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Managing Volunteers and Paid Workers in a Nonprofit Operation

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Abstract. Some nonprofit organizations (NPOs) manage a complex workforce composed of a mix of volunteers, part-time workers, and full-time workers. We study the NPO's finite-horizon staffing problem to determine the optimal initial staff planning decisions and per period optimal hiring and assignment decisions given a budget, capacity constraints, and an uncertain supply of volunteers and part-time workers. Our main goal is to solve this problem in a way that is effective and easy to implement while obtaining interesting managerial insights. To this end, we first demonstrate that the optimal staffing policies are computationally challenging to identify in general. However, we demonstrate that a prioritization assignment policy and a hire-up-to policy for part-time workers can be conveniently applied and are close to optimal. These policies are, in fact, optimal under staff scarcity and staff sufficiency. In our numerical analysis, we study the value and impact of the general optimal solution that considers flexibility and turnover of part-time workers versus the prioritization assignment policy and a constant hire-up-to policy that omit flexibility and turnover behaviors. We further suggest two easy-to-implement heuristics and theoretically analyze them and run a numerical performance study. We observe that both heuristics have low relative optimality gaps. Finally, we extend our analysis by studying how the optimal policy varies under three different practical considerations: a concave social value objective, nonzero volunteer costs, and dynamic volunteer behaviors.

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1. Introduction

Nonprofit organizations (NPOs) experience fundamentally different staffing problems compared with those faced by for-profit organizations (FPOs). The retention of paid employees and the management of volunteers are two of the major challenges of workforce management faced by NPOs (Hatten 1982, Berenguer and Shen 2020). Paid NPO workers have lower salaries and fewer benefits than FPO workers, and so turnover rates are generally higher in the nonprofit sector (Nonprofit HR 2016). NPOs typically engage volunteers largely because they are unpaid, but there is no guarantee that volunteers will always be available. Furthermore, because of different levels of experience and training, the service quality of volunteers is lower on average than the service quality of paid workers for the same nonprofit activity (Brudney and Duncombe 1992). Nevertheless, volunteers comprise a significant part of the workforce. A study across 16 countries (Salamon et al. 2013) reveals that volunteers represent on average 2.2% of the total workforce and generate on average 0.9% of the gross domestic product. In the United States, the nonprofit sector is the third largest single workforce (including paid workers and volunteers), behind retail and manufacturing, representing 10% of the total workforce in 2010 (Lambert 2013).

A typical NPO staffing problem has a mix of full-time workers, part-time workers, and volunteers. Our goal is to analyze how to best staff nonprofit activities, which can include a mix of paid workers and volunteers. Full-time workers are employees who are paid an annual salary and usually are required to work around forty hours per week. Part-time workers are paid based on hours or shifts worked and can be assigned variable time per week. Volunteers are unpaid workers and are not obliged to work.

For example, Noble of Indiana (Noble) supports individuals with disabilities. A few of Noble's services, such as special one-day events or summer day camps, are run by a mix of different types of workers. Noble's managers invite volunteers to help them staff day camps because this activity has a stringent budget and does not require advanced skills. At the same time, Noble experiences a large turnover of part-time workers due to low hourly wage and high competition. However, the supply of volunteers is not an issue for Noble because there is always a good pool of high school and college students willing to donate their time. Our numerical experiments are based on Noble's summer camp setting.

1.1. Main Contributions

This study contributes to the small body of operations management (OM) literature on management of volunteers in NPOs. The two most striking characteristics of volunteers are that they are unpaid and that supply is uncertain due to the free nature of volunteerism. This paper is also one of few works in OM that addresses NPO staffing issues and their implications. Nonprofit operations differ from for-profit ones because they have a not-for-profit objective, which is usually focused on maximizing service, and stringent budget limitations to ensure financial sustainability to the degree possible to run the operation (Berenguer and Shen 2020).

In this paper, we build a dynamic programming model to study the NPO staffing problem with a blended workforce, based on availability of volunteers and part-time workers, budget, and resource capacity. We initially study the optimal solution and find that it does not have a closed form and the computational cost of using a discretization method to find a nearly optimal solution is extremely high. Next, we observe that a simple prioritization assignment policy and a hire-up-to policy for part-time workers are optimal under the two extreme scenarios where either the NPO has sufficient staff availability in each period (staff sufficiency scenario) or the NPO has insufficient staffing time available in every period (staff scarcity scenario). We also provide guidance on the staffing problem that decides how many full-time and/or part-time workers to initially assign. We compare the marginal social value of budget per unit of full-time worker staffing time and per unit part-time worker staffing time to identify conditions where no more than one type of paid worker should be chosen. Our numerical results are based on Noble's real setting and are initially focused on identifying particular settings where NPO managers should be aware of potential optimality gaps. We observe that the prioritization assignment policy (PAP) shows larger optimality gaps in three settings: When service quality of volunteers is high, when volunteer staffing time availability is in the middle range, or when budget level is in the middle range. In contrast, a constant hire-up-to policy (CHP) has a larger optimality gap when there is low volunteer supply, low volunteer service quality, or high hiring costs. We then suggest two easy-to-implement heuristic policies: one composed of the PAP and adjusted CHP policies (PCH) and one based on linear programming (LPH). Our theoretical performance bounds and numerical results suggest that both heuristics perform well relative to the optimal policy, however LPH shows more robustness in nonstationary scenarios. The advantages of PCH are its ease of implementation over LPH and its particularly good performance when PAP is optimal or close to optimal. Finally, we extend our model to consider some specific practical situations.

1.2. Outline

The rest of this paper is organized as follows. We review the related literature in the next section. Section 3 provides a description of the main features and assumptions underlying the problem and lays out the objective and constraints of the model. In Section 4, we analyze the optimal policy, including its monotonicity results, and the assignment, hiring, and staff planning decisions under general and particular cases. In Section 5, we provide numerical experiments, managerial insights, and analysis of the theoretical and empirical performance of two easy-to-implement heuristics. Next, in Section 6, we discuss extensions of our main model. We conclude the paper in Section 7. The proofs of all formal results are organized in the online appendices.

2. Literature Review

This paper is related to three streams of literature: workforce management in OM; resource management in nonprofit operations; and management of volunteers.

2.1. Workforce Management in Operations Management

This review focuses on two (sub)streams of this extensive research: flexible workers and hiring/staff turnover decisions.

Flexible workers can include contingent, part-time, temporary, and contract workers who provide staffing flexibility to their employers. Most of the work on flexible workers studies staffing and scheduling problems in the presence of both flexible and full-time workers.

On the modeling side, some work uses stochastic dynamic programming (SDP) as the modeling tool of choice, as we do. For example, Pinker and Larson (2003) use SDP models to study staffing decisions and pool sizing decisions for full-time workers, contingent workers, and overtime to minimize total wage cost. Their paper considers uncertain demand; backlogging of unmet demand; and absenteeism. In contrast, we study staffing decisions, but overtime and absenteeism are not considered because they are not typical practices for NPOs. Milner and Pinker (2001) study labor supply contracts between an employment agent supplying contingent workers and full-time employers, under productivity uncertainty and supply uncertainty of contingent workers. They assume that the productivity of temporary workers is lower than that of full-time and contract workers. In our paper, we consider the difference in work quality between full-time workers, part-time workers, and volunteers. Management of flexible workers has also been studied from the lens of queueing theory (Bhandari et al. 2008, Dong and Ibrahim 2020). For example, Dong and Ibrahim (2020) study the selection between full-time and flexible workers, where the latter can choose not to show up even if they are assigned a job. They characterize flexible workers in a way that is similar to our paper, except that our volunteer wage cost is negligible.

There is work in empirical OM that studies workforce management with flexible workers. For example, Kesavan et al. (2014) use data from a large retailer to study the relationship between flexible labor resources and financial performance. Their result shows a clear difference between full-time and flexible workers' performance in a for-profit setting. In the context of nursing homes, which are usually NPOs, Bourbonniere et al. (2006) and Bae et al. (2010) find that using flexible workers is associated with decreased service quality. In our paper, we assume the service quality of part-time workers is on average lower than that of full-time workers due to differences in experience and attitude.

The second pool of workforce management literature in OM relates to workforce hiring and turnover decisions. Some researchers study the classic control problem of hiring, firing, and promoting workers to maintain appropriate staff levels (Grinold and Stanford 1974, Gaimon and Thompson 1984). Pinker and Shumsky (2000) construct a general staffing model, including the service process, turnover and career paths of workers, and relationship between service quality and learning time, to study the trade-off between the cost efficiency provided by cross-trained workers and the experience-based quality provided by specialists. Gans and Zhou (2002) study the employee staffing problem with a random nonstationary service requirement in which employees experience learning and turnover. They assume employee turnover only depends on a person's state and not on how many periods the person has been in that state (i.e., it is memoryless). They show that a state-dependent hire-up-to policy is optimal. Ahn et al. (2005) study the hiring and firing of heterogeneous workers who randomly leave. They find, when the number of works is allowed to be noninteger, that the hire-up-to/fire-up-to policy is optimal but that this claim may not hold in the discrete case. In our paper, we model turnover behavior like Gans and Zhou (2002) and allow the number of workers to vary continuously.

2.2. Resource Management in Nonprofit Operations

There is a growing literature on resource management in nonprofit operations (Berenguer et al. 2017, Devalkar et al. 2017). Here, we highlight papers that use similar dynamic modeling tools as the ones we adopt and that also incorporate budget or funding constraints. Lien et al. (2014) study a sequential resource allocation problem for an NPO delivering perishable food from donors to agencies. They use a dynamic programming model and assume random food donations and random demand to study allocation policies that balance equity in fill rates and efficiency. De Véricourt and Lobo (2009) focus on resource allocation between for-profit and not-for-profit activities under a given budget. They also use a dynamic program to maximize the social-impact-to-go objective, which is the sum of the weighted assets spent on not-for-profit and for-profit activities subject to resource constraints. The budget use and objective in our model are similar to their resource use and objective, except that our budget is allocated to two types of staff. In humanitarian operations, both Natarajan and Swaminathan (2014) and Natarajan and Swaminathan (2017) study inventory management problems in the presence of funding constraints, where funding is uncertain and periodically donated over finitely many periods. In our paper, funding is deterministic and given at the beginning of the time horizon.

2.3. Management of Volunteers

The third stream of related work is about the management of volunteers in OM which, to the best of our knowledge, is contained in the following published articles that we relate to our work. Wisner et al. (2005) find that volunteer satisfaction is positively related to interaction with clients, paid staff, and other volunteers, as well as to the volunteer's intent to remain in service and to recommend volunteering for the organization to others. Sampson (2006) optimizes volunteer assignments using mathematical programming and empirical data, and finds solutions that significantly differ from those of the traditional labor assignment problem. The model assumes that the wage cost for volunteers is negligible and that the number of volunteers available to be recruited is limited. Mayorga et al.

(2017) and Lassiter et al. (2015) consider dynamical volunteer assignment problems in disaster relief. The former uses a queueing model to maximize the discounted reward with uncertain volunteer supply and some deterministic demands. The latter uses a robust optimization approach to minimize the cumulative unmet demand and maximize volunteers' preference with uncertain volunteer supply and demand. Falasca et al. (2011) develop a scheduling model for a small development organization's volunteer workforce. Their objectives are to reduce unfilled shifts, schedule costs, and undesired assignments. They assume that volunteer costs are nontrivial because there are payment allowances. In our paper, we assume volunteer wage cost to be negligible, but we also study the nontrivial case as an extension. Sönmez et al. (2015) and Ata et al. (2019) study the staffing decisions of food banks that use volunteers to glean food on donated farms. Sönmez et al. (2015) develop a stochastic optimization model for the schedule of volunteers. Ata et al. (2019) construct a queueing model that regards volunteers as servers. In their problem, volunteers are the only staff to work, and there is no budget constraint. Urrea et al. (2019) study charity storehouse operations that run entirely on volunteer efforts to prepare orders. They use simulation to study the storehouse congestion control problem and the problem of pairing volunteers who have high or low experience levels. Outside OM, interesting papers can be found in sociology (Netting et al. 2004) and in the volunteer and human resources management literature, where most of the work is empirical or case based. We highlight Brudney and Duncombe (1992) and Duncombe and Brudney (1995), who study the optimal mix of volunteers and paid workers using data from municipal fire departments, with three different staffing options: all-paid, mixed-paid, and all-volunteer. In particular, Brudney and Duncombe (1992) find that the departments using paid workers provide better fire protection quality than volunteer departments, which conforms to our assumption on the differences in service quality between staff types.

3. Model Description

Our problem is a finite horizon periodic staff hiring and assignment problem. In period $t=0$, there is an initial staffing problem where the number of full-time workers and the initial number of part-time workers are chosen. Let $[T] := \{1, 2, \dots, T\}$ for any integer $T \geq 1$. Then, the next T periods (indexed by $[T]$) consist of part-time hiring and staff assignment decisions. Each of these periods represents one or two weeks, which is the frequency at which hiring decisions of part-time workers are made. Any remaining cash is returned in the terminal period $T+1$. We begin by describing the major supply (i.e., staff and budget) and demand features; then, we describe the sequence of events (Section 3.1) and present the staffing problem's objective and constraints (Section 3.2).

Our nonprofit operation consists of a mix of volunteers, part-time, and full-time workers. Our problem centers around the management of staffing time, which is a more granular unit of measurement than number of employees, with full-time worker staffing time (FST); part-time worker staffing time (PST); and volunteer staffing time (VST). Full-time workers are employees who are paid a salary, so FST typically corresponds to a fixed quantity. Part-time workers are paid based on their staffing time (e.g., the number of hours or the number of shifts worked), so PST corresponds to a variable quantity, which allows for complete flexibility of PST needed in each period. We let w_f and w_p denote the unit wage cost for FST and PST, respectively. We consider both cases $w_f \geq w_p$ and $w_f \leq w_p$ (Mocan and Tekin 2003). The wage cost absorbs all aspects of compensation, including benefits and other direct personnel costs. In our main model, VST is costless, but we consider various practical settings where VST also incurs costs.

We let n_f^t and n_p^t be the availability for FST and PST in period $t \in [T]$, respectively. We assume that full-time workers do not resign so that the level of FST in every period $t \in [T]$ (n_f^t) is constant, that is, $n_f^t = n_f$ for all $t \in [T]$. Thus, for each period $t \in [T]$, the wage cost for full-time workers is constant at $w_f n_f$. For part-time workers, the decision variable x_p^t represents how much PST is used, where $x_p^t \leq n_p^t$. Then, the wage cost for part-time workers is $w_p x_p^t$. Similarly, for each period $t \in [T]$, x_v^t represents how much VST is used, and y_p^t is the PST-to-hire.

Volunteers. NPOs commonly create and maintain a volunteer pool (Ata et al. 2019). Thus, we assume a constant total available VST of n_v for the entire planning horizon. Because volunteers are not obliged to offer service, the available VST is uncertain, as it is based on volunteers' intentions. However, the manager can observe it by communicating with volunteers. Thus, the available VST at each period $t \in [T]$ is a stochastic proportion of available VST $\tilde{s}^t n_v$, where $\tilde{s}^t$ is a stochastic process with support on $[s_l, s_u] \subseteq [0,1]$ for each period $t \in [T]$ and has cumulative distribution function (CDF) $F_{\tilde{s}^t}(\cdot)$. Because the wage cost for volunteers is negligible (Sampson 2006), this cost is assumed to be zero in our model. We study the case with nonzero volunteer wage cost in Section 6.2.

Service Quality. Because of different levels of experience and attitudes (Cole 1993), the service quality of the three types of staff is assumed to be different. Full-time workers' service quality is the highest (γ_f), followed by that of part-time workers (γ_p), and then volunteers' service quality (γ_v), which is the lowest (i.e., $1 \geq \gamma_f \geq \gamma_p \geq \gamma_v \geq 0$). We assume that service quality is linear in utilized staffing time, and we call the expression $\gamma_f x_f + \gamma_p x_p + \gamma_v x_v$ the "social value function."

Periodic Turnover and Hiring. Full-time workers have long-term contracts with the NPO and high commitment, which implies a stable supply of FST. In contrast, as already mentioned and as a general observable trend in any for-profit or nonprofit industry (Peters et al. 1981, Wotruba 1990), part-time workers have higher turnover rates because of low hourly wages and high competition. Thus, we assume that part-time workers may resign but are required to inform their managers one period in advance. We assume that turnover is not related to how many periods the worker has worked. Let the turnover rate $\tilde{q}^t$ be an independent stochastic process with support on $[q_l, q_u] \subseteq [0,1]$, which represents a "stochastic proportion" of part-time workers who resign in period t. The distribution of $\tilde{q}^t$ is allowed to be time dependent and may vary for different t. This characterization of turnover behavior has already been established (Gans and Zhou 2002). This characterization has also appeared in similar settings, for example, to model reneging behavior of patients on a waitlist (Huh et al. 2013). Regarding hiring, the manager can always hire as much PST as needed. Yet, there is a one period lead time until newly hired PST becomes available. The NPO manager observes $\tilde{q}^t$ at the beginning of each period t and decides on the PST of part-time workers to hire, y_p^t, at the beginning of each period $t \in [T]$ based on the observed $\tilde{q}^t$. Hence, the available PST who keep working in period $t+1$ is $(1-\tilde{q}^t)n_p^t$, so the available PST in the following period is $n_p^{t+1} = (1-\tilde{q}^t)n_p^t + y_p^t$. The hiring cost per PST is c_p, so the total hiring cost for period $t \in [T]$ is $c_p y_p^t$. We base our linear hiring cost on the hiring cost per PST employee divided by the average number of staffing hours a PST employee works.

Budget. As described by Noble's managers, the initial budget b is given and available at the beginning of the entire planning horizon $t=0$. The budget is used for all workforce costs related to running the nonprofit activity, which includes hiring costs ($c_p y_p$) and wage costs ($w_f n_f$ and $w_p x_p$). Let $\tau \geq 0$ represent the social value of one unit of budget, so that the social value of the leftover budget b^{T+1} at the end of the planning horizon is τb^{T+1}. Sometimes the budget designated for this nonprofit activity is exclusively earmarked for this purpose; in this case, leftover budget at the end of the planning horizon cannot be used for other nonprofit activities and so $\tau = 0$. In contrast, if leftover budget can be used for other nonprofit activities (i.e., it is not earmarked), then $\tau > 0$, and it can be regarded as the social value of offering other nonprofit activities beyond the current one. We also assume that it is worth it to hire full-time ($\gamma_f - \tau w_f > 0$) and part-time ($\gamma_p - \tau w_p > 0$) workers. Otherwise, only volunteers should provide services, as observed in, for example, food gleaning (Ata et al. 2019) and food dispensation (Urrea et al. 2019).

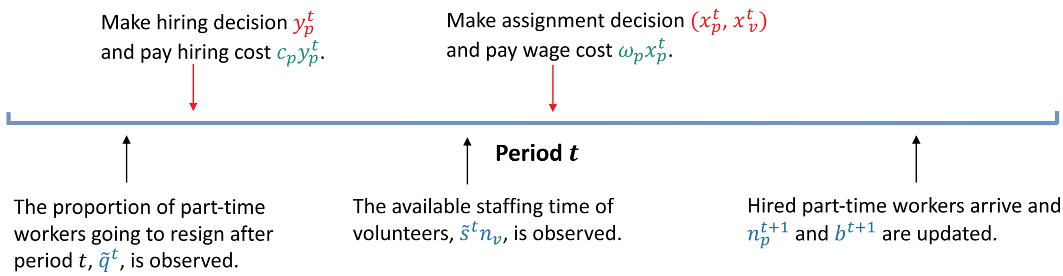
Demand and Capacity. On the demand side, NPOs usually provide service for free or at very low cost (Steinberg and Weisbrod 1998), unlike the for-profit context where demand is usually uncertain and affected by price. This could induce a great amount of demand for the NPO, which is the case for Noble, which solves this issue by creating waitlists. Often a capacity constraint (e.g., room capacity or hours of operation) is set and this is how demand is restricted. Hence, we assume that there exists a known upper bound on total possible staffing time allotted in one period for the offered nonprofit activity, $\overline{d}$, which is constant across all periods. Assigned staffing time that exceeds the upper bound $\overline{d}$ has zero social value. In some cases, we could assume that there is a lower bound on staffing time to guarantee minimal staffing levels, which could easily be incorporated into the model. In practice, however, such minimum staffing levels are usually covered by full-time workers, who are a reliable and experienced type of supply. In our analysis, we do not assume a required minimum staffing level because it does not add any interesting insights. To summarize and clarify, our model does not try to directly match supply with demand. Rather, it is centered around trying to maximize the social value given a certain amount of staffing time available (i.e., maximum capacity), where shortage with respect to the maximum staffing time is not penalized.

3.1. Sequence of Events

At period $t=0$, a one-time staff planning decision (n_f, n_p^1) is made based on the initial budget b. This decision is to determine how much FST and PST to be available at the beginning of period $t=1$. All full-time workers should be assigned in each period because the NPO has to pay a fixed wage cost for the full-time workers regardless of whether they are assigned. In addition, full-time workers have the highest service quality, so we want to avoid having idle full-time workers. Then, the total wage cost for full-time workers ($T w_f n_f$) can be deducted from the budget once n_f is determined. Hence, given the staff planning decision (n_f, n_p^1), we can ignore full-time workers and simplify the hiring and assignment problems by setting the budget to be $b^1 = b - T w_f n_f - w_p n_p^1$ and the remaining available capacity to be $d = \overline{d} - n_f$.

Now we describe the sequence of events in each period $t \in [T]$ (Figure 1). At the beginning of period $t \in [T]$, the available budget b^t and available PST n_p^t are known. Then, the part-time workers' turnover rate $\tilde{q}^t$ is observed, based on which a hiring decision for part-time workers (y_p^t) is made which incurs a hiring cost of $c_p y_p^t$. Next, the available VST ($\tilde{s}^t n_v$) is observed, and the

Figure 1. (Color online) Sequence of Events in Period $t \in [T]$



assignment decision (x_p^t, x_v^t), which incurs assignment cost $w_p x_p^t$. At the end of period $t \in [T]$, new part-time workers are hired and become available at the beginning of the next period.

3.2. Objective Functions and Constraints

Previous work on NPOs has used different objectives such as minimizing expected cost (Natarajan and Swaminathan 2014), maximizing social impact (De Véricourt and Lobo 2009), or maximizing satisfied demand (Berenguer et al. 2017). In our model, we maximize the expected social value from service plus the social value of any leftover budget.

In our staffing problem, the underlying uncertainty is due to PST turnover $(\tilde{q}^t)_{t=1}^{T-1}$ and VST availability $(\tilde{s}^t)_{t=1}^{T}$. In the initial planning stage, n_f and n_p^1 are chosen given the initial budget $b > 0$. The full-time staff level n_f is then fixed for all periods $t \in [T]$. In subsequent periods t, we determine how much VST to assign x_v^t, how much PST to assign x_p^t, and how many new part-time workers to hire y_p^t.

The state at period $t \in [T]$ is given by the current budget b^t and the available PST n_p^t. The state dynamics are $b^1 = b - T w_f n_f - c_p n_p^1$ for the initial period, $b^{t+1} = b^t - c_p y_p^t - w_p x_p^t$ for all $t \in [T]$, and $n_p^{t+1} = (1 - \tilde{q}^t) n_p^t + y_p^t$ for all $t \in [T-1]$ (where n_p^1 is a decision variable in period $t = 0$).

A (deterministic Markov) policy $\pi = (\pi^0, \pi_y^1, \pi_x^1, \ldots, \pi_y^{T-1}, \pi_x^{T-1}, \pi_x^T)$ is a collection of mappings: (i) $\pi^0 : b \to (n_f, n_p^1)$ for the initial hiring problem; (ii) $\pi_y^t : (b^t, n_p^t, q^t) \to y_p^t$ for the period $t \in [T-1]$ hiring problem; (iii) $\pi_x^t : (b^t, n_p^t, q^t, y_p^t, s^t) \to (x_p^t, x_v^t)$ for the period $t \in [T-1]$ assignment problem; and (iv) $\pi_x^T : (b^T, n_p^T, s^T) \to (x_p^T, x_v^T)$ for the period T assignment problem. We define the constraint sets $S^0(b) := \{(n_f, n_p^1) \geq 0 : T w_f n_f + c_p n_p^1 \leq b, n_f, n_p^1 \leq \overline{d}\}$ for the initial hiring problem, $S(b^t) := \{0 \leq y_p^t \leq b^t / c_p\}$ for the hiring problems, and $S^t(b^t - c_p y_p^t, n_p^t, s^t) := \{(x_p^t, x_v^t) \geq 0 : w_p x_p^t \leq b^t - c_p y_p^t,\ x_p^t + x_v^t \leq d,\ x_p^t \leq n_p^t, x_v^t \leq s^t n_v\}$ for the assignment problems. Feasible policies must then satisfy

$$(n_f, n_p^1) \in S^0(b), \quad y_p^t \in S(b^t), \ \forall t \in [T-1],$$
$$(x_v^t, x_p^t) \in S^t(b^t - c_p y_p^t, n_p^t, \tilde{s}^t), \ \forall t \in [T]. \tag{1}$$

Let Π denote the set of feasible Markov policies that respect (1). Our problem is to maximize the expected social value over the planning horizon:

$$\max_{\pi \in \Pi} \mathbb{E}^{\pi} \left[\sum_{t=1}^{T} (\gamma_p x_p^t + \gamma_v x_v^t) + \tau\, b^{T+1} \right] + T \gamma_f n_f, \tag{2}$$

where the expectation is with respect to $(\tilde{q}^t)_{t=1}^{T-1}$, $(\tilde{s}^t)_{t=1}^{T}$, and π.

4. Analysis

In this section, we study the DP decomposition of our staffing problem (Section 4.1), the optimal assignment policies (Section 4.2), the hiring policies (Section 4.3), the staff planning decisions (Section 4.4), and the impact of the random processes on the optimal value and policies (Section 4.5).

4.1. Dynamic Programming Decomposition of the Optimal Policy

We start by deriving the dynamic programming (DP) equations for Problem (2). In period $t = 0$, we make the initial staff planning decisions (n_f, n_p^1). Then in periods $t \in [T-1]$, we (i) first observe the current budget b^t, level n_p^t of available PST, and turnover $\tilde{q}^t$; (ii) make a hiring decision y_p^t for the next period; (iii) observe the level $\tilde{s}^t n_v$ of volunteers; and then (iv) make the staff assignment decisions (x_p^t, x_v^t). There is no hiring in period T, so we (i) observe the level $\tilde{s}^T n_v$ of volunteers; and (ii) then make the staff assignment decisions (x_p^T, x_v^T). In period $T+1$, we collect value from any remaining budget.

We let $\{V^t\}_{t=0}^{T+1}$ denote the optimal value functions (the expected social value-to-go), which satisfy the DP decomposition:

$$V^{T+1}(b^{T+1}, \cdot) = \tau b^{T+1}, \tag{3}$$

$$V^T(b^T, n_p^T) = \mathbb{E}_{\tilde{s}^T}\left[\max_{(x_p^T, x_v^T)\in S^T(b^T, n_p^T, \tilde{s}^T)} \gamma_p x_p^T + \gamma_v x_v^T + V^{T+1}(b^{T+1}, \cdot)\right], \quad (4)$$

$$V^t(b^t, n_p^t) = \mathbb{E}_{\tilde{q}^t}\left[\max_{y_p^t \in S(b^t)} \mathbb{E}_{\tilde{s}^t}\left[\max_{(x_p^t, x_v^t)\in S^t(b^t - c_p y_p^t, n_p^t, \tilde{s}^t)} \gamma_p x_p^t + \gamma_v x_v^t + V^{t+1}(b^{t+1}, n_p^{t+1})\right]\right], \ \forall t \in [T-1], \quad (5)$$

$$V^0(b) = \max_{(n_f, n_p^1)\in S^0(b)} V^1(b - T w_f n_f - c_p n_p^1, n_p^1; \overline{d} - n_f) + T\gamma_f n_f. \quad (6)$$

Note that $V^t(b^t, n_p^t)$ depends on the capacity $d = \overline{d} - n_f$. For simpler expressions, we drop the explicit dependence on d because $d = \overline{d} - n_f$ is fixed for all $t \in [T]$ after n_f is determined. We only indicate the dependence explicitly in the initial staffing problem and let $V^1(b^1, n_p^1; \overline{d} - n_f) \equiv V^1(b^1, n_p^1)$. Next, the decomposition equations are characterized.

Theorem 1.

(i) *For all $t \in [T+1]$, the value function $V^t(b^t, n_p^t)$ is concave and nondecreasing in b^t and n_p^t.*

(ii) *The value function $V^0(b)$ is concave and nondecreasing in b.*

The DP decomposition Equations (3)–(6) has a continuous state space and a continuous action space, and the action selection problem is a convex optimization problem. We find that the value functions and optimal policy for Equations (3)–(6) do not have closed form, and so we cannot compute the optimal policy exactly. This computational cost is extremely high, which is part of our motivation for looking for structured hiring and assignment policies, as well as heuristics. Online Appendix A1 provides further details of the DP decomposition, and Online Appendix A3 studies its computational complexity.

4.2. Assignment Policies

We study the staff planning, hiring, and assignment decisions "backward," that is, in the reverse order from the actual planning problem. Thus, we start by analyzing the assignment decisions. First, the FST assignment decision n_f is fixed in the initial period $t = 0$. Hence, we only need to assign PST and VST depending on the state $(b^t, n_p^t, \tilde{q}^t, y_p^t, \tilde{s}^t)$. In some cases, we find that the optimal assignment policy has "assignment priority," where a certain type of staffing time should be fully assigned before assigning the other type. To indicate these priorities, we let P and V denote part-time workers and volunteers, respectively. In the following definition, the optimal decisions are denoted by a superscript $*$.

Definition 1. *P dominates V* or $P \succ V$ if it is optimal to first fully assign PST and then fully assign VST, up to the capacity constraint. Explicitly, $x_p^{t*} = \min\{n_p^t, d, (b^t - c_p y_p^t)/w_p\}$ and $x_v^{t*} = \min\{(d - x_p^{t*})^+, \tilde{s}^t n_v\}$.

When $P \succ V$, we first fully assign PST depending on the available PST, capacity, and budget. If there is remaining capacity after fully assigning PST, it is optimal to then assign as much VST as possible up to the available VST. The assignment mechanism for the reverse priority is described next.

Definition 2. *V dominates P* or $V \succ P$ if it is optimal to first fully assign VST and then fully assign PST, up to the capacity and/or budget constraints. Explicitly, $x_p^{t*} = \min\{(d - \tilde{s}^t n_v)^+, n_p^t, (b^t - c_p y_p^t)/w_p\}$ and $x_v^{t*} = \min\{d, \tilde{s}^t n_v\}$.

We can now characterize the optimal assignment decisions in terms of these priorities.

Proposition 1 (Assignment Policy). *For each $t \in [T]$, given $\tilde{q}^t$ and $\tilde{s}^t$, there exists an optimal assignment decision (x_p^{t*}, x_v^{t*}) and*

(i) *If $\gamma_v \geq \gamma_p - \tau w_p$, then $V \succ P$;*

(ii) *if $\gamma_v < \gamma_p - \tau w_p$ and $\tilde{s}^t n_v + \min\{n_p^t, (b^t - c_p y_p^t)/w_p\} \leq d$, then $P \succ V$; and*

(iii) *If $\gamma_v < \gamma_p - \tau w_p$ and $\tilde{s}^t n_v + \min\{n_p^t, (b^t - c_p y_p^t)/w_p\} > d$, then $x_p^{t*} + x_v^{t*} = d$ holds.*

This result implies that, if VST has larger net social value than PST (i.e., $\gamma_v \geq \gamma_p - \tau w_p$), it is optimal to fully assign all the VST first because the cost of assigning volunteers (which is zero) does not reduce the social value gained in the future. In contrast, if PST has larger net social value, then the optimal assignment depends on the constraint $\tilde{s}^t n_v + \min\{n_p^t, (b^t - c_p y_p^t)/w_p\} \leq d$. If this constraint is satisfied (Condition (ii)), then $P \succ V$ is optimal. Otherwise (Condition (iii)), it may be better to use VST to replace some PST to save some budget for the future. There could be future periods with such low $\tilde{s}^t$ that $\tilde{s}^t n_v + \min\{n_p^t, (b^t - c_p y_p^t)/w_p\} < d$ may hold. Here, the optimal assignment decision (x_p^{t*}, x_v^{t*}) depends on the state $(b^t, n_p^t, \tilde{q}^t, y_p^t, \tilde{s}^t)$ in general, but the capacity is fully used (i.e., $x_p^{t*} + x_v^{t*} = d$). However, if there is sufficient staff, that is, $\tilde{s}^t n_v + \min\{n_p^t, (b^t - c_p y_p^t)/w_p\} \geq d$ holds for all t, then it is not necessary to save any budget for the future and $P \succ V$ is optimal.

The optimal assignment policy is simple to apply, except under Condition (iii) in Proposition 1 where it does not have a closed form expression. We propose an alternative easy-to-implement policy called the prioritization assignment policy. It simplifies the optimal policy by following $P \succ V$ under Condition (iii).

Definition 3 (Prioritization Assignment Policy). Under the *prioritization assignment policy* (PAP): if $\gamma_p - \tau w_p \le \gamma_v$, then $V \succ P$; and if $\gamma_p - \tau w_p > \gamma_v$, then $P \succ V$.

Next, we observe that this simple assignment policy is optimal under two scenarios: the cases of sufficient and scarce staff. These two extreme scenarios correspond to relaxing the staff availability constraint and the capacity constraint in S^t, respectively. The staff sufficiency scenario occurs when the operation is run with sufficiently large staffing time availability in each period. On the contrary, the staff scarcity scenario occurs when the operation has insufficient staffing availability levels at each period. Both scenarios are formally defined next.

Definition 4. We have *staff sufficiency* when $\tilde{s}^t n_v + \min\{n_p^t, (b^t - c_p y_p^t)/w_p\} \ge d$ for all $t \in [T]$.

There are two examples within the staff sufficiency scenario specific to each type of staff that can be posed as sufficient conditions. The *PST sufficiency example* when $b/(Tw_p + Tq_u c_p) \ge d$, where q_u is the highest turnover rate over all periods and the *VST sufficiency example* when $s_l n_v \ge d$, where $s_l n_v$ is the lowest volunteer supply over all periods.

The second scenario is the opposite extreme where the available PST and VST are below the capacity in every period.

Definition 5. We have *staff scarcity* when $\tilde{s}^t n_v + \min\{n_p^t, (b^t - c_p y_p^t)/w_p\} < d$ for all $t \in [T]$.

In this scenario, the capacity constraint is never binding, and PAP is still optimal. Here, the assignment decisions related to PST and VST are separable, and indeed, it is optimal to assign all available VST and PST in each period.

Corollary 1. *The PAP policy is optimal under staff sufficiency and staff scarcity scenarios.*

Finally, we observe that if the assignment decisions always follow $P \succ V$, then the part-time workers' flexibility is not being used except for the period when the budget is exhausted. In Section 5.1.1, we further study the value and impact of the flexibility of part-time workers.

4.3. Hiring Policies

Part-time hiring decisions are made via Equation (5) and depend on the initial staff planning decisions. The budget is shared by hiring and wage payments, so if more budget is used for hiring then less budget is available for the staff assignment that directly contributes to social value. We thus need to consider all state variables $(b^t, n_p^t, \tilde{q}^t)$ (including the current budget) to balance budget allocation between hiring and assignment.

Proposition 2 (Hiring Policy). *For each period $t \in [T]$, the objective function of the hiring decision is concave and there exists a state-dependent optimal hiring decision $y_p^{t*}(b^t, n_p^t, \tilde{q}^t)$.*

In the general setting, the optimal hiring policy $y_p^{t*}(b^t, n_p^t, \tilde{q}^t)$ is complex and does not have a closed form. Even under the case of deterministic turnover and volunteer availability, the hiring policy does not have closed form. Rather, a linear program needs to be solved (which gives rise to one of our suggested heuristics). Similar to what we did for the assignment policies, next we define two types of hiring policies that are simple and easy-to-implement.

Definition 6 (Hire-up-to Policy). In a hire-up-to policy, there is an n_p^{t+1*} such that the optimal PST to hire is $y_p^{t*} = \max\{0, n_p^{t+1*} - (1 - \tilde{q}^t) n_p^t\}$.

Definition 7 (Constant Hire-up-to Policy). A constant hire-up-to policy (CHP) is a hire-up-to policy where $n_p^{t*} = n_p^{1*}$ for all $t \in [T]$ and n_p^{1*} is the initial staff planning decision for PST.

We observe that if the NPO manager implements the PAP, we can prove that its respective optimal hiring policy is a state-dependent hire-up-to policy. This state dependence is simplified to one state when $P \succ V$ because the assignment of PST does not depend on the realization of volunteer supply such that the hiring is only budget dependent.

Proposition 3. *This is formally described in the following proposition.*

(i) *If the assignment decisions follow $P \succ V$, the optimal hiring decision in period $t \in [T-1]$ follows a state-dependent hire-up-to policy which depends on $b^t - w_p n_p^t + c_p(1 - \tilde{q}^t) n_p^t$.*

(ii) *If the assignment decisions follow $V \succ P$, the optimal hiring decision in period $t \in [T-1]$ follows a state-dependent hire-up-to policy which depends on (b^t, n_p^t).*

Next, we return to a particular subcase of staff sufficiency when there is *PST sufficiency* and connect it with optimal hiring policies.

Corollary 2. *If there are sufficient PST, then there exists a hire-up-to policy that is optimal. Moreover, if $(\tilde{q}^t)_{t=1}^{T-1}$ and $(\tilde{s}^t)_{t=1}^{T}$ are independent and identically distributed (i.i.d.), then a constant hire-up-to policy is optimal as $T \to \infty$, that is, there exists an n_p^* such that the optimal hire-up-to level satisfies $n_p^{t*} = n_p^*$ for all $t \in [T]$ as $T \to \infty$.*

When there are sufficient PST, we can remove the budget state variable and the budget constraint from the general staffing problem (because we always have enough budget). In this case, PAP is optimal, and the hiring problem amounts to choosing the hire-up-to level. In practice, a constant hire-up-to policy might be preferable because it can keep the available PST stable over time (and this

Table 1. Summary of Optimal Policies Under the Extreme Examples

Extreme examples		Assignment policy	Hiring policy	Staff planning policy
Staff sufficiency	Part-time sufficiency	PAP (Corollary 1)	Hire-up-to If $(\tilde{q}^t)_{t=1}^{T-1}$, $(\tilde{s}^t)_{t=1}^{T}$ i.i.d. and $T \to \infty$: Constant hire-up-to (Corollary 2)	—
	Otherwise[a]		State-dependent hire-up-to (Proposition 3)	
	Volunteers sufficiency			No more than one type of paid worker (Corollary 3)
Staff scarcity				

[a]"Otherwise" refers to all staff sufficiency scenarios that are not "part-time sufficiency" or "volunteers sufficiency."

is the optimal policy when $T \to \infty$). In Section 5.1.2, we computationally study the performance of the constant hire-up-to policy in the general setting.

4.4. Staff Planning

NPO managers initially face the staffing problem concerning the FST and the initial PST to hire. We assume that the staff planning decision (n_f, n_p^1) is made before the first planning period. We also assume that the NPO always has enough paid workers available to hire. Full-time workers are only hired in period $t = 0$ (n_f), whereas part-time workers can be hired in every period (y_p^t).

Let (n_f^*, n_p^{1*}) be the optimal staff planning decision for Equation (6). The following result concerns the existence of the optimal staff planning decisions and its characterization under some specific conditions.

Proposition 4 (Staff Planning Decision). *The function $V^1(b - Tw_f n_f - c_p n_p^1, n_p^1; \overline{d} - n_f) + T\gamma_f n_f$ is jointly concave in (b, n_f, n_p^1) and nondecreasing in b. There exists a state-dependent optimal staff planning decision (n_f^*, n_p^{1*}) depending on b.*

1. *If $\gamma_p / w_p \le \gamma_f / w_f$, then it is optimal to not hire part-time workers initially.*
2. *If $\gamma_p - \tau(w_p + c_p) \ge \gamma_f - \tau w_f$, then it is optimal to not hire full-time workers initially.*

The previous proposition states that, if the marginal social value of budget per unit of FST (γ_f / w_f) is larger than the marginal social value of budget per unit of PST (γ_p / w_p), then no part-time workers should be hired initially even when $c_p = 0$. This is because we can generate the same social value with full-time workers at lower cost compared with part-time workers. Furthermore, the remaining capacity can be used to assign VST. Similarly, if the lowest possible net social value of using PST $(\gamma_p - \tau(w_p + c_p))$ is larger than the net social value of FST $(\gamma_f - \tau w_f)$, then no full-time workers should be hired.

In fact, from Proposition 4 we can identify particular cases where hiring at most one type of paid worker is optimal. These are the cases of staff scarcity and VST sufficiency.

Corollary 3. *Let $W_p(b) = \max_{n_p^1 \in [0, b/c_p]} V^1(b - w_p n_p^1, n_p^1; \overline{d})$ be the optimal expected social value from PST and VST at period 0 given b (and $n_f = 0$), and denote $W_p'(b) := \lim_{\delta \downarrow 0} W_p(b + \delta)/\delta$.*

1. *If there is staff scarcity, then $W_p'(b) = W_p'(0)$ for all $b > 0$ satisfying staff scarcity, and*
   (1-i) *It is optimal to not hire part-time workers initially when $W_p'(0) < \gamma_f / w_f$;*
   (1-ii) *It is optimal to not hire full-time workers when $W_p'(0) > \gamma_f / w_f$.*
2. *If there is VST sufficiency (i.e., $s_l n_v \ge \overline{d}$) and $(\gamma_j - \tau w_j) \ge \gamma_v$ for both $j \in \{f, p\}$, then*
   (2-i) *It is optimal to not hire part-time workers initially when $W_p'(0) \le (\gamma_f - \gamma_v)/w_f$;*
   (2-ii) *It is optimal to not hire full-time workers when $W_p'(b) > (\gamma_f - \gamma_v)/w_f$.*

Under both cases, the impact of VST on social value can be set aside such that, the marginal social value of budget for FST becomes a constant (γ_f / w_f under staff scarcity and $(\gamma_f - \gamma_v)/w_f$ under VST sufficiency). In addition, no flexibility of part-time workers would be used because of the optimality of PAP. Note that $W_p'(b)$ is the marginal social value of budget for PST and $W_p'(0) \ge W_p'(b)$ for $b > 0$ because of the concavity of $W_p(\cdot)$. Thus, we can compare $W_p'(0)$ or $W_p'(b)$ with a constant value to exclude one type of paid worker. As for the type of paid staff not excluded, whether there should be any initial hiring depends on if their marginal social value of budget is larger than τ.

In Table 1, we summarize some of the results that connect the optimal assignment, hiring, and staff planning policies with the two particular availability staffing level cases highlighted. Although the optimal policies are complex in the general case, we see that they have a simple form in these practical specific cases.

4.5. Impact of the Stochastic Processes on the Optimal Social Value and Optimal Staffing Policies

We have two sources of uncertainty in this problem: the turnover rate of part-time workers and volunteer availability. First, we study the effect of both stochastic processes on the optimal expected social value over the entire planning horizon. To start, we recall the following stochastic orders.

Definition 8 (Shaked and Shanthikumar 2006). Given two random variables X and Y: X is smaller than Y in the *increasing (decreasing) concave order*, written as $X \le_{icv} (\le_{dcv}) Y$, if $\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(Y)]$ for all increasing (decreasing) concave functions $\phi : \mathbb{R} \to \mathbb{R}$ (such that both expectations exist).

In the following proposition, we explain how the optimal expected social value changes as the distributions of the uncertainty change.

Proposition 5. *Let $V^0(b|(\tilde{q}_i^t, \tilde{s}_i^t)_{t\in[T]})$ be the optimal expected social value where the turnover rate follows $\tilde{q}_i^t$ and the volunteer availability follows $\tilde{s}_i^t$ for all $t \in [T]$, for $i = 1$, 2. If $\tilde{q}_1^t \le_{dcv} \tilde{q}_2^t$ and $\tilde{s}_1^t \le_{icv} \tilde{s}_2^t$ for all $t \in [T]$, then $V^0(b|(\tilde{q}_1^t, \tilde{s}_1^t)_{t\in[T]}) \le V^0(b|(\tilde{q}_2^t, \tilde{s}_2^t)_{t\in[T]})$ for all $b \ge 0$.*

First, we observe that more volunteers are preferred because the cost of assigning VST is zero. In addition, if the expectation of volunteer availability ($\mathbb{E}[\tilde{s}^t]$) is fixed, we prefer a lower variance of VST availability due to the capacity constraint. In fact, we prefer the distribution of VST availability to be larger in the increasing concave order. Second, we see that a higher PST turnover may drive more budget to be allocated to hiring which does not generate social value directly. Similarly, if the distribution of the PST turnover rate is higher in the decreasing concave order, then the optimal expected social value is greater.

Second, we study monotonicity of the staffing problem. The optimal assignment decisions are not monotonic in $\tilde{q}^t$, n_p^t, and b^t and the optimal hiring decisions are not monotonic in $\tilde{q}^t$. Counterexamples are provided in Online Appendix A13. This lack of monotonicity demonstrates the complexity of identifying the optimal policies. However, we do have monotonicity of the optimal PST and VST assignment decisions in $\tilde{s}^t$.

Proposition 6. *The optimal assignment decision x_p^{t*} (x_v^{t*}) is decreasing (increasing) in $\tilde{s}^t$.*

5. Managerial and Computational Discussions

In this section, we base our experiments on the real setting of Noble's summer day camp to test the suboptimality of PAP and CHP and to illustrate interesting managerial insights (Section 5.1). In Section 5.2, we develop two easy-to-implement heuristics, give performance guarantees, and then investigate their numerical performance.

In Noble's summer camp setting, staffing time is set to be one hour. We set one period to equal one week, and the planning horizon is ($T = 8$) periods. We assume $n_f = 0$ so we only employ part-time workers and volunteers. In addition, the budget for summer camps is earmarked to this activity, so $\tau = 0$. The service quality of part-time workers is constant at 0.9, but the service quality of volunteers varies from 0.1 to 0.9. Noble's manager mentioned that some of the part-time workers are volunteers from previous years, to explain why the service quality of part-time workers is higher than for volunteers. The PST wage is 12 dollars/hour and the capacity is $d = \overline{d} - n_f = 180$ hours per period. We consider two cases of the PST hiring cost: low (12 dollars) and high (48 dollars).

In Section 5.1, we first assume the stationary case where the PST turnover $\tilde{q}^t \overset{d}{=} \tilde{q}$ for all $t \in [T-1]$ and the VST availability $\tilde{s}^t \overset{d}{=} \tilde{s}$ for all $t \in [T]$ are modeled by a discretized beta distribution (Gans and Zhou 2002) with *Beta*(0.4, 7.6) and *Beta*(4, 4), respectively. Later, when studying our heuristics' performance in Section 5.2, we also study the nonstationary case. In our upcoming experiments, we assume that all assignment and hiring decisions must be nonnegative integers and all states (b^t, n_p^t) and both $(1 - \tilde{q}^t)n_p^t$ and $\tilde{s}^t n_v^t$ are rounded to the nearest integers. In addition, all hiring policies include the initial staff planning decision n_p^1 because we suppose that $n_f = 0$. We approximate the optimal policy (OP) by exactly solving a discretization of the staffing problem. Online Appendix B4 elaborates on the parameter values used in these numerical experiments. Online Appendix B5 reports further numerical experiments where $\tilde{q}^t$ and $\tilde{s}^t$ have lower variability.

5.1. Managerial Insights

In this section, we analyze two experiments (Figures 7 and 8 in the online appendix) that provide multiple managerial insights that can be useful to Noble's managers and many other NPO managers. In particular, we test the suboptimality of the simple assignment (PAP) and hiring policies (CHP) from the previous section and identify potential optimality gaps. A preview of these recommendations is summarized in Table 2.

Part-time workers have two characteristics that distinguish them from full-time workers: a high turnover rate and hourly wages (instead of a salary). The latter is a flexible payment system that helps to hedge the uncertainty about supply and demand (Milner and Pinker 2001, Pinker and Larson 2003). We study the suboptimality of policies that do not consider the flexibility and the high turnover rate of part-time workers, respectively. We also study the impact of changing budget, VST availability, and volunteer service quality levels.

5.1.1. Flexibility of Part-Time Workers. In Noble's setting, where $\tau = 0$ and $\gamma_v < \gamma_p$, Condition (ii) or (iii) of Proposition 1 is satisfied. If we follow PAP for assignment

Table 2. Summary of Cases with Larger Optimality Gaps

Policies	Particular setting[a]
PAP (i.e., not exploiting part-time flexibility by applying PAP and optimal hiring policy)	High γ_v (Remark 1) Medium level n_v (Remark 1) Midrange budget level (Remark 2)
CHP (i.e., not reacting to turnover of PST by applying the optimal assignment and CHP)	High hiring costs (Remark 3) Low γ_v (Remark 4) Low level n_v (Remark 4)

[a]Each setting is identified separately and corresponds to a condition in which PAP/CHP has a large optimality gap.

decisions and the optimal hiring policy (we call this joint assignment-hiring policy PAP), then all PST is assigned except in periods without enough budget to pay the wages. Thus, PAP is close to the policy that pays part-time workers a fixed payment, and it ignores PST flexibility.

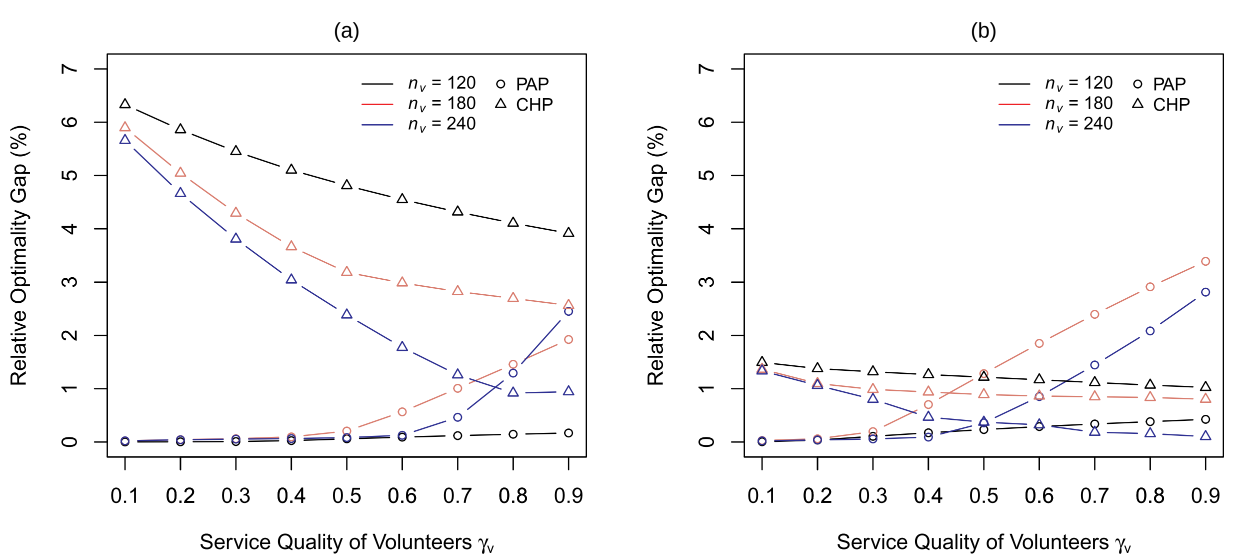
Figure 2(a) and (b), reports the optimality gaps of PAP compared with OP across different volunteer service quality levels (γ_v), different volunteer supply levels (n_v), and worst-case budget levels. We know that PAP is not optimal because OP may not assign all PST to save some budget for future hiring and assignment costs (condition (iii) from Proposition 1). Therefore, PAP may assign fewer volunteers than OP does. Hence, the relative optimality gap of PAP is mainly determined by the social value of the gap between the amount of assigned VST under OP and PAP. This explains why the relative optimality gap increases in γ_v. In Figure 2(a) and (b), we also see that the relative optimality gap of PAP is almost zero under $n_v = 120$ and 240 when $\gamma_v \leq 0.6$, because these cases are close to the staff scarcity and sufficiency cases.

From the analysis of the extreme examples in Section 4, PAP is optimal under very high n_v levels (staff sufficiency) and very low n_v levels (staff scarcity), but not necessarily for medium n_v ($n_v = 180$).

Remark 1. A prioritization assignment policy (PAP) (i.e., not exploiting part-time flexibility) can cause a larger optimality gap when volunteer service quality is higher and when VST availability is medium range.

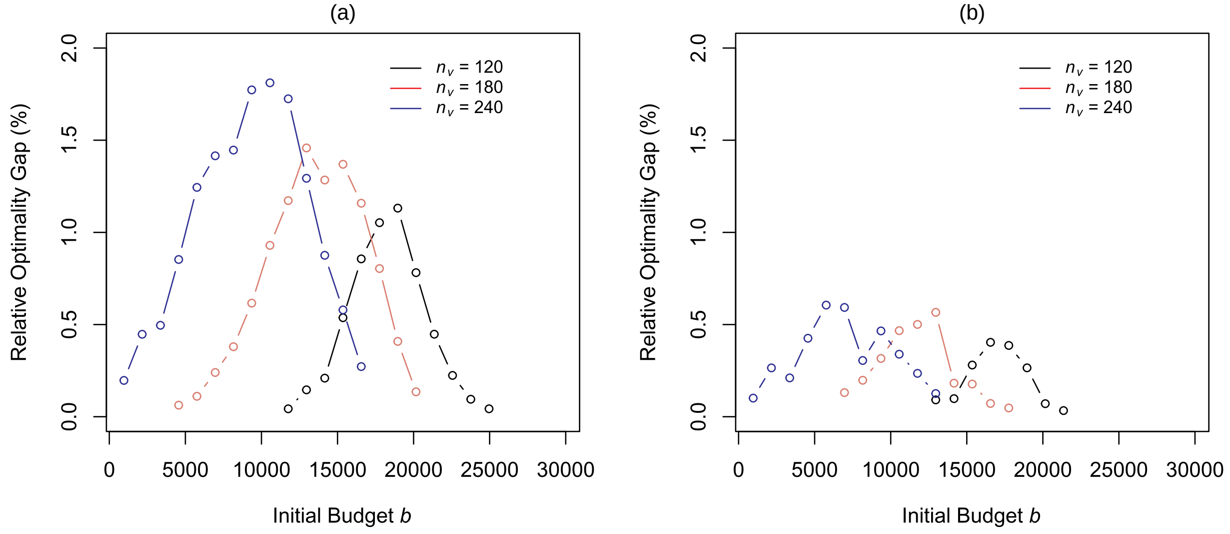
Next, we further study the impact of budget level, VST availability, and level of service quality on the performance of PAP. Figure 3 shows the relative optimality gap of implementing PAP over different budget levels and different levels of VST service quality ($\gamma_v = 0.8$ and $\gamma_v = 0.6$). In this experiment, we observe that there is no optimality gap when the budget is sufficiently low (condition (ii) of Proposition 1) or sufficiently high (the case of staff sufficiency), but the optimality gap is positive for a midrange budget level. In some periods, OP satisfies $V > P$ to save budget for future use when the VST supply is low.

Figure 2. (Color online) Performance of PAP and CHP Against the Optimal Policy ($n_v = 180$, $\tilde{q} \sim Beta(0.4, 7.6)$, $\tilde{s} \sim Beta(4, 4)$)



Notes. (a) High $c_p = 48$ and medium $b = 12{,}960$. (b) Low $c_p = 12$ and medium $b = 9{,}600$. The budget is selected by running an exhaustive search of budget levels and choosing the one that provides the maximum gap% of PAP for highest volunteer social value $\gamma_v = 0{:}8$ and medium volunteer supply $n_v = 180$.

Figure 3. (Color online) Performance of PAP over Different Budgets ($c_p = 48$, $\tilde{q} \sim Beta(0.4, 7.6)$, $\tilde{s} \sim Beta(4, 4)$)



Notes. (a) $\gamma_v = 0.8$. (b) $\gamma_v = 0.6$.

PAP can be optimal even without staff scarcity or sufficiency. For example, PAP is optimal when $b > 20{,}000$ and $n_v = 180$ for which there is neither staff scarcity nor sufficiency. Assigning only part of the available PST to save budget (i.e., not following PAP) takes the risk of PST turnover (some currently available PST may not be available in the future). Thus, PAP is only triggered when there is enough improvement in future social value from saving budget.

In Figure 3, we observe the range of budget values for which there is an optimality gap. The maximum optimality gap increases when n_v increases (condition (iii) of Proposition 1). In addition, as in Remark 1, the optimality gap for PAP is larger for higher volunteer service quality levels.

Remark 2. For midrange budget levels, PAP may be suboptimal. In addition, PAP can be optimal under other cases than staff scarcity and sufficiency.

5.1.2. Turnover of Part-Time Workers. The second policy that we study is based on the assumption that part-time workers have no turnover. The hiring decision follows a CHP, and the optimal assignment policy is used (we call this joint assignment-hiring policy CHP). Under CHP, the available PST in each period is maintained at a fixed level by paying a hiring cost, except for periods when the available budget might not be enough to hire all necessary part-time workers. Thus, CHP is close to the setting that assumes part-time workers have a higher wage but zero turnover and zero hiring cost.

Figure 2(a) and (b), reports the relative optimality gaps of CHP compared with OP. We observe that the gaps in Figure 2(a) are larger than in Figure 2(b).

Remark 3. Higher hiring costs can lead to a higher suboptimality gap under a CHP.

This insight alerts managers that when hiring costs are high, if CHP is applied where it is not actually optimal, the social value lost might be significant because an observable volume of budget dedicated to hiring costs would be misallocated. This issue can be mitigated to some extent by increasing volunteer supply, especially when γ_v is high. Volunteers can replace part-time workers to offer service, and part-time workers depend on available budget. A high volunteer supply allows more volunteers to replace part-time workers. In parallel, a high volunteer social value reduces the per unit loss in social value from replacement. Thus, we observe that the optimality gap decreases with larger γ_v and n_v.

Remark 4. CHPs (i.e., not dynamically responding to turnover behavior) can cause a larger optimality gap when volunteer supply or service quality is low.

As described in Table 2, the particular settings where PAP and CHP have large optimality gaps can be opposed to each other. For example, this occurs for high and low volunteer service quality levels (e.g., Remarks 1 and 4, where large optimality gaps occur when the volunteer quality is high for PAP and low for CHP). These opposite results can be exploited when implementing PAP and CHP together. In Section 5.2, we investigate a practical heuristic that combines both PAP and an adjusted CHP. To conclude, unlike PAP, which only experiences a confined range of relatively small optimality gaps when budget levels are medium range (Figure 3), most cases of CHP under different budget levels are not optimal due to the rigidity of this hiring policy.

5.2. Performance of Easy-to-Implement Heuristic Policies

Given the complexity of our staffing problem, in Section 5.2.1, we upper bound the expected performance of an omniscient decision maker, and in Sections 5.2.2 and 5.2.3, we suggest two practical heuristics and give lower bounds on their competitive ratios. We develop these bounds for the case $P > V$, but they can be extended to the case $V > P$ by applying similar high-level steps. Then, in Section 5.2.4, we run computational experiments to directly compare the performance of these heuristics with OP.

Let $\omega = (q^1, s^1, \ldots, q^{T-1}, s^{T-1}, q^T, s^T)$ denote a trajectory of the uncertainty (where $\tilde{q}^T = 1$ almost surely), and let ALG denote a generic online algorithm. Given budget b, $\text{ALG}(b, \omega)$ denotes the total utility earned by ALG on ω, so the expected performance of ALG is $\mathbb{E}[\text{ALG}(b, \omega)]$.

It is difficult to directly compare ALG against OP (which is hard to compute and has a complicated structure). Instead, we can compare ALG to an omniscient decision maker denoted by OPT, where $\text{OPT}(b, \omega)$ is the optimal total utility given budget b and knowledge of ω in advance. The expected performance of the omniscient decision maker is then $\mathbb{E}[\text{OPT}(b, \omega)]$, which also upper bounds the performance of any achievable policy. We compare ALG to OPT in terms of the competitive ratio $\text{COM}(b, \text{ALG}) := \mathbb{E}[\text{ALG}(b, \omega)]/\mathbb{E}[\text{OPT}(b, \omega)]$ (Stein et al. 2020).

5.2.1. Upper Bound. Let $\bar{q}^t := \mathbb{E}[\tilde{q}^t]$ and $\bar{s}^t := \mathbb{E}[\tilde{s}^t]$ for all $t \in [T]$, and let $\bar{\omega} := (\bar{q}^1, \bar{s}^1, \ldots, \bar{q}^T, \bar{s}^T)$. In the following LP, all uncertainty is replaced by its expectation (corresponding to $\bar{\omega}$):

$$\max \quad T(\gamma_f - \tau\, w_f) n_f + (\gamma_p - \tau\, w_p) \sum_{k=1}^{T} x_p^k + \gamma_v \sum_{k=1}^{T} x_v^k - \tau\, c_p \sum_{k=1}^{T-1} y_p^k \tag{7a}$$

$$\text{s.t.} \quad T\, w_f n_f + c_p \left(n_p^1 + \sum_{t=1}^{T-1} y_p^t \right) + w_p \sum_{t=1}^{T} x_p^t \le b, \tag{7b}$$

$$n_p^{t+1} = (1 - \bar{q}^t) n_p^t + y_p^t, \quad \forall t \in [T-1], \tag{7c}$$

$$x_p^t \le n_p^t,\; x_v^t \le \bar{s}^t n_v,\; x_p^t + x_v^t \le \bar{d} - n_f, \quad \forall t \in [T], \tag{7d}$$

$$n_f, (n_p^t)_{t=1}^T, (y_p^t)_{t=1}^{T-1}, (x_p^t)_{t=1}^T, (x_v^t)_{t=1}^T \ge 0.$$

Let $\text{UB}(b)$ denote the optimal value of Problem (7) as a function of b.

Theorem 2. *For any $b \ge 0$, $\text{UB}(b) \ge \mathbb{E}[\text{OPT}(b, \omega)]$.*

We then get the lower bound $\text{COM}(b, \text{ALG}) \ge \mathbb{E}[\text{ALG}(b, \omega)]/\text{UB}(b)$, which is easier to use in practice.

5.2.2. PAP-CHP Heuristic. Given their ease of implementation and the analysis of the previous section, it is natural to use a heuristic that combines PAP and CHP, which we call PAP-CHP heuristic (PCH) that operates as follows:

- The capacity d is given (e.g., from the optimal solution of Problem (7) with $d = \bar{d} - n_f^*$).
- The assignment decisions follow PAP as defined in Definition 3.
- The hiring decisions follow an adjusted CHP under the assumption that the future turnover rate is deterministic and equal to $\bar{q}^t$ for all $t \in [T-1]$. Let $\text{HC}_t(n) := c_p \sum_{k=t+1}^{T-1} \bar{q}^k n + c_p[n - (1 - q^t) n_p^t]$ be the remaining hiring costs as a function of the constant hire-up-to level $n \ge 0$, current turnover q^t, and forecast turnover $\{\bar{q}^k\}_{k=t+1}^{T-1}$. The constant hire-up-to level is adjusted based on b^t in each period $t \in [T-1]$ so that $n_p^{t+1} = \min\{\max\{n, (1 - q^t) n_p^t\}, d\}$, where n is the solution of the equation $b^t = w_p[n_p^t + (T - t) n] + \text{HC}_t(n)$ if $P > V$ or $b^t = w_p \mathbb{E}[\min\{n_p^t, (d - \tilde{s}^t n_v)^+\}] + \sum_{k=t+1}^{T} \mathbb{E}[\min\{n, (d - \tilde{s}^k n_v)^+\}] + \text{HC}_t(n)$ if $V > P$. PCH budgets for all future PST wages through the term $(T - t)n$ when it makes hiring decisions. Then, we have $b^{t+1} = b^t - w_p n_p^t - c_p(n_p^{t+1} - (1 - q^t) n_p^t)$.
- We solve for the initial PST staff planning decision n_p^1 (using $\bar{q}^1$) in the above equations by setting $n_p^0 = 0$ and $q^0 = 0$.

Let $\text{PCH}(b, \omega)$ denote the total utility earned by PCH given budget b on ω, so its expected performance is $\mathbb{E}[\text{PCH}(b, \omega)]$. Next we give a lower bound on $\text{COM}(b, \text{PCH})$. Define $C_t := c_p d (T - t)/(w_p (T - t) + c_p \sum_{k=t+1}^{T} \bar{q}^k)$ for all $t \in [T-1]$, and also define $L := \max\{\{|\gamma_p - \tau (w_p + \bar{q}^t c_p)|\}_{t \in [T]}, \gamma_v\}$.

Theorem 3 (PCH Competitive Ratio). *Suppose $P > V$. For all $b \ge 0$, we have*

$$\text{COM}(b, \text{PCH}) \ge \left(\text{PCH}(b, \bar{\omega}) - L \sum_{t=1}^{T-1} C_t \mathbb{E}[|\tilde{q}^t - \bar{q}^t|] - \gamma_v n_v \sum_{t=1}^{T-1} \mathbb{E}[\max\{\bar{s}^t - \tilde{s}^t, 0\}] \right) \Big/ \text{UB}(b).$$

5.2.3. Linear Programming Heuristic. Our second heuristic is based on solving a sequence of LP problems. In all these LPs, future PST turnover and future VST availability are forecast by their expected values. We refer to this heuristic as the linear programming heuristic (LPH), which is implemented as follows. Let $[t, T] := \{t, t+1, \ldots, T\}$.

- In period $t = 0$, solve Problem (7) to determine n_f, n_p^1, and the capacity $d = \bar{d} - n_f$.

- In period $t \in [T-1]$ in state (b^t, n_p^t), with observed PST turnover q^t, solve

$$\max \quad (\gamma_p - \tau\, w_p)\sum_{k=t}^{T} x_p^k + \gamma_v \sum_{k=t}^{T} x_v^k - \tau\, c_p \sum_{k=t}^{T-1} y_p^k \tag{8a}$$

$$\text{s.t.} \quad c_p \sum_{k=t}^{T-1} y_p^k + w_p \sum_{k=t}^{T} x_p^k \le b^t, \tag{8b}$$

$$n_p^{t+1} = (1-q^t)n_p^t + y_p^t,\ n_p^{k+1} = (1-\bar{q}^k)n_p^k + y_p^k, \quad \forall k \in [t+1, T-1], \tag{8c}$$

$$0 \le x_p^k \le n_p^k,\ 0 \le x_v^k \le \bar{s}^k n_v,\ x_p^k + x_v^k \le d, \quad \forall k \in [t, T],$$

$$(y_p^k)_{k=t}^{T-1}, (n_p^k)_{k=t+1}^{T}, (x_p^k)_{k=t}^{T}, (x_v^k)_{k=t}^{T} \ge 0, \tag{8d}$$

to determine y_p^t and n_p^{t+1}.

- In period $t \in [T]$ in state $(b^{t+1/2}, n_p^t)$ where $b^{t+1/2} = b^t - c_p y_p^t$, with observed PST turnover q^t and VST availability s^t, and n_p^{t+1} determined in the previous step, solve

$$\max \quad (\gamma_p - \tau\, w_p)\sum_{k=t}^{T} x_p^k + \gamma_v \sum_{k=t}^{T} x_v^k - \tau\, c_p \sum_{k=t+1}^{T-1} y_p^k \tag{9a}$$

$$\text{s.t.} \quad w_p \sum_{k=t}^{T} x_p^k + c_p \sum_{k=t+1}^{T-1} y_p^k \le b^{t+1/2}, \tag{9b}$$

$$n_p^{k+1} = (1-\bar{q}^k)n_p^k + y_p^k, \quad \forall k \in [t+1, T-1], \tag{9c}$$

$$0 \le x_p^t \le n_p^t,\ 0 \le x_v^t \le s^t n_v,\ x_p^t + x_v^t \le d, \tag{9d}$$

$$0 \le x_p^k \le n_p^k,\ 0 \le x_v^k \le \bar{s}^k n_v,\ x_p^k + x_v^k \le d, \quad \forall k \in [t+1, T],$$

$$(x_p^k)_{k=t}^{T}, (x_v^k)_{k=t}^{T}, (y_p^k)_{k=t+1}^{T-1}, (n_p^k)_{k=t+2}^{T} \ge 0, \tag{9e}$$

to determine (x_p^t, x_v^t).

Table 3. Examples of Competitive Ratio Lower Bounds ($c_p = 48$, $n_v = 180$, $\tilde{q} \sim Beta(0.4, 7.6)$, $\tilde{s} \sim Beta(4, 4)$)

	$b = 12{,}960$		$b = 25{,}200$	
	COM(b, PCH)	COM(b, LPH)	COM(b, PCH)	COM(b, LPH)
$\gamma_v = 0.1$	65.3%	78.6%	79.3%	88.1%
$\gamma_v = 0.8$	75.3%	81.9%	79.2%	83.5%

Let LPH(b, ω) denote the total utility earned by LPH given budget b on trajectory ω, so its expected performance is $\mathbb{E}$[LPH(b, ω)]. Next we give a lower bound on COM(b, LPH).

Theorem 4 (LPH Competitive Ratio). *Suppose $P > V$. For all $b \ge 0$, we have*

$$\mathrm{COM}(b, \mathrm{LPH}) \ge 1 - \frac{c_p d\, \gamma_p}{w_p \mathrm{UB}(b)} \sum_{t=1}^{T} \mathbb{E}[\max\{\tilde{q}^t - \bar{q}^t, 0\}] - \frac{\gamma_v n_v}{\mathrm{UB}(b)} \sum_{t=1}^{T} \mathbb{E}[\max\{\bar{s}^t - \tilde{s}^t, 0\}].$$

Table 3 gives lower bounds on COM(b, PCH) and COM(b, LPH) based on different budget levels.

5.2.4. Computational Performance.

In Figures 4 and 5, we discuss the relative performance of PCH and LPH and compare both to OP. First, we observe that PCH is run entirely by solving an equation in each period, whereas LPH requires a total of $2T$ LPs to be solved. Thus, PCH is easier to implement by the NPO manager than LPH. Nonetheless, in general, we expect LPH to perform better than PCH because PCH solutions are constrained by CHP and PAP.

To assess the performance of each heuristic, we generate 20,000 sample trajectories of ω over the entire planning horizon, where each stochastic process $((\tilde{q}^t)_{t=1}^{T-1}$ and

Figure 4. (Color online) Performance of PCH and LPH Against the Optimal Policy ($c_p = 48$, $n_v = 180$)

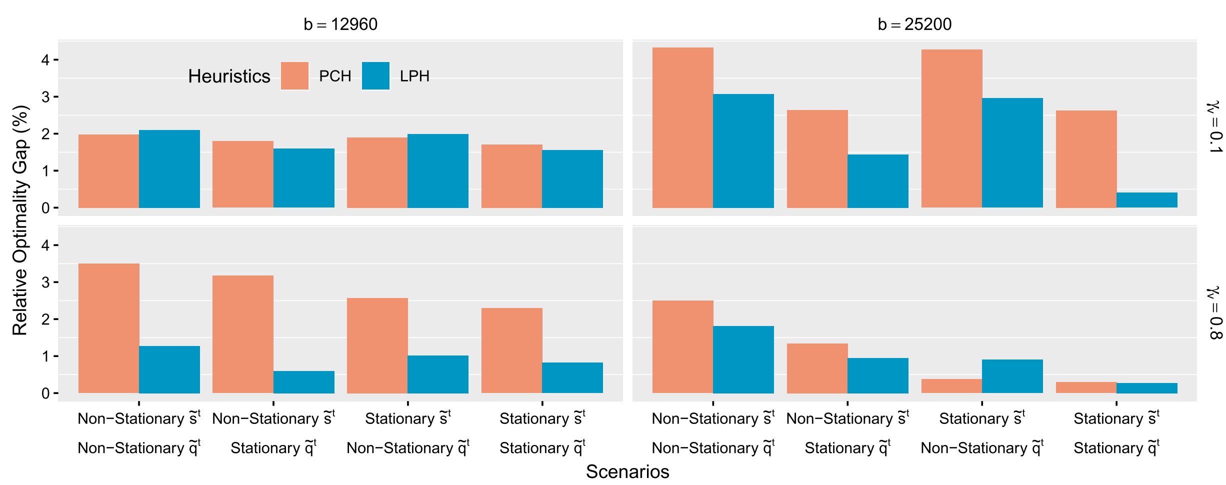
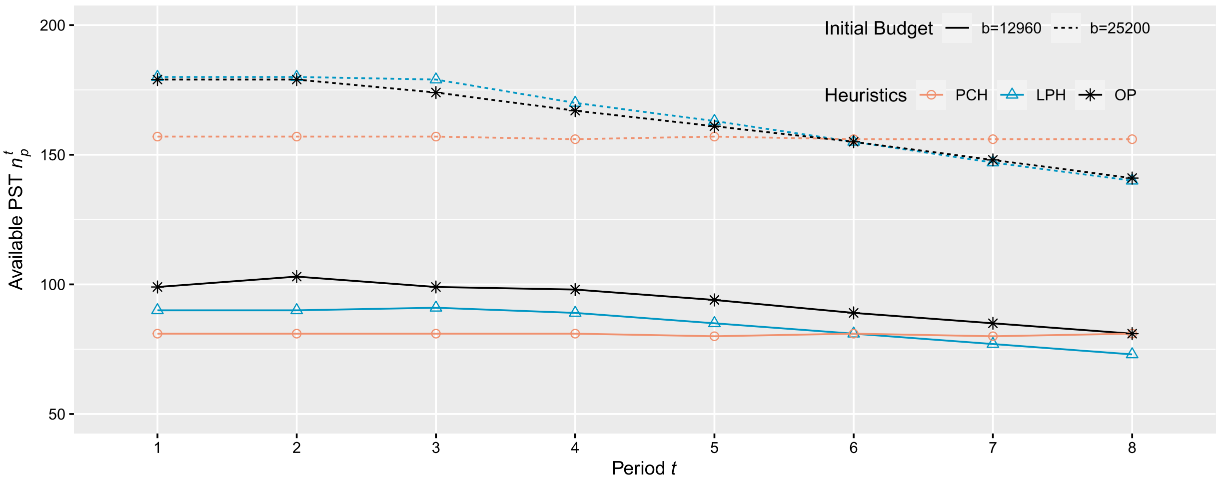


Figure 5. (Color online) Available PST Under PCH and LPH Given a Trajectory in a Stationary Scenario ($c_p = 48$, $n_v = 180$, $\gamma_v = 0.8$, $\tilde{q} \sim Beta(0.4, 7.6)$, $\tilde{s} \sim Beta(4, 4)$)



$(\tilde{s}^t)_{t=1}^T$) follows the corresponding beta distribution in each experiment described later. We run the heuristic on each trajectory to compute the corresponding total utility, and then we take the sample average to estimate the performance of this heuristic.

The relative optimality gaps of both heuristic policies versus OP are shown in Figure 4 under four scenarios with combined stationary or nonstationary $(\tilde{q}^t)_{t=1}^7$ and $(\tilde{s}^t)_{t=1}^8$:

- Stationary $\tilde{q}^t$: $\tilde{q}^t \sim Beta(0.4, 7.6)$, for all $t \in [7]$.
- Nonstationary $\tilde{q}^t$: $\tilde{q}^t \sim Beta(0.4, 7.6)$ for $t \in [4]$ and $\tilde{q}^t \sim Beta(1.2, 6.8)$ for all $t \in \{5, 6, 7\}$.
- Stationary $\tilde{s}^t$: $\tilde{s}^t \sim Beta(4, 4)$, for all $t \in [8]$.
- Nonstationary $\tilde{s}^t$: $\tilde{s}^t \sim Beta(4, 4)$ for $t \in [4]$ and $\tilde{s}^t \sim Beta(1.3, 6.5)$ for $t \in \{5, \ldots, 8\}$.

Note that $\tilde{q}^t \sim Beta(1.2, 6.8)$ and $\tilde{s}^t \sim Beta(1.3, 6.5)$ have higher and lower mean, respectively, and higher variance than the original Beta distributions of $\tilde{q}^t$ and $\tilde{s}^t$. We also select two different VST service quality levels and two different budget levels (by empirical search, under the stationary scenario, the worst optimality gap for PAP occurs for budget $b = 12{,}960$ when $\gamma_v = 0.8$ (Remark 1), and the worst optimality gap for adapted CHP in PCH with budget level higher than 3,036 occurs for budget $b = 25{,}200$ when $\gamma_v = 0.1$ (Remark 4)).

By design of the experiment and related to the performance of PCH, the instance with the largest optimality gap related to the assignment decisions of PCH (PAP) is the one under the stationary $\tilde{q}^t$ and $\tilde{s}^t$ case when $\gamma_v = 0.8$ and $b = 12{,}960$ (Remarks 1 and 2). In the rest of the experiments (and in general), the optimality gap of PCH is mainly caused by staff planning and hiring policies instead of PAP. For PCH, when the volunteer service quality is low (i.e., $\gamma_v = 0.1$), the optimality gap is mainly determined by the gap in social value from PST, which is larger under the high budget level (i.e., $b = 25{,}200$). When volunteer service quality is high (e.g., $\gamma_v = 0.8$), volunteers can almost completely substitute for PST and therefore the optimality gap is lower.

For LPH, although a deterministic volunteer supply is assumed, LPH's assignment decisions partly capture the property of OP of saving budget for the future. When PAP has a high optimality gap (when $\gamma_v = 0.8$; Remark 1), LPH has the lowest optimality gap. This indicates that LPH's optimality gap is mainly caused by its suboptimal hiring and staff planning policies, and, in particular, a larger optimality gap occurs when volunteer service quality is low.

Next, we compare the performance of both heuristics. First, both heuristics perform very well (with less than 5% relative optimality gap) under all four scenarios with different volunteer service qualities and budgets. Nonetheless, LPH shows better performance than PCH in general, except in three experimental instances. In these three instances, PAP is almost optimal; thus, PCH's performance is closer to OP. In addition, PCH always performs worse in nonstationary scenarios mainly because of the constant hire-up-to constraint. In contrast, without this constraint, we observe that LPH has more robust performance in nonstationary scenarios. To sum up, we observe that LPH performs better in general, but PCH outperforms LPH when PAP is close enough to optimality. Moreover, PCH is easier to implement.

Finally, in Figure 5, we show the available PST track under PCH, LPH, and OP, given a representative trajectory (i.e., $\tilde{q}^t = 0.05$ and $\tilde{s}^t = 0.5$ for all $t \in [T]$) in the scenario with stationary $\tilde{q}^t$ and stationary $\tilde{s}^t$. As we can see, the available PST under PCH is almost identical over all periods due to the constant hire-up-to level constraint. Nonetheless, the available PST decreases in t under both LPH and OP, where LPH follows a similar available PST level to OP for both worst-case budget level experiments. Most trajectories show this very similar representation.

6. Extensions

We give three practical extensions of our model in this section. The first assumes concavity of the social value function and is related to the operation of a team of workers. The second covers volunteer costs. Finally, the third considers volunteer fatigue and engagement across time.

6.1. Concave Social Value

In some nonprofit activities, such as the day camps run by Noble, staff work together as a team. In this setting, a concave social value function that represents the decreasing benefit of adding extra staffing hours could be a better fit than a linear one. We suppose that the social value function is $h(x_p^t + x_v^t) - \delta x_v^t$, where $h(\cdot)$ is an increasing concave function and $\delta \geq 0$ represents a penalty due to the lower social value of volunteers compared with part-time workers. Thus, the objective of the assignment problem is $\max_{(x_p^t, x_v^t) \in S^t} h(x_p^t + x_v^t) - \delta x_v^t + V^{t+1}(b^{t+1}, n_p^{t+1})$, where we suppose that $h'(0) > \tau w_p$ and $h'(0) - \delta > 0$ (so assigning part-time workers and volunteers can have positive marginal social value). Let $\frac{\partial V^{t+1}}{\partial b^{t+1}}(p) := \frac{\partial V^{t+1}}{\partial b^{t+1}}\Big|_{(b^t - c_p y_p^t - w_p p, n_p^{t+1})}$, then we can characterize the optimal assignment decisions.

Proposition 7 (Assignment Policy). *For each period $t \in [T]$, given any $\tilde{s}^t$ and $\tilde{q}^t$, there exists an optimal assignment decision (x_p^{t*}, x_v^{t*}), and x_p^{t*} (x_v^{t*}) is nonincreasing (nondecreasing) in $\tilde{s}^t n_v$.*

1. *If $\delta \leq w_p \frac{\partial V^{t+1}}{\partial b^{t+1}}(0)$, then $x_v^{t*} = \min\{v_1, \tilde{s}^t n_v, d\}$ and $x_p^{t*} = \min\{(p_1)^+, n_p^t, d - x_v^{t*}\}$, where $v_1 = \sup\{v : h'(v) \geq \delta\}$ and $p_1 = \sup\left\{p : h'(p + x_v^{t*}) \geq w_p \frac{\partial V^{t+1}}{\partial b^{t+1}}(p)\right\}$.*
2. *If $\delta > w_p \frac{\partial V^{t+1}}{\partial b^{t+1}}(0)$ and $\tilde{s}^t n_v + n_p^t \leq d$, then $x_p^{t*} = \min\{p_2, p_3, n_p^t\}$ and $x_v^{t*} = \min\{(v_2)^+, \tilde{s}^t n_v\}$, where $p_2 = \sup\left\{p : h'(p) \geq w_p \frac{\partial V^{t+1}}{\partial b^{t+1}}(p)\right\}$, $p_3 = \sup\left\{p : \delta \geq w_p \frac{\partial V^{t+1}}{\partial b^{t+1}}(p)\right\}$, and $v_2 = \sup\{v : h'(x_p^{t*} + v) \geq \delta\}$.*
3. *If $\delta > w_p \frac{\partial V^{t+1}}{\partial b^{t+1}}(0)$ and $\tilde{s}^t n_v + n_p^t > d$, then $x_p^{t*} + x_v^{t*} = \min\{\min\{p_2, p_3, n_p^t\} + \min\{v_2, \tilde{s}^t n_v\}, d\}$.*

The structure of this optimal assignment policy is equivalent to the one in Proposition 1, where δ and $\frac{\partial V^{t+1}}{\partial b^{t+1}}(0)$ correspond to $\gamma_p - \gamma_v$ and τ in Proposition 1, respectively. Overall, for concave social value, optimal decisions are more prone to reduce current staffing time assignments in favor of saving budget for the future.

6.2. Nonzero Volunteer Cost

Volunteer costs might not be negligible. For example, some NPOs offer in-kind gifts to volunteers after providing service. In the context of volunteer fire departments (Brudney and Duncombe 1992, Duncombe and Brudney 1995), there are administrative costs due to recruitment, training, and supervision. Here we extend our model to include VST cost $w_v > 0$, so the budget b^t now follows $b^{t+1} = b^t - c_p y_p^t - w_p x_p^t - w_v x_v^t$.

The assignment decisions are more complex here because budget can be allocated to both PST and VST. The following proposition analyzes the optimal assignment decisions for this setting, which differ from those in Proposition 1 because of dependence on the budget levels.

Proposition 8. *For each period $t \in [T]$, given any $\tilde{s}^t$ and $\tilde{q}^t$, there exists an optimal assignment policy (x_p^{t*}, x_v^{t*}), and x_p^{t*} (x_v^{t*}) is nondecreasing (nonincreasing) in $b^t - c_p y_p^t$.*

(i) *If $\gamma_v / w_v > \gamma_p / w_p$, then there exist $\underline{b}^t$ and $\overline{b}^t$ for each n_p^{t+1} such that*

(1) *If $b^t - c_p y_p^t \geq \overline{b}^t$ and $\gamma_v - \tau w_v < \gamma_p - \tau w_p$, then $P \succ V$.*

(2) *If $b^t - c_p y_p^t \leq \underline{b}^t$, then it is optimal to assign as much VST as possible.*

(3) *If $\gamma_v - \tau w_v > \gamma_p - \tau w_p$, then it is optimal to assign as much VST as possible.*

(4) *If $\underline{b}^t < b^t < \overline{b}^t$, given x_p^{t*}, then $x_v^{t*} = \{\overline{d} - n_f - x_p^{t*}, \tilde{s}^t n_v\}$.*

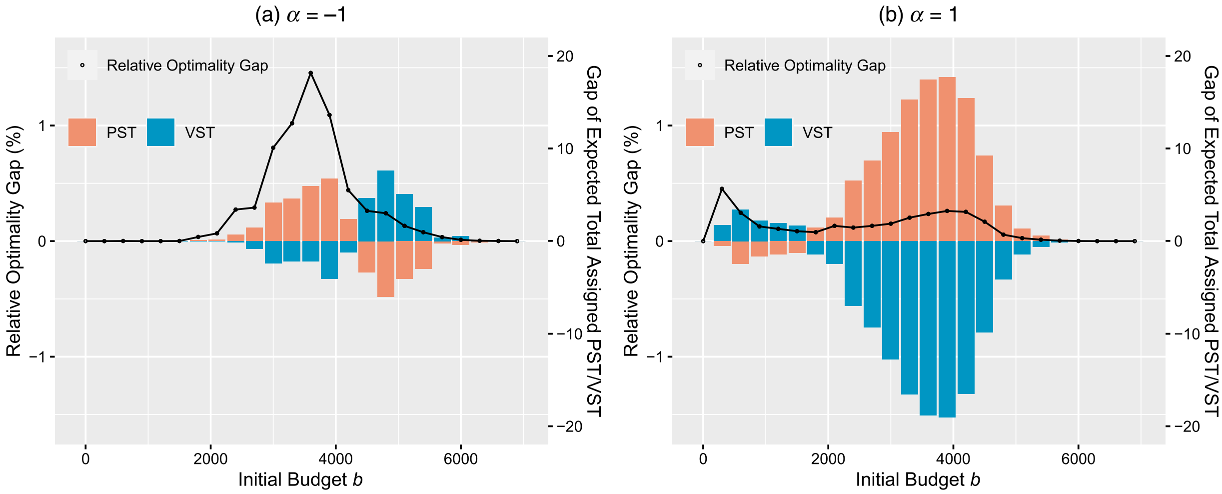
(ii) *If $\gamma_v / w_v < \gamma_p / w_p$, then it is optimal to assign as much PST as possible.*

6.3. Volunteer Behavior Across Time

Sampson (2006) finds that used volunteers may be more likely to volunteer in the future compared with unused volunteers. However, volunteers might also have the opposite behavior and be less willing to participate if they have recently participated (Ata et al. 2019), a phenomenon called "volunteer fatigue." High engagement and fatigue have been investigated in the volunteer management literature, where this future engagement depends on the type of service offered, the environment, and the engagement of the organization with its volunteers (Wisner et al. 2005, Gabbey 2018, Urrea and Yoo 2023). A similar phenomenon is observed in cash donations (Kessler and Milkman 2018).

We incorporate this behavior into our model by assuming that the level of volunteer assignment in the current period has an effect on the volunteer supply in the next period (positively or negatively). This effect can also be regarded as temporarily enlarging or reducing the size of the volunteer pool. Hence, we define

Figure 6. (Color online) Optimality Gap of Social Value and Difference Between Optimal Assigned Staffing Time Under High Hiring Cost $c_p = 48$ ($n_v = 40$, $\tilde{q} \sim Beta(0.4, 7.6)$, $\tilde{s} \sim Beta(4, 4)$)



Note. PST (VST) refers to the gap of expected total assigned staffing time $\mathbb{E}[\sum_{t=1}^{T}(x_{p\alpha}^{t\,*} - x_p^{t\,*})]$ ($\mathbb{E}[\sum_{t=1}^{T}(x_{v\alpha}^{t\,*} - x_v^{t\,*})]$).

$\tilde{s}^t(n_v + \alpha x_{v\alpha}^{t-1})$ to be the available VST in period $t \in [T]$, where $x_{v\alpha}^{t-1}$ is the VST assigned in the last period $t-1$ and $\alpha \in [-1, 1]$. If $\alpha > (<) 0$, then more (less) VST would be available in the next period $t+1$ unless $x_{v\alpha}^t = 0$. Let $V_\alpha^t(\cdot)$ be the optimal expected social value with dynamic volunteer behavior. Then, we have

$$V_\alpha^t(b^t, n_p^t, x_{v\alpha}^{t-1}) = \mathbb{E}_{\tilde{q}^t}\left[\max_{0 \le y_{p\alpha}^t \le b^t/c_p} \mathbb{E}_{\tilde{s}_v^t}\left[\max_{(x_{p\alpha}^t, x_{v\alpha}^t) \in S_\alpha^t} \gamma_p x_{p\alpha}^t + \gamma_v x_{v\alpha}^t + V_\alpha^{t+1}(b^{t+1}, n_p^{t+1}, x_{v\alpha}^t)\right]\right], \quad (10)$$

where $S_\alpha^t = \{(x_{p\alpha}^t, x_{v\alpha}^t) \ge 0 : c_p y_{p\alpha}^t + w_p x_{p\alpha}^t \le b^t, x_{p\alpha}^t + x_{v\alpha}^t \le d, x_{p\alpha}^t \le n_p^t, x_{v\alpha}^t \le \min\{n_v, \tilde{s}_v^t(n_v + \alpha x_{v\alpha}^{t-1})\}\}$. The dynamics and boundary conditions are the same as those defined in Section 3 and $x_{v\alpha}^0 = 0$. The only difference between S_α^t and S^t is the constraint $x_{v\alpha}^t \le \min\{n_v, \tilde{s}_v^t(n_v + \alpha x_{v\alpha}^{t-1})\}$.

Proposition 9. *For each period $t \in [T]$, $V_\alpha^t(\cdot)$ is jointly concave and there exists an optimal hiring and assignment decision.*

A closed form analytical solution is challenging to derive. We numerically compare this solution with the optimal policy of Section 4 (OP; i.e., $\alpha = 0$), and we explore the effect of dynamic volunteer behavior on the optimal social value and assignment decisions. OP should be suboptimal for this problem because it does not consider dynamic volunteer behavior. In this experiment, the parameter values are the same as those in §5 except that the capacity is $d = 40$ hours per period.

We consider two scenarios: $\alpha = -1$ (Figure 6(a)) and $\alpha = 1$ (Figure 6(b)). Figure 6 illustrates the relative optimality gap and the difference between expected total assigned PST and VST (in hours) (i.e., $\sum_{t=1}^{T}\mathbb{E}[x_{p\alpha}^{t\,*} - x_p^{t\,*}]$ and $\sum_{t=1}^{T}\mathbb{E}[x_{v\alpha}^{t\,*} - x_v^{t\,*}]$) under the two scenarios. We draw similar insights from the case of low c_p, shown in Figure 11 in Online Appendix C. First, we observe that when budgets are sufficiently low (i.e., staff scarcity) or high (i.e., staff sufficiency), the OP is optimal because PAP is optimal. Most importantly, in Figure 6(b), we observe that as the budget increases, the optimal policy with dynamic volunteer behavior assigns more VST than OP first and then assigns more PST compared with OP. This is because inducing greater available VST is more important when the budget is low. Positive dependence on past volunteer levels ($\alpha = 1$) increases volunteer supply more than the case $\alpha = 0$. However, higher budget levels reduce the importance of inducing more volunteer supply. With higher budget levels, the case $\alpha = 1$ has probabilistically more volunteer supply in future periods compared with OP. Then, there is less motivation to save some budget for the future by reducing PST to assign. Figure 6(a) shows the opposite results.

7. Conclusions

In this paper we study the staffing problem for NPOs with a blended workforce of full- and part-time employees and volunteers. We use stochastic dynamic programming to build a model that considers an initial staffing problem and periodic hiring and assignment decisions with budget and capacity constraints. Our analysis shows that the optimal solution does not have a closed form and that it is computationally costly to solve this problem exactly. Moreover, we find that a prioritization assignment policy (PAP) and a state-dependent hire-up-to policy for part-time workers are optimal for the extreme scenarios of staff scarcity and staff sufficiency. While analyzing the initial staffing problem, we also find that no more than one type of paid worker should be chosen depending on the marginal social value of budget per unit of full-time and part-time staffing time.

Our computational study is based on a specific NPO's service setting and compares the optimal policy, which considers flexibility and turnover of part-time workers, to simpler policies that do not consider these behaviors. For example, among other results, we observe that a constant hire-up-to policy (that ignores PST turnover) causes larger relative optimality gaps when the service quality of volunteers is low. In contrast, our suggested PAP (which omits PST flexibility) shows larger relative optimality gaps when the service quality of volunteers is high. We propose an easy-to-implement heuristic policy that combines the PAP and adjusted constant hire-up-to policies (PCH) and an LP-based heuristic (LPH). We also provide theoretical performance guarantees for both heuristics. Numerically, we see that both heuristics perform well in a variety of scenarios, where LPH has robust performance even for nonstationary settings and PCH performs well for instances where PAP is close to optimal. Finally, we extend our model to a concave social value function, where it shares the same structure of the assignment policy as in the original linear setting. We also study the cases of nonzero volunteer cost and dynamic volunteer behaviors, both of which have some differences from our original setting that could be relevant.

We point out three future research directions. First, we would like to incorporate the effect of learning for volunteers and paid workers, where service quality is improved by practice. Second, one can model friction between experienced workers and inexperienced volunteers, especially for activities done in teams. Third, our model can be expanded to include dynamic financial donations, which would be made periodically.

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