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STRONG CONVERGENCE RATE OF AN EXPONENTIALLY INTEGRABLE SCHEME FOR STOCHASTIC NONLINEAR WAVE EQUATION

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Abstract. In this paper, we present an exponentially integrable numerical method for stochastic wave equation with cubic nonlinearity and additive space-time noise. We first apply the spectral Galerkin method to discretize the original equation and show that this spatial discretization possesses an energy evolution law and certain exponential integrability property. Then the exponential integrability property of the exact solution is deduced by proving the strong convergence of the semi-discretization. To propose a fully discrete numerical method which could inherit both the energy evolution law and the exponential integrability, we use the splitting technique and averaged vector field method in the temporal direction. Combining these structure-preserving properties with regularity estimates of the exact and the numerical solutions, we obtain the strong convergence rate of the proposed scheme. Finally, numerical experiments verify the theoretical results.

1. Introduction.

As a kind of commonly observed physical phenomena, the wave motions are usually described by stochastic partial differential equations (SPDEs) of hyperbolic type. The main objective of this paper is to numerically investigate the following stochastic wave equation with cubic nonlinearity, driven by an additive

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noise

$$\begin{cases} du(t) = v(t)dt, & \text{in } \mathcal{O} \times (0,T], \\ dv(t) = \Lambda u(t)dt - f(u(t))dt + dW(t), & \text{in } \mathcal{O} \times (0,T], \\ u(0) = u_0, \quad v(0) = v_0, & \text{in } \mathcal{O}, \end{cases} \tag{1}$$

where $\mathcal{O} = (0,1)^d$ with $d \leq 2$, $T \in (0,\infty)$ and $u_0, v_0 : \mathcal{O} \to \mathbb{R}$ are deterministic functions. Assume that $\Lambda = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator with homogeneous Dirichlet boundary condition, and the nonlinear term $f(u) = c_3u^3+c_2u^2+c_1u+c_0$ is assumed to be a polynomial with $c_3 > 0$ and $c_0, c_1, c_2 \in \mathbb{R}$. Throughout this paper, W is an $L^2 := L^2(\mathcal{O};\mathbb{R})$-valued $\mathbf{Q}$-Wiener process with respect to a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, i.e., there exists an orthonormal basis $\{e_k\}_{k\in\mathbb{N}^+}$ of L^2 and a sequence of mutually independent real-valued Brownian motions $\{\beta_k\}_{k\in\mathbb{N}^+}$ such that $W(x,t) = \sum_{k\in\mathbb{N}^+} \mathbf{Q}^{\frac{1}{2}} e_k(x)\beta_k(t)$ with $\mathbf{Q}$ being a self-adjoint, positive definite and finite trace operator.

For the well-posedness of stochastic wave equation, we refer to [8, 9] for the existence and uniqueness of the mild solution with more general polynomial drift coefficients, and to [17] with more general driving noises. As an intrinsic quantity of the wave equation, the evolution law of the energy holds,

$$V_1(u(t), v(t)) = V_1(u_0, v_0) + \frac{1}{2}\mathrm{Tr}\,(\mathbf{Q})\, t + \int_0^t \langle v(s), dW(s)\rangle_{L^2}, \text{a.s.}$$

where $\langle\cdot,\cdot\rangle_{L^2}$ denotes the L^2-inner product and the energy V_1 is defined in (8) (see Lemma 2.3 for more details), has been established in [8]. Besides, by studying the exponential moment of V_1, the exponential integrability property of the solution of the stochastic wave equation has been firstly shown in [1]. Further utilizing the uniform exponential integrability property and regularity estimate of the spectral Galerkin method applied to (1), we prove that the exact solution admits the following exponential integrability property

$$\mathbb{E}\left(\exp\left(\int_0^T c\|u(s)\|_{L^6}^2 ds\right)\right) \leq C,$$

where $c \in \mathbb{R}$ is arbitrary, $T \in (0,\infty)$, $C := C(u_0, v_0, \mathbf{Q}, T, c, d)$ and $d \leq 2$.

The numerical resolution of the stochastic wave equation is a vibrant field of research with many ongoing studies (for example, see [2, 5, 7, 11, 22, 21, 29, 25] and the references cited therein). In the context of the stochastic wave equation with cubic nonlinearity, the literature primarily mentions a unique partial-implicit midpoint-type difference method, suitable for one-dimensional cases (d=1), which maintains the energy functional in a dynamically consistent manner, as proposed in [28]. From a numerical perspective, devising methods that preserve both the energy evolution law and the exponential integrability property of the original system is both natural and crucial. For example, in the realm of energy-preserving numerical methods, we refer to the time splitting method applied to the stochastic nonlinear Schrödinger equation [13], and the exponential integrator used for the stochastic linear wave equation [10]. In terms of methods that ensure exponential integrability, we note the explicit splitting scheme for the stochastic parabolic equation [4], the stopped increment-tamed Euler approximations for stochastic differential equations [20], and both the finite difference method and the splitting fully discrete scheme for the stochastic nonlinear Schrödinger equation [14, 15].

Despite these advancements, there remains a gap in the research for fully discrete schemes that can simultaneously inherit the energy evolution law and exponential integrability in the case of stochastic nonlinear wave equations with non-globally Lipschitz coefficients. In the present paper, we propose a strategy to design numerical methods preserving these two important properties. The idea is based on splitting the original equation in temporal direction into a deterministic Hamiltonian system and a stochastic system. We combine the splitting technique with the averaged vector field (AVF) method, and apply spectral Galerkin method in spatial direction to present the following scheme

$$\begin{aligned} u^N_{m+1} &= u^N_m + h\frac{v^N_m + \bar{v}^N_{m+1}}{2}, \\ \bar{v}^N_{m+1} &= v^N_m + h\Lambda_N \frac{u^N_m + u^N_{m+1}}{2} - hP_N\left(\int_0^1 f(u^N_m + \theta(u^N_{m+1} - u^N_m))d\theta\right), \\ v^N_{m+1} &= \bar{v}^N_{m+1} + P_N \delta W_m, \end{aligned}$$

where $N \in \mathbb{N}^+ = \{1, 2, \cdots\}$, $h = T/M$ with $M \in \mathbb{N}^+$ is the time step-size, $m \in \mathbb{Z}_M := \{0, 1, \cdots, M-1\}$. Here P_N is the spectral Galerkin projection operator defined in (14), and δW_m is the increment of the Wiener process defined in (27).

The averaged vector field method (AVF) can be viewed as a kind of the discrete gradient approach to construct numerical schemes with conservation properties, which has been discussed in the deterministic wave equation (see [18]). Furthermore, we show that the proposed numerical method admits the following energy evolution law

$$\begin{aligned} V_1(u^N_{m+1}, v^N_{m+1}) = & V_1(u^N_m, v^N_m) + \int_{t_m}^{t_{m+1}} \langle v^{N,S}_m(s), P_N dW(s)\rangle_{L^2} \\ & + \int_{t_m}^{t_{m+1}} \frac{1}{2}\mathrm{Tr}\left((P_N \mathbf{Q}^{\frac{1}{2}})(P_N\mathbf{Q}^{\frac{1}{2}})^*\right) ds, \end{aligned}$$

where $v^{N,S}$ is a auxiliary splitting process (we refer to (26) for details), and the following exponential integrability property

$$\mathbb{E}\left(\exp\left(ch\sum_{i=1}^{M}\|u^N_i\|^2_{L^6}\right)\right) \le C,$$

where $c \in \mathbb{R}$ is arbitrary, $C := C(u_0, v_0, \mathbf{Q}, T, c, d)$ and $d \le 2$.

Let $\|(-\Lambda)^{\frac{\beta-1}{2}}\mathbf{Q}^{\frac{1}{2}}\|_{\mathcal{L}_2(L^2)} < \infty$ with $\beta \in [1, 2]$, where the index β quantifies the spatial regularity of the Wiener process (see Propositions 3.2-3.3). Note that we do not assume that Q commutes with A in the numerical analysis. Based on the numerical exponential integrability and energy evolution law, we obtain the following strong convergence result when $X_0 = (u_0, v_0)^\top \in \mathbb{H}^\beta, \beta \in [1, 2]$ (see section 2 for the definition of the Sobolev interpolation space).

Theorem 1.1. *Let $d = 1$, $\beta \ge 1$ or $d = 2, \beta = 2$. Let $\gamma = \min(\beta, 2)$ and $T > 0$. Assume that $X_0 \in \mathbb{H}^\beta$, $\|(-\Lambda)^{\frac{\beta-1}{2}}\mathbf{Q}^{\frac{1}{2}}\|_{\mathcal{L}_2(L^2)} < \infty$. For $d = 1$, there exists $h_0 > 0$ such that for $h \le h_0$ and $p \ge 1$,*

$$\sup_{m\in\mathbb{Z}_{M+1}} \mathbb{E}\left[(\|u(t_m) - u^N_m\|^2_{L^2} + \|v(t_m) - v^N_m\|^2_{\dot{\mathbb{H}}^{-1}})^p\right] \le C\left(h^{\gamma p} + \lambda_N^{-\beta p}\right), \quad (2)$$

where $C = C(X_0, \mathbf{Q}, T, p) > 0$, $N \in \mathbb{N}^+, m \in \mathbb{Z}_{M+1}$, $M \in \mathbb{N}^+, Mh = T$.

When $d=2$, it holds that

$$\sup_{m\in\mathbb{Z}_{M+1}} \mathbb{E}\left[(\|u(t_m)-u_m^N\|_{L^2}^2+\|v(t_m)-v_m^N\|_{\dot{\mathbb{H}}^{-1}}^2)^p\right] \le C_1\left(h^{\gamma p}\lambda_N^{2\epsilon p}+\lambda_N^{-\beta p}\right) \quad (3)$$

for sufficiently small $\epsilon>0$, where $C_1=C_1(p,X_0,\mathbf{Q},T)>0$, $N\in\mathbb{N}^+$, $M\in\mathbb{N}^+, Mh=T$.

To the best of our knowledge, this is the first result regarding both the exponential integrability and the strong convergence rate of full discretizations for stochastic wave equations with cubic nonlinearity.

The rest of this paper is organized as follows. Section 2 presents an abstract formulation of the stochastic wave equation, and introduces some properties of the corresponding group. In section 3, the regularity estimate and exponential integrability property of the mild solution of the spectral Galerkin discretization are studied. The analysis of strong convergence for the spectral Galerkin discretization is also presented. Section 4 is devoted to constructing the numerical method which preserves the energy evolution law and exponential integrability property, and deducing its $L^p(\Omega;\mathbb{H})$ error estimate. Numerical experiments are carried out in section 5 to verify theoretical results.

2. **Preliminary and framework.** In this section, we first set forth an abstract formulation of (1) for the stochastic wave equation, and introduce some properties of the unitary group generated by the dominant operator. Throughout this paper, the constant C may be different from line to line but never depending on N and h.

Assume that the eigenvalues $0<\lambda_1\le\lambda_2\le\cdots$ and that the corresponding eigenfunctions $\{e_i\}_{i=1}^{\infty}$ of the operator $-\Lambda$, i.e., with $-\Lambda e_i=\lambda_i e_i$, $i\in\mathbb{N}^+$, form an orthonormal basis in L^2. Define the interpolation space $\dot{\mathbb{H}}^r:=\mathcal{D}((-\Lambda)^{\frac{r}{2}})$ for $r\in\mathbb{R}$ equipped with the inner product $\langle x,y\rangle_{\dot{\mathbb{H}}^r}=\left\langle(-\Lambda)^{\frac{r}{2}}x,(-\Lambda)^{\frac{r}{2}}y\right\rangle_{L^2}=\sum_{i=1}^{\infty}\lambda_i^r\langle x,e_i\rangle_{L^2}\langle y,e_i\rangle_{L^2}$ and the corresponding norm $\|x\|_{\dot{\mathbb{H}}^r}:=\langle x,x\rangle_{\dot{\mathbb{H}}^r}^{1/2}$. Furthermore, we introduce the product space $\mathbb{H}^r:=\dot{\mathbb{H}}^r\times\dot{\mathbb{H}}^{r-1}$, $r\in\mathbb{R}$, endowed with the inner product $\langle X_1,X_2\rangle_{\mathbb{H}^r}=\langle x_1,x_2\rangle_{\dot{\mathbb{H}}^r}+\langle y_1,y_2\rangle_{\dot{\mathbb{H}}^{r-1}}$ for any $X_1=(x_1,y_1)^\top$ and $X_2=(x_2,y_2)^\top$, and the corresponding norm $\|X\|_{\mathbb{H}^r}:=\langle X,X\rangle_{\mathbb{H}^r}^{1/2}=\left(\|x\|_{\dot{\mathbb{H}}^r}^2+\|y\|_{\dot{\mathbb{H}}^{r-1}}^2\right)^{1/2}$ for $X=(x,y)^\top$.

Given two separable Hilbert spaces $(\mathcal{H},\|\cdot\|_{\mathcal{H}})$ and $(\widetilde{H},\|\cdot\|_{\widetilde{H}})$, $\mathcal{L}(\mathcal{H},\widetilde{H})$ and $\mathcal{L}_1(\mathcal{H},\widetilde{H})$ are the Banach spaces of all linear bounded operators and the nuclear operators from $\mathcal{H}$ to $\widetilde{H}$, respectively. The trace of an nonnegative operator $\mathcal{T}\in\mathcal{L}_1(\mathcal{H}):=\mathcal{L}_1(\mathcal{H},\mathcal{H})$ is $\mathrm{Tr}(\mathcal{T})=\sum_{k\in\mathbb{N}^+}\langle\mathcal{T}f_k,f_k\rangle_{\mathcal{H}}$, where $\{f_k\}_{k\in\mathbb{N}^+}$ is any orthonormal basis of $\mathcal{H}$. In particular, if $\mathcal{T}$ is a nonnegative operator, then $\mathrm{Tr}(\mathcal{T})=\|\mathcal{T}\|_{\mathcal{L}_1(\mathcal{H})}$. Denote by $\mathcal{L}_2(\mathcal{H},\widetilde{H})$ the space of Hilbert–Schmidt operators from $\mathcal{H}$ into $\widetilde{H}$, equipped with the norm $\|\cdot\|_{\mathcal{L}_2(\mathcal{H},\widetilde{H})}=(\sum_{k\in\mathbb{N}^+}\|\cdot f_k\|_{\widetilde{H}}^2)^{\frac{1}{2}}$. For convenience, we denote $\mathcal{L}_2(\mathcal{H}):=\mathcal{L}_2(\mathcal{H},\mathcal{H})$ and $L^p:=L^p(\mathcal{O},\mathbb{R}), p\ge1$.

Denote $X=(u,v)^\top$. The abstract form of (1) is

$$\begin{aligned} dX(t)&=AX(t)dt+\mathbb{F}(X(t))dt+\mathbb{G}dW(t),\quad t\in(0,T],\\ X(0)&=X_0,\end{aligned} \quad (4)$$

where

$$X_0=\begin{bmatrix}u_0\\ v_0\end{bmatrix},\quad A=\begin{bmatrix}0 & I\\ \Lambda & 0\end{bmatrix},\quad \mathbb{F}(X(t))=\begin{bmatrix}0\\ -f(u(t))\end{bmatrix},\quad \mathbb{G}=\begin{bmatrix}0\\ I\end{bmatrix}.$$

Here and below we denote I by the identity operator defined in L^2. Moreover, we define the domain of operator A by

$$\mathcal{D}(A) = \left\{ X \in \mathbb{H} : AX = \begin{bmatrix} v \\ \Lambda u \end{bmatrix} \in \mathbb{H} := L^2 \times \dot{\mathbb{H}}^{-1} \right\},$$

then the operator A generates a unitary group $E(t)$, $t \in \mathbb{R}$, on $\mathbb{H}$, given by

$$E(t) = \exp(tA) = \begin{bmatrix} C(t) & (-\Lambda)^{-\frac{1}{2}} S(t) \\ -(-\Lambda)^{\frac{1}{2}} S(t) & C(t) \end{bmatrix},$$

where $C(t) = \cos(t(-\Lambda)^{\frac{1}{2}})$ and $S(t) = \sin(t(-\Lambda)^{\frac{1}{2}})$ are the cosine and sine operators, respectively.

Unless otherwise specified, throughout this article, we always assume that $X_0 \in \mathbb{H}^1$ and $\mathrm{Tr}(\mathbf{Q}) < \infty$. As a result, the mild solution of (4), that is,

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\mathbb{F}(X(s))ds + \int_0^t E(t-s)\mathbb{G}dW(s), \quad a.s. \tag{5}$$

exists. We refer to [8, 9] for the well-posedness of the mild solution for the stochastic wave equation.

The following lemma concerns with the temporal Hölder continuity of both sine and cosine operators.

Lemma 2.1. *For $r \in [0,1]$, there exists a positive constant $C' := C'(r)$ such that*

$$\|(S(t) - S(s))(-\Lambda)^{-\frac{r}{2}}\|_{\mathcal{L}(L^2)} \le C'(t-s)^r,$$
$$\|(C(t) - C(s))(-\Lambda)^{-\frac{r}{2}}\|_{\mathcal{L}(L^2)} \le C'(t-s)^r$$

for all $t \ge s \ge 0$.

The proof of Lemma 2.1 is analogous to that of [10, (4.1)] and thus is omitted here. It indicates that

$$\|(E(t) - E(s))X\|_{\mathbb{H}} \le C'(t-s)^r \|X\|_{\mathbb{H}^r}$$

for $r \in [0,1]$.

Lemma 2.2. *For all $t \in \mathbb{R}$, $C(t)$ and $S(t)$ satisfy a trigonometric identity in the sense that $\|S(t)x\|_{L^2}^2 + \|C(t)x\|_{L^2}^2 = \|x\|_{L^2}^2$ for any $x \in L^2$.*

The above lemma yields that for all $t \in \mathbb{R}$,

$$\|E(t)X\|_{\mathbb{H}^r} = \|X\|_{\mathbb{H}^r} \text{ for any } X \in \mathbb{H}^r, r \in \mathbb{R}. \tag{6}$$

Denote the smooth potential function by $F(\xi) := \frac{c_3}{4}\xi^4 + \frac{c_2}{3}\xi^3 + \frac{c_1}{2}\xi^2 + c_0\xi$, $\xi \in \mathbb{R}$. Then the function $F : \mathbb{R} \to \mathbb{R}$ satisfies

$$a_1\|u\|_{L^4}^4 - b_1 \le \int_{\mathcal{O}} F(u(x))dx \le a_2\|u\|_{L^4}^4 + b_2 \tag{7}$$

for some positive constants a_1, a_2, b_1, b_2. We define the Lyapunov energy functional $V_1 : \mathbb{H}^1 \to \mathbb{R}$ as

$$V_1(u,v) = \frac{1}{2}\|u\|_{\dot{\mathbb{H}}^1}^2 + \frac{1}{2}\|v\|_{L^2}^2 + \int_{\mathcal{O}} F(u(x))dx + C_1, \quad C_1 \ge b_1, \tag{8}$$

then we have the following energy evolution law of (1) .

Lemma 2.3. *The stochastic wave equation* (1) *admits the energy evolution law*

$$V_1(u(t), v(t)) = V_1(u_0, v_0) + \frac{1}{2}\mathrm{Tr}\,(\mathbf{Q})\,t + \int_0^t \langle v(s), dW(s)\rangle_{L^2},\ a.s. \tag{9}$$

for all $t \in \mathbb{R}^+$. *In particular, it possesses the averaged energy evolution law*

$$\mathbb{E}(V_1(u(t), v(t))) = V_1(u_0, v_0) + \frac{1}{2}\mathrm{Tr}\,(\mathbf{Q})\,t$$

for all $t \in \mathbb{R}^+$.

The proof of the above lemma can be given by similar procedures in [8] by using finite dimensional approximation, Itô's formula and then taking limits. For simplicity, we omit the details.

To derive the exponential integrability property of the exact and numerical solution, we will frequently use the following Gagliardo–Nirenberg inequalities,

$$\|u\|_{L^6} \le C\|\nabla u\|_{L^2}^{a}\|u\|_{L^2}^{1-a},\ \text{with } a = \frac{d}{3}, d \le 2, \tag{10}$$

$$\|u\|_{L^\infty} \le \|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}},\ \text{when } d = 1, \tag{11}$$

$$\|\nabla u\|_{L^4} \le C\|\Lambda u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}, \text{when } d = 2, \tag{12}$$

$$\|u\|_{L^\infty} \le C\|\Lambda u\|_{L^2}^{\frac{1}{4}}\|u\|_{L^6}^{\frac{3}{4}}, \text{when } d = 2, \tag{13}$$

and and the Sobolev embedding $\dot{\mathbb{H}}^1 \hookrightarrow L^\infty$ for $d = 1$ and $\dot{\mathbb{H}}^1 \hookrightarrow L^q, q < \infty$ for $d = 2$. The main assumption for $d \le 2$ in this paper is due to the usage of the above Gagliardo–Nirenberg equalities.

3. Exponential integrability property of stochastic wave equation.

This section is devoted to analyzing the spatial spectral Galerkin method of (4). We present the existence, uniqueness and regularity estimate for the solution to the spectral Galerkin discretization, including the uniform boundedness of the solution in $L^p(\Omega; \mathcal{C}([0,T]; \mathbb{H}^\beta))$-norm and Hölder continuity of the solution in $L^p(\Omega; \mathbb{H})$-norm. By giving the strong convergence of the spectral Galerkin method, we show the exponential integrability property of the exact solution of (1).

3.1. Spectral Galerkin method.

In this subsection, we study the spectral Galerkin method for stochastic wave equation (4). The spectral Galerkin method has been used to discretize SPDEs in spatial direction (see e.g., [15, 21] and references therein). For the considered equation and $N \in \mathbb{N}^+$, we define a finite dimensional subspace U_N of L^2 spanned by $\{e_1, e_2, \cdots, e_N\}$, and the projection operator $P_N : \dot{\mathbb{H}}^r \to U_N$ by

$$P_N\zeta = \sum_{i=1}^{N} \langle \zeta, e_i\rangle_{L^2} e_i, \quad \forall\, \zeta \in \dot{\mathbb{H}}^r, \quad r \ge -1, \tag{14}$$

which satisfies $\|P_N\|_{\mathcal{L}(L^2)} \le 1$. Define $\Lambda_N : U_N \to U_N$ by

$$\Lambda_N\zeta = \Lambda P_N\zeta = P_N\Lambda\zeta = -\sum_{i=1}^{N} \lambda_i \langle \zeta, e_i\rangle_{L^2} e_i, \quad \forall\, \zeta \in U_N. \tag{15}$$

By denoting $X^N = (u^N, v^N)^\top$, the spectral Galerkin method applied to (4) yields

$$\begin{aligned} dX^N(t) &= A_N X^N(t)dt + \mathbb{F}_N(X^N(t))dt + \mathbb{G}_N dW(t), \quad t \in (0,T], \\ X^N(0) &= X_0^N, \end{aligned} \tag{16}$$

where

$$X_0^N = \begin{bmatrix} u_0^N \\ v_0^N \end{bmatrix}, \quad A_N = \begin{bmatrix} 0 & I \\ \Lambda_N & 0 \end{bmatrix}, \quad \mathbb{F}_N(X^N) = \begin{bmatrix} 0 \\ -P_N\left(f(u^N)\right) \end{bmatrix}, \quad \mathbb{G}_N = \begin{bmatrix} 0 \\ P_N \end{bmatrix}$$

with $u_0^N = P_N u_0, v_0^N = P_N v_0$. Similarly, the discrete operator A_N generates a unitary group

$$E_N(t) = \exp(tA_N) = \begin{bmatrix} C_N(t) & (-\Lambda_N)^{-\frac{1}{2}} S_N(t) \\ -(-\Lambda_N)^{\frac{1}{2}} S_N(t) & C_N(t) \end{bmatrix}, \quad t \in \mathbb{R},$$

where $C_N(t) = \cos(t(-\Lambda_N)^{\frac{1}{2}})$ and $S_N(t) = \sin(t(-\Lambda_N)^{\frac{1}{2}})$ are the discrete cosine and sine operators defined in U_N, respectively. It can be verified straightforwardly that

$$C_N(t)P_N\zeta = C(t)P_N\zeta = P_N C(t)\zeta, \quad S_N(t)P_N\zeta = S(t)P_N\zeta = P_N S(t)\zeta$$

for any $\zeta \in \dot{\mathbb{H}}^r$, $r \geq -1$.

Thanks to the Lyapunov function V_1 in (8), one can repeat the arguments in the proof of [8, Theorem 4.2] and obtain the existence and uniqueness of the mild solution of (16) and a priori estimate of $X^N(t)$. As a consequence, the energy evolution law of $X^N(t)$ follows.

Lemma 3.1. *Let $T > 0$. The spectral Galerkin discretization (16) has a unique mild solution given by*

$$X^N(t) = E_N(t)X_0^N + \int_0^t E_N(t-s)\mathbb{F}_N(X^N(s))ds + \int_0^t E_N(t-s)\mathbb{G}_N dW(s) \tag{17}$$

for $t \in [0,T]$. Moreover, for $p \geq 2$, there exists a positive constant $C := C(X_0, T, \mathbf{Q}, p)$ such that

$$\sup_{N \in \mathbb{N}^+} \|X^N\|_{L^p(\Omega; \mathcal{C}([0,T]; \mathbb{H}^1))} \leq C. \tag{18}$$

Proposition 3.1. *The mild solution $X^N(t)$ satisfies the energy evolution law*

$$\begin{aligned} V_1(u^N(t), v^N(t)) =& V_1(u_0^N, v_0^N) + \frac{1}{2}\mathrm{Tr}\left((P_N\mathbf{Q}^{\frac{1}{2}})(P_N\mathbf{Q}^{\frac{1}{2}})^*\right)t \\ &+ \int_0^t \langle v^N(s), dW(s)\rangle_{L^2}, \ a.s. \end{aligned} \tag{19}$$

In particular, it admits the averaged energy evolution law

$$\mathbb{E}(V_1(u^N(t), v^N(t))) = V_1(u_0^N, v_0^N) + \frac{1}{2}\mathrm{Tr}\left((P_N\mathbf{Q}^{\frac{1}{2}})(P_N\mathbf{Q}^{\frac{1}{2}})^*\right)t, \quad t \in \mathbb{R}^+.$$

Due to the definition of P_N and the assumption $\mathrm{Tr}(\mathbf{Q}) < \infty$, it can be verified that $\mathrm{Tr}\left((P_N\mathbf{Q}^{\frac{1}{2}})(P_N\mathbf{Q}^{\frac{1}{2}})^*\right)$ is uniformly bounded with respect to N for any $N \in \mathbb{N}_+$.

3.2. **Exponential integrability and regularity estimates of the spatial discretization.** In this part, we show the exponential integrability property of X^N. In [12, Section 5.4], the authors first obtain the exponential integrability of spectral Galerkin method of 2-dimensional stochastic wave equation driven by multiplicative noise on a non-empty compact domain. We would like to remark that the exponential integrability has been applied to study the large deviation principle and the well-posedness of SPDEs, as well as the strong convergence of the stochastic numerical scheme. For details, we refer to [3, 4, 13, 19] and references therein.

Lemma 3.2. *Let $T > 0$. There exist a constant $\alpha \geq \mathrm{Tr}(\mathbf{Q})$ and a positive constant $C := C(X_0, T, \mathbf{Q}, \alpha)$ such that*

$$\sup_{s\in[0,T]} \mathbb{E}\left[\exp\left(\frac{V_1(u^N(s), v^N(s))}{\exp(\alpha s)}\right)\right] \leq C. \tag{20}$$

Proof. Denote

$$\begin{aligned}(\mathcal{G}_{A_N+\mathbb{F}_N,\mathbb{G}_N}(V_1))(u,v) :=& \langle D_u V_1(u,v), v\rangle_{L^2} + \langle D_v V_1(u,v), \Lambda u - f(u)\rangle_{L^2} \\ &+ \frac{1}{2}\sum_{i=1}^{\infty}\langle D_{vv}V_1(u,v)P_N\mathbf{Q}^{\frac{1}{2}}e_i, P_N\mathbf{Q}^{\frac{1}{2}}e_i\rangle_{L^2}.\end{aligned}$$

Notice that $D_u V_1(u,v) = f(u) - \Lambda u$, $D_v V_1(u,v) = v$ and $D^2_{vv}V_1(u,v) = I$. Similar to the estimates (5.43) in [12, Section 5.4], one has that

$$\begin{aligned}&(\mathcal{G}_{A_N+\mathbb{F}_N,\mathbb{G}_N}(V_1))\left(u^N, v^N\right) \\ =&\langle f(u^N) - \Lambda u^N, v^N\rangle_{L^2} + \langle v^N, P_N(\Lambda u^N - f(u^N))\rangle_{L^2} + \frac{1}{2}\mathrm{Tr}(P_N\mathbf{Q}^{\frac{1}{2}}(P_N\mathbf{Q}^{\frac{1}{2}})^*) \\ =&\frac{1}{2}\mathrm{Tr}(P_N\mathbf{Q}^{\frac{1}{2}}(P_N\mathbf{Q}^{\frac{1}{2}})^*).\end{aligned}$$

Then we get that for $\alpha > 0$,

$$\begin{aligned}&(\mathcal{G}_{A_N+\mathbb{F}_N,\mathbb{G}_N}(V_1))\left(u^N, v^N\right) + \frac{1}{2\exp(\alpha t)}\sum_{i=1}^{\infty}\langle (P_N\mathbf{Q}^{\frac{1}{2}})^* v^N, e_i\rangle^2_{L^2} \\ \leq&\frac{1}{2}\mathrm{Tr}(\mathbf{Q}) + \frac{1}{2\exp(\alpha t)}\sum_{i=1}^{\infty}\langle v^N, \mathbf{Q}^{\frac{1}{2}}e_i\rangle^2_{L^2} \leq \frac{1}{2}\mathrm{Tr}(\mathbf{Q}) + \frac{1}{\exp(\alpha t)}V_1(u^N, v^N)\mathrm{Tr}(\mathbf{Q}).\end{aligned}$$

Let $\bar{U}(s) = -\frac{1}{2}\mathrm{Tr}(\mathbf{Q}), \alpha \geq \mathrm{Tr}(\mathbf{Q})$. To sum up, we have verify the condition (27) in [14, Lemma 3.1], i.e.,

$$(\mathcal{G}_{A_N+\mathbb{F}_N,\mathbb{G}_N}(V_1))(u,v) + \frac{1}{2\exp(\alpha t)}\sum_{i=1}^{+\infty}\langle D_v V_1(u,v), (P_N\mathbf{Q}^{\frac{1}{2}})e_i\rangle^2 + \bar{U}(s) \leq \alpha V_1(u,v),$$

for $u, v \in P^N H$. Thus, we conclude that

$$\mathbb{E}\left[\exp\left(\frac{V_1(u^N(t), v^N(t))}{\exp(\alpha t)} + \int_0^t \frac{\bar{U}(s)}{\exp(\alpha s)}ds\right)\right] \leq \exp(V_1(u_0^N, v_0^N)),$$

which implies (20). □

Corollary 3.1. *Let $d = 1, 2$. For any $c > 0$, it holds that*

$$\sup_{N\in\mathbb{N}^+} \mathbb{E}\Big[\exp\left(\int_0^T c\|u^N(s)\|^2_{L^6}ds\right)\Big] < \infty.$$

Proof. By applying Jensen's inequality, we get

$$\begin{aligned}&\mathbb{E}[\exp(\int_0^T c\|u^N(s)\|_{L^6}^2 ds)] \le \mathbb{E}[\int_0^T \frac{1}{T}\exp(cT\|u^N(s)\|_{L^6}^2)ds]\\ \le&\frac{1}{T}\int_0^T \sup_{s\in[0,T]} \mathbb{E}[\exp(cT\|u^N(s)\|_{L^6}^2)]ds \le \sup_{t\in[0,T]} \mathbb{E}[\exp(cT\|u^N(t)\|_{L^6}^2)].\end{aligned}$$

Based on the Gagliardo–Nirenberg inequality (10) with $a = \frac{d}{3}$, we have that

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\int_0^T c\|u^N(s)\|_{L^6}^2 ds\right)\right)\\ &\le \sup_{t\in[0,T]} \mathbb{E}\left[\exp(cCT\|\nabla u^N(t)\|_{L^2}^{2a}\|u^N(t)\|_{L^2}^{2-2a})\right]\\ &\le \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\frac{\|\nabla u^N(t)\|_{L^2}^2}{2\exp(\alpha t)}\right)\exp\left(\exp(\frac{a}{1-a}\alpha T)\|u^N(t)\|_{L^2}^2(cCT)^{\frac{1}{1-a}}2^{\frac{a}{1-a}}\right)\right].\end{aligned}$$

In the last step, we have used the Young inequality
$|u||v| \le \frac{|u|^{\frac{1}{a}}}{2\exp(\alpha t)} + (2\exp(\alpha t))^{\frac{a}{1-a}}|v|^{\frac{1}{1-a}}$. Then the Hölder and the Young inequalities imply that for some small $\epsilon > 0$,

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\int_0^T c\|u^N(s)\|_{L^6}^2 ds\right)\right]\\ &\le \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\frac{\|\nabla u^N(t)\|_{L^2}^2}{2\exp(\alpha t)}\right)\exp\left(\epsilon\|u^N(t)\|_{L^2}^4 + \frac{1}{4\epsilon}\exp(2\frac{a}{1-a}\alpha T)(cCT)^{\frac{2}{1-a}}4^{\frac{a}{1-a}}\right)\right]\\ &\le C(\epsilon,d)\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\frac{\|\nabla u^N(t)\|_{L^2}^2}{2\exp(\alpha t)}\right)\exp(\epsilon\|u^N(t)\|_{L^4}^4)\right].\end{aligned}$$

Then using (7) and the definition (8) and taking $\epsilon \le a_1 e^{-\alpha T}$, we get

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\int_0^T c\|u^N(s)\|_{L^6}^2 ds\right)\right]\\ \le& C(\epsilon,d)\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\frac{\|\nabla u^N(t)\|_{L^2}^2}{2\exp(\alpha t)}\right)\exp\left(\epsilon\frac{\int_{\mathcal{O}} F(u(x))dx}{a_1} + \frac{b_1}{a_1}\right)\right]\\ \le& C(\epsilon,d)\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\frac{V_1(u^N(t),v^N(t))}{\exp(\alpha t)}\right)\right].\end{aligned}$$

Applying Lemma 3.2, we complete the proof. □

Note that when $d = 1$, using the Gagliardo–Nirenberg inequality (11), one can obtain that for any $c > 0$, $\sup_{N\in\mathbb{N}^+} \mathbb{E}\left[\exp\left(\int_0^T c\|u^N(s)\|_{L^\infty}^2 ds\right)\right] < \infty$.

Now we show the higher regularity estimate of the solution of (16) in two different cases, i.e., the case $d = 1$ in Proposition 3.2 and the case $d = 2$ in Proposition 3.3. The main reason is due to the fact that in the case $d = 1$, one can use the Gagliardo–Nirenberg inequality (11).

Proposition 3.2. *Let $p \ge 1$, $d = 1$, $\beta \in [1,2]$, $\|(-\Lambda)^{\frac{\beta-1}{2}}\mathbf{Q}^{\frac{1}{2}}\|_{\mathcal{L}_2(L^2)} < \infty$, $T > 0$ and $X_0 \in \mathbb{H}^\beta$. Then the mild solution of* (16) *satisfies*

$$\sup_{N\in\mathbb{N}} \|X^N\|_{L^p(\Omega;\mathcal{C}([0,T];\mathbb{H}^\beta))} \le C(X_0,T,\mathbf{Q},p).$$

Proof. For the stochastic convolution, using the unitary property (6) of $E_N(\cdot)$ and then applying the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t E_N(t-s)\mathbb{G}_N dW(s)\right\|_{\mathbb{H}^\beta}^p\\
&\le \mathbb{E}\sup_{t\in[0,T]}\|E_N(t)\|_{\mathcal{L}(\mathbb{H}^\beta)}\sup_{t\in[0,T]}\left\|\int_0^t E_N(s)\mathbb{G}_N dW(s)\right\|_{\mathbb{H}^\beta}^p\\
&\le C\left(\int_0^T \|(-\Lambda)^{\frac{\beta-1}{2}}\mathbf{Q}^{\frac12}\|_{\mathcal{L}_2(L^2)}^2 ds\right)^{\frac{p}{2}}\le C.
\end{aligned}$$

Now it suffices to estimate $\|\int_0^t E_N(t-s)\mathbb{F}_N(X^N(s))ds\|_{L^p(\Omega;\mathcal{C}([0,T];\mathbb{H}^\beta))}$. Since

$$E_N(t-s)\mathbb{F}_N(X^N(s))=\begin{bmatrix}-(-\Lambda)^{-\frac12}S(t-s)P_N(f(u^N(s)))\\ -C(t-s)P_N(f(u^N(s)))\end{bmatrix},$$

it suffices to estimate $\mathbb{E}\Big[\sup_{t\in[0,T]}(\int_0^t\|(-\Lambda)^{\frac{\beta-1}{2}}P_N(f(u^N(s)))\|_{L^2}ds)^p\Big]$. Now we first consider the case of $\beta\in[1,2)$. Using the Sobolev embedding $\dot{H}^1\hookrightarrow L^\infty$ and the argument in the proof of [16, Lemma 4], i.e.,

$$\|f(u^N(s))\|_{\dot{\mathbb{H}}^{\beta-1}}\le C(1+\|u^N(s)\|_{L^\infty}^3+\|u^N(s)\|_{\dot{\mathbb{H}}^{\beta-1}}^3),$$

we have

$$\int_0^t\left\|(-\Lambda)^{\frac{\beta-1}{2}}P_N(f(u^N(s)))\right\|_{L^2}ds\le C\int_0^t(1+\|u^N(s)\|_{\dot{\mathbb{H}}^1}^3+\|u^N(s)\|_{\dot{\mathbb{H}}^{\beta-1}}^3)ds.$$

Based on the Hölder inequality and the Young inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left(\int_0^t\|(-\Lambda)^{\frac{\beta-1}{2}}P_N(f(u^N(s)))\|_{L^2}ds\right)^p\\
\le&C\mathbb{E}\int_0^T(1+\|u^N(s)\|_{\dot{\mathbb{H}}^1}+\|u^N(s)\|_{\dot{\mathbb{H}}^{\beta-1}})^{3p}ds\\
\le&C+C\int_0^T\mathbb{E}(1+\|u^N(s)\|_{\dot{\mathbb{H}}^1}^{3p})ds+C\int_0^T\mathbb{E}\|u^N(s)\|_{\dot{\mathbb{H}}^{\beta-1}}^{3p}ds,
\end{aligned}$$

which, together with Lemma 3.1 shows the desired result for the case $\beta\in[1,2)$. With regard to the case that $\beta=2$, we can use the verified result in the case $\beta\in[1,2)$ and the fact that

$$\|f(u^N(s))\|_{\dot{\mathbb{H}}^1}\le C(1+\|u^N(s)\|_{\dot{\mathbb{H}}^{1+\epsilon}}^3)$$

for any small $\epsilon\in(0,1)$. We omit further tedious details. □

The following regularity estimate of X^N is for the case $d=2$. Compared to Proposition 3.2, one needs to use the Sobolev embedding theorem in 2d to deal with the cubic nonlinearity.

Proposition 3.3. *Let $d=2$, $T>0$, $X_0\in\mathbb{H}^2$ and $\|(-\Lambda)^{\frac12}\mathbf{Q}^{\frac12}\|_{\mathcal{L}_2(L^2)}<\infty$. Then for any $p\ge2$, there exists a positive constant $C:=C(X_0,\mathbf{Q},T,p)$ such that*

$$\sup_{N\in\mathbb{N}}\|X^N\|_{L^p(\Omega;\mathcal{C}([0,T];\mathbb{H}^2))}\le C(X_0,\mathbf{Q},T,p). \tag{21}$$

Proof. We only present the proof for $p = 2$ here, since the proof for general $p > 2$ is similar. Similar to the proof of Proposition 3.2, it only suffices to get a uniform bound of X^N under the $\mathcal{C}([0,T]; L^2(\Omega; \mathbb{H}^2))$-norm. We introduce another Lyapunov functional

$$V_2(u^N, v^N) = \frac{1}{2}\left\|\Lambda u^N\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla v^N\right\|_{L^2}^2 + \frac{1}{2}\langle(-\Lambda)u^N, f(u^N)\rangle_{L^2}.$$

By applying the Itô's formula to V_2 and the commutativity between Λ and P_N, we get

$$\begin{aligned} &dV_2(u^N(t), v^N(t)) \\ =&\mathrm{I}_1(t)dt + \langle \nabla v^N(t), \nabla P_N dW(t)\rangle_{L^2} + \frac{1}{2}\mathrm{Tr}\left((\nabla P_N \mathbf{Q}^{\frac{1}{2}})(\nabla P_N \mathbf{Q}^{\frac{1}{2}})^*\right)dt, \end{aligned} \tag{22}$$

where

$$\mathrm{I}_1(t) = \frac{1}{2}\langle \nabla u^N(t), D^2 f(u^N(t))\nabla u^N(t) v^N(t)\rangle_{L^2}.$$

Making use of the Hölder inequality and the Gagliardo–Nirenberg inequality (12), we have

$$\begin{aligned} \mathrm{I}_1(t) \le& C\|\nabla u^N(t)\|_{L^4}^2(1 + \|u^N(t)\|_{L^\infty})\|v^N\|_{L^2} \\ \le& C\|\Lambda u^N(t)\|_{L^2}\|\nabla u^N(t)\|_{L^2}(1 + \|u^N(t)\|_{L^\infty})\|v^N(t)\|_{L^2}. \end{aligned}$$

By further applying the Gagliardo–Nirenberg inequality (13) and using the Young inequality, we get

$$\begin{aligned} \mathrm{I}_1(t) &\le C\|\Lambda u^N(t)\|_{L^2}\|\nabla u^N(t)\|_{L^2}(1 + \|\Lambda u^N(t)\|_{L^2}^{\frac{1}{4}}\|u^N(t)\|_{L^6}^{\frac{3}{4}})\|v^N(t)\|_{L^2} \\ &\le C\Big(\|\nabla u^N(t)\|_{L^2}^2\|v^N(t)\|_{L^2}^2 + \|\nabla u^N(t)\|_{L^2}^{\frac{8}{3}}\|v^N(t)\|_{L^2}^{\frac{8}{3}}\|u^N(t)\|_{L^6}^2 + \|\Lambda u^N(t)\|_{L^2}^2\Big). \end{aligned}$$

On the other hand, using the Cauchy–Schwarz inequality and the Young inequality and the fact that $\dot{\mathbb{H}}^1 \hookrightarrow L^6$, we deduce that

$$\begin{aligned} &\left|\langle(-\Lambda)u^N(t), f(u^N(t))\rangle_{L^2}\right| \le \|(-\Lambda)u^N(t)\|_{L^2}\|f(u^N(t))\|_{L^2} \\ \le&\frac{1}{2}\|\Lambda u^N(t)\|_{L^2}^2 + \frac{\tilde{C}_1(c_0, c_1, c_2, c_3)}{2}\|u^N(t)\|_{L^6}^6 + \tilde{C}_2(c_0, c_1, c_2, c_3) \\ \le&\frac{1}{2}\|\Lambda u^N(t)\|_{L^2}^2 + \frac{\tilde{C}_1(c_0, c_1, c_2, c_3, d)}{2}\|u^N(t)\|_{\dot{\mathbb{H}}^1}^6 + \tilde{C}_2(c_0, c_1, c_2, c_3). \end{aligned}$$

The above inequality leads to $V_2(u^N, v^N) \ge \frac{1}{4}\|\Lambda u^N\|_{L^2}^2 - \frac{\tilde{C}_1}{4}\|u^N\|_{\dot{\mathbb{H}}^1}^6 - \frac{1}{2}\tilde{C}_2$. Thus, by Young's inequality and the fact that $\dot{\mathbb{H}}^1 \hookrightarrow L^6$, we obtain that

$$\begin{aligned} I_1(t) &\le C\Big(\|\nabla u^N(t)\|_{L^2}^2\|v^N(t)\|_{L^2}^2 + \|\nabla u^N(t)\|_{L^2}^{\frac{8}{3}}\|v^N(t)\|_{L^2}^{\frac{8}{3}}\|u^N(t)\|_{L^6}^2 + \|\Lambda u^N(t)\|_{L^2}^2\Big) \\ &\le C\Big(\|u^N(t)\|_{\dot{\mathbb{H}}^1}^4 + \|v^N(t)\|_{L^2}^4 + \|u^N(t)\|_{\dot{\mathbb{H}}^1}^{\frac{14}{3}}\|v^N(t)\|_{L^2}^{\frac{8}{3}} + \|\Lambda u^N(t)\|_{L^2}^2\Big) \\ &\le C\Big(\|u^N(t)\|_{\dot{\mathbb{H}}^1}^4 + \|v^N(t)\|_{L^2}^4 + \|u^N(t)\|_{\dot{\mathbb{H}}^1}^6 + \|v^N(t)\|_{L^2}^{12} + \|\Lambda u^N(t)\|_{L^2}^2\Big) \\ &\le C(1 + V_2(u^N(t), v^N(t)) + \|u^N(t)\|_{\dot{\mathbb{H}}^1}^6 + \|v^N(t)\|_{L^2}^{12}). \end{aligned}$$

Taking expectation on (22), using the above estimates of I_1 and V_2, it follows that

$$\begin{aligned} &d\mathbb{E}V_2(u^N(t), v^N(t)) \\ &= \mathbb{E}\mathrm{I}_1(t)dt + \frac{1}{2}\mathrm{Tr}\left((\nabla P_N \mathbf{Q}^{\frac{1}{2}})(\nabla P_N \mathbf{Q}^{\frac{1}{2}})^*\right)dt \end{aligned}$$

$$\leq C\left(V_2(u^N(t),v^N(t)) + \|u^N(t)\|^6_{\mathbb{H}^1} + \|v^N(t)\|^{12}_{L^2} + 1\right)dt$$
$$+ \frac{1}{2}\mathrm{Tr}\left((\nabla P_N \mathbf{Q}^{\frac{1}{2}})(\nabla P_N \mathbf{Q}^{\frac{1}{2}})^*\right)dt.$$

By using the inverse equality $\|u^N\|_{\dot{\mathbb{H}}^2} \leq C\lambda_N \|u^N\|_{L^2}$ and the integrability of u^N in $\mathbb{H}^1$ in (18), one could obtain the integrability of V_2 and other terms on the right hand side of the above equality. Taking the expectation on both sides and applying the Gronwall inequality, we have

$$\mathbb{E}V_2(u^N(t),v^N(t)) \leq C\exp(Ct)\Big(\|X_0\|^2_{\mathbb{H}^2} + \frac{1}{2}\mathrm{Tr}\left((\nabla P_N \mathbf{Q}^{\frac{1}{2}})(\nabla P_N \mathbf{Q}^{\frac{1}{2}})^*\right)t$$
$$+ \int_0^t \mathbb{E}(1 + \|v^N(s)\|^{12}_{L^2} + \|u^N(s)\|^6_{\dot{\mathbb{H}}^1})ds\Big),$$

which, combined with Lemma 3.1, shows the desired result. □

Next we derive the Hölder continuity in temporal direction for the numerical solution $\{u^N\}_{N\in\mathbb{N}}$ and $\{X^N\}_{N\in\mathbb{N}}$ with respect to $L^p(\Omega;L^2)$-norm and $L^p(\Omega;\mathbb{H})$-norm, respectively. Both results play a key role in our error analysis in Section 4.

Lemma 3.3. *Assume that conditions in Lemma 3.1 hold. Then there exists* $C := C(X_0, \mathbf{Q}, T, p) > 0$ *such that for any* $0 \leq s \leq t \leq T$,

$$\sup_{N\in\mathbb{N}} \|u^N(t) - u^N(s)\|_{L^p(\Omega;L^2)} \leq C|t-s|, \ \sup_{N\in\mathbb{N}} \|X^N(t) - X^N(s)\|_{L^p(\Omega;\mathbb{H})} \leq C|t-s|^{\frac{1}{2}}.$$

Proof. From (16), we have

$$\begin{aligned}
u^N(t) - u^N(s) =&(C_N(t) - C_N(s))P_N(u_0) + (-\Lambda_N)^{-\frac{1}{2}}(S_N(t) - S_N(s))P_N(v_0)\\
&- \int_0^s (-\Lambda_N)^{-\frac{1}{2}}(S_N(t-r) - S_N(s-r))P_N(f(u^N))dr\\
&- \int_s^t (-\Lambda_N)^{-\frac{1}{2}}S_N(t-r)P_N(f(u^N))dr\\
&+ \int_0^s (-\Lambda_N)^{-\frac{1}{2}}(S_N(t-r) - S_N(s-r))P_N dW(r)\\
&+ \int_s^t (-\Lambda_N)^{-\frac{1}{2}}S_N(t-r)P_N dW(r).
\end{aligned}$$

Using the properties of $C_N(t)$ and $S_N(t)$ in Lemma 2.1 and the Burkholder–Davis–Gundy inequality, as well as the Sobolev embedding theorem, gives

$$\begin{aligned}
&\|u^N(t) - u^N(s)\|_{L^p(\Omega;L^2)}\\
&\leq C|t-s|\left(\|u_0\|_{L^p(\Omega;\dot{\mathbb{H}}^1)} + \|v_0\|_{L^p(\Omega;L^2)}\right)\\
&\quad + C\int_0^s (t-s)\|f(u^N)\|_{L^p(\Omega;L^2)}ds + C\int_s^t \|f(u^N)\|_{L^p(\Omega;\dot{\mathbb{H}}^{-1})}ds\\
&\quad + \left(\int_0^s \|(-\Lambda_N)^{-\frac{1}{2}}(S_N(t-r) - S_N(s-r))P_N\mathbf{Q}^{\frac{1}{2}}\|^2_{\mathcal{L}_2(L^2)}dr\right)^{\frac{1}{2}}\\
&\quad + \left(\int_s^t \|(-\Lambda_N)^{-\frac{1}{2}}S_N(t-r)P_N\mathbf{Q}^{\frac{1}{2}}\|^2_{\mathcal{L}_2(L^2)}dr\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C|t-s|\left(1+\|u_0\|_{L^p(\Omega;\dot{\mathbb{H}}^1)}+\|v_0\|_{L^p(\Omega;L^2)}+\sup_{0\leq t\leq T}\|u^N(t)\|^3_{L^{3p}(\Omega;\dot{\mathbb{H}}^1)}\right)$$
$$\leq C|t-s|,$$

which is the claim for u^N. To obtain the estimate of X^N, it suffices to deal with $v^N(t)-v^N(s)$. Using the mild formulation of v^N,

$$\begin{aligned}
&v^N(t)-v^N(s)\\
=&-\left((-\Lambda_N)^{\frac{1}{2}}S_N(t)-(-\Lambda_N)^{\frac{1}{2}}S_N(s)\right)P_N(u_0)+(C_N(t)-C_N(s))P_N(v_0)\\
&-\int_0^s(C_N(t-r)-C_N(s-r))P_N(f(u^N))dr\\
&-\int_s^t C_N(t-r)P_N(f(u^N))dr\\
&+\int_0^s(C_N(t-r)-C_N(s-r))P_N dW(r)\\
&+\int_s^t C_N(t-r)P_N dW(r).
\end{aligned}$$

Using again the properties of $C_N(t)$ and $S_N(t)$ in Lemma 2.1 and the Burkholder–Davis–Gundy inequality, as well as the Sobolev embedding theorem, gives

$$\begin{aligned}
&\|v^N(t)-v^N(s)\|_{L^p(\Omega;\dot{\mathbb{H}}^{-1})}\\
&\leq C|t-s|\left(\|u_0\|_{L^p(\Omega;\dot{\mathbb{H}}^1)}+\|v_0\|_{L^p(\Omega;L^2)}\right)\\
&\quad+C\int_0^s(t-s)\|f(u^N)\|_{L^p(\Omega;L^2)}ds+C\int_s^t\|f(u^N)\|_{L^p(\Omega;\dot{\mathbb{H}}^{-1})}ds\\
&\quad+\left(\int_0^s\|(-\Lambda_N)^{-\frac{1}{2}}(C_N(t-r)-C_N(s-r))P_N\mathbf{Q}^{\frac{1}{2}}\|^2_{\mathcal{L}_2(L^2)}dr\right)^{\frac{1}{2}}\\
&\quad+\left(\int_s^t\|(-\Lambda_N)^{-\frac{1}{2}}C_N(t-r)P_N\mathbf{Q}^{\frac{1}{2}}\|^2_{\mathcal{L}_2(L^2)}dr\right)^{\frac{1}{2}}\\
&\leq C|t-s|^{\frac{1}{2}}\left(1+\|u_0\|_{L^p(\Omega;\dot{\mathbb{H}}^1)}+\|v_0\|_{L^p(\Omega;L^2)}+\sup_{0\leq t\leq T}\|u^N(t)\|^3_{L^{3p}(\Omega;\dot{\mathbb{H}}^1)}\right)\\
&\leq C|t-s|^{\frac{1}{2}},
\end{aligned}$$

which completes the proof. □

3.3. **Exponential integrability of stochastic wave equation.** In this part, we first prove that the spectral Galerkin approximation X_N converges to the solution of (4) in strong sense based on Lemma 3.2.

Proposition 3.4. *Assume that $d=1$, $\beta\geq 1$ (or $d=2$, $\beta=2$), $X_0\in\mathbb{H}^\beta$, $T>0$ and $\|(-\Lambda)^{\frac{\beta-1}{2}}\mathbf{Q}^{\frac{1}{2}}\|_{\mathcal{L}_2(L^2)}<\infty$. Then for any $p\geq 2$, (16) satisfies that*

$$\|X^N-X\|_{L^p(\Omega;\mathcal{C}([0,T];\mathbb{H}))}=O(\lambda_N^{-\frac{\beta}{2}}).$$

Proof. For the sake of simplicity, we take the first component of X^N as an example to illustrate the desired result.

Step 1: Strong convergence and limit of u^N. We claim that $\{u^N\}_{N\in\mathbb{N}^+}$ is a Cauchy sequence in $L^p(\Omega;\mathcal{C}([0,T];L^2))$. Notice that

$$u^N(t)-u^{N'}(t)=\Big(u^N(t)-P_Nu^{N'}(t)\Big)+\Big((P_N-I)u^{N'}(t)\Big),$$

where $N,N'\in\mathbb{N}^+$. Without loss of generality, it may be assumed that $N'>N$. According to the expression of both u^N and $u^{N'}$, using the definition of P_N, we have

$$\|(P_N-I)u^{N'}(t)\|_{L^2}^2=\sum_{i=N+1}^{\infty}\lambda_i^{-\beta}\langle u^{N'}(t),\lambda_i^{\frac{\beta}{2}}e_i\rangle_{L^2}^2\le\lambda_N^{-\beta}\|u^{N'}(t)\|_{\dot{\mathbb{H}}^\beta}^2$$

with $\beta\ge 1$. With respect to the term $u^N(t)-P_Nu^{N'}(t)$, we have

$$u^N(t)-P_Nu^{N'}(t)=\int_0^t(-\Lambda)^{-\frac12}S(t-s)P_N\Big(f(u^{N'}(s))-f(u^N(s))\Big)ds.$$

From the Sobolev embedding $L^{\frac65}\hookrightarrow\dot{\mathbb{H}}^{-1}$, using the Hölder inequality with the exponent $\frac13+\frac12=\frac56$, we obtain that

$$\begin{aligned}
&\|u^N(t)-P_Nu^{N'}(t)\|_{L^2}\\
\le&\int_0^t\Big\|(-\Lambda)^{-\frac12}S(t-s)P_N\Big(f(u^{N'}(s))-f(u^N(s))\Big)\Big\|_{L^2}ds\\
\le&C\int_0^t\Big\|\Big(1+|u^N(s)|^2+|u^{N'}(s)|^2\Big)\Big(u^N(s)-u^{N'}(s)\Big)\Big\|_{L^{\frac65}}ds\\
\le&C\int_0^t(1+\|u^N(s)\|_{L^6}^2+\|u^{N'}(s)\|_{L^6}^2)\\
&\Big(\|u^N(s)-P_Nu^{N'}(s)\|_{L^2}+\|(P_N-I)u^{N'}(s)\|_{L^2}\Big)ds,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|u^N(t)-P_Nu^{N'}(t)\|_{L^2}\\
\le&C\lambda_N^{-\frac{\beta}{2}}\exp\Big(C\int_0^T\Big(\|u^N(s)\|_{L^6}^2+\|u^{N'}(s)\|_{L^6}^2\Big)ds\Big)\int_0^t(1+\|u^N(s)\|_{L^6}^2+\|u^{N'}(s)\|_{L^6}^2)\\
&\times\|u^{N'}(s)\|_{\dot{\mathbb{H}}^\beta}ds
\end{aligned}$$

due to the Gronwall's inequality. Taking the pth moment and then using the Hölder and the Young inequalities, and the exponential moment bounds given in Corollary 3.1, we obtain

$$\begin{aligned}
&\|u^N-u^{N'}\|_{L^p(\Omega;\mathcal{C}([0,T];L^2))}\\
\le&C\lambda_N^{-\frac{\beta}{2}}\Big\|\exp\Big(\int_0^T\Big(\|u^N(s)\|_{L^6}^2+\|u^{N'}(s)\|_{L^6}^2\Big)ds\Big)\Big\|_{L^{2p}(\Omega;\mathbb{R})}\\
&\times\Big\|\int_0^T(1+\|u^N(s)\|_{L^6}^2+\|u^{N'}(s)\|_{L^6}^2)\|u^{N'}\|_{\dot{\mathbb{H}}^\beta}ds\Big\|_{L^{2p}(\Omega;\mathbb{R})}\\
&+\lambda_N^{-\frac{\beta}{2}}\|u^{N'}(s)\|_{L^p(\Omega;\mathcal{C}([0,T];\dot{\mathbb{H}}^\beta))},
\end{aligned}$$

which leads to

$$\|u^N-u^{N'}\|_{L^p(\Omega;\mathcal{C}([0,T];L^2))}\le C\lambda_N^{-\frac{\beta}{2}}. \tag{23}$$

Similarly, we can prove that $\{v^N\}_{N\in\mathbb{N}^+}$ is a Cauchy sequence in $L^p(\Omega;\mathcal{C}([0,T];\dot{\mathbb{H}}^{-1}))$ which means that $\{X^N\}_{N\in\mathbb{N}^+}$ is a Cauchy sequence in $L^p(\Omega;\mathcal{C}([0,T];\mathbb{H}))$. Indeed, we can first decompose

$$v^N(t)-v^{N'}(t)=\Big(v^N(t)-P_Nv^{N'}(t)\Big)+\Big((P_N-I)v^{N'}(t)\Big),$$

where $\|(P_N-I)v^{N'}(t)\|_{L^p(\Omega;\dot{\mathbb{H}}^{-1})}\le C\lambda_N^{\frac{\beta}{2}}$ due to $v^{N'}\in\dot{\mathbb{H}}^{\beta-1}$ in (18) and (21). Then applying the Sobolev embedding $L^{\frac{6}{5}}\hookrightarrow\dot{\mathbb{H}}^{-1}$ and Hölder's inequality again, we get

$$\begin{aligned}
&\|v^N(t)-P_Nv^{N'}(t)\|_{\dot{\mathbb{H}}^{-1}}\\
&=\|\int_0^t C_N(t-s)\Big(f(u^{N'}(s))-f(u^N(s))\Big)ds\|_{\dot{\mathbb{H}}^{-1}}\\
&\le C\int_0^t(1+\|u^N(s)\|_{L^6}^2+\|u^{N'}(s)\|_{L^6}^2)\Big(\|u^N(s)-P_Nu^{N'}(s)\|_{L^2}+\|(P_N-I)u^{N'}(s)\|_{L^2}\Big)ds.
\end{aligned}$$

We omit further tedious details for applying Gronwall's arguments.

Step 2: Existence and uniqueness of the mild solution.

Denote by $X=(u,v)^\top\in\mathbb{H}$ the limit of $\{X^N\}_{N\in\mathbb{N}^+}$. To show that the strong limit X is the mild solution of (1), it suffices to prove that

$$X(t)=E(t)X_0+\int_0^t E(t-s)\mathbb{F}(X(s))ds+\int_0^t E(t-s)\mathbb{G}dW(s) \tag{24}$$

for any $t\in[0,T]$. We take the convergence of $\{u^N\}_{N\in\mathbb{N}^+}$ as an example to illustrate the details, that is, to show that u satisfies

$$\begin{aligned}
u(t)=C(t)u_0+(-\Lambda)^{-\frac12}S(t)v_0-\int_0^t(-\Lambda)^{-\frac12}S(t-s)f(u(s))ds\\
+\int_0^t(-\Lambda)^{-\frac12}S(t-s)dW(s).
\end{aligned}$$

To this end, we show that the mild form of the exact solution u^N is convergent to that of u. The assumption on X_0 yields that

$$\|C(t)(I-P_N)u_0\|_{L^2}+\|(-\Lambda)^{-\frac12}S(t)(I-P_N)v_0\|_{L^2}\le C\lambda_N^{-\frac{\beta}{2}}(\|u_0\|_{\dot{\mathbb{H}}^\beta}+\|v_0\|_{\dot{\mathbb{H}}^{\beta-1}}).$$

Notice that by Sobolev embedding theorem, it holds that for $d=1,\beta\ge1$,

$$\begin{aligned}
\|(I-P^N)\int_0^t(-\Lambda)^{-\frac12}S(t-s)f(u(s))ds\|_{L^2}&\le C\lambda_N^{-\frac{\beta}{2}}\int_0^t\|f(u(s))\|_{\dot{\mathbb{H}}^{\beta-1}}ds\\
&\le C\lambda_N^{-\frac{\beta}{2}}\int_0^t(1+\|u(s)\|_{\dot{\mathbb{H}}^1}^3)ds,
\end{aligned}$$

and for $d=2,\beta=2$,

$$\|(I-P^N)\int_0^t(-\Lambda)^{-\frac12}S(t-s)f(u(s))ds\|_{L^2}\le C\lambda_N^{-\frac{\beta}{2}}\int_0^t(1+\|u(s)\|_{\dot{\mathbb{H}}^{1+\epsilon}}^3)ds,$$

where $\epsilon\in(0,1)$ is small. Thanks to Propositions 3.2 and 3.3, the above term is convergent. Based on the Sobolev embedding $L^{\frac65}\hookrightarrow\dot{\mathbb{H}}^{-1}$ and (23), we have

$$\begin{aligned}
&\Big\|\int_0^t(-\Lambda)^{-\frac12}S(t-s)P_N\left(f(u(s))-f(u^N(s))\right)ds\Big\|_{L^p(\Omega;\mathcal{C}([0,T];L^2))}\\
&\le C\int_0^T\|(1+\|u(s)\|_{L^6}^2+\|u^N(s)\|_{L^6}^2)\|_{L^{2p}(\Omega;\mathbb{R})}\|u(s)-u^N(s)\|_{L^{2p}(\Omega;L^2)}ds
\end{aligned}$$

$$\leq C\lambda_N^{-\frac{\beta}{2}}.$$

For the stochastic term, by using the fact that $S(t-s) = S(t)C(s) - C(t)S(s)$, the boundedness of $C(\cdot)$ and $S(\cdot)$, and the Burkholder–Davis–Gundy inequality we obtain that for $p \geq 2$,

$$\begin{aligned}
&\|\int_0^t (-\Lambda)^{-\frac{1}{2}} S(t-s)(I-P_N)dW(s)\|_{L^p(\Omega;\mathcal{C}([0,T];L^2))} \\
&\leq C\|\int_0^t (-\Lambda)^{-\frac{1}{2}} C(s)(I-P_N)dW(s)\|_{L^p(\Omega;\mathcal{C}([0,T];L^2))} \\
&\quad + C\|\int_0^t (-\Lambda)^{-\frac{1}{2}} S(s)(I-P_N)dW(s)\|_{L^p(\Omega;\mathcal{C}([0,T];L^2))} \\
&\leq C\sqrt{\int_0^t \|(I-P^N)(-\Lambda)^{-\frac{\beta}{2}}(-\Lambda)^{-\frac{1}{2}+\frac{\beta}{2}}\mathbf{Q}^{\frac{1}{2}}\|^2_{\mathcal{L}_2(L^2)} ds} \leq C\lambda_N^{-\frac{\beta}{2}}.
\end{aligned}$$

Next, we use analogous steps to deal with the convergence of v^N, i.e., to verify that the limit v satisfies

$$\begin{aligned}
v(t) = &-(-\Lambda)^{-\frac{1}{2}} S(t)u_0 + C(t)v_0 - \int_0^t C(t-s)f(u(s))ds \\
&+ \int_0^t C(t-s)dW(s).
\end{aligned}$$

The convergence of $-(-\Lambda)^{-\frac{1}{2}}S(t)u_0^N$ to $-(-\Lambda)^{-\frac{1}{2}}S(t)u_0$ and that of $C(t)v_0^N$ to $C(t)v_0$ is straightforward. The convergence of $\| \int_0^t C(t-s)(I-P^N)dW(s)\|_{\dot{H}^{-1}}$ is similar to that of $\| \int_0^t (-\Lambda)^{-\frac{1}{2}}(-\Lambda)^{-\frac{1}{2}} S(t-s)(I-P^N)dW(s)\|_{L^2}$ due to the fact that $C(t-s) = C(t)C(s) - S(t)S(s)$. It remains to show that

$$\int_0^t C(t-s)P^N f(u^N(s))ds \to \int_0^t C(t-s)f(u(s))ds, \text{ a.s. } N \to \infty.$$

Notice that by Sobolev embedding theorem, it holds that for $d = 1, \beta \geq 1$,

$$\|(I-P^N)\int_0^t C(t-s)f(u(s))ds\|_{\dot{\mathbb{H}}^{-1}} \leq C\lambda_N^{-\frac{\beta}{2}} \int_0^t (1 + \|u(s)\|^3_{\dot{\mathbb{H}}^1})ds,$$

and for $d = 2, \beta = 2$,

$$\|(I-P^N)\int_0^t C(t-s)f(u(s))ds\|_{\dot{\mathbb{H}}^{-1}} \leq C\lambda_N^{-\frac{\beta}{2}} \int_0^t (1 + \|u(s)\|^3_{\dot{\mathbb{H}}^{1+\epsilon}})ds,$$

where $\epsilon \in (0,1)$ is small. To verify the convergence of the term

$$\left\|\int_0^t C(t-s)P_N\left(f(u(s)) - f(u^N(s))\right)ds\right\|_{L^p(\Omega;\mathcal{C}([0,T];\dot{\mathbb{H}}^{-1}))},$$

we need repeat the procedures in the convergence proof of u and thus omit further tedious details.

Combining the above properties of v and u, we complete the proof. □

From the proof of Proposition 3.4, we have the following exponential integrability property of the exact solution.

Proposition 3.5. *Let $d = 1, 2$. For any $c \in \mathbb{R}$ and $T > 0$, it holds that*

$$\mathbb{E}\left(\exp\left(\int_0^T c\|u(s)\|_{L^6}^2 ds\right)\right) < \infty.$$

Proof. Due to Propositions 3.2-3.3 and Fatou's lemma, we have $\mathbb{E}[\|X\|^p_{\mathcal{C}([0,T];\mathbb{H}^1)}] \leq C(X_0, \mathbf{Q}, T, p)$. From the Gagliardo–Nirenberg inequality (10) and the boundedness of X and X^N in $L^p(\Omega; \mathcal{C}([0,T]; \mathbb{H}^1))$, it follows that u^N converges to u in $L^p(\Omega; \mathcal{C}([0,T]; L^6))$. By Jensen's inequality and Fatou's lemma, we have

$$\begin{aligned}\mathbb{E}\left(\exp\left(\int_0^T c\|u(s)\|_{L^6}^2 ds\right)\right) &\leq \frac{1}{T}\int_0^T \mathbb{E}\exp\left(cT\|u(s)\|_{L^6}^2\right) ds \\ &\leq \varliminf_{N\to\infty} \frac{1}{T}\int_0^T \mathbb{E}\exp\left(cT\|u^N(s)\|_{L^6}^2\right) ds.\end{aligned}$$

Then by applying Corollary 3.1, it holds that for any constant $c > 0$,

$$\mathbb{E}\left(\exp\left(\int_0^T c\|u(s)\|_{L^6}^2 ds\right)\right) < \infty.$$

□

4. **An exponentially integrable fully discrete method.** The stochastic wave equation with Lipschitz and regular coefficients has been systematically investigated theoretically and numerically (see e.g., [17, 23, 27] and references therein). However, as far as we know, for stochastic wave equations with non-globally Lipschitz coefficients, there are no results about the fully discrete method preserving both the energy evolution law and the exponential integrability property and the related strong convergence analysis. In this section, we propose an energy-preserving exponentially integrable fully discrete method for stochastic wave equation (1) by applying splitting AVF method to (16), and finally obtain a strong convergence theorem for the fully discrete numerical method.

Let $N, M \in \mathbb{N}^+$ and $T = Mh$ and denote $\mathbb{Z}_{M+1} = \{0, 1, \ldots, M\}$. For any $T > 0$, we partition the time domain $[0, T]$ uniformly with nodes $t_m = mh$, $m = 0, 1, \cdots M$. One may use the non-uniform discretization, and the analysis of both the convergence and structure-preserving properties is similar.

We first decompose (1) into a deterministic system on $[t_m, t_{m+1}]$,

$$du_m^{N,D}(t) = v_m^{N,D}(t)dt, \quad dv_m^{N,D}(t) = \Lambda_N u_m^{N,D}(t)dt - P_N(f(u_m^{N,D}(t)))dt, \tag{25}$$

and a stochastic system on $[t_m, t_{m+1}]$,

$$du_m^{N,S}(t) = 0, \quad dv_m^{N,S}(t) = P_N dW(t). \tag{26}$$

Then on each subinterval $[t_m, t_{m+1}]$, $u^{N,S}(t)$ starting from $u_m^{N,S}(t_m) = u_m^{N,D}(t_{m+1})$ and $v_m^{N,S}(t)$ starting from $v_m^{N,S}(t_m) = v_m^{N,D}(t_{m+1})$ can be formally viewed as approximations of $u^N(t)$ with $u^N(t_m) = u_m^{N,D}(t_m)$ and $v^N(t)$ with $v^N(t_m) = v_m^{N,D}(t_m)$, respectively.

By further using the explicit solution of (26) and the AVF method to discretize (25), we obtain the splitting AVF method

$$\begin{aligned} u^N_{m+1} =& u^N_m + h\bar{v}^N_{m+\frac{1}{2}}, \\ \bar{v}^N_{m+1} =& v^N_m + h\Lambda_N u^N_{m+\frac{1}{2}} - hP_N\left(\int_0^1 f(u^N_m + \theta(u^N_{m+1} - u^N_m))d\theta\right), \\ v^N_{m+1} =& \bar{v}^N_{m+1} + P_N\delta W_m, \end{aligned} \tag{27}$$

where $u^N_0 = P_N u_0, v^N_0 = P_N v_0$, $\bar{v}^N_{m+\frac{1}{2}} = \frac{1}{2}(\bar{v}^N_{m+1} + v^N_m)$, $u^N_{m+\frac{1}{2}} = \frac{1}{2}(u^N_{m+1} + u^N_m)$ and the increment $\delta W_m := W(t_{m+1}) - W(t_m) = \sum_{k=1}^{\infty}(\beta_k(t_{m+1}) - \beta_k(t_m))\mathbf{Q}^{\frac{1}{2}}e_k$.

Denote

$$\mathbb{A}(t) = \begin{pmatrix} I & \frac{t}{2}I \\ \Lambda_N\frac{t}{2} & I \end{pmatrix}, \quad \mathbb{B}(t) = \begin{pmatrix} I & -\frac{t}{2}I \\ -\Lambda_N\frac{t}{2} & I \end{pmatrix}$$

and $\mathbb{M}(t) = I - \Lambda_N\frac{t^2}{4}$. By the spectral expansion of Λ_N, one can verify that $\mathbb{B}^{-1} = \begin{pmatrix} \mathbb{M}^{-1}(t) & 0 \\ 0 & \mathbb{M}^{-1}(t) \end{pmatrix}\mathbb{A}$. Therefore, it holds that

$$\mathbb{B}^{-1}(t)\mathbb{A}(t) = \begin{pmatrix} \mathbb{M}^{-1}(t) & 0 \\ 0 & \mathbb{M}^{-1}(t) \end{pmatrix}\mathbb{A}^2(t) = \begin{pmatrix} 2\mathbb{M}^{-1}(t) - I & \mathbb{M}^{-1}(t)t \\ \mathbb{M}^{-1}(t)\Lambda_N t & 2\mathbb{M}^{-1}(t) - I \end{pmatrix}.$$

This formula yields that (27) can be rewritten as

$$\begin{aligned} \begin{pmatrix} u^N_{m+1} \\ v^N_{m+1} \end{pmatrix} =& \mathbb{B}^{-1}(h)\mathbb{A}(h)\begin{pmatrix} u^N_m \\ v^N_m \end{pmatrix} \\ &+ \mathbb{B}^{-1}(h)\begin{pmatrix} 0 \\ -hP_N\left(\int_0^1 f(u^N_m + \theta(u^N_{m+1} - u^N_m))d\theta\right) \end{pmatrix} + \begin{pmatrix} 0 \\ P_N(\delta W_m) \end{pmatrix}. \end{aligned} \tag{28}$$

Here we omit the dependence on N of $\mathbb{B}, \mathbb{A}, \mathbb{M}$ for simplicity since the error estimate for the spectral Galerkin method has been established in the previous section.

In the appendix, we prove the well-posedness of the proposed scheme. Indeed, by (40), there exists a sufficiently small $h_0 > 0$ which is not depending on N such that the numerical solution of (27) exists and is unique (see more details in appendix). Throughout this section, we always require that the temporal step size $h \le h_0$. When performing the numerical scheme (27), some iterations procedures are used to approximate (27) since the AVF scheme is implicit. However, the overall error analysis involving Picard's iteration or Netwon's iteration and the AVF scheme for SWEs with non-globally Lipschitz coefficients is beyond the scope of this current paper.

To study the strong convergence of the proposed numerical method, we first give some estimates of the matrix $\mathbb{B}^{-1}(\cdot)\mathbb{A}(\cdot)$.

Lemma 4.1. *For any $r \ge 0$, $t \ge 0$, and $w \in \mathbb{H}^r$, $\|\mathbb{B}^{-1}(t)\mathbb{A}(t)w\|_{\mathbb{H}^r} = \|w\|_{\mathbb{H}^r}$.*

Proof. When $t = 0$, $\mathbb{B}^{-1}(0)\mathbb{A}(0) = I$. The desired property holds. When $t > 0$, $\mathbb{B}^{-1}(t)\mathbb{A}(t)$ is unitary in $\mathbb{H}^r$ if and only if the discrete scheme defined by

$$\begin{aligned} u^N_{m+1} =& u^N_m + tv^N_{m+\frac{1}{2}}, \\ v^N_{m+1} =& v^N_m + t\Lambda_N u^N_{m+\frac{1}{2}}, m \in \mathbb{N}^+, \end{aligned}$$

keeps the $\mathbb{H}^r$-norm of $X^N_m := (u^N_m, v^N_m)$. This is true by considering the $\|X^N_{m+1}\|^2_{\mathbb{H}^r} - \|X^N_m\|^2_{\mathbb{H}^r}$. Indeed, we have that

$$\|X^N_{m+1}\|^2_{\mathbb{H}^r} - \|X^N_m\|^2_{\mathbb{H}^r}$$

$$= \|u_{m+1}^N\|_{\dot{\mathbb{H}}^r}^2 - \|u_m^N\|_{\dot{\mathbb{H}}^r}^2 + \|v_{m+1}^N\|_{\dot{\mathbb{H}}^{r-1}}^2 - \|v_m^N\|_{\dot{\mathbb{H}}^{r-1}}^2$$
$$= \langle u_{m+\frac{1}{2}}^N, t\bar{v}_{m+\frac{1}{2}}^N\rangle_{\dot{\mathbb{H}}^r} + \langle v_{m+\frac{1}{2}}^N, t\Lambda_N u_{m+\frac{1}{2}}^N\rangle_{\dot{\mathbb{H}}^{r-1}}.$$

The definition of $\dot{\mathbb{H}}^r$ yields that $\|X_{m+1}^N\|_{\dot{\mathbb{H}}^r}^2 = \|X_m^N\|_{\dot{\mathbb{H}}^r}^2$. □

Following [6, Theorem 3], we give the following lemma which is applied to the error estimate for (27).

Lemma 4.2. *For any $r \geq 0$ and $h \geq 0$, there exists a positive constant $C := C(r)$ such that for any $w \in \mathbb{H}^{r+2}$,*

$$\begin{aligned}\sup_{N\in\mathbb{N}^+} \|(E_N(h) - \mathbb{B}^{-1}(h)\mathbb{A}(h))w\|_{\mathbb{H}^r} &\leq Ch^2\|w\|_{\mathbb{H}^{r+2}},\\ \sup_{N\in\mathbb{N}^+} \|(E_N(h) - \mathbb{B}^{-1}(h))w\|_{\mathbb{H}^r} &\leq Ch\|w\|_{\mathbb{H}^{r+1}}.\end{aligned} \tag{29}$$

4.1. **Useful a priori bounds and exponential moment of numerical solutions.** In this part, we provide several useful bounds on the numerical solution of (26), as well as its energy evolution law. Denote $V_1(u_m^N, v_m^N) := \frac{1}{2}\|u_m^N\|_{\dot{\mathbb{H}}^1}^2 + \frac{1}{2}\|v_m^N\|_{L^2}^2 + \int_{\mathcal{O}} F(u_m^N)dx + C_1$ with $C_1 > b_1$. Recall that b_1 is defined in (7).

Proposition 4.1. *Assume that $T > 0$, $p \geq 1$. Then the solution of (27) satisfies*

$$\sup_{N\in\mathbb{N}^+} \sup_{m\in\mathbb{Z}_{M+1}} \mathbb{E}(V_1^p(u_m^N, v_m^N)) \leq C, \tag{30}$$

where $N \in \mathbb{N}^+$, $Mh = T$, $M \in \mathbb{N}^+$, $C = C(X_0, \mathbf{Q}, T, p) > 0$.

Proof. Fix $t \in T_m := [t_m, t_{m+1}]$ with $m \in \mathbb{Z}_M$. Recall that the splitting process $u_m^{N,S}, v_m^{N,S}$ in (26) defined in T_m. Let $u_m^{N,S}(t) = u_{m+1}^N, v_m^{N,S}(t_m) = \bar{v}_{m+1}^N$. Then on the interval T_m, we have

$$\begin{aligned}V_1(u_{m+1}^N, v_{m+1}^N) =& V_1(u_{m+1}^N, \bar{v}_{m+1}^N) + \int_{t_m}^{t_{m+1}} \langle v_m^{N,S}(t), P_N dW(t)\rangle_{L^2}\\ &+ \int_{t_m}^{t_{m+1}} \frac{1}{2}\mathrm{Tr}\left((P_N\mathbf{Q}^{\frac{1}{2}})(P_N\mathbf{Q}^{\frac{1}{2}})^*\right)dt.\end{aligned}$$

Due to the fact that $du_m^{N,S}(t) = 0$ and $dv_m^{N,S}(t) = P_N dW(t)$, we can apply the Itô's formula to $V_1^p(u_m^{N,S}(t), v_m^{N,S}(t))$ and obtain that for $p \geq 2$

$$\begin{aligned}V_1^p(u_m^{N,S}(t), v_m^{N,S}(t)) &= V_1^p(u_m^{N,S}(t_m), v_m^{N,S}(t_m))\\ &+ \frac{p}{2}\int_{t_m}^t V_1^{p-1}(u_m^{N,S}(r), v_m^{N,S}(r))\mathrm{Tr}\left((P_N\mathbf{Q}^{\frac{1}{2}})(P_N\mathbf{Q}^{\frac{1}{2}})^*\right)dr\\ &+ p\int_{t_m}^t V_1^{p-1}(u_m^{N,S}(r), v_m^{N,S}(r))\langle v_m^{N,S}(s), P_N dW(r)\rangle_{L^2}\\ &+ \frac{p(p-1)}{2}\sum_{i=1}^N \int_{t_m}^t V_1^{p-2}(u_m^{N,S}(r), v_m^{N,S}(r))\langle v_m^{N,S}(r), \mathbf{Q}^{\frac{1}{2}}e_i\rangle_{L^2}^2 dr.\end{aligned}$$

Taking the expectation on both sides of the above equation, using the martingality of the stochastic integral and applying the Hölder and Young inequalities,

$$\begin{aligned}\mathbb{E}(V_1^p(u_m^{N,S}(t), v_m^{N,S}(t))) \leq& \mathbb{E}(V_1^p(u_{m+1}^N, \bar{v}_{m+1}^N))\\ &+ C\int_{t_m}^t (1 + \mathbb{E}(V_1^p(u_m^{N,S}(r), v_m^{N,S}(r))))dr,\end{aligned}$$

which, together with the Gronwall inequality and the property that

$$V_1(u_{m+1}^N, \bar{v}_{m+1}^N) = V_1(u_m^N, v_m^N), \tag{31}$$

leads to $\mathbb{E}(V_1^p(u_{m+1}^N, v_{m+1}^N)) \leq \exp(Ch)\left(\mathbb{E}(V_1^p(u_m^N, v_m^N)) + Ch\right)$. Since $Mh = T$, iteration arguments lead to

$$\sup_{m\in\mathbb{Z}_M} \mathbb{E}(V_1^p(u_{m+1}^N, v_{m+1}^N)) \leq \exp(CT)\mathbb{E}(V_1^p(u_0^N, v_0^N)) + \exp(CT)CT,$$

which implies the estimate (30). □

From the above proof of Proposition 4.1, we get the following theorem which shows that the proposed method admits the following evolution law of the energy V_1.

Theorem 4.1. *Let $T > 0$. Then the solution of (27) satisfies*

$$\begin{aligned} V_1(u_{m+1}^N, v_{m+1}^N) =& V_1(u_m^N, v_m^N) + \int_{t_m}^{t_{m+1}} \langle v_m^{N,S}(t), P_N dW(t)\rangle_{L^2} \\ &+ \int_{t_m}^{t_{m+1}} \frac{1}{2}\mathrm{Tr}\left((P_N\mathbf{Q}^{\frac12})(P_N\mathbf{Q}^{\frac12})^*\right) ds, \end{aligned}$$

where $N \in \mathbb{N}^+, m \in \mathbb{Z}_{M+1}$, $Mh = T$, $M \in \mathbb{N}^+$, $v_m^{N,S}$ is the splitting process defined in (26) with initial value $v_m^{N,S}(t_m) = \bar{v}_{m+\frac12}^N$. In particular, it holds that

$$\mathbb{E}(V_1(u_m^N, v_m^N)) = V_1(u_0^N, v_0^N) + \frac12 \mathrm{Tr}\left((P_N\mathbf{Q}^{\frac12})(P_N\mathbf{Q}^{\frac12})^*\right) t_m,$$

where $N \in \mathbb{N}^+, m \in \mathbb{Z}_{M+1}$, $Mh = T$, $M \in \mathbb{N}^+$.

Beside the energy-preserving property, the proposed numerical method also inherits the exponential integrability property of the original system as following.

Proposition 4.2. *Let $d = 1, 2$, and $T > 0$. Then the solution of (27) satisfies*

$$\sup_{N\in\mathbb{N}^+} \mathbb{E}\left[\exp\left(ch\sum_{i=0}^m \|u_i^N\|_{L^6}^2\right)\right] \leq C \tag{32}$$

for any $c > 0$, where $C := C(X_0, \mathbf{Q}, T, c) > 0$, $N \in \mathbb{N}^+, m \in \mathbb{Z}_{M+1}$, $M \in \mathbb{N}^+, Mh = T$.

Proof. Notice that

$$V_1(u_{m+1}^N, v_{m+1}^N) = V_1(u_m^{N,S}(t_{m+1}), v_m^{N,S}(t_{m+1})),$$

where $v_m^{N,S}(t_{m+1})$ is the solution of (26) defined on $[t_m, t_{m+1}]$ with $v_m^{N,S}(t_m) = \bar{v}_{m+1}^N$ and $u_m^{N,S}(t_m) = u_{m+1}^N$. Denote $\widetilde{\mathbb{F}}_N := (0,0)^\top$. Then for $\alpha > 0$,

$$\begin{aligned} &\left(\mathcal{G}_{\widetilde{\mathbb{F}}_N,\mathbb{G}_N}(V_1)\right)(u_m^{N,S}, v_m^{N,S}) + \frac{1}{2\exp(\alpha t)}\sum_{i=1}^\infty \langle (P_N\mathbf{Q}^{\frac12})^* v_m^{N,S}, e_i\rangle_{L^2}^2 \\ \leq& \frac12\mathrm{Tr}(\mathbf{Q}) + \frac{1}{\exp(\alpha t)} V_1(u_m^{N,S}, v_m^{N,S})\mathrm{Tr}(\mathbf{Q}). \end{aligned}$$

Let $\bar{U} = -\frac12\mathrm{Tr}(\mathbf{Q})$, $\alpha \geq \mathrm{Tr}(\mathbf{Q})$. Then using [14, Lemma 3.1], it holds that for $t \in [t_m, t_{m+1}]$,

$$\mathbb{E}\left[\exp\left(e^{-\alpha t_m}\Big(\frac{V_1(u_m^{N,S}(t), v_m^{N,S}(t))}{\exp(\alpha(t-t_m))} + \int_{t_m}^t \frac{\bar{U}(s)}{\exp(\alpha(r-t_m))}dr\Big)\right)\Big|\mathcal{F}_{t_m}\right]$$

$$\leq \exp\left(\frac{V_1(u_m^{N,S}(t_m), v_m^{N,S}(t_m))}{\exp(\alpha t_m)}\right).$$

Thus, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{V_1(u_m^{N,S}(t), v_m^{N,S}(t))}{\exp(\alpha t)} + \int_{t_m}^{t} \frac{\bar{U}(s)}{\exp(\alpha r)} dr\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\exp\left(\frac{V_1(u_m^{N,S}(t), v_m^{N,S}(t))}{\exp(\alpha t)} + \int_{t_m}^{t} \frac{\bar{U}(s)}{\exp(\alpha r)} dr\right) \Big| \mathcal{F}_{t_m}\right]\right] \\
&\leq \mathbb{E}\Big[\exp\Big(\frac{V_1(u_{m+1}^N, \bar{v}_{m+1}^N)}{\exp(\alpha t_m)}\Big)\Big] = \mathbb{E}\Big[\exp\Big(\frac{V_1(u_m^N, v_m^N)}{\exp(\alpha t_m)}\Big)\Big],
\end{aligned}$$

where we use the fact that on $[t_m, t_{m+1}]$, $v_m^{N,S}(t_m) = \bar{v}_{m+1}^N$ and $u_m^{N,S}(t_m) = u_{m+1}^N$ and that the energy preservation of the AVF method, $V_1(u_{m+1}^N, \bar{v}_{m+1}^N) = V_1(u_m^N, v_m^N)$.

Repeating the above arguments on every subinterval $[t_l, t_{l+1}]$, $l \leq m-1$, we obtain

$$\mathbb{E}\left[\exp\left(\frac{V_1(u_{m+1}^N, v_{m+1}^N)}{\exp(\alpha t_{m+1})}\right)\right] \leq \exp\left(V_1(u_0^N, v_0^N)\right) \exp\left(-\int_0^{t_{m+1}} \frac{\bar{U}(r)}{\exp(\alpha r)} dr\right). \tag{33}$$

Now, we are in a position to show (32). By using Jensen's inequality, the Gagliardo–Nirenberg inequality (10), and the Young inequality, we have that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(ch\sum_{i=0}^{m} \|u_i^N\|_{L^6}^2\right)\right] \leq \sup_{i\in\mathbb{Z}_{M+1}} \mathbb{E}\left[\exp(cT\|u_i^N\|_{L^6}^2)\right] \\
&\leq \sup_{i\in\mathbb{Z}_{M+1}} \mathbb{E}\left[\exp\left(\frac{\|\nabla u_i^N\|_{L^2}^2}{2\exp(\alpha t_i)}\right) \exp\left(\exp(\frac{a}{1-a}\alpha T)\|u_i^N\|_{L^2}^2 (cCT)^{\frac{1}{1-a}} 2^{\frac{a}{1-a}}\right)\right].
\end{aligned}$$

Then the Hölder and the Young inequalities imply that for small enough $\epsilon > 0$,

$$\begin{aligned}
\mathbb{E}\left[\exp\left(ch\sum_{i=0}^{m} \|u_i^N\|_{L^6}^2\right)\right] &\leq C(\epsilon, d) \sup_{i\in\mathbb{Z}_{M+1}} \mathbb{E}\left[\exp\left(\frac{\|\nabla u_i^N\|_{L^2}^2}{2\exp(\alpha t_i)}\right) \exp(\epsilon\|u_i^N\|_{L^4}^4)\right] \\
&\leq C(\epsilon, d) \sup_{i\in\mathbb{Z}_{M+1}} \mathbb{E}\left[\exp\left(\frac{V_1(u_i^N, v_i^N)}{\exp(\alpha t_i)}\right)\right].
\end{aligned}$$

By applying (33), we complete the proof. □

Assume that $X_0 \in \mathbb{H}^2$, $T > 0$ and $\|(-\Lambda)^{\frac{1}{2}}\mathbf{Q}^{\frac{1}{2}}\| < \infty$. By introducing the Lyapunov functional

$$V_2(u_m^N, v_m^N) = \frac{1}{2}\left\|\Lambda u_m^N\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla v_m^N\right\|_{L^2}^2 + \frac{1}{2}\langle(-\Lambda)u_m^N, f(u_m^N)\rangle_{L^2},$$

similar and tedious arguments as in the proof of [15, Lemma 3.3] yield that for any $p \geq 1$, there exists a constant $C = C(X_0, \mathbf{Q}, T, p) > 0$ such that

$$\mathbb{E}\left[\sup_{m\in\mathbb{Z}_{M+1}} \|u_m^N\|_{\dot{\mathbb{H}}^2}^p\right] \leq C.$$

The above estimate could be used to remove the infinitesimal factor in the strong convergence order in (2) when $d = 2$.

4.2. **Strong convergence rate analysis.** Based on the above exponential integrability property of $\{u_i^N\}_{1\le i\le M}$ and Lemma 3.3, we obtain the strong convergence rate in temporal direction as follows. In Proposition 4.3 below, we use the index $\gamma = \min(\beta, 2)$ to measure the convergence order since the order of convergence may not exceed 1 when $\beta < 2$.

Proposition 4.3. *Let $d = 1$, $\beta \ge 1$ or $d = 2, \beta = 2$. Let $\gamma = \min(\beta, 2)$ and $T > 0$. Assume that $X_0 \in \mathbb{H}^\beta$, $\|(-\Lambda)^{\frac{\beta-1}{2}}\mathbf{Q}^{\frac12}\|_{\mathcal{L}_2(L^2)} < \infty$. For $d = 1$, there exists $h_0 > 0$ such that for $h \le h_0$, $p \ge 1$,*

$$\sup_{m\in\mathbb{Z}_{M+1}} \mathbb{E}\left[\|X^N(t_m) - X_m^N\|^{2p}\right] \le Ch^{\gamma p}, \tag{34}$$

where $C := C(p, X_0, \mathbf{Q}, T) > 0$, $N \in \mathbb{N}^+$, $M \in \mathbb{N}^+, Mh = T$.

When $d = 2$, it holds that

$$\sup_{m\in\mathbb{Z}_{M+1}} \mathbb{E}\left[\|X^N(t_m) - X_m^N\|^{2p}\right] \le C_1 h^{\gamma p}\lambda_N^{2\epsilon p}, \tag{35}$$

for sufficiently small $\epsilon > 0$, where $C_1 = C_1(p, X_0, \mathbf{Q}, T) > 0$, $N \in \mathbb{N}^+$, $M \in \mathbb{N}^+, Mh = T$.

Proof. Let $\varepsilon_i = X^N(t_i) - X_i^N$ for $i \in \mathbb{Z}_{M+1}$. Fix $m \in \{1, \cdots, M\}$. Using (28) iteratively, we have

$$\begin{aligned}X_m^N = (\mathbb{B}^{-1}(h)\mathbb{A}(h))^m X_0^N &+ \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)}\begin{pmatrix}0\\ P_N dW(s)\end{pmatrix}\\ &+ \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} -(\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)}\mathbb{B}^{-1}(h)\begin{pmatrix}0\\ P_N(\int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta)ds\end{pmatrix}.\end{aligned}$$

From the above equation and (17), it follows that

$$\begin{aligned}\varepsilon_m =&(E(t_m) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^m)X_0^N\\ &+ \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(E(t_m - s) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)}\right)\begin{pmatrix}0\\ P_N dW(s)\end{pmatrix}\\ &+ \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(E(t_m - s)\begin{pmatrix}0\\ -P_N(f(u^N(s)))\end{pmatrix} - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)}\right.\\ &\left.\cdot\,\mathbb{B}^{-1}(h)\begin{pmatrix}0\\ -P_N(\int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta)\end{pmatrix}\right)ds.\end{aligned}$$

This implies that

$$\begin{aligned}\|\varepsilon_m\|_{\mathbb{H}} \le&\|(E(t_m) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^m)X_0^N\|_{\mathbb{H}}\\ &+ \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(E(t_m - s) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)}\right)\begin{pmatrix}0\\ P_N dW(s)\end{pmatrix}\right\|_{\mathbb{H}}\\ &+ \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(E(t_m - s)\begin{pmatrix}0\\ P_N(f(u^N(s)))\end{pmatrix} - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)}\right.\right.\\ &\left.\left.\cdot\,\mathbb{B}^{-1}(h)\begin{pmatrix}0\\ P_N(\int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta)\end{pmatrix}\right)ds\right\|_{\mathbb{H}}\end{aligned}$$

$$\leq Ch^{\frac{\gamma}{2}}\|X_0^N\|_{\mathbb{H}^\gamma} + Err_m^1 + Err_m^2.$$

We first deal with the term Err_m^2 and decompose it into several parts as following

$$\begin{aligned}
Err_m^2 \leq & \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (E(t_m - s) - E(t_{m-1} - s)\mathbb{B}^{-1}(h)) \begin{pmatrix} 0 \\ P_N(f(u^N(s))) \end{pmatrix} ds \right\|_{\mathbb{H}} \\
& + \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} E(t_{m-1} - s)\mathbb{B}^{-1}(h) \begin{pmatrix} 0 \\ P_N(f(u^N(s))) \end{pmatrix} - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)} \right. \\
& \left. \cdot \mathbb{B}^{-1}(h) \begin{pmatrix} 0 \\ P_N(\int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta) \end{pmatrix} ds \right\|_{\mathbb{H}} =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Using the Hölder inequality, the Young inequality and (29), we have

$$\begin{aligned}
\mathrm{I} \leq & \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| (E(h) - \mathbb{B}^{-1}(h)) \begin{pmatrix} 0 \\ P_N(f(u^N(s))) \end{pmatrix} \right\|_{\mathbb{H}} ds \\
\leq & Ch \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|f(u^N(s))\|_{L^2} ds.
\end{aligned}$$

For the term II, denoting $[\frac{s}{h}]$ the integer part of $\frac{s}{h}$, we obtain

$$\begin{aligned}
\mathrm{II} \leq & \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} E(t_{m-1} - s)\mathbb{B}^{-1}(h) \begin{pmatrix} 0 \\ P_N(f(u^N(s)) - f(u^N(t_{[\frac{s}{h}]}))) \end{pmatrix} ds \right\|_{\mathbb{H}} \\
& + \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} E(t_{m-1} - s)\mathbb{B}^{-1}(h) \begin{pmatrix} 0 \\ P_N(f(u^N(t_{[\frac{s}{h}]})) - f(u^N_{[\frac{s}{h}]})) \end{pmatrix} ds \right\|_{\mathbb{H}} \\
& + \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} E(t_{m-1} - s)\mathbb{B}^{-1}(h) \begin{pmatrix} 0 \\ P_N(f(u^N_{[\frac{s}{h}]}) - \int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta) \end{pmatrix} ds \right\|_{\mathbb{H}} \\
& + \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (E(t_{m-1} - s) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)})\mathbb{B}^{-1}(h) \right. \\
& \left. \cdot \begin{pmatrix} 0 \\ P_N(\int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta) \end{pmatrix} ds \right\|_{\mathbb{H}} =: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3 + \mathrm{II}_4.
\end{aligned}$$

Now we estimate II_i, $i = 1, 2, 3, 4$, separately.

Recall that $\mathbb{B}^{-1}(h) = \begin{pmatrix} \mathbb{M}^{-1}(h) & 0 \\ 0 & \mathbb{M}^{-1}(h) \end{pmatrix} \mathbb{A}$ and $\mathbb{A}(t) = \begin{pmatrix} I & \frac{t}{2}I \\ \Lambda_N \frac{t}{2} & I \end{pmatrix}$, where $\mathbb{M}(h) = I - \Lambda_N \frac{h^2}{4}$. For the first term II_1, using the boundedness of $E(\cdot)$, we have

$$\begin{aligned}
\mathrm{II}_1 \leq & \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| \mathbb{M}^{-1}(h) \frac{h}{2} P_N(f(u^N(s)) - f(u^N(t_{[\frac{s}{h}]}))) \right\|_{L^2} ds \\
& + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| \mathbb{M}^{-1}(h) P_N(f(u^N(s)) - f(u^N(t_{[\frac{s}{h}]}))) \right\|_{\dot{\mathbb{H}}^{-1}} ds.
\end{aligned}$$

By means of the inequalities $|\frac{4}{4+\lambda_i h^2}| \le C$ and $\left|\frac{2h}{4+\lambda_i h^2}\right| \le C\lambda_i^{-\frac{1}{2}}$ for some $C > 1$, we have

$$\begin{aligned}
\mathrm{II}_1 \le & C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \left\|P_N(f(u^N(s)) - f(u^N(t_{[\frac{s}{h}]})))\right\|_{\dot{\mathbb{H}}^{-1}} ds \\
\le & C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \left\|(u^N(s))^2 + (u^N(t_{[\frac{s}{h}]}))^2 + u^N(s)u^N(t_{[\frac{s}{h}]}))(u^N(s) - u^N(t_{[\frac{s}{h}]}))\right\|_{\dot{\mathbb{H}}^{-1}} ds \\
& + C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \left\|(u^N(s) + u^N(t_{[\frac{s}{h}]}))(u^N(s) - u^N(t_{[\frac{s}{h}]}))\right\|_{\dot{\mathbb{H}}^{-1}} ds \\
& + C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \left\|u^N(s) - u^N(t_{[\frac{s}{h}]})\right\|_{\dot{\mathbb{H}}^{-1}} ds.
\end{aligned}$$

Based on the Young inequality, Sobolev embedding $L^{\frac{6}{5}} \hookrightarrow \dot{\mathbb{H}}^{-1}$ and the Hölder inequality,

$$\mathrm{II}_1 \le C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} (\|u^N(s)\|_{L^6}^2 + \|u^N(t_{[\frac{s}{h}]})\|_{L^6}^2 + 1)\|u^N(s) - u^N(t_{[\frac{s}{h}]})\|_{L^2} ds.$$

Similarly, the term II_2 satisfies

$$\mathrm{II}_2 \le C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} (\|u^N(t_{[\frac{s}{h}]})\|_{L^6}^2 + \|u^N_{[\frac{s}{h}]}\|_{L^6}^2 + 1)\|u^N(t_{[\frac{s}{h}]}) - u^N_{[\frac{s}{h}]}\|_{L^2} ds.$$

For the term II_3, using again Sobolev embedding $L^{\frac{6}{5}} \hookrightarrow \dot{\mathbb{H}}^{-1}$ and the properties of $\mathbb{B}$, as well as the mean value theorem, we have

$$\begin{aligned}
\mathrm{II}_3 \le & \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \|P_N(f(u^N_{[\frac{s}{h}]}) - \int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta)\|_{\dot{\mathbb{H}}^{-1}} ds, \\
\le & \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \int_0^1 \|\int_0^\theta f'(\lambda u_j^N + (1-\lambda)\theta(u_{j+1}^N - u_j^N))(u_{j+1}^N - u_j^N) d\lambda\|_{\dot{\mathbb{H}}^{-1}} d\theta ds.
\end{aligned}$$

Then the fact that $u_{j+1}^N - u_j^N = \frac{h}{2}(\bar{v}_{j+1}^N + v_j^N)$ and the property $L^{\frac{6}{5}} \hookrightarrow \dot{\mathbb{H}}^{-1}$ yield that

$$\mathrm{II}_3 \le Ch^2\sum_{j=0}^{m-1}(\|u_j^N\|_{L^6}^2 + \|u_{j+1}^N\|_{L^6}^2 + 1)(\|\bar{v}_{j+1}^N\|_{L^2} + \|v_j^N\|_{L^2}).$$

With respect to the term II_4, using Lemma 4.2 and the boundedness of $E(\cdot)$ and $\mathbb{B}^{-1}(\cdot)\mathbb{A}(\cdot)$, we have

$$\|(E(t_{m-1} - t_j) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)})g\|_{L^2} \le Cmh^2\|g\|_{\dot{\mathbb{H}}^2},\ g \in \dot{\mathbb{H}}^2.$$

On the other hand, it holds that

$$\|(E(t_{m-1} - t_j) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)})g\|_{L^2} \le C\|g\|_{L^2},\ g \in L^2.$$

An interpolation argument yields that for $\gamma \in [0, 2]$,

$$\|(E(t_{m-1} - t_j) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)})g\|_{L^2} \le Ch^{\frac{\gamma}{2}}\|g\|_{\dot{\mathbb{H}}^\gamma}.$$

As a result, using Lemma 2.1, we have

$$
\begin{aligned}
\mathrm{II}_4 \leq & \Bigg\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (E(t_{m-1}-s) - E(t_{m-1}-t_j))\mathbb{B}^{-1}(h) \\
& \cdot \begin{pmatrix} 0 \\ P_N(\int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta) \end{pmatrix} ds \Bigg\|_{\mathbb{H}} \\
& + \Bigg\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (E(t_{m-1}-t_j) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)})\mathbb{B}^{-1}(h) \\
& \cdot \begin{pmatrix} 0 \\ P_N(\int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta) \end{pmatrix} ds \Bigg\|_{\mathbb{H}} \\
\leq & \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (s-t_j) \Bigg\| \mathbb{B}^{-1}(h) \begin{pmatrix} 0 \\ P_N(\int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta) \end{pmatrix} \Bigg\|_{\mathbb{H}^1} ds \\
& + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} h^{\frac{\gamma}{2}} \Bigg\| \mathbb{B}^{-1}(h) \begin{pmatrix} 0 \\ P_N(\int_0^1 f(u_j^N + \theta(u_{j+1}^N - u_j^N))d\theta) \end{pmatrix} \Bigg\|_{\mathbb{H}^\gamma} ds \\
\leq & C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (s-t_j) \int_0^1 \|f(u_j^N + \theta(u_{j+1}^N - u_j^N))\|_{L^2} d\theta ds \\
& + C h^{\frac{\gamma}{2}} \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_0^1 \|f(u_j^N + \theta(u_{j+1}^N - u_j^N))\|_{\dot{\mathbb{H}}^{\gamma-1}} d\theta ds.
\end{aligned}
$$

For the case that $d = 1, \beta \geq 1$, taking $\gamma = \min(\beta, 2)$, the Sobolev embedding $\dot{\mathbb{H}}^{\frac{1}{2}+\epsilon} \hookrightarrow L^\infty$ for a small number $\epsilon > 0$ leads to

$$
\begin{aligned}
& \int_{t_j}^{t_{j+1}} \int_0^1 \|f(u_j^N + \theta(u_{j+1}^N - u_j^N))\|_{\dot{\mathbb{H}}^{\gamma-1}} d\theta ds \\
\leq & C \int_{t_j}^{t_{j+1}} \int_0^1 (1 + \|u_j^N + \theta(u_{j+1}^N - u_j^N)\|_{L^\infty}^2) \|u_j^N + \theta(u_{j+1}^N - u_j^N)\|_{\dot{\mathbb{H}}^{\gamma_1 - 1}} d\theta ds \\
\leq & Ch(1 + \|u_j^N\|_{\dot{\mathbb{H}}^{\frac{1}{2}+\epsilon}}^2 + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^{\frac{1}{2}+\epsilon}}^2)(\|u_j^N\|_{\dot{\mathbb{H}}^{\gamma-1}} + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^{\gamma-1}}) \\
\leq & Ch(1 + \|u_j^N\|_{\dot{\mathbb{H}}^1}^2 + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^1}^2)(\|u_j^N\|_{\dot{\mathbb{H}}^{\gamma-1}} + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^{\gamma-1}}).
\end{aligned}
$$

Note that the energy evolution bound in Proposition 4.1 gives the upper bound of $\|u_j^N\|_{L^p(\Omega;\dot{\mathbb{H}}^1)}$ for any $p \geq 1$.

For the case that $d = 2$, $\beta = 2$, using the Sobolev embedding $\dot{\mathbb{H}}^{1+\epsilon} \hookrightarrow L^\infty$, for $\epsilon > 0$ sufficiently small,

$$
\begin{aligned}
& \int_{t_j}^{t_{j+1}} \int_0^1 \|f(u_j^N + \theta(u_{j+1}^N - u_j^N))\|_{\dot{\mathbb{H}}^{\gamma-1}} d\theta ds \\
\leq & Ch(1 + \|u_j^N\|_{\dot{\mathbb{H}}^{1+\epsilon}}^2 + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^{1+\epsilon}}^2)(\|u_j^N\|_{\dot{\mathbb{H}}^{\gamma-1}} + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^{\gamma-1}}).
\end{aligned} \tag{36}
$$

To sum up, we have

$$
\mathrm{II}_4 \leq C h^{\frac{\gamma}{2}} h \sum_{j=0}^{m-1} (1 + \|u_j^N\|_{\dot{\mathbb{H}}^\kappa}^2 + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^\kappa}^2)(\|u_j^N\|_{\dot{\mathbb{H}}^{\gamma-1}} + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^{\gamma-1}}),
$$

where $\kappa = 1$ for $d = 1$ and $\kappa = 1 + \epsilon$ for $d = 2$. Based on the estimates of I and II, we obtain

$$\|\varepsilon_m\|_{\mathbb{H}} \leq \sum_{j=0}^{m-1} \Phi_j \|\varepsilon_j\|_{\mathbb{H}} + \sum_{j=0}^{m-1} \psi_j + Err_m^1 + Ch^{\frac{\gamma}{2}} \|X_0^N\|_{\mathbb{H}^\gamma}, \tag{37}$$

where for $j = 0, 1, \cdots, (m-1)$, $\Phi_j = Ch(\|u^N(t_j)\|_{L^6}^2 + \|u_j^N\|_{L^6}^2 + 1)$, and

$$\begin{aligned}\psi_j =& Ch \int_{t_j}^{t_{j+1}} \|(f(u^N(s)))\|_{L^2} ds \\ &+ Ch^{\frac{\gamma}{2}} h(1 + \|u_j^N\|_{\dot{\mathbb{H}}^\kappa}^2 + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^\kappa}^2)(\|u_j^N\|_{\dot{\mathbb{H}}^{\gamma-1}} + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^{\gamma-1}}) \\ &+ C \int_{t_j}^{t_{j+1}} (\|u^N(s)\|_{L^6}^2 + \|u^N(t_{[\frac{s}{h}]})\|_{L^6}^2 + 1) \|u^N(s) - u^N(t_{[\frac{s}{h}]})\|_{L^2} ds \\ &+ Ch^2(\|u_j^N\|_{L^6}^2 + \|u_{j+1}^N\|_{L^6}^2 + 1)(\|\bar{v}_{j+1}^N\|_{L^2} + \|v_j^N\|_{L^2}).\end{aligned}$$

By the discrete Gronwall's inequality (see e.g., [15, Lemma 2.6]), we have

$$\|\varepsilon_m\|_{\mathbb{H}} \leq \left(Ch^{\frac{\gamma}{2}} \|X_0^N\|_{\mathbb{H}^\gamma} + Err_m^1 + \sum_{j=0}^{m-1} \psi_j \right) \exp\left(\sum_{j=0}^{m-1} \Phi_j \right).$$

Taking the $2p$th moment and using the Hölder inequality, we have

$$\mathbb{E}\|\varepsilon_m\|_{\mathbb{H}}^{2p} \leq \left[\mathbb{E}\left(Ch^{\frac{\gamma}{2}} \|X_0^N\|_{\mathbb{H}^\gamma} + Err_m^1 + \sum_{j=0}^{m-1} \psi_j \right)^{4p} \right]^{\frac{1}{2}} \left[\mathbb{E}\exp\left(4p \sum_{j=0}^{m-1} \Phi_j \right) \right]^{\frac{1}{2}}.$$

According to the exponential integrability of u^N and u_m^N, the above inequality becomes

$$\mathbb{E}\|\varepsilon_m\|_{\mathbb{H}}^{2p} \leq Ch^{\gamma p} \|X_0^N\|_{\mathbb{H}^\gamma}^{2p} + C\left[\mathbb{E}(Err_m^1)^{4p}\right]^{\frac{1}{2}} + Cm^{2p-\frac{1}{2}} \left[\sum_{j=0}^{m-1} \mathbb{E}\psi_j^{4p} \right]^{\frac{1}{2}}.$$

Thanks to Lemma 2.1 and an interpolation version of Lemma 4.2, we obtain

$$\begin{aligned}\mathbb{E}(Err_m^1)^{4p} \leq& C\left[\int_0^{t_m} \left\| (E(t_m - s) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-[\frac{s}{h}])}) \mathbb{G}_N \mathbf{Q}^{\frac{1}{2}} \right\|_{\mathcal{L}_2(\mathbb{H})}^2 ds \right]^{2p} \\ \leq& C \sup_{0 \leq s \leq t_m} \left\| (E(t_m - s) - E(t_{m-1} - t_{[\frac{s}{h}]})) \mathbb{G}_N \mathbf{Q}^{\frac{1}{2}} \right\|_{\mathcal{L}_2(\mathbb{H})}^{4p} \\ &+ C \sup_{0 \leq s \leq t_m} \left\| (E(t_{m-1} - t_{[\frac{s}{h}]}) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-[\frac{s}{h}])}) \mathbb{G}_N \mathbf{Q}^{\frac{1}{2}} \right\|_{\mathcal{L}_2(\mathbb{H})}^{4p} \\ \leq& C(h^{2\gamma p} + h^{4p}),\end{aligned}$$

where $\mathbb{G}_N \mathbf{Q}^{\frac{1}{2}} = (0, P^N \mathbf{Q}^{\frac{1}{2}})^\top$. This leads to

$$\mathbb{E}\|\varepsilon_m\|_{\mathbb{H}}^{2p} \leq Ch^{\gamma p} \|X_0^N\|_{\mathbb{H}^\gamma}^{2p} + C(h^{\gamma p} + h^{2p}) + Cm^{2p-\frac{1}{2}} \left[\sum_{j=0}^{m-1} \mathbb{E}\psi_j^{4p} \right]^{\frac{1}{2}}.$$

According to the Hölder inequality, the a prior estimates of u^N and u_m^N and the Hölder continuity of u^N, we obtain that for $d = 1$,

$$\begin{aligned}\mathbb{E}\psi_j^{4p} \leq& Ch^{4p}\mathbb{E}\left(\int_{t_j}^{t_{j+1}} \|f(u^N(s))\|_{L^2} ds\right)^{4p} \\ &+ Ch^{2\gamma p}h^{4p}\mathbb{E}\left((1+\|u_j^N\|^2_{\dot{\mathbb{H}}^\kappa} + \|u_{j+1}^N\|^2_{\dot{\mathbb{H}}^\kappa})(\|u_j^N\|_{\dot{\mathbb{H}}^{\gamma-1}} + \|u_{j+1}^N\|_{\dot{\mathbb{H}}^{\gamma-1}})\right)^{4p} \\ &+ C\mathbb{E}\left(\int_{t_j}^{t_{j+1}} (\|u^N(s)\|^2_{L^6} + \|u^N(t_{[\frac{s}{h}]})\|^2_{L^6} + 1)\|u^N(s) - u^N(t_{[\frac{s}{h}]})\|_{L^2} ds\right)^{4p} \\ &+ Ch^{8p}\mathbb{E}\left[(\|u_j^N\|^2_{L^6} + \|u_{j+1}^N\|^2_{L^6} + 1)(\|\bar{v}_{j+1}^N\|_{L^2} + \|v_j^N\|_{L^2})\right]^{4p} \\ \leq& C(h^{8p} + h^{4p+2\gamma p} + h^{8p} + h^{8p}) \leq Ch^{8p} + Ch^{4p+2\gamma p}.\end{aligned}$$

When $d = 2$, using the inverse inequality for spectral Galerkin approximation, i.e.,

$$\|u_j^N\|^2_{\dot{\mathbb{H}}^\kappa} + \|u_{j+1}^N\|^2_{\dot{\mathbb{H}}^\kappa} \leq C\lambda_N^{-\epsilon}(\|u_j^N\|^2_{\dot{\mathbb{H}}^1} + \|u_{j+1}^N\|^2_{\dot{\mathbb{H}}^1}),$$

and the above argument, we obtain that

$$\mathbb{E}\psi_j^{4p} \leq Ch^{8p} + Ch^{4p+2\gamma p}\lambda_N^{4\epsilon p}.$$

As a consequence, when $d = 1$,

$$\mathbb{E}\|\varepsilon_m\|^{2p}_{\mathbb{H}} \leq Ch^{\gamma p}\mathbb{E}\|X_0^N\|^{2p}_{\mathbb{H}^\gamma} + C(h^{\gamma p} + h^{2p}) + Cm^{2p}h^{2p+\gamma p} + Cm^{2p}h^{4p} \leq Ch^{\gamma p},$$

and when $d = 2$,

$$\begin{aligned}\mathbb{E}\|\varepsilon_m\|^{2p}_{\mathbb{H}} &\leq Ch^{\gamma p}\mathbb{E}\|X_0^N\|^{2p}_{\mathbb{H}^\gamma} + C(h^{\gamma p} + h^{2p}) + Cm^{2p}h^{2p+\gamma p}\lambda_N^{2\epsilon p} + Cm^{2p}h^{4p} \\ &\leq Ch^{\gamma p}\lambda_N^{2\epsilon p},\end{aligned}$$

which completes the proof. □

Finally, the above convergence result in Proposition 4.3, together with Proposition 3.4, implies Theorem 1.1.

5. **Numerical experiments.**

In this section we provide some numerical examples to illustrate the accuracy and capability of the method developed in the previous section for the following d-dimensional stochastic nonlinear wave equation

$$\begin{aligned}&du = vdt, \\ &dv = \Lambda udt - u^3dt + dW(t), \qquad \text{in } \mathcal{O}\times(0,T], \\ &u(0,x) = 0, \quad v(0,x) = 1, \qquad \text{in } \mathcal{O}\end{aligned} \tag{38}$$

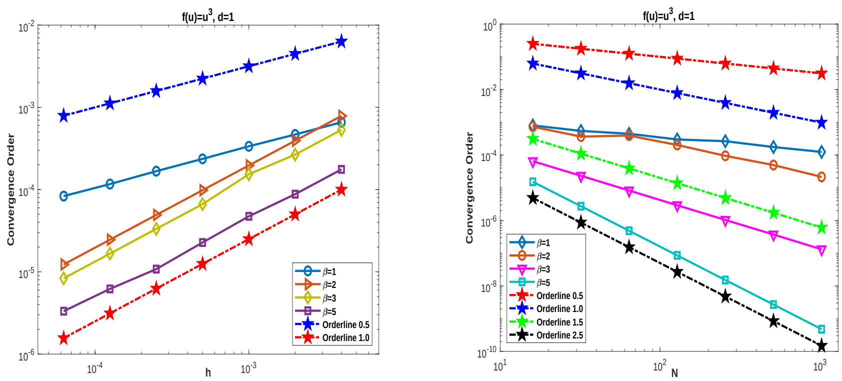
with $\mathcal{O} = (0,1)^d$ for $d = 1, 2$ and $T = 1$.

Example 5.1. We study the stochastic nonlinear wave equation (38) with $\beta = 1$, 2, 3 and 5 in 1-dimensional case. Furthermore, we choose the orthonormal basis $\{e_k\}_{k\in\mathbb{N}^+}$ of $L^2([0,1])$ and the corresponding eigenvalues $\{\eta_k\}_{k\in\mathbb{N}^+}$ of $\mathbf{Q}$ as

$$e_k(x) = \sqrt{2}\sin(k\pi x), \qquad \eta_k = \frac{1}{k^{2\beta-1}}, \quad k = 1, 2, \ldots, \quad x \in [0,1].$$

We fix $N = 100$ (space discretization parameter) and then apply the method (27) with different $M = 2^r$, $r = 2, 3, \cdots, 7$ (time discretization parameter). In general, structure-preserving numerical methods for nonlinear stochastic differential equations are implicit. In the course of numerical implementation, they are approximated by means of the fixed point iteration or the Picard approximation at each step. As a consequence, one needs to solve nonlinear algebraic equations, and the preservation of mathematical structures will be somewhat damaged due to the fixed point iteration. It remains unclear to analyze the overall error of utilizing the fixed point iteration or Newton's method and implicit method for the considered equation (we refer to [24] for a recent related study). Fig. 1(a) shows the temporal error for various parameters β, where the expectation in the error is approximated by the mean of 1000 independent realizations. The splitting AVF method (27) clearly converges with the rate $1/2$ in time when $\beta = 1$ and with the rate 1 in time when $\beta = 2$, 3 and 5, which coincide with the theoretical analysis.

Fig. 1(b) displays the spatial error for various parameters β. Here, we fix $M = 2^{10}$ (time discretization) and apply the method (27) with different $N = 2^s$, $s = 4, 5, \cdots, 9$ (space discretization). It can be observed that the proposed method (27) is of order $\beta/2$ in spatial direction. This coincides with our theoretical findings.



(a) Temporal error (b) Spatial error

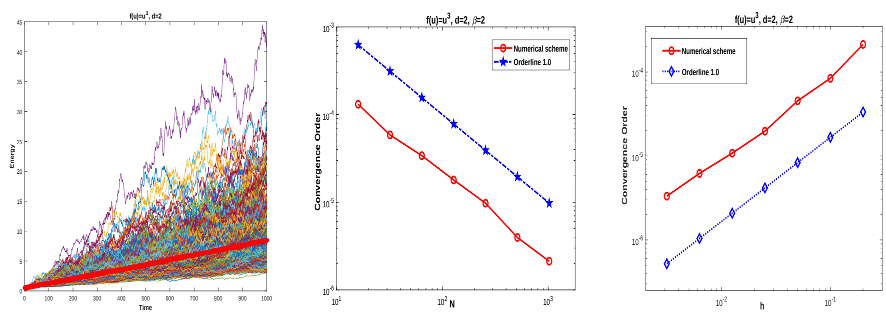
FIGURE 1. Strong convergence order in time and space for the fully discrete numerical method (27) applied to stochastic wave equation (38).

Example 5.2. We study the stochastic nonlinear wave equation (38) with $\beta = 2$ in 2-dimensional case. In the sequel, we choose the orthonormal basis $\{e_{k,l}\}_{k,l\in\mathbb{N}^+}$ of $L^2([0,1]^2)$ and the corresponding eigenvalues $\{\eta_{k,l}\}_{k,l\in\mathbb{N}^+}$ of $\mathbf{Q}$ as

$$e_{k,l} = 2\sin(k\pi x)\sin(l\pi y), \qquad \eta_{k,l} = \frac{1}{(k^2+l^2)^2}, \quad k,l = 1,2,\ldots, \quad (x,y)\in[0,1]^2.$$

Fig. 2(a) presents the evolution of discrete energy using the proposed methods in Section 4, where the red line represents the discrete averaged energy along 500 trajectories. From Theorem 4.1, we noted that, the averaged energy evolution law $\mathbb{E}(V_1(u_m^N, v_m^N))$ follows a linear evolution with growth rate $\frac{1}{2}\mathrm{Tr}((P_N\mathbf{Q}^{\frac{1}{2}})(P_N\mathbf{Q}^{\frac{1}{2}})^*)$. It can be observed from Fig. 2(a) that the discrete averaged energy obeys nearly linear growth over 500 trajectories, which coincides with the theoretical analysis.

Now let us start with tests on the convergence rates. First of all, we consider the spatial convergence rate of the numerical method (27). The middle figure in Fig. 2 displays the spatial approximation errors $\sup\limits_{0\le t\le T}\|X_N(t) - X(t)\|_{L^p(\Omega;\mathbb{H})}$ against N on a log-log scale with $N = 2^s$, $s = 4, 5, \cdots, 9$. It can be observed that the slope is



(a) Energy linear growth (b) Spatial error (c) Temporal error

FIGURE 2. : Energy-preserving property and strong convergence order in space and time for the fully discrete numerical method (27) applied to stochastic wave equation (38).

closed to 1 which is consistent with our previous theoretical result (see Proposition 3.4) on the spatial convergence order. Note that for the temporal discretization we used here the proposed method (27) at a sufficiently small time step-size $h = 2^{-10}$. In addition, $N = 2^{11}$ is used to simulate the reference solution.

To investigate the strong convergence order in temporal direction of (27) by using various step-sizes $h = 2^{-r},\ r = 2, 3, \cdots, 7$, we now fix $N = 100$. Again, the "reference' solution is approximated by the method (27) with a very small time step-size $h = 2^{-12}$. The right figure in Fig. 2 presents the strong approximation errors of the proposed method (27) in temporal direction. It can be seen that this numerical performance coincides with the theoretical assertion (see Proposition 4.3).

6. Appendix. Existence and uniqueness of numerical solution of (27).

To prove the well-posedness of the numerical scheme, we need to show that the existence and uniqueness of numerical solution of the scheme at each $(m+1)$-th step . Assume that $d \le 2$, $\mathrm{Tr}(\mathbf{Q}) < \infty$, $h_0 > 0$ to be determined later, the temporal step size $h < h_0$, and that $(u_m^N, v_m^N)^\top \in \mathbb{H}^1$ is $\mathcal{F}_{t_m}$-measurable and $\mathbb{E}\Big[\exp(V_1(u_m^N, v_m^N)e^{-\alpha t_m})\Big] \le C$ for some $\alpha(Q, T, u_0, v_0, d) \ge 1$ and $C(Q, T, u_0, v_0, d) > 0$. The proof is similar to that of [26, Lemma 8.1] by making use of the structure of the drift coefficient f and the fact that (1) is a separable system.

Since the last equation in (27) has an explicit analytical solution, it suffices to show the existence of a unique solution for the first two equations in (27), that is,

$$
\begin{aligned}
u_{m+1}^N &= u_m^N + h\bar{v}_{m+\frac{1}{2}}^N, \qquad (39)\\
\bar{v}_{m+1}^N &= v_m^N + h\Lambda_N u_{m+\frac{1}{2}}^N - hP_N\left(\int_0^1 f(u_m^N + \theta(u_{m+1}^N - u_m^N))d\theta\right).
\end{aligned}
$$

Finding the solution of the above system is equivalent to the solvability of finding $\widetilde{v}$ such that

$$
\widetilde{v} - v_m^N - \frac{1}{2}h\Lambda_N(u_m^N + \frac{1}{2}h\widetilde{v}) + \frac{1}{2}hP_N\left(\int_0^1 f(u_m^N + \theta h\widetilde{v})d\theta\right) = 0.
$$

Let $y \in \mathbb{R}$ be a real variable. Choosing $\widetilde{F}(\xi, y) > 0$ as the anti-derivative of $\frac{1}{2}h\int_0^1 f(\xi+\theta h y)d\theta$ for fixed $\xi \in \mathbb{R}$, then it suffices to find v^* which minimizes

$$\frac{1}{2}\|\widetilde{v}-v_m^N\|_{L^2}^2+\frac{1}{2}\|u_m^N+\frac{1}{2}h\widetilde{v}\|_{\mathbb{H}^1}^2+\int_{\mathcal{O}}\widetilde{F}(u_m^N,\widetilde{v})dx.$$

The existence of the minimizer is guaranteed by the fact that $f(\cdot)$ is smooth and its anti-derivative is bounded below. Furthermore, the uniqueness is obtained by using the fact $f(u) = c_3u^3 + \cdots + c_1u + c_0$ is a polynomial with $c_3 > 0$. Indeed, there exists $C(c_3, c_2, c_1, c_0) > 0$ such that $(f(x) - f(y)) \cdot (x - y) \geq -C\|x - y\|^2$.

Assume that we have two different numerical solutions $(\widetilde{u}_1, \widetilde{v}_1)$ and $(\widetilde{u}_2, \widetilde{v}_2)$ in $P_N(\mathbb{H})$ of (27) with the same initial condition (u_m^N, v_m^N). Then

$$\widetilde{v}_i-\frac{1}{2}h\Lambda_N(u_m^N+\frac{h}{2}\widetilde{v}_i)+P_N\left(\frac{\partial}{\partial y}\widetilde{F}(u_m^N,\widetilde{v}_i)\right)=v_m^N,\ i=1,2.$$

Using $c_3 > 0$ and the Poincaré inequality, we have that

$$0\geq(1+\lambda_1\frac{h^2}{4}-C'h^2)\|\widetilde{v}_1-\widetilde{v}_2\|_{L^2}^2 \tag{40}$$

for some constant $C'(c_3, c_2, c_1, c_0)$. By taking h_0 small enough such that $1+\lambda_1\frac{h_0^2}{4} > C'h_0^2$, the uniqueness of the numerical solution of (27) follows.

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