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Novel Simplified Practical Method for One-Dimensional Large-Strain Consolidation

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ABSTRACT

A new simplified practical method for one-dimensional nonlinear large-strain consolidation of saturated homogenous soils is proposed. The derivation processes of the proposed method are introduced first, with a modification of Terzaghi's theory from a novel perspective to solve large-strain consolidation problems. Verification checks of the proposed method with other solutions are then conducted. The proposed method is different from Lekha's solution because Lekha's analytical solution is based on the small strain theory. For linear consolidation, the proposed method shows excellent agreement with the Consolidation Settlement 2 (CS2) model. For nonlinear large-strain consolidation, the new method is in good agreement with the CS2 model when $C_c/C_k \leq 1$. After that, optimization of the proposed nonlinear solution is carried out for $C_c/C_k > 1$ with a more precise average constant coefficient of consolidation used in the simplified practical method, and good agreement is obtained between the solutions from the proposed method and the CS2 model. Overall, the proposed simplified method provides practical, reliable, and efficient solutions for analyzing linear and nonlinear large-strain consolidation.

1 | Introduction

Terzaghi [1] first established the one-dimensional consolidation theory of saturated homogeneous soil in 1924. Since then, scholars of geotechnical engineering have been committed to revising and improving the theory, which has led to many remarkable developments in the field of one-dimensional consolidation. Among the diverse interests of researchers, the large-strain deformation and nonlinear response of soil materials, which are frequently encountered in actual projects, have attracted the most attention in the last several decades. Gibson et al. [2], as pioneers, established the one-dimensional large-strain nonlinear consolidation theory, in which material coordinates are introduced to simplify the difficult moving bounding problem, and the finite difference method is usually used to solve the governing equations because of its highly nonlinear character. Experimental studies, theoretical improvements, and numerical analyses have been carried out to enrich the theory of large-strain consolidation [3–18], and various factors affecting the consolidation, such as the self-weight of soil [6, 12], multilayer soil [14], time-dependent loading [15, 17], non-Darcian flow [18], and external hydraulic gradient [16], were considered in these studies. However, the existing analytical solutions of large-strain consolidation are only applicable to some special cases [13], and the calculation processes of numerical solutions considering large strain and nonlinearity of soil are rather complicated [2, 5, 9, 10, 16]. In view of the simplicity and applicability of analytical methods, a new simplified analytical method for the calculation of one-dimensional large-strain linear or nonlinear consolidation is proposed in this paper.

The contribution of the piecewise linear method to the calculation of large-strain nonlinear consolidation is mentioned here to explain why the CS2 model [10] is used for verification. In the piecewise linear method, all variables are updated relative to the fixed coordinate system at each time step [10, 16, 19], and this method was considered to be more versatile than models based on material coordinates [19, 20]. Based on the piecewise linear method, the CS2 model was established by Fox and Berles [10] to solve one-dimensional large-strain nonlinear consolidation problems. The results of this model were in good agreement with the results of other approaches, such as the solution of Gibson et al. [2], solution of Lee and Sills [6], and field tests of McVay et al. [8]. Subsequent studies based on the CS2 model have also proven the reliability and robustness of this method [21–29]. Not surprisingly, we may assume that if the result of an analytical solution is consistent with that of the CS2 model for the same problem, then the solution can be deemed accurate.

This paper presents simplified analytical solutions of one-dimensional large-strain linear and nonlinear consolidation from a new perspective. Theoretical derivations of the analytical solutions under linear and nonlinear material assumptions are first presented, followed by verifications with Lekha's small-strain solution [30] and the CS2 model [10]. Linear soil materials are assumed to have a constant compressibility coefficient (α_v) and vertical permeability coefficient (k_v), and $e - \log \sigma'$ and $e - \log k_v$ relationships are used to represent the nonlinear compressibility and permeability of soil, as recommended by Mesri et al. [3, 31]. Finally, optimization of the proposed nonlinear solution is discussed to improve the shortcomings of this method and existing analytical solutions when $C_c/C_k > 1$.

2 | Simplified Analytical Solutions for Large-Strain Consolidation

2.1 | Linear Materials

The proposed analytical solution for large-strain linear consolidation is based on Terzaghi's one-dimensional small-strain consolidation theory. For a detailed description of the proposed method, the derivation process of Terzaghi's theory is briefly described here. To establish the consolidation governing equation, the following assumptions were made by Terzaghi [1].

1. The soil is homogeneous and saturated.
2. Soil particles and water are incompressible.
3. Fluid seepage and soil compression occur only along the vertical direction, as shown in Figure 1a.
4. Fluid seepage follows Darcy's law, and the permeability coefficient remains unchanged.
5. The void ratio changes linearly with the effective stress, and the compression coefficient remains unchanged.
6. The external load is applied instantaneously and remains constant.
7. There is no movement of the soil boundary throughout the consolidation process.
8. Only the primary consolidation of soil is considered, and no soil creep occurs.

Based on the above assumptions, the governing equation for one-dimensional consolidation was derived as follows

$$C_v \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} \tag{1}$$

where u = excess pore water pressure; $C_v = k_v(1+e_0)/a_v/\gamma_w$ = coefficient of consolidation, k_v = hydraulic conductivity, e_0 = initial void ratio, a_v = coefficient of compressibility, and γ_w = the unit weight of water.

Although the solutions for Equation (1) can be obtained according to different initial and boundary conditions, only the case described in Figure 1a was discussed, as the solutions under other conditions can be easily obtained in a similar way. The initial and boundary conditions of the problem described in Figure 1a can be expressed as follows:

$$\begin{cases} t = 0, 0 \le z \le H_0, u = u_0 = p \\ 0 < t \le \infty, z = 0, u = 0 \\ 0 \le t \le \infty, z = H_0, \dfrac{\partial u}{\partial z} = 0 \\ t = \infty, 0 \le z \le H_0, u = 0 \end{cases} \tag{2}$$

where p is the constant instantaneous load applied to the soil stratum.

The solution of differential Equation (1) meeting the aforementioned conditions can be obtained using the method of variable separation:

$$u_{zt} = \frac{4p}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{m\pi z}{2H_0} e^{-m^2(\pi^2/4)T_v} \qquad (m = 1,\ 3,\ 5, \ldots) \tag{3}$$

where H_0 is the maximum drainage distance (i.e., initial height of the soil layer) and the time factor is $T_v = C_v t/H_0^2$.

The solution of the average degree of consolidation, which is one of the most important references for the design and construction of actual projects, can be derived from Equation (3):

$$\begin{aligned} U_t &= \frac{\int_0^{H_0} u_0 dz - \int_0^{H_0} u_{zt} dz}{\int_0^{H_0} u_0 dz} \\ &= 1 - \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} e^{-m^2\left(\frac{\pi^2}{4}\right)T_v} \qquad (m = 1, 3, 5, \ldots) \end{aligned} \tag{4}$$

The above derivation provides the solution of one-dimensional small-strain linear consolidation. For large-strain linear consolidation, the most significant difference is that the void ratio and seepage path vary greatly during the consolidation process. As shown in Figure 1b, considerable changes in the void ratio occurred in large-strain consolidation transfer of z, dz, and H_0 into z_t, dz_t, and H_t from time 0 to time t. Different from other large-strain theories, in this work, we focus on the seepage path (H_0 and H_t) directly and modify Equation (4) to solve the problem of large-strain consolidation concisely. The proposed

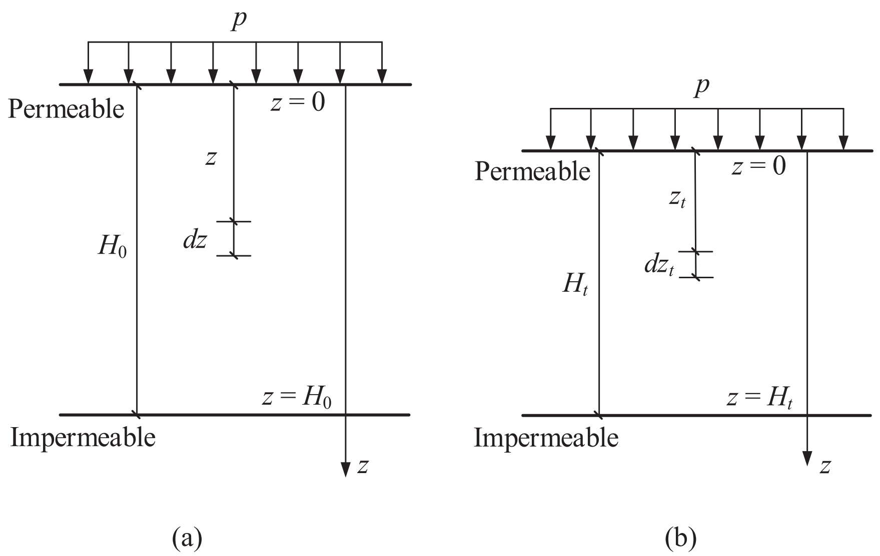

FIGURE 1 | Configuration of the one-dimensional consolidation: (a) initial condition, $t = 0$; (b) after consolidation for time t.

method for calculating the problem of large-strain consolidation, which concisely modifies Terzaghi's theory, is briefly described as follows:

Step 1, the total magnitude settlement at the end of consolidation, S_{total}, is calculated with the linear compressibility relationship and the value of the surcharge load:

$$S_{total} = H_0 \frac{\alpha_v p}{(1 + e_0)} \tag{5}$$

Step 2, the degree of consolidation for time t is calculated by Equation (6), although it may not be definitely accurate:

$$U_t = 1 - \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} e^{-m^2\left(\frac{\pi^2}{4}\right)T_v}, T_v = \frac{C_v}{H_0^2} t \quad (m = 1, 3, 5, \ldots) \tag{6}$$

Step 3, the seepage path (or height of soil) at time t is calculated by U_t:

$$H_t = H_0 - S_{total} U_t \tag{7}$$

Step 4, the seepage path changes from H_0 to H_t in the time interval of t, and their mean value is taken as the equivalent seepage path to calculate the new average degree of consolidation:

$$H_{mean} = \frac{H_0 + H_t}{2} \tag{8}$$

$$U_t = 1 - \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} e^{-m^2\left(\frac{\pi^2}{4}\right)T_{v1}}, T_{v1} = \frac{C_v}{H_{mean}^2} t \quad (m = 1, 3, 5, \ldots) \tag{9}$$

where T_{v1} is the updated time factor for large-strain linear consolidation.

Repeat Steps 3 and 4 until the difference in the average degree of consolidation between continuous iterations can be ignored, for example, when the difference between adjacent U_t values is less than 1×10^{-4}. Following the preceding steps, the degree of consolidation corresponding to a more realistic seepage path is obtained, and the accuracy of this method is illustrated in Section 3.

2.2 | Nonlinear Materials

By analyzing a large number of experimental data, Mesri et al. [3, 31] reported that the nonlinear compressibility and permeability of soil materials can be expressed by the following empirical equations:

$$e = e_1 - C_c \log \frac{\sigma'}{\sigma_1'} \tag{10}$$

$$e = e_1 + C_k \log \frac{k_v}{k_{v1}} \tag{11}$$

where C_c is the compression index and e_1 and σ_1' are the void ratio and corresponding effective stress on the compressibility curve, respectively. C_k is the permeability index, and e_1 and k_{v1} are the corresponding void ratio and permeability coefficient on the permeability curve, respectively. The consolidation coefficient changes significantly during the process of nonlinear consolidation, which can be expressed as follows:

$$C_v = \frac{k_v(1 + e_0)}{\alpha_v \gamma_w} \tag{12}$$

where α_v can be derived from Equation (12) and k_v can be obtained from Equations (10) and (11):

$$\alpha_v = -\frac{\partial e}{\partial \sigma'} = C_c \frac{1}{\sigma' \ln 10} \tag{13}$$

$$k_v = k_{v1} \left(\frac{\sigma'}{\sigma_1'}\right)^{-\frac{C_c}{C_k}} \tag{14}$$

Substituting Equations (13) and (14) into Equation (12), the following equation can be obtained:

$$C_v = \frac{k_{v1}(\sigma'/\sigma'_1)^{-C_c/C_k}}{C_c/\ln 10/\sigma'} \frac{(1+e_0)}{\gamma_w} \tag{15}$$

Consequently, the initial consolidation coefficient is

$$C_{v0} = \frac{k_{v1}(\sigma'/\sigma'_1)^{-C_c/C_k}}{C_c/\ln 10/\sigma'_0} \frac{(1+e_0)}{\gamma_w} = \frac{k_{v0}\ln 10(1+e_0)}{\gamma_w C_c} \tag{16}$$

Combining Equations (15) and (16), we can obtain

$$C_v = C_{v0}\left(\frac{\sigma'}{\sigma'_0}\right)^{(1-C_c/C_k)} \tag{17}$$

From the beginning to the end of consolidation, the consolidation coefficient changes from C_{v0} to $C_{v0}(1+p/\sigma')^{(1-C_c/C_k)}$. The general solution of effective stress is difficult to determine because of the changes in t and z; therefore, the accurate value of C_v is also hard to obtain. Here, the average value of C_{v0} and $C_{v0}(1+p/\sigma'_0)^{(1-C_c/C_k)}$ is used as the equivalent consolidation coefficient throughout the whole consolidation process, which is similar to the practices of Lekha et al. [30], Geng et al. [32], and Indraratna et al. [33]. Briefly, analytical solutions for one-dimensional large-strain nonlinear consolidation can be obtained as follows:

Step 1, the final settlement is calculated by the nonlinear compressibility relationship and the value of the surcharge load:

$$S_{total} = \frac{H_0}{(1+e_0)} C_c \log\left(1+\frac{p}{\sigma'_0}\right) \tag{18}$$

Step 2, the degree of consolidation for time t is calculated as follows, although it may be inaccurate:

$$C_{vm} = \left(C_{v0} + C_{v0}\left(1+\frac{p}{\sigma'_0}\right)^{(1-C_c/C_k)}\right)/2 \tag{19}$$

$$U_t = 1 - \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}e^{-m^2\left(\frac{\pi^2}{4}\right)T_{vm}}, T_{vm} = \frac{C_{vm}}{H_0^2}t \quad (m = 1, 3, 5, \ldots) \tag{20}$$

Step 3, the seepage path for time t can be calculated by U_t:

$$H_t = H_0 - S_{total}U_t \tag{21}$$

Step 4, the new average degree of consolidation is calculated:

$$H_{mean} = \frac{H_0 + H_t}{2} \tag{22}$$

$$U_t = 1 - \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}e^{-m^2\left(\frac{\pi^2}{4}\right)T_{v2}}, T_{v2} = \frac{C_{vm}}{H_{mean}^2}t \quad (m = 1, 3, 5, \ldots) \tag{23}$$

where T_{v2} is the updated time factor for large-strain nonlinear consolidation.

Repeat Steps 3 and 4 until the difference between adjacent values of the average degree of consolidation can be neglected. The schematic diagram of the proposed method is shown in Figure 2. Due to the differences in drainage paths resulting from the coordinate systems used in small-strain and large-strain theories, small-strain theory tends to underestimate the consolidation rate under the same mechanical conditions. This occurs because small-strain theory does not consider changes in drainage paths caused by consolidation settlement, which will be further examined in the Discussion section. The simplified method presented above addresses the effects of large-strain consolidation by estimating approximate drainage paths.

In large-strain consolidation, two average degrees of consolidation are defined: one based on excess pore water pressure and the other on settlement. Research shows that the degree of consolidation based on excess pore water pressure usually lags behind that based on settlements due to the high compressibility at low pressures and the initially limited dissipation of excess pore pressure [34]. The method presented here proposes an average degree of consolidation based on consolidation settlements. Differences between average degrees of consolidation based on settlement and excess pore pressure are beyond the scope of this study, and the average degree of consolidation results are mainly analyzed below.

3 | Method Validation

3.1 | Comparison Against Small-Strain Analytical Solution

Lekha et al. [30] proposed an analytical solution of small-strain nonlinear consolidation and compared the results from that solution with experimental observations. Here, a result comparison of the proposed solution with that of Lekha's solution was performed to reflect the influence of large-strain deformation on one-dimensional consolidation. The saturated homogeneous soil layer used for this comparison example has the following properties: initial height $H_0 = 1$ m, initial void ratio $e_0 = 2.0$, initial effective stress $\sigma'_0 = 20$ kPa, and initial permeability coefficient $k_{v0} = 1\times10^{-9}$ m/s. $C_c = 1$ and $C_k = 1$ and 2 are considered, only the upper boundary is permeable and different values of the surcharge load are applied such that the final vertical strain $\varepsilon_f = 0.01\%$, 10%, and 30%. The comparison of the average degree of consolidation between the proposed solution and Lekha's solution is shown in Figure 3. Results show that the results of Lekha's solution of consolidation under different loadings are the same for $C_c/C_k = 1$, which neglects the influence of large-strain deformation, while the proposed solution does not. It can also be determined from the comparison of the results in Figure 3a,b that the combined effect of large-strain deformation and nonlinearity of soil (Figure 3b) is greater than that of only large-strain deformation (Figure 3a), and this can be attributed to the interaction of the effects of large-strain deformation and a nonlinear soil response. When large-strain deformation occurs, the changes in the compressibility and permeability of soil tend to increase, which in turn significantly influences the consolidation process of soil.

3.2 | Comparison Against Large-Strain Numerical Solution: Linear Problem

In this section, the solution of one-dimensional large-strain linear consolidation is compared with the CS2 model [10] through specific example to verify its accuracy. For the validation example,

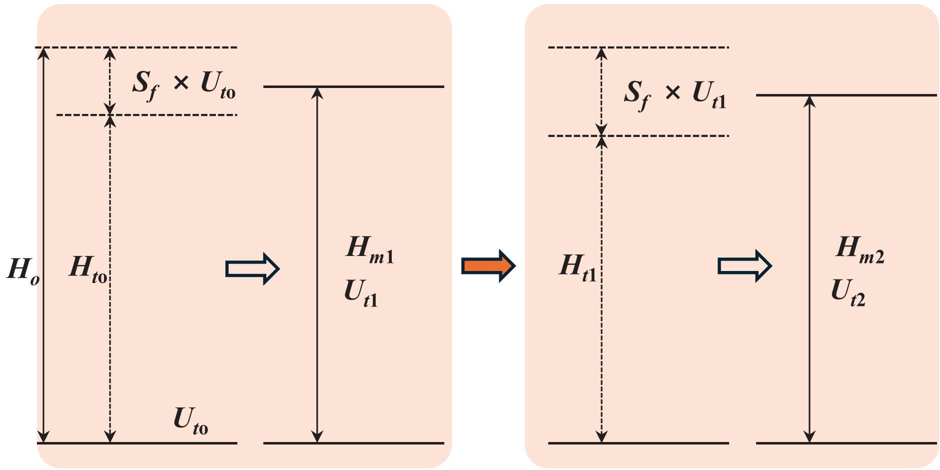

FIGURE 2 | Schematic diagram of the proposed simplified method.

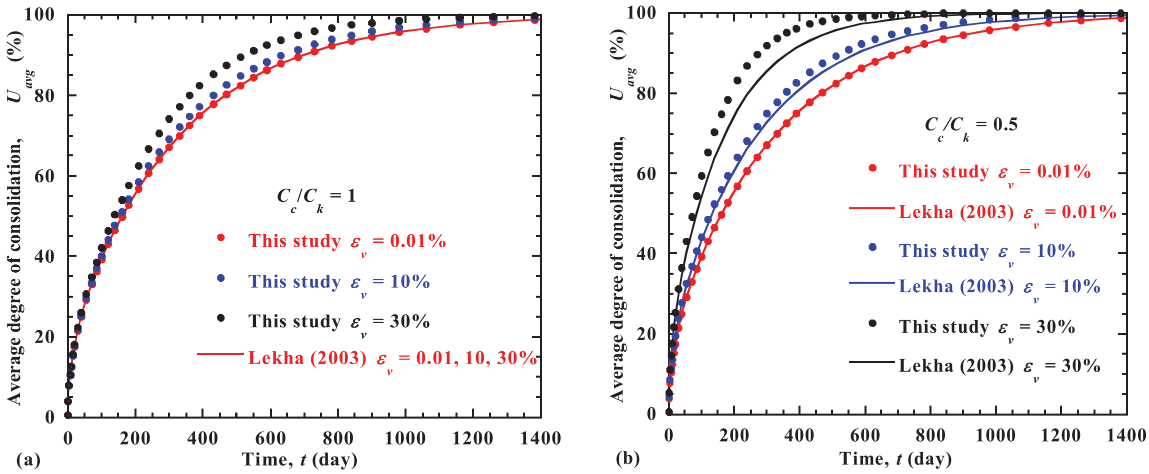

FIGURE 3 | Comparison between the proposed large-strain solution and Lekha's small-strain solution: (a) $C_c/C_k = 1$; (b) $C_c/C_k = 0.5$.

a saturated homogeneous soil layer with the following properties is used: initial void ratio $e_0 = 2.0$, initial effective stress $\sigma_0' = 20$ kPa, constant compressibility coefficient $\alpha_v = 0.01$ kPa^{-1}, permeability coefficient $k_v = 1 \times 10^{-9}$ m/s, and consolidation coefficient $C_v = 3.06 \times 10^{-8}$ m^2/s. Only the upper boundary is permeable, and different values of surcharge load are applied such that the final vertical strain $\varepsilon_f = 0.01\%$, 10%, 20%, and 30%. In addition, $H_0 = 0.5$, 1, 2, and 4 m are considered to show that the initial height of the soil does not affect the accuracy of the proposed solution. For the proposed analytical solution, the first 50 items (Equation 9) are calculated to ensure the accuracy of the average degree of consolidation for very small t. The soil is represented by $R_j = 200$ elements in the CS2 model, which is enough to obtain results with high accuracy. The comparison of the average degree of consolidation versus time between the proposed solution and the CS2 model is shown in Figure 4. Extremely good agreements are observed between the results of this study and the CS2 model for different final vertical strains and different initial heights. Therefore, it can be concluded that the proposed method of solving large-strain linear problems is feasible and reliable. It is also worth mentioning that only four or fewer cycles described in Section 2.1 are required for the results to converge within an error of 1×10^{-4}. Compared with other numerical methods for large-strain consolidation, the proposed method is more convenient and easier to execute. Moreover, because the new method is modified from Terzaghi's theory, which is familiar to researchers and engineers, it is considered to be of great academic and practical interest.

3.3 | Comparison Against Large-Strain Numerical Solution: Nonlinear Problem

The solution of large-strain nonlinear consolidation is compared with the CS2 model in this section. The saturated homogeneous soil layer used for this verification example has the following properties: initial height $H_0 = 1$ m, initial void ratio $e_0 = 2.0$, initial effective stress $\sigma_0' = 20$ kPa, and initial permeability coefficient $k_{v0} = 1 \times 10^{-9}$ m/s. Considering the common range of C_c/C_k, the compression index $C_c = 1$, and permeability index $C_k = 2$, 1.5, 1, 0.75, and 0.5, were selected. Only the upper boundary is permeable, and different values of surcharge load are applied such that the final vertical strain $\varepsilon_f = 0.01\%$, 10%, 20%, and 30%. The comparison of the average degree of consolidation between the proposed solution and the CS2 model is shown in Figure 5. Good agreements are observed between the proposed solution,

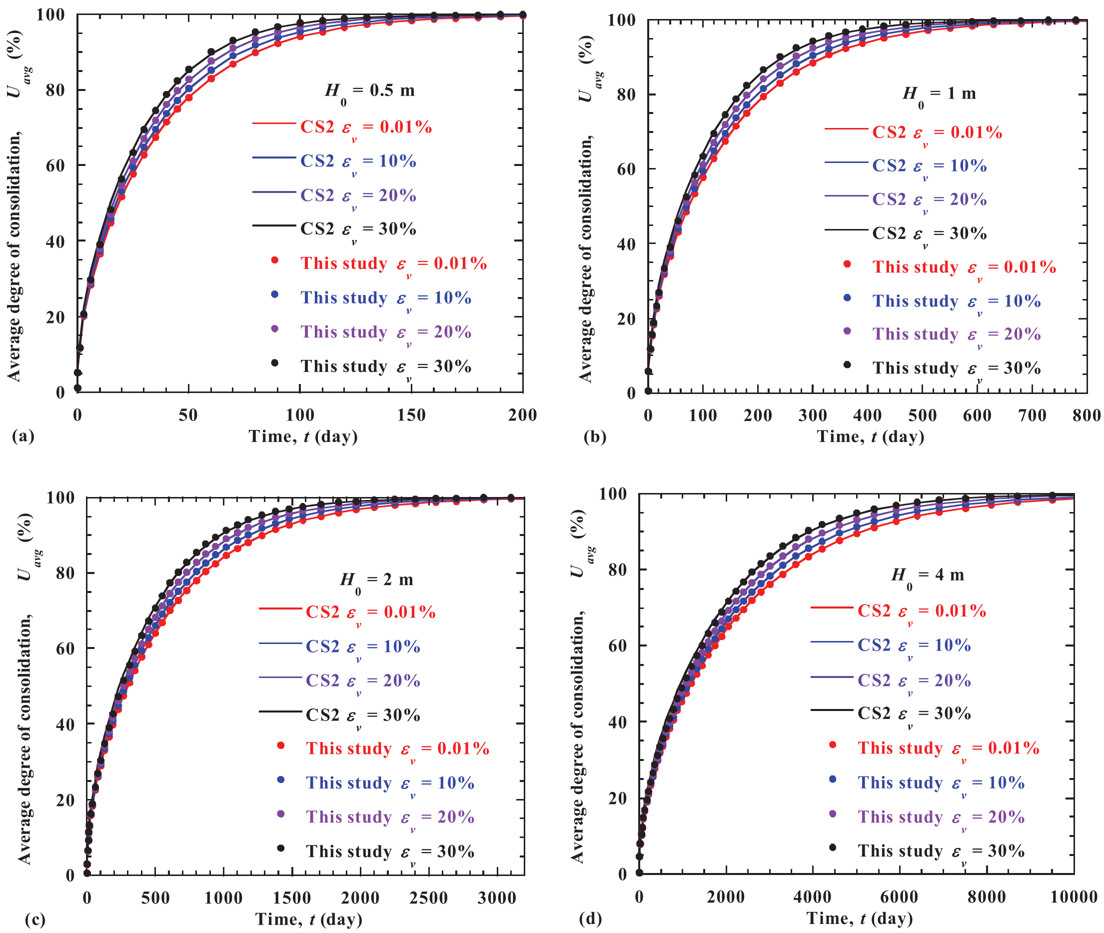

FIGURE 4 | Comparison between the proposed large-strain linear solution and CS2 model: (a) $H_0 = 0.5$ m; (b) $H_0 = 1$ m; (c) $H_0 = 2$ m; (d) $H_0 = 4$ m.

which converges in only four iterations, and the CS2 model for $C_c/C_k \leq 1$, while the results of the two methods are different for $C_c/C_k > 1$. It seems that the average coefficient of consolidation C_{vm} according to Equation (24) is an oversimplification for the whole consolidation process when $C_c/C_k > 1$,

$$C_{vm} = \left[C_{v0} + C_{v0}(1 + p/\sigma'_0)^{(1-C_c/C_k)}\right]/2 \tag{24}$$

and a simple numerical example is listed as follows to explain this problem. When the surcharge load applied to the soil layer is eight times the initial effective stress σ'_0, it is approximately accurate to select

$$\frac{1^{0.5} + (1 + 8\sigma'_0/\sigma'_0)^{0.5}}{2} C_{v0} = \left(1 + \frac{3\sigma'_0}{\sigma'_0}\right)^{0.5} C_{v0} \tag{25}$$

as the equivalent value of C_v, which ranges from C_{v0} to $9^{0.5}C_{v0}$ if $C_c/C_k = 0.5$. However, on the contrary, it is obviously inaccurate to select

$$\frac{1^{-1} + (1 + 8\sigma'_0/\sigma'_0)^{-1}}{2} C_{v0} = \left(1 + \frac{0.8\sigma'_0}{\sigma'_0}\right)^{-1} C_{v0} \tag{26}$$

when C_v ranges from C_{v0} to $9^{-1}C_{v0}$ if $C_c/C_k = 2$, which is too large to precisely represent the whole consolidation process. Consequently, a more reasonable value of the equivalent consolidation coefficient should be used, and more precise consolidation calculation time steps should be divided for cases of $C_c/C_k > 1$.

The relationship between σ'/σ'_0 and C_v/C_{v0} for different C_c/C_k ratios is shown in Figure 6. When $C_c/C_k = 1$, the ratio of C_v/C_{v0} remains constant. When $C_c/C_k = 0.5$, the ratio of C_v/C_{v0} increases almost linearly with the ratio of σ'/σ'_0. During the consolidation process, when the effective stress increases tenfold, the two average values of coefficient of consolidation are relatively close to each other if only two points, the first and the last, are used for calculation, as opposed to using 10 points through the process. However, when $C_c/C_k = 2$, the two average values of coefficient of consolidation differ significantly. Specifically, when $C_c/C_k = 2$, if the Equation (24) is used to calculate the average coefficient of consolidation based on only two points, the results deviates significantly from the true average value. In such case, using multiple data points (>2) to calculate an approximate average coefficient of consolidation can reduce the error.

3.4 | Average Coefficient of Consolidation for Highly Nonlinear Material Parameter

As explained above, the large discrepancy between the results of CS2 and the simplified practical method for $C_c/C_k = 2$ may

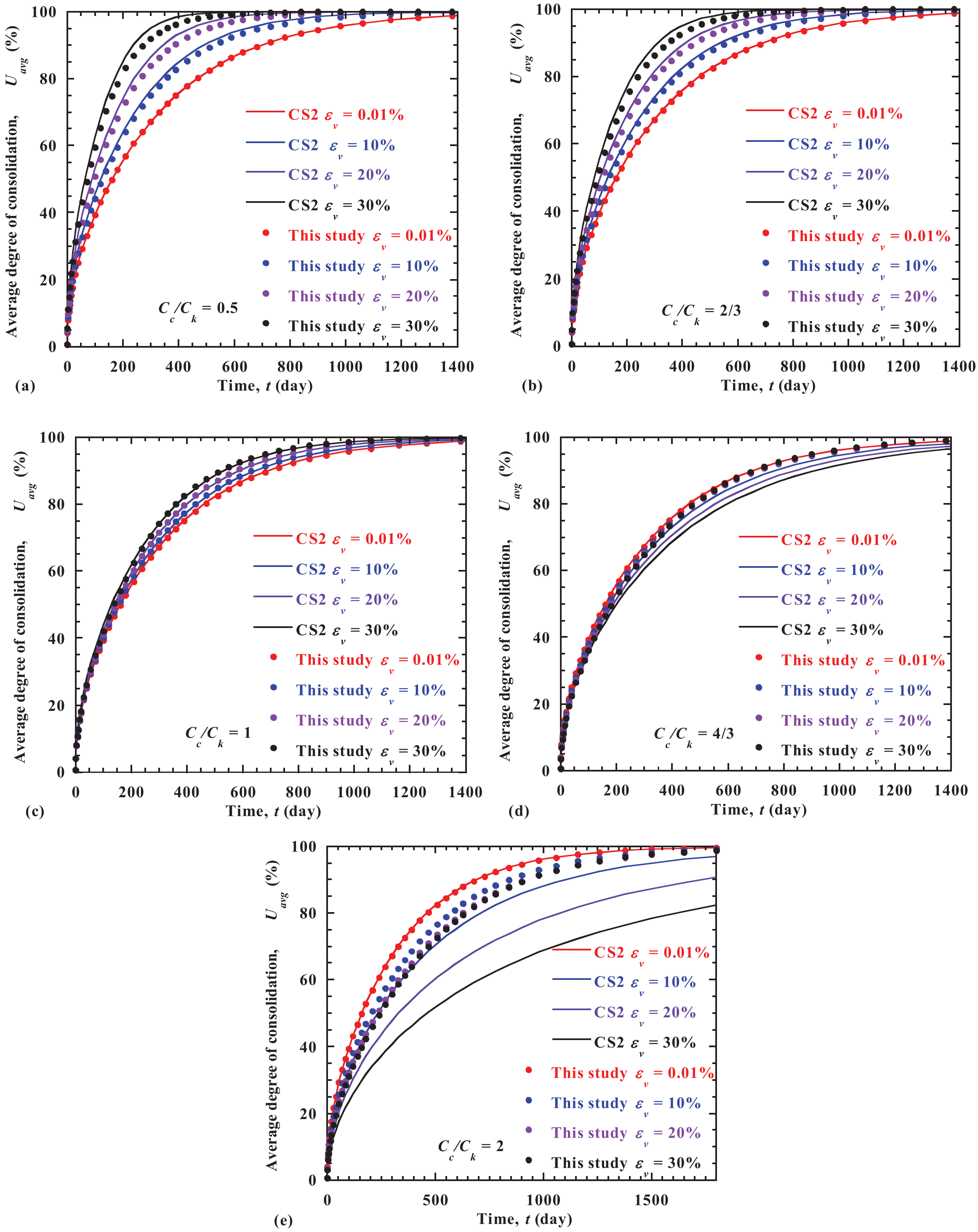

FIGURE 5 | Comparison between the proposed large-strain nonlinear solution and CS2 model: (a) $C_c/C_k = 0.5$; (b) $C_c/C_k = 2/3$; (c) $C_c/C_k = 1$; (d) $C_c/C_k = 4/3$; (e) $C_c/C_k = 2$.

result from inaccuracies in the equivalent value of C_v throughout the consolidation process. In this section, a more precise value of the equivalent consolidation coefficient is used, and smaller consolidation calculation time steps are utilized. The properties of the soil material used in this section have been introduced in Section 3.3, and the situation of $C_c/C_k = 2$, in which case the most remarkable difference occurs, is studied here. In this example, 43 time points are used to ensure that a relatively small-strain deformation occurs during every time interval. While there is no strict rule for selecting the time points, one guideline is that the strain in adjacent intervals should be small enough to ensure the accuracy of the method. Moreover, the mean value of $C_{v0}(1 + p/\sigma_0')^{(1-C_c/C_k)}$ corresponds to the final effective stress of the soil and C_{vpre}, which corresponds to the mean effective stress of the soil at the previous time point, is selected as the current equivalent consolidation coefficient to ensure that it is

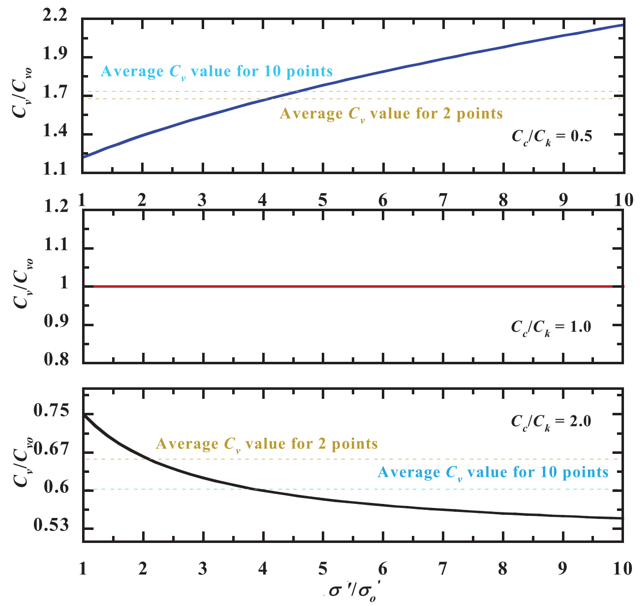

FIGURE 6 | Relationship between σ'/σ'_0 versus C_v/C_{v0} for different C_c/C_k ratios.

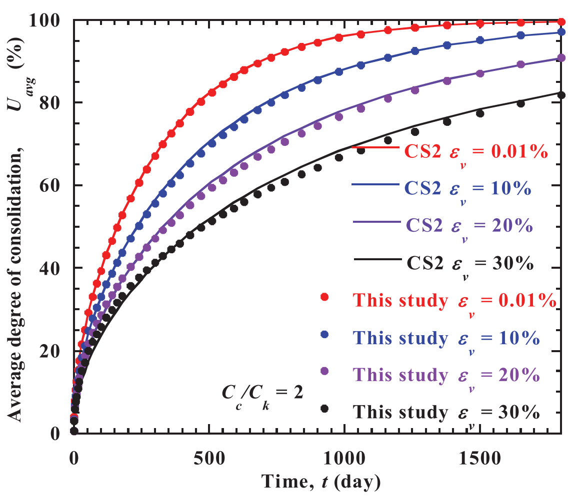

FIGURE 7 | Comparison between the optimized solution of large-strain nonlinear consolidation and the CS2 model.

suitable to accurately reflect the real consolidation process. The comparison of the average degree of consolidation between the optimized solution and the CS2 model is shown in Figure 7. Generally, good agreement is obtained, although relatively small differences are observed in the latter part of the consolidation process. Compared to other numerical methods, the proposed method offers a more convenient approach to deriving solutions. It can also be applied to complex conditions, such as multilayer soil consolidation, though these are not discussed in detail in this study. The simplicity and practicality of the proposed method are its key advantages.

4 | Discussions on the Effects of Nonlinearity

In practical, the nonlinearity in consolidation analysis usually includes two aspects: mechanical nonlinearity (variable hydraulic conductivity and compressibility) and geometrical nonlinearity (large-strain theory) [35]. The widely used Terzaghi's consolidation theory is based on the assumptions of mechanical and geometrical linearity (constant C_v and small-strain theory), which often differ significantly from practical conditions. Several solutions were systematically compared to examine the effects of mechanical and geometrical nonlinearity on consolidation. A hypothetical example with an initial height H_0 = 10 m is used for discussion. The variation in compressibility and hydraulic conductivity during consolidation follows the constitutive relationships in Equations (10) and (11). The initial void ratio ($e_0 = 2.7$) distribution is in equilibrium with the initial effective overburden stress σ'_0 = 20 kPa, and the soil layer has an initial hydraulic conductivity $k_{vo} = 2.0 \times 10^{-8}$ m/s and compression index $C_c = 1$. Four C_k values, 0.6, 0.8, 1.0, and 1.2, are used to explore the effect of C_c/C_k ratio on consolidation. The top boundary is drained, while the bottom is undrained, both with constant hydraulic heads. An instantaneous vertical stress p = 400 kPa is applied to the top of soil layer at the beginning and remains constant thereafter, which yields a final settlement of 3.574 m, corresponding to a final vertical strain of 35.74%.

The effect of mechanical nonlinearity on the consolidation process was investigated using two CS2 simulations: one with constant C_v and the other with variable C_v. CS2 has been extensively validated using analysis solutions, numerical solutions, and experimental data, demonstrating that CS2 can yield high-accuracy results for modeling large-strain consolidation [36]. The coefficient of compressibility and the hydraulic conductivity are both constant for constant C_v case, having values of $a_v = 0.0033$/kPa and $k_v = 1.251 \times 10^{-10}$, 4.449×10^{-10}, 9.524×10^{-10}, and 1.582×10^{-9} m/s for C_k = 0.6, 0.8, 1.0, and 1.2, respectively. The constant value of hydraulic conductivity was determined as the average between the initial and final hydraulic conductivities based on the constitutive relationship in Equation (11). A comparison of the CS2 solutions for constant and variable C_v is presented in Table 1. For each case, the average degree of consolidation U_{avg} is listed as a function of the consolidation time. For each case, the first column gives U_{avg} results for constant C_v, while the second column provides the values for variable C_v. In all cases, the first column values are significantly larger than those in the second column, indicating that using a constant C_v overestimates the consolidation rate. This overestimation results from the significant reduction in hydraulic conductivity as void ratios decrease during consolidation. To further illustrate this, profiles of relative U_{avg} error $E = (U_{avg1} - U_{avg2}) / U_{avg2}$ for the solutions in Table 1 are shown in Figure 8a, where U_{avg1} and U_{avg2} correspond to the first and second column values, respectively. Errors are positive values for all profiles and consolidation time.

Similarly, the effect of geometrical nonlinearity on consolidation process was investigated by comparing solutions from small-strain and large-strain theories. Large-strain consolidation simulation was conducted using CS2. Solutions based on small-strain theory were obtained using FlexPDE, a finite element modeling tool for partial differential equations, with the following equation.

$$\frac{\partial}{\partial z}\left[\frac{k_{vo}}{\gamma_w}\left(\frac{\sigma'_0}{\sigma'}\right)^{C_c/C_k}\frac{\partial u}{\partial z}\right] = m_v\frac{\partial u}{\partial t} = \frac{C_c}{\ln 10\,(1+e_o)\,\sigma'}\frac{\partial u}{\partial t} \quad (27)$$

TABLE 1 | Comparison of U_{avg} solutions for constant and variable C_v (large strain).

Time (year)	$C_c = 1, C_k = 0.6$		$C_c = 1, C_k = 0.8$		$C_c = 1, C_k = 1.0$		$C_c = 1, C_k = 1.2$	
	C-C_v	V-C_v	C-C_v	V-C_v	C-C_v	V-C_v	C-C_v	V-C_v
0.01	4.911	2.303	5.600	3.351	6.165	4.278	6.631	5.075
0.05	10.983	5.166	12.525	7.494	13.787	9.564	14.830	11.344
0.1	15.533	7.310	17.714	10.598	19.499	13.525	20.973	16.042
0.5	34.735	16.353	39.610	23.699	43.600	30.241	46.893	35.870
1	49.110	23.128	55.937	33.516	61.410	42.766	65.790	50.723
2	68.645	32.700	76.802	47.348	82.517	60.278	86.501	71.105
3	81.092	39.965	88.204	57.635	92.414	72.641	94.924	84.021
4	88.806	45.922	94.131	65.712	96.780	81.401	98.131	91.561
5	93.453	50.962	97.114	72.151	98.647	87.487	99.317	95.666
10	99.582	68.030	99.921	89.883	99.983	98.405	99.996	99.864
20	99.998	84.257	100.000	98.602	100.000	99.976	100.000	100.000
30	100.000	91.591	100.000	99.804	100.000	100.000	100.000	100.000
40	100.000	95.348	100.000	99.973	100.000	100.000	100.000	100.000
50	100.000	97.380	100.000	99.996	100.000	100.000	100.000	100.000

Note: C-C_v: Constant C_v; V-C_v: Variable C_v.

TABLE 2 | Comparison of U_{avg} solutions for small-strain and large-strain theories (variable C_v).

Time (year)	$C_c = 1, C_k = 0.6$		$C_c = 1, C_k = 0.8$		$C_c = 1, C_k = 1.0$		$C_c = 1, C_k = 1.2$	
	FlexPDE	CS2	FlexPDE	CS2	FlexPDE	CS2	FlexPDE	CS2
0.01	1.190	2.303	1.633	3.351	2.119	4.278	2.565	5.075
0.05	2.000	5.166	3.142	7.494	4.268	9.564	5.271	11.344
0.1	2.685	7.310	4.344	10.598	5.941	13.525	7.364	16.042
0.5	5.769	16.353	9.535	23.699	13.112	30.241	16.299	35.870
1	8.126	23.128	13.462	33.516	18.542	42.766	23.096	50.723
2	11.560	32.700	19.344	47.348	27.119	60.278	34.729	71.105
3	14.441	39.965	24.667	57.635	35.703	72.641	47.429	84.021
4	17.132	45.922	29.952	65.712	44.584	81.401	60.149	91.561
5	19.721	50.962	35.212	72.151	53.297	87.487	71.294	95.666
10	31.588	68.030	59.045	89.883	84.311	98.405	96.245	99.864
20	50.726	84.257	86.602	98.602	98.848	99.976	99.957	100.000
30	64.796	91.591	96.049	99.804	99.922	100.000	100.000	100.000
40	74.989	95.348	98.869	99.973	99.995	100.000	100.000	100.000
50	82.301	97.380	99.679	99.996	100.000	100.000	100.000	100.000

Note: FlexPDE solutions are obtained using small strain theory, and CS2 solutions are based on large-strain theory.

where $m_v = C_c/[ln10(1+e_o)\sigma']$. Equation (27) represents the governing equation for consolidation, incorporating variable hydraulic conductivity and compressibility (as detailed in Equations 10 and 11) within the context of small-strain theory. Detailed derivations are provided in the Appendix. The comparison results of U_{avg} between FlexPDE (small-strain theory) and CS2 (large-stain theory) for soil layers with variable hydraulic conductivity and compressibility are shown in Table 2. The U_{avg} results from FlexPDE are noticeably smaller than those from CS2, indicating that the small-strain theory underestimates the consolidation rate. Large strains reduce the drainage path length and accelerate the consolidation process, consistent with the findings of Fox and Pu [16] for variable final strains. In large-strain theory, the drainage path progressively shortens, whereas in small-strain theory, the drainage path remains constant. Figure 8b further illustrates the difference between the two solutions.

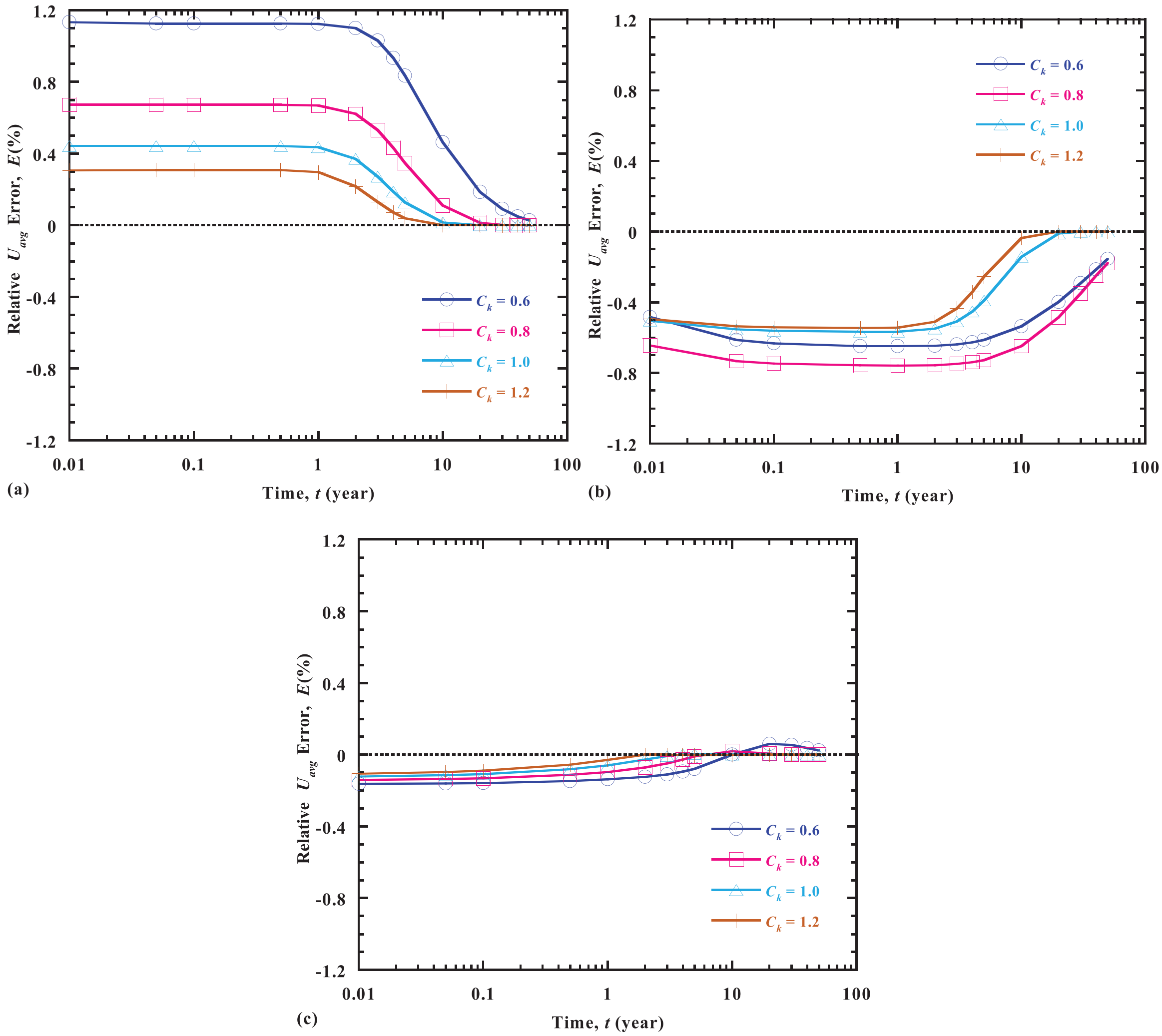

FIGURE 8 | Relative average degree of consolidation error between different solutions: (a) constant C_v and variable C_v; (b) small-strain and large-strain; and (c) proposed method and CS2.

Ignoring the variable hydraulic conductivity during the consolidation process may lead to an overestimation of the consolidation rate, whereas, neglecting large-strain effects may result in an underestimation due to the progressive shortening of the drainage path within the soil layer during large-strain consolidation settlement. Consequently, variable hydraulic conductivity and large-strain effects exert opposing influences on the consolidation process. Table 3 compares the U_{avg} solutions obtained from the proposed method and the CS2 model, while Figure 8c presents the profiles of the relative U_{avg} error. The results show that the proposed method generally agrees with the CS2 numerical solutions. Relative errors in U_{avg} are less than 0.2% across all profiles and the whole consolidation process.

The initial form of governing equation derived by Terzaghi [37] for one-dimensional consolidation is

$$\frac{k_r}{a_v}\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} \tag{28}$$

where $k_r = k(e)/(1 + e)$ is the reduced coefficient of permeability defined in the material coordinate system. A reduced permeability coefficient k_r implies that Darcy's law is expressed in terms of the relative velocities between the fluid and solid phases. By neglecting the self-weight, the governing equation for one-dimensional large-strain consolidation, as derived by Gibson et al. [2], reduces as follows:

$$\frac{(1+e_o)}{\gamma_w}\frac{\partial}{\partial a}\left[\frac{k(e)(1+e_o)}{(1+e)}\frac{d\sigma'}{de}\frac{\partial e}{\partial a}\right] = -\frac{\partial e}{\partial t} \tag{29}$$

where a is the initial coordinate system fixed in space and time, Lagrangian coordinate; and $k(e)$ = hydraulic conductivity dependent on the void ratio. The relationship between the material coordinate z and the Lagrangian coordinate a is

$$z(a) = \int_0^a da/(1+e_o) \tag{30}$$

Using Equation (30), Equation (29) can be rewritten as

$$\frac{\partial}{\partial z}\left[\frac{k_r}{\gamma_w a_v}\frac{\partial e}{\partial z}\right] = \frac{\partial e}{\partial t} \tag{31}$$

TABLE 3 | Comparison of U_{avg} solutions of the proposed method and CS2 (large-strain and variable C_v).

	$C_c = 1, C_k = 0.6$		$C_c = 1, C_k = 0.8$		$C_c = 1, C_k = 1.0$		$C_c = 1, C_k = 1.2$	
Time (year)	**PM**	**CS2**	**PM**	**CS2**	**PM**	**CS2**	**PM**	**CS2**
0.01	1.931	2.303	2.879	3.351	3.761	4.278	4.537	5.075
0.05	4.336	5.166	6.480	7.494	8.481	9.564	10.250	11.344
0.1	6.153	7.310	9.210	10.598	12.073	13.525	14.612	16.042
0.5	13.954	16.353	21.046	23.699	27.793	30.241	33.871	35.870
1	19.954	23.128	30.283	33.516	40.246	42.766	49.309	50.723
2	28.683	32.700	43.955	47.348	58.706	60.278	71.055	71.105
3	35.592	39.965	54.857	57.635	72.172	72.641	84.131	84.021
4	41.570	45.922	63.962	65.712	81.589	81.401	91.391	91.561
5	46.946	50.962	71.490	72.151	87.910	87.487	95.314	95.666
10	68.010	68.030	91.658	89.883	98.489	98.405	99.757	99.864
20	89.380	84.257	99.262	98.602	99.973	99.976	99.999	100.000
30	96.503	91.591	99.931	99.804	100.000	100.000	100.000	100.000
40	98.824	95.348	99.993	99.973	100.000	100.000	100.000	100.000
50	99.599	97.380	99.999	99.996	100.000	100.000	100.000	100.000

Abbreviation: PM: solutions from the proposed method.

where k_r is the same as in Equation (28) from Terzaghi's consolidation theory. Assuming that the k_r and a_v are constants, same as in Terzaghi's consolidation theory, Equation (31) is expressed as

$$\frac{k_r}{\gamma_w a_v}\frac{\partial^2 e}{\partial z^2} = \frac{\partial e}{\partial t} \tag{32}$$

Based on the principle of effective stress and assuming that the total stress remains constant during the consolidation process, that is, $\Delta e = -a_v \Delta\sigma' = -a_v(\Delta\sigma - u)$, Equation (32) simplifies to:

$$\frac{k_r}{\gamma_w a_v}\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} \tag{33}$$

Equation (33) is identical to the governing equation derived by Terzaghi [37] when the unit weight of water is omitted. Notably, the coordinate z in Equation (33) is the material coordinate, whereas Equation (28), when it was introduced, did not explicitly state the diversity of coordinate meanings. In the subsequent publications by Terzaghi, the formation in Equation (1) was used. Equation (28), the governing equation originally proposed by Terzaghi, is considered to be a finite-strain linear formulation, which was pointed out by Znidarcic and Schiffman [38]. Despite the similarities in form between Equations (28) and (33), they differ significantly in their coordinate systems. If the difference in coordinate systems is ignored, the two equations are expected to yield institutionally similar general solutions. Consequently, this study focuses on Terzaghi's consolidation analysis solution, which is widely applied in practical, to address the effect of large strain in a simplified approximate approach.

The simplified method proposed in this paper uses the average coefficient of consolidation C_v to account for nonlinear constitutive relationships (mechanical nonlinearity), and a concise difference format (as shown in Figure 2) to account for the effect of changes in drainage paths during large strain settlement (geometrical nonlinearity). The results indicate that the proposed simplified method provides good, reliable, and practical solutions for analyzing linear and nonlinear large-strain consolidation.

5 | Conclusions

From this study of simplified analytical solutions of one-dimensional large-strain linear and nonlinear consolidation, the following conclusions can be drawn:

1. The proposed solutions for one-dimensional large-strain linear and nonlinear consolidation are based on a simplified method to consider the effect of large-strain deformation. The large-strain linear method can consider soil with a constant compressibility coefficient α_v and permeability coefficient k_v, and the large-strain nonlinear method is aimed at nonlinear soil material with $e - \log\sigma'$ and $e - \log k_v$ relationships. Compared with other large-strain solutions, the proposed method considers the large-strain deformation of the soil from a new perspective, and the results can be obtained through extremely clear and simple iterations.
2. To illustrate the effect of large-strain deformation on one-dimensional consolidation, the results of the proposed method were compared to those of a small-strain nonlinear analytical solution provided by Lekha et al. The comparison between the results of the large-strain nonlinear solution and Lekha's solution shows that a large-strain deformation has an obvious influence on consolidation, especially for cases involving both a large-strain deformation and nonlinear response of the soil.

3. Verification checks of the large-strain linear solution show excellent agreement with the CS2 model. The proposed method, which slightly modifies the well-known Terzaghi's solution, can converge within an error of 1×10^{-4} through only four or fewer iterations, which is one of the great advantages of this method compared with other large-strain methods. Good agreements are obtained between the results of the large-strain nonlinear solution and CS2 model for $C_c/C_k \leq 1$, while the results of the two methods are different for $C_c/C_k > 1$. Further analysis shows that the difference comes from the obviously unreasonable selection of the equivalent value of C_v throughout the consolidation process for cases of $C_c/C_k > 1$.
4. Optimization of the proposed nonlinear solution is carried out for the cases of $C_c/C_k > 1$. A relatively reasonable value of the equivalent consolidation coefficient is used, more precise calculation time steps are utilized, and good agreement is obtained between the optimized solution and the CS2 model for $C_c/C_k > 1$. Although the application of the proposed method is not as simple for the cases of $C_c/C_k > 1$ as that for the cases of $C_c/C_k \leq 1$, compared to other large-strain nonlinear consolidation methods, the proposed method still has inherent advantages in practicality.
5. Large strains, as well as variable hydraulic conductivity and compressibility have opposing effects on consolidation. It is advisable to consider both mechanical and geometrical nonlinearities together in practical engineering design, rather than addressing them separately, to avoid significant errors. The proposed method provides reliable and practical solutions for analyzing the entire consolidation process, accounting for large strains, variable hydraulic conductivity, and compressibility.

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Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Appendix

Derivation of the Governing Equation for Nonlinear Small-Strain Consolidation

Equations (10) and (11) are usually adopted as the compressibility and hydraulic conductivity constitutive relationships in the consolidation analysis. Based on Equations (10) and (11), the following relationships can be obtained.

$$k = k_0 \left(\frac{\sigma'}{\sigma'_0} \right)^{C_c/C_k} \tag{A1}$$

$$m_v = -\frac{1}{1+e_0}\frac{\partial e}{\partial \sigma'} = \frac{C_c}{\ln 10\,(1+e_0)\,\sigma'} \tag{A2}$$

where $m_{vo} = C_c/\ln 10/(1+e_0)/\sigma'_0$. Based on the principle of effective stress, the strain rate is

$$\frac{\partial \varepsilon_v}{\partial t} = -\frac{1}{1+e_0}\frac{\partial e}{\partial t} = m_v \frac{\partial u}{\partial t} \tag{A3}$$

In the context of small strain theory, the rate of pore water volume change within the unit equals to the strain rate of unit, and thus the governing equation can be obtained as follows:

$$\frac{1}{\gamma_w}\nabla \cdot (k \nabla u) = -\frac{\partial \varepsilon_v}{\partial t} \tag{A4}$$

where ε_v is the vertical strain. Substituting Equations (10), (11), (A2), and (A3) into Equation (A4), one-dimensional consolidation governing equation under the framework of small strain theory, incorporating variable hydraulic conductivity and compressibility is derived.

$$\frac{\partial}{\partial z}\left[\frac{k_{vo}}{\gamma_w} \left(\frac{\sigma'_0}{\sigma'} \right)^{C_c/C_k} \frac{\partial u}{\partial z} \right] = \frac{C_c}{\ln 10\,(1+e_0)\,\sigma'}\frac{\partial u}{\partial t} \tag{A5}$$