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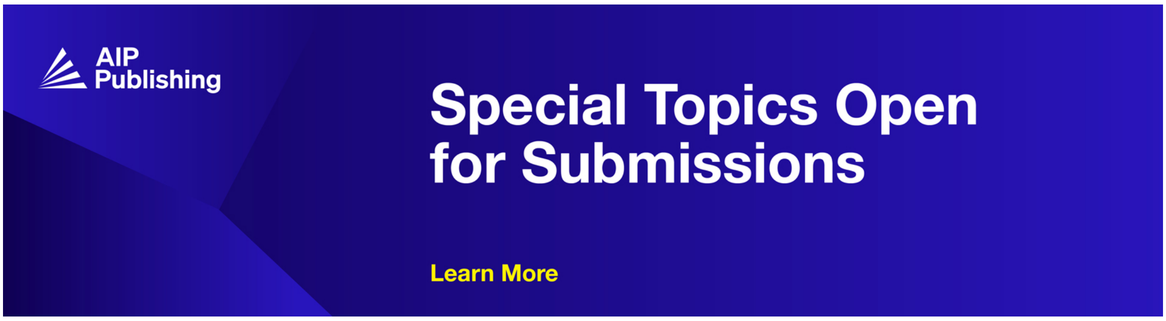

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ABSTRACT

In this paper, we study a one dimensional nonlinear equation with diffusion $-\nu(-\partial_{xx})^{\frac{\alpha}{2}}$ for $0 \le \alpha \le 2$ and $\nu > 0$. We use a viscous-splitting algorithm to obtain global nonnegative weak solutions in space $L^1(\mathbb{R}) \cap H^{1/2}(\mathbb{R})$ when $0 \le \alpha \le 2$. For the subcritical case $1 < \alpha \le 2$ and critical case $\alpha = 1$, we obtain the global existence and uniqueness of nonnegative spatial analytic solutions. We use a fractional bootstrapping method to improve the regularity of mild solutions in the Bessel potential spaces for the subcritical case $1 < \alpha \le 2$. Then, we show that the solutions are spatial analytic and can be extended globally. For the critical case $\alpha = 1$, if the initial data ρ_0 satisfies $-\nu < \inf \rho_0 < 0$, we use the method of characteristics for complex Burgers equation to obtain a unique spatial analytic solution to our target equation in some bounded time interval. If $\rho_0 \ge 0$, the solution exists globally and converges to steady state.

I. INTRODUCTION

In this paper, we are going to study the following nonlinear partial differential equation on the real line $\mathbb{R}$:

$$\begin{cases} \partial_t \rho + \partial_x[\rho(u - \gamma x)] = -\nu \Lambda^\alpha \rho, \quad u = H\rho, \quad t > 0, \quad x \in \mathbb{R}, \\ \rho(x,0) = \rho_0(x), \quad x \in \mathbb{R} \end{cases} \tag{1.1}$$

with $\gamma \ge 0$, $\nu > 0$ and $0 \le \alpha \le 2$. The velocity field $H\rho$ stands for the Hilbert transform of ρ:

$$(H\rho)(x,t) := \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{\rho(y,t)}{x-y} \mathrm{d}y.$$

Here ν is a positive number called the viscosity coefficient, and it controls the strength of the dissipation term. For $0 \le \alpha \le 2$, the fractional Laplacian $\Lambda^\alpha \rho = (-\partial_{xx})^{\frac{\alpha}{2}} \rho$ is defined by its Fourier transform:

$$[\mathcal{F}(\Lambda^\alpha \rho)](\xi,t) = |\xi|^\alpha [\mathcal{F}(\rho)](\xi,t).$$

The parameter α also controls the magnitude of the dissipation term. For $\nu > 0$, let the kernel G_α be the fundamental solution of the linear operator $\partial_t + \nu\Lambda^\alpha$, and

$$\mathcal{F}(G_\alpha)(\xi,t) := e^{-\nu t|\xi|^\alpha}.$$

We call the cases $\alpha > 1$, $\alpha = 1$, and $\alpha < 1$ of (1.1) as subcritical, critical, and supercritical respectively.

When $\nu = 0$, Eq. (1.1) becomes

$$\partial_t \rho + \partial_x[\rho(u - \gamma x)] = 0, \quad u = H\rho, \quad x \in \mathbb{R}, \quad t > 0. \tag{1.2}$$

Equation (1.2) is the mean field equation of the following Dyson Brownian motion:[3,10,30]

$$\mathrm{d}\lambda_j(t) = \frac{1}{\sqrt{N}}\mathrm{d}B_j(t) + \frac{1}{\pi N}\sum_{k \neq j}\frac{\mathrm{d}t}{\lambda_j(t) - \lambda_k(t)} - \gamma\lambda_j(t)\mathrm{d}t, \quad 1 \le j \le N, \tag{1.3}$$

which describes the evolution of eigenvalues $\{\lambda_j\}_{j=1}^N$ of a $N \times N$ Hermitian matrix given by matrix-valued Ornstein–Uhlenbeck process.[13,14,33] Here $B_j(t), j = 1, 2, \ldots, N$, are N independent standard Brownian motions in $\mathbb{R}$. Next, we list three important aspects of Eq. (1.2).

Space-time rescaling: For Eq. (1.2), an important fact is that the linear term $-\gamma\partial_x(x\rho)$ with $\gamma > 0$ can be reformulated into the case $\gamma = 0$ by the following space-time rescaling:

$$\tilde{\rho}(y, \tau)\sqrt{1 + 2\gamma\tau} = \rho(x, t), \quad x = \frac{y}{\sqrt{1 + 2\gamma\tau}}, \quad t = \frac{1}{2\gamma}\log(1 + 2\gamma\tau). \tag{1.4}$$

Then, if ρ is a solution to (1.2), then $\tilde{\rho}$ is a solution to (1.2) with $\gamma = 0$:

$$\partial_\tau\tilde{\rho} + \partial_y(\tilde{\rho}\tilde{u}) = 0, \quad \tilde{u} = H\tilde{\rho}. \tag{1.5}$$

The above rescaling has the same effect for Eq. (1.1) with $\alpha = 2$, but not for $0 < \alpha < 2$.

Gradient flow structure: Eq. (1.1) with $\alpha = 2$ or $\nu = 0$ has a gradient flow structure in the probability measure space with Wasserstein distance for a free energy functional given by

$$\begin{aligned} E(\rho(\cdot, t)) &= \frac{\gamma}{2}\int_{\mathbb{R}} x^2\rho(x,t)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\log|x - y|\rho(x,t)\rho(y,t)\,\mathrm{d}x\,\mathrm{d}y + \nu\int_{\mathbb{R}}\rho(x,t)\log\rho(x,t)\mathrm{d}x \\ &=: E_{\mathrm{h}}(\rho(\cdot,t)) + E_{\mathrm{i}}(\rho(\cdot,t)) + E_{\mathrm{e}}(\rho(\cdot,t)). \end{aligned} \tag{1.6}$$

Here E_{h} is the harmonic trap energy, E_{i} is the interaction energy, and E_{e} is the entropy. Then, Eq. (1.1) with $\alpha = 2$ is recast to

$$\partial_t\rho - \partial_x\left[\rho\partial_x\left(\frac{\delta E}{\delta\rho}\right)\right] = 0, \quad \frac{\delta E}{\delta\rho} = \frac{\gamma}{2}x^2 - \int_{\mathbb{R}}\log|x - y|\rho(y,t)\mathrm{d}y + \nu(1 + \log\rho). \tag{1.7}$$

By the properties of this gradient flow structure, Carrillo *et al.*[8] obtained the existence and uniqueness of global probability measure solutions. They also proved γ-convexity along Wasserstein geodesics of the energy and hence obtained exponential convergence to the steady state given by Wigner's semicircle law

$$\mu(\mathrm{d}x) = \rho(x)\mathrm{d}x := \frac{\sqrt{(4 - x^2)_+}}{2\pi}\mathrm{d}x. \tag{1.8}$$

Complex Burgers equation: From Eq. (1.2), if we define $g(x,t) = H\rho(x,t) - i\rho(x,t) - \gamma x$, then the analytical extention of g on the upper half complex plane $\mathbb{C}_+ := \{z : \Im(z) = \mathrm{Im}(z) > 0\}$ satisfies the following complex Burgers equation with a force term $\gamma^2 z$:[9,15]

$$\partial_t g + g\partial_z g = \gamma^2 z, \quad z \in \mathbb{C}_+, \quad t > 0. \tag{1.9}$$

When $\gamma = 0$, Castro and Córdoba[9] proved global (in time) existence and uniqueness of spatial analytic solutions ($t > 0$) to (1.2) with strictly positive initial data $0 < \rho_0 \in L^2(\mathbb{R}) \cap C^{0,\delta}(\mathbb{R})$ via method of characteristics for (1.9). However, if there is $x_0 \in \mathbb{R}$ such that $\rho_0(x_0) = 0$, then the solution ρ will blow up in $H^s(\mathbb{R})$ for $s > \frac{3}{2}$ in finite time.[9] These two results hold also for $\gamma > 0$ due to the rescaling (1.4), and the global solutions with $\rho_0 > 0$ converge to the steady state pointwise.[15] Global nonnegative weak solutions in $L^\infty(0, T; L^1(\mathbb{R}) \cap H^{\frac{1}{2}}(\mathbb{R}))$ to (1.2) were also obtained.[15]

In this paper, we are going to study Eq. (1.1) with $0 \le \alpha \le 2$. We first use a viscous-splitting algorithm (see, e.g., Ref. 26, Chap. 3) to obtain global weak solutions (see Definition 2.1) for the whole range $0 \le \alpha \le 2$. The following theorem is obtained:

Theorem 1.1. *Let $\gamma \ge 0$, $\nu > 0$ and $0 \le \alpha \le 2$. Assume $0 \le \rho_0 \in L^1(\mathbb{R}) \cap H^{1/2}(\mathbb{R})$ and $\int_{\mathbb{R}} x^2\rho_0(x)\mathrm{d}x < \infty$. Then, there exists a global nonnegative weak solution to (1.1) satisfying*

$$\rho \in L^\infty(0, \infty; L^1(\mathbb{R}) \cap H^{1/2}(\mathbb{R})) \cap W^{1,\infty}(0, \infty; H^{-3}(\mathbb{R})).$$

Moreover, we have:

$$\|\rho(t)\|_{L^1} \le \|\rho_0\|_{L^1}, \quad \|\rho(t)\|_{H^{1/2}} \le e^{\gamma t}\|\rho_0\|_{H^{1/2}}, \quad t > 0. \tag{1.10}$$

The reason to choose the viscous-splitting algorithm is simple because both fractional heat equation and Eq. (1.2) yield global analytic solutions preserving positivity and norms of $L^1(\mathbb{R})$ and $H^{1/2}(\mathbb{R})$ for positive initial data. Hence, we only need to use some compactness argument to derive global weak solutions (see Sec. II). In Ref. 1, global weak solutions to the following general models were studied:

$$\partial_t\rho + \partial_x\rho H\rho + \delta\rho\partial_x H\rho = -\nu\Lambda^\alpha\rho. \tag{1.11}$$

When $\delta = 1$, the above equation becomes Eq. (1.1) when $\gamma = 0$. For different ranges of α and δ, they obtained several results about global weak solutions to Eq. (1.11). Among these results (Ref. 1, Theorem 1.1) shares some similarities with Theorem 1.1 in this paper. For strictly positive initial data $\rho_0 \in L^1(\mathbb{R})\cap H^{1/2}(\mathbb{R})$, global weak solutions were obtained in [Ref. 1, Theorem 1.1] for supercritical case $0<\alpha<1$ and $\delta\geq 1/2$. In comparison, we do not need the strictly positive assumption for initial data and weak solutions are obtained for all $0\leq\alpha\leq 2$.

We will obtain spatial analytic solutions for subcritical and critical cases $1\leq\alpha\leq 2$ by different methods. For the subcritical case $1<\alpha\leq 2$, we only consider (1.1) with $\gamma = 0$. The method here cannot be applied directly for the case $\gamma > 0$ (see Remark 3.1). We have the following theorem:

Theorem 1.2. *Let $0\leq\rho_0\in L^{\frac{1}{\alpha-1}}(\mathbb{R})$. Then, there is a unique nonnegative solution $\rho\in C([0,\infty);L^{\frac{1}{\alpha-1}}(\mathbb{R}))\cap C^\infty((0,\infty);H^{\theta,q}(\mathbb{R}))$ to (1.1) with $\gamma = 0$ for any $\theta\geq 0$ and $\frac{1}{\alpha-1}\leq q\leq\infty$. Moreover, we have*

$$\|\rho(t)\|_{\dot H^{\theta,q}(\mathbb{R})}\leq Ct^{-\frac{\theta}{\alpha}-1+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)},\quad \frac{1}{\alpha-1}\leq q\leq\infty,\quad t>0,$$

and

$$\|\partial_x^n\rho(t)\|_{L^q}\leq K^n n^n t^{-\frac{n}{\alpha}-1+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)},\quad \forall n\in\mathbb{N},\quad t>0,$$

for some constant K independent of n. Consequently, $\rho(\cdot,t)$ is spatially analytic for $t>0$.

Here $H^{\theta,q}$ denotes Bessel potential space (or fractional Sobolev spaces, see Sec. III). Our strategy to prove Theorem 1.2 is as follows. We consider the mild solutions to (1.1) $(1<\alpha\leq 2)$ of the form:

$$\rho(x,t) = G_\alpha(\cdot,t)*\rho_0 - \int_0^t\partial_x G_\alpha(\cdot,t-s)*(\rho(s)H\rho(s))\mathrm{d}s. \tag{1.12}$$

Notice that if $\rho(x,t)$ is a solution to (1.1) with initial data ρ_0, then $\rho_\lambda(x,t) = \lambda^{\alpha-1}\rho(\lambda x,\lambda^\alpha t)$ is also a solution with initial data $\rho_{\lambda,0}(x) = \lambda^{\alpha-1}\rho_0(\lambda x)$. This scaling preserves the $L^{\frac{1}{\alpha-1}}(\mathbb{R})$ norm. It is nature to study mild solutions to (1.1) with initial data $\rho_0\in L^{\frac{1}{\alpha-1}}(\mathbb{R})$. Next, we describe the results for the subcritical case $1<\alpha\leq 2$ in several steps.

Local existence and uniqueness: When $\rho_0\in L^{\frac{1}{\alpha-1}}(\mathbb{R})$, we use Banach fixed point theorem to prove the local existence and uniqueness of mild solutions in the following Banach space (see Theorem 3.1):

$$X_T := \left\{f\in C_b\Big((0,T];L^{\frac{1}{\alpha-1}}(\mathbb{R})\Big),\quad \sup_{0<t\leq T} t^{\frac{\alpha-1}{2\alpha}}\|f(t)\|_{L^{\frac{2}{\alpha-1}}}<\infty\right\}, \tag{1.13}$$

with norm

$$\|f\|_{X_T} := \max\left\{\sup_{0<t\leq T}\|f(t)\|_{L^{\frac{1}{\alpha-1}}},\quad \sup_{0<t\leq T} t^{\frac{\alpha-1}{2\alpha}}\|f(t)\|_{L^{\frac{2}{\alpha-1}}}\right\}.$$

The idea to choose the above space for the contraction argument is well-known. One can refer to Refs. 4, 20, and 34 for some variations of this method for the local existence of solutions to different equations.

Fractional bootstrapping for regularity in Bessel potential spaces: We improve the spatial regularity of solution ρ by a fractional bootstrapping method and obtain time decay estimate in Bessel potential spaces (see Theorem 3.2). Here, we adopt the name "fractional bootstrapping" used in Ref. 12 for fractional Navier–Stokes equations, although the proof of high order regularity and spacial analyticity are different. We first show the hyper-contractivity and prove that $\rho(t)\in L^q(\mathbb{R})$ for any $\frac{1}{\alpha-1}\leq q\leq\infty$ and $t>0$. In comparison with the usual method for hyper-contractivity (see e.g., Refs. 2, 11, 23, and 25), the proof is more direct in the sense that we do not need any kind of *a priori* estimate or contraction argument. Then, we improve the spatial regularity of the mild solutions step by step. From the Proof of Theorem 3.2, we see that in each step, the time integral in the nonlinear term of mild solutions only allows us to increase spatial regularity by some decimal order $0<\ell<\alpha-1\leq 1$. Hence, Bessel potential spaces are natural choices for this method and this is also the reason for the name fractional bootstrapping.

Nonnegativity: For the nonnegativity of mild solutions, we follow the method in Ref. 23, Lemma 2.7, but without the condition $0\leq\rho_0\in L^{\frac{1}{\alpha-1}}(\mathbb{R})\cap L^p(\mathbb{R})$ for some $\frac{1}{\alpha-1}<p<\infty$. In other words, we only need $0\leq\rho_0\in L^{\frac{1}{\alpha-1}}(\mathbb{R})$, which is more compatible with the results for existence and regularity.

Spatial analyticity and global extension: To prove the spatial analyticity of mild solutions, we are going to give a simple generalization of the method in Refs. 16 and 31 for Navier–Stokes equations. As noticed in Ref. 11, Remark 7 (or Ref. 12, Remark 2.4), if we directly use the method in Refs. 16 and 31, we will only obtain

$$|\partial_x^n \rho(x,t)| \le K^{n+1} n^{2n/\alpha} t^{-\frac{n}{\alpha}-1+\frac{1}{\alpha}}$$

for some constant K independent of $n \in \mathbb{N}$. This does not imply the spatial analyticity of ρ if $\alpha < 2$. To overcome this difficulty, we are going to improve the method in Refs. 16 and 31 and use it for fractional diffusion with $1 < \alpha \le 2$ (see Theorem 3.3). Then, by the $L^p(\mathbb{R})$ maximum principle for the nonnegative solutions, we extend the solutions globally (see Lemma 3.3 and Theorem 3.4).

Notice that there is another way to prove spatial analyticity of solutions given by Dong and Li,[11] where some spaces involving the information of high-order derivatives were introduced for contraction argument to obtain spatial analytic solutions to the subcritical dissipative quasi-geostrophic equations. By the same method, Li and Rodrigo[23] studied local existence and finite time blow-up behavior of solutions for the following equation with $1 < \alpha \le 2$:

$$\begin{cases} \partial_t \rho - \partial_x(\rho H\rho) = -\nu\Lambda^\alpha \rho, & t > 0, \quad x \in \mathbb{R}, \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.14}$$

Comparing with Eq. (1.1), the sign of the nonlinear term is different. Studying the nonnegative solutions of (1.1) is equivalent to studying the non-positive solutions of (1.14). For nonnegative solutions of (1.1), we have the $L^p(\mathbb{R})$ maximum principle to extend the mild solutions globally, which is false for nonnegative solutions of (1.14). And finite time blow-up behavior of solutions to (1.14) with some special initial data was proved by [Ref. 23, Theorem 3.1]. The reason for this difference can be easily observed from the particle systems for these two equations. Formally, Eq. (1.14) with $\nu = 0$ corresponds to the mean field equation for the following particle system:

$$\mathrm{d}\lambda_j(t) = \frac{1}{\sqrt{N}}\mathrm{d}B_j(t) - \frac{1}{\pi N}\sum_{k\neq j}\frac{\mathrm{d}t}{\lambda_j(t)-\lambda_k(t)}, \quad 1 \le j \le N. \tag{1.15}$$

The force between particles is attractive. Hence, they try to aggregate together to form singularities. The force between particles in (1.3) is repulsive, and global well-posedness can be obtained; see Ref. 30 for global well-posedness of system (1.3).

For the critical case $\alpha = 1$ of (1.1), we will also prove the global existence and uniqueness of spatial analytic solutions. Due to the following relation

$$\Lambda\rho = \partial_x H\rho = H\partial_x\rho = \frac{1}{\pi}p.v.\int_{\mathbb{R}} \frac{\rho(x,t)-\rho(y,t)}{|x-y|^2}\mathrm{d}y = \partial_x u,$$

Equation (1.1) is rewritten as

$$\partial_t \rho + \partial_x[\rho(u - \gamma x) + \nu u] = 0. \tag{1.16}$$

When $\gamma = 0$, global spatial analytic solutions to (1.16) were given by [Ref. 9, Theorem 4.1] for initial data $\rho_0 > -\nu$ using the same method as the case for $\nu = 0$ (described before). If $\rho_0 \ge -\nu$ and there exists $x_0 \in \mathbb{R}$ such that $\rho_0(x_0) = \nu$, then $\partial_x H\rho$ will blow up in finite time (Ref. 9, Thoerem 4.8). Comparing with the cases $\nu = 0$ or $\alpha = 2$ of (1.1), the rescaling (1.4) does not work for Eq. (1.16) with $\nu > 0$. Hence, we can not derive the spatial analytic solutions to (1.16) with $\gamma > 0$ directly from the results (Ref. 9, Theorem 4.1) by rescaling (1.4). Moreover, Eq. (1.16) also does not have gradient flow structure as (1.7). In this paper, we will use a similar idea as [Ref. 9, Theorem 4.1] to obtain spatial analytic solutions to (1.16) with $\gamma > 0$, and the solutions show some different and interesting properties in comparison with the case $\gamma = 0$. We have the following theorem:

Theorem 1.3. *Let $0 \le \mu < \nu$ and $-\mu \le \rho_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ with $s > 1/2$. Denote $T = \frac{1}{\gamma}\ln\left(\frac{2\nu}{\mu} - 1\right)$. Then, there exists a unique spatial analytic solution $\rho(x,t)$ to (1.1) with $\gamma > 0$ and $\alpha = 1$ in $(0,T)$.*

When $\mu = 0$, the solution $\rho(x,t)$ exists globally and converges to the steady state given by semicircle law:

$$\lim_{t\to\infty}\rho(x,t) = \rho_\infty(x) := \frac{\sqrt{\sqrt{[\gamma^2x^2 - \nu^2 - 2\gamma]^2 + 4\gamma^2x^2\nu^2} - [\gamma^2x^2 - \nu^2 - 2\gamma]} - \sqrt{2}\nu}{\sqrt{2}\pi}. \tag{1.17}$$

As shown in the above theorem, we obtain global spatial analytic solutions to (1.16) when initial data $\rho_0 \ge 0$, and the solutions converge to steady state pointwise. However, if $\rho_0 \ge -\mu$ for some $0 \le \mu < \nu$, we can only obtain spatial analytic solutions in time interval $(0,T)$ with $T = \frac{1}{\gamma}\ln\left(\frac{2\nu}{\mu} - 1\right)$, which is different with the case $\gamma = 0$ given by [Ref. 9, Theorem 4.1].

The rest of this paper is organized as follows. We are going to use a viscous-splitting algorithm to prove Theorem 1.1 in the next section. Local existence and uniqueness of mild solutions to (1.1) with $1 < \alpha \le 2$ are obtained in Sec. III A. Then, we improve the regularity and show the spatial analyticity of solutions in Sec. III B. In Sec. III C we extend the local solution globally by the L^p maximum principle for nonnegative solutions. For the critical case $\alpha = 1$, we first obtain global $\overline{\mathbb{C}_+}$-holomorphic solutions to the corresponding Complex Burgers equation [see (4.5)] in Sec. IV A. Then, we use these $\overline{\mathbb{C}_+}$-holomorphic solutions to recover the solutions to (1.1) with $\alpha = 1$, and derive the pointwise convergence to the steady state when $\rho_0 \ge 0$.

II. GLOBAL NONNEGATIVE WEAK SOLUTIONS FOR $0 \le \alpha \le 2$

In this section, we are going to use a viscous-splitting algorithm to obtain global weak solutions in $L^1(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})$ to Eq. (1.1) with $0 \le \alpha \le 2$ and to prove Theorem 1.1. Note that we have interpolation inequality (see Ref. 32, Lemma 5.3 for instance)

$$\|\rho\|_{L^2} \le 3\|\rho\|_{L^1}^{1/2}\|\rho\|_{\dot{H}^{1/2}}^{1/2},$$

which implies $\rho \in L^1(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})$ is equivalent to $\rho \in L^1(\mathbb{R}) \cap H^{1/2}(\mathbb{R})$. Let us give the definition of weak solutions:

Definition 2.1. For $T > 0$ and $0 \le \rho_0 \in L^1(\mathbb{R}) \cap H^{1/2}(\mathbb{R})$, a nonnegative function $\rho \in L^\infty(0, T; L^1(\mathbb{R}) \cap H^{1/2}(\mathbb{R})) \cap W^{1,\infty}(0, T; H^{-m}(\mathbb{R}))$ for some $m > 0$ is said to be a weak solution of Eq. (1.1) if

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \partial_t \phi(x,t)\rho(x,t)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \phi(x,0)\rho_0(x)\mathrm{d}x \\ = -\frac{1}{2\pi}\int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\partial_x\phi(x,t) - \partial_x\phi(y,t)}{x-y}\rho(x,t)\rho(y,t)\,\mathrm{d}x\,\mathrm{d}y\mathrm{d}t \\ + \gamma\int_0^T\int_{\mathbb{R}} \partial_x\phi(x,t)(\rho(x,t)x)\mathrm{d}x\mathrm{d}t + \nu\int_0^T\int_{\mathbb{R}}\rho(x,t)\Lambda^\alpha\phi(x,t)\mathrm{d}x\mathrm{d}t \qquad (2.1)\end{aligned}$$

holds for any test function $\phi \in C_c^\infty(\mathbb{R} \times [0, T))$.

Next, we describe the viscous-splitting algorithm by utilizing a Trotter formula. Denote the solution operator to Dyson equation by $D(t)$, such that $\omega(x,t) = D(t)\rho_0(x)$ solves

$$\begin{cases} \partial_t\omega + \partial_x[\omega(H\omega - \gamma x)] = 0, & x \in \mathbb{R}, \quad t > 0, \\ \omega(x,0) = \rho_0(x). \end{cases} \qquad (2.2)$$

Also denote $G_\alpha(t)\omega_0 := G_\alpha(\cdot,t) * \omega_0$, so that $v(x,t) = G_\alpha(t)\omega_0(x)$ solves the fractional heat equation

$$\begin{cases} \partial_t v = -\nu\Lambda^\alpha v, & x \in \mathbb{R}, \quad t > 0, \\ v(x,0) = \omega_0(x). \end{cases} \qquad (2.3)$$

By the properties of fractional heat kernel G_α and some easy calculations, we have

$$\|G_\alpha(t)\omega_0\|_{L^1} = \|\omega_0\|_{L^1} \text{ and } \|G_\alpha(t)\omega_0\|_{H^{1/2}} \le \|\omega_0\|_{H^{1/2}}. \qquad (2.4)$$

Let $\varphi_h > 0$ $(h > 0)$ be the standard Friedrichs mollifier. Set

$$\rho_{0,h} = \rho_0 * \varphi_h. \qquad (2.5)$$

Then, for nontrival initial datum $0 \le \rho_0 \in L^1(\mathbb{R}) \cap H^{1/2}(\mathbb{R})$, we have $\rho_{0,h}(x) > 0$ for $x \in \mathbb{R}$ and $\rho_{0,h} \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ $(s > 1/2)$. Moreover, the following estimates

$$\|\rho_{0,h}\|_{L^1} = \|\rho_0\|_{L^1}, \quad \|\rho_{0,h}\|_{L^2} \le \|\rho_0\|_{L^2} \text{ and } \|\rho_{0,h}\|_{\dot{H}^{1/2}} \le \|\rho_0\|_{\dot{H}^{1/2}} \qquad (2.6)$$

hold for any $h > 0$.

The viscous-splitting algorithm by utilizing a Trotter formula is given by

$$\rho_{n,h}(x) = [G_\alpha(h)D(h)]^n\rho_{0,h}(x), \quad x \in \mathbb{R}, \qquad (2.7)$$

where $\rho_{n,h}$ is the approximate value of the solution at time $t_n := nh$ and h is the length of time step. One can also use the Strang's method; see Ref. 26, Chap. 3. Define

$$\tilde{\rho}_h(x,t) = D(s)\rho_{n,h}(x), \quad \rho_h(x,t) = G_\alpha(s)D(s)\rho_{n,h}(x) = G_\alpha(s)\tilde{\rho}_h(x,t) \qquad (2.8)$$

for $t = s + t_n$, $0 \le s \le h$, $n \in \mathbb{N}$. For $t \in (t_n, t_{n+1})$, we have

$$\partial_t\rho_h = -\nu\Lambda^\alpha\rho_h - G_\alpha(s)\partial_x[\tilde{\rho}_h(H\tilde{\rho}_h - \gamma x)]. \qquad (2.9)$$

Hence, for $\phi \in C_c^\infty(\mathbb{R}\times[0,T))$, we have

$$\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}}\rho_h\partial_t\phi\mathrm{d}x\mathrm{d}t-\int_{\mathbb{R}}\rho_h\phi\mathrm{d}x\Big|_{t_n}^{t_{n+1}}$$
$$=-\frac{1}{2\pi}\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\partial_x[G_\alpha(s)\phi(t)](x)-\partial_x[G_\alpha(s)\phi(t)](y)}{x-y}\tilde{\rho}_h(x,t)\tilde{\rho}_h(y,t)\,\mathrm{d}x\,\mathrm{d}y\mathrm{d}t$$
$$+\gamma\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}}\partial_x\phi(\tilde{\rho}_hx)\mathrm{d}x\mathrm{d}t+\nu\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}}\rho_h\Lambda^\alpha\phi\mathrm{d}x\mathrm{d}t. \tag{2.10}$$

Assume $T\in(t_{N_h-1},t_{N_h})$ for some positive integer N_h. Sum (2.10) for $n=0,1,\ldots,N_h$ together and we obtain

$$\int_0^T\int_{\mathbb{R}}\rho_h\partial_t\phi\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}}\rho_{0,h}\phi(x,0)\mathrm{d}x$$
$$=-\frac{1}{2\pi}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\partial_x[G_\alpha(s_t)\phi(t)](x)-\partial_x[G_\alpha(s_t)\phi(t)](y)}{x-y}\tilde{\rho}_h(x,t)\tilde{\rho}_h(y,t)\,\mathrm{d}x\,\mathrm{d}y\mathrm{d}t$$
$$+\gamma\int_0^T\int_{\mathbb{R}}\partial_x\phi(\tilde{\rho}_hx)\mathrm{d}x\mathrm{d}t+\nu\int_0^T\int_{\mathbb{R}}\rho_h\Lambda^\alpha\phi\mathrm{d}x\mathrm{d}t, \tag{2.11}$$

where $s_t=t-t_n$ for n satisfying $t\in(t_n,t_{n+1})$. Hence, $s_t\to 0$ as $h\to 0$.

To obtain the weak solutions to Eq. (1.1) with $0\le\alpha\le 2$ defined by (2.1) by passing h to 0, we first show some compactness results for $\{\tilde{\rho}_h\}_{h>0}$ and $\{\rho_h\}_{h>0}$.

Lemma 2.1. Assume $0\le\rho_0\in L^1(\mathbb{R})\cap H^{1/2}(\mathbb{R})$ and $M_0:=\int_{\mathbb{R}}x^2\rho_0(x)\mathrm{d}x<\infty$. Let $\rho_{0,h}$ be defined by (2.5) for $h>0$. Then, we have

$$\|\rho_{n,h}\|_{L^1}\equiv\|\rho_0\|_{L^1}\ \ \text{and}\ \ \|\rho_{n,h}\|_{H^{1/2}}\le e^{\gamma t}\|\rho_0\|_{H^{1/2}} \tag{2.12}$$

for any $n\in\mathbb{N}$, $h>0$ and $\rho_{n,h}$ given by (2.7).

Moreover, we have

$$\|\tilde{\rho}_h(t)\|_{L^1}\equiv\|\rho_0\|_{L^1},\quad \|\tilde{\rho}_h(t)\|_{H^{1/2}}\le e^{\gamma t}\|\rho_0\|_{H^{1/2}}, \tag{2.13}$$

$$\|\rho_h(t)\|_{L^1}\equiv\|\rho_0\|_{L^1},\quad \|\rho_h(t)\|_{H^{1/2}}\le e^{\gamma t}\|\rho_0\|_{H^{1/2}}, \tag{2.14}$$

and

$$\|\partial_t\tilde{\rho}_h\|_{L^\infty(0,\infty;H^{-3})}\le C,\quad \|\partial_t\rho_h\|_{L^\infty(0,\infty;H^{-3})}\le C \tag{2.15}$$

where $\tilde{\rho}_h$ and ρ_h are given by (2.8) and constant C is independent of h.

Proof. The proof of this lemma shares some similarity with the proof of [Ref. 15, Theorem 2.2]. We provide the details here for completeness.

Step 1: In this step, we prove (2.12)–(2.14). Denote $\omega_{0,h}(x,t):=D(t)\rho_{0,h}(x)$. Then $w_{0,h}(x,t)$ satisfies (2.2), i.e.,

$$\begin{cases}\partial_t\omega_{0,h}+\partial_x[\omega_{0,h}(H\omega_{0,h}-\gamma x)]=0, & x\in\mathbb{R},\quad t>0,\\ \omega_{0,h}(x,0)=\rho_{0,h}(x).\end{cases} \tag{2.16}$$

Because $0<\rho_{0,h}\in L^1(\mathbb{R})\cap H^s(\mathbb{R})$ $(s>1/2)$, $\omega_{0,h}(x,t)>0$ for all $x\in\mathbb{R}$ and $t>0$; see [Ref. 15, Theorem 2.1]. Direct calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_{0,h}(t)\|_{L^1}=\int_{\mathbb{R}}\partial_t\omega_{0,h}(t)\mathrm{d}x=-\int_{\mathbb{R}}\partial_x[\omega_{0,h}(t)(H\omega_{0,h}(t)-\gamma x)]\mathrm{d}x=0.$$

Hence, we have $\|D(t)\rho_{0,h}\|_{L^1}=\|\omega_{0,h}(t)\|_{L^1}=\|\rho_{0,h}\|_{L^1}=\|\rho_0\|_{L^1}$. This together with (2.4) gives

$$\|G_\alpha(h)D(h)\rho_{0,h}\|_{L^1}=\|\rho_0\|_{L^1},$$

which implies the first equality of (2.12) for $n=1$. By the definition of $\rho_{n,h}$, we know that the first equality of (2.12) holds for any $n\in\mathbb{N}$.

Next, we prove the second inequality of (2.12). On the one hand, multiplying (2.16) by $\omega_{0,h}$ and integration by parts implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{|\omega_{0,h}(t)|^2}{2}\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}}|\omega_{0,h}(t)|^2\partial_x(H\omega_{0,h}(t)-\gamma x)\mathrm{d}x=0.$$

By the definition of Hilbert transform, we have

$$\int_{\mathbb{R}}|\omega_{0,h}(t)|^2\partial_x H\omega_{0,h}(t)\mathrm{d}x = \frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\omega_{0,h}(x,t)\frac{|\omega_{0,h}(x,t)-\omega_{0,h}(y,t)|^2}{|x-y|^2}\mathrm{d}y\mathrm{d}x > 0.$$

Hence, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|\omega_{0,h}(t)|^2\mathrm{d}x \le \frac{\gamma}{2}\int_{\mathbb{R}}|\omega_{0,h}(t)|^2\mathrm{d}x,$$

by Grönwall inequality, we have

$$\|D(h)\rho_{0,h}\|_{L^2} = \|\omega_{0,h}(h)\|_{L^2} \le e^{\frac{\gamma}{2}t}\|\rho_{0,h}\|_{L^2} \le e^{\frac{\gamma}{2}t}\|\rho_0\|_{L^2}. \tag{2.17}$$

On the other hand, multiplying (2.16) by $\partial_x H\omega_{0,h}(t)$ and integration by parts to obtain

$$\int_{\mathbb{R}}\partial_x H\omega_{0,h}\partial_t\omega_{0,h}\,\mathrm{d}x + \int_{\mathbb{R}}\partial_x\omega_{0,h}\partial_x H\omega_{0,h}H\omega_{0,h}\,\mathrm{d}x + \int_{\mathbb{R}}(\partial_x H\omega_{0,h})^2\omega_{0,h}\,\mathrm{d}x$$
$$-\gamma\int_{\mathbb{R}}\omega_{0,h}\partial_x H\omega_{0,h}(t)\mathrm{d}x - \gamma\int_{\mathbb{R}}\partial_x\omega_{0,h}x\partial_x H\omega_{0,h}(t)\mathrm{d}x = 0. \tag{2.18}$$

For the first term of (2.18), we have

$$\int_{\mathbb{R}}\partial_x H\omega_{0,h}\partial_t\omega_{0,h}\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\omega_{0,h}\partial_x H\omega_{0,h}\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^{\frac{1}{2}}\omega_{0,h}\right\|_{L^2}^2. \tag{2.19}$$

For the second term of (2.18), we have

$$\begin{aligned}\int_{\mathbb{R}}\partial_x\omega_{0,h}\partial_x H\omega_{0,h}H\omega_{0,h}\,\mathrm{d}x &= -\int_{\mathbb{R}}H(\partial_x\omega_{0,h}\partial_x H\omega_{0,h})\omega_{0,h}\,\mathrm{d}x\\ &= -\frac{1}{2}\int_{\mathbb{R}}\left[(\partial_x H\omega_{0,h})^2 - (\partial_x\omega_{0,h})^2\right]\omega_{0,h}\,\mathrm{d}x.\end{aligned} \tag{2.20}$$

For the fourth term of (2.18), we have

$$-\gamma\int_{\mathbb{R}}\omega_{0,h}\partial_x H\omega_{0,h}(t)\mathrm{d}x = -\gamma\int_{\mathbb{R}}\omega_{0,h}\Lambda\omega_{0,h}\,\mathrm{d}x = -\gamma\left\|\Lambda^{\frac{1}{2}}\omega_{0,h}\right\|_{L^2}^2. \tag{2.21}$$

For the fifth term of (2.18), we can use the fact $H(x\partial_x\omega_{0,h}) = xH(\partial_x\omega_{0,h})$ [see (4.1) below] to obtain

$$-\gamma\int_{\mathbb{R}}\partial_x\omega_{0,h}x\partial_x H\omega_{0,h}(t)\mathrm{d}x = \gamma\int_{\mathbb{R}}H(x\partial_x\omega_{0,h})\partial_x\omega_{0,h}\,\mathrm{d}x = \gamma\int_{\mathbb{R}}\partial_x\omega_{0,h}x\partial_x H\omega_{0,h}(t)\mathrm{d}x,$$

which implies

$$\gamma\int_{\mathbb{R}}\partial_x\omega_{0,h}x\partial_x H\omega_{0,h}(t)\mathrm{d}x = 0. \tag{2.22}$$

As a consequence of (2.18)–(2.22) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^{\frac{1}{2}}\omega_{0,h}\right\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}}\left[(\partial_x H\omega_{0,h})^2 - (\partial_x\omega_{0,h})^2\right]\omega_{0,h}\,\mathrm{d}x$$
$$+\int_{\mathbb{R}}(\partial_x H\omega_{0,h})^2\omega_{0,h}\,\mathrm{d}x - \gamma\left\|\Lambda^{\frac{1}{2}}\omega_{0,h}\right\|_{L^2}^2 = 0,$$

which shows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^{\frac{1}{2}}\omega_{0,h}\right\|_{L^2}^2 \le \gamma\left\|\Lambda^{\frac{1}{2}}\omega_{0,h}\right\|_{L^2}^2.$$

By Grönwall inequality and (2.6), we have

$$\|D(h)\rho_{0,h}\|_{\dot{H}^{\frac{1}{2}}} = \|\omega_{0,h}(h)\|_{\dot{H}^{\frac{1}{2}}} = \|\Lambda^{\frac{1}{2}}\omega_{0,h}(h)\|_{L^2} \le e^{\gamma t}\|\rho_{0,h}\|_{\dot{H}^{\frac{1}{2}}} \le e^{\gamma t}\|\rho_0\|_{\dot{H}^{\frac{1}{2}}}.$$

Combining the above inequality, (2.17) and (2.4) implies the second inequality of (2.12) for $n = 1$. By the definition of $\rho_{n,h}$, we know that the second inequality of (2.12) holds for any $n \in \mathbb{N}$. For the same reason, we also have (2.13) and (2.14).

Step 2: In this step, we prove (2.15). For any $s \in [0, h]$, we define

$$M_n(s) = \int_{\mathbb{R}} x^2 \tilde{\rho}_h(x, s + t_n) \mathrm{d}x.$$

Since $\tilde{\rho}_h(x,t)$ satisfies (2.2) with initial data $\rho_{n,h}(x)$ for $t \in (t_n, t_{n+1})$, direct calculation shows

$$M_n(s) = \begin{cases} \dfrac{\|\rho_0\|_{L^1}^2}{2\gamma\pi} - \dfrac{\|\rho_0\|_{L^1}^2 - 2\gamma\pi M_n(0)}{2\gamma\pi} e^{-2\gamma s}, & \gamma > 0, \\ M_n(0) + \dfrac{1}{\pi}\|\rho_0\|_{L^1}^2 s, & \gamma = 0. \end{cases} \tag{2.23}$$

Notice that $M_{n-1}(h) = M_n(0)$. Set $M(t) = \int_{\mathbb{R}} x^2 \tilde{\rho}_h(x,t)\mathrm{d}x$, $t > 0$. Then, for any $t > 0$, using (2.23) and iteration yields

$$M(t) = \begin{cases} \dfrac{\|\rho_0\|_{L^1}^2}{2\gamma\pi} - \dfrac{\|\rho_0\|_{L^1}^2 - 2\gamma\pi M_0}{2\gamma\pi} e^{-2\gamma t} \le \dfrac{\|\rho_0\|_{L^1}^2}{2\gamma\pi} + M_0, & \gamma > 0, \\ M_0 + \dfrac{1}{\pi}\|\rho_0\|_{L^1}^2 t, & \gamma = 0. \end{cases} \tag{2.24}$$

Let $\phi \in C_c^\infty(\mathbb{R})$. The following estimate holds for any n and $t \in (t_n, t_{n+1})$:

$$\begin{aligned} &\int_{\mathbb{R}} \phi(x)\partial_t\tilde{\rho}_h(x,t)\mathrm{d}x \\ &= -\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\partial_x\phi(x) - \partial_x\phi(y)}{x-y}\tilde{\rho}_h(x,t)\tilde{\rho}_h(y,t)\,\mathrm{d}x\,\mathrm{d}y + \gamma\int_{\mathbb{R}}\partial_x\phi(x)(x\tilde{\rho}_h)\mathrm{d}x \\ &\le \frac{1}{2\pi}\|\rho_0\|_{L^1}^2\|\partial_{xx}\phi\|_{L^\infty} + \frac{\gamma}{2}\big(M(t) + \|\tilde{\rho}_h\|_{L^1}\big)\|\partial_x\phi(x)\|_{L^\infty} \le C\|\phi\|_{H^3}, \end{aligned} \tag{2.25}$$

which implies $\|\partial_t\tilde{\rho}_h\|_{H^{-3}} \le C$ for some constant $C > 0$ independent of h. When $\gamma = 0$ the above constant C is obviously independent of t. When $\gamma > 0$, from (2.24) we can also tell that C is independent of t.

For the estimate of $\partial_t\rho_h$, we use Eqs. (2.9) and (2.25) to obtain

$$\begin{aligned} \int_{\mathbb{R}} \phi(x)\partial_t\rho_h(x,t)\mathrm{d}x &= -\nu\int_{\mathbb{R}}\phi(x)\Lambda^\alpha\rho_h(x,t)\mathrm{d}x - \int_{\mathbb{R}} G_\alpha(t-t_n)\partial_x[\tilde{\rho}_h(H\tilde{\rho}_h - \gamma x)]\phi(x)\mathrm{d}x \\ &\le \nu\|\Lambda^\alpha\phi\|_{L^\infty}\|\rho_h\|_{L^1} + \int_{\mathbb{R}}[G_\alpha(t-t_n) * \phi_x]\tilde{\rho}_h(H\tilde{\rho}_h - \gamma x)\mathrm{d}x \le C\|\phi\|_{H^3} \end{aligned}$$

for any $t \in (t_n, t_{n+1})$ and some constant $C > 0$ independent of h or t. Here the last step is estimated by the same manner as (2.25) and the fact that $\|G_\alpha(t-t_n) * \phi\|_{H^3} \le \|\phi\|_{H^3}$. The above estimate shows the second inequality of (2.15). □

Remark 2.1. Notice that the second momentum assumption $\int_{\mathbb{R}} x^2\rho_0(x)\mathrm{d}x < \infty$ is only used in the estimate of $\partial_t\tilde{\rho}_h$ (or $\partial_t\rho_h$) [see (2.25)], which is not necessary for the case $\gamma = 0$.

With the uniform estimates in Lemma 2.1, we prove Theorem 1.1.

Proof of Theorem 1.1. From Lemma 2.1, there exist subsequences of $\{\tilde{\rho}_h\}_{h>0}$ and $\{\rho_h\}_{h>0}$ (still denoted as $\{\tilde{\rho}_h\}_{h>0}$ and $\{\rho_h\}_{h>0}$) such that they converge to functions $\tilde{\rho}, \rho \in L^\infty(0,T;H^{1/2}(\mathbb{R})) \cap W^{1,\infty}(0,T;H^{-3}(\mathbb{R}))$ respectively:

$$\tilde{\rho}_h \xrightarrow{*} \tilde{\rho}, \quad \rho_h \xrightarrow{*} \rho \text{ in } L^\infty(0,T;H^{1/2}(\mathbb{R})) \text{ as } h \to 0,$$

and

$$\partial_t\tilde{\rho}_h \xrightarrow{*} \partial_t\tilde{\rho}, \quad \partial_t\rho_h \xrightarrow{*} \partial_t\rho \text{ in } L^\infty(0,T;H^{-3}(\mathbb{R})) \text{ as } h \to 0.$$

Combining Lemma 2.1 and Aubin-Lions lemma, we also know

$$\tilde{\rho}_h \to \tilde{\rho}, \quad \rho_h \to \rho \text{ in } L^\infty(0,T;L^2_{loc}(\mathbb{R})) \text{ as } h \to 0,$$

and as a consequence of Hölder inequality on compact set, we have

$$\tilde{\rho}_h \to \tilde{\rho}, \quad \rho_h \to \rho \text{ in } L^\infty(0,T;L^1_{loc}(\mathbb{R})) \text{ as } h \to 0. \tag{2.26}$$

Let $h > 0$ and $t \in (t_n, t_{n+1})$. From (2.8), we have

$$\|\rho_h(t) - \tilde{\rho}_h(t)\|_{L^1} = \|G_\alpha(t - t_n)\tilde{\rho}_h(t) - \tilde{\rho}_h(t)\|_{L^1} \to 0 \text{ as } h \to 0.$$

The proof of the above convergence result is the same as the estimate (B2) in Appendix B. Hence, we have

$$\rho(x,t) = \tilde{\rho}(x,t), \quad a.e.\ x \in \mathbb{R}, \tag{2.27}$$

for all $t \in (0,T)$. Notice that $\partial_x[G_\alpha(s_t)\phi(t)](x) \to \partial_x\phi(x,t)$ as $h \to 0$ for any $x \in \mathbb{R}$ and $t \in [0,T]$. By the strong convergence of $\tilde{\rho}_h$, ρ_h in (2.26), we can take the limit as $h \to 0$ in (2.11) and conclude that ρ satisfies (2.1). Hence, ρ is a global weak solution to (1.1). As a consequence of (2.26) and (2.27), we have (1.10). □

Remark 2.2. Notice that we do not have the uniqueness of weak solutions. For $\alpha = 0, 2$, one can use the convexity along Wasserstein geodesics of the energy (1.6) to show uniqueness (see Ref. 8).

Also notice that we cannot obtain spatial analyticity even for solutions to the corresponding linear equations when $0 < \alpha < 1$, which can be verified by the estimate (3.4) for fractional heat kernel in the next section.

III. GLOBAL SPATIAL ANALYTIC SOLUTIONS FOR THE SUBCRITICAL CASE $1 < \alpha \le 2$

In this section, we are going to obtain global spatial analytic solutions to (1.1) with $\gamma = 0$, $1 < \alpha \le 2$, and initial data $\rho_0 \in L^{\frac{1}{\alpha-1}}(\mathbb{R})$. For the case $\gamma > 0$, the method used here is not applicable (see Remark 3.1). First, let us introduce Bessel potential spaces and give some properties of fractional heat kernel G_α. For more details about Bessel potential spaces, one can refer to Ref. 18, Chap. 6.

The Bessel potential $(I - \partial_{xx})^{\ell/2}$ and Riesz potential $\Lambda^\ell := (-\partial_{xx})^{\ell/2}$ for $\ell \in \mathbb{R}$ are defined via the Fourier transform:

$$[\mathcal{F}((I-\partial_{xx})^{\ell/2}\rho)](\xi,t) = (1+|\xi|^2)^{\ell/2}[\mathcal{F}(\rho)](\xi,t), \quad [\mathcal{F}(\Lambda^\ell\rho)](\xi,t) = |\xi|^\ell[\mathcal{F}(\rho)](\xi,t).$$

For $\ell \ge 0$ and $1 \le q \le \infty$, the Bessel potential spaces are defined by

$$H^{\ell,q}(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}),\ \ (I-\partial_{xx})^{\ell/2} f \in L^q(\mathbb{R})\} = \{f \in L^q(\mathbb{R}),\ \ \Lambda^\ell f \in L^q(\mathbb{R})\},$$

where $\mathcal{S}'(\mathbb{R})$ stands for the space of tempered distributions. When ℓ is a positive integer and $1 < q < \infty$, $H^{\ell,q}(\mathbb{R})$ coincides with the usual Sobolev space $W^{\ell,q}(\mathbb{R})$. For $f \in H^{\ell,q}(\mathbb{R})$, the homogeneous semi-norm is given by

$$\|f\|_{\dot{H}^{\ell,q}} = \|\Lambda^\ell f\|_{L^q}.$$

Let us recall the following useful L^p $(1 < p < \infty)$ bound for the Hilbert transform (see Ref. 29 or Ref. 17, Remark 5.1.8):

$$\|Hf\|_{L^p} \le C_p\|f\|_{L^p}, \quad \forall f \in L^p, \tag{3.1}$$

where the best constant C_p is given by

$$C_p := \begin{cases} \tan\dfrac{\pi}{2p}, & 1 < p \le 2; \\ \cot\dfrac{\pi}{2p}, & 2 \le p < \infty. \end{cases} \tag{3.2}$$

In the rest of this paper, for $\ell \ge 0$ we denote

$$C_{\ell,\alpha} := \max\left\{\sup_{1\le p\le\infty}\|\Lambda^\ell G_\alpha(\cdot,1)\|_{L^p},\ \sup_{1\le p\le\infty}\|\Lambda^\ell\partial_x G_\alpha(\cdot,1)\|_{L^p}\right\}. \tag{3.3}$$

According to [Ref. 27, Lemma 2.2], we have

$$|\Lambda^\ell G_\alpha(\cdot,1)| \le C(1+|x|)^{-1-\ell}, \quad |\Lambda^\ell\partial_x G_\alpha(\cdot,1)| \le C(1+|x|)^{-2-\ell},$$

which implies $C_{\ell,\alpha} < \infty$. Later on we will use $C_{0,\alpha}$ and $C_{1,\alpha}$ for $\ell = 0$ and $\ell = 1$ separately. Moreover, since $\Lambda \sim \partial_x$, we also use $C_{1,\alpha}$ as the upper bound for $\sup_{1\le p\le\infty}\|\partial_x^2 G_\alpha(\cdot,1)\|_{L^p}$.

We have the following useful estimates for fractional heat kernel. The proofs for similar estimates can be found in other papers (e.g., Refs. 7 and 11). We put the proof in Appendix A.

Lemma 3.1. Let $f \in L^p(\mathbb{R})$ *for* $p \geq 1$. *Assume* $k \in \mathbb{N}$, $\ell \geq 0$ *and* $1 \leq p \leq q \leq +\infty$. *We have the following estimates:*

$$\|\Lambda^{k\ell} G_\alpha(\cdot,t)\|_{L^q} \leq C^k_{\ell,\alpha} k^{\frac{k\ell+1}{\alpha}-\frac{1}{\alpha q}} t^{-\frac{k\ell+1}{\alpha}+\frac{1}{\alpha q}}, \tag{3.4}$$

$$\|\Lambda^{k\ell} \partial_x G_\alpha(\cdot,t)\|_{L^q} \leq C^k_{\ell,\alpha} k^{\frac{k\ell+2}{\alpha}-\frac{1}{\alpha q}} t^{-\frac{k\ell+2}{\alpha}+\frac{1}{\alpha q}}, \tag{3.5}$$

$$\|[\Lambda^{k\ell} G_\alpha(\cdot,t)] * f\|_{L^q} \leq C^k_{\ell,\alpha} k^{\frac{k\ell}{\alpha}+\frac{1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} t^{-\frac{k\ell}{\alpha}+\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{L^p}, \tag{3.6}$$

$$\|[\Lambda^{k\ell} \partial_x G_\alpha(\cdot,t)] * f\|_{L^q} \leq C^k_{\ell,\alpha} k^{\frac{k\ell+1}{\alpha}+\frac{1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} t^{-\frac{k\ell+1}{\alpha}+\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{L^p}, \tag{3.7}$$

and

$$\lim_{t\to 0} t^{\frac{1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} \|G_\alpha(\cdot,t) * f\|_{L^q} = 0, \quad \forall q > p. \tag{3.8}$$

Moreover, the estimates for $\|G_\alpha(\cdot,t)\|_{L^q}$, $\|\partial_x G_\alpha(\cdot,t)\|_{L^q}$, $\|G_\alpha(\cdot,t) * f\|_{L^q}$, *and* $\|\partial_x G_\alpha(\cdot,t) * f\|_{L^q}$ *can be obtained by setting* $\ell = 0$ *and* $k = 1$ *in the above estimates.*

In the following of this section, we show the local well-posedness, spatial analyticity and global existence of the solution to Eq. (1.1).

A. Local existence and uniqueness of mild solutions

Next, we are going to prove the local existence and uniqueness of mild solutions to (1.1) of the form (1.12) in space X_T defined by (1.13). For $1 < \alpha \leq 2$, assume the initial data $\rho_0 \in L^{\frac{1}{\alpha-1}}(\mathbb{R})$. Take $q = \frac{2}{\alpha-1}$ in (3.8) and we have

$$\lim_{t\to 0} t^{\frac{\alpha-1}{2\alpha}} \|G_\alpha(\cdot,t) * \rho_0\|_{L^{\frac{2}{\alpha-1}}} = 0. \tag{3.9}$$

Define the operator S:

$$(S\rho)(x,t) := G_\alpha(\cdot,t) * \rho_0 - \int_0^t \partial_x G_\alpha(\cdot,t-s) * (\rho(s) H\rho(s)) \mathrm{d}s.$$

We have the following theorem:

Theorem 3.1. *Let* $0 \leq \rho_0 \in L^{\frac{1}{\alpha-1}}(\mathbb{R})$. *For* $a > 0$ *small enough and* $T > 0$ *satisfying*

$$\sup_{0<t\leq T} t^{\frac{\alpha-1}{2\alpha}} \|G_\alpha(\cdot,t) * \rho_0\|_{L^{\frac{2}{\alpha-1}}} \leq a, \tag{3.10}$$

there exists a unique mild solution ρ *to (1.1) in the following subset of* X_T:

$$X_T^a := \left\{ f \in X_T : \sup_{0<t\leq T} \|f(t)\|_{L^{\frac{1}{\alpha-1}}} \leq 2\|\rho_0\|_{L^{\frac{1}{\alpha-1}}}, \quad \sup_{0<t\leq T} t^{\frac{\alpha-1}{2\alpha}} \|f(t)\|_{L^{\frac{2}{\alpha-1}}} \leq 2a \right\}.$$

Moreover, we have $\rho \in C([0,T]; L^{\frac{1}{\alpha-1}}(\mathbb{R}))$ *and* $\rho(x,0) = \rho_0(x)$, $x \in \mathbb{R}$.

There are different versions of the local existence of mild solutions for different equations, and the main ideas are similar; see, e.g., Refs. 4, 5, and 7. Since some estimates in the proof of this theorem are useful in the rest of this paper, we are going to provide a complete proof here.

Proof. To show the existence, we only need to prove that $S : X_T^a \to X_T^a$ is a contraction mapping for a small enough.

Step 1: Assume $\rho \in X_T^a$ and we are going to show that $S\rho \in X_T^a$ for a small enough.

Estimate of $\|S\rho(t)\|_{L^{\frac{1}{\alpha-1}}}$: From (3.7) with $\ell = 0$ and $q = p = \frac{1}{\alpha-1}$, by the M. Riesz theorem for L^p $(1 < p < \infty)$ boundedness of Hilbert transform we obtain

$$\begin{aligned} \|\partial_x G_\alpha(\cdot,t-s) * (\rho(s) H\rho(s))\|_{L^{\frac{1}{\alpha-1}}} &\leq C(t-s)^{-\frac{1}{\alpha}} \|\rho(s) H\rho(s)\|_{L^{\frac{1}{\alpha-1}}} \\ &\leq C(t-s)^{-\frac{1}{\alpha}} \|\rho(s)\|^2_{L^{\frac{2}{\alpha-1}}} \\ &\leq C(t-s)^{-\frac{1}{\alpha}} s^{-\frac{\alpha-1}{\alpha}} \sup_{0<t\leq T} s^{\frac{\alpha-1}{\alpha}} \|\rho(s)\|^2_{L^{\frac{2}{\alpha-1}}}, \end{aligned}$$

which implies

$$\int_0^t \|\partial_x G_\alpha(\cdot, t-s) * (\rho(s)H\rho(s))\|_{L^{\frac{1}{\alpha-1}}} \,\mathrm{d}s \le Ca^2 \int_0^t (t-s)^{-\frac{1}{\alpha}} s^{-\frac{\alpha-1}{\alpha}} \,\mathrm{d}s = C\mathcal{B}\Big(\frac{1}{\alpha}, \frac{\alpha-1}{\alpha}\Big)a^2, \tag{3.11}$$

where $\mathcal{B}(\frac{1}{\alpha}, \frac{\alpha-1}{\alpha})$ is a Beta function $\mathcal{B}(\beta_1, \beta_2) = \int_0^1 s^{\beta_1-1}(1-s)^{\beta_2-1}\mathrm{d}s$ with $\beta_1 = 1/\alpha$ and $\beta_2 = \frac{\alpha-1}{\alpha}$. For a small enough, we have

$$\|S\rho(t)\|_{L^{\frac{1}{\alpha-1}}} \le \|G_\alpha(\cdot, t) * \rho_0\|_{L^{\frac{1}{\alpha-1}}} + C\mathcal{B}\Big(\frac{1}{\alpha}, \frac{\alpha-1}{\alpha}\Big)a^2 < 2\|\rho_0\|_{L^{\frac{1}{\alpha-1}}}, \quad 0 \le t \le T.$$

Estimate of $\sup_{0<t\le T} t^{\frac{\alpha-1}{2\alpha}} \|S\rho(t)\|_{L^{\frac{2}{\alpha-1}}}$: From (3.7) with $\ell = 0$ and $q = \frac{2}{\alpha-1}$, $p = \frac{1}{\alpha-1}$, we have

$$\begin{aligned}\|\partial_x G_\alpha(\cdot, t-s) * (\rho(s)H\rho(s))\|_{L^{\frac{2}{\alpha-1}}} &\le C(t-s)^{-\frac{\alpha+1}{2\alpha}} \|\rho(s)H\rho(s)\|_{L^{\frac{1}{\alpha-1}}} \\ &\le C(t-s)^{-\frac{\alpha+1}{2\alpha}} s^{-\frac{\alpha-1}{\alpha}} \sup_{0<t\le T} s^{\frac{\alpha-1}{\alpha}} \|\rho(s)\|^2_{L^{\frac{2}{\alpha-1}}},\end{aligned}$$

which implies

$$\begin{aligned}&\sup_{0<t\le T} t^{\frac{\alpha-1}{2\alpha}} \int_0^t \|\partial_x G_\alpha(\cdot, t-s) * (\rho(s)H\rho(s))\|_{L^{\frac{2}{\alpha-1}}} \,\mathrm{d}s \\ &\qquad \le Ca^2 \cdot \sup_{0<t\le T} t^{\frac{\alpha-1}{2\alpha}} \int_0^t (t-s)^{-\frac{\alpha+1}{2\alpha}} s^{-\frac{\alpha-1}{\alpha}} \,\mathrm{d}s \le C\mathcal{B}\Big(\frac{1}{\alpha}, \frac{\alpha-1}{2\alpha}\Big)a^2.\end{aligned}$$

Hence, there exists $a > 0$ small enough such that

$$\sup_{0<t\le T} t^{\frac{\alpha-1}{2\alpha}} \|S\rho(t)\|_{L^{\frac{2}{\alpha-1}}} \le \sup_{0<t\le T} t^{\frac{\alpha-1}{2\alpha}} \|G_\alpha(\cdot, t) * \rho_0\|_{L^{\frac{2}{\alpha-1}}} + C\mathcal{B}\Big(\frac{1}{\alpha}, \frac{\alpha-1}{2\alpha}\Big)a^2 < 2a. \tag{3.12}$$

Since the proof of the continuity of $S\rho$ with respect to time $t \in (0, T]$ in $L^{\frac{1}{\alpha-1}}(\mathbb{R})$ is routine and tedious, we put it in Appendix B.

Step 2. We are going to show that S is a contraction mapping.

Consider $\rho_1, \ \rho_2 \in X_T^a$. We have

$$\begin{aligned}\|S\rho_1 - S\rho_2\|_{X_T} \le &\Big\| \int_0^t \partial_x G_\alpha(\cdot, t-s) * [(\rho_1(s) - \rho_2(s))H\rho_1(s)]\mathrm{d}s \Big\|_{X_T} \\ &+ \Big\| \int_0^t \partial_x G_\alpha(\cdot, t-s) * [\rho_2(s)H(\rho_1(s) - \rho_2(s))]\mathrm{d}s \Big\|_{X_T}.\end{aligned} \tag{3.13}$$

Similarly to Step 1, we have the following estimates for the first term on the right-hand side of (3.13):

$$\begin{aligned}&\sup_{0\le t\le T} \Big\| \int_0^t \partial_x G_\alpha(\cdot, t-s) * [(\rho_1(s) - \rho_2(s))H\rho_1(s)]\mathrm{d}s \Big\|_{L^{\frac{1}{\alpha-1}}} \\ &\le C\mathcal{B}\Big(\frac{1}{\alpha}, \frac{\alpha-1}{\alpha}\Big)a\|\rho_1 - \rho_2\|_{X_T},\end{aligned}$$

and

$$\begin{aligned}&\sup_{0\le t\le T} t^{\frac{\alpha-1}{2\alpha}} \Big\| \int_0^t \partial_x G_\alpha(\cdot, t-s) * [(\rho_1(s) - \rho_2(s))H\rho_1(s)]\mathrm{d}s \Big\|_{L^{\frac{2}{\alpha-1}}} \\ &\le C\mathcal{B}\Big(\frac{1}{\alpha}, \frac{\alpha-1}{2\alpha}\Big)a\|\rho_1 - \rho_2\|_{X_T}.\end{aligned}$$

We have similar estimate for the second term in the right hand of (3.13). Hence

$$\|S\rho_1 - S\rho_2\|_{X_T} \le Ca\|\rho_1 - \rho_2\|_{X_T}. \tag{3.14}$$

This shows S is a contraction mapping for small enough $a > 0$.

Step 3. In this step, we are going to show the time continuity at $t = 0$. Since

$$\lim_{t\to 0} \|G_\alpha(\cdot, t) * \rho_0 - \rho_0\|_{L^{\frac{1}{\alpha-1}}} = 0,$$

we only need to show

$$\lim_{t\to 0}\left\| \int_0^t \partial_x G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s))\mathrm{d}s \right\|_{L^{\frac{1}{\alpha-1}}} = 0. \tag{3.15}$$

Let ρ be the solution constructed by Step 1 and Step 2. Consider another two positive numbers $\tilde{a} < a$ and $\tilde{T} < T$ such that (3.10) holds for $\tilde{a}$ and $\tilde{T}$. We could obtain another solution $\tilde{\rho}$ in the corresponding space $X_{\tilde{T}}^{\tilde{a}}$. Similarly to (3.14), we obtain

$$\|\tilde{\rho}-\rho\|_{X_{\tilde{T}}} \le Ca\|\tilde{\rho}-\rho\|_{X_{\tilde{T}}},$$

which implies $\tilde{\rho}(t) = \rho(t)$ for $0 < t \le \tilde{T}$. Due to (3.9), as $\tilde{T} \to 0$, we could choose $\tilde{a} \to 0$ Combining (3.11) for $\tilde{a} \to 0$, we have (3.15). □

B. Regularity, nonnegativity, and analyticity

In this section, we improve the regularity of mild solutions step by step and obtain the nonnegativity. Then, we show the spatial analyticity of solutions.

For the regularity of mild solutions, the strategy is as follows. We first show the hypercontrativity estimate that the mild solutions belong to L^q spaces for any $\frac{1}{\alpha-1} \le q \le \infty$. Then, we estimate the derivative of mild solutions. The time decay property can only ensure the improvement of fractional step $0 < \ell < \alpha - 1$ for the derivative. Step by step, we could improve the regularity of solutions to any order we want. For the nonegativity, we follow the same idea as in [Ref. 23, Lemma 2.7]. However, comparing with [Ref. 23, Lemma 2.7] where the initial data $\rho_0 \in L^{\frac{1}{\alpha-1}}(\mathbb{R}) \cap L^p(\mathbb{R})$ for some $\frac{1}{\alpha-1} < p < \infty$, we only need the initial data $\rho_0 \in L^{\frac{1}{\alpha-1}}(\mathbb{R})$. We have the following theorem:

Theorem 3.2 *(Regularity and nonnegativity). Let $0 \le \rho_0 \in L^{\frac{1}{\alpha-1}}(\mathbb{R})$. Then, the mild solution ρ obtained by Theorem 3.1 is a strong solution for $t > 0$ belonging to $C^\infty((0,T];H^{\theta,q}(\mathbb{R}))$ for any $\theta > 0$ and $1/(\alpha-1) \le q \le \infty$. The following time decay estimates for derivatives hold:*

$$\|\rho(t)\|_{\dot{H}^{\theta,q}(\mathbb{R})} \le Ct^{-\frac{\theta}{\alpha}-1+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)}, \quad \frac{1}{\alpha-1} \le q \le \infty, \quad 0 < t \le T, \tag{3.16}$$

and

$$\|\partial_x^n \rho(t)\|_{L^\infty} \le Ct^{-\frac{n-1}{\alpha}-1}, \quad \forall n \in \mathbb{N}, \quad 0 < t \le T. \tag{3.17}$$

Moreover, $\rho(x,t) \ge 0$ for any $t \in [0,T]$, $x \in \mathbb{R}$.

Proof. Denote

$$\rho(x,t) = \rho_1(x,t) - \rho_2(x,t) := G_\alpha(\cdot,t) * \rho_0 - \int_0^t \partial_x G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s))\mathrm{d}s. \tag{3.18}$$

The first term $\rho_1(x,t) = G_\alpha(\cdot,t) * \rho_0$ is the solution to the fractional heat equation with initial date ρ_0. Due to instantaneous regularization of the fractional heat equation, we have $\rho_1 \in C^\infty(\mathbb{R}\times(0,\infty))$. From (3.6) with $p = 1/(\alpha-1)$, we obtain

$$\|\rho_1(t)\|_{\dot{H}^{k\ell,q}} = \|\Lambda^{k\ell}G_\alpha(\cdot,t) * \rho_0\|_{L^q} \le C_{\ell,\alpha}^k k^{1+\frac{k\ell}{\alpha}-\frac{1}{\alpha}\left(1+\frac{1}{q}\right)} t^{-1-\frac{k\ell}{\alpha}+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)} \tag{3.19}$$

for $1/(\alpha-1) \le q \le \infty$, $0 < t \le T$. Next, we separate the proof into several steps.

Step 1. In this step, we are going to prove

$$\|\rho(t)\|_{L^q} \le C_0 t^{-1+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)}, \quad \frac{1}{\alpha-1} \le q \le \infty, \quad 0 < t \le T \tag{3.20}$$

for some constant C_0 independent of q.

For $\rho_1(t)$, using (3.9) with $\ell = 0$ and $k = 1$, we have

$$\|\rho_1(t)\| \le C_{0,\alpha}\|\rho_0\|_{L^{\frac{1}{\alpha-1}}} t^{-1+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)}. \tag{3.21}$$

Next we deal with ρ_2. From (3.7) with $\ell = 0$ and $1/(\alpha-1) = p \le q < \infty$, we have

$$\begin{aligned}
\|\rho_2(t)\|_{L^q} &\le C_{0,\alpha}\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\alpha-\frac{1}{q}\right)}\|\rho(s)H\rho(s)\|_{L^{\frac{1}{\alpha-1}}}\mathrm{d}s \\
&\le C_{0,\alpha}C_{2/(\alpha-1)}\int_0^t (t-s)^{-1+\frac{1}{\alpha q}} s^{-\frac{\alpha-1}{\alpha}}\mathrm{d}s \cdot \sup_{0<t\le T} s^{\frac{\alpha-1}{\alpha}}\|\rho(s)\|_{L^{\frac{2}{\alpha-1}}}^2 \\
&= 4a^2 C_{0,\alpha}C_{2/(\alpha-1)}\,\mathcal{B}\left(\frac{1}{\alpha q},\frac{1}{\alpha}\right)t^{-1+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)},
\end{aligned} \tag{3.22}$$

where we have used (3.1) for $p = 2/(\alpha-1)$. Hence, (3.20) holds for $1/(\alpha-1) \le q < \infty$.

From (3.7) with $\ell = 0$, $p = 2/(\alpha-1)$ and $q = \infty$, we have

$$\begin{aligned}\|\rho_2(t)\|_{L^\infty} &\le C_{0,\alpha}\int_0^t (t-s)^{-\frac{\alpha+1}{2\alpha}}\|\rho(s)H\rho(s)\|_{L^{\frac{2}{\alpha-1}}}\,\mathrm{d}s\\ &\le C_{0,\alpha}C_{4/(\alpha-1)}\int_0^t (t-s)^{-\frac{\alpha+1}{2\alpha}}s^{-\frac{3(\alpha-1)}{2\alpha}}\,\mathrm{d}s\cdot\sup_{0<t\le T} s^{\frac{3(\alpha-1)}{2\alpha}}\|\rho(s)\|^2_{L^{\frac{4}{\alpha-1}}}.\end{aligned}$$

Because $-1 < -\frac{\alpha+1}{2\alpha} < 0$ and $-1 < -\frac{3(\alpha-1)}{2\alpha} < 0$, from (3.22) with $q = 4/(\alpha-1)$ we obtain

$$\|\rho_2(t)\|_{L^\infty} \le 16a^4 C_{0,\alpha}^3 C_{2/(\alpha-1)}^2 C_{4/(\alpha-1)}\,\mathcal{B}^2\left(\frac{\alpha-1}{4\alpha},\frac{1}{\alpha}\right)\mathcal{B}\left(\frac{\alpha-1}{2\alpha},\frac{3-\alpha}{2\alpha}\right)t^{-1+\frac{1}{\alpha}}. \tag{3.23}$$

Because of the definitions of $\mathcal{B}(\cdot,\cdot)$ and C_p by (3.2), from (3.22) and (3.23), there exits C_0 (independent of q) big enough such that

$$\|\rho_2(t)\|_{L^q} \le \frac{C_0}{2}t^{-1+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)},\qquad \frac{1}{\alpha-1}\le q\le\frac{4}{\alpha-1},$$

and

$$\|\rho_2(t)\|_{L^\infty} \le \frac{C_0}{2}t^{-1+\frac{1}{\alpha}},\qquad 0<t\le T.$$

For $4/(\alpha-1) = s < q < \infty$, we could use the interpolation to obtain

$$\|\rho_2(t)\|_{L^q} \le \|\rho_2(t)\|_{L^{q_1}}^{q_1/q}\|\rho_2(t)\|_{L^\infty}^{1-q_1/q} \le \frac{C_0}{2}t^{-1+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)}. \tag{3.24}$$

Combining (3.21) and (3.24) gives (3.20) for C_0 large enough.

Step 2. In this step, we are going to prove $\rho(t)\in H^{\ell,q}(\mathbb{R})$ for any $0<\ell<\alpha-1$, $1/(\alpha-1)\le q\le\infty$, and

$$\|\rho(t)\|_{\dot H^{\ell,q}} \le Ct^{-1-\frac{\ell}{\alpha}+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)},\qquad \frac{1}{\alpha-1}\le q\le\infty,\qquad 0<t\le T. \tag{3.25}$$

Because of (3.19), we only need to show (3.25) for ρ_2.

From (3.7) with $p = q$, we have

$$\begin{aligned}\|\rho_2(t)\|_{\dot H^{\ell,q}} = \|\Lambda^\ell\rho_2(t)\|_{L^q} &\le \int_0^t \|\Lambda^\ell\partial_x G_\alpha(\cdot,t-s)*(\rho(s)H\rho(s))\|_{L^q}\,\mathrm{d}s\\ &\le C\int_0^t (t-s)^{-\frac{\ell+1}{\alpha}}\|\rho(s)H\rho(s)\|_{L^q}\,\mathrm{d}s\\ &\le C\int_0^t (t-s)^{-\frac{\ell+1}{\alpha}}s^{-2+\frac{1}{\alpha}\left(2+\frac{1}{q}\right)}\mathrm{d}s\cdot\sup_{0<s\le T}s^{2-\frac{1}{\alpha}\left(2+\frac{1}{q}\right)}\|\rho(s)\|^2_{L^{2q}}.\end{aligned}$$

For any $1/(\alpha-1)\le q<\infty$, we have $-2+\frac{1}{\alpha}(2+\frac{1}{q}) > -1$. Hence, for $0<\ell<\alpha-1$, we have

$$\|\rho_2(t)\|_{\dot H^{\ell,q}} \le C\int_0^t (t-s)^{-\frac{\ell+1}{\alpha}}s^{-2+\frac{1}{\alpha}\left(2+\frac{1}{q}\right)}\mathrm{d}s \le Ct^{-1-\frac{\ell}{\alpha}+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)}. \tag{3.26}$$

From (3.7) with $q = \infty$, we have

$$\begin{aligned}\|\rho_2(t)\|_{\dot H^{\ell,\infty}} = \|\Lambda^\ell\rho_2(t)\|_{L^\infty} &\le \int_0^t \|\Lambda^\ell\partial_x G_\alpha(\cdot,t-s)*(\rho(s)H\rho(s))\|_{L^\infty}\,\mathrm{d}s\\ &\le C\int_0^t (t-s)^{-\frac{\ell+1}{\alpha}-\frac{1}{p\alpha}}\|\rho(s)H\rho(s)\|_{L^p}\,\mathrm{d}s\\ &\le C\int_0^t (t-s)^{-\frac{\ell+1}{\alpha}-\frac{1}{p\alpha}}s^{-2+\frac{1}{\alpha}\left(2+\frac{1}{p}\right)}\mathrm{d}s\cdot\sup_{0<s\le T}s^{2-\frac{1}{\alpha}\left(2+\frac{1}{p}\right)}\|\rho(s)\|^2_{L^{2p}}.\end{aligned}$$

Hence, for $0<\ell<\alpha-1$ and $p<\infty$ big enough, we have $-\frac{\ell+1}{\alpha}-\frac{1}{p\alpha} > -1$ and (3.25) holds for $q=\infty$.

Step 3. In this step, we are going to prove that if $\rho(t)\in H^{\beta,q}(\mathbb{R})$ for any $\beta>0$ and $\frac{1}{\alpha-1}\le q\le\infty$ satisfying

$$\|\rho(t)\|_{\dot H^{\beta,q}} \le Ct^{-1-\frac{\beta}{\alpha}+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)}, \tag{3.27}$$

we have $\rho(t) \in H^{\beta+\ell,q}(\mathbb{R})$ and

$$\|\rho(t)\|_{\dot{H}^{\beta+\ell,q}} \le Ct^{-1-\frac{\beta+\ell}{\alpha}+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)}, \quad \frac{1}{\alpha-1} \le q \le \infty, \quad 0 < t \le T. \tag{3.28}$$

Since (3.19), we only need to show that $\rho_2(x,t)$ satisfies (3.28). Notice that $\Lambda^\beta(H\rho(s)) = H(\Lambda^\beta\rho(s))$ and hence for $1/(\alpha-1) \le q < \infty$,

$$\|\Lambda^\beta(H\rho(s))\|_{L^q} = \|H(\Lambda^\beta\rho(s))\|_{L^q} \le \|\Lambda^\beta\rho(s)\|_{L^q} < \infty.$$

This implies $H\rho \in H^{\beta,q}(\mathbb{R})$ for $1/(\alpha-1) \le q < \infty$. By the Sobolev embedding for q big enough, we know $H\rho(s) \in L^\infty(\mathbb{R})$. Therefore,

$$\rho(s),\ H\rho(s) \in H^{\beta,q}(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

From fractional Leibniz inequality (see, for instance Ref. 21), we have

$$\|\Lambda^\beta(\rho(s)H\rho(s))\|_{L^q} \le C\Big(\|\Lambda^\beta\rho(s)\|_{L^{q_1}}\|H\rho_2(s)\|_{L^{q_2}} + \|\Lambda^\beta H\rho_2(s)\|_{L^{q_3}}\|\rho_2(s)\|_{L^{q_4}}\Big) \tag{3.29}$$

for

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}, \quad q < q_i, \quad i = 1,\dots,4.$$

Combining (3.20), (3.27), and (3.29) gives

$$\|\rho(s)H\rho(s)\|_{\dot{H}^{\beta,q}} = \|\Lambda^\beta(\rho(s)H\rho(s))\|_{L^q} \le Ct^{-2-\frac{\beta}{\alpha}+\frac{1}{\alpha}\left(2+\frac{1}{q}\right)}. \tag{3.30}$$

From (3.6), (3.7), and (3.30), the following holds for $1/(\alpha-1) \le q \le \infty$

$$\begin{aligned}
&\|\Lambda^{\beta+\ell}\rho_2(t)\|_{L^q}\\
&\le \int_\delta^t \|\Lambda^\ell\partial_x G_\alpha(\cdot,t-s) * [\Lambda^\beta(\rho(s)H\rho(s))]\|_{L^q}\,\mathrm{d}s + \int_0^\delta \|\Lambda^{\beta+\ell}\partial_x G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s))\|_{L^q}\,\mathrm{d}s\\
&\le C\int_\delta^t (t-s)^{-\frac{\ell+1}{\alpha}+\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|\Lambda^\beta(\rho(s)H\rho(s))\|_{L^p}\,\mathrm{d}s + C\int_0^\delta (t-s)^{-\frac{\beta+\ell+1}{\alpha}+\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|\rho(s)H\rho(s)\|_{L^p}\,\mathrm{d}s\\
&\le C\int_\delta^t (t-s)^{-\frac{\ell+1}{\alpha}+\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}s^{-2-\frac{\beta}{\alpha}+\frac{1}{\alpha}\left(2+\frac{1}{p}\right)}\mathrm{d}s + C\int_0^\delta (t-s)^{-\frac{\beta+\ell+1}{\alpha}+\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}s^{-2+\frac{1}{\alpha}\left(2+\frac{1}{p}\right)}\mathrm{d}s\\
&\le C(t-\delta)^{1-\frac{\ell+1}{\alpha}+\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\delta^{-2-\frac{\beta}{\alpha}+\frac{1}{\alpha}\left(2+\frac{1}{p}\right)} + C(t-\delta)^{-\frac{\beta+\ell+1}{\alpha}+\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\delta^{-1+\frac{1}{\alpha}\left(2+\frac{1}{p}\right)}.
\end{aligned}$$

Choose $\delta = t/2$ and we obtain the estimate for $\rho_2(x,t)$.

Step 4. Notice that the Bessel potential space $H^{\theta,\infty}(\mathbb{R})$ is not the same as Sobolev space $W^{\theta,\infty}$. Hence we give the details of the proof of (3.17) in this step. We only need to estimate $\rho_2(t)$, and the estimate for $\rho_1(t)$ can be obtained by Lemma 3.1. By the same method as for (3.30), we also have

$$\|\partial_x^n(\rho(s)H\rho(s))\|_{L^q} \le Ct^{-2-\frac{n}{\alpha}+\frac{1}{\alpha}\left(2+\frac{1}{q}\right)}. \tag{3.31}$$

From (3.6), (3.7), and (3.31), the following holds for $1/(\alpha-1) < p < \infty$ and $1 \le n \le \infty$:

$$\begin{aligned}
&\|\partial_x^n\rho_2(t)\|_{L^\infty}\\
&\le \int_\delta^t \|\partial_x G_\alpha(\cdot,t-s) * [\partial_x^n(\rho(s)H\rho(s))]\|_{L^\infty}\,\mathrm{d}s + \int_0^\delta \|\partial_x^{n+1}G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s))\|_{L^\infty}\,\mathrm{d}s\\
&\le C\int_\delta^t (t-s)^{-\frac{1}{\alpha}\left(1+\frac{1}{p}\right)}\|\partial_x^n(\rho(s)H\rho(s))\|_{L^p}\,\mathrm{d}s + C\int_0^\delta (t-s)^{-\frac{n}{\alpha}-\frac{1}{\alpha}\left(1+\frac{1}{p}\right)}\|\rho(s)H\rho(s)\|_{L^p}\,\mathrm{d}s\\
&\le C\int_\delta^t (t-s)^{-\frac{1}{\alpha}\left(1+\frac{1}{p}\right)}s^{-2-\frac{n}{\alpha}+\frac{1}{\alpha}\left(2+\frac{1}{p}\right)}\mathrm{d}s + C\int_0^\delta (t-s)^{-\frac{n}{\alpha}-\frac{1}{\alpha}\left(1+\frac{1}{p}\right)}s^{-2+\frac{1}{\alpha}\left(2+\frac{1}{p}\right)}\mathrm{d}s\\
&\le C(t-\delta)^{1-\frac{1}{\alpha}\left(1+\frac{1}{p}\right)}\delta^{-2-\frac{n}{\alpha}+\frac{1}{\alpha}\left(2+\frac{1}{p}\right)} + C(t-\delta)^{-\frac{n}{\alpha}-\frac{1}{\alpha}\left(1+\frac{1}{p}\right)}\delta^{-1+\frac{1}{\alpha}\left(2+\frac{1}{p}\right)}.
\end{aligned}$$

Choose $\delta = t/2$ and we obtain the estimate for $\rho_2(x,t)$.

Step 5. In this step, we are going to show the regularity of time for the mild solution ρ. First, let us prove $\rho \in C((0,T]; H^{\theta,q}(\mathbb{R}))$ for any $\theta > 0$ and $q \ge \frac{1}{\alpha-1}$. For any $t > s > 0$, we have

$$\rho(t) = G_\alpha(\cdot,t-s) * \rho(s) - \int_s^t \partial_x G_\alpha(\cdot,t-\tau) * (\rho(\tau)H\rho(\tau))\mathrm{d}\tau.$$

Therefore,

$$\begin{aligned}&\|\rho(t)-\rho(s)\|_{H^{\theta,q}}\\&\leq\|G_\alpha(\cdot,t-s)*\rho(s)-\rho(s)\|_{H^{\theta,q}}+\int_s^t\|G_\alpha(\cdot,t-\tau)*\partial_x(\rho(\tau)H\rho(\tau))\|_{H^{\theta,q}}\mathrm{d}\tau.\end{aligned}\tag{3.32}$$

The first term in the right hand side of (3.32) goes to zero as $|s-t|\to 0$, because the solution of fractional heat equation is continuous at the initial data $\rho(s)$ in $H^{\theta,q}(\mathbb{R})$. For the second term in (3.32), due to (3.6) for $\ell=0$, we have the following estimate:

$$\|G_\alpha(\cdot,t-\tau)*\partial_x(\rho(\tau)H\rho(\tau))\|_{H^{\theta,q}}\leq C\|\rho(\tau)H\rho(\tau)\|_{H^{\theta+1,q}}.$$

From Step 3, we know that $\|\rho(\tau)\|_{H^{\theta+1,q}}$ is uniformly bounded for $\tau\in(s,t)$. Therefore, the second term in the right hand side of (3.32) also goes to zero as $|t-s|\to 0$.

Next, we improve the time regularity. Choose an arbitrary $t_0\in(0,T)$ and set the new initial data

$$\tilde{\rho}_0:=\rho(t_0).$$

With this new initial data, we have a mild solution

$$\tilde{\rho}(t)=\rho(t+t_0),\quad t\in[0,T-t_0],$$

which satisfies

$$\tilde{\rho}(x,t)=G_\alpha(\cdot,t)*\tilde{\rho}_0-\int_0^t\partial_xG_\alpha(\cdot,t-s)*(\tilde{\rho}(s)H\tilde{\rho}(s))\mathrm{d}s.\tag{3.33}$$

Notice that for $f\in L^p(\mathbb{R})$, because $G_\alpha(\cdot,t)*f$ is the solution of fractional heat equation $u_t=-\nu\Lambda^\alpha u$ with initial data f, the following holds:

$$G_\alpha(\cdot,t)*f-f=-\int_0^t\nu\Lambda^\alpha G_\alpha(\cdot,s)*f\,\mathrm{d}s.\tag{3.34}$$

Combining (3.33) and (3.34) yields

$$\begin{aligned}&\int_0^t[-\nu\Lambda^\alpha\tilde{\rho}(\tau)-\partial_x(\tilde{\rho}(\tau)H\tilde{\rho}(\tau))]\mathrm{d}\tau\\&=\int_0^t-\nu\Lambda^\alpha G_\alpha(\cdot,\tau)*\tilde{\rho}_0\mathrm{d}\tau-\int_0^t\int_0^\tau-\nu\Lambda^\alpha[\partial_xG_\alpha(\cdot,\tau-s)*(\tilde{\rho}(s)H\tilde{\rho}(s))]\mathrm{d}s\mathrm{d}\tau\\&\qquad-\int_0^t\partial_x(\tilde{\rho}(\tau)H\tilde{\rho}(\tau))\mathrm{d}\tau\\&=(G_\alpha(\cdot,t)*\tilde{\rho}_0-\tilde{\rho}_0)-\int_0^t\int_s^t-\nu\Lambda^\alpha[\partial_xG_\alpha(\cdot,\tau-s)*(\tilde{\rho}(s)H\tilde{\rho}(s))]\mathrm{d}\tau\mathrm{d}s\\&\qquad-\int_0^t\partial_x(\tilde{\rho}(\tau)H\rho(\tau))\mathrm{d}\tau\\&=G_\alpha(\cdot,t)*\tilde{\rho}_0-\int_0^t\partial_xG_\alpha(\cdot,t-s)*(\tilde{\rho}(s)H\rho(s))\mathrm{d}s-\tilde{\rho}_0,\end{aligned}$$

which implies

$$\int_0^t[-\nu\Lambda^\alpha\tilde{\rho}(\tau)-\partial_x(\tilde{\rho}(\tau)H\tilde{\rho}(\tau))]\mathrm{d}\tau=\tilde{\rho}(t)-\tilde{\rho}_0.\tag{3.35}$$

Because $\partial_x(\tilde{\rho}H\tilde{\rho})\in C([0,T-t_0];H^{\theta,q}(\mathbb{R})),\ \forall\theta>0,\ q\geq\frac{1}{\alpha-1}$ we have

$$\tilde{\rho}\in C^\infty([0,T-t_0];H^{\theta,q}(\mathbb{R})),\quad\forall\theta>0.\tag{3.36}$$

Since t_0 is chosen arbitrarily, the time regularity is obtained.

Step 6. In this step, we are going to show the nonnegativity of solutions.

Consider a sequence of smooth positive functions ρ_{0n} such that $\rho_{0n}\to\rho_0$ in $L^{\frac{1}{\alpha-1}}(\mathbb{R})$ as $n\to\infty$. By (A6) for $p=\frac{1}{\alpha-1}$ and $q=\frac{2}{\alpha-1}$, we have

$$\|G_\alpha(\cdot,t)*\rho_{0n}-G_\alpha(\cdot,t)*\rho_0\|_{X_T}\to 0,\quad n\to\infty.$$

Recall Theorem 3.1. For a small enough, there exists (a uniform) $T>0$ such that (3.10) holds for all $n\in\mathbb{N}$, i.e.,

$$\sup_{0<t\leq T}t^{\frac{\alpha-1}{2\alpha}}\|G_\alpha(\cdot,t)*\rho_{0n}\|_{L^{\frac{2}{\alpha-1}}}\leq a.$$

Hence, Theorem 3.1 gives a sequence of solutions $\rho_n \in X_T^a$ with initial data ρ_{0n} in a uniform time interval $[0,T]$. Due to Ref. 24, Lemma 2.7, we also have $\rho_n \geq 0$. Similarly to (3.14), we obtain

$$\|\rho_n - \rho\|_{X_T} \leq \|G_\alpha(\cdot,t) * \rho_{0n} - G_\alpha(\cdot,t) * \rho_0\|_{X_T} + Ca\|\rho_n - \rho\|_{X_T}, \tag{3.37}$$

which implies

$$\|\rho_n - \rho\|_{X_T} \leq \frac{1}{1-Ca}\|G_\alpha(\cdot,t) * \rho_{0n} - G_\alpha(\cdot,t) * \rho_0\|_{X_T} \to 0, \quad n \to \infty. \tag{3.38}$$

Because $\rho_n,\ \rho \in C([0,T]; L^{\frac{1}{\alpha-1}}(\mathbb{R}))$, we have $\|\rho_n(t) - \rho(t)\|_{L^{\frac{1}{\alpha-1}}} \to 0$ as $n \to \infty$ for any $0 < t < T$. Since $\rho_n(t) \geq 0$, we obtain $\rho(t) \geq 0$ for a.e. $x \in \mathbb{R}$. Since $\rho(t) \in C^\infty(\mathbb{R})$ for $t > 0$, we obtain $\rho(x,t) \geq 0$ for all $x \in \mathbb{R}$.

This is the end of the proof. □

To prove the spatial analyticity of mild solutions, we need to obtain more explicit estimate for the constant in (3.17). We are going to generalize the method used in Refs. 16 and 31 for the case of fractional diffusion. First, let us introduce a useful lemma about an estimate for multiplication of sequences, which was proved by Kahane,[19] Lemma 2.1. The original lemma is for multi-index, and here we only need to use the following one dimensional version for integers.

Lemma 3.2. Let $\delta > \frac{1}{2}$. Then there exists a positive constant λ depending only on δ such that

$$\sum_{0\leq j\leq k} \binom{k}{j} j^{j-\delta}(k-j)^{k-j-\delta} \leq \lambda k^{k-\delta}, \quad \forall k \in \mathbb{N}.$$

Here, we use $0^p = 1$ for any $p \in \mathbb{R}$.

We have the following more explicit estimate for (3.17):

Theorem 3.3 (Spatial Analyticity). *Let $\rho(t)$ be a mild solution given in Theorem 3.2. Then*

$$\|\partial_x^n \rho(t)\|_{L^q(\mathbb{R})} \leq K^n n^n t^{-\frac{n}{\alpha}-1+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)}, \quad \forall n \in \mathbb{N}, \quad 0 < t \leq T \tag{3.39}$$

for some constant K independent of n and $\frac{1}{\alpha-1} \leq q \leq \infty$. Consequently, $\rho(\cdot,t)$ is spatial analytic for $0 < t \leq T$.

Proof. Let $n \in \mathbb{N}$. Notice that we only need to prove (3.39) for n big enough. We use induction to prove this. Assume that there exist constants K (independent of q) and $\delta > \frac{1}{2}$ such that

$$\|\partial_x^m \rho(t)\|_{L^q} \leq K^{m-\delta} m^{m-\delta} t^{-\frac{m}{\alpha}-1+\frac{1}{\alpha}\left(1+\frac{1}{q}\right)}, \qquad 0 < t \leq T \tag{3.40}$$

holds for any $\frac{1}{\alpha-1} < q < \infty$ and $m < n$. Then, we prove that (3.40) also holds for $m = n$. The cases for $q = \infty$ and $q = \frac{1}{\alpha-1}$ follow easily after we obtain the results for $\frac{1}{\alpha-1} < q < \infty$.

Due to the regularity results Theorem 3.2, we only need to show (3.40) for n large enough. We have

$$\begin{aligned}
\|\partial_x^n \rho(t)\|_{L^q} &\leq \|\partial_x^n \rho_1(t)\|_{L^q} + \int_0^t \|\partial_x^n \partial_x G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s))\|_{L^q}\,ds \\
&\leq \|\partial_x^n \rho_1(t)\|_{L^q} + \sum_{m=2}^{n-1}\sum_{j=1}^{m}\binom{m}{j}\int_{\frac{m-1}{n}t}^{\frac{m}{n}t} \|\partial_x^{n-m}\partial_x G_\alpha(\cdot,t-s) * (\partial_x^j\rho(s)\partial_x^{m-j}H\rho(s))\|_{L^q}\,ds \\
&\quad + \int_0^{\frac{1}{n}t} \|\partial_x^n \partial_x G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s))\|_{L^q}\,ds \\
&\quad + \int_{\frac{n-1}{n}t}^{t} \sum_{1<j<n}\binom{n}{j}\|\partial_x G_\alpha(\cdot,t-s) * (\partial_x^j\rho(s)\partial_x^{n-j}H\rho(s))\|_{L^q}\,ds \\
&\quad + \int_{\frac{n-1}{n}t}^{t} \|\partial_x G_\alpha(\cdot,t-s) * [\rho(s)\partial_x^n H\rho(s) + \partial_x^n\rho(s)H\rho(s)]\|_{L^q}\,ds \\
&=: A_1 + A_2 + A_3 + A_4 + A_5.
\end{aligned} \tag{3.41}$$

Let $\mu := \frac{n}{\alpha} + 1 - \frac{1}{\alpha}(1+\frac{1}{q})$. Next, we are going to estimate A_i respectively, $1 \leq i \leq 5$. From (3.19), choose K big enough and we have

$$A_1 = \|\partial_x^n \rho_1(t)\|_{L^q} \leq C_{0,\alpha}^n n^\mu t^{-\mu} \leq t^{-\mu} K^{n-2\delta} n^{n-\delta}. \tag{3.42}$$

For A_2, we use (3.7) for $\frac{1}{\alpha-1} \le p \le q$ to obtain

$$A_2 \le \sum_{m=2}^{n-1} C_{1,\alpha}^{n-m}(n-m)^{\frac{n-m}{\alpha}+a}\int_{\frac{m-1}{n}t}^{\frac{m}{n}t}(t-s)^{-\frac{n-m}{\alpha}-a}\sum_{j=1}^{m}\binom{m}{j}\|\partial_x^j\rho(s)\partial_x^{m-j}H\rho(s)\|_{L^p}\,\mathrm{d}s,$$

where we choose p close enough to q such that $a := \frac{1}{\alpha}(1+\frac{1}{p}-\frac{1}{q}) < 1$. Using Hölder's inequality (for $\frac{1}{p} = \frac{1}{2p}+\frac{1}{2p}$), (3.1), (3.40), and Lemma 3.2, we obtain

$$\sum_{j=1}^{m}\binom{m}{j}\|\partial_x^j\rho(s)\partial_x^{m-j}H\rho(s)\|_{L^p} \le C_{2p}K^{m-2\delta}m^{m-\delta}s^{-\frac{m}{\alpha}-b}$$

for $0 < b := 2-\frac{1}{\alpha}(2+\frac{1}{p}) < 1$. Therefore, choose K big enough and we have

$$\begin{aligned}
A_2 &\le \sum_{m=2}^{n-1} C_{1,\alpha}^{n-m}(n-m)^{\frac{n-m}{\alpha}+a}C_{2p}K^{m-2\delta}m^{m-\delta}\int_{\frac{m-1}{n}t}^{\frac{m}{n}t}(t-s)^{-\frac{n-m}{\alpha}-a}s^{-\frac{m}{\alpha}-b}\mathrm{d}s\\
&\le t^{-\mu}C_{2p}K^{n-2\delta}\sum_{m=2}^{n-1}(n-m)^{\frac{n-m}{\alpha}+a}m^{m-\delta}\int_{\frac{m-1}{n}}^{\frac{m}{n}}(1-\tau)^{-\frac{n-m}{\alpha}-a}\tau^{-\frac{m}{\alpha}-b}\mathrm{d}\tau\\
&\le t^{-\mu}C_{2p}K^{n-2\delta}\sum_{m=2}^{n-1}\frac{1}{n}(n-m)^{\frac{n-m}{\alpha}+a}m^{m-\delta}\left(\frac{n}{n-m}\right)^{\frac{n-m}{\alpha}+a}\left(\frac{n}{m-1}\right)^{\frac{m}{\alpha}+b}\\
&= t^{-\mu}C_{2p}K^{n-2\delta}\sum_{m=2}^{n-1}n^{\mu}m^{m-\delta}\left(\frac{1}{m-1}\right)^{\frac{m}{\alpha}+b}.
\end{aligned}$$

We claim that for n big enough we have

$$\sum_{m=2}^{n-1}n^{\mu}m^{m-\delta}\left(\frac{1}{m-1}\right)^{\frac{m}{\alpha}+b} \le n^{n-\delta}. \tag{3.43}$$

See the proof of (3.43) in Appendix C. Hence

$$A_2 \le t^{-\mu}C_{2p}K^{n-2\delta}n^{n-\delta}. \tag{3.44}$$

For A_3 and A_4 in (3.41), choose K big enough and we have

$$\begin{aligned}
A_3 &= \int_0^{\frac{1}{n}t}\|\partial_x^n\partial_x G_\alpha(\cdot,t-s)*(\rho(s)H\rho(s))\|_{L^q}\,\mathrm{d}s \le t^{-\mu}C_{2p}C_{1,\alpha}^n n^{\frac{n}{\alpha}+a}\int_0^{\frac{1}{n}}(1-\tau)^{-\frac{n}{\alpha}-a}\tau^{-b}\mathrm{d}\tau\\
&\le t^{-\mu}C_{2p}C_{1,\alpha}^n n^{\frac{n}{\alpha}+a}\int_0^{\frac{1}{n}}\frac{1}{1-b}\left[(1-\tau)^{-\frac{n}{\alpha}-a}\tau^{1-b}\right]'\mathrm{d}\tau\\
&\le t^{-\mu}C_{2p}C_{1,\alpha}^n n^{\frac{n}{\alpha}+a}\frac{1}{1-b}\left(1+\frac{1}{n-1}\right)^{\frac{n}{\alpha}+a}n^{b-1} \le t^{-\mu}C_{2p}K^{n-2\delta}n^{n-\delta}.
\end{aligned} \tag{3.45}$$

When n big enough, it holds that

$$\begin{aligned}
A_4 &\le C_{2p}C_{0,\alpha}t^{-\mu}K^{n-2\delta}n^{n-\delta}\int_{\frac{n-1}{n}}^{1}(1-\tau)^{-a}\tau^{-\frac{n}{\alpha}-b}\mathrm{d}\tau\\
&\le \frac{1}{1-a}C_{2p}C_{0,\alpha}t^{-\mu}K^{n-2\delta}n^{n-\delta}\left(1+\frac{1}{n-1}\right)^{\frac{n}{\alpha}+b}\left(\frac{1}{n}\right)^{1-a} \le t^{-\mu}C_{2p}K^{n-2\delta}n^{n-\delta}.
\end{aligned} \tag{3.46}$$

Next, we separate the proof for three cases to detal with A_5.

Case 1: $2/(\alpha-1) \le q \le 4/(\alpha-1)$. In this case, set $p = q/2 \ge 1/(\alpha-1)$. By Hölder inequality and (3.1), there holds

$$\|\partial_x^n\rho(s)H\rho(s)+\rho(s)\partial_x^n H\rho(s)\|_{L^p} \le 2C_q\|\partial_x^n\rho(s)\|_{L^q}\|\rho(s)\|_{L^q}.$$

Denote

$$B_n(t) := \sup_{0<s\le t}s^{\mu}\|\partial_x^n\rho(s)\|_{L^q}.$$

We have the following estimate for A_5 when n big enough:

$$\begin{aligned} A_5 &= \int_{\frac{n-1}{n}t}^{t} \|\partial_x G_\alpha(\cdot, t-s) * [\partial_x^n \rho(s) H\rho(s) + \rho(s)\partial_x^n H\rho(s)]\|_{L^q} \, ds \\ &\le C_{0,\alpha} \int_{\frac{n-1}{n}t}^{t} (t-s)^{-a} \|\partial_x^n \rho(s) H\rho(s) + \rho(s)\partial_x^n H\rho(s)\|_{L^p} \, ds \\ &\le 2C_q C_0 C_{0,\alpha} \int_{\frac{n-1}{n}t}^{t} (t-s)^{-a} s^{-1+a} \|\partial_x^n \rho(s)\|_{L^q} \, ds \\ &\le 2C_q C_0 C_{0,\alpha} t^{-\mu} \int_{\frac{n-1}{n}}^{1} (1-\tau)^{-a} \tau^{-\frac{n}{\alpha}-b} d\tau \cdot B_n(t) \\ &\le 2C_q C_0 C_{0,\alpha} t^{-\mu} \left(1 + \frac{1}{n-1}\right)^{\frac{n}{\alpha}+b} \left(\frac{1}{n}\right)^{1-a} B_n(t) \le \frac{C_q}{n^{\frac{1-a}{2}}} t^{-\mu} B_n(t), \end{aligned} \tag{3.47}$$

where C_0 is given in (3.20), and we used

$$\frac{2C_0 C_{0,\alpha}\left(1 + \frac{1}{n-1}\right)^{\frac{n}{\alpha}+b}}{n^{\frac{1-a}{2}}} < 1$$

for n large enough (due to $a < 1$). Notice that $C_{2p} = C_q$. Combining the above estimates yields

$$t^\mu \|\partial_x^n \rho(t)\|_{L^q} \le t^\mu (A_1 + A_2 + A_3 + A_4 + A_5) \le 4C_q K^{n-2\delta} n^{n-\delta} + \frac{C_q}{n^{\frac{1-a}{2}}} B_n(t).$$

Since $2/(\alpha - 1) \le q \le 4/(\alpha - 1)$, by definition (3.2), C_q is uniformly bounded. Therefore, for n, K large enough, the above inequality implies

$$B_n(t) \le K^{-\delta/2} K^{n-\delta} n^{n-\delta}.$$

Case 2: $1/(\alpha - 1) \le q < 2/(\alpha - 1)$. In this case, let $p = 1/(\alpha - 1)$. Then, we have

$$\begin{aligned} \|\partial_x^n \rho(s) H\rho(s) + \rho(s)\partial_x^n H\rho(s)\|_{L^p} &\le 2C_{2p} \|\partial_x^n \rho(s)\|_{L^{2p}} \|\rho(s)\|_{L^{2p}} \\ &\le 2C_0 C_{2p} K^{-\delta/2} K^{n-\delta} n^{n-\delta} s^{-\frac{n}{\alpha}-b}, \end{aligned}$$

where we used Case 1 for $L^{2p}(\mathbb{R})$ with $2p = 2/(\alpha - 1)$, and C_0 is given in (3.20). Then A_5 can be estimated as follows:

$$\begin{aligned} A_5 &= \int_{\frac{n-1}{n}t}^{t} \|\partial_x G_\alpha(\cdot, t-s) * [\partial_x^n \rho(s) H\rho(s) + \rho(s)\partial_x^n H\rho(s)]\|_{L^q} \, ds \\ &\le C_{0,\alpha} \int_{\frac{n-1}{n}t}^{t} (t-s)^{-a} \|\partial_x^n \rho(s) H\rho(s) + \rho(s)\partial_x^n H\rho(s)\|_{L^p} \, ds \\ &\le 2C_0 C_{0,\alpha} C_{2p} K^{-\delta/2} K^{n-\delta} n^{n-\delta} \int_{\frac{n-1}{n}t}^{t} (t-s)^{-a} s^{-\frac{n}{\alpha}-b} ds \\ &= 2C_0 C_{0,\alpha} C_{2p} K^{-\delta/2} K^{n-\delta} n^{n-\delta} t^{-\mu} \int_{\frac{n-1}{n}}^{1} (1-\tau)^{-a} \tau^{-\frac{n}{\alpha}-b} d\tau \\ &\le 2C_0 C_{0,\alpha} C_{2p} K^{-\delta/2} K^{n-\delta} n^{n-\delta} t^{-\mu} \left(1 + \frac{1}{n-1}\right)^{\frac{n}{\alpha}+b} \left(\frac{1}{n}\right)^{1-a}. \end{aligned}$$

Because $a < 1$, by $2p = 2/(\alpha - 1)$ and definition (3.2) of C_p, for n big enough, we have

$$2C_0 C_{0,\alpha} C_{2p} \left(1 + \frac{1}{n-1}\right)^{\frac{n}{\alpha}+b} \left(\frac{1}{n}\right)^{1-a} \le 1,$$

which implies

$$A_5 \le K^{-\delta/2} K^{n-\delta} n^{n-\delta} t^{-\mu}. \tag{3.48}$$

Combining (3.42), (3.44)–(3.46), and (3.48), we obtain

$$\|\partial_x^n \rho(t)\|_{L^q} \le \sum_{i=1}^{5} A_i \le 4C_{2p} K^{n-2\delta} n^{n-\delta} t^{-\mu} + K^{-\delta/2} K^{n-\delta} n^{n-\delta} t^{-\mu} \le K^{n-\delta} n^{n-\delta} t^{-\mu}.$$

Case 3: $\frac{4}{\alpha-1} < q \le \infty$. In this case, let $p = 2/(\alpha-1)$. Then with the same manner as Case 2, we could prove the results by using Case 1 for $L^{2p}(\mathbb{R})$ with $2p = 4/(\alpha-1)$.

From the above proof, we see that only the information of C_q for $2/(\alpha-1) \le q \le 4/(\alpha-1)$ was used, where C_q has a uniform bound. Hence the constant K is independent of q. □

C. Maximum principle in $L^p(\mathbb{R})$ $(p \ge 1)$ and global extension

In this subsection, we are going to finish the Proof of Theorem 1.2 by extending the solutions in Theorem 3.3 globally. We have the following maximum principle results:

Lemma 3.3. Let ρ be a nonnegative strong solution to (1.1). For $p \ge 1$, we have

$$\|\rho(t)\|_{L^p(\mathbb{R})} \le \|\rho(s)\|_{L^p(\mathbb{R})}, \quad t > s > 0. \tag{3.49}$$

Proof. For $p = 1$, we have

$$\|\rho(t)\|_{L^1} \equiv \|\rho_0\|_{L^1}, \quad t > 0.$$

For $p > 1$, we have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \rho^p(x,t)\mathrm{d}x = \int_{\mathbb{R}} \rho^{p-1}\partial_t\rho \mathrm{d}x = -\int_{\mathbb{R}} \rho^{p-1}\partial_x(\rho H\rho)\mathrm{d}x - \int_{\mathbb{R}} \rho^{p-1}\nu\Lambda^\alpha\rho \mathrm{d}x. \tag{3.50}$$

For the first term in the right-hand side of (3.50), we have

$$\begin{aligned} -\int_{\mathbb{R}} \rho^{p-1}\partial_x(\rho H\rho)\mathrm{d}x &= \frac{p-1}{p}\int_{\mathbb{R}} \partial_x\rho^p H\rho \mathrm{d}x \\ &= -\frac{p-1}{p}\int_{\mathbb{R}} \rho^p(x,t)\int_{\mathbb{R}} \frac{\rho(x,t)-\rho(y,t)}{|x-y|^2}\mathrm{d}y\mathrm{d}x \\ &= -\frac{p-1}{2p}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\rho^p(x,t)-\rho^p(y,t))(\rho(x,t)-\rho(y,t))}{|x-y|^2}\mathrm{d}y\mathrm{d}x \\ &\le 0. \end{aligned}$$

For the second term in the right-hand side of (3.50), we have

$$-\nu\int_{\mathbb{R}} \rho^{p-1}\Lambda^\alpha\rho \mathrm{d}x = -\frac{\nu}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\rho^{p-1}(x,t)-\rho^{p-1}(y,t))(\rho(x,t)-\rho(y,t))}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x \le 0.$$

Combining the above two inequalities and (3.50), we obtain (3.49). □

Theorem 3.4. *Assume $1 < \alpha \le 2$ and $0 \le \rho_0 \in L^{\frac{1}{\alpha-1}}(\mathbb{R})$. Then, the local mild solution ρ given by Theorem 3.1 can be extended globally.*

Proof. Due to Theorem 3.1, there exist $a > 0$, $T > 0$ and a unique local mild solution ρ to (1.1) in X_T^a such that

$$\sup_{0<t\le T} t^{\frac{\alpha-1}{2\alpha}}\|G_\alpha(\cdot,t)*\rho_0\|_{L^{\frac{2}{\alpha-1}}(\mathbb{R}^d)} < a, \quad \sup_{0<t\le T} t^{\frac{\alpha-1}{2\alpha}}\|\rho(t)\|_{L^{\frac{2}{\alpha-1}}} \le 2a.$$

Fix $0 < t_0 < T$, and combining Theorem 3.2 and Lemma 3.3 yields

$$\|\rho(s)\|_{L^{\frac{2}{\alpha-1}}} \le \|\rho(t_0)\|_{L^{\frac{2}{\alpha-1}}}, \quad \forall s \ge t_0. \tag{3.51}$$

From (3.51) and (3.6) with $k = 0$ and $p = q = \frac{2}{\alpha-1}$, we obtain

$$\|G_\alpha(\cdot,t)*\rho(s)\|_{L^{\frac{2}{\alpha-1}}} \le \|\rho(t_0)\|_{L^{\frac{2}{\alpha-1}}}, \quad s \ge t_0.$$

Set

$$T_0 := \left(\frac{a}{\|\rho(t_0)\|_{L^{\frac{2}{\alpha-1}}}}\right)^{\frac{2\alpha}{\alpha-1}}.$$

For $s \in [t_0, T]$, we have

$$\sup_{0<t\le T_0} t^{\frac{\alpha-1}{2\alpha}} \|G_\alpha(\cdot,t) * \rho(s)\|_{L^{\frac{2}{\alpha-1}}} \le \sup_{0<t\le T_0} t^{\frac{\alpha-1}{2\alpha}} \cdot \|\rho(t_0)\|_{L^{\frac{2}{\alpha-1}}} \le a.$$

Due to Theorem 3.1, we can extend our solution to $s + T_0$ for any $t_0 \le s \le T$. Moreover, this time span T_0 is uniform for any $s > t_0$. This proves global existence. □

Remark 3.1 (The case for $\gamma > 0$). When $\gamma > 0$ and $1 < \alpha \le 2$, the above method for $\gamma = 0$ is not directly applicable. However, we believe the solution to (1.1) is also analytic. For example, consider (1.1) with $\gamma > 0$ and $\alpha = 2$, i.e.,

$$\begin{cases} \partial_t\tilde\rho + \partial_x[\tilde\rho(H\tilde\rho - \gamma x)] = \nu\Delta\tilde\rho, & t > 0, \quad x \in \mathbb{R}, \\ \tilde\rho(x,0) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.52}$$

Due to transformation (1.4), we have

$$\tilde\rho(x,t) = e^{\gamma t}\rho\left(e^{\gamma t}x, \frac{e^{2\gamma t} - 1}{2\gamma}\right),$$

where ρ is the solution for (1.1) with $\gamma = 0$ and $\alpha = 2$. Hence

$$\partial_x^n\tilde\rho(x,t) = e^{(n+1)\gamma t}\partial_y^n\rho\left(y, \frac{e^{2\gamma t} - 1}{2\gamma}\right), \quad y = e^{\gamma t}x.$$

By Theorem 3.3 and the above relation, we have

$$\|\partial_x^n\tilde\rho(t)\|_{L^\infty} = e^{(n+1)\gamma t}\left\|\partial_x^n\tilde\rho\left(\frac{e^{2\gamma t} - 1}{2\gamma}\right)\right\|_{L^\infty} \le K^n n^n\left(\frac{2\gamma e^{2\gamma t}}{e^{2\gamma t} - 1}\right)^{\frac{n+1}{2}},$$

Because

$$\lim_{\gamma\to 0}\frac{2\gamma e^{2\gamma t}}{e^{2\gamma t} - 1} = \frac{1}{t},$$

the above estimate is compatible with (3.39) for $\alpha = 2$ as $\gamma \to 0$, and it also implies the spatial analyticity of the solutions to (3.52).

IV. GLOBAL SPATIAL ANALYTIC SOLUTIONS FOR THE CRITICAL CASE $\alpha = 1$

In this section, we are going to prove the existence and uniqueness of spatial analytic solutions to the critical equation (1.1) with $\alpha = 1$, $\gamma > 0$ for initial data $-\nu < \rho_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ $(s > 1/2)$. When $\rho_0 \ge -\mu$ for $0 \le \mu < \nu$, the spatial analytic solutions exist at least in the time interval $(0, T)$ for $T = \frac{1}{\gamma}\ln\left(\frac{2\nu}{\mu} - 1\right)$ ($T = \infty$ when $\mu = 0$). When $\rho_0 \ge 0$, the spatial analytic solutions exist globally and pointwise convergent to the steady state will also be shown.

A. Well-posedness of complex Burgers equation on the upper half plane

First, we derive the complex Burgers equation from Eq. (1.1) with $\alpha = 1$. For $f, g \in L^p(\mathbb{R})$ $(p > 1)$, the Hilbert transform has the following properties (see e.g., Ref. 28):

$$H(Hf) = -f, \quad \partial_x(Hf) = H\partial_x f, \text{ and } H(fHg + gHf) = HfHg - fg.$$

Applying the Hilbert transform to the Eq. (1.1) yields

$$\partial_t(H\rho) + H\rho H\partial_x\rho - \rho\partial_x\rho - \gamma\partial_x H(\rho x) = \nu\partial_x\rho.$$

Moreover, for $g \in L^1(\mathbb{R})$, we have

$$\begin{aligned} H(xg(x)) &= \frac{1}{\pi}p.v.\int_{\mathbb{R}}\frac{yg(y)}{x-y}\mathrm{d}y = \frac{1}{\pi}p.v.\int_{\mathbb{R}}\frac{(y-x)g(y)}{x-y}\mathrm{d}y + \frac{1}{\pi}p.v.\int_{\mathbb{R}}\frac{xg(y)}{x-y}\mathrm{d}y \\ &= xHg(x) - \frac{1}{\pi}\int_{\mathbb{R}}g(x)\mathrm{d}x, \end{aligned} \tag{4.1}$$

which implies

$$H(\rho x) = -\frac{\|\rho(t)\|_{L^1}}{\pi} + ux. \tag{4.2}$$

Combining the above two equations yields

$$\partial_t u + u\partial_x u - \rho\partial_x\rho - \gamma\partial_x(ux) = \nu\partial_x\rho. \tag{4.3}$$

Set

$$f = u - i\rho, \qquad u = H\rho.$$

Then f gives the trace of a Holomorphic function in the upper half-plane. Combining (1.1) and (4.3) yields

$$\partial_t f + f\partial_x f - \gamma\partial_x(fx) = i\nu\partial_x f, \quad x \in \mathbb{R}, \quad t > 0.$$

This corresponds to the following complex equation in $\mathbb{C}_+$:

$$\partial_t f + f\partial_z f - \gamma\partial_z(fz) = \partial_t f + f\partial_z f - \gamma z\partial_z f - \gamma f = i\nu\partial_z f, \quad t > 0. \tag{4.4}$$

By the linear transformation $g(z,t) = f(z,t) - \gamma z$, we have

$$\begin{aligned}\partial_t g + g\partial_z g - \gamma^2 z &= \partial_t f + (f - \gamma z)(\partial_z f - \gamma) - \gamma^2 z\\ &= \partial_t f + f\partial_z f - \gamma z\partial_z f - \gamma f = i\nu(\partial_z g + \gamma),\end{aligned}$$

which is

$$\partial_t g + (g - i\nu)\partial_z g = \gamma^2 z + i\nu\gamma. \tag{4.5}$$

Next, we derive the initial data for (4.5) with initial data ρ_0 for Eq. (1.1). Let $\rho_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ with $s > 1/2$ be the initial data for Eq. (1.1). The initial data ρ_0 can be extended to a $\mathbb{C}_+$-holomorphic function by Hilbert transform (also called Stieltjes transform, Borel transform or Markov function) for positive measures:

$$f_0(z) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{\rho_0(s)}{z - s}\mathrm{d}s, \quad z = x + iy \in \mathbb{C}_+. \tag{4.6}$$

Direct calculation shows that

$$\begin{aligned}f_0(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\rho_0(s)}{z - s}\mathrm{d}s &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{x - s}{y^2 + (x - s)^2}\rho_0(s)\mathrm{d}s - i\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{y^2 + (x - s)^2}\rho_0(s)\mathrm{d}s\\ &=: R\rho_0(x,y) - iP\rho_0(x,y),\end{aligned}$$

where $P\rho_0(x,y)$ and $R\rho_0(x,y)$ are given by the convolution of ρ_0 with the Poisson kernel and the conjugate Poisson kernel given by

$$P_y(x) := \frac{1}{\pi}\frac{y}{y^2 + x^2} \text{ and } R_y(x) := \frac{1}{\pi}\frac{x}{y^2 + x^2}. \tag{4.7}$$

Furthermore, we have

$$\lim_{y\to 0+}[R\rho_0(x,y) - iP\rho_0(x,y)] = H\rho_0(x) - i\rho_0(x) \text{ for a.e. } x \in \mathbb{R}.$$

Hence, $f_0(x) = H\rho_0(x) - i\rho_0(x)$ for a.e. $x \in \mathbb{R}$. Let

$$g_0(z) := f_0(z) - \gamma z, \quad z = x + iy \in \mathbb{C}_+. \tag{4.8}$$

Then, g_0 is a $\mathbb{C}_+$-holomorphic function. Consider the following Cauchy problem of the Burgers type equation in $\mathbb{C}_+$:

$$\begin{cases}[\partial_t g + (g - i\nu)\partial_z g](z,t) = \gamma^2 z + i\nu\gamma, & z = x + iy \in \mathbb{C}_+,\\ g(z,0) = g_0(z).\end{cases} \tag{4.9}$$

Next, we prove the existence and uniqueness of $\mathbb{C}_+$-holomorphic solutions to (4.9) by the method of characteristics. Consider the characteristics given by

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(w,t) = g(Z(w,t),t) - i\nu, \quad Z(w,0) = w \in \mathbb{C}_+. \tag{4.10}$$

Then,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}Z(w,t) = \frac{\mathrm{d}}{\mathrm{d}t}g(Z(w,t),t) = [\partial_t g + (g - i\nu)\partial_z g](Z(w,t),t) = \gamma^2 Z(w,t) + i\nu\gamma,$$

with initial date

$$Z(w,0) = w, \quad \frac{\mathrm{d}}{\mathrm{d}t} Z(w,t)\Big|_{t=0} = g_0(w) - iv, \quad w \in \mathbb{C}_+.$$

Equation (4.10) gives the following complex trajectories:

$$Z(w,t) = \begin{cases} \left(w + i\frac{v}{\gamma}\right)\cosh \gamma t + \frac{1}{\gamma}(g_0(w) - iv)\sinh \gamma t - i\frac{v}{\gamma}, & \gamma > 0, \\ (g_0(w) - iv)t + w = (f_0(w) - iv)t + w, & \gamma = 0. \end{cases} \tag{4.11}$$

Here, we only treat the case for $\gamma > 0$ and the proof of the case $\gamma = 0$ is similar. Let

$$Z(w,t) = Z_1(x,y,t) + iZ_2(x,y,t), \quad w = x + iy \in \mathbb{C}_+,$$

and we have real part:

$$\begin{aligned} Z_1(x,y,t) &= x\cosh \gamma t + \frac{1}{\gamma} R\rho_0(x,y)\sinh \gamma t - x\sinh \gamma t \\ &= xe^{-\gamma t} + \frac{1}{\gamma} R\rho_0(x,y)\sinh \gamma t, \end{aligned} \tag{4.12}$$

and imaginary part:

$$\begin{aligned} Z_2(x,y,t) &= \left(y + \frac{v}{\gamma}\right)\cosh \gamma t - \left(\frac{1}{\gamma}P\rho_0(x,y) + \frac{v}{\gamma}\right)\sinh \gamma t - y\sinh \gamma t - \frac{v}{\gamma} \\ &= \left(y + \frac{v}{\gamma}\right)e^{-\gamma t} - \frac{1}{\gamma}P\rho_0(x,y)\sinh \gamma t - \frac{v}{\gamma}. \end{aligned} \tag{4.13}$$

Because the initial data $g_0(w)$ in (4.9) is a $\mathbb{C}_+$-holomorphic function, $Z(w,t)$ given by (4.11) is $\mathbb{C}_+$-holomorphic of w for any $t \geq 0$. Next, we give a lemma to show that for any fixed time $t > 0$ the backward characteristics of (4.11) are well defined on the set $\overline{\mathbb{C}_+}$. This result is an analogy of [Ref. 9, Lemma 2.2]. We have:

Lemma 4.1. Let $0 \leq \mu < v$ *and* $-\mu \leq \rho_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ *with* $s > 1/2$. *Denote* $T = \frac{1}{\gamma}\ln\left(\frac{2v}{\mu} - 1\right)$ ($T = \infty$ *when* $\mu = 0$). *Then for fixed* $0 < t_0 < T$ *and fixed* $Z = Z_1 + iZ_2 \in \overline{\mathbb{C}_+}$, *there exists a unique* $w = x + iy \in \mathbb{C}_+$ *such that* (4.12) *and* (4.13) *hold.*

Proof. Given $t_0 > 0$, denote

$$a := e^{-\gamma t_0}, \quad b := \frac{1}{\gamma}\sinh \gamma t_0.$$

Then (4.12) and (4.13) become

$$Z_1 = ax + bR\rho_0(x,y), \quad Z_2 = ay - bP\rho_0(x,y) - (1-a)\frac{v}{\gamma}.$$

Step 1. In this step, we prove that for any $x \in \mathbb{R}$, there exists a unique $y > 0$ satisfies (4.13) for $Z_2 \geq 0$ and $0 < t_0 < T$. Notice that

$$\frac{1-a}{b}\frac{v}{\gamma} = 2v\frac{1 - e^{-\gamma t_0}}{e^{\gamma t_0} - e^{-\gamma t_0}} = \frac{2v}{e^{\gamma t_0} + 1} > \frac{2v}{e^{\gamma T} + 1} = \mu. \tag{4.14}$$

Because $\rho_0 \in L^\infty$, we know $P\rho_0(x,y)$ is a bounded function on $\mathbb{R}^2_+$. By the property of Poisson kernel, we have $\lim_{y\to+\infty} P\rho_0(x,y) = 0$ and hence

$$\begin{aligned} &\lim_{y\to+\infty} Z_2(x,y,t_0) = +\infty, \\ &\lim_{y\to 0+} Z_2(x,y,t_0) = -b\rho_0(x) + (a-1)\frac{v}{\gamma} = -b\left[\rho_0(x) + \frac{1-a}{b}\frac{v}{\gamma}\right] < 0. \end{aligned} \tag{4.15}$$

Hence, for any fixed $Z_2 \geq 0$, there exists a point $y > 0$ depending on x such that

$$Z_2 = a\left(y + \frac{v}{\gamma}\right) - bP\rho_0(x,y) - \frac{v}{\gamma}.$$

Next, we prove the uniqueness of y. Suppose there exist $y_1 > y_2$ such that

$$Z_2 = ay_1 - bP\rho_0(x, y_1) + (a-1)\frac{\nu}{\gamma} = ay_2 - bP\rho_0(x, y_2) + (a-1)\frac{\nu}{\gamma}.$$

Because $P\rho_0(x,y) + \mu = P(\rho_0 + \mu)(x,y) > 0$, we have

$$y_1, y_2 > Z_2/a - (1 - 1/a)\nu/\gamma - \mu b/a,$$

and

$$\frac{P(\rho_0+\mu)(x,y_1)}{y_1 - Z_2/a + (1-1/a)\nu/\gamma + \mu b/a} = \frac{P(\rho_0+\mu)(x,y_2)}{y_2 - Z_2/a + (1-1/a)\nu/\gamma + \mu b/a} = \frac{a}{b}.$$

Because function

$$h(y) = \frac{y}{y - Z_2/a + (1-1/a)\nu/\gamma + \mu b/a} \cdot \frac{1}{y^2 + (x-s)^2}$$

is a decreasing function for $y > Z_2/a - (1-1/a)\nu/\gamma + \mu b/a$, we obtain a contradiction.

Now we denote by $y_{Z_2}(x) > 0$ the solution of (4.13) with fixed $Z_2 \geq 0$, $t_0 > 0$ and $x \in \mathbb{R}$. Hence, we obtain

$$a\left(y_{Z_2}(x) + \frac{\nu}{\gamma}\right) - \frac{\nu}{\gamma} - Z_2 = bP\rho_0(x, y_{Z_2}(x)). \tag{4.16}$$

Step 2. In this step, we prove there exits a unique x satisfies (4.12) for fixed Z_1, Z_2 and t_0. Since $\rho_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ $(s > 1/2)$, it follows that $H\rho_0 \in L^\infty(\mathbb{R})$ and therefore $R\rho_0 = PH\rho_0$ is a bounded function over $\mathbb{R}^2_+$. Furthermore,

$$\lim_{x\to\pm\infty}[ax + bR\rho_0(x, y_{Z_2}(x))] = \pm\infty. \tag{4.17}$$

Hence, for any $Z_1 \in \mathbb{R}$, we can find a $x \in \mathbb{R}$ such that

$$Z_1 = ax + bR\rho_0(x, y_{Z_2}(x)).$$

To prove the uniqueness, we only have to prove the following function

$$q(x) = ax + bR\rho_0(x, y_{Z_2}(x))$$

is an increasing function. Taking derivative of (4.16) with respect to x gives

$$\frac{\mathrm{d}}{\mathrm{d}x} y_{Z_2}(x) = \frac{\partial_x P\rho_0(x, y_{Z_2}(x))}{a/b - \partial_y P\rho_0(x, y_{Z_2}(x))}. \tag{4.18}$$

Use (4.18) and the Cauchy–Riemann equations

$$\partial_x R\rho_0 = -\partial_y P\rho_0, \quad \partial_x P\rho_0 = \partial_y R\rho_0, \tag{4.19}$$

and taking derivative of $q(x)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}x} q(x) = \frac{b(a/b + \partial_x R\rho_0)^2 + b(\partial_x P\rho_0)^2}{a/b + \partial_x R\rho_0}(x, y_{Z_2}(x)).$$

To prove the increasing of $q(x)$, it suffices to show

$$a/b + \partial_x R\rho_0(x,y) > 0 \tag{4.20}$$

for any $(x,y) \in \mathbb{R}^2_+$ satisfying $y > 0$ and $a(y + \frac{\nu}{\gamma}) - bP\rho_0(x,y) - \frac{\nu}{\gamma} \geq 0$, i.e., $a(y + \frac{\nu}{\gamma}) - bP(\rho_0 + \mu)(x,y) - \frac{\nu}{\gamma} + b\mu \geq 0$. We prove this by a contradiction argument. Suppose that

$$a/b + \partial_x R\rho_0(x_0, y_0) \leq 0$$

for some point $(x_0, y_0) \in \mathbb{R}^2_+$ with

$$ay_0 - bP(\rho_0 + \mu)(x_0, y_0) \geq (1-a)\frac{\nu}{\gamma} - b\mu > 0, \tag{4.21}$$

where we used (4.14) in the last inequality. Due to

$$\int_{\mathbb{R}} \frac{-y^2 + s^2}{(y^2 + s^2)^2} \mathrm{d}s = 0, \quad y > 0,$$

we have

$$
\begin{aligned}
-a/b \geq \partial_x R\rho_0(x_0,y_0) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{y_0^2-(x_0-s)^2}{[y_0^2+(x_0-s)^2]^2}\rho_0(s)\mathrm{d}s\\
&= \frac{1}{\pi}\int_{\mathbb{R}} \frac{y_0^2-(x_0-s)^2}{[y_0^2+(x_0-s)^2]^2}[\rho_0(s)+\mu]\mathrm{d}s\\
&> \frac{1}{\pi}\int_{\mathbb{R}} \frac{-y_0^2-(x_0-s)^2}{[y_0^2+(x_0-s)^2]^2}[\rho_0(s)+\mu]\mathrm{d}s = -\frac{P(\rho_0+\mu)(x_0,y_0)}{y_0},
\end{aligned}
$$

which is a contradiction with (4.21). □

From the above lemma, we know that the backward characteristics are well defined on $\overline{\mathbb{C}_+}$ in the time interval $(0,T)$. More importantly, for any $Z\in\overline{\mathbb{C}_+}$ the initial point w must be an interior point in $\mathbb{C}_+$. For any $t\geq 0$, we denote the backward characteristics as:

$$Z^{-1}(\cdot,t):\overline{\mathbb{C}_+}\to\mathbb{C}_+.$$

From the uniqueness in Lemma 4.1, $Z^{-1}(\cdot,t)$ is an 1–1 map.

B. Spatial analytic solutions to Eq. (1.1) with $\alpha = 1$

By Lemma 4.1, we have the following theorem which covers the results of Theorem 1.3:

Theorem 4.1. *Let $0\leq\mu<\nu$ and $-\mu\leq\rho_0\in L^1(\mathbb{R})\cap H^s(\mathbb{R})$ with $s>1/2$. Denote $T=\frac{1}{\gamma}\ln\left(\frac{2\nu}{\mu}-1\right)$. Then, we have*

(i) *The complex Burgers equation (4.9) has a unique $\overline{\mathbb{C}_+}$-holomorphic solution $g(\cdot,t)$ for $t\in(0,T)$, and $\frac{\partial^k}{\partial t^k}g(\cdot,t)$ is a holomorphic function of z on $\overline{\mathbb{C}_+}$ for any positive integer k and $t>0$.*

(ii) *For any $t>0$, the trace of $f(z,t)=g(z,t)+\gamma z$ on the real line gives an spatial analytic solution $\rho(x,t)\geq-\mu e^{\gamma t}$ to the Eq. (1.1) with $\rho(x,0)=\rho_0(x)$ and $\frac{\partial^k}{\partial t^k}\rho(x,t)$ is an analytic function of $x\in\mathbb{R}$ for any positive integer k. Moreover, the total mass $\|\rho(t)\|_{L^1}$ is conserved:*

$$\|\rho(t)\|_{L^1}=\|\rho_0\|_{L^1}. \tag{4.22}$$

(iii) *For $\gamma>0$ and $\mu=0$, the solution $g(z,t)$ exists globally and converges to the steady state:*

$$\lim_{t\to\infty}g(z,t)=i\nu-\sqrt{(\gamma z+i\nu)^2-2\gamma},\quad \forall z\in\mathbb{C}_+,$$

and (1.17) in Theorem 1.3 holds.

Proof. **Step 1.** Proof of (i). From Lemma 4.1, we have $\overline{\mathbb{C}_+}\subset\{Z(w,t): w\in\mathbb{C}_+\}$ and $Z^{-1}(\cdot,t)$ is well defined on $\overline{\mathbb{C}_+}$ for any fixed time $t>0$. Denote the preimage of $Z(\cdot,t)$ as:

$$Z^{-1}(\overline{\mathbb{C}_+},t):=\{w\in\mathbb{C}_+;\quad Z(w,t)\in\overline{\mathbb{C}_+}\}.$$

Denote

$$a(t):=e^{-\gamma t},\quad b(t):=\frac{1}{\gamma}\sinh\gamma t.$$

For $(x,y)\in\mathbb{R}^2_+$ and $Z_2(x,y,t)\geq 0$, by the Cauchy–Riemann equations (4.19), we have

$$
\begin{aligned}
|Z_w(w,t)| &= \left|\frac{\partial(Z_1,Z_2)}{\partial(x,y)}\right|(x,y) = \begin{vmatrix}\partial_x Z_1 & \partial_y Z_1\\ \partial_x Z_2 & \partial_y Z_2\end{vmatrix} = \begin{vmatrix} a(t)+b(t)\partial_x R\rho_0 & b(t)\partial_y R\rho_0\\ -b(t)\partial_x P\rho_0 & a(t)-b(t)\partial_y P\rho_0\end{vmatrix}\\
&= [a(t)+b(t)\partial_x R\rho_0]^2+[b(t)\partial_x P\rho_0]^2\big|_{(x,y)}>0.
\end{aligned} \tag{4.23}
$$

Due to (4.15) and (4.17), we obtain

$$|Z(w,t)|\to+\infty \text{ as } |w|\to+\infty,$$

which means $Z(\cdot,t)$ is proper (Ref. 22, Definition 6.2.2). By the Hadamard's global inverse function theorem (Ref. 22, Theorem 6.2.8), there exists an inverse function $Z^{-1}(\cdot,t)$ such that

$$Z^{-1}(\cdot,t):\ \overline{\mathbb{C}_+}\to Z^{-1}(\overline{\mathbb{C}_+},t)$$

is a bijection. We also know Z^{-1} is $\overline{\mathbb{C}_+}$-holomorphic since Z is $\mathbb{C}_+$-holomorphic. Moreover, for any $z \in \overline{\mathbb{C}_+}$, there exists $w = Z^{-1}(z,t) \in \mathbb{C}_+$. Due to $z = Z(Z^{-1}(z,t),t) \in \overline{\mathbb{C}_+}$ and $|Z_w(w,t)| \neq 0$ [by ((4.23)], we have

$$\partial_t Z^{-1}(z,t) = -\frac{\partial_t Z(w,t)}{\partial_w Z(w,t)}, \quad w = Z^{-1}(z,t).$$

Because of (4.11), we know $\frac{\partial^k}{\partial t^k} Z(w,t)$ is $\mathbb{C}_+$-holomorphic for any positive integer k. Hence, $\frac{\partial^k}{\partial t^k} Z^{-1}(z,t)$ is $\overline{\mathbb{C}_+}$-holomorphic for any positive integer k. From (4.11), we have

$$z = \left(Z^{-1}(z,t) + i\frac{\nu}{\gamma}\right)\cosh \gamma t + \frac{1}{\gamma}(g_0(Z^{-1}(z,t)) - i\nu)\sinh \gamma t - i\frac{\nu}{\gamma}, \quad z \in \overline{\mathbb{C}_+}. \tag{4.24}$$

By (4.10), we obtain

$$g(Z(w,t),t) = \frac{\mathrm{d}}{\mathrm{d}t}Z(w,t) + i\nu = (\gamma w + i\nu)\sinh \gamma t + (g_0(w) - i\nu)\cosh \gamma t + i\nu.$$

Hence,

$$g(z,t) = \gamma Z^{-1}(z,t)\sinh \gamma t + g_0(Z^{-1}(z,t))\cosh \gamma t + i\nu(1 - e^{-\gamma t}), \tag{4.25}$$

which is a $\overline{\mathbb{C}_+}$-holomorphic solution to the complex Burgers equation (4.9) satisfying $g(z,0) = g_0(z)$. Moreover, due to the time regularity for $Z^{-1}(z,t)$, we know that $\frac{\partial^k}{\partial t^k} g(z,t)$ is $\overline{\mathbb{C}_+}$-holomorphic for any positive integer k and $t > 0$.

Step 2. Proof of (ii). A $\overline{\mathbb{C}_+}$-holomorphic solution to (4.4) is given by

$$f(z,t) := g(z,t) + \gamma z, \quad z \in \overline{\mathbb{C}_+}, \quad t > 0, \tag{4.26}$$

with initial data $f_0(z) = R\rho_0(x,y) - iP\rho_0(x,y)$, $z = x + iy \in \mathbb{C}_+$. Combining (4.24) and (4.25), we obtain for $z \in \overline{\mathbb{C}_+}$:

$$\begin{aligned} z &= e^{-\gamma t}Z^{-1}(z,t) + \frac{1}{\gamma}f_0(Z^{-1}(z,t))\sinh \gamma t + i\frac{\nu}{\gamma}(e^{-\gamma t} - 1), \\ f(z,t) &= f_0(Z^{-1}(z,t))e^{\gamma t}. \end{aligned} \tag{4.27}$$

Consider the trace of $f(z,t)$ on the real line and define:

$$f(x,t) =: u(x,t) - i\rho(x,t).$$

Due to Lemma 4.1, for any $x \in \mathbb{R}$, we have $Z^{-1}(x,t) =: a_x + ib_x \in \mathbb{C}_+$ with some positive real number $b_x > 0$. From (4.27), we have

$$f(x,t) = f_0(a_x + ib_x)e^{\gamma t} = R\rho_0(a_x,b_x)e^{\gamma t} - iP\rho_0(a_x,b_x)e^{\gamma t}$$

Therefore,

$$\rho(x,t) = P\rho_0(a_x,b_x)e^{\gamma t} = P(\rho_0 + \mu)(a_x,b_x)e^{\gamma t} - \mu e^{\gamma t} \geq -\mu e^{\gamma t}, \quad x \in \mathbb{R}. \tag{4.28}$$

Hence, $\rho(x,t)$ is a spatial analytic solution of (1.1). Moreover, by the uniqueness of solutions to the characteristics equation (4.9) we know solutions to Eq. (1.1) is unique.

Step 3. The proof of (iii) follows from the method in Ref. 30 and we put it into Appendix D. □

Remark 4.1. (1) When $\nu = 0$, the function ρ_∞ given by (1.17) becomes [Ref. 15, Eq. (2.15)]. For $\gamma = 0$, we have $\rho_\infty = 0$.

(2) Comparing with Theorem 4.1 (Ref. 9, Theorem 4.1) and (Ref. 9, Theorem 4.8), a nature conjecture is that the $\|\partial_x H\rho\|_{L^\infty}$ blows up in finite time when initial data satisfies $\rho_0(x_0) < 0$ for some $x_0 \in \mathbb{R}$. According to Ref. 15, Remark 2.1, the blow-up behavior is much more complicated for $\gamma > 0$ and $\nu = 0$ (blow-up along a curve), while the blow-up behavior for $\gamma = \nu = 0$ is simpler (blow up along a straight line).

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Yu Gao: Conceptualization (equal); Methodology (equal); Writing – original draft (equal). **Cong Wang**: Conceptualization (equal); Formal analysis (equal); Methodology (equal). **Xiaoping Xue**: Conceptualization (equal); Supervision (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: PROOF OF LEMMA 3.1

Proof. For $\ell \geq 0$, we claim that

$$\Lambda^{k\ell} G_\alpha(x,t) = t^{-\frac{k\ell+1}{\alpha}} (\Lambda^{k\ell} G_\alpha)\Big(t^{-\frac{1}{\alpha}} x, 1\Big), \quad \Lambda^{k\ell} \partial_x G_\alpha(x,t) = t^{-\frac{k\ell+2}{\alpha}} (\Lambda^{k\ell} \partial_x G_\alpha)\Big(t^{-\frac{1}{\alpha}} x, 1\Big). \tag{A1}$$

Actually, we have

$$[\mathcal{F}(\Lambda^{k\ell} G_\alpha)](\xi,t) = |\xi|^{k\ell} e^{-\nu t|\xi|^\alpha},$$

and by changing of variable $\eta = t^{\frac{1}{\alpha}} \xi$ we obtain

$$\begin{aligned}\Lambda^{k\ell} G_\alpha(x,t) &= \mathcal{F}^{-1}[\mathcal{F}(\Lambda^{k\ell} G_\alpha)](x,t) = \int_{\mathbb{R}} e^{i\xi x} |\xi|^{k\ell} e^{-\nu t|\xi|^\alpha} \mathrm{d}\xi \\ &= t^{-\frac{k\ell+1}{\alpha}} \int_{\mathbb{R}} e^{it^{-\frac{1}{\alpha}} x\cdot\eta} |\eta|^{k\ell} e^{-\nu|\eta|^\alpha} \mathrm{d}\eta = t^{-\frac{k\ell+1}{\alpha}} (\Lambda^{k\ell} G_\alpha)\Big(t^{-\frac{1}{\alpha}} x, 1\Big).\end{aligned}$$

The proof for the second equality in (A1) is the same. Hence, for $1 \leq q \leq \infty$, we have

$$\|\Lambda^{k\ell} G_\alpha(\cdot,t)\|_{L^q} = t^{-\frac{k\ell+1}{\alpha}+\frac{1}{\alpha q}} \|\Lambda^{k\ell} G_\alpha(\cdot,1)\|_{L^q} \tag{A2}$$

and

$$\|\Lambda^{k\ell} \partial_x G_\alpha(\cdot,t)\|_{L^q} = t^{-\frac{k\ell+2}{\alpha}+\frac{1}{\alpha q}} \|\Lambda^{k\ell} \partial_x G_\alpha(\cdot,1)\|_{L^q}, \tag{A3}$$

which implies

$$\left\|\Lambda^{\ell} G_\alpha\Big(\cdot,\frac{1}{k}\Big)\right\|_{L^p} = \Big(\frac{1}{k}\Big)^{-\frac{\ell+1}{\alpha}+\frac{1}{\alpha p}} \|\Lambda^{\ell} G_\alpha(\cdot,1)\|_{L^p} \leq C_{\ell,\alpha} k^{\frac{\ell+1}{\alpha}-\frac{1}{\alpha p}} \tag{A4}$$

and

$$\left\|\Lambda^{\ell} \partial_x G_\alpha\Big(\cdot,\frac{1}{k}\Big)\right\|_{L^p} = \Big(\frac{1}{k}\Big)^{-\frac{\ell+2}{\alpha}+\frac{1}{\alpha p}} \|\Lambda^{\ell} \partial_x G_\alpha(\cdot,1)\|_{L^p} \leq C_{\ell,\alpha} k^{\frac{\ell+2}{\alpha}-\frac{1}{\alpha p}}. \tag{A5}$$

Due to

$$\Lambda^{k\ell} G_\alpha(\cdot,1) = \Lambda^{\ell} G_\alpha\Big(\cdot,\frac{1}{k}\Big) * \Lambda^{\ell} G_\alpha\Big(\cdot,\frac{1}{k}\Big) * \cdots * \Lambda^{\ell} G_\alpha\Big(\cdot,\frac{1}{k}\Big)$$

and

$$\Lambda^{k\ell} \partial_x G_\alpha(\cdot,1) = \Lambda^{\ell} G_\alpha\Big(\cdot,\frac{1}{k}\Big) * \cdots * \Lambda^{\ell} G_\alpha\Big(\cdot,\frac{1}{k}\Big) * \Lambda^{\ell} \partial_x G_\alpha\Big(\cdot,\frac{1}{k}\Big),$$

by Young's convolution inequality, we obtain (3.4) and (3.5). Combining (3.4) and Young's inequality for convolution, we obtain (3.6) and (3.7).

The proof of (3.8) follows similarly as [Ref. 7, Lemma 2.1]. Assume $f_n \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ such that $f_n \to f$ in $L^p(\mathbb{R})$. From (3.6) with $\ell = 0$, we have

$$\|G_\alpha(\cdot,t) * f_n\|_{L^q} \le C\|f_n\|_{L^q}, \quad \forall n \in \mathbb{N}.$$

Hence, for $q > p$ we obtain

$$\lim_{t\to 0} t^{\frac{1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)}\|G_\alpha(\cdot,t) * f_n\|_{L^q} = 0, \quad \forall n \in \mathbb{N}.$$

According to (3.6) with $\ell = 0$, we also have

$$t^{\frac{1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)}\|G_\alpha(\cdot,t) * f_n - G_\alpha(\cdot,t) * f\|_{L^q} \le C\|f_n - f\|_{L^p}, \tag{A6}$$

which converges to zero independent of $t > 0$. This implies (3.8). □

APPENDIX B: PROOF OF TIME CONTINUITY OF $S\rho(t)$ IN THEOREM 3.1

Proof. Since the first part $G_\alpha(\cdot,t) * \rho_0$ corresponds to the solution of the fractional heat equation, it is continuous concerning t in space $L^{\frac{1}{\alpha-1}}(\mathbb{R})$. Hence, we only need to show the continuity of the second term

$$\rho_2(x,t) := \int_0^t \partial_x G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s))\mathrm{d}s.$$

Let $t > \tau > 0$ and we have

$$\begin{aligned}\|\rho_2(t) - \rho_2(\tau)\|_{L^{\frac{1}{\alpha-1}}} \le& \left\|\int_\tau^t \partial_x G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s))\mathrm{d}s\right\|_{L^{\frac{1}{\alpha-1}}}\\ &+ \left\|\int_0^\tau \partial_x G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s)) - \partial_x G_\alpha(\cdot,\tau-s) * (\rho(s)H\rho(s))\mathrm{d}s\right\|_{L^{\frac{1}{\alpha-1}}} =: I_1 + I_2.\end{aligned}$$

For I_1, we have

$$I_1 = \left\|\int_\tau^t \partial_x G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s))\mathrm{d}s\right\|_{L^{\frac{1}{\alpha-1}}} \le Ca^2\int_{\tau/t}^1 (1-s)^{-\frac{1}{\alpha}} s^{-\frac{\alpha-1}{\alpha}}\mathrm{d}s \to 0 \text{ as } t \to \tau.$$

For I_2, set $g(x,s) := \partial_x G_\alpha(\cdot,\tau-s) * (\rho(s)H\rho(s))$ for $0 < s < \tau$, and then

$$\partial_x G_\alpha(\cdot,t-s) * (\rho(s)H\rho(s)) = G_\alpha(\cdot,t-\tau) * g(s).$$

We have

$$I_2 = \left\|\int_0^\tau G_\alpha(\cdot,t-\tau) * g(s) - g(s)\mathrm{d}s\right\|_{L^{\frac{1}{\alpha-1}}} \le \int_0^\tau \|G_\alpha(\cdot,t-\tau) * g(s) - g(s)\|_{L^{\frac{1}{\alpha-1}}}\mathrm{d}s. \tag{B1}$$

Next, we estimate the integrand $\|G_\alpha(\cdot,t-\tau) * g(s) - g(s)\|_{L^{\frac{1}{\alpha-1}}}$. For arbitrary $r > 0$, by Jensen's inequality we have

$$\begin{aligned}\|G_\alpha(\cdot,t-\tau) * g(s) - g(s)\|_{L^{\frac{1}{\alpha-1}}}^{\frac{1}{\alpha-1}} &= \int_{\mathbb{R}}\left|\int_{\mathbb{R}} G_\alpha(x-y,t-\tau)[g(y,s) - g(x,s)]\mathrm{d}y\right|^{\frac{1}{\alpha-1}}\mathrm{d}x\\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} G_\alpha(x-y,t-\tau)|g(y,s) - g(x,s)|^{\frac{1}{\alpha-1}}\mathrm{d}y\mathrm{d}x\\ &\le \int_{\mathbb{R}}\int_{B(x,r)} G_\alpha(x-y,t-\tau)|g(y,s) - g(x,s)|^{\frac{1}{\alpha-1}}\mathrm{d}y\mathrm{d}x\\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}\setminus B(x,r)} G_\alpha(x-y,t-\tau)|g(y,s) - g(x,s)|^{\frac{1}{\alpha-1}}\mathrm{d}y\mathrm{d}x =: I_{21} + I_{22}.\end{aligned} \tag{B2}$$

For I_{21}, we have

$$\begin{aligned} I_{21} &= \int_{\mathbb{R}}\int_{B(x,r)} G_\alpha(x-y,t-\tau)|g(y,s)-g(x,s)|^{\frac{1}{\alpha-1}}\,\mathrm{d}y\mathrm{d}x \\ &= \int_{\mathbb{R}}\int_{B(0,r)} G_\alpha(z,t-\tau)|g(x+z,s)-g(x,s)|^{\frac{1}{\alpha-1}}\,\mathrm{d}z\mathrm{d}x \\ &= \int_{B(0,r)} G_\alpha(z,t-\tau)\int_{\mathbb{R}}|g(x+z,s)-g(x,s)|^{\frac{1}{\alpha-1}}\,\mathrm{d}x\mathrm{d}z \\ &\le \int_{B(0,r)} G_\alpha(z,t-\tau)\sup_{|h|\le r}\int_{\mathbb{R}}|g(x+h,s)-g(x,s)|^{\frac{1}{\alpha-1}}\,\mathrm{d}x\mathrm{d}z \\ &\le \sup_{|h|\le r}\int_{\mathbb{R}}|g(x+h,s)-g(x,s)|^{\frac{1}{\alpha-1}}\,\mathrm{d}x. \end{aligned} \tag{B3}$$

Notice that $\partial_x G_\alpha(x,t) = t^{-\frac{2}{\alpha}}\partial_x G_\alpha(t^{-\frac{1}{\alpha}}x,1)$. Denote

$$f(x) = (\tau-s)^{-\frac{2}{\alpha}}\left|\partial_x G_\alpha\Big((\tau-s)^{-\frac{1}{\alpha}}(x+h),1\Big)-\partial_x G_\alpha\Big((\tau-s)^{-\frac{1}{\alpha}}x,1\Big)\right|.$$

By the definition of g, we have

$$\begin{aligned} I_{21}^{\alpha-1} &\le \sup_{|h|\le r}\|f*(\rho(s)H\rho(s))\|_{L^{\frac{1}{\alpha-1}}} \le \sup_{|h|\le r}\|f\|_{L^1}\|\rho(s)H\rho(s)\|_{L^{\frac{1}{\alpha-1}}} \\ &\le Ca^2(\tau-s)^{-\frac{1}{\alpha}}s^{-\frac{\alpha-1}{\alpha}}\sup_{|h|\le r}\|\partial_x G_\alpha(\cdot+h,1)-\partial_x G_\alpha(\cdot,1)\|_{L^1}. \end{aligned} \tag{B4}$$

Due to $G_\alpha(x,t) = t^{-\frac{1}{\alpha}}G(t^{-\frac{1}{\alpha}}x,1)$, we obtain

$$\begin{aligned} I_{22} &= \int_{\mathbb{R}}\int_{\mathbb{R}\setminus B(x,r)} G_\alpha(x-y,t-\tau)|g(y,s)-g(x,s)|^{\frac{1}{\alpha-1}}\,\mathrm{d}y\mathrm{d}x \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}\setminus B\left(0,r/(t-\tau)^{\frac{1}{\alpha}}\right)} G_\alpha(z,1)\left|g\Big(x+(t-\tau)^{\frac{1}{\alpha}}z,s\Big)-g(x,s)\right|^{\frac{1}{\alpha-1}}\,\mathrm{d}z\mathrm{d}x \\ &\le 2\|g(s)\|_{L^{\frac{1}{\alpha-1}}}^{\frac{1}{\alpha-1}}\int_{\mathbb{R}\setminus B\left(0,r/(t-\tau)^{\frac{1}{\alpha}}\right)} G_\alpha(z,1)\mathrm{d}z. \end{aligned} \tag{B5}$$

From (A3), we know $\|\partial_x G_\alpha(\cdot,t)\|_{L^1} = t^{-\frac{1}{\alpha}}\|\partial_x G_\alpha(\cdot,1)\|_{L^1}$. By Young's convolution inequality, we obtain

$$\begin{aligned} \|g(s)\|_{L^{\frac{1}{\alpha-1}}} &\le \|\partial_x G_\alpha(\cdot,\tau-s)\|_{L^1}\|\rho(s)H\rho(s)\|_{L^{\frac{1}{\alpha-1}}} \\ &\le Ca^2(\tau-s)^{-\frac{1}{\alpha}}s^{-\frac{\alpha-1}{\alpha}}. \end{aligned} \tag{B6}$$

Combining (B1)–(B6), we obtain

$$\begin{aligned} I_2 &\le \int_0^\tau \|G_\alpha(\cdot,t-\tau)*g(s)-g(s)\|_{L^{\frac{1}{\alpha-1}}}\,\mathrm{d}s \le \int_0^\tau (I_{21}+I_{22})^{\alpha-1}\mathrm{d}s \\ &\le Ca^2\int_0^\tau (\tau-s)^{-\frac{1}{\alpha}}s^{-\frac{\alpha-1}{\alpha}}\,\mathrm{d}s\sup_{|h|\le r}\|\partial_x G_\alpha(\cdot+h,1)-\partial_x G_\alpha(\cdot,1)\|_{L^1} \\ &\quad + Ca^2\int_0^\tau (\tau-s)^{-\frac{1}{\alpha}}s^{-\frac{\alpha-1}{\alpha}}\,\mathrm{d}s\left(\int_{\mathbb{R}\setminus B\left(0,r/(t-\tau)^{\frac{1}{\alpha}}\right)} G_\alpha(z,1)\mathrm{d}z\right)^{\alpha-1}. \end{aligned} \tag{B7}$$

By Ref. 6, Lemma 4.3, letting $t\to\tau$ first and then $r\to 0$, we have $I_2\to 0$. □

APPENDIX C: PROOF OF (3.43)

Proof of (3.43). We prove (3.43) for n big enough. Notice that

$$\mu = \frac{n}{\alpha}+a+b-1.$$

We have

$$\sum_{m=2}^{n-1} n^{\mu} m^{m-\delta}\left(\frac{1}{m-1}\right)^{\frac{m}{\alpha}+b} = \sum_{m=2}^{n-1} m^{m-\delta}\left(\frac{1}{m-1}\right)^{\frac{m}{\alpha}+b} n^{\frac{n}{\alpha}+a+b-1}$$
$$= \sum_{m=2}^{n-1}\left(\frac{m}{n}\right)^{m}\left(\frac{n}{m-1}\right)^{\frac{m}{\alpha}} \frac{1}{m^{\delta}(m-1)^{b}} n^{\frac{n}{\alpha}+a+b-1+m-\frac{m}{\alpha}}.$$

Because there exists some constant M independent of n such that

$$\left(\frac{m}{n}\right)^{m}\left(\frac{n}{m-1}\right)^{\frac{m}{\alpha}} \le \left(\frac{m}{n}\right)^{\frac{m}{\alpha}}\left(\frac{n}{m-1}\right)^{\frac{m}{\alpha}} \le \left(1+\frac{1}{m-1}\right)^{\frac{m}{\alpha}} \le M,$$

we have

$$\sum_{m=2}^{n-1} n^{\mu} m^{m-\delta}\left(\frac{1}{m-1}\right)^{\frac{m}{\alpha}+b} \le M\sum_{m=2}^{n-1} \frac{1}{m^{\delta}(m-1)^{b}} n^{\frac{n}{\alpha}+a+b-1+m-\frac{m}{\alpha}}$$
$$= Mn^{n-\delta}\sum_{m=2}^{n-1} \frac{1}{m^{\delta}(m-1)^{b}} n^{-(n-m)\left(1-\frac{1}{\alpha}\right)+a+b-1+\delta}$$
$$= Mn^{n-\delta}\left[\sum_{2\le m\le \frac{n}{2}} \frac{n^{a+b-1+\delta}}{m^{\delta}(m-1)^{b}} n^{-(n-m)\left(1-\frac{1}{\alpha}\right)} + \sum_{\frac{n}{2}<m\le n-1} \frac{n^{a+b-1+\delta}}{m^{\delta}(m-1)^{b}} n^{-(n-m)\left(1-\frac{1}{\alpha}\right)}\right]$$
$$=: Mn^{n-\delta}(I_1+I_2).$$

To prove (3.43), it suffices to prove $I_1 \le \frac{1}{2M}$ and $I_2 \le \frac{1}{2M}$ for n big enough. For simplicity, we assume n to be an even number. For I_1, we have

$$I_1 = \sum_{2\le m\le \frac{n}{2}} \frac{n^{a+b-1+\delta}}{m^{\delta}(m-1)^{b}} n^{-(n-m)\left(1-\frac{1}{\alpha}\right)} \le n^{a+b-1+\delta}\sum_{2\le m\le \frac{n}{2}} n^{-(n-m)\left(1-\frac{1}{\alpha}\right)} \le \frac{2n^{a+b-1+\delta}}{n^{\frac{n}{2}\left(1-\frac{1}{\alpha}\right)}} \le \frac{1}{2M}.$$

For I_2, we have

$$I_2 = \sum_{\frac{n}{2}<m\le n-1} \frac{n^{a+b-1+\delta}}{m^{\delta}(m-1)^{b}} n^{-(n-m)\left(1-\frac{1}{\alpha}\right)} \le \sum_{\frac{n}{2}<m\le n-1} \frac{2^{\delta+b} n^{a+b-1+\delta}}{n^{\delta}(n-2)^{b}} n^{-(n-m)\left(1-\frac{1}{\alpha}\right)}$$
$$\le \frac{2^{\delta+b} n^{a+b-1}}{(n-2)^{b}} \sum_{\frac{n}{2}<m\le n-1} n^{-(n-m)\left(1-\frac{1}{\alpha}\right)}.$$

For n big enough, we have

$$\sum_{\frac{n}{2}<m\le n-1} n^{-(n-m)\left(1-\frac{1}{\alpha}\right)} \le \frac{2}{n^{1-\frac{1}{\alpha}}},$$

and hence

$$I_2 \le 2^{\delta+b+1}\left(\frac{n}{n-2}\right)^{b} \frac{1}{n^{1-a} n^{1-\frac{1}{\alpha}}} \le \frac{1}{2M}.$$

□

APPENDIX D: PROOF OF THEOREM 4.1 (iii)

Proof of Theorem 4.1 (iii). To prove the convergence result (iii). Recall formula (4.27). For fixed $z \in \overline{\mathbb{C}_+}$, denote

$$e^{-\gamma t} Z^{-1}(z,t) =: z_r(t) + i z_i(t).$$

Next, we prove that $z_r(t) + i z_i(t)$ converges to a point $w = z_r^* + i z_i^* \in \mathbb{C}_+$ as $t \to \infty$. To this end, we first prove $|z_r(t)|$ and $z_i(t)$ are all bounded from above and below uniformly in time t.

Because

$$f_0(Z^{-1}(z,t)) = R\rho_0(e^{\gamma t} z_r(t), e^{\gamma t} z_i(t)) - iP\rho_0(e^{\gamma t} z_r(t), e^{\gamma t} z_i(t)),$$

by (4.27), we have

$$z = z_r(t) + R\rho_0(e^{\gamma t}z_r(t), e^{\gamma t}z_i(t))\frac{\sinh \gamma t}{\gamma} + i\left[z_i(t) - P\rho_0(e^{\gamma t}z_r(t), e^{\gamma t}z_i(t))\frac{\sinh \gamma t}{\gamma} + \frac{\nu}{\gamma}(e^{-\gamma t} - 1)\right]. \tag{D1}$$

Due to $-P\rho_0(e^{\gamma t}z_r(t), e^{\gamma t}z_i(t))\frac{\sinh \gamma t}{\gamma} + \frac{\nu}{\gamma}(e^{-\gamma t} - 1) \le 0$, we have

$$z_i(t) \ge \Im(z) > 0.$$

Moreover, we have

$$\begin{aligned}\Im(z) &= z_i(t) - P\rho_0(e^{\gamma t}z_r(t), e^{\gamma t}z_i(t))\frac{\sinh \gamma t}{\gamma} + \frac{\nu}{\gamma}(e^{-\gamma t} - 1)\\ &= z_i(t) - \int_{\mathbb{R}}\frac{e^{\gamma t}z_i(t)}{e^{2\gamma t}z_i^2(t) + (e^{\gamma t}z_r(t) - s)^2}\rho_0(s)\mathrm{d}s\frac{\sinh \gamma t}{\gamma} + \frac{\nu}{\gamma}(e^{-\gamma t} - 1)\\ &\ge z_i(t) - \frac{1}{\gamma}\int_{\mathbb{R}}\frac{e^{2\gamma t}z_i(t)}{2e^{2\gamma t}z_i^2(t) + 2(e^{\gamma t}z_r(t) - s)^2}\rho_0(s)\mathrm{d}s + \frac{\nu}{\gamma}(e^{-\gamma t} - 1)\\ &\ge z_i(t) - \frac{1}{2\gamma z_i(t)} + \frac{\nu}{\gamma}(e^{-\gamma t} - 1),\end{aligned}$$

which implies

$$z_i(t) \le \Im(z) + 1/\sqrt{2\gamma} - \frac{\nu}{\gamma}(e^{-\gamma t} - 1).$$

Hence, $z_i(t)$ is bounded as

$$0 < \Im(z) \le z_i(t) \le \Im(z) + 1/\sqrt{2\gamma} - \frac{\nu}{\gamma}(e^{-\gamma t} - 1).$$

Next, we prove

$$\sup_{t\ge 0}|z_r(t)| < +\infty.$$

We prove this by a contradiction argument. If there exists $t_n \to \infty$ such that $z_r(t_n) \to \infty$, then by the dominated convergence theorem we have

$$R\rho_0(e^{\gamma t_n}z_r(t_n), e^{\gamma t_n}z_i(t_n))\frac{\sinh \gamma t_n}{\gamma} = \frac{1}{\pi}\int_{\mathbb{R}}\frac{(e^{\gamma t_n}z_r(t_n) - s)\rho_0(s)}{e^{2\gamma t_n}z_i^2(t_n) + (e^{\gamma t_n}z_r(t_n) - s)^2}\mathrm{d}x\frac{\sinh \gamma t_n}{\gamma} \to 0$$

as n goes to ∞. By (D1), we obtain a contradiction that

$$\Re(z) = z_r(t_n) + R\rho_0(e^{\gamma t_n}z_r(t_n), e^{\gamma t_n}z_i(t_n))\frac{\sinh \gamma t_n}{\gamma} \to \infty.$$

Since $|z_r(t)|$ and $z_i(t)$ are bounded, there exist $t_n \to \infty$ and two constant $z_r^*, z_i^* > 0$ such that

$$z_r(t_n) \to z_r^*, \quad z_i(t_n) \to z_i^*, \quad n \to \infty.$$

For any $s \in \mathbb{R}$, we have

$$\frac{e^{\gamma t_n}z_r(t_n) - s}{e^{2\gamma t_n}z_i^2(t_n) + (e^{\gamma t_n}z_r(t_n) - s)^2}\sinh \gamma t_n \to \frac{z_r^*}{2(z_i^*)^2 + 2(z_r^*)^2}, \quad n \to \infty.$$

Then, by the dominated convergence theorem we have

$$\begin{aligned}&\lim_{n\to\infty} R\rho_0(e^{\gamma t_n}z_r(t_n), e^{\gamma t_n}z_i(t_n))\frac{\sinh \gamma t_n}{\gamma}\\ &= \frac{1}{\gamma\pi}\lim_{n\to\infty}\int_{\mathbb{R}}\frac{e^{\gamma t_n}z_r(t_n) - s}{e^{2\gamma t_n}z_i^2(t_n) + (e^{\gamma t_n}z_r(t_n) - s)^2}\rho_0(s)\mathrm{d}s \sinh \gamma t_n\\ &= \frac{1}{2\gamma\pi}\frac{z_r^*}{(z_i^*)^2 + (z_r^*)^2}.\end{aligned}$$

Similarly, we have

$$\lim_{n\to\infty} P\rho_0\big(e^{\gamma t_n} z_r(t_n), e^{\gamma t_n} z_i(t_n)\big)\frac{\sinh \gamma t_n}{\gamma} = \frac{1}{2\gamma\pi}\frac{z_i^*}{(z_i^*)^2 + (z_r^*)^2}.$$

Let $w := z_r^* + iz_i^*$. Then, let $t = t_n$ in (D1) and sending $n \to \infty$ gives

$$z = w + \frac{1}{2\gamma\pi}\frac{z_r^* - iz_i^*}{(z_i^*)^2 + (z_r^*)^2} - i\frac{\nu}{\gamma} = w + \frac{1}{2\gamma\pi w} - i\frac{\nu}{\gamma}.$$

We obtain a unique solution in $\mathbb{C}_+$ (with positive imaginary part):

$$w = \frac{1}{\gamma\pi z + i\nu\pi - \sqrt{(\gamma\pi z + i\nu\pi)^2 - 2\gamma\pi}}.$$

Hence, we have

$$e^{-\gamma t} Z^{-1}(z,t) = z_r(t) + iz_i(t) \to w = \frac{1}{\gamma\pi z + i\nu\pi - \sqrt{(\gamma\pi z + i\nu\pi)^2 - 2\gamma\pi}}, \quad t\to\infty.$$

By (4.27) and using the dominated convergence theorem again, we have

$$\begin{aligned} f(z,t) &= f_0(Z^{-1}(z,t))e^{\gamma t} \\ &= \int_{\mathbb{R}} \frac{e^{2\gamma t} z_r(t) - s}{e^{2\gamma t} z_i^2(t) + [e^{\gamma t} z_r(t) - s]^2}\rho_0(s)\mathrm{d}s - i\int_{\mathbb{R}} \frac{e^{2\gamma t} z_i(t)}{e^{2\gamma t} z_i^2(t) + [e^{\gamma t} z_r(t) - s]^2}\rho_0(s)\mathrm{d}s \\ &\to \frac{z_r^* - iz_i^*}{(z_i^*)^2 + (z_r^*)^2} = \frac{1}{w} = \gamma\pi z + i\nu\pi - \sqrt{(\gamma\pi z + i\nu\pi)^2 - 2\gamma\pi}, \quad t\to\infty. \end{aligned}$$

Let $z = x + iy$, $y \ge 0$, and the imaginary part be given by

$$\Im\left(\frac{1}{w}\right) = \pi\nu - \frac{\sqrt{\sqrt{[\pi^2\gamma^2x^2 - \pi^2(\gamma y + \nu)^2 - 2\pi\gamma]^2 + 4\pi^4\gamma^2x^2(\gamma y + \nu)^2} - [\pi^2\gamma^2x^2 - \pi^2(\gamma y + \nu)^2 - 2\pi\gamma]}}{\sqrt{2}} < 0. \tag{D2}$$

Consider the trace on the real line $x \in \mathbb{R}$ and $y = 0$, and we obtain

$$\rho(x,t) = -\Im(f(x,t)) \to \rho_\infty(x), \quad t\to\infty.$$

For $\nu = 0$, it is the same as in Ref. 15, Eq. (2.15). For $\gamma = 0$, we have $\rho_\infty = 0$. □

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