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Article

Dynamic programming principle for backward doubly stochastic recursive optimal control problem and sobolev weak solution of the stochastic Hamilton-Jacobi-Bellman equation

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ABSTRACT

In this paper, we investigate a backward doubly stochastic recursive optimal control problem wherein the cost function is expressed as the solution to a backward doubly stochastic differential equation. We present the dynamical programming principle for this type of optimal control problem and establish that the value function is the unique Sobolev weak solution to the associated stochastic Hamilton-Jacobi-Bellman equation.

1. Introduction

Backward stochastic differential equation (BSDE in short) has been introduced by Pardoux and Peng [1]. Independently, Duffie and Epstein [2] introduced BSDE from economic background. In ref. [2], they presented a stochastic differential recursive utility which is an extension of the standard additive utility with the instantaneous utility depending not only on the instantaneous consumption rate but also on the future utility. The recursive optimal control problem is presented as a kind of optimal control problem whose cost function is described by the solution of BSDE. In ref. [3], Karoui, Peng and Quenez gave the formulation of recursive utilities and their properties from the BSDE point of view. In 1992, Peng [4] got the Bellman's dynamic programming principle for this kind of problem and proved that the value function is a viscosity solution of one kind of quasi-linear second-order partial differential equation (PDE in short) which is the well-known as Hamilton-Jacobi-Bellman equation. Later in 1997, Peng [5] virtually generalized these results to a much more general situation, under Markovian and even Non-Markovian framework. In this Chinese version, Peng used the backward semigroup property introduced by a BSDE under Markovian and Non-Markovian framework. He also proved that the value function is a viscosity solution of a generalized Hamilton-Jacobi-Bellman equation. In 2007, Wu and Yu [6] gave the dynamic programming principle for one kind of stochastic recursive optimal control problem with the obstacle constraint for the cost functional described by the solution of a reflected BSDE and showed that the value function is the unique viscosity solution of the obstacle problem for the corresponding Hamilton-Jacobi-Bellman equation.

In 1994, the study of backward doubly stochastic differential equations (BDSDE in short) was initiated by Pardoux and Peng [7]. The

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equations involve two independent standard Brownian motions with two different directions of stochastic integral: a standard forward dW_t and a backward dB_t. They proved the existence and uniqueness of solutions to this equation and established a connection between BDSDE and a classical solution for stochastic partial differential equations (SPDE in short) under smoothness assumptions on the coefficients. Bally and Matoussi [8] provided a probabilistic representation of the solution in the Sobolev space of semilinear stochastic PDEs in terms of BDSDE. This kind of Sobolev weak solution result concerning stochastic PDEs can also be found in refs. [9] and [10]. In 2005, Shi, Gu, and Liu [11] established a comparison theorem for BDSDEs, while Aman [12] obtained the uniqueness and existence of solutions to reflected BDSDEs.

In our research, we investigate a stochastic recursive optimal control problem where the control system is governed by the classical stochastic differential equation, while the cost function is represented by the solution of a backward doubly stochastic differential equation. This type of recursive optimal control problem is of practical significance, particularly in arbitrage-free incomplete financial markets where "insider trading" exists. An individual with access to insider information can gain an unfair advantage over other investors and potentially generate larger profits, which could be modeled using a BDSDE in financial market models. Specifically, there are two types of investors with varying levels of information about the future price evolution in a market that is influenced by an additional source of randomness. The ordinary trader only has access to "public information" i.e., market prices of underlying assets contained in the filtration $\mathcal{F}^W_{0,t}$. However, an insider who has access to a larger filtration $\mathcal{F}^W_{0,t} \vee \mathcal{F}^B_{t,T}$, including insider information, can gain a significant advantage. For instance, an insider may know the functional law of the price process, or he may be aware of a significant change in the business policy or scope of a security issue, or he may be able to estimate whether his portfolio is better than others. It is worth noting that BDSDE techniques offer powerful tools to analyze the problem of portfolio optimization for an insider trader, where the investment strategy still satisfies the property that locally optimal is equal to globally optimal.

Our primary interest is to determine whether the dynamic programming principle holds for this recursive optimal control problem. Fortunately, the BDSDE properties allow us to achieve this objective. Unlike the HJB equation in previous work [4,6], the corresponding HJB equation that we obtain is a SPDE in a Markovian framework. In the stochastic case where the diffusion may be degenerate, the HJB equation may not have a classical solution. To overcome this issue, Crandall, Ishii, and Lions [13] introduced the concept of viscosity solutions in the early 1980s, which has yielded fruitful results. However, the viscosity solution of the HJB equation cannot provide a reasonable probabilistic interpretation of a solution pair (Y, Z) of BSDE, as there is no established relationship between the Z component of the solution and the HJB equation. In this paper, we propose a different type of weak solution for HJB equations in a Sobolev space, in which the Z component is implicitly included in the weak definition. Wei, Wu, and Zhao [14] have demonstrated that the value function is the unique Sobolev weak solution of the related HJB equation using the nonlinear Doob-Meyer decomposition theorem introduced in the study of BSDEs.

In our paper, we address the problem of establishing a connection between the Sobolev weak solution of the HJB equation and BDSDE. Unlike in the case of BSDE, there is no Doob-Meyer decomposition theorem available for BDSDEs, and hence it is not immediately clear how to derive equations similar to Lemma 4.1 and 4.2 in ref. [14]. To tackle this issue, we draw inspiration from the reflected solution of BDSDE in [12,15]. Specifically, we introduce an increasing process into the equation, which acts as a minimal force that drives the cost function upwards. By doing so, we are able to establish the desired connection between the Sobolev weak solution of the HJB equation and BDSDE.

The paper is structured as follows. In Section 2, we present the preliminaries and assumptions. In Section 3, we formulate a stochastic recursive optimal control problem, where the cost function is defined by the solution of a BDSDE. We demonstrate that the dynamic programming principle is valid for this optimal control problem. In Section 4, we establish that the value function of the problem is the sole weak solution in a Sobolev space for the related stochastic Hamilton-Jacobi-Bellman equation.

2. Preliminaries and assumption

In this section, we present preliminary results on the BDSDEs which provide the foundation for the recursive optimal control problem.

Let us first introduce the setting in which we want to study stochastic optimal control problem. We consider a probability space $(\Omega, \mathcal{F}, P)$, where $T > 0$ is a fixed constant throughout this paper. We define two independent standard Brownian Motion processes $\{W_t; 0 \le t \le T\}$ and $\{B_t; 0 \le t \le T\}$ with values respectively in R^d and R^l, defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define $\mathcal{F}_t := \mathcal{F}^W_{0,t} \vee \mathcal{F}^B_{t,T}$, where $\mathcal{F}^W_{s,t} = \mathcal{N} \vee \sigma\{W_r - W_s; s \le r \le t\}$, $\mathcal{F}^B_{t,T} = \mathcal{N} \vee \sigma\{B_r - B_t; t \le r \le T\}$. It is important to note that $\{\mathcal{F}_t; t \in [0, T]\}$ is neither increasing nor decreasing, and it does not constitute a filtration. However, $\{\mathcal{F}^W_{0,t}, t \in [0, T]\}$ is a filtration.

We shall also introduce the following spaces of processes, which will be frequently utilized in the forthcoming discussion

$$\mathcal{L}^2 = \{\xi \text{ is an } \mathcal{F}_T - \text{measurable random variable s.t.} E(|\xi|^2) < +\infty\}$$
$$\mathcal{H}^2 = \left\{ \{\psi_t, 0 \le t \le T\} \text{ is a predictable process s.t.} E\int_0^T |\varphi_t|^2 dt < +\infty \right\}$$
$$\mathcal{S}^2 = \left\{ \{\varphi_t, 0 \le t \le T\} \text{ is a predictable process s.t.} E(\sup_{0\le t\le T} |\varphi_t|^2) < +\infty \right\}$$

Let us now consider two functions $f : [0, T] \times R \times R^d \to R$, and $g : [0, T] \times R \times R^d \to R^l$ with the property that $(f(t, y, z))_{t\in[0,T]}$ and $(g(t, y, z))_{t\in[0,T]}$ are $\mathcal{F}_t$- progressively measurable for each (y, z) in $R \times R^d$, and we also make the following assumptions on f and g throughout the paper:

(H2.1) For any $(y, z) \in R \times R^d$

$$f(\cdot, y, z) \in \mathcal{H}^2(0, T; R); g(\cdot, y, z) \in \mathcal{H}^2(0, T; R^l)$$

(H2.2) There exist constants $L > 0$ and $0 < \alpha < 1$, such that for any $y, y' \in R, z, z' \in R^d$,

$$|f(t, y, z) - f(t, y', z')| \le L(|y - y'| + |z - z'|)$$
$$|g(t, y, z) - g(t, y', z')| \le L|y - y'| + \alpha|z - z'|$$

There exists constants C such that for all $(t, y, z) \in [0, T] \times R \times R^d$

$$|f(t, y, z)| \le |f(t, 0, 0)| + C(|y| + |z|), |g(t, y, z)| \le |g(t, 0, 0)| + C(|y| + |z|)$$

The proof of the following well-known result on BDSDEs can be found in Pardoux and Peng [7]

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \quad 0 \le t \le T \tag{1}$$

where $\xi \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P; R)$ is a $\mathcal{F}_T$-measurable random variables, and $\mathcal{F}_T = \mathcal{F}^W_{0,T} \vee \mathcal{F}^B_{T,T} = \mathcal{F}^W_{0,T}$.

We observe that Eq. 1 involves two independent Brownian motions W and B. The dW integral is an Itô's integral, whereas the $d\overleftarrow{B}$ integral is a backward Itô's integral. The additional noise B in the equation can be interpreted as extra information that is generally not observable in the market, but is accessible to a specific investor. Consequently, the central problem is to demonstrate how this investor can exploit the additional information to optimize the utility, while adhering to the admissible portfolios' standard class, that is, by selecting an optimal strategy that is entirely "legal".

Then from Theorem 1.1 in ref. [7], there exists a unique solution $\{(Y_t, Z_t), 0 \le t \le T\} \in \mathcal{S}^2(0, T; R) \times \mathcal{H}^2(0, T; R^d)$.

We shall recall the fundamental results on BDSDEs, starting with the two precise estimates of the solutions.

Proposition 2.1. *Let* $\{(Y_t, Z_t) \in \mathcal{S}^2(0,T;R) \times \mathcal{H}^2(0,T;R^d), 0 \le t \le T\}$ *be the solution of the above BDSDE (1), then for some* $\xi \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P; R)$ *and*

$$E\int_0^T (|f(t,0,0)|^2 + |g(t,0,0)|^2)dt < \infty$$

we have

$$E\left\{\sup_{0\le s\le T} |Y_s|^2 + \left(\int_0^T |Z_s|^2 dt\right)\right\} < \infty \tag{2}$$

Proposition 2.2. *Let* (ξ, f, g) *and* (ξ', f', g') *be two triplets satisfying the above assumption (H2.1) and (H2.2). Suppose* (Y, Z) *is the solution of the BDSDE* (ξ, f, g) *and* (Y', Z') *is the solution of the BDSDE* (ξ', f', g')*. Define*

$$\triangle\xi = \xi - \xi', \quad \triangle f = f - f', \quad \triangle g = g - g'$$

$$\triangle Y = Y - Y', \quad \triangle Z = Z - Z'$$

Then there exists a constant C such that

$$E\left\{\sup_{0\le s\le T} |\triangle Y_s|^2 + \left(\int_0^T |\triangle Z_s|^2 dt\right)\right\} \le CE\{|\triangle \xi|^2\} \tag{3}$$

3. Formulation of the problem and the dynamic programming principle

In this section, we present a formulation of a backward doubly stochastic recursive optimal control problem. We then establish that the dynamic programming principle holds for this type of optimization problem.

We define the admissible control set $\mathcal{U}$ as follows

$$\mathcal{U} := \{v(\cdot) \in \mathcal{H}^p | \, v(\cdot) \text{ take value in } U \subset R^k\}.$$

An admissible control is an element of the admissible control set $\mathcal{U}$. Note that U is a compact subset of R^k, a condition that is often met in practical applications.

For a given admissible control, we consider the following control system

$$\begin{cases} dX_s^{t,\zeta;v} = b(s, X_s^{t,\zeta;v}, v_s)ds + \sigma(s, X_s^{t,\zeta;v}, v_s)dW_s, s \in [t,T] \\ X_t^{t,\zeta;v} = \zeta \end{cases} \tag{4}$$

Where $t \ge 0$ is regarded as the initial time and $\zeta \in \mathcal{L}^2(\Omega, \mathcal{F}_{0,t}^W, P; R^n)$ as the initial state, the mappings

$$b : [0,T] \times R^n \times U \longrightarrow R^n$$
$$\sigma : [0,T] \times R^n \times U \longrightarrow R^{n\times d}$$

satisfy the following conditions:

(H3.1) For every fixed $x \in R^n$, $b(\cdot, x, \cdot)$ and $\sigma(\cdot, x, \cdot)$ are continuous in (t, v);

(H3.2) there exists a $L > 0$, such that, for all $t \in [0,T]$, $x, x' \in R^n$, $v, v' \in U$

$$|b(t,x,v) - b(t,x',v')| + |\sigma(t,x,v) - \sigma(t,x',v')| \le L(|x - x'| + |v - v'|)$$

Proposition 3.1. *It is worth noting that the conditions (H3.1) and (H3.2) guarantee the existence of a unique strong solution* $\{X_s^{t,\zeta;v} \in \mathcal{H}^2(0,T;\mathbb{R}^n), 0 \le t \le s \le T\}$ *for the control system (4). Furthermore, for any* $t \in [0,T]$, $\zeta, \zeta' \in \mathcal{L}^2(\Omega, \mathcal{F}_{0,t}^W, P; R^n)$, $v(\cdot), v'(\cdot) \in \mathcal{U}$ *and* $\delta \in [0, T-t]$, *the following estimates hold*

$$E^{\mathcal{F}_t^W}\left\{\sup_{t\le s\le T} |X_s^{t,\zeta;v}|^2\right\} \le C(1 + |\zeta|^2)$$

$$E^{\mathcal{F}_t^W}\left\{\sup_{t\le s\le T} |X_s^{t,\zeta;v} - X_s^{t,\zeta';v'}|^2\right\}$$
$$\le C|\zeta - \zeta'|^2 + CE^{\mathcal{F}_t^W}\left\{\int_t^T |v_s - v_s'|^2 ds\right\}$$

$$E\left\{\sup_{t\le s\le t+\delta} |X_s^{t,\zeta;v} - \zeta|^2\right\} \le C\delta$$

where the constant C depends on L, T, and the compact set U.

For a given admissible control $v(\cdot)$, we consider the following BDSDE

$$\begin{aligned} Y_s^{t,\zeta;v} &= \Phi(X_T^{t,\zeta;v}) + \int_s^T f(r, X_r^{t,\zeta;v}, Y_r^{t,\zeta;v}, Z_r^{t,\zeta;v}, v_r)dr \\ &\quad + \int_s^T g(r, X_r^{t,\zeta;v}, Y_r^{t,\zeta;v}, Z_r^{t,\zeta;v})d\overleftarrow{B}_r - \int_s^T Z_r^{t,\zeta;v} dW_r, \quad t \le s \le T \end{aligned} \tag{5}$$

where

$$\Phi : R^n \to R,$$
$$f : [0,T] \times R^n \times R \times R^d \times U \to R$$
$$g : [0,T] \times R^n \times R \times R^d \to R^l$$

and they satisfy the following conditions:

(H3.3) f and h are continuous in t.

(H3.4) for some $L > 0$ and $0 < \alpha < 1$ all $x, x' \in R^n$; $y, y' \in R$; $z, z' \in R^d$; $v, v' \in U$

$$|f(t,x,y,z,v) - f(t,x',y',z',v')| + |\Phi(x) - \Phi(x')|$$
$$\le L(|x - x'| + |y - y'| + |z - z'| + |v - v'|)$$

$$|g(t,x,y,z) - g(t,x',y',z')| \le L(|x - x'| + |y - y'|) + \alpha|z - z'|$$

(H3.5) The function $\Phi \in \mathcal{L}^2(\Omega, \mathcal{F}_{0,T}^W, P; R)$.

(H3.6) $\forall (y,z) \in R \times R^d$, $f(\cdot, y, z, \cdot) \in \mathcal{H}^2$, $g(\cdot, y, z) \in \mathcal{H}^2$.

(H3.7) f is measurable in (t, x, y, z, v) and for any $r \in [t,T]$

$$E\int_0^T |f(r,0,0,0,v_r)|^2 dr \le M$$

functions f and g are continuous and controlled by $C(1 + |x| + |y| + |z|)$.

Then there exists a unique solution $\{(Y_s^{t,\zeta;v}, Z_s^{t,\zeta;v}) \in \mathcal{S}^2(0,T;R) \times \mathcal{H}^2(0,T;R^d), t \le s \le T\}$. Furthermore, the solution satisfies the following estimates, which follow from the estimates in Proposition 3.1.

Proposition 3.2.

$$E^{\mathcal{F}_t^W}\left\{\sup_{t\le s\le T} |Y_s^{t,\zeta;v}|^2 + \int_t^T |Z_r^{t,\zeta;v}|^2 dr\right\} \le C(1 + |\zeta|^2)$$

$$E^{\mathcal{F}_t^W}\left\{\sup_{t\le s\le T} |Y_s^{t,\zeta;v} - Y_s^{t,\zeta';v'}|^2 + \int_t^T |Z_r^{t,\zeta;v} - Z_r^{t,\zeta';v'}|^2 dr\right\}$$
$$\le C|\zeta - \zeta'|^2 + CE^{\mathcal{F}_t^W}\left\{\int_t^T |v_r - v_r'|^2 dr\right\}$$

Where the constant C depends on L and T.

Given a control process $v(\cdot) \in \mathcal{U}$, we introduce the associated cost functional

$$J(t,x;v(\cdot)) := Y_s^{t,x;v}|_{s=t}, \qquad (t,x) \in [0,T] \times R^n \tag{6}$$

and we define the value function of the stochastic optimal control problem

$$u(t,x) := ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)), \qquad (t,x)\in[0,T]\times R^n \tag{7}$$

Remark 3.1. In this stochastic optimal control problem, the term B in the cost function represents additional information that is not generally available in the market but is known to a specific investor. For example, an insider may have knowledge of the functional law of the price process or may be aware of significant changes in the business policy before they occur, but they are unable to influence these factors. The admissible control $v(\cdot)$ is independent of the Brownian motion B, and the process $\{X_s^{t,\zeta;v}\}$ is a $\mathcal{F}_{0,s}^W$-measurable random process. However, the solution of the BDSDE (5), given by $(Y_s^{t,\zeta;v}, Z_s^{t,\zeta;v})$, is the big σ-field $\mathcal{F}_t$-measurable random process.

Now we continue our study of the control problem (7) and prove that the celebrated dynamic programming principle still holds for this optimization problem. Our proof draws inspiration from the work of Peng on the dynamic programming principle for recursive problems, as presented in the Chinese version [4], as well as that of Wu and Yu in ref. [6].

Let us now consider the following subspace of $\mathcal{U}$

$$\mathcal{U}^t := \left\{ v(\cdot)\in\mathcal{U} \mid v(s) \text{ is } \mathcal{F}_{t,s}^W \text{ progressively measurable}, \forall t\le s\le T \right\}$$

$$\overline{\mathcal{U}}^t := \left\{ v_s = \sum_{j=1}^N v_s^j I_{A_j} | v_s^j \in \mathcal{U}^t, \{A_j\}_{j=1}^N \text{ is a partition of } (\Omega, \mathcal{F}_{0,t}^W) \right\}$$

Firstly, we will establish that:

Proposition 3.3. *Under the assumptions (H3.1)-(H3.7), the value function $u(t,x)$ defined in (7) is $\mathcal{F}_{t,T}^B$ measurable.*

Proof. First we can prove

$$ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)) = ess \sup_{v(\cdot)\in\overline{\mathcal{U}}^t} J(t,x;v(\cdot)) \tag{8}$$

$\overline{\mathcal{U}}^t$ is the subset of $\mathcal{U}$, then

$$ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)) \ge ess \sup_{v(\cdot)\in\overline{\mathcal{U}}^t} J(t,x;v(\cdot)).$$

To show Eq. 8, we consider the inverse inequality. For any $v(\cdot)$, $\widetilde{v}(\cdot)\in\mathcal{U}$, by Proposition 3.2, we have

$$E\left\{ |Y_t^{t,x;v} - Y_t^{t,x;\widetilde{v}}|^2 \right\} \le CE\int_t^T |v_r - \widetilde{v}_r|^2 ds$$

Note that $\overline{\mathcal{U}}^t$ is dense in $\mathcal{U}$, then for each $v(\cdot)\in\mathcal{U}$, there exists a sequence $\{v_n(\cdot)\}_{n=1}^\infty \in \overline{\mathcal{U}}^t$ such that

$$\lim_{n\to\infty} E\left\{ |Y_t^{t,x;v_n} - Y_t^{t,x;v}|^2 \right\} = 0$$

So there exists a subsequence, we denote without loss of generality $\{v_n(\cdot)\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} Y_t^{t,x;v_n} = Y_t^{t,x;v}, \qquad a.s.$$

so that

$$\lim_{n\to\infty} J(t,x,v_n(\cdot)) = J(t,x,v(\cdot)), \qquad a.s.$$

By the arbitrariness of $v(\cdot)$ and the definition of essential supremum, we get

$$ess \sup_{v(\cdot)\in\overline{\mathcal{U}}^t} J(t,x;v(\cdot)) \ge ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot))$$

then we obtain (8).

Secondly, we want to prove

$$ess \sup_{v(\cdot)\in\overline{\mathcal{U}}^t} J(t,x;v(\cdot)) = ess \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v(\cdot))$$

Obviously,

$$ess \sup_{v(\cdot)\in\overline{\mathcal{U}}^t} J(t,x;v(\cdot)) \ge ess \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v(\cdot))$$

Next we will proof the inverse inequality by considering the partition of probability space

$$X^{t,x;\sum_{j=1}^N v^j I_{A_j}} = \sum_{j=1}^N I_{A_j} X^{t,x;v^j}, \qquad Y^{t,x;\sum_{j=1}^N v^j I_{A_j}} = \sum_{j=1}^N I_{A_j} Y^{t,x;v^j}$$

$$Z^{t,x;\sum_{j=1}^N v^j I_{A_j}} = \sum_{j=1}^N I_{A_j} Z^{t,x;v^j}$$

$\forall v(\cdot)\in\overline{\mathcal{U}}^t$, we have

$$J(t,x;v(\cdot)) = J(t,x;\sum_{j=1}^N v^j(\cdot) I_{A_j}) = \sum_{j=1}^N I_{A_j} J(t,x;v^j(\cdot))$$

Note that $v^j(\cdot)(j=1,2,\cdots,N)$ are $\mathcal{F}_{t,s}^W$ progressively measurable, then $J(t,x,v^j(\cdot))(j=1,2,\cdots,N)$ are $\mathcal{F}_{t,T}^B$ measurable. By the comparison theorem of the BDSDE in ref. [11], we assume that

$$J(t,x,v^1) \ge J(t,x,v^j) \qquad \forall j=2,3,\cdots,N$$

So

$$ess \sup_{v(\cdot)\in\overline{\mathcal{U}}^t} J(t,x;v(\cdot)) = ess \sup_{v(\cdot)\in\overline{\mathcal{U}}^t} \sum_{j=1}^N I_{A_j} J(t,x;v^j(\cdot))$$

$$\le \sum_{j=1}^N ess \sup_{v(\cdot)\in\overline{\mathcal{U}}^t} I_{A_j} J(t,x;v^j(\cdot))$$

$$\le ess \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v^1(\cdot)) = ess \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v(\cdot))$$

then we can get

$$ess \sup_{v(\cdot)\in\overline{\mathcal{U}}^t} J(t,x;v(\cdot)) \le ess \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v(\cdot))$$

However, when $v(\cdot)\in\mathcal{U}^t$, the cost functional $J(t,x;v(\cdot))$ is $\mathcal{F}_{t,T}^B$ measurable. So

$$u(t,x) = ess \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v(\cdot))$$

is $\mathcal{F}_{t,T}^B$ measurable. □

Next, we shall investigate the continuity properties of the value function $u(t,x)$ with respect to x and t. We establish the following estimates:

Lemma 3.4. *For each $t\in[0,T]$, x and $x'\in R^n$, we have*

$$\begin{aligned} &(i) \quad E|u(t,x)-u(t,x')|^2 \le C|x-x'|^2 \\ &(ii) \quad E|u(t,x)| \le C(1+|x|). \end{aligned} \tag{9}$$

Proof. utilizing the estimate: $E(\sup_{t\le s\le T} |Y_s^{t,x;v}|^2) \le C(1+|x|^2)$, for each admissible control $v(\cdot)\in\mathcal{U}$, we have

$$E|J(t,x;v(\cdot))| \le C(1+|x|) \tag{10}$$

and

$$E|J(t,x;v(\cdot)) - J(t,x';v(\cdot))|^2 \le C|x-x'|^2$$

On the other hand, utilizing the comparison theorem of the BDSDE presented in ref. [11], for each $\varepsilon>0$, $\exists v(\cdot)$, $v'(\cdot)\in\mathcal{U}$ such that

$$J(t,x;v'(\cdot)) \le u(t,x) \le J(t,x,v(\cdot)) + \varepsilon$$
$$J(t,x';v(\cdot)) \le u(t,x') \le J(t,x',v'(\cdot)) + \varepsilon$$

From the estimate (10) we can get

$$-C(1+|x|) - \varepsilon \le E|J(t,x;v'(\cdot))| \le E|u(t,x)|$$
$$\le E|J(t,x;v(\cdot))| + \varepsilon \le C(1+|x|) + \varepsilon$$

From the arbitrariness of ε, we can obtain (ii).

Similarly

$$J(t,x;v'(\cdot))-J(t,x';v'(\cdot))-\varepsilon\le u(t,x)-u(t,x')$$
$$\le J(t,x;v(\cdot))-J(t,x';v(\cdot))+\varepsilon$$

$$|u(t,x)-u(t,x')|$$
$$\le \max\{|J(t,x;v(\cdot))-J(t,x';v'(\cdot))|,|J(t,x;v(\cdot))-J(t,x';v(\cdot))|\}+\varepsilon$$

$$E|u(t,x)-u(t,x')|^2$$
$$\le 2\max\{E|J(t,x;v'(\cdot))-J(t,x';v'(\cdot))|^2,E|J(t,x;v(\cdot))$$
$$-J(t,x';v(\cdot))|^2\}+2\varepsilon^2$$
$$\le 2C|x-x'|^2+2\varepsilon^2$$

Then we can obtain (9). □

Additionally, we have:

Lemma 3.5. $\forall t\in[0,T]$, $\forall v(\cdot)\in\mathcal{U}$, *for all* $\zeta\in\mathcal{L}^2(\Omega,\mathcal{F}^W_{0,t},P;R^n)$, *we have*

$$J(t,\zeta;v(\cdot))=Y_t^{t,\zeta;v(\cdot)}$$

Proof. We first study a simple case: ζ is the following form: $\zeta=\sum_{i=1}^N I_{A_i}x_i$, where $\{A_i\}_{i=1}^N$ is a finite partition of $(\Omega,\mathcal{F}^W_{0,t})$, and $x_i\in R^n$ for $1\le i\le N$, so

$$Y_s^{t,\zeta;v}=Y_s^{t,\sum_{i=1}^N I_{A_i}x_i;v}=\sum_{i=1}^N I_{A_i}Y_s^{t,x_i;v}$$

From the definition of cost functional. We deduce that

$$Y_t^{t,\zeta;v}=\sum_{i=1}^N I_{A_i}Y_t^{t,x_i;v}=\sum_{i=1}^N I_{A_i}J(t,x_i;v(\cdot))$$
$$=J(t,\sum_{i=1}^N I_{A_i}x_i;v(\cdot))=J(t,\zeta;v(\cdot))$$

Therefor, for simple functions, we get the desired result.

Given a general $\zeta\in\mathcal{L}^2(\Omega,\mathcal{F}^W_{0,t},P;R^n)$, we can choose a sequence of simple function $\{\zeta_i\}$ which converges to ζ in $\mathcal{L}^2(\Omega,\mathcal{F}^W_{0,t},P;R^n)$. Consequently, by Proposition 3.2 we have

$$E\{|Y_t^{t,\zeta;v}-Y_t^{t,\zeta_i;v}|^2\}\le E\{C|\zeta-\zeta_i|^2\}\to 0,\quad as\ i\to\infty$$

So

$$E\{|J(t,\zeta;v(\cdot))-J(t,\zeta_i;v(\cdot))|^2\}\le E\{C|\zeta-\zeta_i|^2\}\to 0,\quad as\ i\to\infty$$

With the help of $Y_t^{t,\zeta;v}=J(t,\zeta;v(\cdot))$, the proof is completed. □

We present the following result for the value function of our recursive optimal control problem:

Lemma 3.6. *Fixed* $t\in[0,T)$ *and* $\zeta\in\mathcal{L}^2(\Omega,\mathcal{F}^W_{0,t},P;R^n)$, *for each* $v(\cdot)\in\mathcal{U}$, *we have*

$$u(t,\zeta)\ge Y_t^{t,\zeta;v(\cdot)},a.s. \tag{11}$$

On the other hand, for each $\varepsilon>0$, *there exists an admissible control* $v(\cdot)\in\mathcal{U}$ *such that*

$$u(t,\zeta)\le Y_t^{t,\zeta,;v(\cdot)}+\varepsilon,\quad a.s. \tag{12}$$

Proof. In order to prove this lemma, Proposition 3.1 and Proposition 3.2 are of paramount importance.

We first proof (11). When ζ is a simple function: $\zeta=\sum_{i=1}^N I_{A_i}x_i$, for all $v(\cdot)\in\mathcal{U}$, We have

$$Y_t^{t,\zeta;v}=Y_t^{t,\sum_{i=1}^N I_{A_i}x_i;v}=\sum_{i=1}^N I_{A_i}Y_t^{t,xi;v}=\sum_{i=1}^N I_{A_i}J(t,x_i;v(\cdot))\le u(t,\zeta)$$

If $\zeta\in\mathcal{L}^2(\Omega,\mathcal{F}^W_{0,t},P;R^n)$, we can choose a sequence of simple functions $\{\zeta_i\}$ which converges to ζ in $\mathcal{L}^2(\Omega,\mathcal{F}^W_{0,t},P;R^n)$. Consequently, similar to Lemma 3.5, we have

$$E\{|Y_t^{t,\zeta;v}-Y_t^{t,\zeta_i;v}|^2\}\le E\{C|\zeta-\zeta_i|^2\}\to 0,\quad as\ i\to\infty$$

$$E\{|u(t,\zeta)-u(t,\zeta_i)|^2\}\to 0, as\quad i\to\infty$$

Then there exists a subsequence, without loss of generality we use same notation, such that

$$\lim_{i\to\infty}Y_t^{t,\zeta_i;v}=Y_t^{t,\zeta;v},\quad a.s,\qquad \lim_{i\to\infty}u(t,\zeta_i)=u(t,\zeta),\quad a.s.$$

Here $Y_t^{t,\zeta_i;v}\le u(t,\zeta_i), i=1,2,\cdots$, so $Y_t^{t,\zeta;v}\le u(t,\zeta), a.s.$.

We now turn to (12). We first consider the case that ζ is a bound random variable, suppose that $|\zeta|\le M$, and construct a simple random variable $\eta=\sum_{i=1}^N I_{A_i}x_i$, where $\{A_i\}_{i=1}^N$ is a finite partition of $(\Omega,\mathcal{F}^W_{0,t})$, such that

(*i*) $|\eta|\le|\zeta|$

(*ii*) $|\eta-\zeta|\le\min\{\frac{\varepsilon}{6\sqrt{C}},\frac{\varepsilon^2}{36C(1+2M)}\}$

For any $v(\cdot)\in\mathcal{U}$, By the comparison theorem of the BDSDE in ref. [11] and the Proposition 3.2, we have

$$|Y_t^{t,\zeta;v}-Y_t^{t,\eta,v}|\le\frac{\varepsilon}{3},\quad |u(t,\zeta)-u(t,\eta)|\le\frac{\varepsilon}{3}$$

Then, for each x_i, we can choose an $\mathcal{F}^W_{t,s}$-adapted admissible control $v^i(\cdot)$ such that

$$u(t,x_i)\le Y_t^{t,x_i;v_i}+\frac{\varepsilon}{3}$$

We denote $v(\cdot):=\sum_{i=1}^N I_{A_i}v^i(\cdot)$, then

$$Y_t^{t,\zeta;v}\ge-|Y_t^{t,\zeta;v}-Y_t^{t,\eta;v}|+Y_t^{t,\eta;v}\ge-\frac{\varepsilon}{3}+\sum_i^N I_{A_i}Y_t^{t,x_i;v_i}$$
$$\ge-\frac{\varepsilon}{3}+\sum_i^N I_{A_i}(u(t,x_i)-\frac{\varepsilon}{3})=-\frac{2}{3}\varepsilon+u(t,\eta)$$
$$\ge-\varepsilon+u(t,\zeta)$$

Therefore, for a bounded random variable ζ, we have the desired result (12).

Given a general $\zeta\in\mathcal{L}^2(\Omega,\mathcal{F}^W_{0,t},P;R^n)$, we note that ζ has the following form

$$\zeta=\sum_{i=1}^\infty I_{A_i}\zeta_i$$

where $\{A_i\}_{i=1}^\infty$ is a partition of $(\Omega,\mathcal{F}^W_{0,t})$, ζ_i is a bounded random variable. So, for every ζ_i, there exists $v^i(\cdot)\in\mathcal{U}$, such that

$$u(t,\zeta_i)\le Y_t^{t,\zeta_i;v_i}+\varepsilon$$

We denote $v(\cdot)=\sum_{i=1}^\infty I_{A_i}v^i(\cdot)$ and get

$$u(t,\zeta)=u(t,\sum_{i=1}^\infty I_{A_i}\zeta_i)=\sum_{i=1}^\infty I_{A_i}u(t,\zeta_i)\le\sum_{i=1}^\infty I_{A_i}(Y_t^{t,\zeta_i;v_i}+\varepsilon)$$
$$=\sum_{i=1}^\infty I_{A_i}Y_t^{t,\zeta_i;v_i}+\varepsilon=Y_t^{t,\zeta;v}+\varepsilon$$

The proof is now completed. □

Now we start to discuss the (generalized) dynamic programming principle for our recursive optimal control problem.

Firstly we introduce a family of (backward) semigroups which is original from Peng's idea in ref. [4]. Given the initial condition (t,x), an admissible control $v(\cdot)\in\mathcal{U}$, a positive number $\delta\le T-t$ and a real-value random variable $\eta\in\mathcal{L}^2(\Omega,\mathcal{F}_{t+\delta},P;\mathbb{R})$, we denote

$$G^{t,x;v}_{t,t+\delta}[\eta]:=Y_t$$

where (Y_s, Z_s) is the solution of the following BDSDE with the horizon $t+\delta$

$$Y_s = \eta + \int_s^{t+\delta} f(r, Y_r, Z_r)dr + \int_s^{t+\delta} g(r, Y_r, Z_r)d\overline{B}_r - \int_s^{t+\delta} Z_r dW_r,\ t \le s \le t+\delta$$

Obviously,

$$G^{t,x;v}_{t,T}[\Phi(X^{t,x;v}_T)] = G^{t,x;v}_{t,t+\delta}[Y^{t,x;v}_{t+\delta}]$$

Then our (generalized) dynamic programming principle holds.

Theorem 3.7. *Under the assumption (H3.1)-(H3.7), the value function $u(t,x)$ obeys the following dynamic programming principle: for each $0 < \delta \le T-t$,*

$$u(t,x) = ess \sup_{v(\cdot)\in\mathcal{U}} G^{t,x;v}_{t,t+\delta}[u(t+\delta, X^{t,x;v}_{t+\delta})]$$

Proof. We have

$$u(t,x) = ess \sup_{v(\cdot)\in\mathcal{U}} G^{t,x;v}_{t,T}[\Phi(X^{t,x;v}_T)] = ess \sup_{v(\cdot)\in\mathcal{U}} G^{t,x;v}_{t,t+\delta}[Y^{t,x;v}_{t+\delta}]$$
$$= ess \sup_{v(\cdot)\in\mathcal{U}} G^{t,x;v}_{t,t+\delta}[Y^{t+\delta, X^{t,x;v}_{t+\delta};v}_{t+\delta}]$$

Form Lemma 3.6 and the comparison theorem of double BDSDE

$$u(t,x) \le ess \sup_{v(\cdot)\in\mathcal{U}} G^{t,x;v}_{t,t+\delta}[u(t+\delta, X^{t,x;v}_{t+\delta})]$$

On the other hand, from Lemma 3.6, for every $\varepsilon > 0$, we can find an admissible control $\overline{v}(\cdot) \in \mathcal{U}$ such that

$$u(t+\delta, X^{t,x;\overline{v}}_{t+\delta}) \le Y^{t+\delta, X^{t,x;\overline{v}}_{t+\delta};\overline{v}}_{t+\delta} + \varepsilon$$

For each $v(\cdot) \in \mathcal{U}$, we denote $\widetilde{v}(s) = I_{\{s\le t+\delta\}}v(s) + I_{\{s>t+\delta\}}\overline{v}(s)$. From the above inequality and the comparison theorem, we get

$$Y^{t+\delta, X^{t,x;\widetilde{v}}_{t+\delta};\widetilde{v}}_{t+\delta} \ge u(t+\delta, X^{t,x;\widetilde{v}}_{t+\delta}) - \varepsilon, \quad u(t,x) \ge ess \sup_{\widetilde{v}(\cdot)\in\mathcal{U}} G^{t,x;\widetilde{v}}_{t,t+\delta}[u(t+\delta, X^{t,x;\widetilde{v}}_{t+\delta}) - \varepsilon]$$

By Proposition 2.2, there exists a positive constant C_0 such that

$$u(t,x) \ge ess \sup_{\widetilde{v}\in\mathcal{U}} G^{t,x;\widetilde{v}}_{t,t+\delta}[u(t+\delta, X^{t,x;\widetilde{v}}_{t+\delta})] - C_0\varepsilon$$

Therefore, letting $\varepsilon \downarrow 0$, we obtain

$$u(t,x) \ge ess \sup_{\widetilde{v}\in\mathcal{U}} G^{t,x;\widetilde{v}}_{t,t+\delta}[u(t+\delta, X^{t,x;\widetilde{v}}_{t+\delta})]$$

Because $\widetilde{v}(\cdot)$ acts only on $[t, t+\delta]$ for $G^{t,x;\widetilde{v}}_{t,t+\delta}$, from the definition of $\widetilde{v}(\cdot)$ and the arbitrariness of $\widetilde{v}(\cdot)$, we know that the above inequality can be written as

$$u(t,x) \ge ess \sup_{v\in\mathcal{U}} G^{t,x;v}_{t,t+\delta}[u(t+\delta, X^{t,x;v}_{t+\delta})]$$

which is our desired conclusion. □

4. Sobolev weak solution for the stochastic HJB equation corresponding to the stochastic recursive control problem

In this section we consider the Sobolev weak solution for the SHJB equation related to the stochastic recursive optimal control problem.

We give some preliminary results of the BDSDE which are useful for the sobolev weak solutions for the recursive optimal control problem. In order to facilitate understanding and narration, we divided it into several parts.

***Part* I**

Consider the control system defined by (4)

$$\begin{cases} dx^{t,x;v}_s = b(s, x^{t,x;v}_s, v_s)ds + \sigma(s, x^{t,x;v}_s, v_s)dW_s, & s\in[t,T] \\ x^{t,x;v}_t = x. \end{cases} \tag{13}$$

satisfying the following conditions:

(H4.1) The coefficient b is 2 times continuously differentiable in x and all their partial derivatives are uniformly bounded, σ is 3 times continuously differentiable in x and all their partial derivatives are uniformly bounded, and $|b(t,x,v)| + |\sigma(t,x,v)| \le L(1+|x|)$, where L is a constant.

And the cost function defined by the following BDSDE

$$y^{t,x;v}_s = \Phi(x^{t,x;v}_T) + \int_s^T f(r, x^{t,x;v}_r, y^{t,x;v}_r, z^{t,x;v}_r, v_r)dr + \int_s^T g(r, x^{t,x;v}_r, y^{t,x;v}_r, z^{t,x;v}_r)d\overline{B}_r - \int_s^T z^{t,x;v}_r dW_r \tag{14}$$

where

$$\Phi : R^n \to R,$$
$$f : [0,T]\times R^n \times R \times R^d \times U \to R$$
$$g : [0,T]\times R^n \times R \times R^d \to R^l$$

satisfying the conditions as same as that denoted in Section 3.

Obviously, under the above assumptions (H3.1)-(H3.7) and (H4.1), for a given control $v(\cdot)\in\mathcal{U}$, there exists a unique solution $(y^{t,x;v}_s, z^{t,x;v}_s) \in \mathcal{S}^2(0,T;R)\times\mathcal{H}^2(0,T;R^d)$. We introduce the associated cost function

$$J(t,x;v) := y^{t,x;v}_s|_{s=t}, \qquad (t,x)\in[0,T]\times R^n \tag{15}$$

and define the value function of the stochastic optimal control problem

$$u(t,x) := ess \sup_{v\in\mathcal{U}} J(t,x;v), \qquad (t,x)\in[0,T]\times R^n \tag{16}$$

According to the conclusion in previous section, we know that the celebrated dynamic programming principle still holds for this recursive stochastic optimal control problem. We therefore deduce the following stochastic HJB equation

$$u(s,x) = \Phi(x) + \sup_{v\in\mathcal{U}} \left\{ \int_s^T (\mathcal{L}(r,x,v)u(r,x) + f(r,x,u(r,x)), \sigma\nabla u(r,x), v)dr + \int_s^T g(r,x,u(r,x),\sigma\nabla u(r,x))d\overline{B}_r \right\} \tag{17}$$

where $\mathcal{L}$ is a family of second order linear partial differential operators

$$\mathcal{L}(r,x,q)\varphi = \frac{1}{2}tr[\sigma(r,x,v_t)\sigma(r,x,v_t)^T D^2\varphi] + \langle b(r,x,v_t), D\varphi\rangle$$

***Part* II**

We define the weight function ρ is continuous positive on R^n satisfying $\int_{R^n}\rho(x)dx = 1$ and $\int_{R^n}|x|^2\rho(x)dx < \infty$.

Denote by $L^2(R^n, \rho(x)dx)$ the weighted L^2-space with weight function endowed with the norm

$$\|u\|_{L^2(R^n,\rho(x)dx)} = \left[\int_{R^n} |u(x)|^2\rho(x)dx\right]^{\frac{1}{2}}$$

We set $D := \{u : R^n \to R \ such\ that\ u\in L^2(R^n,\rho(x)dx)\ and\ \frac{\partial u}{\partial x_i} \in L^2(R^n,\rho(x)dx)\}$, where $\frac{\partial u}{\partial x_i}$ is derivative with respect to x in the weak sense. Note that D equipped with the norm

$$\|u\|_D = \left[\int_{R^n}|u(x)|^2\rho(x)dx + \sum_{1\le i\le n}\int_{R^n}\left|\frac{\partial u}{\partial x_i}\right|^2\rho(x)dx\right]^{\frac{1}{2}}$$

is a Hilbert space, which is a classical Dirichlet space. Moreover, D is a subset of the Sobolev space $H_1(R^n)$.

We set $H := \{u : u\in L^2(R^n,\rho(x)dx)\ and\ (\sigma^*\nabla u)\in L^2(R^n,\rho(x)dx)\}$ equipped with the norm

$$\|u\|_H = \left[\int_{R^n}|u(x)|^2\rho(x)dx + \int_{R^n}\left|(\sigma^*\nabla u(x))\right|^2\rho(x)dx\right]^{\frac{1}{2}}$$

We say $u \in L^2([0,T],H)$ if $\int_0^T \|u(t)\|_H^2 dt < \infty$.

Let T be a strictly positive real number and U a nonempty compact set of R^k.

***Part* III**

Then, we introduce some equivalence norm.

The solution of SDE (13) generates a stochastic flow, and the inverse flow is denoted by $\hat{x}_s^{t,x,v}$. It is known from [16] that $x \to \hat{x}_s^{t,x,v}$ is differentiable and we denote by $J(\hat{x}_s^{t,x,v})$ the determinant of the Jacobian matrix of $\hat{x}_s^{t,x,v}$, which is positive and $J(\hat{x}_t^{t,x,v}) = 1$. For $\varphi \in C_c^\infty(R^n)$ we define a process $\varphi_t : \Omega \times [0,T] \times R^n \to R$ by $\varphi_t(s,x) = \varphi(\hat{x}_s^{t,x,v}) J(\hat{x}_s^{t,x,v})$. Following Kunita [17], we can define the composition of $u \in L^2(R^n)$ with the stochastic flow by $(u \circ x_s^{t,\cdot,v}, \varphi) = (u, \varphi_t(s,\cdot))$. Indeed, by a change of variable, we have

$$(u \circ x_s^{t,\cdot,v}, \varphi) = \int_{R^n} u(y)\varphi(\hat{x}_s^{t,x,v}) J(\hat{x}_s^{t,x,v}) dy = \int_{R^n} u(x_s^{t,x,v})\varphi(x)dx$$

In ref. [8], V. Bally and A. Matoussi proved that $\varphi_t(s,x)$ is a semimartingale and admits the following lemma 4.1 and lemma 4.2.

Lemma 4.1. *For $\varphi \in C_c^2(R^n)$, we have*

$$\varphi_t(s,x) = \varphi(x) - \sum_{j=1}^d \int_t^s \sum_{i=1}^d \frac{\partial}{\partial x_i}(\sigma_{i,j}(r,x)\varphi_t(r,x))dW_r^j + \int_t^s L_r^* \varphi_t(r,x)dr$$

where L_t^* is the adjoint operator of L_t.

The next lemma, known as the norm equivalence result and proved in ref. [8] plays an important role in the proof of the main result.

Lemma 4.2. *Assume that (H4.1) holds. Then for any $v \in \mathcal{U}$ there exist two constants $c > 0$ and $C > 0$ such that for every $t \le s \le T$ and $\varphi \in L^1(R^n; \rho(x)dx)$*

$$c\int_{R^n} |\varphi(x)|\rho(x)dx \le \int_{R^n} E(|\varphi(x_s^{t,x;v})|)\rho(x)dx \le C\int_{R^n} |\varphi(x)|\rho(x)dx$$

Moreover, for every $\psi \in L^1([0,T]\times R^n; dt \otimes \rho(x)dx)$

$$c\int_{R^n}\int_t^T |\psi(s,x)|ds\rho(x)dx \le \int_{R^n}\int_t^T E(|\psi(s,x_s^{t,x;v})|)ds\rho(x)dx \le C\int_{R^n}\int_t^T |\psi(s,x)|ds\rho(x)dx$$

The constants c and C depend on T, on ρ and on the bounds of derivatives of the b and σ. The proof is similar to the proof of Proposition 5.1 in [8], hence we omit it.

We now provide the definition of a Sobolev solution for the SHJB Eq. 17.

Definition 4.1. We say that $V(t,x)$ is a weak solution of the Eq. 17, if $V(t,x)$ is $\mathcal{F}_{t,T}^B$-measurable stochastic variable and

(i) $E^{\mathcal{F}_{t,T}^B}(V(t,x)) \in L^2([0,T];H)$, *i.e.*

$$E^{\mathcal{F}_{t,T}^B}\left\{\int_0^T \|V(t)\|_H^2 dt\right\} = E^{\mathcal{F}_{t,T}^B}(V(t,x))\left\{\int_0^T \left(\int_{R^n} |V(t,x)|^2\rho(x)dx\right\} + E^{\mathcal{F}_{t,T}^B}(V(t,x))\left\{\int_{R^n} |(\sigma^*\nabla V)(t,x)|^2\rho(x)dx\right)dt\right\} < \infty \tag{18}$$

(ii) For any non-negative $\varphi \in C_c^{1,\infty}([0,T]\times R^n)$ and for any $v \in \mathcal{U}$

$$\begin{aligned}&\int_{R^n}\int_s^T (V(r,x),\partial_r\varphi(r,x))drdx + \int_{R^n}(V(s,x),\varphi(s,x))dx \\ &\ge \int_{R^n}(\Phi(x),\varphi(T,x))dx + \int_{R^n}\int_s^T (f(r,x,V,\sigma^*\nabla V,v_r),\varphi(r,x))drdx \\ &+ \int_{R^n}\int_s^T (g(r,x,V,\sigma^*\nabla V),\varphi(r,x))d\overleftarrow{B}_r dx \\ &+ \int_{R^n}\int_s^T (\mathcal{L}_r^v V(r,x),\varphi(r,x))drdx \end{aligned}\tag{19}$$

where $(L_r V(r,x),\varphi(r,x)) = \int_{R^n}(\frac{1}{2}(\nabla V\sigma)(\sigma^*\nabla\varphi) + V div(b-A)\varphi)dx$ with $A_i = \frac{1}{2}\sum_{k=1}^n \frac{\partial a_{k,i}}{\partial x_k}$.

(iii) For any non-negative $\varphi \in C_c^{1,\infty}([0,T]\times R^n)$ and for any small $\varepsilon > 0$, there exists a control $v' \in \mathcal{U}$, such that

$$\begin{aligned}&\int_{R^n}\int_s^T (V(r,x),\partial_r\varphi(r,x))drdx + \int_{R^n}(V(s,x),\varphi(s,x))dx - \varepsilon \\ &\le \int_{R^n}(\Phi(x),\varphi(T,x))dx + \int_{R^n}\int_s^T (f(r,x,V,\sigma^*\nabla V,v'_r),\varphi(r,x))\mathbf{drdx} \\ &+ \int_{R^n}\int_s^T (g(r,x,V,\sigma^*\nabla V),\varphi(r,x))d\overleftarrow{B}_r dx \\ &+ \int_{R^n}\int_s^T (\mathcal{L}_r^{v'} V(r,x),\varphi(r,x))\mathbf{drdx} \end{aligned}\tag{20}$$

Prior to presenting the main result of this section, we shall first recall the comparison theorem for BDSDE in [11].

Lemma 4.3. *Let (ξ,f,g) and (ξ',f',g) be two parameters of BDSDEs. Assuming that each one satisfies all the assumptions (H2.1) and (H2.2), we further suppose the following*

$$\xi \le \xi', a.s, \quad f(t,y,z) \le f'(t,y,z), a.s.a.e., \quad \forall (y,z) \in R \times R^d$$

Let (Y,Z) be the solution of the BDSDE with parameter(ξ,f,g) and (Y',Z') is the solution of the BDSDE with parameter (ξ',f',g). Then

$$Y_t \le Y'_t, \quad a.e. \quad \forall\, 0 \le t \le T$$

Lemma 4.4. *Let (H3.1)-(H3.7) and (H4.1) hold, then for any $v \in \mathcal{U}$, the value function satisfies*

$$V(s,x_s^{t,x,v}) \ge E\left\{\int_s^{s'} f(r,x_r^{t,x,v},y_r^{t,x,v},z_r^{t,x,v},v_r)dr + g(r,x_r^{t,x,v},y_r^{t,x,v},z_r^{t,x,v})d\overleftarrow{B}_r + V(s',x_{s'}^{t,x,v})|\mathcal{F}_{t,s}^W \vee \mathcal{F}_{s,T}^B\right\} \quad \forall t \le s \le s' \le T \tag{21}$$

and for any small $\varepsilon > 0$, there exists a $v' \in \mathcal{U}$, such that

$$V(s,x_s^{t,x,v'}) - \varepsilon \le E\left\{\int_s^{s'} f(r,x_r^{t,x,v'},y_r^{t,x,v'},z_r^{t,x,v'},v_r)dr + g(r,x_r^{t,x,v'},y_r^{t,x,v'},z_r^{t,x,v'})d\overleftarrow{B}_r + V(s',x_{s'}^{t,x,v'})|\mathcal{F}_{t,s}^W \vee \mathcal{F}_{s,T}^B\right\} \quad \forall t \le s \le s' \le T \tag{22}$$

Proof. According to the dynamic programming principle, as derived above

$$V\left(s,x_s^{t,x,v}\right) = \operatorname*{ess\,sup}_{v\in\mathcal{U}} G_{s,s'}^{t,x,v}\left[V\left(s',x_{s'}^{t,x,v}\right)\right], \quad \forall t \le s \le s' \le T$$

Let

$$G_{s,s'}^{t,x,v}\left[V\left(s',x_{s'}^{t,x,v}\right)\right] := \tilde{y}_s^{t,x,v}$$

be the solution to the following BDSDE

$$\tilde{y}_s^{t,x,v} = V(s', x_{s'}^{t,x,v}) + \int_s^{s'} f(r, x_r^{t,x,v}, \tilde{y}_r^{t,x,v}, \tilde{z}_r^{t,x,v}, v_r)dr + \int_s^{s'} g(r, x_r^{t,x,v}, \tilde{y}_r^{t,x,v}, \tilde{z}_r^{t,x,v})d\overline{B}_r - \int_s^{s'} \tilde{z}_r^{t,x,v} dW_r, \quad i.e.$$

$$\tilde{y}_s^{t,x,v} = E\left\{ \int_s^{s'} f(r, x_r^{t,x,v}, \tilde{y}_r^{t,x,v}, \tilde{z}_r^{t,x,v}, v_r)dr + \int_s^{s'} g(r, x^{t,x,v}, \tilde{y}_r^{t,x,v}, \tilde{z}_r^{t,x,v})d\overline{B}_r + V\left(s', x_{s'}^{t,x,v}\right) | \mathcal{F}_{t,s}^W \vee \mathcal{F}_{s,T}^B \right\} \tag{23}$$

By utilizing the comparison theorem for BDSDE in Lemma 4.3, obtaining our desired result can be achieved effortlessly. □

To introduce the next lemma, we first define the following BDSDE. Let $A_s^{t,x,v}$ be a continuous increasing process with $A_t^{t,x,v} = 0$ and $A_s^{t,x,v} \in \mathcal{L}^2$, we define the BDSDE as follows

$$y_s^{t,x;v} = \Phi(x_T^{t,x;v}) + \int_s^T f(r, x_r^{t,x;v}, y_r^{t,x;v}, y_r^{t,x;v}, v_r)dr + A_T^{t,x,v} - A_s^{t,x,v} + \int_s^T g(r, x_r^{t,x;v}, y_r^{t,x;v}, y_r^{t,x;v})d\overline{B}_r - \int_s^T z_r^{t,x;v} dW_r \tag{24}$$

The following proposition provides the existence and uniqueness of the solution to Eq. 24.

Proposition 4.5. *We assume (H3.1)-(H3.7) and (H4.1), then there exists a unique pair of processes $(y_s^{t,x,v}, z_s^{t,x,v}) \in \mathcal{L}^2$ of solution (24) such that $y_s^{t,x,v} + A_s^{t,x,v}$ is continuous and that*

$$E \sup_{t \le s \le T} |y_s^{t,x,v}|^2 < \infty \tag{25}$$

Proof. In the case where $A_t^{t,x,v} = 0$, we can make the change of variable $\overline{y}_s^{t,x,v} := y_s^{t,x,v} + A_s^{t,x,v}$ and treat the equivalent BDSDE

$$\overline{y}_s^{t,x;v} = \Phi(x_T^{t,x;v}) + A_T^{t,x,v} + \int_s^T f(r, x_r^{t,x;v}, \overline{y}_r^{t,x;v} - A_r^{t,x,v}, z_r^{t,x;v}, v_r)dr + \int_s^T g(r, x_r^{t,x;v}, \overline{y}_r^{t,x;v} - A_r^{t,x,v}, z_r^{t,x;v})d\overline{B}_r - \int_s^T z_r^{t,x;v} dW_r \tag{26}$$

The BDSDE (26) has a unique solution. □

In this stochastic optimal control problem, the process $y_s^{t,x,v}$ is controlled by $V(s, x_s^{t,x,v})$ and $y_s^{t,x,v} \le V(s, x_s^{t,x,v}), a.s.$. In this problem, $u(t,x) := ess\sup_{v \in \mathcal{U}} y_s^{t,x,v}|_{s=t}$, so for any $v(\cdot)$, $y_t^{t,x,v}$ is controlled by $u(t,x)$ and $y_t^{t,x,v} \le u(t,x)$. To express $u(t,x)$ we employ the penalization method and the comparison theorem of BDSDE to construct a BDSDE with an increasing process.

Lemma 4.6. *We assume (H3.1)-(H3.7) and (H4.1), then the solution of equation (14) is controlled by $V(s, x_s^{t,x,v})$ and $E|V(s, x_s^{t,x,v})|^2 < \infty$. Moreover there exists a unique increasing process $(A_r^{t,x,v})$ with $A_t^{t,x,v} = 0$ and $E[(A_T^{t,x,v})^2] < \infty$ such that $V(s, x_s^{t,x,v})$ coincides with the unique solution $y_s^{t,x,v}$ of the BDSDE*

$$y_t^{t,x,v} = V(T, x_T^{t,x,v}) + \int_t^T f(r, x_r^{t,x,v}, y_r^{t,x,v}, z_r^{t,x,v}, v_r)dr + A_T^{t,x,v} - A_t^{t,x,v} + \int_t^T g(r, x_r^{t,x,v}, y_r^{t,x,v}, z_r^{t,x,v})d\overline{B}_r - \int_t^T z_r^{t,x,v} dW_r \tag{27}$$

where $z_r^{t,x,v} = \sigma^* \nabla V(r, x_r^{t,x,v})$ in the sense of Definition 4.1.

To prove Lemma 4.6, we consider the following family of BDSDEs parameterized by $n = 1, 2, \cdots$

$$y_t^{t,x,v;n} = V(T, x_T^{t,x,v}) + \int_t^T f\left(s, x_s^{t,x,v}, y_s^{t,x,v;n}, z_s^{t,x,v;n}, v_s\right)ds + n\int_t^T \left(V(s, x_s^{t,x,v}) - y_s^{t,x,v;n}\right)ds + \int_t^T g\left(s, x_s^{t,x,v}, y_s^{t,x,v;n}, z_s^{t,x,v;n}\right)d\overline{B}_s - \int_t^T z_s^{t,x,v;n} dW_s \tag{28}$$

An important observation is that, for each $n > 0$, the process $y_s^{t,x,v;n}$ is bounded above by $V(s, x_s^{t,x,v})$.

Lemma 4.7. *We have, for each $n = 1, 2, \cdots$*

$$V(s, x_s^{t,x,v}) \ge y_s^{t,x,v;n}$$

Proof. If it is not the case, then there exist $\delta > 0$ and a positive integer n such that the measure of $\{(\omega, s); y_s^{t,x,v;n} - V(s, x_s^{t,x,v}) - \delta \ge 0\} \subset \Omega \times [t,T]$ is nonzero. We then can define the following stopping times

$$\sigma := \min[T, \inf\{s; y_s^{t,x,v;n} \ge V(s, x_s^{t,x,v}) + \delta\}]$$

$$\tau := \inf\{s \ge \sigma; y_s^{t,x,v;n} \le V(s, x_s^{t,x,v})\}$$

It is seen that $\sigma \le \tau$ and $P(\tau > \sigma) > 0$. Since $V(s, x_s^{t,x,v}) - y_s^{t,x,v;n}$ is right-continuous, we have

$$y_\sigma^{t,x,v;n} \ge V(\sigma, x_\sigma^{t,x,v}) + \delta \tag{29}$$

$$y_\tau^{t,x,v;n} \le V(\tau, x_\tau^{t,x,v}) \tag{30}$$

Now let $y_\tau^{t,x,v;n} = V(\tau, x_\tau^{t,x,v})$, consider the following BDSDEs

$$y_s^{t,x,v;n} = V(\tau, x_\tau^{t,x,v}) + \int_s^\tau f\left(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n}, v_r\right)dr + n\int_s^\tau \left(V(r, x_r^{t,x,v}) - y_r^{t,x,v;n}\right)dr + \int_s^\tau g\left(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n}\right)d\overline{B}_r - \int_s^\tau z_r^{t,x,v;n} dW_r \tag{31}$$

$$y_s^{t,x,v} = V(\tau, x_\tau^{t,x,v}) + \int_s^\tau f\left(r, x_r^{t,x,v}, y_r^{t,x,v}, z_r^{t,x,v}, v_r\right)dr + \int_s^\tau g\left(r, x_r^{t,x,v}, y_r^{t,x,v}, z_r^{t,x,v}\right)d\overline{B}_r - \int_s^\tau z_r^{t,x,v} dW_r \tag{32}$$

By Comparison Theorem implies $y_s^{t,x,v;n} \le y_s^{t,x,v} \le V(s, x_s^{t,x,v})$, where $\sigma \le s \le \tau$. This is contrary with (29). The proof is complete. □

Proof of Lemma 4.6.. Since the solution of the BDSDE is no longer a super-martingale, the proof method used in Lemma 4.1 [14] is not applicable in our situation. Instead, we rely on the properties of BDSDE and limitation theory to develop our proof strategy.

We first consider the BDSDEs (28), and we denote

$$A_r^{t,x,v;n} := n\int_t^r \left(V(s, x_s^{t,x,v}) - y_s^{t,x,v;n}\right)ds \tag{33}$$

and

$$f_n(s, x_s^{t,x,v}, y_s^{t,x,v;n}, z_s^{t,x,v;n}, v_s) := f(s, x_s^{t,x,v}, y_s^{t,x,v;n}, z_s^{t,x,v;n}, v_s) + n(V(s, x_s^{t,x,v}) - y_s^{t,x,v;n}) \tag{34}$$

For each n, $(y_s^{t,x,v;n}, z_s^{t,x,v;n})$ is the unique solution of Eq. 28 with $\mathcal{F}_t$ measurable process valued in $R \times R^d$.

We begin by establishing the existence of a limit $(y_s^{t,x,v}, z_s^{t,x,v})$ for the sequence $(y_s^{t,x,v;n}, z_s^{t,x,v;n})$. By observing that $f_n(s, x_s^{t,x,v}, y_s^{t,x,v;n}, z_s^{t,x,v;n}, v_s) \le f_{n+1}(s, x_s^{t,x,v}, y_s^{t,x,v;n}, z_s^{t,x,v;n}, v_s)$, it follows from the comparison theorem that $y_t^{t,x,v;n} \le y_t^{t,x,v;n+1}, 0 \le t \le T$. Moreover, according to Lemma 4.7 the BDSDE solution $y_s^{t,x,v;n}$ is bounded above by $V(s, x_s^{t,x,v})$. Hence, by the dominated convergence theorem, we can establish the convergence of

$$y_t^{t,x,v;n} \uparrow y_t^{t,x,v}, \quad 0 \le t \le T, \quad \text{a.e.} \tag{35}$$

From the BDSDE (28), we have

$$
\begin{aligned}
&A_T^{t,x,v;n} - A_t^{t,x,v;n} \\
&= y_t^{t,x,v;n} - V(T, x_T^{t,x,v}) - \int_t^T f(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n}, v_r)dr \\
&\quad - \int_t^T g(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n})d\overleftarrow{B}_r + \int_t^T z_r^{t,x,v;n} dW_r \\
&\le |y_t^{t,x,v;n}| + |V(T, x_T^{t,x,v})| + \int_t^T f(r,0,0,0,v_r)dr \\
&\quad + \int_t^T (L|x_r^{t,x,v}| + L|y_r^{t,x,v;n}| + L|z_r^{t,x,v;n}|)dr \\
&\quad + |\int_t^T g(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n})d\overleftarrow{B}_r| + |\int_t^T z_r^{t,x,v,n} dW_r| \\
&\le |V(t,x)| + |V(T, x_T^{t,x,v})| + \int_t^T f(r,0,0,0,v_r)dr \\
&\quad + \int_t^T (L|x_r^{t,x,v}| + L|V(r, x_r^{t,x,v})| + L|z_r^{t,x,v,n}|)dr \\
&\quad + |\int_t^T g(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n})d\overleftarrow{B}_r| + |\int_t^T z_r^{t,x,v,n} dW_r|
\end{aligned}
$$

Because for any v_r, $E\int_0^T |f(r,0,0,0,v_r)|^2 dr \le M$, we have

$$
\begin{aligned}
&E|A_T^{t,x,v;n}|^2 \\
&\le 8|V(t,x)|^2 + 8E|g(x_T^{t,x,v})|^2 + 8E\int_t^T |f(r,0,0,0,v_r)|^2 dr \\
&\quad + 8E\int_t^T (L^2|x_r^{t,x,v}|^2 + L^2|V(r, x_r^{t,x,v})|^2 + L^2|z_r^{t,x,v,n}|^2)dr \\
&\quad + 8E\int_t^T |z_r^{t,x,v,n}|^2 dr
\end{aligned}
$$

Thus we can define a $C_1(t,T,x,v)$ independent of n, such that

$$
E|A_T^{t,x,v;n}|^2 \le C_1(t,T,x,v) + 8(L^2+1)E\int_t^T |z_r^{t,x,v;n}|^2 dr \tag{36}
$$

On the other hand, we use Itô's formula to $|y_r^{t,x,v;n}|^2$

$$
\begin{aligned}
&E|y_t^{t,x,v;n}|^2 + E\int_t^T |z_r^{t,x,v;n}|^2 dr \\
&= E|V(T, x_T^{t,x,v})|^2 + 2E\int_t^T y_r^{t,x,v;n} f(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n}, v_r)dr \\
&\quad + E\int_t^T |g(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n})|^2 dr + 2E\int_t^T y_r^{t,x,v;n} dA_r^{t,x,v;n} \\
&\le E|V(T, x_T^{t,x,v})|^2 \\
&\quad + 2E\int_t^T |y_r^{t,x,v;n}|(|(f(r,0,0,0,v_r)| + L|x_r^{t,x,v}| + L|y_r^{t,x,v;n}| + L|z_r^{t,x,v;n}|)dr \\
&\quad + E\int_t^T (|g(r,0,0,0) + L|x_r^{t,x,v}| + L|y_r^{t,x,v;n}| + \alpha|z_r^{t,x,v;n}|)^2 dr + 2E\int_t^T y_r^{t,x,v;n} dA_r^{t,x,v;n} \\
&\le E|V(T, x_T^{t,x,v})|^2 + \int_t^T |f(r,0,0,0,v_r)|^2 dr + E\int_t^T |y_r^{t,x,v;n}|^2 dr \\
&\quad + E\int_t^T L^2|y_r^{t,x,v;n}|^2 + |x_r^{t,x,v;n}|^2 dr + E\int_t^T (2L + \frac{L^2}{\varepsilon})|y_r^{t,x,v;n}|^2 + \varepsilon|z_r^{t,x,v;n}|^2 dr \\
&\quad + E\int_t^T (4 + \frac{\alpha^2}{\varepsilon})|g(r,0,0,0)|^2 + L^2(4 + \frac{\alpha^2}{\varepsilon})|x_r^{t,x,v}|^2 + L^2(4 + \frac{\alpha^2}{\varepsilon})|y_r^{t,x,v;n}|^2 \\
&\quad + (\alpha^2 + 3\varepsilon)|z_r^{t,x,v;n}|^2|dr + 2E[A_T^{t,x,v;n} \sup_{t\le s\le T} |y_s^{t,x,v;n}|] \\
&\le E|V(T, x_T^{t,x,v})|^2 + E\int_t^T |f(r,0,0,0,v_r)|^2 dr + (4L^2 + \frac{L^2\alpha^2}{\varepsilon} + 1)E\int_t^T |x_r^{t,x,v}|^2 dr \\
&\quad + (5L^2 + \frac{L^2(\alpha^2+1)}{\varepsilon} + 2L + 1)E\int_t^T |V(r, x_r^{t,x,v})|^2 dr + (\alpha^2 + 4\varepsilon)E\int_t^T |z_r^{t,x,v;n}|^2 dr \\
&\quad + (4 + \frac{\alpha^2}{\varepsilon})E\int_t^T |g(r,0,0,0)|^2 dr + \frac{1}{32(L^2+1)}E|A_T^{t,x,v;n}|^2 \\
&\quad + 64(L^2+1)E\sup_{t\le s\le T}[|y_s^{t,x,v;1}|^2 + |V(s, x_s^{t,x,v})|^2].
\end{aligned}
$$

Where $0 < \varepsilon < \frac{1}{4}(1-\alpha^2)$, so that $\alpha^2 + 4\varepsilon < 1$. Then, we can define a $C_2(t,T,x,v)$ satisfying

$$
E\int_t^T |z_r^{t,x,v;n}|^2 dr \le C_2(t,T,x,v) + \frac{1}{16(L^2+1)}E\left|A_T^{t,x,v;n}\right|^2
$$

So that we have

$$
E\left|A_T^{t,x,v;n}\right|^2 \le 2C_1(t,T,x,v) + 16(L^2+1)C_2(t,T,x,v) \tag{37}
$$

$$
\begin{aligned}
&E\{n^2\int_t^r \left(V(s, x_s^{t,x,v}) - y_s^{t,x,v;n}\right)^2 ds\} \\
&\le 2C_1(t,T,x,v) + 16(L^2+1)C_2(t,T,x,v).
\end{aligned} \tag{38}
$$

For any n, Inequality (38) continues to hold, with $E|A_T^{t,x,v;n}|^2$ bounded by the constant $2C_1(t,T,x,v) + 16(L^2+1)C_2(t,T,x,v)$. As $i \to \infty$, we observe that $y_s^{t,x,v;n} \uparrow y_s^{t,x,v}$ and $y_s^{t,x,v} = V(s, x_s^{t,x,v})$.

Next, we prove uniqueness. If there is another solution $A't,x,v_r$ and $Z't,x,v_r$ satisfying (27), then we apply Itô's formula to $(y_r^{t,x,v} - y_r^{t,x,v})^2 \equiv 0$ on $[t,T]$ and take expectation

$$
\begin{aligned}
&E\int_t^T \left|z_r^{t,x,v} - z't,x,v_r\right|^2 dr + E[(A_T^{t,x,v} - A't,x,v_T) \\
&-(A_t^{t,x,v} - A't,x,v_t)]^2 = 0
\end{aligned}
$$

for any $t \in [0,T]$. Thus $z_r^{t,x,v} = z't,x,v_r$, $A_r^{t,x,v} = A't,x,v_r$.

Now we consider BDSDE (27). We have that $n(V(r, x_r^{t,x,v}) - y_r^{t,x,v;n})$ is Lipschitz about $y_r^{t,x,v;n}$, and that

$$
E\int_t^T \int_{R^n} (|f(r,x,0,0,v_r)|^2 + |V(r,x)|^2)\rho(x)dxdr < \infty
$$

and b, σ satisfy assumption (H3.1) and (H3.2). Thus by Proposition 2.3 in [7], we know that $z_r^{t,x,v;n} = \sigma^*\nabla y_r^{t,x,v;n}$. Applying Itô's formula to the process $|y_t^{t,x,v;n} - y_t^{t,x,v;p}|^2$

$$
\begin{aligned}
&\left|y_t^{t,x,v;n} - y_t^{t,x,v;p}\right|^2 + \int_t^T |z_r^{t,x,v;n} - z_r^{t,x,v;p}|^2 dr \\
&= 2\int_t^T \left[f\left(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n}, v_r\right)\right. \\
&\quad \left.-f\left(r, x_r^{t,x,v}, y_r^{t,x,v;p}, z_r^{t,x,v;p}, v_r\right)\right]\left(y_r^{t,x,v;n} - y_r^{t,x,v;p}\right)dr \\
&\quad + \int_t^T \left|g\left(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n}\right) - g\left(r, x_r^{t,x,v}, y_r^{t,x,v;p}, z_r^{t,x,v;p}\right)\right|^2 dr \\
&\quad + 2\int_t^T \left[g\left(r, x_r^{t,x,v}, y_r^{t,x,v;n}, z_r^{t,x,v;n}\right) - g\left(r, x_r^{t,x,v}, y_r^{t,x,v;p}, z_r^{t,x,v;p}\right)\right] \\
&\quad \left(y_r^{t,x,v;n} - y_r^{t,x,v;p}\right)d\overleftarrow{B}_r \\
&\quad - 2\int_t^T \left(y_r^{t,x,v;n} - y_r^{t,x,v;p}\right)\left(z_r^{t,x,v;n} - z_r^{t,x,v;p}\right)dW_r \\
&\quad + \int_t^T \left(y_r^{t,x,v;n} - y_r^{t,x,v;p}\right)d\left(A_r^{t,x,v;n} - A_r^{t,x,v;p}\right)
\end{aligned}
$$

So

$$
\begin{aligned}
&E\left(\left|y_t^{t,x,v;n} - y_t^{t,x,v;p}\right|^2\right) + E\int_t^T \left|z_r^{t,x,v;n} - z_r^{t,x,v;p}\right|^2 dr \\
&\le 2LE\int_t^T \left(\left|y_r^{t,x,v;n} - y_r^{t,x,v;p}\right|^2 + \left|y_r^{t,x,v;n} - y_r^{t,x,v;p}\right| \cdot \left|z_r^{t,x,v;n} - z_r^{t,x,v;p}\right|\right)ds \\
&\quad + \left(L^2 + \frac{L^2\alpha^2}{\varepsilon}\right)E\int_t^T \left|y_r^{t,x,v;n} - y_r^{t,x,v;p}\right|^2 ds \\
&\quad + (\alpha^2 + \varepsilon)E\int_t^T \left|z_r^{t,x,v;n} - z_r^{t,x,v;p}\right|^2 ds \\
&\quad + E\sup_{t\le r\le T} |y_r^{t,x,v;n} - y_r^{t,x,v;p}|(A_T^{t,x,v;n} - A_T^{t,x,v;p}) \\
&\le \left(L^2 + \frac{L^2(\alpha^2+1)}{\varepsilon} + 2L\right)E\int_t^T \left|y_r^{t,x,v;n} - y_r^{t,x,v;p}\right|^2 dr
\end{aligned}
$$

$$+(\alpha^2+2\varepsilon)E\int_t^T \left|z_r^{t,x,v;n}-z_r^{t,x,v;p}\right|^2dr$$
$$+(E\sup_{t\le r\le T}|y_r^{t,x,v;n}-y_r^{t,x,v;p}|^2)^{\frac14}(E(A_T^{t,x,v;n}-A_T^{t,x,v;p})^2)^{\frac12}$$

Where $0<\varepsilon<\frac12(1-\alpha^2)$, so that $\alpha^2+2\varepsilon<1$. We have

$$E\int_t^T \left|z_r^{t,x,v;n}-z_r^{t,x,v;p}\right|^2dr$$
$$\le CE\int_t^T |y_r^{t,x,v;n}-y_r^{t,x,v;p}|^2dr$$
$$+(E\sup_{t\le r\le T}|y_r^{t,x,v;n}-y_r^{t,x,v;p}|^2)^{\frac14}(E(A_T^{t,x,v;n}-A_T^{t,x,v;p})^2)^{\frac12}$$

Since $A_T^{t,x,v;n}$ is bounded and $y_r^{t,x,v;n}$ is convergent

$$\lim_{n,p\to\infty}E\int_t^T |z_r^{t,x,v;n}-z_r^{t,x,v;p}|^2dr=0$$

Thus, we conclude that $z_r^{t,x,v}=\sigma^*\nabla y_r^{t,x,v}=\sigma^*\nabla V(r,x_r^{t,x,v})$. □

On the other hand, for any small $\varepsilon>0$, there exists a control $v'\in\mathcal{U}$, $V\left(r,x_r^{t,x,v'}\right)$ satisfying

$$V\left(s,x_s^{t,x,v'}\right)\le y_s^{s,x_s^{t,x,v'},v'}+\varepsilon$$

Lemma 4.8. *We assume (H3.1)-(H3.7) and (H4.1), then $V(s,x_s^{t,x,v'})$ is controlled by a special $y_s^{s,x_s^{t,x,v'},v'}+\varepsilon$. Same as the proof of Lemma 4.6, there exists a unique increasing process $(A_r^{t,x,v'})$ with $A_t^{t,x,v'}=0$ and $E[(A_T^{t,x,v'})^2]<\infty$ such that $V(s,x_s^{t,x,v'})$ coincides with the unique solution $y_s^{t,x,v}$ of the BDSDE*

$$y_t^{t,x,v'}=V(T,x_T^{t,x,v'})+\varepsilon+\int_t^T f(r,x_r^{t,x,v'},y_r^{t,x,v'},Z_r^{t,x,v'},v_r')dr$$
$$-(A_T^{t,x,v'}-A_t^{t,x,v'})+\int_t^T g(r,x_r^{t,x,v'},y_r^{t,x,v'},Z_r^{t,x,v'})d\overleftarrow{B}_r-\int_t^T Z_r^{t,x,v'}dW_r$$

where $Z_r^{t,x,v'}=\sigma^*\nabla V(r,x_r^{t,x,v'})$ in the sense of Definition 4.1

The proof of Lemma 4.8 is similar to that of Lemma 4.6.

Theorem 4.9. *Under the assumption (H3.1)-(H3.7) and (H4.1), the value function $V(t,x)$ defined in (16) is the unique Sobolev solution of the SPDE (17).*

Proof. Existence: In the context of stochastic recursive optimal control, the value function $V(t,x)$ defined in (16) satisfies the Bellman dynamic programming principle. Based on Lemma 4.6 and Lemma 4.8, for any $v\in\mathcal{U}$, there have a unique increasing process $A_s^{t,x,v}$, $V(s,x_s^{t,x,v})$ satisfies the following BDSDE

$$V(s,x_s^{t,x,v})=V(T,x_T^{t,x,v})+\int_s^T f(r,x_r^{t,x,v},V(r,x_r^{t,x,v}),\sigma^*\nabla V(r,x_r^{t,x,v}),v_r)dr$$
$$+(A_T^{t,x,v}-A_s^{t,x,v})+\int_s^T g(r,x_r^{t,x,v},V(r,x_r^{t,x,v}),\sigma^*\nabla V(r,x_r^{t,x,v}))d\overleftarrow{B}_r$$
$$-\int_s^T \sigma^*\nabla V(r,x_r^{t,x,v})dW_r. \quad (39)$$

Hence, it follows readily that

$$V(s,x_s^{t,x,v})\ge V(T,x_T^{t,x,v})+\int_s^T f(r,x_r^{t,x,v},V(r,x_r^{t,x,v}),\sigma^*\nabla V(r,x_r^{t,x,v}),v_r)dr$$
$$+\int_s^T g(r,x_r^{t,x,v},V(r,x_r^{t,x,v}),\sigma^*\nabla V(r,x_r^{t,x,v}))d\overleftarrow{B}_r$$
$$-\int_s^T \sigma^*\nabla V(r,x_r^{t,x,v})dW_r \quad (40)$$

Moreover, for any small $\varepsilon>0$ there exists a control $v'\in\mathcal{U}$, such that $V(s,x_s^{t,x,v'})$ satisfies the following BDSDE

$$V(s,x_s^{t,x,v'})-\varepsilon$$
$$=V(T,x_T^{t,x,v'})+\int_s^T f(r,x_r^{t,x,v'},V(r,x_r^{t,x,v'}),\sigma^*\nabla V(r,x_r^{t,x,v'}),v_r')dr$$
$$-(A_T^{t,x,v'}-A_s^{t,x,v'})+\int_s^T g(r,x_r^{t,x,v'},V(r,x_r^{t,x,v'}),\sigma^*\nabla V(r,x_r^{t,x,v'}))d\overleftarrow{B}_r$$
$$-\int_s^T \sigma^*\nabla V(r,x_r^{t,x,v'})dW_r \quad (41)$$

Then we have

$$V(s,x_s^{t,x,v'})-\varepsilon$$
$$\le V(T,x_T^{t,x,v'})+\int_s^T f(r,x_r^{t,x,v'},V(r,x_r^{t,x,v'}),\sigma^*\nabla V(r,x_r^{t,x,v'}),v_r')dr$$
$$+\int_s^T g(r,x_r^{t,x,v'},V(r,x_r^{t,x,v'}),\sigma^*\nabla V(r,x_r^{t,x,v'}))d\overleftarrow{B}_r$$
$$-\int_s^T \sigma^*\nabla V(r,x_r^{t,x,v'})dW_r \quad (42)$$

By applying the equivalence of norm result (Lemma 4.6) we can deduce that $E(V)\in L^2([t,T],H)$. Indeed, in the stochastic recursive optimal control problem, the cost function can be viewed as a solution of BDSDE

$$y_s^{t,x,v}=\Phi(x_T^{t,x,v})+\int_s^T f(r,x_r^{t,x,v},y_r^{t,x,v},z_r^{t,x,v},v_r)dr$$
$$+\int_s^T g(r,x_r^{t,x,v},y_r^{t,x,v},z_r^{t,x,v})d\overleftarrow{B}_r-\int_s^T z_r^{t,x,v}dW_r$$

Based on the standard estimates of BDSDEs, along with assumptions (H3.1)-(H3.7) and (H4.1), it is clear that there exist constants K and C such that

$$\int_{R^n}E(|y_t^{t,x,v}|^2+\int_t^T|z_r^{t,x,v}|^2dr)\rho(x)dx$$
$$\le K\int_{R^n}E|\Phi(x_T^{t,x,v})|^2\rho(x)dx+K\int_{R^n}\int_t^T E|f(r,x_r^{t,x,v},0,0,v_r)|^2dr\rho(x)dx$$
$$\le KC\int_{R^n}|\Phi(x)|^2\rho(x)dx+KC\int_{R^n}\int_t^T|f(r,x,0,0,v_r)|^2dr\rho(x)dx$$
$$\le KC\int_{R^n}|\Phi(x)|^2\rho(x)dx+KC\int_{R^n}\int_t^T Mdr\rho(x)dx$$
$$=KC\int_{R^n}|\Phi(x)|^2\rho(x)dx+(T-t)MCK<\infty \quad (43)$$

Thus, for any v, $y_r^{t,x,v}\in H$, where H is a Hilbert space. Note that $V(t,x)=\sup_{v\in\mathcal{U}}y_t^{t,x,v}$ and since U is compact set of R^k, we can conclude that $V(t,\cdot)$ is also an element of H. Moreover, due to the validity of Eq. 40, it follows that for any non-negative $\varphi\in C_c^\infty(R^n)$, we have

$$\int_{R^n}(V(s,x_s^{t,x,v}),\varphi(x))dx$$
$$\ge\int_{R^n}(V(T,x_T^{t,x,v}),\varphi(x))dx$$
$$+\int_{R^n}\int_s^T(f(r,x_r^{t,x,v},V(r,x_r^{t,x,v}),\sigma^*\nabla V(r,x_r^{t,x,v}),v_r),\varphi(x))drdx$$
$$+\int_{R^n}\int_s^T(g(r,x,V(r,x_r^{t,x,v}),\sigma^*\nabla V(r,x_r^{t,x,v})),\varphi(x))d\overleftarrow{B}_rdx$$
$$-\int_{R^n}\int_s^T(\sigma^*\nabla V(r,x_r^{t,x,v}),\varphi(x))dW_rdx \quad (44)$$

It turns out that

$$\int_{R^n}(V(s,x),\varphi(s,x))dx$$
$$\ge\int_{R^n}(\Phi(x),\varphi(T,x))dx+\int_{R^n}\int_s^T(f(r,x,V(r,x),\sigma^*\nabla V(r,x),v_r),\varphi(r,x))drdx$$
$$+\int_{R^n}\int_s^T(g(r,x,V,\sigma^*\nabla V(r,x)),\varphi(r,x))d\overleftarrow{B}_rdx$$

$$-\int_{R^n}\int_s^T(\sigma^*\nabla V(r,x),\varphi(r,x))dW_r dx \tag{45}$$

Moreover, by applying Lemma 4.1, we obtain that

$$\begin{aligned}&-\int_{R^n}\int_s^T\sigma^*\nabla V(r,x)\varphi(r,x)dW_r dx\\&=-\int_{R^n}\sum_{j=1}^d\int_s^T(\sum_{i=1}^n\sigma_{i,j}(r,x)\frac{\partial V}{\partial x_i}(r,x),\varphi(r,x))dW_r^j\\&=\int_{R^n}\int_s^T(\mathcal{L}^{\upsilon}V(r,x),\varphi(r,x))dr-\int_{R^n}\int_s^T(V(r,x),\partial_r\varphi(r,x))drdx\end{aligned} \tag{46}$$

Taking (46) into (45), we have that

$$\begin{aligned}&\int_{R^n}\int_s^T(V(r,x),\partial_r\varphi(r,x))drdx+\int_{R^n}(V(s,x),\varphi(s,x))dx\\&\geq\int_{R^n}(\Phi(x),\varphi(T,x))dx+\int_{R^n}\int_s^T(f(r,x,V(r,x),\sigma^*\nabla V(r,x),\upsilon_r),\varphi(r,x))drdx\\&+\int_{R^n}\int_s^T(g(r,x,V(r,x),\sigma^*\nabla V),\varphi(r,x))d\overleftarrow{B}_r dx\\&+\int_{R^n}\int_s^T(\mathcal{L}_r^{\upsilon}V(r,x),\varphi(r,x))drdx\end{aligned} \tag{47}$$

By virtue of the same techniques, because (42) holds, so for any non-negative $\varphi\in C_c^\infty(R^n)$, we take $\varepsilon=\frac{\varepsilon'}{\int_{R^n}\varphi(x)dx}$, then

$$\begin{aligned}&\int_{R^n}(V(s,x_s^{t,x,\upsilon'}),\varphi(x))dx-\varepsilon'\\&\leq\int_{R^n}(\Phi(x_T^{t,x,\upsilon'}),\varphi(x))dx\\&+\int_{R^n}\int_s^T(f(r,x_r^{t,x,\upsilon'},V(r,x_r^{t,x,\upsilon'}),\sigma^*\nabla V(r,x_r^{t,x,\upsilon'}),\upsilon_r'),\varphi(x))drdx\\&+\int_{R^n}\int_s^T(g(r,x_r^{t,x,\upsilon'},V(r,x_r^{t,x,\upsilon'}),\sigma^*\nabla V(r,x_r^{t,x,\upsilon'})),\varphi(x))d\overleftarrow{B}_r dx\\&-\int_{R^n}\int_s^T(\sigma^*\nabla V(r,x_r^{t,x,\upsilon'}),\varphi(x))dW_r dx\end{aligned} \tag{48}$$

This is equivalent to

$$\begin{aligned}&\int_{R^n}(V(s,x),\varphi(s,x))dx-\varepsilon'\\&\leq\int_{R^n}(\Phi(x),\varphi(T,x))dx+\int_{R^n}\int_s^T(f(r,x,V(r,x),\sigma^*\nabla V(r,x),\upsilon_r'),\varphi(r,x))drdx\\&+\int_{R^n}\int_s^T(g(r,x,V(r,x),\sigma^*\nabla V(r,x)),\varphi(r,x))d\overleftarrow{B}_r dx\\&-\int_{R^n}\int_s^T(\sigma^*\nabla V(r,x),\varphi(r,x))dW_r dx\end{aligned} \tag{49}$$

Taking (46) into (49), we obtain

$$\begin{aligned}&\int_{R^n}\int_s^T(V(r,x),\partial_r\varphi(r,x))drdx+\int_{R^n}(V(s,x),\varphi(s,x))dx-\varepsilon'\\&\leq\int_{R^n}(\Phi(x),\varphi(T,x))dx+\int_{R^n}\int_s^T(f(r,x,V(r,x),\sigma^*\nabla V(r,x),\upsilon_r'),\varphi(r,x))drdx\\&+\int_{R^n}\int_s^T(g(r,x,V(r,x),\sigma^*\nabla V(r,x)),\varphi(r,x))d\overleftarrow{B}_r dx\\&+\int_{R^n}\int_s^T(\mathcal{L}_r^{\upsilon'}V(r,x),\varphi(r,x))drdx\end{aligned} \tag{50}$$

Uniqueness: Let $\overline{V}$ be another solution of the SPDE (17). By Definition 4.1, one gets that for any $\upsilon\in\mathcal{U}$

$$\begin{aligned}&\int_{R^n}\int_s^T(\overline{V}(r,x),\partial_r\varphi(r,x))drdx+\int_{R^n}(\overline{V}(s,x),\varphi(s,x))dx\\&\geq\int_{R^n}(\Phi(x),\varphi(T,x))dx+\int_{R^n}\int_s^T(f(r,x,\overline{V}(r,x),\sigma^*\nabla\overline{V}(r,x),\upsilon_r),\varphi(r,x))drdx\\&+\int_{R^n}\int_s^T(g(r,x,\overline{V}(r,x),\sigma^*\nabla\overline{V}(r,x)),\varphi(r,x))d\overleftarrow{B}_r dx\\&+\int_{R^n}\int_s^T(\mathcal{L}_t^{\upsilon}\overline{V}(r,x),\varphi(r,x))drdx\end{aligned} \tag{51}$$

By Lemma 4.5 in [18], we have

$$\begin{aligned}&\int_{R^n}\int_s^T(\overline{V}(r,x),\partial_r\varphi(r,x))drdx\\&=\int_{R^n}\Sigma_{j=1}^d\int_s^T(\Sigma_{i=1}^n(\sigma_{i,j}\frac{\partial\overline{V}}{\partial x_i}(r,x),\varphi(r,x)))dW_r^j dx\\&+\int_{R^n}\int_s^T(L_r^{\upsilon}\overline{V}(r,x),\varphi(r,x))drdx\\&=\int_{R^n}\int_s^T((\sigma^*\nabla\overline{V})(r,x),\varphi(r,x))dW_r dx\\&+\int_{R^n}\int_s^T(L_r^{\upsilon}\overline{V}(r,x),\varphi(r,x))drdx\end{aligned} \tag{52}$$

Taking (52) into (51), we get

$$\begin{aligned}&\int_{R^n}(\overline{V}(s,x),\varphi(s,x))dx+\int_{R^n}\int_s^T(\sigma^*\nabla\overline{V})(r,x)\varphi(r,x)dW_r dx\\&+\int_{R^n}\int_s^T(L_r^{\upsilon}\overline{V}(r,x),\varphi(r,x))drdx\\&\geq\int_{R^n}(\Phi(x),\varphi(T,x))dx+\int_{R^n}\int_s^T(f(r,x,\overline{V}(r,x)\\&\sigma^*\nabla\overline{V}(r,x),\upsilon_r),\varphi(r,x))drdx\\&+\int_{R^n}\int_s^T(g(r,x,\overline{V}(r,x),\sigma^*\nabla\overline{V}(r,x)),\varphi(r,x))d\overleftarrow{B}_r dx\\&+\int_{R^n}\int_s^T(L_r^{\upsilon}\overline{V}(r,x),\varphi(r,x))drdx\end{aligned}$$

So

$$\begin{aligned}&\int_{R^n}(\overline{V}(s,x),\varphi(s,x))dx\\&\geq\int_{R^n}(\Phi(x),\varphi(T,x))dx+\int_{R^n}\int_s^T(f(r,x,\overline{V}(r,x),\sigma^*\nabla\overline{V}(r,x),\upsilon_r),\varphi(r,x))drdx\\&+\int_{R^n}\int_s^T(g(r,x,\overline{V}(r,x),\sigma^*\nabla\overline{V}(r,x)),\varphi(r,x))d\overleftarrow{B}_r dx\\&-\int_{R^n}\int_s^T(\sigma^*\nabla\overline{V})(r,x)\varphi(r,x)dW_r dx\end{aligned} \tag{53}$$

According to Definition 4.1, it follows that for any sufficiently small $\varepsilon>0$, there exists a control $\upsilon'\in\mathcal{U}$ such that

$$\begin{aligned}&\int_{R^n}\int_s^T(\overline{V}(r,x),\partial_r\varphi(r,x))drdx+\int_{R^n}(\overline{V}(s,x),\varphi(s,x))dx-\varepsilon\\&\leq\int_{R^n}(\Phi(x),\varphi(T,x))dx+\int_{R^n}\int_s^T(f(r,x,\overline{V}(r,x),\sigma^*\nabla\overline{V}(r,x),\upsilon_r'),\varphi(r,x))drdx\\&+\int_{R^n}\int_s^T(g(r,x,\overline{V}(r,x),\sigma^*\nabla\overline{V}(r,x)),\varphi(r,x))d\overleftarrow{B}_r dx\\&+\int_{R^n}\int_s^T(\mathcal{L}_r^{\upsilon'}\overline{V}(r,x),\varphi(r,x))drdx\end{aligned} \tag{54}$$

Taking (52) into (54), we have

$$\begin{aligned}&\int_{R^n}(\overline{V}(s,x),\varphi(s,x))dx-\varepsilon\\&\leq\int_{R^n}(\Phi(x),\varphi(T,x))dx+\int_{R^n}\int_s^T(f(r,x,\overline{V}(r,x),\sigma^*\nabla\overline{V}(r,x),\upsilon_r'),\varphi(r,x))drdx\\&+\int_{R^n}\int_s^T(g(r,x,\overline{V}(r,x),\sigma^*\nabla\overline{V}(r,x)),\varphi(r,x))d\overleftarrow{B}_r dx\\&-\int_{R^n}\int_s^T(\sigma^*\nabla\overline{V})(r,x)\varphi(r,x)dW_r dx\end{aligned} \tag{55}$$

Let us make the change of variable $y = \hat{x}_r^{t,x,v}$ in each term of (53), then

$$\int_{R^n} (\overline{V}(s,x), \varphi(s,x))dx = \int_{R^n} (\overline{V}(s, x_s^{t,y,v}), \varphi(y))dy \tag{56}$$

$$\int_{R^n} (\Phi(x), \varphi(T,x))dx = \int_{R^n} (\Phi(x_T^{t,y,v}), \varphi(y))dy \tag{57}$$

$$\begin{aligned}&\int_{R^n}\int_s^T (f(r,x,\overline{V}(r,x), \sigma^*\nabla\overline{V}(r,x), v_r), \varphi(r,x))drdx \\ &= \int_{R^n}\int_s^T (f(r,x_r^{t,y,v},\overline{V}(r,x_r^{t,y,v}), (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v}), v_r), \varphi(y))drdy\end{aligned} \tag{58}$$

$$\begin{aligned}&\int_{R^n}\int_s^T (g(r,x,\overline{V}(r,x), \sigma^*\nabla\overline{V}(r,x)), \varphi(r,x))d\overleftarrow{B}_r dx \\ &= \int_{R^n}\int_s^T (g(r,x_r^{t,y,v},\overline{V}(r,x_r^{t,y,v}), \sigma^*\nabla\overline{V}(r,x_r^{t,y,v})), \varphi(y))d\overleftarrow{B}_r dy\end{aligned} \tag{59}$$

$$\begin{aligned}&\int_{R^n}\int_s^T ((\sigma^*\nabla\overline{V})(r,x), \varphi(r,x))dW_r dx \\ &= \int_{R^n}\int_s^T ((\sigma^*\nabla\overline{V})(r,x_r^{t,y,v}), \varphi(y))dW_r dy\end{aligned} \tag{60}$$

Thus, inequality (53) can be simplified as follows

$$\begin{aligned}&\int_{R^n} \overline{V}(s,x_s^{t,y,v})\varphi(y)dy \\ &\geq \int_{R^n} \Phi(x_T^{t,y,v})\varphi(y)dy + \int_s^T\int_{R^n} (f(r,x_r^{t,y,v},\overline{V}(r,x_r^{t,y,v}), \\ &\quad (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v}), v_r), \varphi(y))dydr \\ &\quad + \int_{R^n}\int_s^T (g(r,x_r^{t,y,v},\overline{V}(r,x), \sigma^*\nabla\overline{V}(r,x)), \varphi(y))d\overleftarrow{B}_r dy \\ &\quad - \int_s^T\int_{R^n} ((\sigma^*\nabla\overline{V})(r,x_r^{t,y,v}), \varphi(y))dydW_r\end{aligned} \tag{61}$$

Since φ is arbitrary, we have proven that for almost every y

$$\begin{aligned}&\overline{V}(s,x_s^{t,y,v}) \\ &\geq \Phi(x_T^{t,y,v}) + \int_s^T f(r,x_r^{t,y,v},\overline{V}(r,x_r^{t,y,v}), (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v}), v_r)dr \\ &\quad + \int_s^T g(r,x_r^{t,y,v},\overline{V}(r,x_r^{t,y,v}), (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v}))d\overleftarrow{B}_r \\ &\quad - \int_s^T (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v})dW_r\end{aligned} \tag{62}$$

Let

$$\begin{aligned}\overline{y}_s^{t,y,v} &= \Phi(x_T^{t,y,v}) + \int_s^T f(r,x_r^{t,y,v},\overline{V}(r,x_r^{t,y,v}), (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v}), v_r)dr \\ &\quad + \int_s^T g(r,x_r^{t,y,v},\overline{V}(r,x_r^{t,y,v}), (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v}))d\overleftarrow{B}_r \\ &\quad - \int_s^T (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v})dW_r\end{aligned} \tag{63}$$

Then

$$\begin{aligned}\overline{V}(s,x_s^{t,y,v}) &= \Phi(x_T^{t,y,v}) + \int_s^T f(r,x_r^{t,y,v},\overline{V}(r,x_r^{t,y,v}), (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v}), v_r)dr \\ &\quad + \int_s^T g(r,x_r^{t,y,v},\overline{V}(r,x_r^{t,y,v}), (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v}))d\overleftarrow{B}_r \\ &\quad + (\overline{V}(s,x_s^{t,y,v}) - \overline{y}_s^{t,y,v}) - \int_s^T (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v})dW_r\end{aligned} \tag{64}$$

Here, note that $\overline{V}(s,x_s^{t,y,v}) - \overline{y}_s^{t,y,v} \geq 0$, By the comparison theorem for BDSDEs, we conclude that the solution of the BDSDE (14) is bounded by $\overline{V}(r,x_r^{t,y,v})$. Therefore, we have

$$\overline{V}(s,x_s^{t,y,v}) \geq y_s^{s,x_s^{t,y,v},v} \tag{65}$$

Let us make the same change of variable $y = \hat{x}_r^{t,x,v'}$ in each term of (55), then (55) becomes

$$\begin{aligned}&\int_{R^n} \overline{V}(s,x_s^{t,y,v'})\varphi(y)dy - \varepsilon \\ &\leq \int_{R^n} \Phi(x_T^{t,y,v'})\varphi(y)dy + \int_s^T\int_{R^n} (f(r,x_r^{t,y,v'},\overline{V}(r,x_r^{t,y,v'}), \\ &\quad (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v'}), v'_r), \varphi(y))dydr \\ &\quad + \int_s^T\int_{R^n} (g(r,x_r^{t,y,v'},\overline{V}(r,x_r^{t,y,v'}), (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v'})), \varphi(y))dyd\overleftarrow{B}_r \\ &\quad - \int_s^T\int_{R^n} (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v'})\varphi(y)dydW_r\end{aligned}$$

Since φ is arbitrary, we have proven that for almost every y

$$\begin{aligned}&\overline{V}(s,x_s^{t,y,v'}) - \frac{\varepsilon}{\int_{R^n}\varphi(y)dy} \\ &\leq \Phi(x_T^{t,y,v'}) + \int_s^T f(r,x_r^{t,y,v'},\overline{V}(r,x_r^{t,y,v'}), (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v'}), v'_r)dr \\ &\quad + \int_s^T g(r,x_r^{t,y,v'},\overline{V}(r,x_r^{t,y,v'}), (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v'}))d\overleftarrow{B}_r \\ &\quad - \int_s^T (\sigma^*\nabla\overline{V})(r,x_r^{t,y,v'})dW_r\end{aligned} \tag{66}$$

Let $\widetilde{V}(s,x_s^{t,y,v'}) = \overline{V}(s,x_s^{t,y,v'}) - \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}$. Then

$$\begin{aligned}&\widetilde{V}(s,x_s^{t,y,v'}) + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy} \\ &\leq \Phi(x_T^{t,y,v'}) + \int_s^T f(r,x_r^{t,y,v'},\widetilde{V}(r,x_r^{t,y,v'}) \\ &\quad + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}, (\sigma^*\nabla\widetilde{V})(r,x_r^{t,y,v'}), v'_r)dr \\ &\quad + \int_s^T g(r,x_r^{t,y,v'},\widetilde{V}(r,x_r^{t,y,v'}) + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}, (\sigma^*\nabla\widetilde{V})(r,x_r^{t,y,v'}))d\overleftarrow{B}_r \\ &\quad - \int_s^T (\sigma^*\nabla\widetilde{V})(r,x_r^{t,y,v'})dW_r + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}\end{aligned} \tag{67}$$

Define

$$\begin{aligned}K_s^{t,y,v'} &= \Phi(x_T^{t,y,v'}) + \int_s^T f(r,x_r^{t,y,v'},\widetilde{V}(r,x_r^{t,y,v'}) \\ &\quad + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}, (\sigma^*\nabla\widetilde{V})(r,x_r^{t,y,v'}), v'_r)dr \\ &\quad + \int_s^T g(r,x_r^{t,y,v'},\widetilde{V}(r,x_r^{t,y,v'}) + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}, (\sigma^*\nabla\widetilde{V})(r,x_r^{t,y,v'}))dBr \\ &\quad - \int_s^T (\sigma^*\nabla\widetilde{V})(r,x_r^{t,y,v'})dW_r + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}\end{aligned} \tag{68}$$

By (67), we can know that

$$\begin{aligned}&\widetilde{V}(s,x_s^{t,y,v'}) + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy} \\ &= \Phi(x_T^{t,y,v'}) - (K^{t,y,v'} - \widetilde{V}(s,x_s^{t,y,v'}) - \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}) \\ &\quad + \int_s^T f(r,x_r^{t,y,v'},\widetilde{V}(r,x_r^{t,y,v'}) + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}, (\sigma^*\nabla\widetilde{V})(r,x_r^{t,y,v'}), v'_r)dr \\ &\quad + \int_s^T g(r,x_r^{t,y,v'},\widetilde{V}(r,x_r^{t,y,v'}) + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}, (\sigma^*\nabla\widetilde{V})(r,x_r^{t,y,v'}))d\overleftarrow{B}_r\end{aligned}$$

$$-\int_s^T (\sigma^*\nabla\widetilde{V})(r, x_r^{t,y,v'})dW_r + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy} \tag{69}$$

Because $K_s^{t,y,v'} - \widetilde{V}(s, x_s^{t,y,v'}) - \frac{\varepsilon}{\int_{R^n}\varphi(y)dy} \geq 0$, so by the comparison theorem of BDSDEs, we know that

$$\widetilde{V}(s, x_s^{t,y,v'}) + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy} \leq y_s^{s,x_s^{t,y,v'},v'} + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy}$$

So

$$\overline{V}(s, x_s^{t,y,v'}) \leq y_s^{s,x_s^{t,y,v'},v'} + \frac{\varepsilon}{\int_{R^n}\varphi(y)dy} \tag{70}$$

Finally combining (65) and (70), we know that

$$\overline{V}(t,y) = \sup_{v\in\mathcal{U}} y_t^{t,y,v}$$

Thus $\overline{V}(t,y)$ is also the value of $\sup_{v\in\mathcal{U}} J(t,y,v)$, from uniqueness of the solution of cost functional and the uniqueness of supremum, we get uniqueness of weak solution for SPDE (17), i.e. $\overline{V}(t,x) = V(t,x)$. □

Declaration of competing interest

The authors declare that they have no conflicts of interest in this work.

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