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Steady States of a Diffusive Population-Toxicant Model with Negative Toxicant-Taxis

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Abstract
This paper is dedicated to studying the steady state problem of a population-toxicant model with negative toxicant-taxis, subject to homogeneous Neumann boundary conditions. The model captures the phenomenon in which the population migrates away from regions with high toxicant density towards areas with lower toxicant concentration. This paper establishes sufficient conditions for the non-existence and existence of non-constant positive steady state solutions. The results indicate that in the case of a small toxicant input rate, a strong toxicant-taxis mechanism promotes population persistence and engenders spatially heterogeneous coexistence (see, Theorem 2.3). Moreover, when the toxicant input rate is relatively high, the results unequivocally demonstrate that the combination of a strong toxicant-taxis mechanism and a high natural growth rate of the population fosters population persistence, which is also characterized by spatial heterogeneity (see, Theorem 2.4).

Keywords Existence · Non-existence · Non-constant steady states · Toxicant-taxis

Mathematics Subject Classification 35B09 · 35J15 · 35K57 · 35Q92

1 Introduction

The investigation of population-toxicant interactions within polluted aquatic environments has garnered significant attention. Extensive research has been conducted on this topic, with studies such as [12–14, 16, 17, 26, 34] focusing on ordinary differential equation models, and [9, 15, 32, 33] concerning matrix population models. Additionally, reaction-advection-diffusion equation models have been considered in works like [35, 38, 39], which involve the influence of unidirectional water flow on population dispersal. However, the movement of populations can also be influenced by toxicants. For instance, populations may exhibit a tendency to migrate from regions with high toxicant concentrations to areas with lower concentrations, thereby enhancing their chances of survival [37].

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To describe this phenomena, Deng et al. [6] introduced the following diffusive population-toxicant model with negative toxicant-taxis:

$$\begin{cases} u_t = d_1\Delta u + \chi\nabla\cdot(u\nabla w) + u(r - mw) - u^2, & x\in\Omega,\ t>0,\\ w_t = d_2\Delta w + h - gw - buw, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\\ (u, v)(x,0) = (u_0, v_0)(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain with a smooth boundary. $u := u(x,t)$ and $w := w(x,t)$ represent the population density and toxicant concentration, respectively, at position x and time t. The positive constants d_1 and d_2 describe the diffusion rates of the population and toxicant, respectively. The term $+\chi\nabla\cdot(u\nabla w)$ denotes the negative toxicant-taxis, which indicates that the population retreats from the area of high density of toxicant to the region with lower toxicant concentration with a taxis coefficient $\chi > 0$. The function $(r - mw)$ (where r and m are both positive constants) represents the toxicant-dependent intrinsic growth rate of the population, and r corresponds to the natural growth rate of the population. The term $-u^2$ describes the competition between populations. Additionally, the positive constant h accounts for the input rate of the toxicant. The parameters $g > 0$ and $b > 0$ represent the decay rate of the toxicant and the uptake rate by the population, respectively.

The solution behaviors of (1.1), including boundedness, globally asymptotical stability, and pattern formation, were explored in the work [6]. More precisely, when h is large enough, the population will go extinct. Conversely, for a small h, [6] demonstrated that the population and the toxicant would reach a spatially homogeneous coexistence state. This coexistence state is achieved when $\chi > 0$ is small. For the case of $\chi > 0$ being large, the authors numerically illustrated the occurrence of spatially heterogeneous coexistence, characterized by non-constant positive steady states. However, the rigorous proof of the existence of such non-constant positive steady state solutions of (1.1) remains an open problem. Additionally, it is still unclear whether the population and the toxicant will achieve a (spatially homogeneous/inhomogeneous) coexistence state when the toxicant input rate is at a moderate level.

To address the above-mentioned questions, this paper considers the stationary problem of (1.1)

$$\begin{cases} d_1\Delta u + \chi\nabla\cdot(u\nabla w) + u(r - mw) - u^2 = 0, & x\in\Omega,\\ d_2\Delta w + h - gw - buw = 0, & x\in\Omega,\\ \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega, \end{cases} \tag{1.2}$$

and focuses on the following mathematical question:

- The non-existence and existence of non-constant positive solutions for (1.2).

Investigating the (non-) existence of non-constant positive steady state solutions can shed light on the intricate interactions between toxicant input, degradation, population growth, and toxicant-taxis, and how these factors collectively impact population persistence. The results are a new and meaningful endeavor in this direction. Specifically, Theorem 2.3 rigorously proves that in the case of a small toxicant input rate, a strong toxicant-taxis mechanism promotes population persistence and engenders spatially heterogeneous coexistence of the toxicant and the population, which solves an open question in [6]. Furthermore, Theorem 2.4

unequivocally demonstrates that, under relatively high toxicant input rates, the combined effect of a strong toxicant-taxis mechanism and a high natural growth rate of population fosters population persistence characterized by spatial heterogeneity. This finding is new.

Numerous studies have been conducted on the existence of non-constant positive solutions to elliptic systems. For example, the works [18, 19] involved the application of the singular perturbation method, and research [2, 4, 5, 10, 22] used the bifurcation technique. In addition, the Leray-Schauder degree theorem, as a variation of the bifurcation approach, is also a powerful technique to investigate the existence of non-constant positive solutions (see [3, 7, 8, 20, 23–25, 28, 29, 36] and references therein). In this work, the existence of non-constant positive solutions to (1.2) is established by employing this approach.

In the sequel, all solutions under consideration are assumed to be classical solutions. The rest of this paper will be organized as follows: Sect. 2 presents the main results, while Sect. 3 establishes a priori estimates for positive solutions of (1.2). The sufficient conditions for the non-existence and existence of non-constant positive steady state solutions of (1.1) are discussed in Sect. 4 and Sect. 5, respectively. Section 6 provides simulations to validate the theoretical findings.

2 Main Results

The main results for (1.1) encompass two parts: the sufficient conditions for non-existence and existence of non-constant positive steady states of (1.1) (i.e., non-constant positive solutions of (1.2)). Before presenting the main results, we note that the non-negative constant solutions of (1.2) fulfill

$$\begin{cases} 0 = u(r - mw - u), \\ 0 = h - gw - buw, \end{cases} \tag{2.1}$$

which can be solved to obtain the following cases:

- semi-coexistence steady state: $(0, \frac{h}{g})$;
- coexistence steady states: (u_1, w_1), (u_2, w_2) and (u_3, w_3)

with (u_i, w_i) $(i = 1, 2, 3)$ are defined as:

$$u_1 = \frac{br - g - \sqrt{(g + rb)^2 - 4bmh}}{2b}, \ w_1 = \frac{h}{bu_1 + g}, \tag{2.2}$$

$$u_2 = \frac{br - g + \sqrt{(g + rb)^2 - 4bmh}}{2b}, \ w_2 = \frac{h}{bu_2 + g}, \tag{2.3}$$

$$u_3 = \frac{br - g}{2b}, \ w_3 = \frac{h}{bu_3 + g}. \tag{2.4}$$

The conditions of existence for these constant steady states are presented in Table 1.

In the sequel, the non-existence results are first presented. Subsequently, the notations employed in the existence theorems are introduced, followed by the existence results.

Table 1 Constant Equilibria of (1.2)

<table>
<tr><th></th><th>$0 < h < \frac{gr}{m}$</th><th>$h = \frac{gr}{m}$</th><th>$\frac{gr}{m} < h < \frac{(g+br)^2}{4bm}$</th><th>$h = \frac{(g+br)^2}{4bm}$</th><th>$h > \frac{(g+br)^2}{4bm}$</th></tr>
<tr><td>$r < \frac{g}{b}$</td><td rowspan="3">$(0, \frac{h}{g})$
(u_2, w_2)</td><td rowspan="2">$(0, \frac{h}{g})$</td><td>$(0, \frac{h}{g})$</td><td rowspan="2">$(0, \frac{h}{g})$</td><td rowspan="3">$(0, \frac{h}{g})$</td></tr>
<tr><td>$r = \frac{g}{b}$</td><td>Not applicable $\left(\text{since } \frac{gr}{m} = \frac{(g+br)^2}{4bm}\right)$</td></tr>
<tr><td>$r > \frac{g}{b}$</td><td>$(0, \frac{h}{g})$
(u_2, w_2)</td><td>$(0, \frac{h}{g})$
(u_1, w_1)
(u_2, w_2)</td><td>$(0, \frac{h}{g})$
(u_3, w_3)</td></tr>
</table>

2.1 Non-existence of Non-constant Positive Steady States

This subsection first gives the non-existence result in the regions $\frac{gr}{m} \leq h$ with $r < \frac{g}{b}$ and $\frac{(g+br)^2}{4bm} \leq h$ with $r \geq \frac{g}{b}$, where only admits constant semi-coexistence steady state. Theorem 2.2 (i) in [6] shows that $(0, \frac{h}{g})$ is globally asymptotically stable, which implies the following statements directly.

Proposition 2.1 *Assume one of the following conditions holds*

(1) $0 < r < (\sqrt{2} - 1)\frac{g}{b}$ *and* $h \in \left[\frac{gr}{m}, \infty\right)$;
(2) $(\sqrt{2} - 1)\frac{g}{b} \leq r < \frac{g}{b}$ *and* $h \in \left[\frac{gr}{m}, \frac{g^2}{bm}\right) \cup \left(\frac{2gr+br^2}{m}, \infty\right)$;
(3) $\frac{g}{b} \leq r$ *and* $h \in \left(\frac{2gr+br^2}{m}, \infty\right)$.

Then for any fixed parameters $\chi, b, g, r, m, h, d_1, d_2$, (1.1) *has no non-constant positive steady state solution.*

In the left cases $0 < h < \frac{gr}{m}$ and $\frac{gr}{m} \leq h \leq \frac{(g+br)^2}{4bm}$ with $r > \frac{g}{b}$, the following results show that (1.1) has no non-constant positive steady state solutions if d_1 is large.

Theorem 2.2 *Assume* $0 < h < \frac{gr}{m}$ *or* $\frac{gr}{m} \leq h \leq \frac{(g+br)^2}{4bm}$ *with* $r > \frac{g}{b}$*, and let* λ_1 *be the smallest positive eigenvalue of* $-\Delta$ *on* Ω *under the homogeneous Neumann boundary condition. Then for any fixed parameters* χ, b, g, r, m, h, d_2*, there admits a positive constant* $D := D(d_2, \chi, b, g, m, r, h, \lambda_1)$ *such that* (1.1) *has no non-constant positive steady state solution if* $d_1 \geq D$.

Remark 2.1 In fact, for $h \in \left[\frac{g^2}{bm}, \frac{2gr+br^2}{m}\right]$ with $(\sqrt{2} - 1)\frac{g}{b} \leq r < \frac{g}{b}$ and $h \in \left(\frac{(g+br)^2}{4bm}, \frac{2gr+br^2}{m}\right]$ with $r \geq \frac{g}{b}$, Theorem 2.2 remains applicable. However, when $d_1 < D$, the non-existence/existence of non-constant steady states in these remaining regions is unclear due to technical obstacles, which remain open for future investigations.

Hence, it is natural to ask whether the non-constant positive steady state solution will exist when parameters outside the non-existence regimes found in Theorem 2.2? To explore this question, we shall apply the Leray-Schauder degree theorem for the parameters regions where exist constant positive steady state solutions (i.e., $0 < h < \frac{gr}{m}$ and $\frac{gr}{m} \leq h \leq \frac{(g+br)^2}{4bm}$ with $r > \frac{g}{b}$) and get positive answers.

2.2 Existence of Non-constant Positive Steady States

Before stating the existence results, some notations are introduced for clarity and simplicity.

- Denote

$$X_i^{\pm}(d_1) = \frac{J_i \pm \sqrt{J_i^2 - 4K_i}}{2}, \ i = 1, 2, 3, \tag{2.5}$$

where

$$\begin{aligned} J_i &:= \frac{g + bu_i}{d_2} - \frac{u_i}{d_1 d_2}(\chi b w_i - d_2), \\ K_i &:= \frac{u_i}{d_1 d_2}(g + bu_i - mbw_i). \end{aligned} \tag{2.6}$$

- Let

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_j < \cdots \to \infty$$

be the eigenvalues of $-\Delta$ on Ω subject to the homogeneous Neumann boundary condition.

- Denote

$$\begin{aligned} \ell_1 &:= \frac{u_2}{(g + bu_2)^2} \left\{ \sqrt{(g + bu_2 - bmw_2)d_2} - \sqrt{bw_2(g\chi + bu_2\chi - md_2)} \right\}^2, \\ \ell_2 &:= \frac{u_3(\chi b w_3 - d_2)}{g + bu_3}, \\ j_0 &:= \min\{j \in \mathbb{N} : q_2 < \lambda_j\} (\geq 1), \\ k_0 &:= \min\{k \in \mathbb{N} : q_1 < \lambda_k\} (\geq 1). \end{aligned}$$

Here

$$\begin{aligned} q_1 &:= \frac{g + bu_2 - bmw_2}{\chi b w_2 - d_2}, \\ q_2 &:= \frac{g + bu_2 - bmw_2 + \sqrt{(g + bu_2 - bmw_2)(g\chi + bu_2\chi - md_2)bw_2/d_2}}{\chi b w_2 - d_2}. \end{aligned} \tag{2.7}$$

- Define the following sequences

$$D_j := \begin{cases} \sup\left\{d_1 > 0 : X_2^-(d_1) < -\lambda_j\right\}, & \text{for } j = j_0, j_0 + 1, \ldots \\ \ell_1, & \text{for } j = j_0 - 1, \end{cases}$$

$$\widetilde{D}_k := \begin{cases} \inf\left\{d_1 > 0 : X_2^+(d_1) < -\lambda_k\right\}, & \text{for } k_0 \leq k \leq j_0 - 1, \\ 0, & \text{for } k = k_0 - 1, \\ \ell_1, & \text{for } k = j_0, \end{cases}$$

$$E_i := \begin{cases} \sup\left\{d_1 > 0 : X_1^-(d_1) < -\lambda_i\right\}, & \text{for } i = 1, 2, \dots \\ \infty, & \text{for } i = 0, \end{cases}$$

$$\widetilde{E}_n := \begin{cases} \sup\left\{d_1 > 0 : X_3^-(d_1) < -\lambda_n\right\}, & \text{for } n = 1, 2, \dots \\ \ell_2, & \text{for } n = 0. \end{cases}$$

Based on the aforementioned notations, we shall show the existence results.

Theorem 2.3 *If* $0 < h < \frac{gr}{m}$ *and* $\chi > \frac{d_2}{bw_2}$, *then* (1.1) *admits at least one non-constant positive steady state solution provided that* $d_1 \in (D_{j+1}, D_j) \cap (\widetilde{D}_k, \widetilde{D}_{k+1})$ *and* $j+k+2$ *is odd, where the integers* $j \geq j_0 - 1 \geq k \geq k_0 - 1$.

Theorem 2.4 *Let* $\chi > \frac{d_2}{bw_2}$ *and* $r > \frac{g}{b}$. *Then* (1.1) *admits at least one non-constant positive steady state solution provided that one of the following conditions holds*:

(c1) $\frac{gr}{m} < h < \frac{(g+br)^2}{4bm}$, *and* $d_1 \in (E_{i+1}, E_i) \cap (D_{j+1}, D_j) \cap (\widetilde{D}_k, \widetilde{D}_{k+1})$ *as well as* $i + j + k + 2$ *is odd, where the integers* $i \geq 0$, $j \geq j_0 - 1 \geq k \geq k_0 - 1$;
(c2) $h = \frac{(g+br)^2}{4bm}$, *and* $d_1 \in (\widetilde{E}_{n+1}, \widetilde{E}_n)$ *as well as* $n + 1$ *is odd* $(n \geq 0)$.

Remark 2.2 The following statements hold:

- Let $\chi > \frac{d_2}{bw_2}$. If $0 < h < \frac{gr}{m}$ or $\frac{gr}{m} < h < \frac{(g+br)^2}{4bm}$ with $r > \frac{g}{b}$, one can check that $\ell_1,\ q_1,\ q_2 > 0$, $j_0 \geq k_0$ and

$$0 \leftarrow \cdots < D_j < \cdots < D_{j_0+1} < D_{j_0} < \ell_1 =: D_{j_0-1},$$
$$0 = \widetilde{D}_{k_0-1} < \widetilde{D}_{k_0} < \widetilde{D}_{k_0+1} < \cdots < \widetilde{D}_{j_0-1} < \widetilde{D}_{j_0} = \ell_1 \text{ for } j_0 > k_0.$$

- When $\frac{gr}{m} < h < \frac{(g+br)^2}{4bm}$, $r > \frac{g}{b}$ and $\chi > \frac{d_2}{bw_2}$, the monotonicity of $X_1^-(d_1)$ and λ_k indicates that

$$0 \leftarrow \cdots < E_i < \cdots < E_2 < E_1 < \infty =: E_0.$$

- If $h = \frac{(g+br)^2}{4bm}$ with $r > \frac{g}{b}$ and $\chi > \frac{d_2}{bw_3}$, one can verify that $\ell_2 > 0$ and the sequence $\{\widetilde{E}_n\}_{n=1}^{\infty}$ satisfies

$$0 \leftarrow \cdots < \widetilde{E}_n < \cdots < \widetilde{E}_2 < \widetilde{E}_1 < \ell_2 =: \widetilde{E}_0.$$

Furthermore, one has $u_1 = u_2 = u_3$, $\ell_2 = \ell_1$, $X_1^-(d_1) = X_2^-(d_1) = X_3^-(d_1)$, $q_1 = q_2 = 0$, $j_0 = 1$ and hence $E_j = D_j = \widetilde{E}_j$ for $j \geq 1$.

Remark 2.3 Based on the above results and Theorem 2.2 (ii) in [6], there is a constant

$$0 < d_1 < \frac{d_2 h(br - g + \sqrt{(g+br)^2 - 4bmh})}{4m(br + g + \sqrt{(g+br)^2 - 4bmh})}$$

such that

$$\chi_c := \sqrt{\frac{4md_1d_2(br + g + \sqrt{(g+br)^2 - 4bmh})}{h(br - g + \sqrt{(g+br)^2 - 4bmh})}} < \frac{d_2}{bw_2}.$$

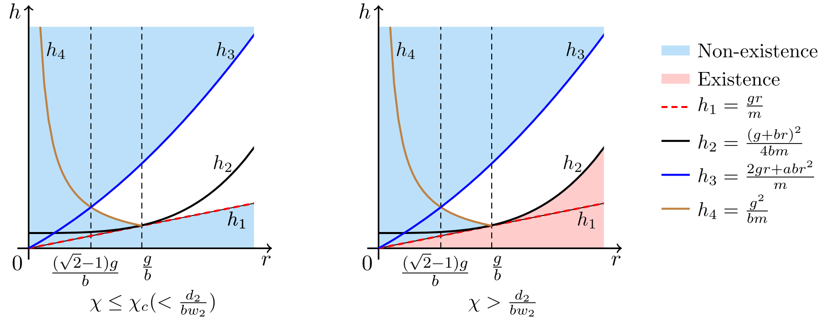

Fig. 1 The schematic of existence/non-existence regions of non-constant positive steady states

For such d_1 that also satisfies the conditions outlined in Theorem 2.3 - 2.4, one can visually depict the regions where non-constant positive steady state solutions to (1.1) exist or do not exist, for $\chi \le \chi_c$ or $\chi > \frac{ad_2}{bw_2}$, see in Fig. 1.

3 A Priori Estimates

This section aims to give a priori estimates of solutions to (1.2). The first lemma shows the upper bound.

Lemma 3.1 *Let (u, w) be any positive solution to (1.2). Then (u, w) satisfies*

$$\max_{\bar{\Omega}} u(x) \le r e^{\frac{\chi h}{d_1 g}}, \ \max_{\bar{\Omega}} w(x) \le \frac{h}{g} \text{ and } \bar{u} := \frac{1}{|\Omega|}\int_{\Omega} u \le r \tag{3.1}$$

for all $x \in \bar{\Omega}$.

Proof Let $x_1 \in \bar{\Omega}$ satisfy $w(x_1) = \max_{\bar{\Omega}} w(x)$. Then applying the maximum principle [23], one has

$$h - gw(x_1) - bu(x_1)w(x_1) \ge 0,$$

which implies $w(x_1) \le \frac{h}{g}$. Hence, $\max_{\bar{\Omega}} w(x) \le \frac{h}{g}$ for all $x \in \bar{\Omega}$.

Note that

$$d_1 \nabla \cdot (\nabla u + \frac{\chi}{d_1} u \nabla w) = d_1 \nabla \cdot (e^{-\frac{\chi w}{d_1}} \nabla (u e^{\frac{\chi w}{d_1}})).$$

Then setting $V = u e^{\frac{\chi w}{d_1}}$, (1.2) can be reduced to

$$\begin{cases} -d_1 \nabla \cdot (e^{-\frac{\chi w}{d_1}} \nabla V) = V e^{-\frac{\chi w}{d_1}} (r - mw - V e^{-\frac{\chi w}{d_1}}), & x \in \Omega, \\ -d_2 \Delta w = h - gw - bwV e^{-\frac{\chi w}{d_1}}, & x \in \Omega, \\ \frac{\partial V}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Suppose $x_2 \in \bar{\Omega}$ is a maximum point of $V : V(x_2) = \max_{\bar{\Omega}} V(x)$.

Case 1: $x_2 \in \Omega$. Since $V(x_2) = \max_{\bar{\Omega}} V(x)$, one has $\nabla V(x_2) = 0$ and $\Delta V(x_2) \le 0$, therefore

$$\nabla \cdot \left(e^{-\frac{\chi w}{d_1}} \nabla V\right)\Big|_{x=x_2} = e^{-\frac{\chi w}{d_1}} \Delta V\Big|_{x=x_2} - \frac{\chi}{d_1} e^{-\frac{\chi w}{d_1}} \nabla w \cdot \nabla V\Big|_{x=x_2} \le 0.$$

It follows from the first equation of (3.2) that

$$V(x_2) \le (r - mw(x_2)) e^{\frac{\chi w(x_2)}{d_1}} \le r e^{\frac{\chi h}{d_1 g}},$$

which gives $\max_{\bar{\Omega}} V(x) \le r e^{\frac{\chi h}{d_1 g}}$ for all $x \in \bar{\Omega}$. Since $V(x) = u(x) e^{\frac{\chi w(x)}{d_1}}$, one obtains

$$\max_{\bar{\Omega}} u(x) = \max_{\bar{\Omega}} V(x) e^{-\frac{\chi w(x)}{d_1}} \le r e^{\frac{\chi h}{d_1 g}} \text{ for all } x \in \bar{\Omega}.$$

Case 2: $x_2 \in \partial\Omega$. Suppose $q(x_2) := r - mw(x_2) - V(x_2) e^{-\frac{\chi w}{d_1}} < 0$. By the continuity of w, V and $q(x)$, one can find a small ball U in $\bar{\Omega}$ with $\partial U \cap \partial\Omega = \{x_2\}$ such that

$$r - m - V(x) e^{-\frac{\chi w}{d_1}} < 0 \text{ for all } x \in U.$$

Hence, $\nabla \cdot (e^{-\frac{\chi w}{d_1}} \nabla V) > 0$ for all $x \in U$. Due to $V(x_2) = \max_{\bar{\Omega}} V(x)$, then the Hopf Boundary lemma [31] yields $\frac{\partial V}{\partial \nu}\Big|_{x=x_2} > 0$. This is a contradiction to the condition $\frac{\partial V}{\partial \nu}\Big|_{x \in \partial\Omega} = 0$. Therefore, one has

$$r - mw(x_2) - V(x_2) e^{-\frac{\chi w(x_2)}{d_1}} \ge 0.$$

Proceeding the same procedure as Case 1, one gets $\max_{\bar{\Omega}} u(x) \le r e^{\frac{\chi h}{d_1 g}}$ for all $x \in \bar{\Omega}$ directly.

Integrating the first equation of (1.2) and applying Young's inequality, one obtains $\frac{1}{|\Omega|} \int_\Omega u \le r$. Hence, the proof of Lemma 3.1 is completed. □

Lemma 3.2 *Let* $\varepsilon > 0$ *be any fixed positive constant*, $d_1, d_2 \ge \varepsilon$ *and* (u, w) *be any positive solution to* (1.2). *Then there admits a constant* $M := M(\varepsilon, \chi, r, m, g, h, b, |\Omega|) > 0$ *such that for any* $p \ge 1$

$$\|(u, w)\|_{W^{2,p}(\Omega)} \le M.$$

Proof Set positive constants $M_i := M_i(\varepsilon, \chi, r, m, g, h, b, |\Omega|)$ with $i = 1, 2, \ldots, 7$. It follows from Lemma 3.1 that

$$\left\| \frac{1}{d_2}(h - gw - buw) \right\|_{L^p(\Omega)} \le \frac{1}{\varepsilon} \left(2h + \frac{brh}{g} e^{\frac{\chi h}{\varepsilon g}}\right) |\Omega|^{\frac{1}{p}} =: M_1$$

for any $p \ge 1$. Then the L^p-theory of elliptic equations (see, e.g., [11, Theorem 8.33]) implies that there is a constant $M_2 > 0$ such that

$$\|w\|_{W^{2,p}(\Omega)} \le M_2, \tag{3.3}$$

which, along with Sobolev embedding, gives $\|w\|_{C^{1,\alpha}(\Omega)} \le M_3$ with some constant $0 < \alpha < 1$.

Next, we shall show the $W^{2,p}$ – bound of u. Applying the elliptic regularity to (3.2), one has

$$\|V\|_{C^1(\bar{\Omega})} = \|ue^{\frac{\chi w}{d_1}}\|_{C^1(\bar{\Omega})} \leq M_4.$$

The fact $V = ue^{\frac{\chi w}{d_1}}$ implies that

$$\nabla u = \nabla(Ve^{-\frac{\chi w}{d_1}}) = e^{-\frac{\chi w}{d_1}}\nabla V - \frac{\chi}{d_1}Ve^{-\frac{\chi w}{d_1}}\nabla w,$$

and hence

$$|\nabla u| \leq |\nabla V| + \frac{\chi}{d_1}|V\nabla w| \leq M_5. \tag{3.4}$$

On the other hand, we can rewrite the first equation in (1.2) as

$$\begin{cases} -\Delta u = \frac{1}{d_1}(\chi\nabla u\cdot\nabla w + \chi u\Delta w + u(r-mw) - u^2), & x\in\Omega,\\ \frac{\partial u}{\partial \nu} = 0, & x\in\partial\Omega. \end{cases} \tag{3.5}$$

Then Lemma 3.1, (3.3) and (3.4) ensure that we can find a constant $M_6 > 0$ such that

$$\left\|\frac{1}{d_1}(\chi\nabla u\cdot\nabla w + \chi u\Delta w + u(r-mw) - u^2)\right\|_{L^p(\Omega)} \leq M_6$$

for any $p \geq 1$. By utilizing the L^p-theory of elliptic equations (see, e.g., [11, Theorem 8.33]) once again, we can obtain a constant $M_7 > 0$ such that $\|u\|_{W^{2,p}(\Omega)} \leq M_7$. This allows us to complete the proof of Lemma 3.2. □

4 Non-existence of Non-constant Positive Steady States

In this section, we shall show the sufficient condition for the non-existence of non-constant positive steady state solutions in the parameter regions where $0 < h < \frac{gr}{m}$ and $\frac{gr}{m} \leq h \leq \frac{(g+br)^2}{4bm}$ with $r > \frac{g}{b}$.

Proof of Theorem 2.2 Let $(\bar{u}, \bar{w}) := \frac{1}{|\Omega|}(\int_\Omega u, \int_\Omega w)$. Multiplying the first equation of (1.2) by $u - \bar{u}$, noting $\int_\Omega (u-\bar{u}) = 0$ and using Young's inequality, $0 < u \leq re^{\frac{\chi h}{d_1 g}}$ and $\bar{u} \leq r$, we have

$$\begin{aligned} d_1\int_\Omega |\nabla u|^2 &= \chi\int_\Omega \nabla\cdot(u\nabla w)(u-\bar{u}) + \int_\Omega u(r-mw-u)(u-\bar{u})\\ &= -\chi\int_\Omega u\nabla w\cdot\nabla u + \int_\Omega (r-mw-u)(u-\bar{u})^2 + \bar{u}\int_\Omega (r-mw-u)(u-\bar{u})\\ &= -\chi\int_\Omega u\nabla w\cdot\nabla u + \int_\Omega (r-mw-u-\bar{u})(u-\bar{u})^2 - \bar{u}m\int_\Omega (w-\bar{w})(u-\bar{u})\\ &\leq r\int_\Omega |\nabla u|^2 + \frac{\chi^2 r}{4}e^{\frac{2\chi h}{d_1 g}}\int_\Omega |\nabla w|^2 + r\int_\Omega (u-\bar{u})^2 + \frac{rm^2}{4}\int_\Omega (w-\bar{w})^2. \end{aligned} \tag{4.1}$$

By Poinaré-Wirtinger inequality $\lambda_1\|v-\bar{v}\|^2_{L^2(\Omega)} \le \|\nabla v\|_{L^2(\Omega)}$ for any $v \in H^1(\Omega)$, where λ_1 is the smallest positive eigenvalue of $-\Delta$ on Ω under the homogeneous Neumann boundary condition and $\bar{v} := \frac{1}{|\Omega|}\int_\Omega v$, it follows from (4.1) that

$$\left(d_1 - \frac{r(\lambda_1+1)}{\lambda_1}\right)\|\nabla u\|^2_{L^2(\Omega)} \le \frac{r}{4}\left(\chi^2 e^{\frac{2\chi h}{d_1 g}} + \frac{m^2}{\lambda_1}\right)\|\nabla w\|^2_{L^2(\Omega)}. \tag{4.2}$$

The second equation of (1.2) gives

$$\begin{aligned} d_2\int_\Omega |\nabla w|^2 &= \int_\Omega (h - gw - buw)(w-\bar{w}) \\ &= -g\int_\Omega w(w-\bar{w}) - b\int_\Omega uw(w-\bar{w}) \\ &= (-b\bar{u} - g)\int_\Omega (w-\bar{w})^2 - b\int_\Omega w(u-\bar{u})(w-\bar{w}) \\ &\le -b\bar{u}\int_\Omega (w-\bar{w})^2 + \frac{b^2h^2}{4g^3}\int_\Omega (u-\bar{u})^2, \end{aligned}$$

where we have used Young's inequality and (3.1). Applying Poinaré-Wirtinger inequality again, we update the above inequality as

$$\|\nabla w\|^2_{L^2(\Omega)} \le \frac{b^2h^2}{4g^3 d_2\lambda_1}\|\nabla u\|^2_{L^2(\Omega)}, \tag{4.3}$$

which substituted into (4.2) indicates

$$H(d_1)\|\nabla u\|^2_{L^2(\Omega)} \le 0$$

with $H(d_1) := d_1 - \frac{rb^2h^2\chi^2}{16g^3d_2\lambda_1}e^{\frac{2\chi h}{d_1 g}} - \frac{r}{\lambda_1}\left(\frac{b^2h^2m^2}{16g^3d_2\lambda_1} + \lambda_1 + 1\right)$.

Denote

$$F(d_1) := \frac{rb^2h^2\chi^2}{16g^3d_2\lambda_1}e^{\frac{2\chi h}{d_1 g}} + \frac{r}{\lambda_1}\left(\frac{b^2h^2m^2}{16g^3d_2\lambda_1} + \lambda_1 + 1\right),$$

then $F(d_1) > 0$ and $F_{d_1}(d_1) < 0$. Therefore, there admits a unique positive constant $D := D(d_2, \chi, b, g, m, r, h, \lambda_1)$ such that $d_1 > F(d_1)$ for all $d_1 \ge D$. This implies $0 \le H(d_1)\|\nabla u\|^2_{L^2(\Omega)} \le 0$ and hence u is constant. It follows from (4.3) that w is also constant. The proof is completed. □

5 Existence of Non-constant Positive Steady States

In this section, we shall prove the existence of non-constant positive solutions of (1.2). Before proceeding, we introduce some notations used later. Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the eigenvalues of $-\Delta$ on Ω subject to the homogeneous Neumann boundary condition, $E(\lambda_k)$ be the eigenspace respect to λ_k in $C^1(\bar{\Omega})$ and $\{\varphi_{kj} : j = 1, 2, \ldots, \dim E(\lambda_k)\}$ be an orthonormal basis of $E(\lambda_k)$. Then

$$\left\{(u,w)\in\left[C^1(\bar\Omega)\right]^2\,\middle|\,\partial_\nu u=\partial_\nu w=0 \text{ on } \partial\Omega\right\}=: Y=\bigoplus_{k=0}^{\infty} Y_k \text{ and } Y_k=\bigoplus_{j=1}^{\dim E(\lambda_k)} Y_{kj},$$

where $Y_{kj}=\left\{\mathbf{c}\varphi_{kj},\mathbf{c}\in\mathbb{R}^2\right\}$.

Plugging $\Delta w=\frac{gw+buw-h}{d_2}$ into the first equation of (1.2), we can reduce (1.2) as

$$\begin{cases}-\Delta u=\frac{\chi}{d_1}\nabla u\cdot\nabla w+\frac{\chi}{d_1d_2}u(gw+buw-h)+\frac{u(r-mw-au)}{d_1}, & x\in\Omega,\\ -\Delta w=\frac{h-gw-buw}{d_2}, & x\in\Omega,\\ \frac{\partial u}{\partial\nu}=\frac{\partial w}{\partial\nu}=0, & x\in\partial\Omega,\end{cases}\tag{5.1}$$

which is equivalent to

$$\mathbf{u}=(I-\Delta)^{-1}\cdot[\mathbf{u}+\mathbf{G}(\mathbf{u},d_1)]=:\mathbf{H}(\mathbf{u}).$$

Here $\mathbf{u}:=(u,w)\in Y$, $(I-\Delta)^{-1}$ denotes the inverse of $I-\Delta$ under homogeneous Neumann boundary condition and the operator

$$\mathbf{G}(\mathbf{u},d_1):=\begin{pmatrix}\frac{\chi}{d_1}\nabla u\cdot\nabla w+\frac{\chi}{d_1d_2}u(gw+buw-h)+\frac{u(r-mw-au)}{d_1}\\ \frac{h-gw-buw}{d_2}\end{pmatrix}^{\mathcal{T}}.\tag{5.2}$$

Then $\mathbf{u}$ is a solution to (1.2) iff $\mathbf{u}$ satisfies

$$\mathbf{F}(\mathbf{u},d_1)=0,\tag{5.3}$$

where $\mathbf{F}(\cdot):=I-\mathbf{H}(\cdot)$ and $\mathbf{H}(\cdot)$ is a compact operator from Y to itself.

To get a non-constant positive solution of (5.3), we shall use the Leray-Schauder degree theory to the operator $\mathbf{F}$ in a subset of Y. To this end, we need to introduce a bounded set and check the condition for the application of the Leray-Schauder degree theory.

Lemma 5.1 *Let $\varepsilon>0$ be any fixed constant, assume $d_1,\ d_2\geq\varepsilon$ and $h\neq\frac{gr}{m}$. Then there exist two positive constants $R:=R(\varepsilon)$ and $\delta:=\delta(\varepsilon)$ such that any positive solution $\mathbf{u}$ of $\mathbf{F}(\mathbf{u},d_1)=0$ fulfills $\mathbf{u}\in\Sigma$ ($\mathbf{u}\notin\partial\Sigma$), where the set Σ is defined by*

$$\Sigma:=\left\{(u,w)\in U_R:\min_{\bar\Omega}u(x)>\delta,\ \min_{\bar\Omega}w(x)>\delta\right\}\tag{5.4}$$

with

$$U_R:=\left\{(u,w)\in Y:\|u\|_{C^1(\bar\Omega)}<R,\ \|w\|_{C^1(\bar\Omega)}<R\right\}.\tag{5.5}$$

Proof For any fixed constant $\varepsilon>0$, let $\mathbf{u}$ be the solution of (5.3) with $d_1,\ d_2\geq\varepsilon$. Then Lemma 3.2 ensures that there admits a positive constant $R:=R(\varepsilon)$ such that

$$\|u\|_{C^1(\bar\Omega)}<R,\ \|w\|_{C^1(\bar\Omega)}<R.$$

Thus, we can introduce a bounded set U_R (see (5.5)) and any positive solution $\mathbf{u}$ of $\mathbf{F}(\mathbf{u},d_1)=0$ belongs to U_R.

We next prove that there admits a constant $\delta := \delta(\varepsilon) > 0$ such that the solution $\mathbf{u}$ of (5.3) with $d_1,\ d_2 \geq \varepsilon$ satisfies $\min_{\bar{\Omega}} u(x) > \delta$, $\min_{\bar{\Omega}} w(x) > \delta$. Let $x_1 \in \bar{\Omega}$ be a minimum point of w: $w(x_1) = \min_{\bar{\Omega}} w(x)$. We apply the maximum principle [23, Proposition 2.2] to the second equation of (1.2), and get

$$h - gw(x_1) - bu(x_1)w(x_1) \leq 0,$$

which combined with (3.1) gives

$$w(x) \geq \min_{\bar{\Omega}} w(x) = w(x_1) \geq \frac{h}{g + bu(x_1)} > \frac{h}{2(g + bre^{\frac{\chi h}{\varepsilon g}}/a)} =: \delta_1(\varepsilon)$$

for all $x \in \bar{\Omega}$.

Now, assume for contradiction that for any $\delta > 0$, we can find $d_{1,\delta} \geq \varepsilon$ such that the corresponding positive solution $\mathbf{u}_\delta := (u_\delta, w_\delta)^{\mathcal{T}}$ of $\mathbf{F}(\mathbf{u}, d_1) = 0$ satisfy $0 \leq \min_{\bar{\Omega}} u_\delta(x) \leq \delta$. Fixing $d_2 \geq \varepsilon$ for some fixed small $\varepsilon > 0$, we only need to consider the case when $\varepsilon \leq d_{1,\delta} < D$ by Theorem 2.2. It follows from Lemma 3.2 that we can find a sequence $\{\delta_j\}_{j=1}^{\infty}$ satisfying

$$\lim_{j \to \infty} \delta_j = 0$$

such that $(u_j, w_j, d_{1,j}) := (u_{\delta_j}, w_{\delta_j}, d_{1,\delta_j})$ fulfills that as $j \to \infty$

$$\min_{\bar{\Omega}} u_j \to 0$$

and

$$(u_j, w_j, d_{1,j}) \to (u_\infty, w_\infty, d_{1,\infty}) \text{ in } C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \times [\varepsilon, D),$$

where functions $0 \leq u_\infty,\ w_\infty \in C^1(\bar{\Omega})$ and the contant $d_{1,\infty} \geq \varepsilon > 0$. Then, we have

$$\min_{\bar{\Omega}} u_\infty = 0.$$

Applying the strong maximum principle to the first equation of (1.2), we obtain $u_\infty \equiv 0$. Then the second equation of (1.2) implies

$$-d_2 \Delta w_\infty = h - gw_\infty \text{ in } \Omega, \quad \frac{\partial w_\infty}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Integrating the above equation over Ω, one has

$$0 = \int_\Omega (h - gw_\infty),$$

which, along with (3.1), gives $w_\infty = \frac{h}{g}$.

One the other hand, we define $\widetilde{u}_j := \frac{u_j}{\|u_j\|_{L^\infty}}$. Then it satisfies that $\|\widetilde{u}_j\|_{L^\infty} = 1$ for any $j \geq 1$ and

$$\begin{cases} -d_{1,j} \Delta \widetilde{u}_j = \chi \nabla \cdot (\widetilde{u}_j \nabla w_j) + \widetilde{u}_j (r - mw_j - u_j), & x \in \Omega, \\ \frac{\partial \widetilde{u}_j}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Lemma 3.2 shows that the $W^{2,p}$-bound of $\widetilde{u}_j$ can be taken uniformly with respect to $d_{1,j} \to d_{1,\infty}$, given that $d_{1,j} \geq \varepsilon > 0$. Therefore, by combining the Sobolev embedding theorem with the standard compactness argument for elliptic equations, we can find a non-negative function $\widetilde{u}_\infty \in C^1(\bar{\Omega})$ with $\|\widetilde{u}_\infty\|_{L^\infty} = 1$ fulfilling (after passing to a further subsequence, if necessary)

$$\widetilde{u}_j \to \widetilde{u}_\infty \text{ in } C^1(\bar{\Omega}),$$

and

$$-d_1\Delta\widetilde{u}_\infty = \widetilde{u}_\infty(r - \frac{mh}{g}) \text{ in } \Omega, \quad \frac{\partial \widetilde{u}_\infty}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{5.6}$$

as $j \to \infty$. Applying the Harnack – type inequality (see, [21] or [30, Lemma 2.2]]) to (5.6), we know that $\widetilde{u}_\infty > 0$. This fact together with $\int_\Omega \widetilde{u}_\infty(r - \frac{mh}{g}) = 0$ implies $h = \frac{gr}{m}$, which contradicts the assumption $h \neq \frac{gr}{m}$.

Consequently, for any fixed $\varepsilon > 0$ and $d_1, d_2 \geq \varepsilon$, there admits some $\delta_2 := \delta_2(\varepsilon) > 0$ such that $\min_{\bar{\Omega}} u > \delta_2$. Taking $\delta := \min\{\delta_1, \delta_2\}$ in (5.4), we finish the proof of Lemma 5.1. □

Lemma 5.1 yields that the Leray-Schauder degree $\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0)$ is well defined if $h \neq \frac{gr}{m}$ because of $\mathbf{F}(\cdot, d_1) \neq 0$ on $\partial\Sigma$ (i.e., $0 \notin \mathbf{F}(\partial\Sigma, d_1)$). The Leray-Schauder index formula indicates

$$\text{index } (\mathbf{F}(\cdot, d_1), \mathbf{u}_i) = (-1)^{\gamma_i}, \ i = 1, \ 2, \ 3,$$

where γ_i is the number of negative eigenvalues (counting the algebraic multiplying) of $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}_i, d_1)$.

Next, we compute the number γ_i. A straightforward calculation gives

$$D_{\mathbf{u}}\mathbf{F}(\mathbf{u}_i, d_1) = I - (I - \Delta)^{-1}(I + M_i),$$

where the matrix

$$M_i = \begin{pmatrix} \frac{\chi b}{d_1 d_2} u_i w_i - \frac{u_i}{d_1} & \frac{\chi u_i}{d_1 d_2}(g + bu_i) - \frac{mu_i}{d_1} \\ -\frac{bw_i}{d_2} & -\frac{g+bu_i}{d_2} \end{pmatrix}. \tag{5.7}$$

Thus, the linearized eigenvalue problem

$$D_{\mathbf{u}}\mathbf{F}(\mathbf{u}_i, d_1) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sigma^{(i)} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

can be rewritten as

$$\begin{cases} -(1 - \sigma^{(i)})\Delta\phi = \left[\left(\frac{\chi b}{d_1 d_2} u_i w_i - \frac{u_i}{d_1}\right) + \sigma^{(i)}\right]\phi + \left(\frac{\chi u_i}{d_1 d_2}(g + bu_i) - \frac{mu_i}{d_1}\right)\psi, & x \in \Omega, \\ -(1 - \sigma^{(i)})\Delta\psi = -\frac{bw_i}{d_2}\phi + (\sigma^{(i)} - \frac{g+bu_i}{d_2})\psi, & x \in \Omega, \\ \frac{\partial\psi}{\partial\nu} = \frac{\partial\phi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{5.8}$$

For every $(\phi, \psi) \in Y$, they can be uniquely expressed as

$$\phi = \sum_{k=0}^{\infty} \phi_k, \ \psi = \sum_{k=0}^{\infty} \psi_k \tag{5.9}$$

with

$$\phi_k = \sum_{j=1}^{\dim E(\lambda_k)} c_{kj}\varphi_{kj}, \ \psi_k = \sum_{j=1}^{\dim E(\lambda_k)} b_{kj}\varphi_{kj}.$$

Substituting (5.9) into (5.8) gives that for each $k \geq 0$ and $1 \leq j \leq \dim E(\lambda_k)$,

$$\begin{pmatrix} X + \frac{\chi b u_i w_i}{d_1 d_2} - \frac{u_i}{d_1} & \frac{\chi u_i}{d_1 d_2}(g + bu_i) - \frac{mu_i}{d_1} \\ -\frac{bw_i}{d_2} & X - \frac{g+bu_i}{d_2} \end{pmatrix} \begin{pmatrix} c_{kj} \\ b_{kj} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.10}$$

where

$$X := \sigma^{(i)} - (1 - \sigma^{(i)})\lambda_k. \tag{5.11}$$

Consequently, (5.10) has a nontrivial solution (c_{kj}, b_{kj}) iff

$$X^2 - J_i X + K_i = 0, \tag{5.12}$$

where J_i and K_i $(i = 1, 2, 3)$ are defined in (2.6). Moreover, (5.12) has two roots

$$X_i^{\pm}(d_1) = \frac{J_i \pm \sqrt{J_i^2 - 4K_i}}{2}.$$

It follows from (5.11) that all eigenvalues of $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}_i, d_1)$ can be denoted by

$$\sigma_k^{(i)} = \frac{X_i^{\pm}(d_1) + \lambda_k}{1 + \lambda_k} \tag{5.13}$$

with $k \in \mathbb{N} \cup \{0\}$ and $i = 1, 2, 3$. Let # – sign denote the cardinal number, then

$$\gamma_i = \#\left\{k \in \mathbb{N} \cup \{0\} : X_i^{+}(d_1) < -\lambda_k\right\} + \#\left\{k \in \mathbb{N} \cup \{0\} : X_i^{-}(d_1) < -\lambda_k\right\}. \tag{5.14}$$

Therefore, to compute γ_i $(i = 1, 2, 3)$, we need to analyse the signs of J_i and K_i as well as the properties of $X_i^{\pm}(d_1)$.

Lemma 5.2 *The following statements hold.*

(1) *If* $0 < h < \frac{gr}{m}$, *then*

$$K_i = K_2 = \frac{u_2}{d_1 d_2}(g + bu_2 - mbw_2) > 0; \tag{5.15}$$

(2) *If* $\frac{gr}{m} < h < \frac{(g+br)^2}{4bm}$ *and* $r > \frac{g}{b}$, *then*

$$K_i = \begin{cases} K_1 < 0, \\ K_2 > 0; \end{cases} \tag{5.16}$$

(3) *If* $h = \frac{(g+br)^2}{4bm}$ *and* $r > \frac{g}{b}$, *then*

$$K_i = K_3 = 0. \tag{5.17}$$

Proof *Case* 1: $0 < h < \frac{gr}{m}$. In this case, (1.2) only has a unique constant positive solution (u_2, w_2) and hence $K_i = K_2$. We use $r - u_2 - mw_2 = 0$ and $h - bu_2w_2 - gw_2 = 0$ to update K_2 as

$$\begin{aligned} K_2 &= \frac{1}{d_1d_2}\left(bu_2^2 + gu_2 - mbu_2w_2\right) \\ &= \frac{1}{d_1d_2}\left(bu_2^2 + gu_2 - mh + mgw_2\right) \\ &= \frac{1}{d_1d_2}\left(bu_2^2 - mh + gr\right), \end{aligned}$$

which, along with $0 < h < \frac{gr}{m}$, gives $K_2 > 0$.

Case 2: $\frac{gr}{m} < h < \frac{(g+br)^2}{4bm}$ and $r > \frac{g}{b}$. For this case, (1.2) has two constant positive solutions (u_1, w_1) and (u_2, w_2). Applying $w_i = \frac{h}{bu_i+g}$ $(i = 1, 2)$, we rewrite K_i as

$$K_i = \frac{u_i}{d_1d_2(bu_i + g)}((g + bu_i)^2 - mbh). \tag{5.18}$$

For $i = 1$, $h < \frac{(g+br)^2}{4bm}$ implies

$$\begin{aligned} (g + bu_i)^2 - mbh &= \frac{(g + br)^2 - 4bmh - (br + g)\sqrt{(g + rb)^2 - 4bmh}}{2} \\ &= \frac{\sqrt{(g + br)^2 - 4bmh}\left(\sqrt{(g + br)^2 - 4bmh} - (br + g)\right)}{2} < 0, \end{aligned}$$

which substituted into (5.18) gives $K_1 < 0$.

For $i = 2$, applying $h < \frac{(g+br)^2}{4bm}$ again, one has

$$(g + bu_i)^2 - mbh = \frac{(g + br)^2 - 4bmh + (br + g)\sqrt{(g + rb)^2 - 4bmh}}{2} > 0.$$

This shows $K_2 > 0$.

Case 3: $h = \frac{(g+br)^2}{4bmh}$ and $r > \frac{g}{b}$. In this case, (1.2) only has a unique constant positive solution (u_3, w_3). Using (2.1), (2.4) and $h = \frac{(g+br)^2}{4bmh}$, one obtains $K_3 = 0$ directly. We complete the proof of Lemma 5.2. □

Next, we shall prove the existence of non-constant positive solutions of (1.2) in three cases for h.

5.1 The Case of $0 < h < \frac{gr}{m}$

Lemma 5.3 *For any fixed* $\varepsilon > 0$, *assume* $d_1,\ d_2 \geq \varepsilon$. *Then* $\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = 1$.

Proof For $d_1,\ d_2 \geq \varepsilon$, let $\mathbf{u}$ be the corresponding positive solution of $\mathbf{F}(\mathbf{u}, d_1) = 0$. Then it follows from Lemma 5.1 that $\mathbf{F}(\mathbf{u}, d_1) \neq 0$ on $\partial\Sigma$. By the homotopy invariance of the topological degree [1, Theorem 11.1], we obtain

$$\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) \text{ is a constant for any } d_1 \geq \varepsilon. \tag{5.19}$$

By Table 1 and Theorem 2.2, we know that $\mathbf{u}_2$ is the unique solution of $\mathbf{F}(\mathbf{u}, d_1) = 0$ in Σ for $d_1 \geq D$, and hence the excision property [1, Corollary 11.2] gives

$$\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = \text{ index}(\mathbf{F}(\cdot, d_1), \mathbf{u}_2) = (-1)^{\gamma_2} \text{ for } d_1 \geq D,$$

where γ_2 is defined in (5.14).

On the other hand, (5.15) shows $K_2 > 0$. By the definitions in (2.5) and (2.6), one can easily check that

$$0 < X_2^-(d_1) < X_2^+(d_1)$$

for $d_1 \geq D$ is sufficiently large. Therefore, we have $\gamma_2 = 0$. This implies

$$\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = 1 \text{ for } d_1 \geq D \text{ large enough,}$$

which together with (5.19) implies

$$\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = 1 \text{ for } d_1 \geq \varepsilon.$$

Thus, the proof of Lemma 5.3 is finished. □

Proof of Theorem 2.3 Assume that there is no non-constant positive solution of (1.2), then $\mathbf{u}_2$ is the unique solution of (1.2). Hence, the excision property [1, Corollary 11.2] yields

$$\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = \text{ index}(\mathbf{F}(\cdot, d_1), \mathbf{u}_2) = (-1)^{\gamma_2} \text{ for } d_1 \geq \varepsilon, \tag{5.20}$$

where the constant $\varepsilon > 0$ is any fixed.

Next, we shall calculate γ_2. Based on (2.5), (2.6), (5.15) and the condition $\chi > \frac{d_2}{bw_2}$, we can obtain a number

$$\ell_1 := \frac{u_2}{(g + bu_2)^2} \left\{ \sqrt{(g + bu_2 - bmw_2)d_2} - \sqrt{bw_2(g\chi + bu_2\chi - md_2)} \right\}^2$$

such that

$$X_2^-(d_1) < X_2^+(d_1) < 0 \text{ for all } d_1 \in (0, \ell_1),$$

and they satisfy

$$\lim_{d_1 \to 0} X_2^-(d_1) = -\infty,$$
$$\lim_{d_1 \to 0} X_2^+(d_1) =: -q_1 < 0,$$
$$\lim_{d_1 \to \ell_1} X_2^-(d_1) = \lim_{d_1 \to \ell_1} X_2^+(d_1) = -q_2,$$

where q_1 and q_2 are positive constants and defined in (2.7). Moreover, one can check that $X_2^-(d_1)$ and $X_2^+(d_1)$ are monotone increasing and decreasing with respect to $d_1 \in (0, \ell_1)$, respectively.

Denote

$$j_0 := \min\{j \in \mathbb{N} : q_2 < \lambda_j\}, \; k_0 := \min\{k \in \mathbb{N} : q_1 < \lambda_k\} \leq j_0. \tag{5.21}$$

Since $X_2^-(d_1)$ is monotone increasing with respect to d_1 on $(0, \ell_1)$, let

$$D_j := \sup\{d_1 > 0 : X_2^-(d_1) < -\lambda_j\} \text{ for } j = j_0,\ j_0+1, \dots. \tag{5.22}$$

Then the monotone increasing property of λ_j for $j \in \mathbb{N}$ yields that the sequence $\{D_j\}_{j=j_0}^{\infty}$ fullfills

$$0 \leftarrow \cdots D_j < \cdots < D_{j_0+1} < D_{j_0} < \ell_1 (:= D_{j_0-1}).$$

If $j_0 > k_0$, define

$$\widetilde{D}_k := \inf\{d_1 > 0 : X_2^+(d_1) < -\lambda_k\},\ k = k_0,\ k_0+1, \dots, j_0-1.$$

Take $0 < \varepsilon < \widetilde{D}_{k_0}$, it follows from the monotonicity of $X_2^+(d_1)$ and λ_k that

$$0 < \varepsilon < \widetilde{D}_{k_0} < \widetilde{D}_{k_0+1} < \cdots < \widetilde{D}_{j_0-1} < \ell_1 (:= \widetilde{D}_{j_0} = D_{j_0-1}).$$

Consequently, (5.14) implies

$$\begin{aligned}\gamma_2 &= \#\left\{m \in \mathbb{N}\cup\{0\} : X_2^+(d_1) < -\lambda_k\right\} + \#\left\{m \in \mathbb{N}\cup\{0\} : X_2^-(d_1) < -\lambda_k\right\}\\ &= \begin{cases}(j+1)+(k+1), & \text{if } d_1 \in (D_{j+1}, D_j)\cap(\widetilde{D}_k, \widetilde{D}_{k+1}),\ j \ge j_0-1 \ge k \ge k_0,\\ (j+1)+(k_0-1)+1, & \text{if } d_1 \in (D_{j+1}, D_j)\cap(\varepsilon, \widetilde{D}_{k_0}),\ j \ge j_0-1.\end{cases}\end{aligned}$$

For the case of $j_0 = k_0$, one can check that

$$\gamma_2 = (j+1)+(k_0-1)+1,\ \text{if } d_1 \in (D_{j+1}, D_j),\ j \ge j_0-1.$$

Let $\widetilde{D}_{k_0-1} := \varepsilon$. Hence, if $d_1 \in (D_{j+1}, D_j)\cap(\widetilde{D}_k, \widetilde{D}_{k+1})$ and $j+k+2$ is odd ($j \ge j_0-1 \ge k \ge k_0-1$), then

$$\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = \text{index}\ (\mathbf{F}(\cdot, d_1), \mathbf{u}_2) = (-1)^{j+k+2} = -1.$$

This contradicts Lemma 5.3. Since $\varepsilon > 0$ is arbitrary, the proof of Theorem 2.3 is finished. □

5.2 The Case of $\frac{gr}{m} < h < \frac{(g+br)^2}{4bm}$ with $r > \frac{g}{b}$

In this case, (1.2) has two constant positive solutions: $\mathbf{u}_1$ and $\mathbf{u}_2$.

Lemma 5.4 *For any fixed $\varepsilon > 0$, assume $d_1,\ d_2 \ge \varepsilon$. Then* $\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = 0$.

Proof We use the same manner as the proof of Lemma 5.3 to obtain

$$\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) \text{ is constant for any } d_1 \ge \varepsilon. \tag{5.23}$$

On the other hand, Table 1 and Theorem 2.2 show that $\mathbf{F}(\mathbf{u}, d_1) = 0$ only has two constant positive solutions $\mathbf{u}_1, \mathbf{u}_2$ in set Σ for $d_1 \ge D$. Then the excision property [1, Corollary 11.2] indicates that for $d_1 \ge D$

$$\begin{aligned}\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) &= \text{index}(\mathbf{F}(\cdot, d_1), \mathbf{u}_1) + \text{index}(\mathbf{F}(\cdot, d_1), \mathbf{u}_2)\\ &= (-1)^{\gamma_1} + (-1)^{\gamma_2},\end{aligned} \tag{5.24}$$

where γ_i $(i = 1, 2)$ are defined in (5.14).

For $i=1$, (5.16) implies $K_1<0$. One can verify that

$$\lim_{d_1\to\infty} X_1^-(d_1)=0.$$

Hence, we can find a contant $d_*>0$ such that $X_1^+(d_1)>0$ and $0>X_1^-(d_1)>-\lambda_1$ for all $d_1\geq d_*\geq D$. This, along with the fact $0=\lambda_0<\lambda_1<\lambda_2<\cdots$, gives

$$\gamma_1=1+0=1. \tag{5.25}$$

For $i=2$, it follows from (5.16) that $K_2>0$. Proceeding a similar procedure as the proof of Lemma 5.3, one obtains $\gamma_2=0$. This together with (5.23), (5.24) and (5.25) yields that

$$\deg(\mathbf{F}(\cdot,d_1),\Sigma,0)=(-1)^1+(-1)^0=0$$

for $d_1\geq\varepsilon$. The proof of Lemma 5.4 is finished. □

Proof of Theorem 2.4 under condition (c1) Assume that (1.2) has no non-constant positive solution. Then (1.2) has only two positive solutions: $\mathbf{u}_1$ and $\mathbf{u}_2$. Hence the excision property [1, Corollary 11.2] and the Leray-Schauder index formula [27, Theorem 2.8.1] yield that

$$\begin{aligned}\deg(\mathbf{F}(\cdot,d_1),\Sigma,0)&=\text{index }(\mathbf{F}(\cdot,d_1),\mathbf{u}_1)+\text{ index}(\mathbf{F}(\cdot,d_1),\mathbf{u}_2)\\&=(-1)^{\gamma_1}+(-1)^{\gamma_2}\end{aligned} \tag{5.26}$$

for $d_1\geq\varepsilon$, where the constant $\varepsilon>0$ is any fixed. We next compute γ_1 and γ_2.

Since $\chi>\frac{d_2}{bw_2}(>\frac{d_2}{bw_1})$, a direct calculation shows that $X_1^-(d_1)$ is monotone increasing about $d_1\in(0,\infty)$ and

$$X_1^-(d_1)<0<X_1^+(d_1), \tag{5.27}$$

as well as

$$\lim_{d_1\to 0} X_1^-(d_1)=-\infty,\ \lim_{d_1\to\infty} X_1^-(d_1)=0. \tag{5.28}$$

Define

$$E_i:=\sup\left\{d_1>0: X_1^-(d_1)<-\lambda_i\right\}\ \text{ for } i=1,2,\cdots. \tag{5.29}$$

Then, (5.27), (5.28) and the monotonicity of $X_1^-(d_1)$ and λ_k enable us to get a sequence $\{E_i\}_{i=1}^\infty$ defined by (5.29), such that

$$0\leftarrow\cdots<E_i<\cdots<E_2<E_1<\infty=:E_0.$$

By (5.14), we get

$$\gamma_1=i+1,\ \text{if } d_1\in(E_{i+1},E_i),\ i=0,1,2,\cdots. \tag{5.30}$$

On the other hand, using the same manner as the proof of Theorem 2.3, one has

$$\gamma_2=\begin{cases} j+k+2, & \text{if } d_1\in(D_{j+1},D_j)\cap(\widetilde{D}_k,\widetilde{D}_{k+1}),\ j\geq j_0-1>k\geq k_0,\\ j+k_0+1, & \text{if } d_1\in(D_{j+1},D_j)\cap(\varepsilon,\widetilde{D}_{k_0}),\ j\geq j_0-1,\\ j+j_0+1, & \text{if } d_1\in(D_{j+1},D_j),\ j\geq j_0-1=k_0-1.\end{cases} \tag{5.31}$$

Hence, let $k_0 - 1 \le k \le j_0 - 1 \le j$ and denote $\widetilde{D}_{k_0-1} := \varepsilon$. Then if $\chi > \frac{d_2}{bw_2}$ and

$$d_1 \in (E_{i+1}, E_i) \cap (D_{j+1}, D_j) \cap (\widetilde{D}_k, \widetilde{D}_{k+1})$$

as well as $i + j + k + 2$ is odd, (5.26), (5.30) and (5.31) yield

$$\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = (-1)^{i+1} + (-1)^{j+k+2} = -2 \text{ or } 2,$$

which contradicts Lemma 5.4. Since the constant $\varepsilon > 0$ is any fixed, the contradiction argument enables us to obtain Theorem 2.4 under condition (c1). □

5.3 The Case of $h = \frac{(g+br)^2}{4bm}$ with $r > \frac{g}{b}$

In this case, $\mathbf{u}_3$ is the unique constant positive solution of (1.2).

Lemma 5.5 *For any fixed* $\varepsilon > 0$, *assume* $d_1,\ d_2 \ge \varepsilon$. *Then* $\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = 1$.

Proof Proceeding the same manner as the proof of Lemma 5.3 and Lemma 5.4, we obtain $\deg\,(\mathbf{F}(\cdot, d_1), \Sigma, 0) = (-1)^{\gamma_3} = (-1)^0 = 1$ readily. □

Proof of Theorem 2.4 under condition (c2) Assume that there is no non-constant positive solution of (1.2), then $\mathbf{u}_3$ is the unique positive solution of (1.2). Hence, the excision property [1, Corollary 11.2] yields

$$\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = \text{index}\ (\mathbf{F}(\cdot, d_1), \mathbf{u}_3) = (-1)^{\gamma_3} \text{ for } d_1 \ge \varepsilon, \tag{5.32}$$

where the constant $\varepsilon > 0$ is any fixed.

Since $h = \frac{(g+br)^2}{4bm}$ and $r > \frac{g}{b}$, it follows from (5.17) that $K_3 = 0$. Then the fact $\chi > \frac{d_2}{bw_2} = \frac{d_2}{bw_3}$ enables us to find a positive constant

$$\ell_2 := \frac{u_3(\chi b w_3 - d_2)}{g + bu_3}$$

such that

$$J_3 = X_3^-(d_1) < 0 = X_3^+(d_1) \text{ for } d_1 \in (0, \ell_2), \tag{5.33}$$

and

$$\lim_{d_1 \to 0} X_3^-(d_1) = -\infty,\ \lim_{d_1 \to \ell_2} X_3^-(d_1) = 0. \tag{5.34}$$

Denote

$$\widetilde{E}_n := \sup\left\{d_1 > 0 : X_3^-(d_1) < -\lambda_n\right\} \text{ for } n = 1, 2, \cdots.$$

Since $X_3^-(d_1)$ is monotone increasing with respect to $d_1 \in (0, \ell_2)$, the sequence $\{\widetilde{E}_n\}_{n=1}^{\infty}$ fulfills

$$0 \leftarrow \cdots < \widetilde{E}_n < \cdots < \widetilde{E}_2 < \widetilde{E}_1 < \ell_2 := \widetilde{E}_0,$$

which combined with the definition of γ_3 in (5.14) and (5.32) implies that

$$\deg(\mathbf{F}(\cdot, d_1), \Sigma, 0) = (-1)^{n+1} = -1 \tag{5.35}$$

when $d_1 \in (\widetilde{E}_{n+1}, \widetilde{E}_n)$ and $n+1$ $(n \geq 0)$ is odd. Therefore, (5.35) is contradicted with Lemma 5.5. Then the proof of Theorem 2.4 under condition (c2) is finished. □

6 Numerical Simulations for Spatial Patterns

This section aims to numerically verify the theoretical results, and show the distribution of populations (patterns generated by (1.1)) in one-dimensional space. In all simulations, we set

$$r = m = 1, \ b = 0.2, \ g = d_1 = 0.1, \tag{6.1}$$

which gives

$$h_1 = \frac{gr}{m} = 0.1, \ h_2 = \frac{(g+br)^2}{4bm} = 0.1125.$$

We begin by considering the case of $0 < h < h_1$. Taking $h = 0.08 < h_1$ and $d_2 = 2$, one obtains $(u_2, w_2) = (0.653113, 0.346887)$. As shown in Fig. 2, we do obtain the stationary

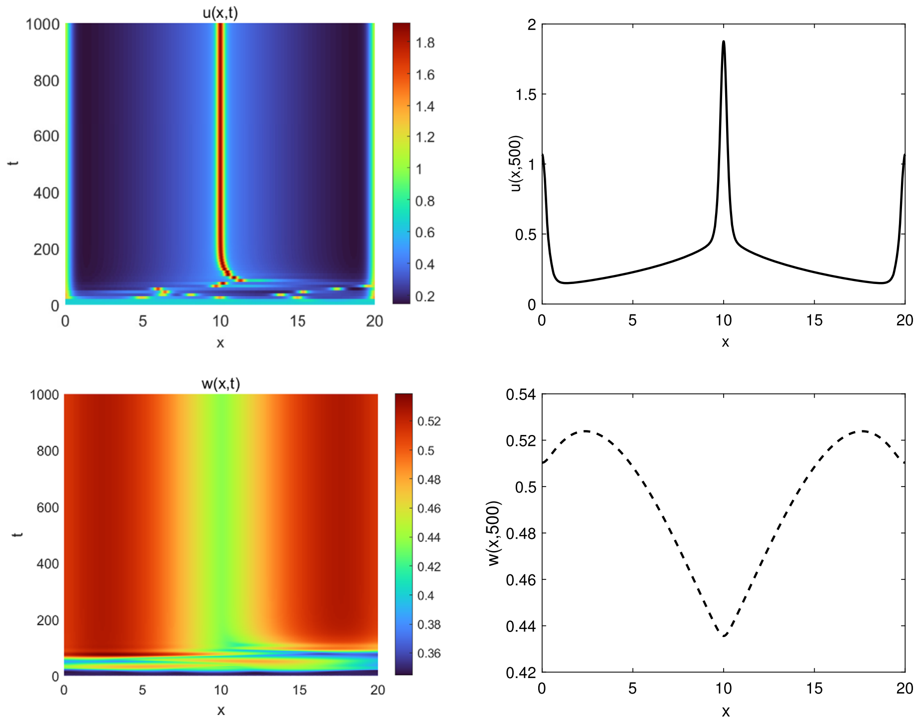

Fig. 2 Pattern formation generated by (1.1) with $d_2 = 2$, $h = 0.08 < h_1$ and $\chi = 60$ in $\Omega = (0, 20)$. Other parameter values are given by (6.1); The initial datum (u_0, w_0) is set as a small random perturbation of the homogeneous coexistence steady state (0.653113, 0.346887)

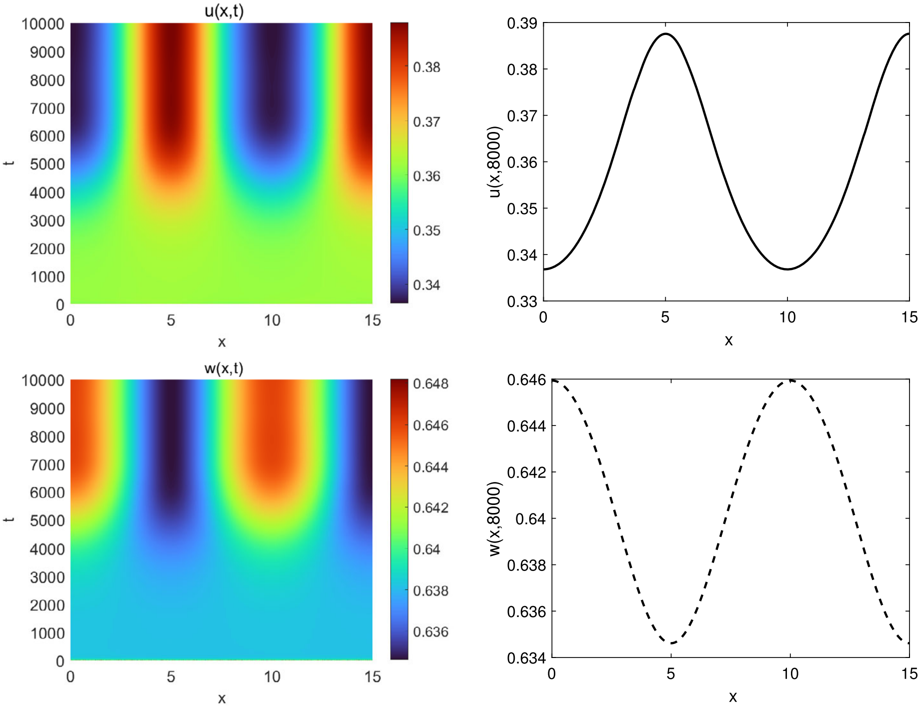

Fig. 3 Pattern formation generated by (1.1) with $d_2 = 1$, $h_1 < h = 0.11 < h_2$ and $\chi = 10$ in $\Omega = (0, 15)$. Other parameter values are given by (6.1); The initial datum (u_0, w_0) is set as a small random perturbation of the homogeneous coexistence steady state $(0.361803, 0.638197)$

spatial patterns in the domain $\Omega = (0, 20)$. Moreover, the simulation in Fig. 2 is consistent with the result shown in Theorem 2.3. In fact, by the parameter values chosen in Fig. 2 and (6.1), one has $\chi = 60 > \frac{ad_2}{bw_2} = 28.8278$, $q_1 = 0.0745592$, $q_2 = 0.193541$, $\ell_1 = 0.160555$. It follows from $\lambda_1 = \frac{1^2\pi^2}{400} = 0.024674$, $\lambda_2 = \frac{2^2\pi^2}{400} = 0.098696$, $\lambda_3 = \frac{3^2\pi^2}{400} = 0.222066$, $\lambda_4 = \frac{4^2\pi^2}{400} = 0.394784$, $\lambda_{16} = \frac{16^2\pi^2}{400} = 6.31655$, $\lambda_{17} = \frac{17^2\pi^2}{400} = 7.13079$ and (5.21) that $j_0 = 3$, $k_0 = 2$. According to the definitions of D_j and $\widetilde{D}_k$, it is easy to verify that $j = 16$ satisfies that $d_1 = 0.1 \in (D_{17}, D_{16}) \cap (0, \widetilde{D}_2) \cap (0, \ell_1)$ and $j + k_0 + 1 = 19$ is odd. Hence, the conditions in Theorem 2.3 are fulfilled and there admits a non-constant steady state solution as given in Fig. 2.

Next, we explore the numerical spatial patterns in the case of $h_1 < h \leq h_2$. We take $h_1 < h = 0.11 < h_2$, $d_2 = 1$ and get $(u_1, w_1) = (0.138197, 0.861803)$, $(u_2, w_2) = (0.361803, 0.638197)$. As shown in Fig. 3, (1.1) does induce stationary spatial patterns in one-dimensional space $\Omega = (0, 15)$. Moreover, the simulation in Fig. 3 is also consistent with the result shown in Theorem 2.4. Indeed, by the parameter values chosen in Fig. 3 and (6.1), one has $\chi = 10 > \frac{ad_2}{bw_2} = 7.83458$, $q_1 = 0.161803$, $q_2 = 0.176748$, $\ell_1 = 0.104056$. Then the facts $\lambda_1 = \frac{1^2\pi^2}{225} = 0.0438649$, $\lambda_2 = \frac{2^2\pi^2}{225} = 0.17546$, $\lambda_3 = \frac{3^2\pi^2}{225} = 0.394784$, $\lambda_4 = \frac{4^2\pi^2}{225} = 0.701839$, $\lambda_5 = \frac{5^2\pi^2}{225} = 1.09662$ along with (5.21) gives $j_0 = 3$, $k_0 = 2$. Moreover, by the definitions of D_j, $\widetilde{D}_k$ and E_i $(i \geq 0, j \geq j_0 - 1 \geq k \geq k_0 - 1)$, we can choose $i = 4$, $j = 3$, $k = 2$ such that $d_1 = 0.1 \in (E_5, E_4) \cap (D_4, D_3) \cap (\widetilde{D}_2, \ell_1)$ and $i + j + k + 2 = 11$

is odd. Thus, these conditions in Theorem 2.4 are satisfied and there admits a non-constant steady state solution as given in Fig. 3.

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