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Event-triggered Distributed Observer for Rigid Body Systems over Jointly Connected Acyclic Switching Networks ⋆

Tianqi Wang * **Jie Huang** **

* *The Hong Kong Polytechnic University, Hong Kong (e-mail: tianqi-aae.wang@polyu.edu.hk).*

** *The Chinese University of Hong Kong, Hong Kong (e-mail: jhuang@mae.cuhk.edu.hk)*

Abstract: A drawback of the existing event-triggered distributed observer for a rigid body leader system over jointly connected switching networks is that the upper bounds of two key design parameters were only shown to exist without giving an explicit estimate of the upper bounds. In this paper, by assuming that the communication network is acyclic, we further show that these two design parameters can take any positive value by choosing other parameters appropriately. We will also apply our event-triggered distributed observer to the leader-following consensus problem of multiple rigid body systems and illustrate our design by a numerical example.

Keywords: Rigid body systems; event-triggered control; distributed observer.

1. INTRODUCTION

The cooperative control of multiple rigid body systems has attracted extensive attentions over the past decade Cai and Huang (2014); Dimarogonas et al. (2009); Liu and Huang (2018); Wang and Huang (2020). An effective tool to solve this problem is the so-called distributed observer approach summarized by Cai et al. (2022). The distributed observer is a distributed dynamic compensator that estimates the leader system's information over the communication network. Based on the distributed observer, one may systematically synthesize a distributed control law by composing a distributed observer and a purely decentralized control law. As pointed out in Cai et al. (2022); Wang and Huang (2022), the distributed observer approach has two advantages over other approaches. First, the distributed observer approach can handle the jointly connected communication network which can be disconnected at every time. Second, the distributed observer and the purely decentralized control law can be separately designed.

The event-triggered control technology, which generates the samplings and control actuation by some event-triggered mechanisms, has been gaining momentum over the past two decades. Some representative papers are Abdelrahim et al. (2017); Åström and Bernhardsson (1999); Borgers and Heemels (2014); Chen et al. (2020); Donkers and Heemels (2012); Eqtami et al. (2010); Heemels et al. (2012); Nowzari et al. (2019); Tabuada (2007). Compared with time-triggered sampling, event-triggered control can significantly reduce the consumption of communication and actuation energy. This advantage has motivated the study of the event-triggered distributed observer. Specifically, Dong and Lin (2022) proposed an event-triggered distributed observer for a linear leader system and applied this observer to the cooperative linear output regulation and the leader-following consensus of Euler-Lagrange systems. Under the assumption that the communication network is every time connected, Wang et al. (2021) presented an approach to implement the continuous-time distributed observer for rigid body systems used in Cai and Huang (2014); Liu and Huang (2018). Our recent work Wang and Huang (2022) has extended the result in Wang et al. (2021) to the case where the communication network is

Table 1. Table of Notations

Symbol	*Meaning*
$\\|\cdot\\|$	The Euclidean norm of a vector; or the induced Euclidean norm of a matrix.
$\mathrm{col}(\cdot)$	$\mathrm{col}(A_1,...,A_m)=\left[A_1^\top,...,A_m^\top\right]^\top \in \mathbb{R}^{(n_1+\cdots+n_m)\times p}$, for $A_i\in\mathbb{R}^{n_i\times p}$.
$\mathbb{Q}$	The set of all quaternions: $\mathbb{Q}=\{q: q=\mathrm{col}(\hat{q},\bar{q}), \hat{q}\in\mathbb{R}^3, \bar{q}\in\mathbb{R}\}$.
$\mathbb{Q}_u$	$\mathbb{Q}_u=\{q\in\mathbb{Q}: \\|q\\|=1\}$.
q^*	Quaternion conjugate: $q^*=\left[-\hat{q}^\top,\bar{q}\right]^\top$, for $q\in\mathbb{Q}$.
$\odot$	$q_i\odot q_j=\begin{bmatrix}\bar{q}_i\hat{q}_j+\bar{q}_j\hat{q}_i+\hat{q}_i^\times\hat{q}_j\\ \bar{q}_i\bar{q}_j-\hat{q}_i^\top\hat{q}_j\end{bmatrix}$, for $q_i,q_j\in\mathbb{Q}$.
$(\cdot)^\times$	$x^\times=\begin{bmatrix}0&-x_3&x_2\\ x_3&0&-x_1\\ -x_2&x_1&0\end{bmatrix}$, for $x=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$.
$\mathbf{Q}(\cdot)$	$\mathbf{Q}(\alpha)=\mathrm{col}(\alpha,0)$ where $\alpha\in\mathbb{R}^3$.
$\mathbf{C}(\cdot)$	$\mathbf{C}(q)=(\bar{q}^2-\hat{q}^\top\hat{q})I_3+2\hat{q}\hat{q}^\top-2\bar{q}\hat{q}^\times, q\in\mathbb{Q}$.

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jointly connected. Nevertheless, one drawback of the result in Wang and Huang (2022) is that it only guarantees the existence of the upper bounds of two key design parameters β_v and β_η without giving an explicit estimate of the upper bounds of these two parameters.

In this paper, we will further investigate the problem of designing an event-triggered distributed observer for rigid body systems studied in Wang and Huang (2022). We will show that, for the class of the acyclic communication network, the above-mentioned two design parameters can take any positive value by properly choosing other parameters. For this purpose, we need to study the stability properties of some classes of linear time-varying systems.

The rest of this paper is organized as follows. In Section 2, we will review the result in Wang and Huang (2022) and formulate the problem. Then, we present some stability results on some classes of linear time-varying systems in Section 3, which will be used to establish our main result. Our main result will be presented in Section 4. In Section 5, we will apply our event-triggered distributed observer to the leader-following consensus problem of multiple rigid body systems together with a numerical example. Finally, we close this paper with some concluding remarks in Section 6. The notation we will use in this paper are collected in Table 1. In what follows, given a system $\dot{x} = f(x,t)$, we say it is *globally exponentially stable at the rate of at least* λ if there exists a positive constant α such that

$$\|x(t)\| \leq \alpha \|x(t_0)\| e^{-\lambda(t-t_0)}, \ \forall t \geq t_0 \tag{1}$$

for any $x(t_0)$; if a time function $x(t)$ satisfies (1) with $t_0 = 0$, we say $x(t)$ *tends to zero* ($x(t) \to 0$) *exponentially at the rate of at least* λ.

2. PROBLEM FORMULATION

Consider a group of N rigid body systems as follows:

$$\dot{q}_i(t) = \frac{1}{2} q_i(t) \odot \mathbf{Q}(\omega_i(t)), \tag{2a}$$

$$J_i \dot{\omega}_i(t) = -\omega_i(t)^\times J_i \omega_i(t) + u_i(t), \tag{2b}$$

where $q_i \in \mathbb{Q}_u$ describes the attitude of follower i; $\omega_i \in \mathbb{R}^3$ is the angular velocity; $J_i \in \mathbb{R}^{3\times 3}$ is the uncertain positive definite inertia matrix of the ith follower; $u_i \in \mathbb{R}^3$ is the control torque; $i = 1, ..., N$.

Also, assume that the desired attitude and angular velocity for system (2) are generated by:

$$\dot{q}_0(t) = \frac{1}{2} q_0(t) \odot \mathbf{Q}(\omega_0(t)), \tag{3a}$$

$$\dot{v}(t) = Sv(t), \ \omega_0(t) = Ev(t), \tag{3b}$$

where $q_0 \in \mathbb{Q}_u$ is the desired attitude to be synchronized by the followers; $\omega_0 \in \mathbb{R}^3$ is the desired angular velocity; $v \in \mathbb{R}^n$; $S \in \mathbb{R}^{3\times 3}$ and $E \in \mathbb{R}^{3\times n}$ are two constant matrices.

Systems (2) and (3) together can be viewed as a leader-follower multi-agent system with (3) as the leader system and the N subsystems of (2) as N followers. The communication network among these $N+1$ agents is described by a switching digraph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$, where $\sigma : [0,\infty) \to \mathcal{P} = \{1, ..., n_0\}$ is a piecewise constant switching signal with dwell time $\tau > 0$; the node set is defined as $\mathcal{V} = \{0, 1, ..., N\}$ with node 0 represents the leader and nodes 1 to N represent followers 1 to N, respectively; the edge set $\mathcal{E}_{\sigma(t)}$ is such that $(i,j) \in \mathcal{E}_{\sigma(t)}$ if and only if node j can receive the information from node i at time t. We say node i_k is reachable from node i_1 at time t if $\{(i_1,i_2),(i_2,i_3),...,(i_{k-1},i_k)\} \subset \mathcal{E}_{\sigma(t)}$. The neighbor set of node i is defined as $\mathcal{N}_i(t) = \{j : (j,i) \in \mathcal{E}_{\sigma(t)}, j \neq i\}$. The weighted adjacency matrix associated with $\mathcal{G}_{\sigma(t)}$ is defined as $\mathcal{A}_{\sigma(t)} = [a_{ij}(t)]_{i,j=0}^N$ where $a_{ij}(t) = 1 \Leftrightarrow (j,i) \in \mathcal{E}_{\sigma(t)}, i \neq j$ and $a_{ij} = 0$ otherwise. Define $H_{\sigma(t)} = [h_{ij}(t)]_{i,j=1}^N$ where $h_{ii}(t) = \sum_{j=0}^N a_{ij}(t)$ and $h_{ij}(t) = -a_{ij}(t)$ for $i \neq j$.

Let us recall the following event-triggered distributed observer for the leader system (3) proposed by Wang and Huang (2022):

$$\begin{cases} \dot{v}_i(t) = Sv_i(t) + v_{ei}(t_k^{v,i}), \\ v_{ei}(t) = \mu_v \sum\limits_{j\in\mathcal{N}_i(t)} (v_j(t) - v_i(t)), \ t \in [t_k^{v,i}, t_{k+1}^{v,i}), \end{cases} \tag{4a}$$

$$\begin{cases} \dot{\eta}_i(t) = \frac{1}{2}\eta_i(t) \odot \mathbf{Q}(Ev_i(t)) + \eta_{ei}(t_k^{\eta,i}), \\ \eta_{ei}(t) = \mu_\eta \sum\limits_{j\in\mathcal{N}_i(t)} (\eta_j(t) - \eta_i(t)), \ t \in [t_k^{\eta,i}, t_{k+1}^{\eta,i}), \end{cases} \tag{4b}$$

$$t_{k+1}^{v,i} = \inf\left\{ t > t_k^{v,i} : \|e_{vi}(t)\| \geq \alpha_v e^{-\beta_v t} \right\}, \tag{4c}$$

$$t_{k+1}^{\eta,i} = \inf\left\{ t > t_k^{\eta,i} : \|e_{\eta i}(t)\| \geq \alpha_\eta e^{-\beta_\eta t} \right\}, \tag{4d}$$

where (4a) and (4b) are respectively the event-triggered distributed observer for leader's states v and q_0; (4c) and (4d) are the event-triggered mechanism respectively for the triggering times $t_k^{v,i}$ and $t_k^{\eta,i}$, with

$$e_{vi}(t) = v_{ei}(t_k^{v,i}) - v_{ei}(t), \ t \in [t_k^{v,i}, t_{k+1}^{v,i}), \tag{5a}$$

$$e_{\eta i}(t) = \eta_{ei}(t_k^{\eta,i}) - \eta_{ei}(t), \ t \in [t_k^{\eta,i}, t_{k+1}^{\eta,i}). \tag{5b}$$

Two standard assumptions are as follows:

Assumption 1. All the eigenvalues of S are semi-simple with zero real parts.

Assumption 2. The switched digraph $\mathcal{G}_{\sigma(t)}$ is jointly connected with node 0 as the root. That is, there exists a subsequence $\{j_k : k = 0,1,2,...\}$ of $\{j : j = 0,1,2,...\}$ with $t_{j_{k+1}} - t_{j_k} < T$ for some positive T such that node 0 can reach every other node in the union graph $\bigcup_{t_{j_k} \leq t < t_{j_{k+1}}} \mathcal{G}_{\sigma(t)}$.

Remark 1. Under Assumption 1, the leader system is able to generate a large class of reference trajectories including step functions and sinusoidal functions with arbitrarily unknown amplitudes. Assumption 2 is the mildest restriction on the digraph $\mathcal{G}_{\sigma(t)}$ which allows the graph to be disconnected at every time.

The main result of Wang and Huang (2022) is summarized as follows:

Lemma 1. Under Assumptions 1 and 2, there exist $\beta_v^* > 0$ and $\beta_\eta^* > 0$ such that, for any $\mu_v > 0, \mu_\eta > 0, \alpha_v > 0, \alpha_\eta > 0$, for any $0 < \beta_v < \beta_v^*$, $0 < \beta_\eta < \beta_\eta^*$, and, for any initial condition with $q_0(0) \in \mathbb{Q}_u$, the trajectories of (3) and (4) exist for all $t \geq 0$ and are such that for all $i = 1, ..., N$, $\lim_{t\to\infty}(v_i(t) - v(t)) = 0$ and $\lim_{t\to\infty}(\eta_i(t) - q_0(t)) = 0$ both exponentially.

Remark 2. Due to Lemma 1, the dynamic compensator (4) is able to exponentially estimate the leader's states v and q_0; and from (4), the inter-agent signals $\mu_v \sum_{j\in\mathcal{N}_i(t)}(v_j(t) - v_i(t))$ and $\mu_\eta \sum_{j\in\mathcal{N}_i(t)}(\eta_j(t) - \eta_i(t))$ are only updated at the triggering time $t_k^{v,i}$ and $t_k^{\eta,i}$, respectively. Thus, we call (4) the event-triggered distributed observer for the leader system (3). However, one weakness of Lemma 1 is that it only guarantees the existence of the upper bounds β_v^* and β_η^* without knowing their specific values.

In this paper, we will further show that β_v and β_η may take any positive value if we strengthen Assumption 2 to the following form:

Assumption 3. The digraph $\mathcal{G}_{\sigma(t)}$ is jointly connected with node 0 as the root, and every node in the union graph $\bigcup_{p\in\mathcal{P}} \mathcal{G}_p$ is not reachable from itself.

In what follows, we call a switching graph $\mathcal{G}_{\sigma(t)}$ satisfying Assumption 3 *jointly connected acyclic switching graph.*

3. SOME STABILITY RESULTS

In this section, we present some stability results on some classes of linear time-varying systems, which will be used to establish our main results in the subsequent sections.

Lemma 2. Suppose that the following system

$$\dot{x}(t) = A(t)x(t) \tag{6}$$

is exponentially stable at the rate of at least β_1, i.e., its state transition matrix $\Phi(t,\tau)$ satisfies $\|\Phi(t,\tau)\| \le \alpha_1 e^{-\beta_1(t-\tau)}$ for some $\alpha_1, \beta_1 > 0$. Let $b(t) \to 0$ exponentially at the rate of at least β_2, i.e., $\|b(t)\| \le \alpha_2 e^{-\beta_2 t}$ for $\alpha_2, \beta_2 > 0$. Then, the solution of the following system

$$\dot{x}(t) = A(t)x(t) + b(t) \tag{7}$$

is such that $\|x(t)\| \le \alpha e^{-\beta t}$, where $\alpha > 0$ and

$$\beta = \begin{cases} \min\{\beta_1, \beta_2\} & \text{if } \beta_1 \ne \beta_2; \\ \beta_1 - \epsilon & \text{if } \beta_1 = \beta_2. \end{cases}$$

for any $\epsilon \in (0, \beta_1)$.

Proof: For any $x(0)$, the general solution of the system (7) is given by

$$x(t) = \Phi(t,0)x(0) + \int_0^t \Phi(t,\tau)b(\tau)\mathrm{d}\tau. \tag{8}$$

Taking the norm on both sides gives

$$\begin{aligned}
\|x(t)\| \le& \|\Phi(t,0)\|\|x(0)\| + \int_0^t \|\Phi(t,\tau)\|\|b(\tau)\|\mathrm{d}\tau \\
\le& \alpha_1 e^{-\beta_1 t}\|x(0)\| + \int_0^t \alpha_1 e^{-\beta_1(t-\tau)}\alpha_2 e^{-\beta_2\tau}\mathrm{d}\tau \\
=& \alpha_1 e^{-\beta_1 t}\|x(0)\| + \alpha_1\alpha_2 e^{-\beta_1 t}\int_0^t e^{(\beta_1-\beta_2)\tau}\mathrm{d}\tau \\
=& \begin{cases} \alpha_1 e^{-\beta_1 t}\|x(0)\| + \dfrac{\alpha_1\alpha_2}{\beta_1-\beta_2}\left(e^{-\beta_2 t} - e^{-\beta_1 t}\right) \\ \qquad\qquad\qquad\qquad \text{if } \beta_1 \ne \beta_2; \\ \alpha_1 e^{-\beta_1 t}\|x(0)\| + \alpha_1\alpha_2 t e^{-\beta_1 t} \\ \qquad\qquad\qquad\qquad \text{if } \beta_1 = \beta_2, \end{cases}
\end{aligned}$$

which implies $\|x(t)\| \le \alpha e^{-\beta t}$. □

Now, consider the following system

$$\dot{x}(t) = \big(\text{block diag}(G_1(t), ..., G_N(t)) - \mu(F(t) \otimes I_4)\big)x(t), \tag{9}$$

where $\mu > 0$; $x \in \mathbb{R}^{4N}$ is the state; $G_1(t), ..., G_N(t) \in \mathbb{R}^{4\times 4}$ are bounded skew-symmetric matrices; $F(t) = [f_{ij}(t)]_{i,j=1}^N \in \mathbb{R}^{N\times N}$ is a lower-triangular, piecewise continuous, and bounded matrix satisfying the following assumption:

Assumption 4. For all $i = 1, ..., N$ and all $t \ge 0$, $f_{ii}(t) \ge 0$, and there exists positive constants δ_0 and T_0 such that $\int_t^{t+T_0} f_{ii}(\tau)\mathrm{d}\tau \ge \delta_0 T_0$.

Before presenting the stability result of (9), we first establish a technical lemma as follows:

Lemma 3. Under Assumption 4, for any $\mu > 0$ and for all $i = 1, ..., N$, the origin of the following system

$$\dot{\tilde{x}}(t) = \big(G_i(t) - \mu f_{ii}(t)I_4\big)\tilde{x}(t), \tag{10}$$

where $\tilde{x} \in \mathbb{R}^4$, is globally exponentially stable at the rate of at least $\mu\delta_0$.

Proof: Inspired from Su and Huang (2012), let $\Phi_i(t,\tau)$ be the state transition matrix of the following system:

$$\dot{\bar{x}}(t) = G_i(t)\bar{x}(t). \tag{11}$$

Performing the state transformation $z(t) = \Phi_i(0,t)\tilde{x}(t)$ gives

$$\begin{aligned}
\dot{z}(t) =& \dot{\Phi}_i(0,t)\tilde{x}(t) + \Phi_i(0,t)\dot{\tilde{x}}(t) \\
=& -\Phi_i(0,t)G_i(t)\tilde{x}(t) + \Phi_i(0,t)(G_i(t) - \mu f_{ii}(t)I_4)\tilde{x}(t) \\
=& -\mu f_{ii}(t)\Phi_i(0,t)\tilde{x}(t) = -\mu f_{ii}(t)I_4 z(t).
\end{aligned} \tag{12}$$

Keeping in mind the diagonal structure of $-\mu f_{ii}(t)I_4$, one may conclude from the proof of Lemma 3.2 of He and Huang (2021) that there exists a constant $\alpha_3 > 0$ such that

$$\|z(t)\| \le \alpha_3 e^{-\mu\delta_0 t}\|z(0)\|. \tag{13}$$

For (11), using the skew-symmetry of $G_i(t)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{x}(t)\|^2 = 2\bar{x}(t)^\top\dot{\bar{x}}(t) = 2\bar{x}(t)^\top G_i(t)\bar{x}(t) = 0, \tag{14}$$

thus, $\Phi_i(t,0)$ is bounded for all $t \ge 0$. As a result,

$$\begin{aligned}
\|\tilde{x}(t)\| =& \|\Phi_i(t,0)z(t)\| \le \alpha_3 e^{-\mu\delta_0 t}\|\Phi_i(t,0)\|\|z(0)\| \\
\le& \alpha_4 e^{-\mu\delta_0 t}\|\tilde{x}(0)\|
\end{aligned} \tag{15}$$

for some $\alpha_4 > 0$. □

Now we are ready to present the convergence result of system (9).

Lemma 4. Under Assumption 4, for any initial condition, the trajectory of (9) satisfies

$$\|x(t)\| \le \alpha_5 e^{-(\mu\delta_0-\epsilon)t}\|x(0)\| \tag{16}$$

for any $\epsilon \in (0, \mu\delta_0)$, where $\alpha_5 > 0$.

Proof: Let $x = [x_1^\top, x_2^\top, ..., x_N^\top]^\top$ where $x_i \in \mathbb{R}^4$, $i = 1, ..., N$. Then, (9) is equivalent to the following system:

$$\dot{x}_1(t) = \big(G_1(t) - \mu f_{11}(t)I_4\big)x_1(t), \tag{17a}$$

$$\dot{x}_2(t) = \big(G_2(t) - \mu f_{22}(t)I_4\big)x_2(t) - \mu f_{21}(t)x_1(t), \tag{17b}$$

$$\begin{aligned}
\dot{x}_3(t) =& \big(G_3(t) - \mu f_{33}(t)I_4\big)x_3(t) - \mu f_{31}(t)x_1(t) \\
& - \mu f_{32}(t)x_2(t),
\end{aligned} \tag{17c}$$

$$\vdots$$

$$\dot{x}_N(t) = \big(G_N(t) - \mu f_{NN}(t) I_4\big) x_N(t) - \sum_{i=1}^{N-1} \mu f_{Ni}(t) x_i(t). \tag{17d}$$

In what follows, we will show the claim of this lemma by induction. First of all, (17a) is globally exponentially stable at the rate of at least $\mu\delta_0$ by invoking Lemma 3 with $i = 1$. Then consider (17b), which takes the form (7) with $A(t) = G_2(t) - \mu f_{22}(t) I_4$ and $b(t) = -\mu f_{21}(t) x_1(t)$. Since by Lemma 3, system

$$\dot{x}_2(t) = \big(G_2(t) - \mu f_{22}(t) I_4\big) x_2(t)$$

is globally exponentially stable at the rate of at least $\mu\delta_0$; and from our previous argument, $-\mu f_{21}(t) x_1(t)$ also tends to zero exponentially at the rate of at least $\mu\delta_0$, invoking Lemma 2 concludes that $x_2(t) \to 0$ exponentially at the rate of at least $\mu\delta_0 - \epsilon$ for any $\epsilon \in (0, \mu\delta_0)$ as $t \to \infty$.

Now consider (17c), which takes the form (7) with $A(t) = G_3(t) - \mu f_{33}(t) I_4$ and $b(t) = -\mu f_{31}(t) x_1(t) - \mu f_{32}(t) x_2(t)$. Since, by Lemma 3, system

$$\dot{x}_3(t) = \big(G_3(t) - \mu f_{33}(t) I_4\big) x_3(t)$$

is globally exponentially stable at the rate of at least $\mu\delta_0$; and from the previous argument, $-\mu f_{31}(t) x_1(t) - \mu f_{32}(t) x_2(t)$ tends to zero exponentially at the rate of at least $\mu\delta_0 - \epsilon$ for any $\epsilon \in (0, \mu\delta_0)$, Lemma 2 implies that $x_3(t) \to 0$ exponentially at the rate of at least $\mu\delta_0 - \epsilon$ for any $\epsilon \in (0, \mu\delta_0)$ as $t \to \infty$.

Repeating the above steps concludes that, for all $i = 2, 3, ..., N$, $x_i(t) \to 0$ exponentially at the rate of at least $\mu\delta_0 - \epsilon$ for any $\epsilon \in (0, \mu\delta_0)$ as $t \to \infty$. Thus, the origin of (9) is globally exponentially stable at the rate of at least $\mu\delta_0 - \epsilon$ for any $\epsilon \in (0, \mu\delta_0)$, in other words, (16) holds for any initial condition. □

4. EVENT-TRIGGERED DISTRIBUTED OBSERVER OVER ACYCLIC SWITCHING GRAPHS

In this section, we will establish our main result. Let us first note that, under Assumption 3, by properly numbering the nodes, the weighted adjacency matrix $\mathcal{A}_{\sigma(t)}$ and the corresponding $H_{\sigma(t)}$ are both in lower-triangular form with $h_{ii}(t) \geq 0$ for all $t \geq 0$ and $i = 1, ..., N$. Moreover, by Remark 4.1 of He and Huang (2021), there exists $\delta_1 > 0$ such that, for any $t \geq 0$,

$$\int_t^{t+2T} h_{ii}(\tau) \mathrm{d}\tau \geq 2\delta_1 T. \tag{18}$$

In other words, $H_{\sigma(t)}$ satisfies Assumption 4 with $\delta_0 = \delta_1$ and $T_0 = 2T$.

Let $\tilde{v}_i = v_i - v$ and $\tilde{\eta}_i = \eta_i - q_0$ be the tracking errors of v and q_0 of follower i. Define $\tilde{v} = \mathrm{col}(\tilde{v}_1, ..., \tilde{v}_N)$ and $\tilde{\eta} = \mathrm{col}(\tilde{\eta}_1, ..., \tilde{\eta}_N)$. Also, for any $y = [y_1, y_2, y_3]^\top$, let $\mathcal{M} : \mathbb{R}^3 \to \mathbb{R}^{4\times 4}$ be such that

$$\mathcal{M}(y) = \begin{bmatrix} 0 & y_3 & -y_2 & y_1 \\ -y_3 & 0 & y_1 & y_2 \\ y_2 & -y_1 & 0 & y_3 \\ -y_1 & -y_2 & -y_3 & 0 \end{bmatrix}. \tag{19}$$

Then, from Wang and Huang (2022), the error system of (4) can be put as follows:

$$\dot{\tilde{v}}(t) = A_v(t)\tilde{v}(t) + e_v(t), \tag{20a}$$

$$\dot{\tilde{\eta}}(t) = A_\eta(t)\tilde{\eta}(t) + F_d(t) + e_\eta(t), \tag{20b}$$

where $A_\eta(t) = \frac{1}{2}\text{block diag}\left(\mathcal{M}(Ev_1), ..., \mathcal{M}(Ev_N)\right) - \mu_\eta (H_{\sigma(t)} \otimes I_4)$, $A_v(t) = I_N \otimes S - \mu_v (H_{\sigma(t)} \otimes I_n)$, $F_d(t) = \mathrm{col}\left(\frac{1}{2}\mathcal{M}(E\tilde{v}_1)q_0, ..., \frac{1}{2}\mathcal{M}(E\tilde{v}_N)q_0\right)$, $e_v = \mathrm{col}(e_{v1}, ..., e_{vN})$, $e_\eta = \mathrm{col}(e_{\eta 1}, ..., e_{\eta N})$.

Our main result is summarized as follows:

Theorem 1. Under Assumption 1 and Assumption 3, for any $\mu_\eta, \alpha_v, \alpha_\eta > 0$, let $\mu_v > \frac{2\|S\|}{\delta_1}$, $\beta_v \in (0, \frac{\mu_v \delta_1}{2} - \|S\|)$, and $\beta_\eta \in (0, \min\{\beta_v, \mu_\eta \delta_1\})$. Then, for any initial condition with $q_0(0) \in \mathbb{Q}_u$, the trajectories of (3) and (4) exist for all $t \geq 0$ and are such that for all $i = 1, ..., N$, $\lim_{t\to\infty}(v_i(t) - v(t)) = 0$ and $\lim_{t\to\infty}(\eta_i(t) - q_0(t)) = 0$ both exponentially.

Proof: First consider (20a). Let $[0, T_M^v)$ with $0 < T_M^v \leq \infty$ be the maximally defined interval for the solution of (20a) with the triggering mechanism (4c). By Theorem 4.1 of He and Huang (2021), under Assumptions 1 and 3, let $\mu_v > \frac{2\|S\|}{\delta_1}$, the origin of the following system

$$\dot{\tilde{v}}(t) = A_v(t)\tilde{v}(t) \tag{21}$$

is globally exponentially stable at the rate of at least $\frac{\mu_v\delta_1}{2} - \|S\|$. Set $0 < \beta_v < \frac{\mu_v\delta_1}{2} - \|S\|$. Invoking Lemma 2 to (20a) gives, along the trajectory of (20a),

$$\|\tilde{v}(t)\| \leq c_{v1} e^{-\beta_v t}, \ t \in [0, T_M^v), \tag{22}$$

for some $c_{v1} > 0$.

Next, we show that there is a positive lower bound of the inter-event time $t_{k+1}^{v,i} - t_k^{v,i}$, which implies that $T_M^v = \infty$. First note that, for $t \in [t_k^{v,i}, t_{k+1}^{v,i})$, $\frac{\mathrm{d}}{\mathrm{d}t}\|e_{vi}(t)\| = \frac{e_{vi}(t)^\top \dot{e}_{vi}(t)}{\|e_{vi}(t)\|} \leq \|\dot{e}_{vi}(t)\|$, and (4a), (4c), (22) imply that there exists $c_{v2} > 0$ such that, for $t \in [t_k^{v,i}, t_{k+1}^{v,i})$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|e_{vi}(t)\| \leq \|\dot{e}_{vi}(t)\| = \|\dot{v}_{ei}(t)\| \\
=&\left\| \mu_v \sum_{j\in\mathcal{N}_i(t)} \left(S v_j(t) + v_{ej}(t_k^{v,j}) - S v_i(t) - v_{ei}(t_k^{v,i}) \right) \right\| \\
\leq&\left\| \mu_v \sum_{j\in\mathcal{N}_i(t)} (S\tilde{v}_j(t) - S\tilde{v}_i(t)) \right\| \\
&+ \left\| \mu_v \sum_{j\in\mathcal{N}_i(t)} (e_{vj}(t) - e_{vi}(t)) + (v_{ej}(t) - v_{ei}(t)) \right\| \\
\leq& c_{v2} e^{-\beta_v t} \ t \in [0, t_M^v).
\end{aligned} \tag{23}$$

Since $e_{vi}(t_k^{v,i}) = 0$ for all i, k, (23) further yields

$$\|e_{vi}(t)\| \leq \int_{t_k^{v,i}}^t c_{v2} e^{-\beta_v \tau} \mathrm{d}\tau = \frac{c_{v2}}{\beta_v} e^{-\beta_v t}\left(e^{\beta_v (t - t_k^{v,i})} - 1\right), \tag{24}$$

for $t \in [t_k^{v,i}, t_{k+1}^{v,i})$.

Note that the right-hand side of (24) is monotonically increasing with respect to t and the next triggering time instant $t_{k+1}^{v,i}$ must satisfy $\|e_{vi}((t_{k+1}^{v,i})^-)\| \geq \alpha_v e^{-\beta_v t_{k+1}^{v,i}}$, which, together with (24) gives $t_{k+1}^{v,i} - t_k^{v,i} \geq \frac{1}{\beta_v}\ln\left(\frac{\alpha_v\beta_v}{c_{v2}} + 1\right)$. As $\frac{1}{\beta_v}\ln\left(\frac{\alpha_v\beta_v}{c_{v2}} + 1\right)$ is a positive constant independent of t, we conclude that there is a dwell time no smaller than $\frac{1}{\beta_v}\ln\left(\frac{\alpha_v\beta_v}{c_{v2}}+1\right)$ between any two triggering time instants of (4a).

Now we turn our attention to (20b). First consider the following time-varying system

$$\begin{aligned}\dot{\tilde{\eta}}(t) =& \Big(\frac{1}{2}\text{block diag}\left(\mathcal{M}(Ev_1),...,\mathcal{M}(Ev_N)\right) \\ & - \mu_\eta(H_{\sigma(t)}\otimes I_4)\Big)\tilde{\eta}(t), \end{aligned} \tag{25}$$

which takes the form (9) with $G_i = \frac{1}{2}\mathcal{M}(Ev_i)$, $\mu = \mu_\eta$, $F(t) = H_{\sigma(t)}$. Since under Assumptions 1 and 3, (25) satisfies all the conditions of Lemma 4, it is globally exponentially stable at the rate of at least $\mu_\eta\delta_1 - \epsilon$ for all $\epsilon \in (0, \mu_\eta\delta_1)$.

Now consider (20b). Let $[0, T_M^\eta)$ with $0 < T_M^\eta \leq \infty$ be the maximally defined interval for the solution of (20b) with the triggering mechanism (4d). Since $\|e_\eta(t)\| \leq N\alpha_\eta e^{-\beta_\eta t}$ by (4d) and $\|F_d(t)\| \to 0$ exponentially at the rate of at least β_v by (22), letting $0 < \beta_\eta < \min\{\beta_v, \mu_\eta\delta_1\}$ and making use of Lemma 2 gives

$$\|\tilde{\eta}(t)\| \leq c_{\eta 1}e^{-\beta_\eta t},\ t\in[0, T_M^\eta), \tag{26}$$

for some $c_{\eta 1} > 0$.

Next, we show $T_M^\eta = \infty$. First note that, for $t \in [t_k^{\eta,i}, t_{k+1}^{\eta,i})$, $\frac{\mathrm{d}}{\mathrm{d}t}\|e_{\eta i}(t)\| = \frac{e_{\eta i}(t)^\top \dot{e}_{\eta i}(t)}{\|e_{\eta i}(t)\|} \leq \|\dot{e}_{\eta i}(t)\|$. By (4b), (4d), (5b), for $t \in [t_k^{\eta,i}, t_{k+1}^{\eta,i})$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|e_{\eta i}(t)\| \leq \|\dot{e}_{\eta i}(t)\| = \|\dot{\eta}_{ei}(t)\| \\ \leq & \left\|\mu_\eta\sum_{j\in\mathcal{N}_i(t)}\left(\frac{1}{2}\eta_j\odot\mathbf{Q}(Ev_j) - \frac{1}{2}\eta_i\odot\mathbf{Q}(Ev_i)\right)\right\| \\ &+ \left\|\mu_\eta\sum_{j\in\mathcal{N}_i(t)}\left(\eta_{ej}(t_k^{\eta,j}) - \eta_{ei}(t_k^{\eta,i})\right)\right\| \\ \leq & \left\|\mu_\eta\sum_{j\in\mathcal{N}_i(t)}\left(\frac{1}{2}(\tilde{\eta}_j - \tilde{\eta}_i)\odot\mathbf{Q}(Ev)\right)\right\| \\ &+ \left\|\mu_\eta\sum_{j\in\mathcal{N}_i(t)}\left(\frac{1}{2}(\tilde{\eta}_j - \tilde{\eta}_i)\odot\mathbf{Q}(E\tilde{v}_j)\right)\right\| \\ &+ \left\|\mu_\eta\sum_{j\in\mathcal{N}_i(t)}\left(\frac{1}{2}q_0\odot\mathbf{Q}(E\tilde{v}_j - E\tilde{v}_i)\right)\right\| \\ &+ \left\|\mu_\eta\sum_{j\in\mathcal{N}_i(t)}\left(\frac{1}{2}\tilde{\eta}_i\odot\mathbf{Q}(E\tilde{v}_j - E\tilde{v}_i)\right)\right\| \\ &+ \left\|\mu_\eta\sum_{j\in\mathcal{N}_i(t)}\left((e_{\eta j}(t) - e_{\eta i}(t)) + (\eta_{ej}(t) - \eta_{ei}(t))\right)\right\|. \end{aligned} \tag{27}$$

Since, under Assumptions 1 and 3, for any $\beta_\eta < \min\{\lambda_\eta, \beta_v\}$, each term on the right-hand side of (27) exponentially converges to zero at the rate of at least β_η, there exists some $c_{\eta 2} > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e_{\eta i}(t)\| \leq c_{\eta 2}e^{-\beta_\eta t},\ t\in[0, T_M^\eta). \tag{28}$$

Like the previous argument, (28) further yields

$$\|e_{\eta i}(t)\| \leq \int_{t_k^{\eta,i}}^{t} c_{\eta 2}e^{-\beta_\eta\tau}\mathrm{d}\tau = \frac{c_{\eta 2}}{\beta_\eta}e^{-\beta_\eta t}\left(e^{\beta_\eta(t - t_k^{\eta,i})} - 1\right), \tag{29}$$

for $t \in [t_k^{\eta,i}, t_{k+1}^{\eta,i})$ since $e_{\eta i}(t_k^{\eta,i}) = 0$; and the event-triggered mechanism (4d) implies that $\|e_{\eta i}((t_{k+1}^{\eta,i})^-)\| \geq \alpha_\eta e^{-\beta_\eta t_{k+1}^{\eta,i}}$, which, combined with (29) yields $t_{k+1}^{\eta,i} - t_k^{\eta,i} \geq \frac{1}{\beta_\eta}\ln\left(\frac{\alpha_\eta\beta_\eta}{c_{\eta 2}}+1\right)$. Thus, there is a lower bound, which is independent of t, between two triggering times of (4b). As a result, the solution of the event-triggered distributed observer (4) is well-defined over $[0,\infty)$. This fact together with equations (22) and (26) concludes that $\lim_{t\to\infty}(v_i(t) - v(t)) = 0$ and $\lim_{t\to\infty}(\eta_i(t) - q_0(t)) = 0$ both exponentially. The proof is thus completed. □

Remark 3. Since both μ_v and μ_η can be arbitrarily large, β_v and β_η can take any positive numbers. Thus, Theorem 1 effectively overcomes the drawback of Lemma 1.

Remark 4. It can be easily shown that the positive lower bounds of $t_{k+1}^{v,i} - t_k^{v,i}$ and $t_{k+1}^{\eta,i} - t_k^{\eta,i}$ are monotonically decreasing respectively with respect to β_v and β_η. Thus, even though large β_v and β_η will lead to faster convergence rate, they may incur shorter dwell time for the sequence $t_k^{v,i}$ and $t_k^{\eta,i}$, respectively.

5. AN APPLICATION TO THE COOPERATIVE RIGID BODY CONTROL PROBLEM

In this section, we will apply the event-triggered distributed observer to the leader-following attitude consensus of multiple rigid body systems as studied in Liu and Huang (2018). Define the attitude tracking error $q_{\epsilon i}$ and the angular velocity tracking error $\omega_{\epsilon i}$ as follows:

$$q_{\epsilon i} = q_0^*\odot q_i, \tag{30a}$$
$$\omega_{\epsilon i} = \omega_i - \mathbf{C}(q_{\epsilon i})\omega_0. \tag{30b}$$

By Proposition 1 of Yuan (1988), the body frame of the ith follower coincide with the body frame of the leader if and only if $\hat{q}_{\epsilon i} = 0$. Thus, the leader-following consensus problem is formulated as follows:

Problem 1. Given the leader system (3), the N followers (2), and a digraph $\mathcal{G}_{\sigma(t)}$, design a distributed control law such that, for any initial condition with $q_i(0) \in \mathbb{Q}_u$, there hold $\lim_{t\to\infty}\hat{q}_{\epsilon i}(t) = 0$ and $\lim_{t\to\infty}\omega_{\epsilon i}(t) = 0$, $i = 0, ..., N$.

Following the same argument as that in Wang and Huang (2022), the solution of Problem 1 under jointly connected acyclic graphs is summarized as follows:

Theorem 2. Under Assumptions 1 and 3, for any $\mu_\eta > 0$, $\alpha_v > 0$, $\alpha_\eta > 0$, $k_{1i} > 0$, $k_{2i} > 0$, let $\mu_v > \frac{2\|S\|}{\delta_1}$, $\beta_v \in (0, \frac{\mu_v\delta_1}{2} - \|S\|)$, and $\beta_\eta \in (0, \min\{\beta_v, \mu_\eta\delta_1\})$. Then Problem 1 is solvable by combining the event-triggered distributed observer (4) and the control input

$$\begin{aligned} u_i(t) =& -k_{2i}z_i + \omega_i(t)^\times J_i\omega_i(t) \\ & - J_i((z_i(t) - k_{1i}\hat{q}_{ei}(t))^\times\mathbf{C}(q_{ei}(t))Ev_i(t) \\ & + \frac{1}{2}k_{1i}(\bar{q}_{ei}(t)I_3 + \hat{q}_{ei}(t)^\times)(z_i(t) - k_{1i}\hat{q}_{ei}(t)) \\ & - \mathbf{C}(q_{ei}(t))ESv_i(t)). \end{aligned}$$

Next, we use an example to illustrate our approach. Consider the case where $N = 4$. The inertia matrices of follower i is given by $J_i = \begin{bmatrix} 11-i & 0.3 & -0.2 \\ 0.3 & 9-i & 0.1 \\ -0.2 & 0.1 & 6-i \end{bmatrix}$; and the leader takes the form of (3) with $S = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E = I_3$. It is clear that Assumption 1 is satisfied.

The switching communication network is generated by the switching signal being given as follows:

$$\sigma(t) = \begin{cases} 1, \text{ if } kT_1 \leq t < (k+0.25)T_1 \\ 2, \text{ if } (k+0.25)T_1 \leq t < (k+0.5)T_1 \\ 3, \text{ if } (k+0.5)T_1 \leq t < (k+0.75)T_1 \\ 4, \text{ if } (k+0.75)T_1 \leq t < (k+1)T_1 \end{cases} \tag{31}$$

with $T_1 = 1$ second and $k = 0, 1, 2, \ldots$. The edge sets for $\sigma(t) = 1, 2, 3, 4$ are respectively given by $\mathcal{E}_1 = \{(1,3)\}$, $\mathcal{E}_2 = \{(0,1)\}$, $\mathcal{E}_3 = \emptyset$, $\mathcal{E}_4 = \{(1,2),(3,4)\}$. It can be verified that the switching graph satisfies Assumption 3. Moreover, by simple calculation, (18) holds with $\delta_1 = 0.25$.

As Assumptions 1 and 3 are all satisfied, a distributed control law in Theorem 2 can be designed to solve Problem 1. The design parameters are $\mu_v = 20$, $\alpha_v = 5$, $\beta_v = 0.45$, $\mu_\eta = 10$, $\alpha_\eta = 5$, $\beta_\eta = 0.4$, $k_{1i} = 10$, $k_{2i} = 3$, $i = 1, 2, 3, 4$. It can be verified that the above setting satisfies the conditions in Theorem 2.

The performance is evaluated under random chosen initial conditions. As expected, the estimation errors approach 0 asymptotically, and the attitude and angular velocity tracking errors both converge to zero as $t \to \infty$. Due to the space limit, all figures are removed.

6. CONCLUSION

In this paper, we have investigated the design of the event-triggered distributed observer problem in Wang and Huang (2022) by assuming that the communication network is acyclic. As a result, we have shown that the two design parameters β_v and β_η can take any positive value by choosing other parameters appropriately. We have then applied our event-triggered distributed observer to the leader-following consensus problem of multiple rigid body systems and illustrated our design by a numerical example.

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