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ABSTRACT

Using a generalized framework that consists of evolution of the solution to the Camassa–Holm equation and its energy measure, we establish the global-in-time orbital stability of peakons with respect to the perturbed (energy) conservative solutions to the Camassa–Holm equation. In particular, we extend the H^1 -stability result obtained by Constantin and Strauss [Commun. Pure Appl. Math. 53(5), 603–610 (2000)] globally-in-time even after the perturbed solutions experience wave breaking. In addition, our result also shows that the singular part of the energy measure of the perturbed solutions will remain stable for all times.

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I. INTRODUCTION

In this paper, we study the global-in-time orbital stability of peakons of the Camassa–Holm equation on the whole real line \mathbb{R} . For any $x, t \in \mathbb{R}$, the Camassa–Holm equation is given by

$$u_t + uu_x + P_x = 0, \quad (1.1)$$

where $u = u(x, t)$ is the unknown function. Here, the function $P := P(x, t)$ is a nonlocal source term given by

$$P(x, t) = \frac{1}{2} \left[\varphi * \left(u^2 + \frac{u_x^2}{2} \right) \right](x, t) = \frac{1}{2} \int_{\mathbb{R}} \varphi(x - y) \left(u^2 + \frac{u_x^2}{2} \right)(y, t) dy,$$

where $\frac{1}{2}\varphi(x) := \frac{1}{2}e^{-|x|}$ is the fundamental solution to the Helmholtz operator $I - \partial_{xx}$. Equation (1.1) is an integrable system that has a bi-Hamiltonian structure and infinitely many conservation laws.^{6,15} It is used to model the propagation of unidirectional shallow water waves over a flat bottom.^{6,19} Similar to the Korteweg–de Vries (KdV) equation, Eq. (1.1) has a special class of soliton solutions known as peakons, which are in the form

$$c\varphi(x - ct) = ce^{-|x-ct|},$$

where $c \in \mathbb{R}$ is any nonzero constant. We are going to study the global-in-time orbital stability of these peakons. To illustrate our results clearly, let us first introduce the results of (energy) conservative solutions, as well as previous results of the stability of peakons.

For any initial data $\tilde{u} \in H^s(\mathbb{R})$ with $s > 3/2$, the Camassa–Holm equation has a unique solution in $u \in C([0, T]; H^s(\mathbb{R}))$ for some $T > 0$, and for these solutions, the following two quantities are conserved:^{20,23}

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2) dx \quad \text{and} \quad F(u) = \int_{\mathbb{R}} (u^3 + uu_x^2) dx. \quad (1.2)$$

It was then proved in Ref. 12 that if the initial datum $\bar{u} \in H^1(\mathbb{R})$ satisfies $\bar{u} - \bar{u}_{xx} \in \mathcal{M}_+(\mathbb{R})$, the Camassa–Holm equation has a unique global-in-time solution $u \in C([0, \infty); H^1(\mathbb{R}))$ with $E(u)$ and $F(u)$ conserved. Here, $\mathcal{M}_+(\mathbb{R})$ stands for the set of all finite positive Radon measures on \mathbb{R} . On the other hand, even for smooth initial data, the solutions to the Camassa–Holm equation might blow up in finite time,^{1,6,7,10,11,21} and at the blow-up time, the solution u remains continuous, while its derivative u_x tends to negative infinity. This blow-up phenomenon is known as the wave breaking in the literature. Before the perturbed solution u experiences wave breaking, the orbital stability of peakons was proved by Constantin and Strauss in Ref. 13 based on a very clever use of the two conserved quantities $E(u)$ and $F(u)$ defined by (1.2). Indeed, it was commented in Ref. 13 (Sec. 3) that their stability result is applicable only up to the wave breaking time of the perturbed solution u , as at the blow-up time, there might be some “energy loss” in the sense that the energy $E(u)$ defined by (1.2) is strictly smaller than the initial one, and the perturbed solution $u(\cdot, t)$ is not continuous in $H^1(\mathbb{R})$ at the blow-up time (see Refs. 3, 4, and 18 for instance). Therefore, the stability results in Ref. 13 cannot be used beyond the wave breaking time of the perturbed solution u .

However, the peakons exist globally, and for any initial data $\bar{u} \in H^1(\mathbb{R})$, the solutions $u(\cdot, t)$ can be extended globally-in-time and uniquely after the wave breaking time in a conservative way so that the solution $u(\cdot, t) \in H^1(\mathbb{R})$ for all $t \in \mathbb{R}$ and the energy $E(u)$ is conserved for a.e. $t \in \mathbb{R}$; see Refs. 3, 4, and 18 for instance. Moreover, the energy $E(u)$ is strictly dissipative at some zero measure set of time t . Hence, the natural question is do we still have the orbital stability of peakons with respect to these global conservative solutions for all time t , especially after the wave breaking time. As mentioned in Refs. 14 and 22 for instance, this still remains to be an open question since these conservative solutions fail to be continuous in $H^1(\mathbb{R})$ at the aforementioned zero measure set of time t . The aim of this paper is to provide an affirmative answer to this question by using a generalized framework of the Camassa–Holm equation.

As mentioned above, the energy $E(u)$ might be strictly dissipative at the blow-up time; however, the conservations of $E(u)$ and $F(u)$ are essential for obtaining the orbital stability of peakons in Ref. 13 before the wave breaking. Then, it is quite natural to find some substitutes of $E(u)$ and $F(u)$ so that they are conserved for all time, and we can utilize them in the same way as $E(u)$ and $F(u)$. The strict dissipation of the energy $E(u)$ at the blow up time is because some of the total energy is transferred into a singular measure, which cannot be measured/detected by the solution u itself; see Refs. 3 and 18 for instance. To describe the evolution of the total energy, we shall introduce the energy measure $\mu(t)$, which is conserved globally-in-time [i.e., $\mu(t)(\mathbb{R}) = \mu(0)(\mathbb{R})$ for all $t \in \mathbb{R}$], and $d\mu_{ac} = (u^2 + u_x^2)dx$, where μ_{ac} is the absolutely continuous part of μ with respect to the Lebesgue measure; see Ref. 18 (Sec. 3). Based on this idea, the generalized framework of the Camassa–Holm equation is described as follows: Assume that u is sufficiently smooth, such as $u \in C_t^1 C_x^1 \cap C_t^0 C_x^2$. Differentiating Eq. (1.1) with respect to the spatial variable x yields

$$u_{xt} + u_x^2 + uu_{xx} + P_{xx} = 0. \quad (1.3)$$

Multiplying (1.1) by $2u$ and multiplying (1.3) by $2u_x$ and then summing them up, we formally obtain the conservation equation for the energy density,

$$(u^2 + u_x^2)_t + [u(u^2 + u_x^2)]_x = [u(u^2 - 2P)]_x. \quad (1.4)$$

To describe the evolution of the total energy, we may replace $u^2 + u_x^2$ in (1.4) with $\mu(t)$. Then, supplementing the original Camassa–Holm equation (1.1) with the resulting equation, the generalized framework for the Camassa–Holm equation is given by

$$u_t + uu_x + P_x = 0, \quad (1.5)$$

$$(d\mu)_t + (u d\mu)_x = [u(u^2 - 2P)dx]_x, \quad (1.6)$$

$$d\mu_{ac}(t) = (u^2 + u_x^2)(\cdot, t)dx, \quad (1.7)$$

where the function P now becomes $P := \frac{1}{4}\varphi * \mu + \frac{1}{4}\varphi * u^2$, and $\varphi(x) := e^{-|x|}$. Here, we understand Eq. (1.6) in the distributional sense (1.10). The conservative solutions to the above generalized framework (1.5)–(1.7) is considered in the following space:

$$\mathcal{D} = \{(u, \mu) : u \in H^1(\mathbb{R}), \mu \in \mathcal{M}_+(\mathbb{R}), d\mu_{ac} = (u^2 + u_x^2)dx\}. \quad (1.8)$$

When the energy measure $\mu(t)$ is absolutely continuous with respect to the Lebesgue measure dx , the above framework is the same as Eqs. (1.1) and (1.4). The above generalized framework is similar to the generalized framework for the Hunter–Saxton equation; see Ref. 16 [Eqs. (1.4)–(1.6)] for instance. Similar ideas were also used in other models, such as the two component Camassa–Holm equation,¹⁷ the Hunter–Saxton equation,^{5,8} and the Novikov equation.⁹ The definition of conservative solutions to the generalized framework (1.5)–(1.7) is as follows:

Definition 1.1 (conservative solutions). For any initial datum $(\bar{u}, \bar{\mu}) \in \mathcal{D}$, the pair $(u(\cdot, t), \mu(t))$ is said to be a global-in-time conservative solution to the generalized framework (1.5)–(1.7) subject to the initial datum $(\bar{u}, \bar{\mu})$ if the pair $(u(\cdot, t), \mu(t))$ satisfies the following:

- (i) $u \in L^\infty(\mathbb{R}; H^1(\mathbb{R}))$ and $u(\cdot, t) \in H^1(\mathbb{R})$ for all $t \in \mathbb{R}$. Moreover, for any $T > 0$, the map $t \mapsto u(\cdot, t)$ is Lipschitz continuous from $[-T, T]$ into $L^2(\mathbb{R})$, and $\mu \in C(\mathbb{R}; \mathcal{M}_+(\mathbb{R}))$.
- (ii) $(u(\cdot, 0), \mu(0)) = (\bar{u}, \bar{\mu})$, and $d\mu(t) = (u^2 + u_x^2)(\cdot, t)dx$ for a.e. $t \in \mathbb{R}$.
- (iii) Equation (1.5) is satisfied in the weak sense: for any test function $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} u \phi_t - \phi(uu_x + P_x) dx dt = 0. \quad (1.9)$$

(iv) (Conservation of energy). Eq. (1.6) is satisfied in the distributional sense: for any test function $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$, the identity

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (\phi_t + u \phi_x) d\mu(t) dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_x u (u^2 - 2P) dx dt \quad (1.10)$$

holds.

(v) Equation (1.7) holds for all $t \in \mathbb{R}$.

The last condition is equivalent to say that $(u(t), \mu(t)) \in \mathcal{D}$ for all times $t \in \mathbb{R}$.

The existence and uniqueness of conservative solutions (u, μ) in the sense of Definition 1.1 with initial datum $(\bar{u}, \bar{\mu}) \in \mathcal{D}$ has been obtained based on the characteristics method; see Refs. 2, 3, and 18 for example. Our main result for stability of peakons is as follows:

Theorem 1.1. Let $0 < \varepsilon < 1$ and $c > 0$ be any given constants, and the initial data $(\bar{u}, \bar{\mu}) \in \mathcal{D}$, where the space \mathcal{D} was defined in (1.8) above, satisfy

$$\bar{\mu}_s(\mathbb{R}) + \|\bar{u} - c\varphi\|_{H^1}^2 \leq \frac{c^2}{2} \left[\frac{\varepsilon}{6(1+c)} \right]^8,$$

where $\varphi(x) := e^{-|x|}$ and $\bar{\mu}_s$ is the singular part of the initial energy measure $\bar{\mu}$. If (u, μ) is the conservative solution to the generalized framework (1.5)–(1.7) of the Camassa–Holm equation subject to the initial datum $(\bar{u}, \bar{\mu})$ in the sense of Definition 1.1, then for any $t \in \mathbb{R}$, we have

$$\mu_s(t)(\mathbb{R}) + \|u(\cdot, t) - c\varphi(\cdot - \xi(t))\|_{H^1}^2 \leq \varepsilon^2,$$

where $\mu_s(t)$ is the singular part of the energy measure $\mu(t)$, and the spatial location $\xi(t) \in \mathbb{R}$ is any point where the function $u(\cdot, t)$ attains its maximum.

Since the Camassa–Holm equation (1.1) is invariant under the transformation: $u(x, t) \mapsto -u(-x, t)$, one can obtain similar results for the anti-peakon cases, i.e., $c < 0$. Compared with previous results for the stability of peakons, our result shows the following novelties:

- (i) The orbital stability of peakons in the $H^1(\mathbb{R})$ norm is obtained for all times t , so it applies even after the wave breaking time of the perturbed solution u .
- (ii) Our results give the estimates for u and $\mu_s(t)$ together; in particular, the singular part $\mu_s(t)$ of the energy measure will also remain stable for all times t .

The rest of this paper is organized as follows: In Sec. II, we will introduce the characteristics method for this generalized framework and present an equivalent semi-linear system. Using the energy measure μ , we will then define the extensions of $E(u)$ and $F(u)$ in the generalized framework; see (2.8). In addition, we will also show that these extensions are actually conserved for all times. The main result for stability of peakons will be proved in Sec. III by utilizing these two conserved quantities.

II. GENERALIZED FRAMEWORK AND CONSERVATIVE QUANTITIES

A. Generalized framework and system of α -variable

The existence and uniqueness of conservative solutions (u, μ) in the sense of Definition 1.1 with initial datum $(\bar{u}, \bar{\mu}) \in \mathcal{D}$ has been obtained in Refs. 2, 3, and 18. The idea is to use the characteristics method to transfer the Camassa–Holm equation into a semilinear system. Following the ideas in Ref. 18, we briefly introduce the semilinear system below. For any initial data $(\bar{u}, \bar{\mu}) \in \mathcal{D}$, define the function \tilde{x} via

$$\tilde{x}(\alpha) + \bar{\mu}((-\infty, \tilde{x}(\alpha))) \leq \alpha \leq \tilde{x}(\alpha) + \bar{\mu}((-\infty, \tilde{x}(\alpha)]) \quad (2.1)$$

for every $\alpha \in \mathbb{R}$. The use of the α -variable transfers the initial energy measure $\bar{\mu}$ into another measure with the density function $1 - \tilde{x}'$ in $L^1(\mathbb{R})$, namely, $d\bar{\mu} = \tilde{x}\#[(1 - \tilde{x}')d\alpha]$. Here, the symbol $\#$ stands for push-forward of a measure, i.e., for any Lebesgue measurable set $A \subset \mathbb{R}$, we have $\bar{\mu}(A) = \int_{\tilde{x}^{-1}(A)} (1 - \tilde{x}')(\alpha) d\alpha$. This function $\tilde{x} := \tilde{x}(\alpha)$ is Lipschitz continuous with a Lipschitz constant bounded by 1; see Ref. 16, Proposition 2.1, for instance. Consider the characteristics $y(\alpha, t)$ satisfying $y_t(\alpha, t) = u(y(\alpha, t), t)$. Then, the nonlocal source term P and its derivative in α -coordinate are in the following forms:

$$P(y(\alpha, t), t) = \frac{1}{4} \int_{\mathbb{R}} e^{-|y(\alpha, t) - y(\theta, t)|} [H_\theta(\theta, t) + u^2(y(\theta, t), t) y_\theta(\theta, t)] d\theta \quad (2.2)$$

and

$$P_x(y(\alpha, t), t) = -\frac{1}{4} \int_{\mathbb{R}} \operatorname{sgn}(\alpha - \theta) e^{-|y(\alpha, t) - y(\theta, t)|} [H_\theta(\theta, t) + u^2(y(\theta, t), t) y_\theta(\theta, t)] d\theta, \quad (2.3)$$

where $H(\alpha, t) = \int_{-\infty}^{y(\alpha, t)} (u^2 + u_x^2) dx$ is the cumulative energy distribution in α -coordinate. Then, the semilinear system of ordinary differential equations is given by {see Ref. 18 [Eq. (2.10)] and details in Ref. 18 [Sec. 2.1]}

$$\frac{d}{dt} y(\alpha, t) = u(y(\alpha, t), t), \quad y(\alpha, 0) = \bar{x}(\alpha), \quad (2.4)$$

$$\frac{d}{dt} u(y(\alpha, t), t) = -P_x(y(\alpha, t), t), \quad u(y(\alpha, 0), 0) = \bar{u}(\bar{x}(\alpha)), \quad (2.5)$$

$$\frac{d}{dt} H(\alpha, t) = (u^3 - 2uP)(y(\alpha, t), t), \quad H(\alpha, 0) = \alpha - \bar{x}(\alpha). \quad (2.6)$$

For any initial datum $(\bar{u}, \bar{\mu}) \in \mathcal{D}$, the global solutions $(y(\alpha, t), u(y(\alpha, t), t), H(\alpha, t))$ to this system, namely, Eqs. (2.4)–(2.6), was obtained in Ref. 18.

Remark 2.1. Let $\zeta(\alpha, t) := y(\alpha, t) - \alpha$ and $U(\alpha, t) := u(y(\alpha, t), t)$. Then, it follows from Ref. 18 (Theorem 2.8) that

$$(\zeta, U, H) \in C^1(\mathbb{R}; E \cap [W^{1,\infty}(\mathbb{R})]^3),$$

where $E = V \times H^1(\mathbb{R}) \times V$ and $V = \{f \in C_b(\mathbb{R}) : f_x \in L^2(\mathbb{R})\}$. Moreover, $y_\alpha(\cdot, t) \geq 0$, $H_\alpha(\cdot, t) \geq 0$, and P, P_x defined by (2.2) and (2.3) [as functions of (α, t)] are in the space $C(\mathbb{R}; H^1(\mathbb{R}))$; see Ref. 18 (Lemma 2.1) for details.

For any given global solution $(y(\alpha, t), u(y(\alpha, t), t), H(\alpha, t))$ to system (2.4)–(2.6), the corresponding global conservative solutions $(u(t), \mu(t))$ to the generalized framework (1.5)–(1.7) will be given by $u(x, t) = u(y(\alpha, t), t)$ for $x = y(\alpha, t)$ and global energy measure $\mu(t)$ will be given by

$$\mu(t) = y(\cdot, t) \# [H_\alpha(\cdot, t) d\alpha]. \quad (2.7)$$

The uniqueness of conservative solutions was proved via characteristics methods in Ref. 2. Since the global solution $(y(\alpha, t), u(y(\alpha, t), t), H(\alpha, t))$ to (2.4)–(2.6) can immediately provide the global conservative solution $(u(t), \mu(t))$ to the generalized framework (1.5)–(1.7), this is equivalent to say that any (energy) conservative solution in the sense of Definition 1.1 can be represented by the solution to (2.4)–(2.6).

B. Conservative quantities under generalized framework

As mentioned in Sec. I, the two quantities $E(u)$ and $F(u)$ defined in (1.2) are not conserved, in general, due to the potential wave breaking. Under the generalized framework, for a pair $(u, \mu) \in \mathcal{D}$, the natural generalizations of $E(u)$ and $F(u)$ are defined by

$$\tilde{E}(u, \mu) = \mu(\mathbb{R}) \quad \text{and} \quad \tilde{F}(u, \mu) = \int_{\mathbb{R}} u d\mu. \quad (2.8)$$

Here, the generalized quantity \tilde{E} indeed depends on the variable μ only; however, we still use the pair (u, μ) in its argument to emphasis that \tilde{E} is the total energy for the generalized solution $(u(t), \mu(t))$. For a conservative solution (u, μ) to the Camassa–Holm equation in the sense of Definition 1.1, we know that the (generalized) energy $\tilde{E}(u(t), \mu(t))$ is conserved, namely, $\tilde{E}(u(t), \mu(t)) = \mu(t)(\mathbb{R}) = \bar{\mu}(\mathbb{R}) = \tilde{E}(\bar{u}, \bar{\mu})$ for all $t \in \mathbb{R}$. This energy conservation can be easily verified by using (1.10) and the fact that $\mu \in C(\mathbb{R}; \mathcal{M}_+(\mathbb{R}))$; for the methodology, see the argument in Ref. 16 (Remark 1.1) for instance. Next, we are going to show the conservation of $\tilde{F}(u(t), \mu(t))$, which will play an essential role for proving the orbital stability of peakons. It is worth noting that for a classical solution u , the following conservation law holds:

$$[u(u^2 + u_x^2)]_t + [u^2(u^2 + u_x^2)]_x + \left(Pu^2 - \frac{3}{4}u^4 + P^2 - P_x^2 \right)_x = 0. \quad (2.9)$$

This motivates us to consider the following lemma for weak solutions:

Lemma 2.1. Let (u, μ) be a conservative solution to the generalized framework (1.5)–(1.7) subject to the initial datum $(\bar{u}, \bar{\mu}) \in \mathcal{D}$ in the sense of Definition 1.1. Then, for any $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (\phi_t + u\phi_x) u d\mu(t) dt + \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_x \left(Pu^2 - \frac{3}{4}u^4 + P^2 - P_x^2 \right) dx dt = 0, \quad (2.10)$$

where $P = \frac{1}{4}\varphi * \mu + \frac{1}{4}\varphi * u^2$.

Proof. Using (2.4) and (2.7), we can apply a change of variables and obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} (\phi_t + u\phi_x) u \, d\mu(t) dt &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{d}{dt} \phi(y(\alpha, t), t) u(y(\alpha, t), t) H_\alpha(\alpha, t) d\alpha dt \\ &= - \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(y(\alpha, t), t) \frac{d}{dt} [u(y(\alpha, t), t) H_\alpha(\alpha, t)] d\alpha dt, \end{aligned} \quad (2.11)$$

where we integrated by parts in the last equality. It follows from Ref. 18 (Lemma 2.4) that for a.e. $(\alpha, t) \in \mathbb{R}^2$,

$$\frac{\partial}{\partial t} H_\alpha(\alpha, t) = \frac{\partial}{\partial \alpha} (u^3 - 2uP)(y(\alpha, t), t). \quad (2.12)$$

Combining (2.6) and (2.12) yields that for a.e. $(\alpha, t) \in \mathbb{R}^2$,

$$\begin{aligned} \frac{\partial}{\partial t} [u(y(\alpha, t), t) H_\alpha(\alpha, t)] &= -P_x(y(\alpha, t), t) H_\alpha(\alpha, t) + u(y(\alpha, t), t) \frac{\partial}{\partial \alpha} (u^3 - 2uP)(y(\alpha, t), t) \\ &= -P_x(y(\alpha, t), t) H_\alpha(\alpha, t) + \frac{3}{4} \frac{\partial}{\partial \alpha} [u^4(y(\alpha, t), t)] - 2u(y(\alpha, t), t) \frac{\partial}{\partial \alpha} [(uP)(y(\alpha, t), t)]. \end{aligned}$$

Using the above identity, we can re-write (2.11) as follows:

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} (\phi_t + u\phi_x) u \, d\mu(t) dt &= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(y(\alpha, t), t) P_x(y(\alpha, t), t) H_\alpha(\alpha, t) d\alpha dt \\ &\quad - \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(y(\alpha, t), t) \left(\frac{3}{4} \frac{\partial}{\partial \alpha} [u^4(y(\alpha, t), t)] - 2u(y(\alpha, t), t) \frac{\partial}{\partial \alpha} [(uP)(y(\alpha, t), t)] \right) d\alpha dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(y(\alpha, t), t) P_x(y(\alpha, t), t) H_\alpha(\alpha, t) d\alpha dt + \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_x \left(\frac{3}{4} u^4 - Pu^2 \right) dx dt + \int_{\mathbb{R}} \int_{\mathbb{R}} \phi u^2 P_x dx dt. \end{aligned} \quad (2.13)$$

Comparing (2.13) with (2.10), we only need to show

$$- \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_x (P^2 - P_x^2) dx dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(y(\alpha, t), t) P_x(y(\alpha, t), t) H_\alpha(\alpha, t) d\alpha dt + \int_{\mathbb{R}} \int_{\mathbb{R}} \phi u^2 P_x dx dt. \quad (2.14)$$

It follows from part (ii) of Definition 1.1 that $d\mu(t) = (u^2 + u_x^2)(\cdot, t) dx$ for a.e. t . At these times t , we have

$$P(x, t) = \frac{1}{4} \int_{\mathbb{R}} \varphi(x - y) d\mu(t) + \frac{1}{4} \int_{\mathbb{R}} \varphi(x - y) u^2(y, t) dy = \frac{1}{2} \int_{\mathbb{R}} \varphi(x - y) \left(u^2 + \frac{u_x^2}{2} \right) (y, t) dy,$$

which implies

$$(P - P_{xx})(x, t) = \left(u^2 + \frac{u_x^2}{2} \right) (x, t).$$

Hence,

$$\begin{aligned} - \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_x (P^2 - P_x^2) dx dt &= - \int_A \int_{\mathbb{R}} \phi_x (P^2 - P_x^2) dx dt \\ &= \int_A \int_{\mathbb{R}} 2\phi P_x (P - P_{xx}) dx dt = \int_A \int_{\mathbb{R}} 2\phi P_x \left(u^2 + \frac{u_x^2}{2} \right) (x, t) dx dt \\ &= \int_A \int_{\mathbb{R}} \phi u^2 P_x dx dt + \int_A \int_{\mathbb{R}} \phi P_x (u^2 + u_x^2) dx dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi u^2 P_x dx dt + \int_A \int_{\mathbb{R}} \phi(y(\alpha, t), t) P_x(y(\alpha, t), t) H_\alpha(\alpha, t) d\alpha dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi u^2 P_x dx dt + \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(y(\alpha, t), t) P_x(y(\alpha, t), t) H_\alpha(\alpha, t) d\alpha dt. \end{aligned} \quad (2.15)$$

This verifies (2.14) and, hence, completes the proof of (2.10). \square

Using Lemma 2.1, we have the following lemma for conservation of \tilde{F} :

Lemma 2.2. *Let (u, μ) be a conservative solution to the generalized framework (1.5)–(1.7) subject to the initial datum $(\tilde{u}, \tilde{\mu}) \in \mathcal{D}$ in the sense of Definition 1.1. Then, $\tilde{F}(u, \mu)$ is conserved, namely, $\tilde{F}(u(t), \mu(t)) = \tilde{F}(\tilde{u}, \tilde{\mu})$ for all $t \in \mathbb{R}$.*

The Proof of Lemma 2.2 basically follows from (2.10) and shares some similarity with Ref. 16, Remark 1.1. For the sake of self-containedness, we will provide the details here.

Proof. Without loss of generality, let us consider any arbitrary time $t > 0$; the case for $t < 0$ will be basically similar, so we will leave this to interested readers. We choose non-negative smooth functions $\chi_\varepsilon(s)$ and $g_R(x)$ for $\varepsilon > 0$ and $R > 0$, where $\chi_\varepsilon(s) = 0$ for $s \leq -\varepsilon$ and $s \geq t + \varepsilon$, $\chi_\varepsilon(s) = 1$ for $s \in (0, t)$, $\chi'_\varepsilon(s) \geq 0$ for $s \in (-\varepsilon, 0)$, and $\chi'_\varepsilon(s) \leq 0$ for $s \in (t, t + \varepsilon)$. The function g_R satisfies $g_R(x) = 1$ for $|x| \leq R$, $g_R(x) = 0$ for $|x| \geq 2R$, and $|g'_R(x)| \leq \frac{2}{R}$. Finally, let $\phi(x, t) = \chi_\varepsilon(t)g_R(x)$, and substituting this ϕ into Eq. (2.10), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi'_\varepsilon(s) g_R(x) u(x, s) d\mu(s) ds + \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_\varepsilon(s) g'_R(x) u^2(x, s) d\mu(s) ds \\ + \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_\varepsilon(s) g'_R(x) \left(Pu^2 - \frac{3}{4} u^4 + P^2 - P_x^2 \right) (x, s) dx ds = 0. \end{aligned}$$

Passing to the limit as $R \rightarrow \infty$ yields

$$\int_{-\varepsilon}^0 \chi'_\varepsilon(s) \int_{\mathbb{R}} u(x, s) d\mu(s) ds + \int_t^{t+\varepsilon} \chi'_\varepsilon(s) \int_{\mathbb{R}} u(x, s) d\mu(s) ds = 0. \quad (2.16)$$

It is worth noting that

$$\begin{aligned} \left| \int_{-\varepsilon}^0 \chi'_\varepsilon(s) \int_{\mathbb{R}} u(x, s) d\mu(s) ds - \tilde{F}(\tilde{u}, \tilde{\mu}) \right| &= \left| \int_{-\varepsilon}^0 \chi'_\varepsilon(s) \int_{\mathbb{R}} u(x, s) d\mu(s) ds - \int_{-\varepsilon}^0 \chi'_\varepsilon(s) \int_{\mathbb{R}} \tilde{u}(x) d\tilde{\mu} ds \right| \\ &\leq \int_{-\varepsilon}^0 \chi'_\varepsilon(s) \left| \int_{\mathbb{R}} u(x, s) d\mu(s) - \int_{\mathbb{R}} \tilde{u}(x) d\tilde{\mu} \right| ds \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}} u(x, s) d\mu(s) - \int_{\mathbb{R}} \tilde{u}(x) d\tilde{\mu} \right| &\leq \left| \int_{\mathbb{R}} [u(x, s) - \tilde{u}(x)] d\mu(s) \right| + \left| \int_{\mathbb{R}} \tilde{u}(x) d\mu(s) - \int_{\mathbb{R}} \tilde{u}(x) d\tilde{\mu} \right| \\ &=: I_1(s) + I_2(s). \end{aligned}$$

Since $\mu \in C(\mathbb{R}; \mathcal{M}_+(\mathbb{R}))$, we have $I_2(s) \rightarrow 0$ as $s \rightarrow 0$. For the first term I_1 , we use the fact that the solutions $u(\cdot, t)$ is continuous [see Ref. 18 (Theorem 4.2 and Proposition 5.2 for instance)] in time in the space $L^\infty(\mathbb{R})$ and conclude that $I_1(s) \rightarrow 0$ as $s \rightarrow 0$. This shows that $\int_{-\varepsilon}^0 \chi'_\varepsilon(s) \int_{\mathbb{R}} u(x, s) d\mu(s) ds \rightarrow \tilde{F}(\tilde{u}, \tilde{\mu})$ as $\varepsilon \rightarrow 0^+$. On the other hand, one can also verify that $\int_t^{t+\varepsilon} \chi'_\varepsilon(s) \int_{\mathbb{R}} u(x, s) d\mu(s) ds \rightarrow -\tilde{F}(u(t), \mu(t))$ as $\varepsilon \rightarrow 0^+$ in a similar manner, and hence, passing to the limit $\varepsilon \rightarrow 0^+$ in (2.16) yields $\tilde{F}(u(t), \mu(t)) = \tilde{F}(\tilde{u}, \tilde{\mu})$. Since the time t is arbitrarily chosen, this completes the proof. \square

Remark 2.2 (a formal proof). Ignoring the regularity issue, one could formally prove the conservation of \tilde{F} more directly as follows:

$$\begin{aligned} \frac{d}{dt} \tilde{F}(u, \mu) &= \frac{d}{dt} \int_{\mathbb{R}} u(y(\alpha, t), t) H_\alpha(\alpha, t) d\alpha \\ &= - \int_{\mathbb{R}} P_x(y(\alpha, t), t) H_\alpha(\alpha, t) d\alpha + \int_{\mathbb{R}} [3u^3 u_x - 2u(uP)_x](y(\alpha, t), t) y_\alpha(\alpha, t) d\alpha \\ &= - \int_{\mathbb{R}} P_x(y(\alpha, t), t) H_\alpha(\alpha, t) d\alpha + \frac{3}{4} \int_{\mathbb{R}} \partial_\alpha u^4(y(\alpha, t), t) d\alpha + \int_{\mathbb{R}} \partial_\alpha [u^2(y(\alpha, t), t)] P(y(\alpha, t), t) d\alpha \\ &= - \int_{\mathbb{R}} P_x(y(\alpha, t), t) H_\alpha(\alpha, t) d\alpha - \int_{\mathbb{R}} u^2(y(\alpha, t), t) P_x(y(\alpha, t), t) y_\alpha(\alpha, t) d\alpha. \end{aligned} \quad (2.17)$$

Now, using (2.3), we have

$$\begin{aligned} \frac{d}{dt} \tilde{F}(u, \mu) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \operatorname{sgn}(\alpha - \theta) e^{-|y(\alpha, t) - y(\theta, t)|} [H_\theta(\theta, t) + u^2(y(\theta, t), t) y_\theta(\theta, t)] \\ &\quad \times [H_\alpha(\alpha, t) + u^2(y(\alpha, t), t) y_\alpha(\alpha, t)] d\alpha d\theta = 0. \end{aligned}$$

However, the regularity of weak solutions seems not enough for taking the time derivative under the integral sign in the second line of (2.17). This is the main reason why we had to use the “integral form” of this computation in the rigorous Proof of Lemma 2.2.

III. STABILITY OF PEAKONS

We will follow the ideas in Ref. 13 to show the stability of one peakon under the generalized framework introduced in Sec. II; in particular, we will have to control the singular part of the energy measure. More precisely, we have the following lemma:

Lemma 3.1. Let c be a positive constant and $(u, \mu) \in \mathcal{D}$. Then,

(i) for any $\xi \in \mathbb{R}$,

$$\tilde{E}(u, \mu) - E(c\varphi) = \mu_s(\mathbb{R}) + \|u - c\varphi(\cdot - \xi)\|_{H^1}^2 + 4c(u(\xi) - c);$$

(ii) let $M := \max_{x \in \mathbb{R}} u(x)$, then

$$\tilde{F}(u, \mu) \leq M\tilde{E}(u, \mu) - \frac{2}{3}M^3. \quad (3.1)$$

In addition, if there exists a constant $0 < \delta < \min\{c/30, \sqrt{2}\}$ such that $\mu_s(\mathbb{R}) + \|u - c\varphi\|_{H^1}^2 < \delta^2/2$, then

$$|\tilde{E}(u, \mu) - E(c\varphi)| \leq 4c\delta \quad \text{and} \quad |\tilde{F}(u, \mu) - F(c\varphi)| \leq 8c^2\delta; \quad (3.2)$$

(iv)

$$|M - c| \leq \sqrt{6c\delta}.$$

Compared with Lemmas 1–4 in Ref. 13, the key point is that our energy $\tilde{E}(u, \mu)$ and $\tilde{F}(u, \mu)$ are defined for all time t and are both conserved, and hence, the estimates can be applied for all time t . Moreover, our estimates contain the singular part $\mu_s(\mathbb{R})$, and this improves the estimates in Lemmas 1–4 in Ref. 13. It is worth noting that in (3.2), we use the coefficients 4 and 8 for convenience, although they are not optimal according to the proof below. One could also improve the coefficient $\frac{1}{30}$ in $0 < \delta < \min\{c/30, \sqrt{2}\}$, but we will not pursue this here. These technical details will be left for interested readers.

Proof. (i) It follows from the definition of \tilde{E} that

$$\tilde{E}(u, \mu) - E(c\varphi) = \mu_s(\mathbb{R}) + E(u) - E(c\varphi).$$

A direct differentiation yields that $\varphi_x(x) = \varphi(x)$ for $x < 0$ and $\varphi_x(x) = -\varphi(x)$ for $x > 0$, so we have

$$\begin{aligned} \|u - c\varphi(\cdot - \xi)\|_{H^1}^2 &= E(u) + E(c\varphi(\cdot - \xi)) - 2c \int_{\mathbb{R}} u(x)\varphi(x - \xi)dx - 2c \int_{\mathbb{R}} u_x(x)\varphi_x(x - \xi)dx \\ &= E(u) + E(c\varphi(\cdot - \xi)) - 2c \int_{\mathbb{R}} u(x)\varphi(x - \xi)dx \\ &\quad - 2c \int_{-\infty}^{\xi} u_x(x)\varphi(x - \xi)dx + 2c \int_{\xi}^{\infty} u_x(x)\varphi(x - \xi)dx \\ &= E(u) - E(c\varphi) - 4cu(\xi) + 4c^2. \end{aligned} \quad (3.3)$$

Combining the above two identities, we obtain the desired result.

(ii) It follows from Ref. 13, Lemma 2 that for any $u \in H^1(\mathbb{R})$, the quantities $E(u)$ and $F(u)$ defined by (1.2) satisfy

$$F(u) \leq ME(u) - \frac{2}{3}M^3, \quad M = \max_{x \in \mathbb{R}} u(x);$$

then, for any $(u, \mu) \in \mathcal{D}$, we have

$$\tilde{F}(u, \mu) = \int_{\mathbb{R}} u d\mu_s + F(u) \leq M\mu_s(\mathbb{R}) + ME(u) - \frac{2}{3}M^3 = M\tilde{E}(u, \mu) - \frac{2}{3}M^3.$$

(iii) Due to the hypothesis $\mu_s(\mathbb{R}) + \|u - c\varphi\|_{H^1}^2 < \delta^2/2 < 1$, we also have

$$\mu_s^{1/2}(\mathbb{R}) + \|u - c\varphi\|_{H^1} < \delta, \quad (3.4)$$

and by Morrey's inequality,

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|u\|_{H^1} \leq \frac{1}{\sqrt{2}} (\|c\varphi\|_{H^1} + \delta) = c + \frac{\delta}{\sqrt{2}}. \quad (3.5)$$

A direct computation yields $\|c\varphi\|_{H^1} = \sqrt{2}c$. Since $\delta < c/2$ and (3.4), we have

$$\begin{aligned} |\tilde{E}(u, \mu) - E(c\varphi)| &= |\mu_s(\mathbb{R}) + E(u) - E(c\varphi)| \leq \mu_s(\mathbb{R}) + |(\|u\|_{H^1} - \|c\varphi\|_{H^1})(\|u\|_{H^1} + \|c\varphi\|_{H^1})| \\ &\leq \mu_s(\mathbb{R}) + (2\sqrt{2}c + \delta)\|u - c\varphi\|_{H^1} \leq \frac{\delta^2}{2} + \delta(2\sqrt{2}c + \delta) \leq 4c\delta, \end{aligned}$$

and by Hölder's inequality,

$$\begin{aligned} |\tilde{F}(u, \mu) - F(c\varphi)| &\leq \left| \int_{\mathbb{R}} u d\mu_s \right| + |F(u) - F(c\varphi)| \\ &\leq \left(c + \frac{\delta}{\sqrt{2}} \right) \mu_s(\mathbb{R}) + \left| \int_{\mathbb{R}} (u - c\varphi)(u^2 + u_x^2) dx + \int_{\mathbb{R}} c\varphi(u^2 + u_x^2 - c^2\varphi^2 - c^2\varphi_x^2) dx \right| \\ &\leq \left(c + \frac{\delta}{\sqrt{2}} \right) \frac{\delta^2}{2} + \left| \int_{\mathbb{R}} (u - c\varphi)(u^2 + u_x^2) dx + \int_{\mathbb{R}} c\varphi[(u - c\varphi)^2 + (u_x - c\varphi_x)^2] dx \right. \\ &\quad \left. + \int_{\mathbb{R}} c\varphi[2(u - c\varphi)c\varphi + 2(u_x - c\varphi_x)c\varphi_x] dx \right| \\ &\leq \left(c + \frac{\delta}{\sqrt{2}} \right) \frac{\delta^2}{2} + \|u - c\varphi\|_{L^\infty} E(u) + \|c\varphi\|_{L^\infty} \|u - c\varphi\|_{H^1}^2 + 2\|c\varphi\|_{L^\infty} \|u - c\varphi\|_{H^1} \|c\varphi\|_{H^1}. \end{aligned}$$

It follows from (3.4) that $\|u - c\varphi\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|u - c\varphi\|_{H^1} \leq \frac{\delta}{\sqrt{2}}$ and $E(u) = \|u\|_{H^1}^2 \leq (\sqrt{2}c + \delta)^2$. Furthermore, a direct computation yields $\|c\varphi\|_{L^\infty} = c$ and $\|c\varphi\|_{H^1} = \sqrt{2}c$, so we finally have

$$|\tilde{F}(u, \mu) - F(c\varphi)| \leq \left(c + \frac{\delta}{\sqrt{2}} \right) \frac{\delta^2}{2} + \frac{\delta}{\sqrt{2}} (2c^2 + 2\sqrt{2}c\delta + \delta^2) + c \frac{\delta^2}{2} + 2c\delta\sqrt{2}c \leq 8c^2\delta.$$

(iv) A direct calculation yields that

$$E(c\varphi) = 2c^2 \quad \text{and} \quad F(c\varphi) = \frac{4}{3}c^3. \quad (3.6)$$

Since (3.6) and $\delta < c/30$, using (3.1) and (3.2), we have

$$M \geq \frac{\tilde{F}(u, \mu)}{\tilde{E}(u, \mu)} \geq \frac{c}{2}.$$

Define

$$Q(y) := y^3 - \frac{3}{2}y\tilde{E}(u, \mu) + \frac{3}{2}\tilde{F}(u, \mu)$$

and

$$Q_0(y) := y^3 - \frac{3}{2}yE(c\varphi) + \frac{3}{2}F(c\varphi).$$

Then, it follows from part (ii) that

$$Q(M) \leq 0. \quad (3.7)$$

Furthermore, using (3.6), we also have

$$Q_0(y) = (y - c)^2(y + 2c).$$

Hence, using (3.7) and part (iii), we finally have

$$\begin{aligned}(M-c)^2(M+2c) &= Q_0(M) = Q(M) + \frac{3}{2}M(\tilde{E}(u, \mu) - E(c\varphi)) - \frac{3}{2}(\tilde{F}(u, \mu) - F(c\varphi)) \\ &\leq \frac{3}{2}M \cdot 4c\delta + \frac{3}{2} \cdot 8c^2\delta = 6c(M+2c)\delta,\end{aligned}$$

which implies $|M-c| \leq \sqrt{6c\delta}$. □

Using Lemma 3.1, we can immediately show our main result, namely, Theorem 1.1, for the stability of peakons as follows:

Proof of Theorem 1.1. Let $\delta := \frac{c\varepsilon^4}{6^4(1+c)^4}$ and $M_t := \max_{x \in \mathbb{R}} u(t, x)$ for any $t \in \mathbb{R}$. Using Lemma 3.1 and the fact that $u(\xi(t), t) = M_t$, we have, for any $t \in \mathbb{R}$,

$$\begin{aligned}\mu_s(t)(\mathbb{R}) + \|u(\cdot, t) - c\varphi(\cdot - \xi(t))\|_{H^1}^2 &\leq |\tilde{E}(u(t), \mu(t)) - E(c\varphi)| + 4c|M_t - c| \\ &= |\tilde{E}(\tilde{u}, \tilde{\mu}) - E(c\varphi)| + 4c|M_t - c| \\ &\leq 4c \frac{c\varepsilon^4}{6^4(1+c)^4} + 4c\sqrt{6c \frac{c\varepsilon^4}{6^4(1+c)^4}} \leq \varepsilon^2.\end{aligned}$$

□

Remark 3.1. Let (u, μ) be a conservative solution to the generalized framework (1.5)–(1.7) of the Camassa–Holm equation subject to the initial data $(\tilde{u}, \tilde{\mu}_x^2 dx)$, which means that the initial energy measure $\tilde{\mu}$ does not contain any singular part. If

$$\|\tilde{u} - c\varphi\|_{H^1}^2 \leq \frac{c^2}{2} \left[\frac{\varepsilon}{6(1+c)} \right]^8,$$

then for any $t \in \mathbb{R}$, we obviously have

$$\mu_s(t)(\mathbb{R}) + \|u(\cdot, t) - c\varphi(\cdot - \xi(t))\|_{H^1}^2 \leq \varepsilon^2$$

because of Theorem 1.1, where $\xi(t) \in \mathbb{R}$ is any point where the function $u(\cdot, t)$ attains its maximum. In particular, we know that for the usual $H^1(\mathbb{R})$ initial data, the stability of the peakon holds even after wave breaking. This extends the stability results of Constantin and Strauss¹³ globally-in-time.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Yu Gao: Writing – review & editing (equal). **Hao Liu:** Writing – review & editing (equal). **Tak Kwong Wong:** Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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