^{1/3}\geq O(1). \tag{92}$$

The proof is completed. ■

We notice that our blowup rate in the above theorem is different form the results of the Constantin and Escher's papers.[7,4]

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