AN OPTIMAL PID TUNING METHOD FOR A TWO-LINK MANIPULATOR VIA AN EXACT PENALTY FUNCTION METHOD

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Abstract. In this paper, an optimal PID tuning method is proposed for a class of two-link manipulators. In particular, the control specifications of the manipulators are considered. By modeling the control specifications into state constraints, the optimal PID tuning problem is converted to an optimal parameters selection problem with state constraints. An exact penalty function based method is then utilized to handle the state constraints. The superior of the proposed method over an existing method is verified by carrying out an numerical example.

1. Introduction.

The manipulators have been widely used in the industry, and it has many successful applications in which the manipulators are required to perform various types of complex tasks. Thus, the controller design of the manipulators is challenging. The most fundamental task for the controller design of the manipulators is to drive the manipulator to follow a desired reference input. In addition, some control performance specifications have to be met. This is because most of the complex tasks for a manipulator can be achieved by tacking desired several joint angles and desired torques.

In the literature, various types of control strategies has been developed for controlling the manipulators [14]. For example, the PID control [41, 8], the adaptive control [28, 9], the robust control [32, 33], the neural networks [26] based control and the fuzzy control [15], and the iterative learning control [35]. Among all of these control strategies, PID control is the most used one in real world applications. In fact, it has been adopted for more than 90% of the control systems [39] in engineering practice. The key problem for a PID controller design is to determine the parameters for the proportional, the integral and the differential terms, which motivates the study of this paper.

The problem of PID tuning has been intensively studied. For example, the Ziegler-Nichols (Z-N) method [51] has been regarded as one of the most classical PID tuning schemes. Followed by the Z-N method, many modified versions were developed by exploring the features of the plants [1, 27, 34] or by incorporating a relay feedback [40, 49]. Intelligent tuning scheme [3, 4] was proposed by Astrom in 1988. Some are based on the plant model and some are developed based on the rules. The former one tunes the PID parameters by incorporating the Z-N method with identifying the system input and the system output [5, 6]. The latter one is a model free tuning method [50, 10, 31]. By introducing some rules, which are manually set by experienced engineers, the PID parameters can be obtained by according to the transient responses, and the changes of the set value and the disturbance [17, 7]. Taking the position and trajectory tracking errors as the optimization indexes, a multi-objective particle swarm optimization algorithm is proposed for fractional-order fuzzy PID controller of double link manipulator [2]. An adaptive neural network controller design method based on PID structure is proposed in [29]. Furthermore, the problem of PID tuning can be regarded as the optimal parameter selection problem subject to state constraints. The control parametrization method is an efficient approach for solving such problem [38], and it has been widely applied in spacecraft attitude control [20], UAV formation control [22] and trajectory optimization [19].

In this paper, an exact penalty function based PID tuning method [43] is proposed for a two-link manipulator [37]. By formulating the control specifications into state constraints, the PID tuning problem can be treated as an optimal parameters selection problem subject to state constraints. Particularly, the state constraints are difficult to handle since there are infinite number of constraints on the time horizon to satisfy. The constraint transcription method [45, 48, 24, 46, 47, 23, 25] was regarded an effective method for tackling the state constraints. However, the exact penalty function [43, 25, 44, 12, 13] has shown to be more efficient in terms of optimality and numerical computations. Thus, it is adopted in this paper to handle the state constraints. By appending the state constraints into the cost function, an unconstrained optimal parameters selection problem, which is a nonlinear program and can be solved by many existing computational methods, is obtained. For example, the sequential quadratic program (SQP) method. There are even off-the-shelf optimal control softwares available for solving such unconstrained optimal parameters selection problem. For example, MISER 3.2 [16] and Visual MISER [42].

The rest of the paper is organized as follows. The PID tuning problem is converted into the state constrained optimal parameters selection problem in Section 2. The proposed computational method is proposed in Section 3, and some numerical examples are proved in Section 4 to test the effectiveness of the proposed method. In Section 5, the paper is concluded by making some remarks.

2. **Problem formulation.**

We consider a two-link rigid manipulator with the following dynamic equations:

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{1}$$

where $\boldsymbol{q} \in \mathbb{R}^2$ denotes the joint displacements, $\dot{\boldsymbol{q}} \in \mathbb{R}^2$ denotes the joint velocities, $\boldsymbol{\tau} \in \mathbb{R}^2$ is the torques, $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{2\times 2}$ represents the manipulator inertia matrix which is a symmetric positive definite matrix, $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathbb{R}^{2\times 2}$ is the 2×2 matrix of centripetal and Coriolis torques,

$$\boldsymbol{g}(\boldsymbol{q}) = \frac{\partial \boldsymbol{U}(\boldsymbol{q})}{\partial \boldsymbol{q}}$$

denotes the gravitational torques and $\boldsymbol{U}(\boldsymbol{q})$ is potential energy.

Let $\boldsymbol{q}_d = [q_{d_1}, q_{d_2}]^{\mathrm{T}}$ be the desired joint position and define $\boldsymbol{e} = \boldsymbol{q} - \boldsymbol{q}_d$ as the position error, respectively, then the PID controller can be expressed as

$$\boldsymbol{\tau}(t) = \boldsymbol{K}_p \boldsymbol{e}(t) + \boldsymbol{K}_i \int_0^t \boldsymbol{e}(\tau)\mathrm{d}\tau + \boldsymbol{K}_d \dot{\boldsymbol{e}}(t) \tag{2}$$

where $\boldsymbol{K}_p = [K_{p_1}, K_{p_2}] > 0 \in \mathbb{R}^2$, $\boldsymbol{K}_i = [K_{i_1}, K_{i_2}] > 0 \in \mathbb{R}^2$ and $\boldsymbol{K}_d = [K_{d_1}, K_{d_2}] > 0 \in \mathbb{R}^2$ are the proportional, the integral and the derivative gains of the PID controller, respectively. The objective of this paper is to choose these gain matrices $\boldsymbol{K}_p$, $\boldsymbol{K}_i$ and $\boldsymbol{K}_d$ such that the required control specifications of the system is satisfied.

By defining the following system state

$$x_1 = q_1, x_2 = \dot{q}_1, x_3 = q_2, x_4 = \dot{q}_2 \tag{3}$$

the dynamics equation (1) can be rewritten into the following sate-space model:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{d_{12}}{w}\tau_2 - \dfrac{d_{22}}{w}h_1 - \dfrac{d_{22}c_{12}}{w}x_4^2 - 2\dfrac{d_{22}c_{11}}{w}x_2x_4 \\ \qquad + \dfrac{c_{21}d_{12}}{w}x_2^2 + \dfrac{d_{12}}{w}h_2 + \dfrac{d_{22}}{w}\tau_1 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dfrac{d_{11}}{w}\tau_2 + \dfrac{d_{21}}{w}h_1 + \dfrac{d_{21}c_{12}}{w}x_4^2 + 2\dfrac{d_{22}c_{11}}{w}x_2x_4 \\ \qquad - \dfrac{c_{12}d_{11}}{w}x_2^2 - \dfrac{d_{11}}{w}h_2 - \dfrac{d_{21}}{w}\tau_1 \end{cases} \tag{4}$$

where $c_{11} = c_{12} = -m_3\sin(x_3)$, $c_{21} = -c_{11}$, $d_{11} = m_1 + m_2 + 2m_3\cos\big(x_3\big)$, $d_{12} = m_2 + m_3\cos(x_3)$, $d_{21} = d_{12}$, $d_{22} = m_2$, and $h_1 = m_4 g\cos(x_1) + m_5 g\cos(x_1 + x_3)$, $h_2 = m_5 g\cos(x_1 + x_3)$. Here, m_i, $i = 1, 2, \ldots, 5$, are constant numbers.

It is well known that the performance of the controller is described by the control specifications. For example, the overshoot and the setting time. In real world applications, a large overshoot may lead to serious consequences for the physical system. In order to prevent such situation from happening, we impose the following constraints:

$$g_1 = x_1 - 1.05q_{d_1} \leq 0, \quad t \in [0, 5s] \tag{5}$$

$$g_2 = x_3 - 1.05q_{d_2} \leq 0, \quad t \in [0, 5s] \tag{6}$$

Since the rise time and the setting time are two important performance specifications for a control system, then we set the rise time as $2s$ and the setting time as $3s$, respectively. Hence, the following constraints are introduced:

$$g_3 = l - x_1 \leq 0, \quad t \in [0, 5s] \tag{7}$$

$$g_4 = l - x_3 \leq 0, \quad t \in [0, 5s] \tag{8}$$

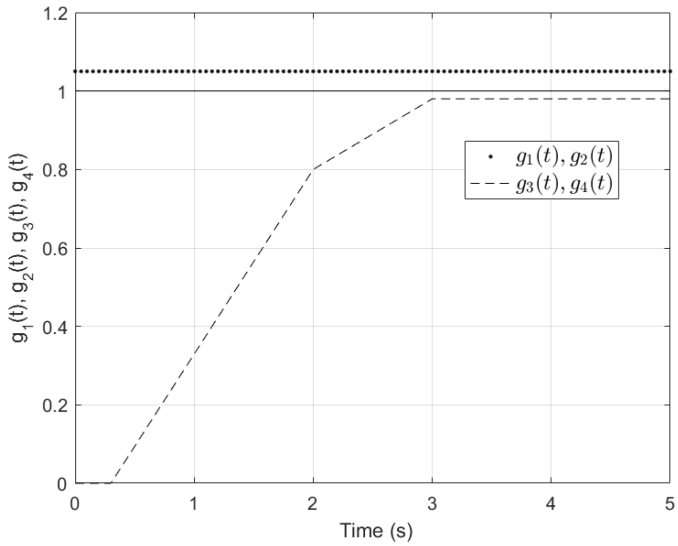

FIGURE 1. Performance Specifications

where

$$l = \begin{cases} 0, & 0 < t \leq 0.3 \\ 0.5t - 0.15, & 0.3 < t \leq 2 \\ 0.15t + 0.59, & 2 < t \leq 3 \\ 0.98, & 3 < t \leq 5 \end{cases}$$

These performance specifications are illustrated as in Figure 1. With the state constraints (5)-(8), it can be guaranteed that the manipulator joint position could satisfy the requirement of overshoot, rise time and setting time.

By considering the limit of the actuator [30], the following box constraints are imposed:

$$g_5 = \tau_1 - 150 \leq 0, \quad t \in [0, 5s] \tag{9}$$

$$g_6 = -\tau_1 - 150 \leq 0, \quad t \in [0, 5s] \tag{10}$$

$$g_7 = \tau_2 - 15 \leq 0, \quad t \in [0, 5s] \tag{11}$$

$$g_8 = -\tau_2 - 15 \leq 0, \quad t \in [0, 5s] \tag{12}$$

where (9)-(12) imply that the torque on the first link is required to be less than $150\, N \cdot m$ and the torque on the second link has to be less than $15\, N \cdot m$.

Note that g_1, g_2, g_3 and g_4 are state constraints while g_5, g_6, g_7 and g_8 are not. In what follows, we shall reformulate g_7 and g_8 into state constraints. For this, we define

$$x_5(t) = \int_0^t e_1(\tau)\mathrm{d}\tau \tag{13}$$

$$x_6(t) = \int_0^t e_2(\tau)\mathrm{d}\tau \tag{14}$$

Obviously, $x_5(0) = 0$ and $x_6(0) = 0$. Then, by substituting (13)-(14) into (2), system (4) can be rewritten as:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{d_{12}}{w}\tau_2 - \frac{d_{22}}{w}g_1 - \frac{d_{22}c_{12}}{w}x_4^2 - 2\frac{d_{22}c_{11}}{w}x_2x_4 \\ & +\frac{c_{21}d_{12}}{w}x_2^2 + \frac{d_{12}}{w}g_2 + \frac{d_{22}}{w}\tau_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{d_{11}}{w}\tau_2 + \frac{d_{21}}{w}g_1 + \frac{d_{21}c_{12}}{w}x_4^2 + 2\frac{d_{22}c_{11}}{w}x_2x_4 \\ & -\frac{c_{12}d_{11}}{w}x_2^2 - \frac{d_{11}}{w}g_2 - \frac{d_{21}}{w}\tau_1 \\ \dot{x}_5 &= x_1 - q_{d_1} \\ \dot{x}_6 &= x_3 - q_{d_2} \end{cases} \tag{15}$$

where

$$\tau_1 = K_{p_1}(x_1 - q_{d_1}) + K_{i_1}x_5 + K_{d_1}x_2 \tag{16}$$

$$\tau_2 = K_{p_2}(x_3 - q_{d_2}) + K_{i_2}x_6 + K_{d_2}x_4 \tag{17}$$

and the initial state is given by

$$\boldsymbol{x}(0) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^\top \tag{18}$$

Remark 2.1. By observing (16) and (17), the control inputs τ_1 and τ_2 are functions of the states. Hence, g_5, g_6, g_7 and g_8 are state constraints. Interestingly, the PID control (16)-(17) becomes state feedback control by introducing the transform (13)-(14).

In order to track the reference input q_{d_1} and q_{d_2}, we define the objective function as below:

$$J = \int_0^T \left[(x_1(t) - q_{d_1})^2 + (x_3(t) - q_{d_2})^2\right] \mathrm{d}t \tag{19}$$

The optimal PID parameters tuning problem for the two-link manipulator can be formulated as the follows:

$$\begin{aligned} \min_{\boldsymbol{K}} \quad & J \\ s.t. \quad & (15) \\ & (5) - (12) \end{aligned}$$

where $\boldsymbol{K} = [K_{p_1}\ K_{i_1}\ K_{d_1}\ K_{p_2}\ K_{i_2}\ K_{d_2}]^\top$. We refer to this problem as Problem Q.

Remark 2.2. Problem Q is a stat constrained optimal parameter selection problem. The state constraints are difficult to handle since the number of constraints on the time horizon is infinite. An exact penalty function method [21] will be introduced in the next section to tackle this difficulty.

3. **Computational method.** In this section, an exact penalty function method [21] will be introduced to handle the state constraints. Inspired by [21], we construct a new objective function as below:

$$J_\delta = J + \epsilon^{-\alpha}J_\Delta + \delta\epsilon^\beta \tag{20}$$

where

$$J_\Delta = \sum_{i=1}^{8} \int_0^T \left[\max\{0, g_i - \epsilon^\gamma W_i\}\right]^2 \mathrm{d}t \tag{21}$$

$0 < W_i < 1$, $\alpha > 0$, $\gamma > 0$, $\beta > 2$, $\delta > 0$ are fixed constants, and $\epsilon > 0$ is the penalty parameter.

By replacing the objective function (19) with (20), we obtain a new problem which is referred to as **Problem $\boldsymbol{Q_\delta}$**: Given system (15), find a pair $(\boldsymbol{K}, \epsilon) \in \mathbb{R}^6 \times [0, +\infty)$ such that the cost function (20) is minimized.

Remark 3.1. According to Theorem 5.1, Theorem 5.2, Theorem 5.3 and Theorem 5.4 in [21], there exits a finite number δ^* such that the optimal solution of Problem Q_{δ^*} is the optimal solution of Problem Q . This implies that Problem Q can be solved by solving a sequence of Problem Q_δ by increasing δ.

The idea of the exact penalty function can be interpreted as follows. By observing the third term of (20), if δ increases, then ϵ^β will decrease in order to minimize the objective function J_δ. Therefore, ϵ will decrease since β is a constant. This may lead to the increase of $\epsilon^{-\alpha}$ in the second term of (20), which will push the constraint violation function J_Δ to decrease.

Problem Q_δ is an unconstrained optimal parameters selection problem, which can be regarded a nonlinear program. There exists many efficient methods for solving a nonlinear program. For example, the sequential quadratic program (SQP) method. In addition, some off-the-shelf optimal control software packages are available for solving Problem Q_δ. For example, MISER 3.2 (Matlab Version) [16] and Visual MISER (Fortran Version) [42].

In this paper, the SQP method, which is a gradient based method, is adopted to solve Problem Q_δ. Thus, the gradient formulas of the objective function are required. In the following theorem, the corresponding gradient formulas are derived.

Theorem 3.2. *The gradient formulas of the cost function J_δ with respect to $\boldsymbol{K}$ and ϵ are*

$$\begin{aligned}\frac{\partial J_\delta}{\partial \epsilon} \;=\;& \epsilon^{-\alpha-1}\Bigg\{-\alpha\sum_{i=1}^{8}\int_0^T\Big[\max\left\{0, g_i-\epsilon^\gamma W_i\right\}\Big]^2\mathrm{d}t \\ &+2\gamma\sum_{i=1}^{8}\int_0^T \max\Big\{0, g_i-\epsilon^\gamma W_i\Big\}(-\epsilon W_i)\mathrm{d}t\Bigg\} \\ &+\delta\beta\epsilon^{\beta-1} \end{aligned} \tag{22}$$

$$\frac{\partial J_\delta}{\partial \boldsymbol{K}} = \int_0^T \left(\epsilon^{-\alpha}\frac{\partial J_\Delta}{\partial \boldsymbol{K}} + \frac{\partial\left(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{K})^\top\right)}{\partial \boldsymbol{K}}\boldsymbol{\lambda}(t)\right)\mathrm{d}t \tag{23}$$

where

$$\frac{\partial J_\Delta}{\partial \boldsymbol{K}} = \begin{bmatrix} 2\max\left\{0, g_5-\epsilon^\gamma W_i\right\} - 2\max\left\{0, g_6-\epsilon^\gamma W_i\right\} \\ 2\max\left\{0, g_7-\epsilon^\gamma W_i\right\} - 2\max\left\{0, g_8-\epsilon^\gamma W_i\right\}\end{bmatrix}$$

$$\frac{\partial \boldsymbol{f}(\boldsymbol{x},\boldsymbol{K})}{\partial \boldsymbol{K}} =$$

$$\begin{bmatrix} 0 & 0 & 0 \\ \frac{d_{22}}{w}\left(x_1-q_{d_1}\right)-\frac{d_{12}}{w}\left(x_3-q_{d_2}\right) & \frac{d_{22}}{w}x_5-\frac{d_{12}}{w}x_6 & \frac{d_{22}}{w}x_2-\frac{d_{12}}{w}x_4 \\ 0 & 0 & 0 \\ \frac{d_{21}}{w}\left(x_1-q_{d_1}\right)-\frac{d_{11}}{w_{11}}\left(x_3-q_{d_2}\right) & \frac{d_{21}}{w}x_5-\frac{d_{11}}{w_{11}}x_6 & \frac{d_{21}}{w}x_2-\frac{d_{11}}{w_{11}}x_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, $\boldsymbol{\lambda}(t)$ is the solution of the following system of co-state differential equations:

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{x}} \tag{24}$$

with the boundary condition

$$\boldsymbol{\lambda}(T) = \mathbf{0} \tag{25}$$

where

$$\frac{\partial H}{\partial \boldsymbol{x}} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{x}} + \left(\frac{\partial \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{K})}{\partial \boldsymbol{x}} \right)^{\top} \boldsymbol{\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}} = 2 \begin{bmatrix} \Omega \\ K_{d_1} \max\{0, g_5 - \epsilon^{\gamma} W_i\} - K_{d_1} \max\{0, g_6 - \epsilon^{\gamma} W_i\} \\ \Xi \\ K_{d_2} \max\{0, g_7 - \epsilon^{\gamma} W_i\} - K_{d_2} \max\{0, g_8 - \epsilon^{\gamma} W_i\} \\ K_{i_1} \max\{0, g_5 - \epsilon^{\gamma} W_i\} - K_{i_1} \max\{0, g_6 - \epsilon^{\gamma} W_i\} \\ K_{i_2} \max\{0, g_7 - \epsilon^{\gamma} W_i\} - K_{i_2} \max\{0, g_8 - \epsilon^{\gamma} W_i\} \end{bmatrix} \tag{26}$$

$$\begin{aligned} \Omega = {} & (x_1(t) - q_{d_1}) + \max\{0, g_1 - \epsilon^{\gamma} W_i\} - \max\{0, g_3 - \epsilon^{\gamma} W_i\} \\ & + K_{p_1} \max\{0, g_5 - \epsilon^{\gamma} W_i\} - K_{p_1} \max\{0, g_6 - \epsilon^{\gamma} W_i\} \end{aligned}$$

$$\begin{aligned} \Xi = {} & (x_3(t) - q_{d_2}) + 1.05 q_{d_2} \max\{0, g_3 - \epsilon^{\gamma} W_i\} - \max\{0, g_4 - \epsilon^{\gamma} W_i\} \\ & + K_{p_2} \max\{0, g_7 - \epsilon^{\gamma} W_i\} - K_{p_2} \max\{0, g_8 - \epsilon^{\gamma} W_i\} \end{aligned}$$

$$\frac{\partial \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{K})}{\partial \boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2\frac{d_{22}c_{11}}{w}x_4 + 2\frac{c_{21}d_{12}}{w}x_2 & 0 & 2\frac{d_{22}c_{11}}{w}x_4 - 2\frac{c_{12}d_{11}}{w}x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\frac{d_{22}c_{12}}{w}x_4 - 2\frac{d_{22}c_{11}}{w}x_4 & 1 & 2\frac{d_{21}c_{12}}{w_1}x_4 + 2\frac{d_{22}c_{11}}{w}x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\top}$$

Proof. (22) can be proved straight forwardly by differentiating (20) with respective to ϵ. To prove (23), we need to consider the following optimal parameters selection problem.

Problem P. *Given the following system:*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{\zeta}) \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0(\boldsymbol{\zeta}) \end{aligned} \tag{27}$$

find a system parameter $\boldsymbol{\zeta} \in \mathbb{R}^s$ such that the cost functional:

$$g_0(\boldsymbol{\zeta}) = \Phi_0(\boldsymbol{x}(t_f|\boldsymbol{\zeta}), \boldsymbol{\zeta}) + \int_0^T \mathcal{L}_0(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta}) \mathrm{d}t \tag{28}$$

is minimized and subject to the equality constraints:

$$\begin{aligned} g_i(\boldsymbol{\zeta}) = \Phi_i(\boldsymbol{x}(t_f|\boldsymbol{\zeta}), \boldsymbol{\zeta}) + \int_0^T \mathcal{L}_i(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta}) \mathrm{d}t = 0 \\ i = 1, 2, \cdots, N_e \end{aligned}$$

and inequality constraints:

$$\begin{aligned} g_i(\boldsymbol{\zeta}) = \Phi_i(\boldsymbol{x}(t_f|\boldsymbol{\zeta}), \boldsymbol{\zeta}) + \int_0^T \mathcal{L}_i(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta}) \mathrm{d}t \geq 0 \\ i = N_e + 1, 2, \cdots, N \end{aligned}$$

Lemma 1 gives the gradient formulas of the cost functional and the constraint functionals.

Lemma 3.3. *(Theorem 5.2.1 of [36]) Consider Problem P. For each $i = 1, 2, \cdots, N$, the gradient of the functional is given as follows:*

$$\frac{\partial g_i(\boldsymbol{\zeta})}{\partial \boldsymbol{\zeta}} = \frac{\Phi_i(\boldsymbol{x}(T|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \boldsymbol{\zeta}} + (\boldsymbol{\lambda}_i(0|\boldsymbol{\zeta}))^\top \frac{\partial \boldsymbol{x}^0(\boldsymbol{\zeta})}{\partial \boldsymbol{\zeta}} + \int_0^T \frac{\partial H_i\left(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta}, \boldsymbol{\lambda}_i(t|\boldsymbol{\zeta})\right)}{\partial \boldsymbol{\zeta}} \mathrm{d}t \tag{29}$$

where, for each $i = 0, 1, 2, \cdots, N$,

$$H_i\left(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta}, \boldsymbol{\lambda}_i(t|\boldsymbol{\zeta})\right) = \mathcal{L}_i(t, \boldsymbol{x}, \boldsymbol{\zeta}) + \boldsymbol{\lambda}^\top \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\zeta})$$

is the corresponding Hamiltonian and $\boldsymbol{\lambda}_i(t)$ is the corresponding co-state variable that satisfies the following differential equations

$$\dot{\boldsymbol{\lambda}}_i(t) = -\frac{H_i\left(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta}, \boldsymbol{\lambda}_i(t|\boldsymbol{\zeta})\right)}{\partial \boldsymbol{x}}, \ t \in [0, T)$$

with

$$\boldsymbol{\lambda}_i(T) = \frac{\partial \Phi_i(\boldsymbol{x}(t_f|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \boldsymbol{x}}$$

To prove (23), we define the corresponding Hamiltonian of Problem Q_δ as follows:

$$H = \mathcal{L} + \boldsymbol{\lambda}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{K}) \tag{30}$$

where

$$\mathcal{L} = (x_1(t) - q_{d_1})^2 + (x_3(t) - q_{d_2})^2 + \epsilon^\alpha \sum_{i=1}^{8} \left(\max\left\{0, g_i - \epsilon^\gamma W_i\right\}\right)^2 \tag{31}$$

Since $\Phi(\boldsymbol{x}(T)) = \delta\epsilon^\beta$ and $\boldsymbol{x}^0$ does not depend on $\boldsymbol{K}$ in Problem Q_δ, then it follows that

$$\frac{\partial \Phi(\boldsymbol{x}(T))}{\partial \boldsymbol{K}} = 0 \tag{32}$$

and

$$\frac{\partial \boldsymbol{x}^0}{\partial \boldsymbol{K}} = 0 \tag{33}$$

Considering (30) and differentiating H with respective to $\boldsymbol{K}$, we have

$$\frac{\partial H}{\partial \boldsymbol{K}} = \epsilon^{-\alpha} \frac{\partial J_\Delta}{\partial \boldsymbol{K}} + \frac{\partial \left(\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{K})^\top\right)}{\partial \boldsymbol{K}} \boldsymbol{\lambda} \tag{34}$$

By applying Lemma 1, it follows that

$$\frac{\partial J_\delta}{\partial \boldsymbol{K}} = \frac{\partial \Phi(\boldsymbol{x}(T))}{\partial \boldsymbol{K}} + (\boldsymbol{\lambda}_0(0))^\top \frac{\partial \boldsymbol{x}^0}{\partial \boldsymbol{K}} + \int_0^T \frac{\partial H}{\partial \boldsymbol{K}} \mathrm{d}t \tag{35}$$

(23) can be proved by substituting (32), (33) and (34) into (35). This completes the proof. □

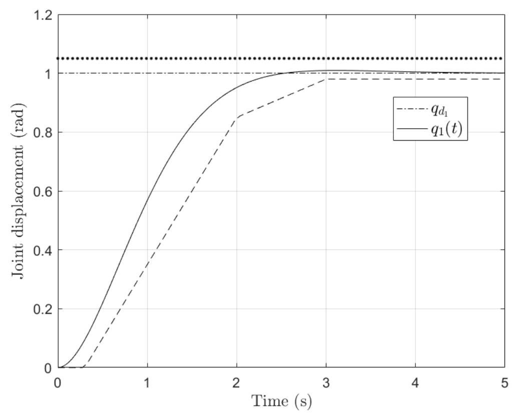

FIGURE 2. Joint displacement of link 1 with proposed method

4. **Numerical example.** In this section, we shall test the proposed method by solving a numerical example. The goal of the example is to design a PID controller such that the joint displacement tracks a constant angle.

The details of the two-link manipulator are given below. The length of link 1 and link 2 are $l_1 = 1$, $l_2 = 1$, respectively. The desired tracking inputs are $q_{d_1} = q_{d_2} = 1$. In (4), $m_1 = 2.90$, $m_2 = 0.78$, $m_3 = 0.87$, $m_4 = 3.04$, $m_5 = 0.87$.

The parameters of the exact penalty function based method are given below. We set $\alpha = 2, \gamma = 3, W_1 = W_2 = W_3 = W_4 = W_5 = W_6 = 0.30$. By the applying the proposed tuning method, the obtained results are shown below. $\delta = 1 \times 10^3$, $\epsilon = 9.5366 \times 10^{-4}$, and the optimal PID parameters are $\boldsymbol{K}_p = \text{diag}\{19.30, 6.51\}$, $\boldsymbol{K}_i = \text{diag}\{3.43 \times 10^{-5}, 1.67 \times 10^{-5}\}$, $\boldsymbol{K}_d = \text{diag}\{18.50, 5.71\}$.

By using the obtained tuning parameters, we plot the joint displacement for both of the links in Figure 2 and Figure 3, respectively. The corresponding torque τ_1 and τ_2 are plotted in Figure 4. As shown in Figure 2 and Figure 3, all of the control specifications are satisfied since the $q_1(t)$ and $q_2(t)$ are lower than the bound of the overshoot constraint and higher than boundaries of the other constraints. From Figure 4 we can see that the torque constraints on both of the links are satisfied .

For comparison, the PID tuning method by using the constraint transcription method [18] is applied. The resulted joint displacement cures are plotted in Figure 5 and Figure 6, and the corresponding torque is plotted in Figure 7. As shown in Figure 6, the overshoot constraint is violated.

By using the Z-N tuning method [11], the resulted joint displacement cures and torques on both of the links are plotted in Figure 8-Figure 10. From Figure 9, we observe that the overshoot constraint is also violated on the second link.

5. **Conclusions.** An exact penalty function based PID tuning method was proposed for the two-link manipulator. The PID tuning problem was described as a state constrained optimal parameters selection problem by formulating the control specifications into state constraints. By utilizing the exact penalty function method, the formulated problem was transformed into an unconstrained nonlinear program. In the simulations, the proposed method has shown to be superior than the existing methods [18] and [11].

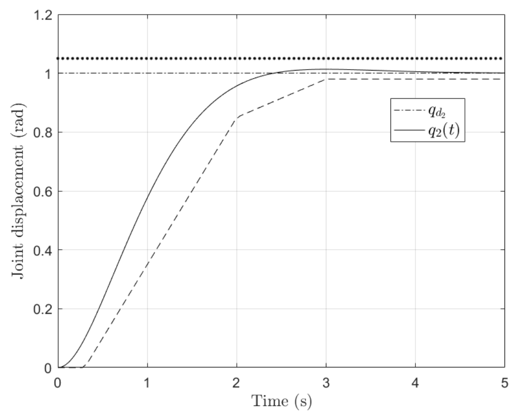

FIGURE 3. Joint displacement of link 2 with proposed method

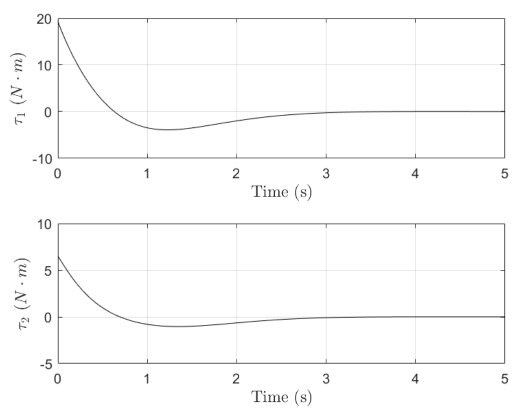

FIGURE 4. Motor torque with proposed method

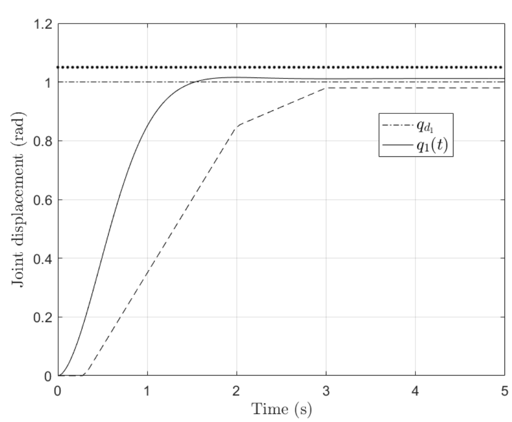

FIGURE 5. Joint displacement of link 1 with constraint transcription method

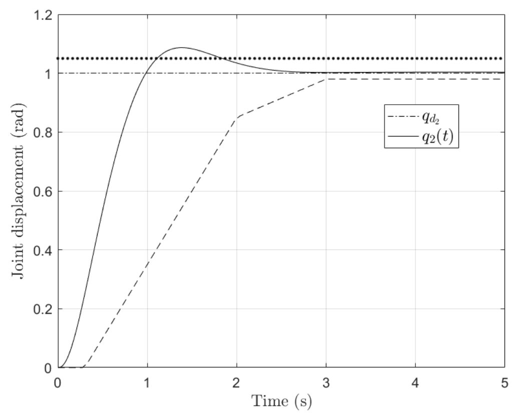

FIGURE 6. Joint displacement of link 2 with constraint transcription method

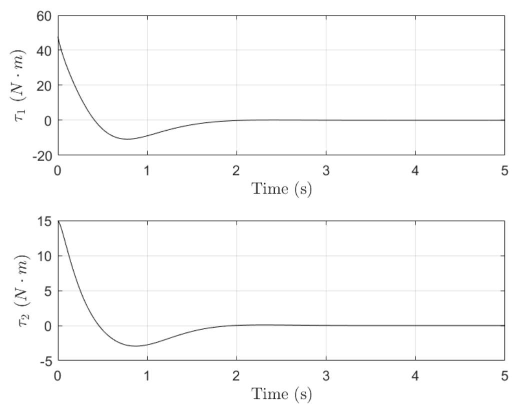

FIGURE 7. Motor torque with constraint transcription method

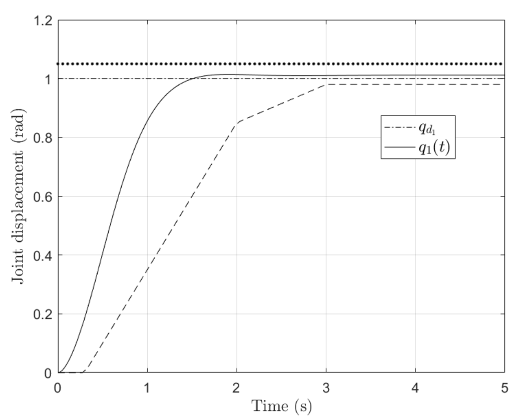

FIGURE 8. Joint displacement of link 1 with Z-N tuning method

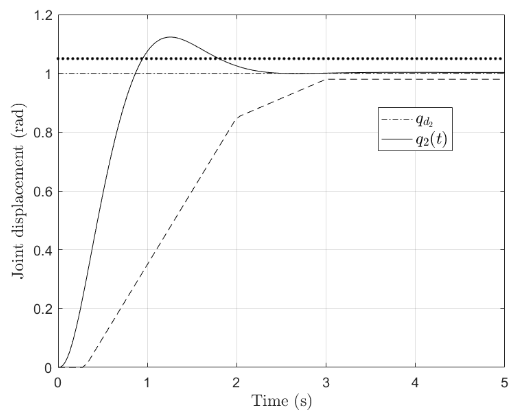

FIGURE 9. Joint displacement of link 2 with Z-N tuning method

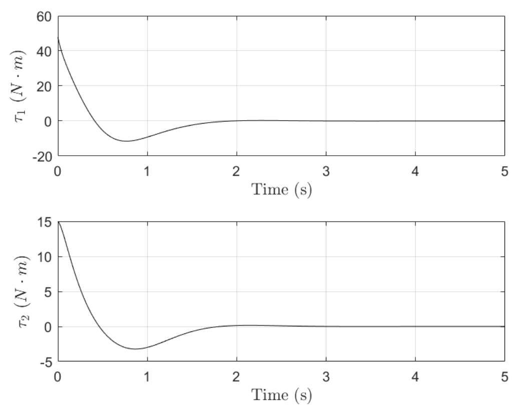

FIGURE 10. Motor torque with Z-N tuning method

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