

A bipolar-valued fuzzy set is an intersected interval-valued fuzzy set

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Abstract

This paper shows that a bipolar-valued fuzzy set and a special interval-valued fuzzy set, whose cut-off point of left and right ends of the interval-valued membership is λ ($0 < \lambda < 1$), are identical from a mathematical point of view. All results on bipolar-valued fuzzy sets can be obtained by corresponding results of interval-valued fuzzy sets or fuzzy sets. This paper does not completely reject bipolar-valued fuzzy sets, but points the way for such research.

Keywords: bipolar-valued fuzzy sets; intersected interval-valued fuzzy sets; interval-valued fuzzy sets

1. Introduction

Zhang initiated the concept of bipolar-valued fuzzy sets in 1994 [46] as a generalization of fuzzy sets whose membership degree range is extended from $[0,1]$ to $[-1,0] \times [0,1]$, which called bipolar fuzzy sets in [47] or YinYang fuzzy sets in [49] or YinYang bipolar fuzzy set in [48] by Zhang. However, it should be noted that some authors also used the name, bipolar fuzzy set, but in fact it is an Atanassov's intuitionistic fuzzy set [8]. This paper discusses bipolar-valued fuzzy sets introduced by Zhang and uses the name of bipolar-valued fuzzy set throughout the text.

After Zhang introduced the concept of bipolar-valued fuzzy sets, it has been attracting the attention of a large number of scholars whose made a series of work both in theory and applications on bipolar-valued fuzzy sets. These studies are inseparable from the shadow of research on fuzzy set theory. Figuring out the relationship with fuzzy sets is very useful for the study of bipolar-valued fuzzy sets.

1.1 A brief review on theory research of bipolar-valued fuzzy sets

Theoretical research on bipolar-valued fuzzy sets is mainly in the following aspects.

- Promote bipolar-valued fuzzy sets by imitating the promotion route of fuzzy sets.

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In the study of fuzzy sets, their membership functions play an important role. Different fuzzy sets are commensurate with different descriptions of membership functions. Bipolar-valued fuzzy sets, which have bipolar membership functions, also are considered various generalizations of membership functions like fuzzy sets, such as bipolar-valued fuzzy soft sets, bipolar single-valued neutrosophic sets, hesitant bipolar-valued fuzzy sets, bipolar-valued fuzzy soft expert sets, fuzzy parameterized bipolar-valued fuzzy soft expert sets, bipolar-valued hesitant fuzzy sets, hesitant bipolar-valued neutrosophic sets, bipolar complex fuzzy sets [7], bipolar-valued fuzzy and soft expert sets, rough hesitant bipolar valued neutrosophic set, bipolar intuitionistic fuzzy soft expert sets [40] and so on.

- Study logic algebra of bipolar-valued fuzzy sets by imitating the research route of fuzzy logic.

Like fuzzy logic, a lot of bipolar fuzzy logic work has been carried out, such as bipolar aggregation operators, bipolar-valued fuzzy logic, t-norms and t-conorms [50], BCK-algebras, bipolar aggregation using the uninorms, hesitant bipolar-valued fuzzy aggregation operators, bipolar-valued fuzzy Dombi aggregation operators, soft hyper BCK ideals in hyper BCK algebras, ideals in BCI-algebras [1] and so on.

- Combine bipolar-valued fuzzy sets with other by imitating the linking route of fuzzy sets.

Akram first discussed bipolar-valued fuzzy graphs in 2011 [2], and brought out the applications in 2013 [3]. Subsequently, properties of bipolar-valued fuzzy graphs were studied, such as f-morphism on bipolar-valued fuzzy graphs, product bipolar-valued fuzzy graphs [33], bipolar-valued fuzzy graph structures [29], perfect bipolar-valued fuzzy graphs, energy of bipolar-valued fuzzy graphs, inverse domination of bipolar-valued fuzzy graphs, total dominating set of bipolar-valued fuzzy graphs, the dominant vertex edge set of bipolar-valued fuzzy graphs and so on. The concept of bipolar-valued fuzzy graphs was extended by the authors, such as bipolar intuitionistic fuzzy graphs, bipolar picture fuzzy graphs [22], bipolar complex intuitionistic fuzzy graphs [32], bipolar single-valued neutrosophic graphs [45], bipolar interval-valued fuzzy graphs [15]. There are also some authors who discuss some special bipolar-valued fuzzy graphs, such as regular bipolar-valued fuzzy graphs, bipolar-valued fuzzy competition graphs, dominating bipolar-valued fuzzy graphs and double dominating bipolar-valued fuzzy graphs, cubic bipolar-valued fuzzy graphs [21], bipolar-valued hesitant fuzzy graph [37] and so on.

Han, Shi and Chen, in 2015, introduced bipolar-valued fuzzy sets into rough sets and gave their applications to the decision information system [18]. Mandal and Ranadive introduced bipolar-valued fuzzy sets into decision-theoretic rough sets and presented their applications in 2019 [28]. In 2020, Ali, Akram and Alcantud investigated the attribute reduction problems for bipolar-valued fuzzy relation systems. In 2019, Malik and Shabir linked the Pawlak's rough sets with the fuzzy bipolar soft sets [27]. In 2021, Kumari and Thirucheran introduced rough hesitant bipolar neutrosophic sets into game theory [23]. In 2024, Hu discussed three-way decisions based on bipolar-valued fuzzy sets over three-way decision spaces [19].

In addition, there are other combination studies, such as K-algebras, concept lattices, UP-algebras, fuzzy topologies, fuzzy concept lattices, semigroups, fuzzy groups, topological structures [38] and so on.

1.2 A brief review on applications of bipolar-valued fuzzy sets

In terms of applications, many authors considered applications of bipolar-valued fuzzy sets. There are the following applications on bipolar-valued fuzzy sets.

- Decision analysis

Many authors promoted bipolar-valued fuzzy sets through considering decision-making problem, such as bipolar fuzzy softs' and bipolar fuzzy soft expert set's applications in decision-making problem, extended bipolar fuzzy EDAS approach and bipolar fuzzy Dombi aggregation operators for multi-criteria group decision-making process, attributes reductions of bipolar fuzzy relation decision systems, bipolar fuzzy metric spaces with application [44], distance and similarity measures for bipolar fuzzy soft sets with application [39], VIKOR method for MCDM based on bipolar

fuzzy soft beta-covering based bipolar fuzzy rough set model [42], a robust hybrid decision making model for human-computer interaction in the environment of bipolar complex picture fuzzy soft sets [20] and so on.

- System analysis

Some authors used bipolar-valued fuzzy information, such as YinYang dynamic neurobiological modeling and diagnostic analysis, Polar representation of bipolar information, Determination of journeys order based on graph's Wiener absolute index with bipolar fuzzy information, a method for solving bipolar fuzzy complex systems [4], ARAS technique for prioritizing COVID-19 vaccines [34], decision-making system in selecting COVID-19 vaccines [35] and so on.

- Bipolar cognitive

In 1996, Zhang studied bipolar cognitive [47]. Zhang considered bipolar-valued fuzzy cognitive mapping and bipolar visualization, established bipolar quantum-fuzzy cognitive models. There are other authors who have discussed bipolar-valued fuzzy cognitive.

1.3 The motivation of our research

In [9], Bustince, Barrenechea Pagola, et al. referred to bipolar-valued fuzzy sets and pointed out Zhang's bipolar-valued fuzzy set is just a particular case of Lee's bipolar-valued fuzzy set [24-26] and Atanassov's intuitionistic fuzzy set [8, 11] can be transformed a bipolar-valued fuzzy set. However, the "transformation" method is wrong. In [9], Authors wrote "even more, and regarding Zhang's approach, if we have an Atanassov intuitionistic pair (μ, ν) , with $(\mu, \nu) \in [0, 1]$ and $\mu + \nu \leq 1$, since we can consider the linear transformation $F : [0, 1] \rightarrow [-1, 0]$ given by $F(t) = t - 1$, we see that the pair (μ, ν) becomes a new pair $(p, q) \in [0, 1] \times [-1, 0]$ with $p + q = \mu + \nu - 1 \leq 0$ ". Zhang's bipolar-valued fuzzy set does not have this requirement $p + q \leq 0$ for $(p, q) \in [0, 1] \times [-1, 0]$. So, they did not describe a clear relationship between bipolar-valued fuzzy sets and other similar concepts, such as interval-valued fuzzy sets. Our research has found that bipolar-valued fuzzy sets are equivalent to certain type of interval-valued fuzzy sets.

In [26], Lee, Lee and Cios made comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets and investigated the similarities and differences among these fuzzy set representations. However, they did not give the precise relationship between interval-valued fuzzy sets and bipolar-valued fuzzy sets.

In [12], Dubois and Prade discussed gradualness, uncertainty and bipolarity though making sense of fuzzy sets and considered a bipolar scale, $[-1, 1]$, instead of Zhang's bipolar scale, $[0, 1] \times [-1, 0]$. They pointed out "Applying the notion of bipolarity to the fuzzy set framework, it is clear that the membership scale $[0, 1]$ of a fuzzy set is a univariate bipolar scale. The cross-over point 0.5 is neutral, in the sense that in case a final Boolean membership decision is to be made, it is indifferent to consider an element with membership grade 0.5 as a member of the set or not". We think that 0 and 1 are two polar for a fuzzy set, and 0.5, or any number in $(0, 1)$, is the dividing line distinguishing between positive and negative sides. Based on this idea, in this paper, the concept of intersected interval-valued fuzzy set is introduced through considering the given number in $(0, 1)$ as a cut-off point of left and right ends of the interval-valued membership. This paper further shows that a bipolar-valued fuzzy set and an intersected interval-valued fuzzy set are identical from a mathematical point of view.

The rest of this paper is organized as follows. In Section 2, we present the concepts of bipolar-valued fuzzy set and interval-valued fuzzy set. In Section 3, we propose a new concept, intersected interval-valued fuzzy set and isomorphic theorem between bipolar-valued fuzzy sets and intersected interval-valued fuzzy sets. In Section 4, we state some facts that the existing concepts on bipolar-valued fuzzy sets can be induced from the corresponding concepts on interval-valued fuzzy sets. In the final section, our researches are concluded.

2. Preliminaries

In this section, we give a brief introduction to some basic concepts, bipolar-valued fuzzy sets and interval-valued fuzzy sets, which are used throughout the paper.

2.1 Truth value algebras of bipolar-valued fuzzy sets and interval-valued fuzzy sets

It starts with the definition of order relations over $[-1,0] \times [0,1]$ and closed interval number of $[0,1]$.

In this paper, the following notions on truth values are used

$$B_F = [-1,0] \times [0,1] = \{(a,b) \mid -1 \leq a \leq 0, 0 \leq b \leq 1\},$$

$$B_{FA} = \{(a,b) \mid -1 \leq a \leq 0, 0 \leq b \leq 1, -a + b \leq 1\},$$

$$I^{(2)} = \{[a,b] \mid 0 \leq a \leq b \leq 1\},$$

$$I_\lambda^{(2)} = \{[a,b] \mid 0 \leq a \leq \lambda \leq b \leq 1, 0 < \lambda < 1\},$$

$$I^2 = \{(a,b) \mid 0 \leq a, b \leq 1\},$$

$$I_A^2 = \{(a,b) \mid 0 \leq a, b \leq 1, a + b \leq 1\}.$$

In B_F , we define

$$(a,b) \leq_{bv} (c,d) \text{ if } a \leq c, b \leq d \text{ and}$$

$$(a,b) \leq_{ibv} (c,d) \text{ if } a \geq c, b \leq d,$$

for all $(a,b), (c,d) \in B_F$. And one uses notations $\underline{(a,b)} = a$ and $\overline{(a,b)} = b$, for all $(a,b) \in B_F$.

In $I^{(2)}$, we define

$$[a,b] \leq_{iv} [c,d] \text{ if } a \leq c, b \leq d, \text{ and}$$

$$[a,b] \leq_{iiv} [c,d] \text{ if } a \geq c, b \leq d,$$

for all $[a,b], [c,d] \in I^{(2)}$. And one uses notations $\underline{[a,b]} = a$ and $\overline{[a,b]} = b$, for all $[a,b] \in I^{(2)}$.

In I^2 , we define

$$(a,b) \leq_{av} (c,d) \text{ if } a \leq c, b \leq d, \text{ and}$$

$$(a,b) \leq_{iav} (c,d) \text{ if } a \leq c, b \geq d,$$

for all $(a,b), (c,d) \in I^2$. And one uses notations $\underline{(a,b)} = a$ and $\overline{(a,b)} = b$, for all $(a,b) \in I^2$.

In truth value sets, subscript ‘‘A’’ represents Atanassac’s truth value peculiarity, in which the sum of the two values does not exceed 1. In order relations, subscript ‘‘bv’’, ‘‘iv’’ and ‘‘av’’ represent ordinary order of bipolar-valued truth values, interval-valued truth values and Atanassac’s truth values, respectively, and ‘‘ibv’’, ‘‘iiv’’ and ‘‘iav’’ represents logic order of bipolar-valued truth values, interval-valued truth values and Atanassac’s truth values, respectively.

Obviously, (B_F, \leq_{bv}) , (B_F, \leq_{ibv}) , $(I^{(2)}, \leq_{iv})$, $(I^{(2)}, \leq_{iiv})$, $(I_\lambda^{(2)}, \leq_{iv})$, $(I_\lambda^{(2)}, \leq_{iiv})$, (I^2, \leq_{av}) and (I^2, \leq_{iav}) are posets. Their minimum (if exist) and maximum are listed in Table 2.1.

Table 2.1Minimum and maximum of eight posets ($0 < \lambda < 1$)

Posets	Minimum	Maximum
(B_F, \leq_{bv})	$(-1, 0)$	$(0, 1)$
(B_F, \leq_{ibv})	$(0, 0)$	$(-1, 1)$
$(I^{(2)}, \leq_{iv})$	$[0, 0]$	$[1, 1]$
$(I^{(2)}, \leq_{iiv})$	no	$[0, 1]$
$(I_\lambda^{(2)}, \leq_{iv})$	$[0, \lambda]$	$[\lambda, 1]$
$(I_\lambda^{(2)}, \leq_{iiv})$	$[\lambda, \lambda]$	$[0, 1]$
(I^2, \leq_{av})	$(0, 0)$	$(1, 1)$
(I^2, \leq_{iav})	$(0, 1)$	$(1, 0)$

Table 2.2 lists out notations and operations of maximum lower bound and minimum upper bound of posets (B_F, \leq_{bv}) , (B_F, \leq_{ibv}) , $(I^{(2)}, \leq_{iv})$, $(I^{(2)}, \leq_{iiv})$, $(I_\lambda^{(2)}, \leq_{iv})$, $(I_\lambda^{(2)}, \leq_{iiv})$, (I^2, \leq_{av}) and (I^2, \leq_{iav}) .

Table 2.2

Notations and operations of maximum lower bound and minimum upper bound of eight posets

Posets	Maximum lower bound / Minimum upper bound	
(B_F, \leq_{bv})	Maximum lower bound	$(a, b) \wedge_{bv} (c, d) = (\min(a, c), \min(b, d))$
	Minimum upper bound	$(a, b) \vee_{bv} (c, d) = (\max(a, c), \max(b, d))$
(B_F, \leq_{ibv})	Maximum lower bound	$(a, b) \wedge_{ibv} (c, d) = (\max(a, c), \min(b, d))$
	Minimum upper bound	$(a, b) \vee_{ibv} (c, d) = (\min(a, c), \max(b, d))$
$(I^{(2)}, \leq_{iv})$	Maximum lower bound	$[a, b] \wedge_{iv} [c, d] = [\min(a, c), \min(b, d)]$
	Minimum upper bound	$[a, b] \vee_{iv} [c, d] = [\max(a, c), \max(b, d)]$
$(I^{(2)}, \leq_{iiv})$	Maximum lower bound	does not necessarily exist in $I^{(2)}$
	Minimum upper bound	$[a, b] \vee_{iiv} [c, d] = [\min(a, c), \max(b, d)]$
$(I_\lambda^{(2)}, \leq_{iv})$	Maximum lower bound	$[a, b] \wedge_{iv} [c, d] = [\min(a, c), \min(b, d)]$
	Minimum upper bound	$[a, b] \vee_{iv} [c, d] = [\max(a, c), \max(b, d)]$
$(I_\lambda^{(2)}, \leq_{iiv})$	Maximum lower bound	$[a, b] \wedge_{iiv} [c, d] = [\max(a, c), \min(b, d)]$
	Minimum upper bound	$[a, b] \vee_{iiv} [c, d] = [\min(a, c), \max(b, d)]$
(I^2, \leq_{av})	Maximum lower bound	$(a, b) \wedge_{av} (c, d) = (\min(a, c), \min(b, d))$
	Minimum upper bound	$(a, b) \vee_{av} (c, d) = (\max(a, c), \max(b, d))$
(I^2, \leq_{iav})	Maximum lower bound	$(a, b) \wedge_{iav} (c, d) = (\min(a, c), \max(b, d))$
	Minimum upper bound	$(a, b) \vee_{iav} (c, d) = (\max(a, c), \min(b, d))$

It's worth noting that, in fact, \leq_{iiv} is the set contain relationship \subseteq in $I^{(2)}$. For the sake of unity, we use symbol \leq_{iiv} . The complement of $[a, b]$ in $I^{(2)}$ and (a, b) in B_F are defined as $[a, b]^c = [1-b, 1-a]$ and $(a, b)^c = (1-b, 1-a)$, respectively. In [47, 48], author used notation $\leq\leq$, i.e., in B_F , $(a, b) \leq\leq (c, d)$ iff $|a| \leq |c|, b \leq d$, where $|a|$ stands for “the absolute value of a ”, which is consistent with our notation \leq_{ibv} in B_F .

2.2 Negation operators on truth value algebras of bipolar-valued fuzzy sets and interval-valued fuzzy sets

Bilattice structures were studied in [10]. Truth algebras discussed above seems to have similarities with bilattice, which all have two partial orders. Do truth value algebras of bipolar-valued fuzzy sets and interval-valued fuzzy sets

also have bilattice structures? The following answers this question.

Definition 2.1. (See Ginsberg, [14]) A bilattice is a structure $\mathcal{B} = (B, \leq_l, \leq_k, \neg)$, such that B is a nonempty set, (B, \leq_l) and (B, \leq_k) are complete lattices, and \neg (the “negation”) is a unary operation on B satisfying, for every x, y in B , the following properties:

- 1) involution: $\neg\neg x = x$;
- 2) inverse order: if $x \leq_l y$ the $\neg x \geq_l \neg y$;
- 3) order preserving (istone): if $x \leq_k y$ then $\neg x \leq_k \neg y$.

It follows from Definition 2.1 that (I^2, \leq_{av}) , (I^2, \leq_{iav}) , $(I^{(2)}, \leq_{iv})$, $(I_\lambda^{(2)}, \leq_{iv})$, $(I_\lambda^{(2)}, \leq_{iiv})$, (B_F, \leq_{bv}) and (B_F, \leq_{ibv}) are complete lattices, but $(I^{(2)}, \leq_{iiv})$ not.

In I^2 , $I^{(2)}$ and B_F , we define

$$\begin{aligned} N_{I^2}((a,b)) &= (b,a), \quad \forall (a,b) \in I^2, \\ N_{I^{(2)}}([a,b]) &= [1-b, 1-a], \quad \forall [a,b] \in I^{(2)}, \\ N_{B_F}((a,b)) &= (-1-a, 1-b), \quad \forall (a,b) \in B_F, \end{aligned}$$

respectively. It is easy to know N_{I^2} , $N_{I^{(2)}}$ and N_{B_F} are involution operator over I^2 , $I^{(2)}$ and B_F , respectively. And also, N_{I^2} is an order preserving operator over (I^2, \leq_{av}) and an inverse order operator over (I^2, \leq_{iav}) . Thus, we have the following theorem.

Theorem 2.1. $(I^2, \leq_{iav}, \leq_{av})$ is a bilattice.

Proof. $\forall (a,b) \in I^2$, $N_{I^2}(N_{I^2}((a,b))) = (a,b)$. $\forall (a,b), (c,d) \in I^2$,

$$(a,b) \leq_{iav} (c,d) \Leftrightarrow a \leq c, b \geq d \Leftrightarrow (b,a) \geq_{iav} (d,c) \Leftrightarrow N_{I^2}((a,b)) \geq_{iav} N_{I^2}((c,d)),$$

$$(a,b) \leq_{av} (c,d) \Leftrightarrow a \leq c, b \leq d \Leftrightarrow (b,a) \leq_{av} (d,c) \Leftrightarrow N_{I^2}((a,b)) \leq_{av} N_{I^2}((c,d)). \quad \blacksquare$$

Further, $N_{I^{(2)}}$ is an inverse order operator over $(I^{(2)}, \leq_{iv})$ and $(I^{(2)}, \leq_{iiv})$. In fact, $\forall [a,b], [c,d] \in I^{(2)}$,

$$[a,b] \leq_{iv} [c,d] \Leftrightarrow a \leq c, b \leq d \Leftrightarrow 1-a \geq 1-c, 1-b \geq 1-d \Leftrightarrow N_{I^{(2)}}([a,b]) \geq_{iv} N_{I^{(2)}}([c,d]),$$

$$[a,b] \leq_{iiv} [c,d] \Leftrightarrow a \geq c, b \leq d \Leftrightarrow 1-a \leq 1-c, 1-b \geq 1-d \Leftrightarrow N_{I^{(2)}}([a,b]) \geq_{iiv} N_{I^{(2)}}([c,d]).$$

In addition, N_{B_F} is an inverse order operator over (B_F, \leq_{bv}) and (B_F, \leq_{ibv}) . In fact, $\forall (a,b), (c,d) \in B_F$,

$$(a,b) \leq_{bv} (c,d) \Leftrightarrow a \leq c, b \leq d \Leftrightarrow -1-a \geq -1-c, 1-b \geq 1-d \Leftrightarrow N_{B_F}((a,b)) \geq_{bv} N_{B_F}((c,d)),$$

$$(a,b) \leq_{ibv} (c,d) \Leftrightarrow a \geq c, b \leq d \Leftrightarrow -1-a \leq -1-c, 1-b \geq 1-d \Leftrightarrow N_{B_F}((a,b)) \geq_{ibv} N_{B_F}((c,d)).$$

The above discussion shows $(I^{(2)}, \leq_{iv}, \leq_{iiv})$, $(I_\lambda^{(2)}, \leq_{iv}, \leq_{iiv})$ and $(B_F, \leq_{bv}, \leq_{ibv})$ are not bilattices. For a clearer comparison, the properties of the “negation” operators on the sets of truth values on bipolar-valued fuzzy sets and interval-valued fuzzy sets are listed in Table 2.3.

Table 2.3

The properties of the “negation” operators on the sets of truth values on bipolar-valued fuzzy sets and interval-valued fuzzy sets

Truth sets	Partial order	Complete lattice	The properties of the “negation” operators		
			Involution	Inverse order	Order preserving
I^2	\leq_{av}	✓		×	✓
	\leq_{iav}	✓	✓	✓	×
$I^{(2)}$	\leq_{iv}	✓		✓	×
	\leq_{iiv}	×	✓	✓	×
$I_{\lambda}^{(2)}$	\leq_{iv}	✓		✓	×
	\leq_{iiv}	✓	✓	✓	×
B_F	\leq_{bv}	✓		✓	×
	\leq_{ibv}	✓	✓	✓	×

2.3 Bipolar-valued fuzzy sets and interval-valued fuzzy sets

Zhang [46] introduced the concept of bipolar-valued fuzzy set as follows.

Definition 2.2. (See Zhang, [46, 47]) A bipolar-valued fuzzy set H over X is a function

$$H : X \rightarrow B_F,$$

where $H(x) = (\varphi_H^-, \varphi_H^+)(x) = (\varphi_H^-(x), \varphi_H^+(x))$ with

$$\varphi_H^- : X \rightarrow [-1, 0],$$

$$\varphi_H^+ : X \rightarrow [0, 1].$$

Here, the bipolar-valued membership degree $(\varphi_H^-(x), \varphi_H^+(x))$ represents the bipolar of x to the property according to the bipolar-valued fuzzy set H .

A mapping $R : X \times Y \rightarrow B_F$ is a bipolar-valued fuzzy relation from X to Y .

Zadeh suggested the notion of interval-valued fuzzy set in [43] as a particular case of type-2 fuzzy sets.

Definition 2.3. (See Zadeh, [43]) An interval-valued fuzzy set H over X is a function

$$H : X \rightarrow I^{(2)},$$

where $H(x) = [\mu_H^-, \mu_H^+](x) = [\mu_H^-(x), \mu_H^+(x)]$ with

$$\mu_H^- : X \rightarrow [0, 1],$$

$$\mu_H^+ : X \rightarrow [0, 1],$$

and $\mu_H^-(x) \leq \mu_H^+(x)$, $\forall x \in X$.

Here, the interval-valued membership degree $[\mu_H^-(x), \mu_H^+(x)]$ represents the range of intervals of x to the property according to the interval-valued fuzzy set H .

Antanassov suggested the notion of intuitionistic fuzzy set in [8].

Definition 2.4. (See Antanassov, [8]) An intuitionistic fuzzy set H on X is a function

$$H : X \rightarrow I_A^2,$$

where $H(x) = (v_H^+, v_H^-)(x) = (v_H^+(x), v_H^-(x))$ with

$$v_H^+ : X \rightarrow [0, 1],$$

$$v_H^- : X \rightarrow [0, 1],$$

where $v_H^+(x) + v_H^-(x) \leq 1$.

Here, the intuitionistic membership degree $(v_H^+(x), v_H^-(x))$ represents the degree of approval and opposition of x to the property according to the intuitionistic fuzzy set H .

In order to discuss the relationship between bipolar-valued fuzzy sets and interval-valued fuzzy sets, a new class of interval-valued fuzzy sets is defined in the following section.

3. The relationship between bipolar-valued fuzzy sets and interval-valued fuzzy sets

In this section, we prove that bipolar-valued fuzzy sets and a special class of interval-valued fuzzy sets are equivalent concepts, which shall be used in Section 4. We first give out the special class of interval-valued fuzzy sets as follows.

Definition 3.1. For the given $\lambda \in (0,1)$, a mapping $H : X \rightarrow I_\lambda^{(2)}$ is called a λ -intersected interval-valued fuzzy set on X with membership function $H(x) = [\mu_H^-, \mu_H^+](x) = [\mu_H^-(x), \mu_H^+(x)]$, where

$$\mu_H^- : X \rightarrow [0, \lambda],$$

$$\mu_H^+ : X \rightarrow [\lambda, 1].$$

Without confusion, a λ -intersected interval-valued fuzzy set is simply referred to as an intersected interval-valued fuzzy set.

An intersected interval-valued fuzzy set $R : X \times Y \rightarrow I_\lambda^{(2)}$ is named an intersected interval-valued fuzzy relation from X to Y .

An intersected interval-valued fuzzy set is a special interval-valued fuzzy set, which occupies a pivotal place in this paper.

In order to facilitate uniform description, the following concept is introduced.

Definition 3.2. A mapping $H : X \rightarrow I^2$ is called a two-valued fuzzy set on X with membership function $H(x) = (\mu_H^-, \mu_H^+)(x) = (\mu_H^-(x), \mu_H^+(x))$, where

$$\mu_H^- : X \rightarrow [0, 1],$$

$$\mu_H^+ : X \rightarrow [0, 1].$$

A two-valued fuzzy set $R : X \times Y \rightarrow I^2$ is named a two-valued fuzzy relation from X to Y .

An intuitionistic fuzzy set is a special two-valued fuzzy set, which is introduced to facilitate discussions about relationships among bipolar-valued fuzzy sets, interval-valued fuzzy sets and intuitionistic fuzzy sets.

BVFS(X), IVFS(X), IIVFS(X), TVFS(X) and IFS(X) are used to denote the class of bipolar-valued fuzzy sets, interval-valued fuzzy sets, intersected interval-valued fuzzy sets, two-valued fuzzy sets and intuitionistic fuzzy sets on the universe X , respectively. As thus IFS(X) is a subset of TVFS(X).

In BVFS(X), for $A, B \in \text{BVFS}(X)$, we define

$$A \subseteq_{bv} B \text{ if } A(x) \leq_{bv} B(x) \text{ and}$$

$$A \subseteq_{ibv} B \text{ if } A(x) \leq_{ibv} B(x),$$

for all $x \in X$.

In IVFS(X), for $A, B \in \text{IVFS}(X)$, we define

$$A \subseteq_{iv} B \text{ if } A(x) \leq_{iv} B(x) \text{ and}$$

$$A \subseteq_{iiv} B \text{ if } A(x) \leq_{iiv} B(x),$$

for all $x \in X$.

In TVFS(X), for $A, B \in \text{TVFS}(X)$, we define

$$A \subseteq_{av} B \text{ if } A(x) \leq_{av} B(x) \text{ and}$$

$$A \subseteq_{iav} B \text{ if } A(x) \leq_{iav} B(x),$$

for all $x \in X$.

Obviously, $(\text{BVFS}(X), \subseteq_{bv})$, $(\text{BVFS}(X), \subseteq_{ibv})$, $(\text{IVFS}(X), \subseteq_{iv})$, $(\text{IVFS}(X), \subseteq_{iiv})$, $(\text{IIVFS}(X), \subseteq_{iv})$, $(\text{IIVFS}(X), \subseteq_{iiv})$, $(\text{TVFS}(X), \subseteq_{av})$ and $(\text{TVFS}(X), \subseteq_{iav})$ are posets. Their minimum (if exist) and maximum are

such sets which membership functions always take the corresponding minimum and maximum in Table 2.1.

Table 3.1 lists out notations and operations of union and intersection of bipolar-valued fuzzy sets and interval-valued fuzzy sets.

Table 3.1

Notations and operations of maximum lower bound / minimum upper bound of bipolar-valued fuzzy sets, interval-valued fuzzy sets, intersected interval-valued fuzzy sets and intuitionistic fuzzy sets

Posets	Maximum lower bound / Minimum upper bound
$(\text{BVFS}(X), \subseteq_{bv})$	$A \cap_{bv} B(x) = (\min(\varphi_A^-(x), \varphi_B^-(x)), \min(\varphi_A^+(x), \varphi_B^+(x)))$ $A \cup_{bv} B(x) = (\max(\varphi_A^-(x), \varphi_B^-(x)), \max(\varphi_A^+(x), \varphi_B^+(x)))$
$(\text{BVFS}(X), \subseteq_{ibv})$	$A \cap_{ibv} B(x) = (\max(\varphi_A^-(x), \varphi_B^-(x)), \min(\varphi_A^+(x), \varphi_B^+(x)))$ $A \cup_{ibv} B(x) = (\min(\varphi_A^-(x), \varphi_B^-(x)), \max(\varphi_A^+(x), \varphi_B^+(x)))$
$(\text{IVFS}(X), \subseteq_{iv})$	$A \cap_{iv} B(x) = [\min(\mu_A^-(x), \mu_B^-(x)), \min(\mu_A^+(x), \mu_B^+(x))]$ $A \cup_{iv} B(x) = [\max(\mu_A^-(x), \mu_B^-(x)), \max(\mu_A^+(x), \mu_B^+(x))]$
$(\text{IVFS}(X), \subseteq_{iiv})$	Maximum lower bound does not necessarily exist in IIVFS(X) $A \cup_{iiv} B(x) = [\min(\mu_A^-(x), \mu_B^-(x)), \max(\mu_A^+(x), \mu_B^+(x))]$
$(\text{IIVFS}(X), \subseteq_{iv})$	$A \cap_{iv} B(x) = [\min(\mu_A^-(x), \mu_B^-(x)), \min(\mu_A^+(x), \mu_B^+(x))]$ $A \cup_{iv} B(x) = [\max(\mu_A^-(x), \mu_B^-(x)), \max(\mu_A^+(x), \mu_B^+(x))]$
$(\text{IIVFS}(X), \subseteq_{iiv})$	$A \cap_{iiv} B(x) = [\max(\mu_A^-(x), \mu_B^-(x)), \min(\mu_A^+(x), \mu_B^+(x))]$ $A \cup_{iiv} B(x) = [\min(\mu_A^-(x), \mu_B^-(x)), \max(\mu_A^+(x), \mu_B^+(x))]$
$(\text{TVFS}(X), \subseteq_{av}), (\text{IFS}(X), \subseteq_{av})$	$A \cap_{av} B(x) = (\min(\nu_A^+(x), \nu_B^+(x)), \min(\nu_A^-(x), \nu_B^-(x)))$ $A \cup_{av} B(x) = (\max(\nu_A^+(x), \nu_B^+(x)), \max(\nu_A^-(x), \nu_B^-(x)))$
$(\text{TVFS}(X), \subseteq_{iav}), (\text{IFS}(X), \subseteq_{iav})$	$A \cap_{iav} B(x) = (\min(\nu_A^+(x), \nu_B^+(x)), \max(\nu_A^-(x), \nu_B^-(x)))$ $A \cup_{iav} B(x) = (\max(\nu_A^+(x), \nu_B^+(x)), \min(\nu_A^-(x), \nu_B^-(x)))$

The complements of A in $\text{IVFS}(X)$ and $\text{BVFS}(X)$ are defined as

$$N_{I^{(2)}}(A)(x) = N_{I^{(2)}}([\mu_A^-(x), \mu_A^+(x)]) = [1 - \mu_A^+(x), 1 - \mu_A^-(x)] \quad \text{and}$$

$$N_{B_F}(A)(x) = N_{B_F}((\varphi_A^-(x), \varphi_A^+(x))) = (-1 - \varphi_A^+(x), 1 - \varphi_A^-(x)),$$

respectively. By the discussion in Section 2.2, it is easy to know $(\text{IVFS}(X), \subseteq_{iv}, \subseteq_{iiv})$, $(\text{IIVFS}(X), \subseteq_{iv}, \subseteq_{iiv})$ and $(\text{BVFS}(X), \subseteq_{bv}, \subseteq_{ibv})$ are not bilattices. However, $(\text{TVFS}(X), \subseteq_{iav}, \subseteq_{av})$ $(\text{IFS}(X), \subseteq_{iav}, \subseteq_{av})$ is a bilattice.

The following lemma communicates the relationship between (B_F, \leq_{bv}) and $(I_\lambda^{(2)}, \leq_{iv})$, (B_F, \leq_{ibv}) and $(I_\lambda^{(2)}, \leq_{iiv})$.

Lemma 3.1. *Given $0 < \lambda < 1$, then the following hold.*

(1) *The mapping*

$$f : B_F \rightarrow I_\lambda^{(2)}$$

$$(x, y) \mapsto [f(x), f(y)], \quad f(x) = \lambda x + \lambda, \quad f(y) = (1 - \lambda)y + \lambda,$$

is a bijection, and (B_F, \leq_{bv}) is order isomorphic to $(I_\lambda^{(2)}, \leq_{iv})$ and (B_F, \leq_{ibv}) is order isomorphic to $(I_\lambda^{(2)}, \leq_{iiv})$.

(2) *The mapping*

$$g : B_F \rightarrow I_\lambda^{(2)}$$

$$(x, y) \mapsto [g(x), g(y)], \quad g(x) = -\lambda x, \quad g(y) = (1 - \lambda)y + \lambda$$

is a bijection, and (B_F, \leq_{bv}) is order isomorphic to $(I_\lambda^{(2)}, \leq_{iiv})$, and (B_F, \leq_{ibv}) is order isomorphic to $(I_\lambda^{(2)}, \leq_{iv})$.

Proof. Only prove (1). Proof of (2) is similar to that of (1).

(i) For any $[u, v] \in I_\lambda^{(2)}$, let $x = \frac{u - \lambda}{\lambda}$, $y = \frac{v - \lambda}{1 - \lambda}$. Then $(x, y) \in B_F$ and $f(x, y) = [u, v]$. Namely, f is a surjection.

(ii) Let $(x, y), (x', y') \in B_F$ and $(x, y) \neq (x', y')$. Then

$$f(x, y) = [\lambda x + \lambda, (1 - \lambda)y + \lambda] \neq [\lambda x' + \lambda, (1 - \lambda)y' + \lambda] = f(x', y'),$$

i.e., f is an injection.

(iii) Let $(x, y), (x', y') \in B_F$ and $(x, y) \leq_{bv} (x', y')$, i.e., $x \leq x', y \leq y'$. Then

$$f(x, y) = [\lambda x + \lambda, (1 - \lambda)y + \lambda] \leq_{iv} [\lambda x' + \lambda, (1 - \lambda)y' + \lambda] = f(x', y').$$

In a similar way, $(x, y) \leq_{ibv} (x', y')$, i.e., $x \geq x'$ and $y \leq y'$ imply $f(x, y) \leq_{iv} f(x', y')$.

Namely, f is istone. ■

Lemma 3.1 illustrates (B_F, \leq_{bv}) is order isomorphic to $(I_\lambda^{(2)}, \leq_{iv})$, $(I_\lambda^{(2)}, \leq_{iiv})$ and (B_F, \leq_{ibv}) is order isomorphic to $(I_\lambda^{(2)}, \leq_{iv})$, $(I_\lambda^{(2)}, \leq_{iiv})$. They are transformed by the bijections f and g . To make their relationship clearer, it is shown in Figure 3.1.

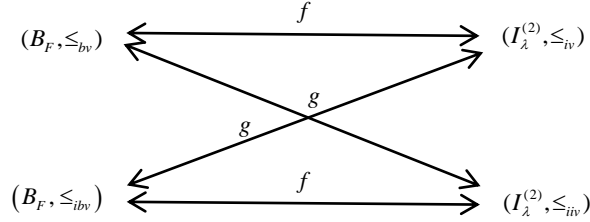


Figure 3.1. Order isomorphic structure between (B_F, \leq_{bv}) , (B_F, \leq_{ibv}) and $(I_\lambda^{(2)}, \leq_{iv})$, $(I_\lambda^{(2)}, \leq_{iiv})$ ($0 < \lambda < 1$)

The following is a bipolar-valued order structure in [48], which can be equivalently replaced by an interval-valued order structure.

Example 3.1. For the sake of simplicity, we consider $\lambda = 0.5$. In [49], the poset (B_F, \leq_{ibv}) was discussed and a type of finite subsets of it was shown in Figure 3.2 which is equivalent to $(I_{0.5}^{(2)}, \leq_{iiv})$, where $(x, y) \in B_F$.

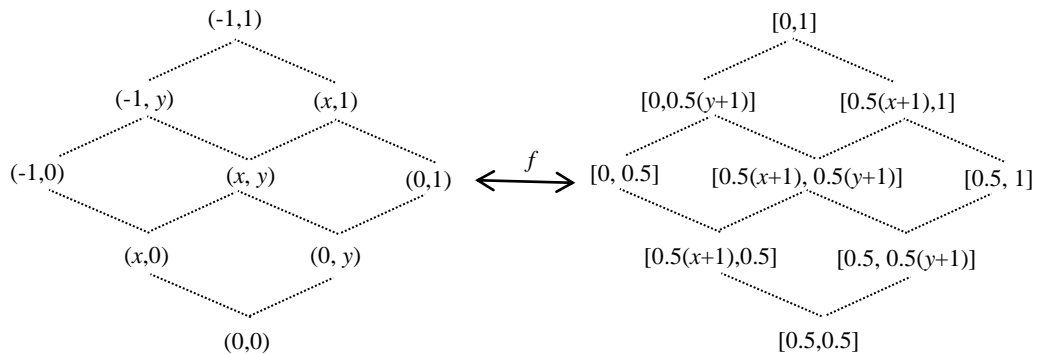


Figure 3.2. Order isomorphic structure between (B_F, \leq_{ibv}) (left) and $(I_{0.5}^{(2)}, \leq_{iiv})$ (right)

There are also direct relationships between B_F and $I^{(2)}$, B_F and I^2 . The following lemma communicates the

relationships between (B_F, \leq_{ibv}) and $(I^{(2)}, \leq_{iiv})$, (B_F, \leq_{bv}) and $(I^{(2)}, \leq_{iv})$, (B_F, \leq_{bv}) and (I^2, \leq_{iav}) , (B_F, \leq_{ibv}) and (I^2, \leq_{av}) .

Lemma 3.2. (1) *The mapping*

$$h: B_F \rightarrow I^{(2)}$$

$$(x, y) \mapsto [h(x), h(y)], \quad h(x) = \min(x+1, y), \quad h(y) = \max(x+1, y)$$

is a surjection and istone from (B_F, \leq_{ibv}) to $(I^{(2)}, \leq_{iiv})$ and from (B_F, \leq_{bv}) to $(I^{(2)}, \leq_{iv})$.

(2) *The mapping*

$$t: B_F \rightarrow I^2$$

$$(x, y) \mapsto (-x, y)$$

is a bijection, and (B_F, \leq_{ibv}) is order isomorphic to (I^2, \leq_{av}) and (B_F, \leq_{bv}) is order isomorphic to (I^2, \leq_{iav}) .

(3) *The mapping*

$$t: B_{FA} \rightarrow I_A^2$$

$$(x, y) \mapsto (-x, y)$$

is a bijection, (B_{FA}, \leq_{ibv}) is order isomorphic to (I_A^2, \leq_{av}) . And the mapping

$$t^T: B_{FA} \rightarrow I_A^2$$

$$(x, y) \mapsto (y, -x)$$

is a bijection, and (B_{FA}, \leq_{bv}) is order isomorphic to (I_A^2, \leq_{iav}) .

Proof. (1) (i) For any $[u, v] \in I^{(2)}$, let $x = u - 1$, $y = v$. Then $(x, y) \in B_F$ and

$$h((x, y)) = [\min(x+1, y), \max(x+1, y)] = [\min(u, v), \max(u, v)] = [u, v].$$

Namely, h is a surjection.

(ii) Let $(x, y), (x', y') \in B_F$ and $(x, y) \leq_{ibv} (x', y')$, i.e., $x \geq x', y \leq y'$. Then

$$h((x, y)) = [\min(x+1, y), \max(x+1, y)] \leq_{iiv} [\min(x'+1, y'), \max(x'+1, y')] = h((x', y')).$$

Namely, h is istone from (B_F, \leq_{ibv}) to $(I^{(2)}, \leq_{iiv})$. In a similar way, h is istone from (B_F, \leq_{bv}) to $(I^{(2)}, \leq_{iv})$.

(2) (i) For any $(u, v) \in I^2$, let $x = -u$, $y = v$. Then $(x, y) \in B_F$ and

$$t((x, y)) = (-x, y) = (u, v).$$

Namely, t is a surjection from B_F to I^2 .

(ii) Let $(x, y), (x', y') \in B_F$ and $(x, y) \neq (x', y')$. Then

$$t(x, y) = (-x, y) \neq (-x', y') = t(x', y'),$$

i.e., h is an injection.

(iii) Let $(x, y), (x', y') \in B_F$ and $(x, y) \leq_{ibv} (x', y')$, i.e., $x \geq x', y \leq y'$. Then

$$t((x, y)) = (-x, y) \leq_{av} (-x', y') = t((x', y')).$$

Namely, h is istone from (B_F, \leq_{ibv}) to (I^2, \leq_{av}) . In a similar way, t is istone from (B_F, \leq_{bv}) to (I^2, \leq_{iav}) .

(3) Specially, $(u, v) \in I_A^2$, i.e., $u + v \leq 1$, implies $-x + y \leq 1$ with $x = -u$ and $y = v$, i.e., $(x, y) \in B_{FA}$. Thus, t is a surjection from B_{FA} to I_A^2 . It follows from the injection and istone of t in (2) that t is a bijection and istone from (B_{FA}, \leq_{ibv}) to (I_A^2, \leq_{av}) . In similar way, one can be proved that t^T is a bijection and istone from (B_{FA}, \leq_{bv}) to (I_A^2, \leq_{iav}) . ■

Lemma 3.2 illustrates (B_F, \leq_{ibv}) (resp. (B_F, \leq_{bv})) is order surjective homeomorphic to $(I^{(2)}, \leq_{iiv})$ (resp. $(I^{(2)}, \leq_{iv})$) and is order isomorphic to (I^2, \leq_{av}) (resp. (I^2, \leq_{iav})). They are transformed by the bijections f and g . To make their

relationship clearer, it is shown in Figure 3.3.

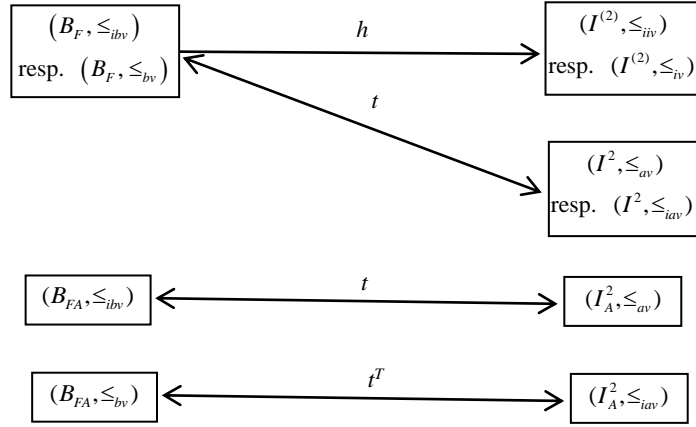


Figure 3.3. Order isomorphic structure between (B_F, \leq_{ibv}) , (B_F, \leq_{bv}) and (I^2, \leq_{iv}) , (I^2, \leq_{iav})

Remark 3.1. Consider $(-0.5, 0.6), (-0.4, 0.5) \in B_F$ and $(-0.5, 0.6) \neq (-0.4, 0.5)$. Then for h in Lemma 3.2, one has

$$h((-0.5, 0.6)) = [0.5, 0.6] = h((-0.4, 0.5)),$$

i.e., h is not an injection.

The following theorem is straightforward from the Lemma 3.1.

Theorem 3.1. Given $0 < \lambda < 1$, then the following hold.

(1) The mapping

$$\begin{aligned} \Phi_f : \text{BVFS}(X) &\rightarrow \text{IIVFS}(X) \\ H &\mapsto \Phi_f(H) \end{aligned}$$

where $\Phi_f(H)(x) = [\lambda\varphi_H^-(x) + \lambda, (1-\lambda)\varphi_H^+(x) + \lambda]$ for $H = (\varphi_H^-, \varphi_H^+) \in \text{BVFS}(X)$, is a bijection, and $(\text{BVFS}(X), \subseteq_{bv})$ is order isomorphic to $(\text{IIVFS}(X), \subseteq_{iv})$ and $(\text{BVFS}(X), \subseteq_{ibv})$ is order isomorphic to $(\text{IIVFS}(X), \subseteq_{iiv})$.

(2) The mapping

$$\begin{aligned} \Phi_g : \text{BVFS}(X) &\rightarrow \text{IIVFS}(X) \\ H &\mapsto \Phi_g(H) \end{aligned}$$

where $\Phi_g(H)(x) = [-\lambda\varphi_H^-(x), (1-\lambda)\varphi_H^+(x) + \lambda]$ for $H = (\varphi_H^-, \varphi_H^+) \in \text{BVFS}(X)$, is a bijection, and $(\text{BVFS}(X), \subseteq_{bv})$ is order isomorphic to $(\text{IIVFS}(X), \subseteq_{iiv})$ and $(\text{BVFS}(X), \subseteq_{ibv})$ is order isomorphic to $(\text{IIVFS}(X), \subseteq_{iv})$.

Remark 3.2. Through Lemma 3.1 and Theorem 3.1, the results on bipolar-valued fuzzy sets [46] (respectively, bipolar-valued fuzzy relations [46]) can be induced from the existing results of interval-valued fuzzy sets (respectively, interval-valued fuzzy relations). The following are some examples.

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4\}$ and

$$H = \frac{(-0.6, 0.7)}{x_1} + \frac{(-0.3, 0)}{x_2} + \frac{(-0.5, 0.6)}{x_3} + \frac{(-0.4, 0.2)}{x_4}$$

be a bipolar-valued fuzzy set over X . Then it follows from Theorem 3.1 (consider $\lambda = 0.5$) that

$$\Phi_f(H) = \frac{[0.2, 0.85]}{x_1} + \frac{[0.35, 0.5]}{x_2} + \frac{[0.25, 0.8]}{x_3} + \frac{[0.3, 0.6]}{x_4}$$

$$\Phi_g(H) = \frac{[0.3, 0.85]}{x_1} + \frac{[0.15, 0.5]}{x_2} + \frac{[0.25, 0.8]}{x_3} + \frac{[0.2, 0.6]}{x_4}$$

are two intersected interval-valued fuzzy sets on X .

Example 3.3. In [50], author discussed bipolar-valued fuzzy relation. The following is an example given in [50].

$$R_1 = \begin{bmatrix} (-0.6, 0.7) & (-0.9, 0.3) & (-0.6, 0) \\ (-0.5, 0) & (0, 0.8) & (-0.7, 0.5) \end{bmatrix}$$

Then it follows from Theorem 3.1 (consider $\lambda = 0.5$) that

$$\Phi_f(R_1) = \begin{bmatrix} [0.2, 0.85] & [0.05, 0.65] & [0.2, 0.5] \\ [0.25, 0.5] & [0.50, 0.9] & [0.15, 0.75] \end{bmatrix}$$

$$\Phi_g(R_1) = \begin{bmatrix} [0.3, 0.85] & [0.45, 0.65] & [0.3, 0.5] \\ [0.25, 0.5] & [0, 0.9] & [0.35, 0.75] \end{bmatrix}$$

are two intersected interval-valued fuzzy relations. In [36], author discussed bipolar-valued fuzzy matrices (bipolar-valued fuzzy relation on finite universe of discourse). The following is an example given in [36].

$$R_2 = \begin{bmatrix} (-0.3, 0.5) & (-0.5, 0.6) & (-0.4, 0.4) \\ (-0.1, 0.8) & (-0.2, 0.7) & (-0.3, 0.6) \\ (-0.6, 0.3) & (-0.7, 0.2) & (-0.8, 0.1) \end{bmatrix}$$

Then it follows from Theorem 3.1 (consider $\lambda = 0.5$) that

$$\Phi_f(R_2) = \begin{bmatrix} [0.35, 0.75] & [0.25, 0.8] & [0.3, 0.7] \\ [0.45, 0.9] & [0.4, 0.85] & [0.35, 0.8] \\ [0.2, 0.65] & [0.15, 0.6] & [0.1, 0.55] \end{bmatrix}$$

$$\Phi_g(R_2) = \begin{bmatrix} [0.15, 0.75] & [0.25, 0.8] & [0.2, 0.7] \\ [0.05, 0.9] & [0.1, 0.85] & [0.15, 0.8] \\ [0.3, 0.65] & [0.35, 0.6] & [0.4, 0.55] \end{bmatrix}$$

are two intersected interval-valued fuzzy relations.

Example 3.4. In [2, 24], the operations of bipolar-valued fuzzy sets are considered.

For every two bipolar-valued fuzzy sets $A = (\varphi_A^-, \varphi_A^+)$ and $B = (\varphi_B^-, \varphi_B^+)$ in X , consider $\lambda = 0.5$, then

$$\begin{aligned} \Phi_f(A \cup_{bv} B)(x) &= [0.5(\max(\varphi_A^-(x), \varphi_B^-(x)) + 1), 0.5(\max(\varphi_A^+(x), \varphi_B^+(x)) + 1)] \\ &= [\max(0.5(\varphi_A^-(x) + 1), 0.5(\varphi_B^-(x) + 1)), \max(0.5(\varphi_A^+(x) + 1), 0.5(\varphi_B^+(x) + 1))] \\ &= (\Phi_f(A) \cup_{iv} \Phi_f(B))(x). \end{aligned}$$

$$\begin{aligned} \Phi_g(A \cup_{ibv} B)(x) &= [-0.5 \min(\varphi_A^-(x), \varphi_B^-(x)), 0.5(\max(\varphi_A^+(x), \varphi_B^+(x)) + 1)] \\ &= [\max(-0.5\varphi_A^-(x), -0.5\varphi_B^-(x)), \max(0.5(\varphi_A^+(x) + 1), 0.5(\varphi_B^+(x) + 1))] \\ &= (\Phi_g(A) \cup_{iv} \Phi_g(B))(x). \end{aligned}$$

In a similar way, we have

$$\Phi_f(A \cap_{bv} B)(x) = (\Phi_f(A) \cap_{iv} \Phi_f(B))(x),$$

$$\Phi_g(A \cap_{ibv} B)(x) = (\Phi_g(A) \cap_{iv} \Phi_g(B))(x).$$

This means that the operations of bipolar-valued fuzzy sets can be defined through the corresponding operations of

interval-valued fuzzy sets.

The following theorem is straightforward from the Lemma 3.2. Use the following notation,

$$\text{BVFS}^*(X) = \{A \in \text{BVFS}(X) \mid -\varphi_A^-(x) + \varphi_A^+(x) \leq 1, \forall x \in X\}.$$

Theorem 3.2. (1) *The mapping*

$$\begin{aligned} \Phi_h : \text{BVFS}(X) &\rightarrow \text{IVFS}(X) \\ H &\mapsto \Phi_h(H) \end{aligned}$$

where

$$\Phi_h(H)(x) = \left[\min(\varphi_H^-(x) + 1, \varphi_H^+(x)), \max(\varphi_H^-(x) + 1, \varphi_H^+(x)) \right]$$

for $H = (\varphi_H^-, \varphi_H^+) \in \text{BVFS}(X)$, is a surjection and istone from $(\text{BVFS}(X), \subseteq_{ibv})$ to $(\text{IVFS}(X), \subseteq_{iv})$ and from $(\text{BVFS}(X), \subseteq_{bv})$ to $(\text{IVFS}(X), \subseteq_{iv})$.

(2) *The mapping*

$$\begin{aligned} \Phi_t : \text{BVFS}(X) &\rightarrow \text{TVFS}(X) \\ H &\mapsto \Phi_t(H) \end{aligned}$$

where $\Phi_t(H)(x) = (-\varphi_H^-(x), \varphi_H^+(x))$ for $H = (\varphi_H^-, \varphi_H^+) \in \text{BVFS}(X)$, is a bijection, and $(\text{BVFS}(X), \subseteq_{ibv})$ is order isomorphic to $(\text{TVFS}(X), \subseteq_{av})$.

(3) *The mapping*

$$\begin{aligned} \Phi_{t'} : \text{BVFS}^*(X) &\rightarrow \text{IFS}(X) \\ H &\mapsto \Phi_{t'}(H) \end{aligned}$$

where $\Phi_{t'}(H)(x) = (\varphi_H^+(x), -\varphi_H^-(x))$ for $H = (\varphi_H^-, \varphi_H^+) \in \text{BVFS}^*(X)$, is a bijection, and $(\text{BVFS}^*(X), \subseteq_{ibv})$ is order isomorphic to $(\text{IFS}(X), \subseteq_{iav})$.

Remarks 3.3. Through Lemma 3.2 and Theorem 3.2, the results on bipolar-valued fuzzy sets [46] (resp. bipolar-valued fuzzy relations [46]) can be induced from the corresponding results of two fuzzy sets (resp. two fuzzy relations). The following are some examples.

Example 3.5. Consider $X = \{x_1, x_2, x_3, x_4\}$ and a bipolar-valued fuzzy set X in Example 3.2, then it follows from Theorem 3.2 that

$$\Phi_h(H) = \frac{[0.4, 0.7]}{x_1} + \frac{[0, 0.7]}{x_2} + \frac{[0.5, 0.6]}{x_3} + \frac{[0.2, 0.6]}{x_4} \in \text{IVFS}(X),$$

$$\Phi_t(H) = \frac{(0.6, 0.7)}{x_1} + \frac{(0.3, 0)}{x_2} + \frac{(0.5, 0.6)}{x_3} + \frac{(0.4, 0.2)}{x_4} \in \text{TVFS}(X).$$

Let $G = \frac{(-0.2, 0.7)}{x_1} + \frac{(-0.3, 0.2)}{x_2} + \frac{(-0.4, 0.6)}{x_3} + \frac{(-0.4, 0.2)}{x_4} \in \text{BVFS}^*(X)$. Then

$$\Phi_t(G) = \frac{(0.2, 0.7)}{x_1} + \frac{(0.3, 0.2)}{x_2} + \frac{(0.4, 0.6)}{x_3} + \frac{(0.4, 0.2)}{x_4} \in (\text{IFS}(X), \subseteq_{av}),$$

$$\Phi_{t'}(G) = \frac{(0.7, 0.2)}{x_1} + \frac{(0.2, 0.3)}{x_2} + \frac{(0.6, 0.4)}{x_3} + \frac{(0.2, 0.4)}{x_4} \in (\text{IFS}(X), \subseteq_{iav}).$$

Remarks 3.4. Whether there is an order isomorphism mapping from B_F to $I^{(2)}$ or from I^2 to $I^{(2)}$? Whether it is or not, this does not affect the original intention of this paper, which it wants to illustrate that a bipolar-valued fuzzy set is an interval-valued fuzzy set and all results on bipolar-valued fuzzy sets can be obtained by corresponding results of interval-valued fuzzy sets or fuzzy sets. Therefor we chose a simple isomorphism between B_F and the subset classes of $I^{(2)}$, such as $I_\lambda^{(2)}$, even more succinctly, $I_{0.5}^{(2)}$.

4. The existing concepts on bipolar-valued fuzzy sets induced from interval-valued fuzzy sets

This section communicates the relationships between bipolar-valued fuzzy graphs and interval-valued fuzzy graphs, bipolar-valued rough fuzzy sets and interval-valued rough fuzzy sets, bipolar-valued t-norm (t-conorm) and t-norm (t-conorm) on $[0,1]$.

4.1 Bipolar-valued fuzzy graphs

In [2, 3], Akram introduced the concept of bipolar-valued fuzzy graphs in the following.

Definition 4.1. (See Akram, [2, 3]) A bipolar-valued fuzzy graph of $G^* = (V, E)$ with an underlying set V is defined to be a pair $G = (A, B)$ where $A = (\varphi_A^-, \varphi_A^+)$ is a bipolar-valued fuzzy set in V and $B = (\varphi_B^-, \varphi_B^+)$ is a bipolar-valued fuzzy set in $E \subseteq V \times V$ such that

$$\varphi_B^-(x, y) \geq \max\{\varphi_A^-(x), \varphi_A^-(y)\} \quad \text{and} \quad \varphi_B^+(x, y) \leq \min\{\varphi_A^+(x), \varphi_A^+(y)\}$$

for all $(x, y) \in E$. We call A the bipolar-valued fuzzy vertex set of V , B the bipolar-valued fuzzy edge set of E , respectively. Note that B is a symmetric bipolar-valued fuzzy relation on V .

A generalization of fuzzy graph known as the interval-valued fuzzy graph was introduced by Akram and Dudek in [5] where the nodes and edges are in the form of interval-valued fuzzy numbers. Akram and Dudek [5] discussed operations on interval-valued fuzzy graphs.

Definition 4.2. (See Akram and Dudek, [5]) By an interval-valued fuzzy of a graph $G^* = (V, E)$ we mean a pair $G = (A, B)$, where $A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set on V and $B = [\mu_B^-, \mu_B^+]$ is an interval-valued fuzzy set on E such that

$$\mu_B^-(x, y) \leq \min\{\mu_A^-(x), \mu_A^-(y)\} \quad \text{and} \quad \mu_B^+(x, y) \leq \min\{\mu_A^+(x), \mu_A^+(y)\}$$

for all $(x, y) \in E$.

If A and B are an intersected interval-valued fuzzy set and an intersected interval-valued fuzzy relation, respectively, then $G = (A, B)$ is named an intersected interval-valued fuzzy graph.

Theorem 4.1. Given $0 < \lambda < 1$. If $G = (A, B)$ is a bipolar-valued fuzzy graph of a graph $G^* = (V, E)$, then $\Phi_g(G) = (\Phi_g(A), \Phi_g(B))$ is an intersected interval-valued fuzzy graph of a graph $G^* = (V, E)$, where $\Phi_g(A)(x) = [-\lambda\varphi_A^-(x), (1-\lambda)\varphi_A^+(x) + \lambda]$ and $\Phi_g(B)(x, y) = [-\lambda\varphi_B^-(x, y), (1-\lambda)\varphi_B^+(x, y) + \lambda]$. Conversely, If $G = (C, D)$ is an intersected interval-valued fuzzy graph of a graph $G^* = (V, E)$, then $\Phi_g^{-1}(G) = (\Phi_g^{-1}(C), \Phi_g^{-1}(D))$ is

a bipolar-valued fuzzy graph of a graph $G^* = (V, E)$, where $\Phi_g^{-1}(C)(x) = \left[-\frac{\mu_C^-(x)}{\lambda}, \frac{\mu_C^+(x) - \lambda}{1 - \lambda} \right]$ and

$$\Phi_g^{-1}(D)(x, y) = \left[-\frac{\mu_D^-(x)}{\lambda}, \frac{\mu_D^+(x) - \lambda}{1 - \lambda} \right].$$

Proof. For all $(x, y) \in E$, since $\varphi_B^-(x, y) \geq \max\{\varphi_A^-(x), \varphi_A^-(y)\}$, we have

$$\begin{aligned} \mu_{\Phi_g(B)}^-(x, y) &= -\lambda\varphi_B^-(x, y) \\ &\leq -\lambda \max\{\varphi_A^-(x), \varphi_A^-(y)\} \\ &= \min\{-\lambda\varphi_A^-(x), -\lambda\varphi_A^-(y)\} \\ &= \min\{\mu_{\Phi_g(A)}^-(x), \mu_{\Phi_g(A)}^-(y)\} \end{aligned}$$

and since $\varphi_B^+(x, y) \leq \min\{\varphi_A^+(x), \varphi_A^+(y)\}$, it follows

$$\begin{aligned}
\mu_{\Phi_g(B)}^+(x, y) &= (1 - \lambda)\varphi_B^+(x, y) + \lambda \\
&\leq \min\{(1 - \lambda)\varphi_A^+(x) + \lambda, (1 - \lambda)\varphi_A^+(y) + \lambda\} \\
&= \min\{\mu_{\Phi_g(A)}^+(x), \mu_{\Phi_g(A)}^+(y)\}.
\end{aligned}$$

It follows from Definition 4.2 that $\Phi_g(G) = (\Phi_g(A), \Phi_g(B))$ is an intersected interval-valued fuzzy graph of a graph $G^* = (V, E)$.

Conversely, for all $(x, y) \in E$, since $\mu_D^-(x, y) \leq \min\{\mu_C^-(x), \mu_C^-(y)\}$, we have

$$\begin{aligned}
\varphi_{\Phi_g^{-1}(D)}^-(x, y) &= -\frac{\mu_D^-(x, y)}{\lambda} \\
&\geq -\frac{\min\{\mu_C^-(x), \mu_C^-(y)\}}{\lambda} \\
&= \max\left\{-\frac{\mu_C^-(x)}{\lambda}, -\frac{\mu_C^-(y)}{\lambda}\right\} \\
&= \max\{\varphi_{\Phi_g^{-1}(C)}^-(x), \varphi_{\Phi_g^{-1}(C)}^-(y)\},
\end{aligned}$$

and since $\mu_D^+(x, y) \leq \min\{\mu_C^+(x), \mu_C^+(y)\}$, it follows

$$\begin{aligned}
\varphi_{\Phi_g^{-1}(D)}^+(x, y) &= \frac{\mu_D^+(x, y) - \lambda}{1 - \lambda} \\
&\leq \frac{\min\{\mu_C^+(x), \mu_C^+(y)\} - \lambda}{1 - \lambda} \\
&= \min\left\{\frac{\mu_C^+(x) - \lambda}{1 - \lambda}, \frac{\mu_C^+(y) - \lambda}{1 - \lambda}\right\} \\
&= \min\{\varphi_{\Phi_g^{-1}(C)}^+(x), \varphi_{\Phi_g^{-1}(C)}^+(y)\}.
\end{aligned}$$

It follows from Definition 4.1 that $\Phi_g^{-1}(G) = (\Phi_g^{-1}(C), \Phi_g^{-1}(D))$ is a bipolar-valued fuzzy graph of a graph $G^* = (V, E)$. ■

In Theorem 4.1, When $\lambda = 0.5$, we have the following concise formula,

$$\Phi_g(A)(x) = [-0.5\varphi_A^-(x), 0.5\varphi_A^+(x) + 0.5] \quad \text{and} \quad \Phi_g(B)(x, y) = [-0.5\varphi_B^-(x, y), 0.5\varphi_B^+(x, y) + 0.5],$$

$$\Phi_g^{-1}(C)(x) = [-2\mu_C^-(x), 2\mu_C^+(x) - 1] \quad \text{and} \quad \Phi_g^{-1}(D)(x, y) = [-2\mu_D^-(x), 2\mu_D^+(x) - 1].$$

Theorem 4.1 communicates the relationships between bipolar-valued fuzzy graphs and intersected interval-valued fuzzy graphs, and they can be transformed into each other. Figure 4.1 shows the relationships among bipolar-valued fuzzy graphs, intersected interval-valued fuzzy graphs and interval-valued fuzzy graphs.

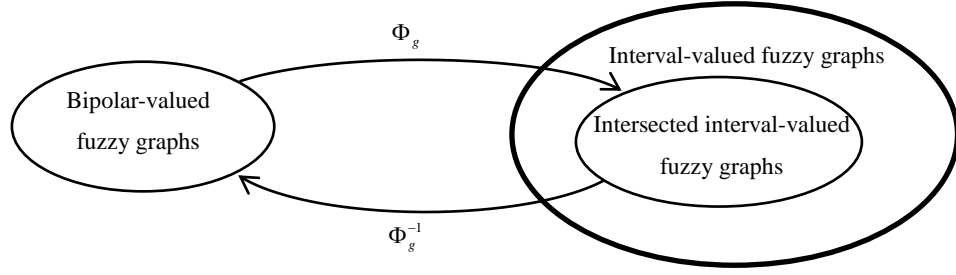


Figure 4.1. The relationship among bipolar-valued fuzzy graphs, intersected interval-valued fuzzy graphs and interval-valued fuzzy graphs

Example 4.1. Consider Example 3.2 in [2], a graph $G^*=(V,E)$ with $V = \{a, b, c\}$, $E = \{ab, bc, ca\}$. Let $A=(\varphi_A^-, \varphi_A^+)$ be a bipolar-valued fuzzy set in V and $B=(\varphi_B^-, \varphi_B^+)$ be a bipolar-valued fuzzy set in $E \subseteq V \times V$ defined by the following Table 4.1 and Figure 4.2.

Table 4.1

A bipolar-valued fuzzy set A in V and a bipolar-valued fuzzy set B in $E \subseteq V \times V$

	a	b	c		ab	bc	ca
φ_A^-	-0.7	-0.5	-0.7	φ_B^-	-0.3	-0.2	-0.1
φ_A^+	0.5	0.6	0.4	φ_B^+	0.4	0.3	0.3

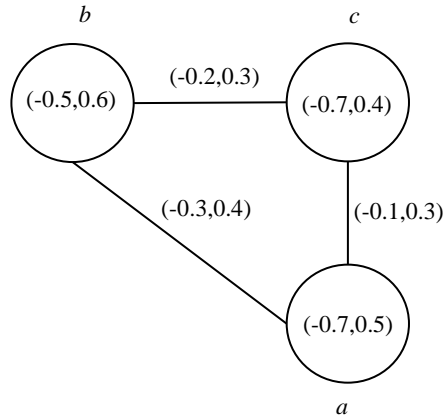


Figure 4.2. Bipolar-valued fuzzy graph $G = (A, B)$

It follows from Theorem 4.1 (consider $\lambda = 0.5$) that an interval-valued fuzzy set in V , $\Phi_g(A)=[\Phi_g(A)^-, \Phi_g(A)^+]$, and an interval-valued fuzzy set in $E \subseteq V \times V$, $\Phi_g(B)=[\Phi_g(B)^-, \Phi_g(B)^+]$ defined by the following Table 4.2 and Figure 4.3.

Table 4.2

The interval-valued fuzzy set $[\Phi_g(A)^-, \Phi_g(A)^+]$ in V and interval-valued fuzzy set $[\Phi_g(B)^-, \Phi_g(B)^+]$ in $E \subseteq V \times V$ corresponding to Table 4.1

	a	b	c		ab	bc	ca
$\Phi_g(A)^-$	0.35	0.25	0.35	$\Phi_g(B)^-$	0.15	0.1	0.05
$\Phi_g(A)^+$	0.75	0.8	0.7	$\Phi_g(B)^+$	0.7	0.65	0.65

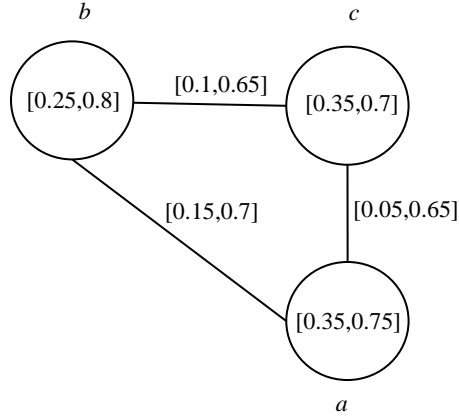


Figure 4.3. Interval-valued fuzzy graph $\Phi_g(G) = (\Phi_g(A), \Phi_g(B))$

The operations on bipolar-valued fuzzy graphs discussed in [2], such as the operations of the Cartesian product, composition, the union and join, can be induced from interval-valued fuzzy graphs. The following Table 4.3 shows corresponding equivalence relations on definitions and properties of these operations between bipolar-valued fuzzy graphs in [2] and interval-valued fuzzy graphs in [5] if interval values are limited in $I_{0.5}^{(2)}$. Therefore, definitions and properties of operations on bipolar-valued fuzzy graphs in [2] can be induced from corresponding definitions and properties of operations on interval-valued fuzzy graphs in [5].

Table 4.3

The list of corresponding equivalence relations on definitions and operations between [2] and [5] limited in $I_{0.5}^{(2)}$

Operation	Literature [2]	Literature [5] limited in $I_{0.5}^{(2)}$
Cartesian Product:	Definition 3.3	Definition 3.3
	Proposition 3.4	Proposition 3.5
Composition:	Definition 3.5	Definition 3.6
	Proposition 3.6	Proposition 3.8
Union:	Definition 3.7	Definition 3.9
	Proposition 3.9	Proposition 3.11
Join:	Definition 3.11	Definition 3.12
	Proposition 3.12	Proposition 3.13
	Proposition 3.14	Proposition 3.15

Similarly, automorphic bipolar-valued fuzzy graphs in [2, Section 4] can be induced by isomorphisms of interval-valued fuzzy graphs in [5, Section 4]. Strong bipolar-valued fuzzy graphs in [2, Section 5] can be induced by interval-valued fuzzy complete graphs in [5, Section 5].

Let's give an example of union operations to illustrate the equivalence between bipolar-valued fuzzy graphs and intersected interval-valued fuzzy graphs.

Example 4.2. Consider Example 3.8 in [2] and $\lambda = 0.5$. Two bipolar-valued fuzzy graph G_1 and G_2 are shown in Figure 4.4 (also appeared in Example 3.2 in [6]) and Figure 4.5, respectively.

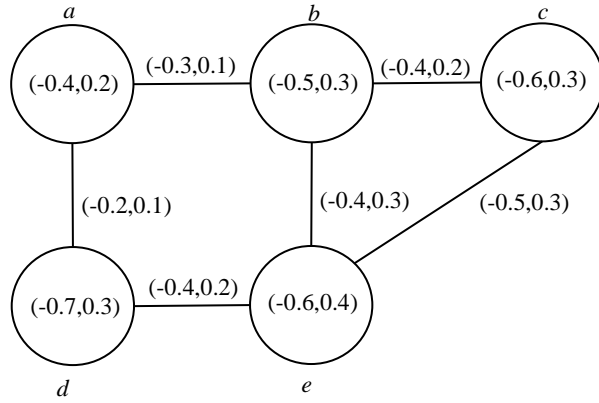


Figure 4.4. Bipolar-valued fuzzy graph G_1

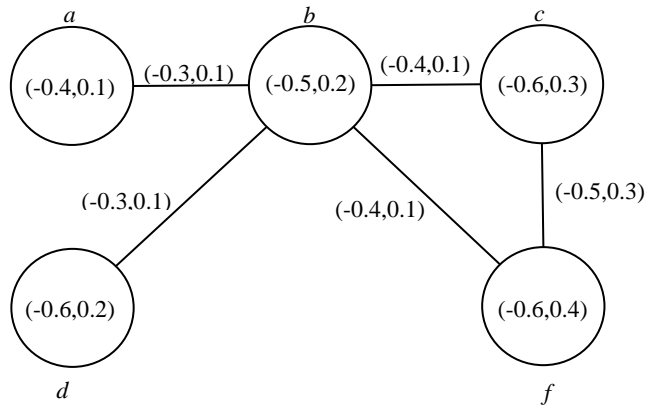


Figure 4.5. Bipolar-valued fuzzy graph G_2

The union of G_1 and G_2 , written as $G_1 \cup_{ibv} G_2$, are shown in Figure 4.6.

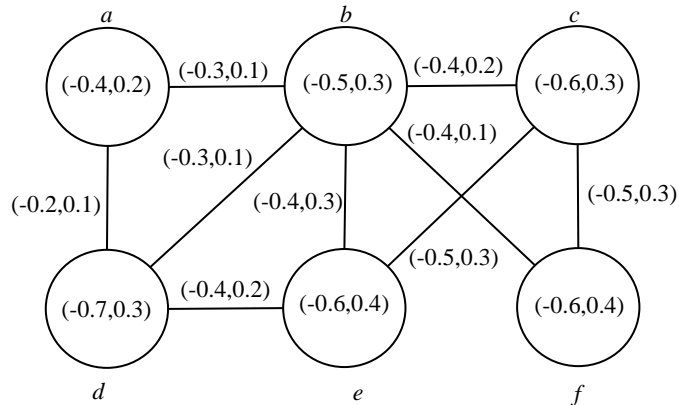


Figure 4.6. Bipolar-valued fuzzy graph $G_1 \cup_{ibv} G_2$

$\Phi_g(G_1)$ and $\Phi_g(G_2)$ are interval-valued fuzzy graphs, shown in Figure 4.7 and Figure 4.8, respectively.

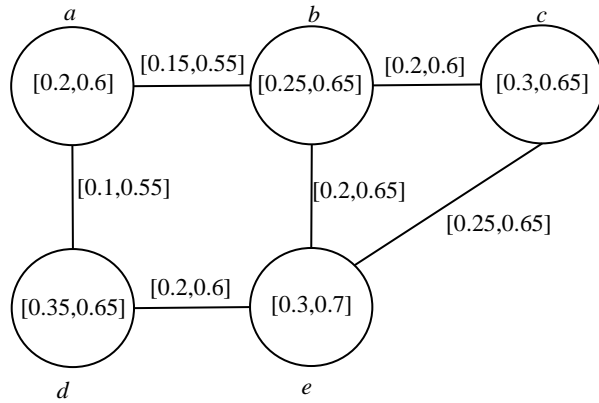


Figure 4.7. Bipolar-valued fuzzy graph $\Phi_g(G_1)$

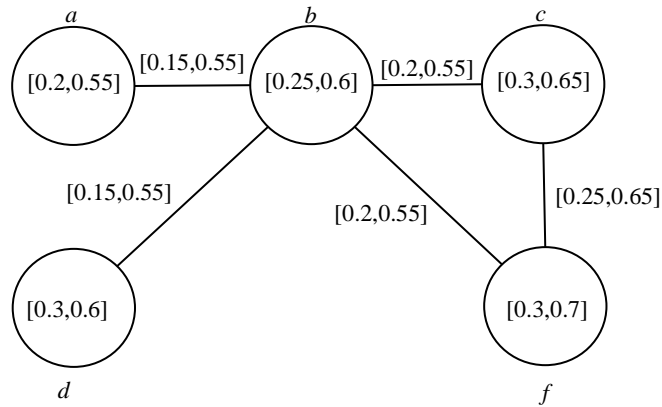


Figure 4.8. Bipolar-valued fuzzy graph $\Phi_g(G_2)$

The union of $\Phi_g(G_1)$ and $\Phi_g(G_2)$, written as $\Phi_g(G_1) \cup_{iv} \Phi_g(G_2)$, are shown in Figure 4.9.

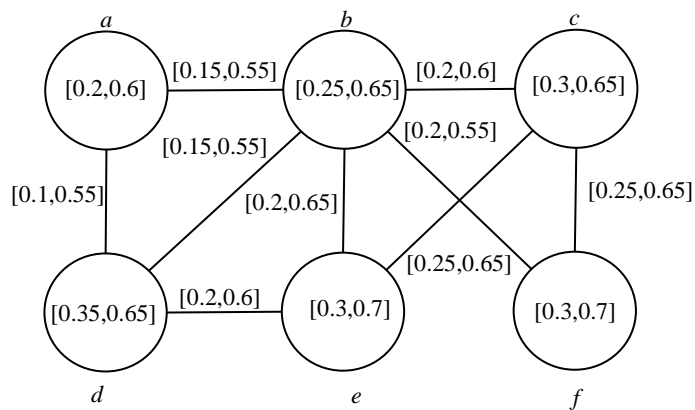


Figure 4.9. Bipolar-valued fuzzy graph $\Phi_g(G_1) \cup_{iv} \Phi_g(G_2)$

It's easy to get the facts $\Phi_g(G_1 \cup_{ibv} G_2) = \Phi_g(G_1) \cup_{iv} \Phi_g(G_2)$.

The above discussions are the operation relationships between the bipolar-valued fuzzy graphs and the

interval-valued fuzzy graphs. Other studies of bipolar-valued fuzzy graphs can be found in the corresponding of interval-valued fuzzy graphs.

Definition 4.3. (see Ghorai, [13]) Let $G = (A, B)$ be a bipolar-valued fuzzy graph of a graph $G^* = (V, E)$. If all the vertices have the same open neighborhood degree $n = (n_1, n_2)$, then G is called an n -regular bipolar-valued fuzzy graph. The open neighborhood degree of a vertex x in G is defined by $\deg(x) = (\deg^-(x), \deg^+(x))$, where

$$\deg_G^-(x) = \sum_{xy \in E, y \neq x} \varphi_B^-(x, y) \quad \text{and} \quad \deg_G^+(x) = \sum_{xy \in E, y \neq x} \varphi_B^+(x, y).$$

Definition 4.4. (see Talebi, Rashmanlou, Davvaz, [41]) Let $G = (C, D)$ be an interval-valued fuzzy graph of a graph $G^* = (V, E)$. The open neighborhood degree of a vertex x is defined as $\deg(x) = (\deg^-(x), \deg^+(x))$, where

$$\deg^-(x) = \sum_{xy \in E, y \neq x} \mu_D^-(x, y) \quad \text{and} \quad \deg^+(x) = \sum_{xy \in E, y \neq x} \mu_D^+(x, y).$$

If all the vertices have the same open neighborhood degree $n = (n_1, n_2)$, then G is called an n -regular interval-valued fuzzy graph.

Theorem 4.2. Given $0 < \lambda < 1$, then the following hold.

(1) If $G = (A, B)$ is a bipolar-valued fuzzy graph of a graph $G^* = (V, E)$ and the open neighborhood degree of a vertex x in G is $\deg_G(x) = (\deg_G^-(x), \deg_G^+(x))$, then $\deg_{\Phi_g(G)}(x) = (\deg_{\Phi_g(G)}^-(x), \deg_{\Phi_g(G)}^+(x))$ is the open neighborhood degree of a vertex x in intersected interval-valued fuzzy graph $\Phi_g(G) = (\Phi_g(A), \Phi_g(B))$ of a graph $G^* = (V, E)$, where

$$\deg_{\Phi_g(G)}^-(x) = \sum_{xy \in E, y \neq x} (-\lambda \varphi_B^-(x, y)) = -\lambda \deg_G^-(x) \quad \text{and}$$

$$\deg_{\Phi_g(G)}^+(x) = \sum_{xy \in E, y \neq x} ((1-\lambda)\varphi_B^+(x, y) + \lambda) = (1-\lambda)\deg_G^+(x) + \lambda |\{y \in V : xy \in E, y \neq x\}|.$$

(2) If $G = (C, D)$ is an intersected interval-valued fuzzy graph of a graph $G^* = (V, E)$ and the open neighborhood degree of a vertex x in G is $\deg_G(x) = (\deg_G^-(x), \deg_G^+(x))$, then $\deg_{\Phi_g^{-1}(G)}(x) = (\deg_{\Phi_g^{-1}(G)}^-(x), \deg_{\Phi_g^{-1}(G)}^+(x))$ is the open neighborhood degree of a vertex x in bipolar-valued fuzzy graph $\Phi_g^{-1}(G) = (\Phi_g^{-1}(C), \Phi_g^{-1}(D))$ of a graph $G^* = (V, E)$, where

$$\deg_{\Phi_g^{-1}(G)}^-(x) = \sum_{xy \in E, y \neq x} \left(-\frac{\mu_D^-(x, y)}{\lambda} \right) = -\frac{\deg_G^-(x)}{\lambda} \quad \text{and}$$

$$\deg_{\Phi_g^{-1}(G)}^+(x) = \sum_{xy \in E, y \neq x} \frac{\mu_D^+(x, y) - \lambda}{1 - \lambda} = \frac{\deg_G^+(x) - \lambda |\{y \in V : xy \in E, y \neq x\}|}{1 - \lambda}.$$

Proof. It is obtained directly from Definition 4.3, Definition 4.4 and Theorem 3.2.

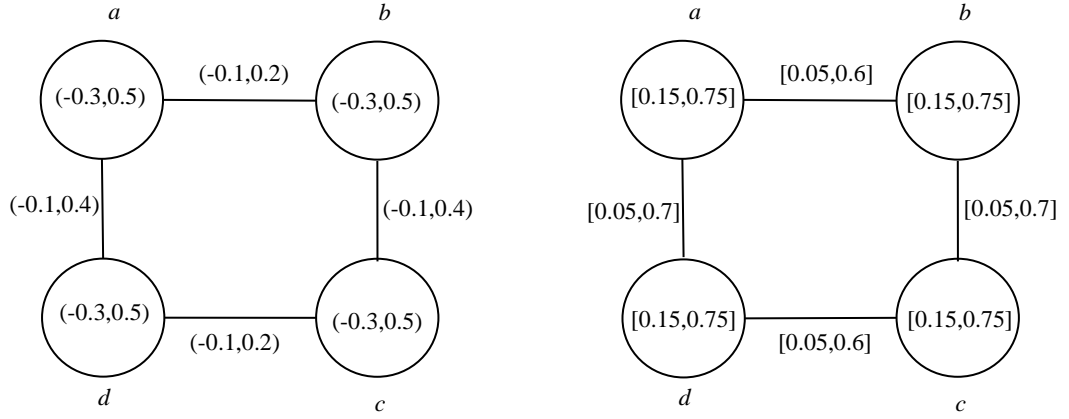
It follows from Theorem 4.2 that an n -regular bipolar-valued fuzzy graph is equivalent to an n -regular interval-valued fuzzy graph limited in intersected interval-valued fuzzy sets.

Example 4.3. Consider Example 3.6 in [6] and $\lambda = 0.5$, a graph $G^* = (V, E)$ with $V = \{a, b, c, d\}$, $E = \{ab, bc, cd, ad\}$. Let $A = (\varphi_A^-, \varphi_A^+)$ be a bipolar-valued fuzzy set in V and $B = (\varphi_B^-, \varphi_B^+)$ be a bipolar-valued fuzzy set in $E \subseteq V \times V$ defined by the following Table 4.4 and Figure 4.10. Then $\Phi_g(A) = [\mu_{\Phi_g(A)}^-, \mu_{\Phi_g(A)}^+]$ be an interval-valued fuzzy set in V and $\Phi_g(B) = [\mu_{\Phi_g(B)}^-, \mu_{\Phi_g(B)}^+]$ be an interval-valued fuzzy set in $E \subseteq V \times V$ defined by the following Table 4.4 and Figure 4.10.

Table 4.4

A bipolar-valued fuzzy set (interval-valued fuzzy set) in V and a bipolar-valued fuzzy set (interval-valued fuzzy set) in $E \subseteq V \times V$

	a	b	c	d		ab	bc	cd	da
φ_A^-	-0.3	-0.3	-0.3	-0.3	φ_B^-	-0.1	-0.1	-0.1	-0.1
φ_A^+	0.5	0.5	0.5	0.5	φ_B^+	0.2	0.4	0.2	0.4
$\mu_{\Phi_g(A)}^-$	0.15	0.15	0.15	0.15	$\mu_{\Phi_g(B)}^-$	0.05	0.05	0.05	0.05
$\mu_{\Phi_g(A)}^+$	0.75	0.75	0.75	0.75	$\mu_{\Phi_g(B)}^+$	0.6	0.7	0.6	0.7

**Figure 4.10.** Bipolar-valued fuzzy graph $G = (A, B)$ (left) and interval-valued fuzzy graph $\Phi_g(G) = (\Phi_g(A), \Phi_g(B))$ (right)

Routine computations show that a bipolar-valued fuzzy graph $G = (A, B)$ is $(-0.2, 0.6)$ -regular bipolar-valued fuzzy graph and an intersected interval-valued fuzzy graph $\Phi_g(G) = (\Phi_g(A), \Phi_g(B))$ is $(0.1, 1.3)$ -regular intersected interval-valued fuzzy graph.

4.2 Bipolar-valued rough fuzzy sets

In [18], Han, Shi and Chen proposed the concept of bipolar-valued rough fuzzy sets and gave their applications. In this section, X is a finite nonempty set, R is an equivalence relation over X , $[x]_R$ is an equivalence class including x and (X, R) comes to be known as a crisp approximation space.

Definition 4.5. (See Han et al., [18]) Let (X, R) be a crisp approximation space and $A = (\varphi_A^-, \varphi_A^+) \in \text{BVFS}(X)$, define, $\forall x \in X$,

$$\underline{R}_{bv}(A)(x) = \left(\bigwedge_{y \in [x]_R} \varphi_A^-(y), \bigwedge_{y \in [x]_R} \varphi_A^+(y) \right)$$

$$\overline{R}_{bv}(A)(x) = \left(\bigvee_{y \in [x]_R} \varphi_A^-(y), \bigvee_{y \in [x]_R} \varphi_A^+(y) \right)$$

Then, $\underline{R}_{bv}(A)$ and $\overline{R}_{bv}(A)$ are gone by the name of the lower approximation and the upper approximation of bipolar-valued fuzzy set A about the crisp approximation space (X, R) , respectively. The pair $(\underline{R}_{bv}(A), \overline{R}_{bv}(A))$ is named a bipolar-valued rough fuzzy set of A .

Definition 4.6. (See Gong et al., [16]) Let (X, R) be a crisp approximation space and $B = [\mu_B^-, \mu_B^+] \in \text{IVFS}(X)$, define, $\forall x \in X$,

$$\underline{R}_{iv}(B)(x) = \left[\bigwedge_{y \in [x]_R} \mu_B^-(y), \bigwedge_{y \in [x]_R} \mu_B^+(y) \right]$$

$$\bar{R}_{iv}(B)(x) = \left[\bigvee_{y \in [x]_R} \mu_B^-(y), \bigvee_{y \in [x]_R} \mu_B^+(y) \right]$$

Then, $\underline{R}_{iv}(B)$ and $\bar{R}_{iv}(B)$ are known as the lower approximation and the upper approximation of interval-valued fuzzy set B about the crisp approximation space (X, R) , respectively. The pair $(\underline{R}_{iv}(B), \bar{R}_{iv}(B))$ is termed an interval-valued rough fuzzy set of B .

Theorem 4.3. Let (X, R) be a crisp approximation space and $0 < \lambda < 1$. If $A = (\varphi_A^-, \varphi_A^+)$ is a bipolar-valued fuzzy set of X , then $\Phi_f(\underline{R}_{bv}(A)) = \underline{R}_{iv}(\Phi_f(A))$ and $\Phi_f(\bar{R}_{bv}(A)) = \bar{R}_{iv}(\Phi_f(A))$. If $B = [\mu_B^-, \mu_B^+]$ is an intersected interval-valued fuzzy set of X , then $\Phi_f^{-1}(\underline{R}_{iv}(B)) = \underline{R}_{bv}(\Phi_f^{-1}(B))$ and $\Phi_f^{-1}(\bar{R}_{iv}(B)) = \bar{R}_{bv}(\Phi_f^{-1}(B))$.

Proof. Suppose $A = (\varphi_A^-, \varphi_A^+)$ is a bipolar-valued fuzzy set of X , then $\forall x \in X$,

$$\begin{aligned} \Phi_f(\underline{R}_{bv}(A))(x) &= \left[f\left(\bigwedge_{y \in [x]_R} \varphi_A^-(y)\right), f\left(\bigwedge_{y \in [x]_R} \varphi_A^+(y)\right) \right] \\ &= \left[\lambda\left(\bigwedge_{y \in [x]_R} \varphi_A^-(y) + 1\right), (1-\lambda)\left(\bigwedge_{y \in [x]_R} \varphi_A^+(y) + \lambda\right) \right] \\ &= \left[\bigwedge_{y \in [x]_R} (\lambda\varphi_A^-(y) + \lambda), \bigwedge_{y \in [x]_R} ((1-\lambda)\varphi_A^+(y) + \lambda) \right] \\ &= \underline{R}_{iv}(\Phi_f(A))(x), \end{aligned}$$

i.e., $\Phi_f(\underline{R}_{bv}(A)) = \underline{R}_{iv}(\Phi_f(A))$. Similarly, $\Phi_f(\bar{R}_{bv}(A)) = \bar{R}_{iv}(\Phi_f(A))$.

Suppose $B = [\mu_B^-, \mu_B^+]$ is an intersected interval-valued fuzzy set of X , then $\forall x \in X$,

$$\begin{aligned} \Phi_f^{-1}(\underline{R}_{iv}(B))(x) &= \left(f^{-1}\left(\bigwedge_{y \in [x]_R} \mu_B^-(y)\right), f^{-1}\left(\bigwedge_{y \in [x]_R} \mu_B^+(y)\right) \right) \\ &= \left(\frac{\left(\bigwedge_{y \in [x]_R} \mu_B^-(y)\right) - \lambda}{\lambda}, \frac{\left(\bigwedge_{y \in [x]_R} \mu_B^+(y)\right) - \lambda}{1-\lambda} \right) \\ &= \left(\bigwedge_{y \in [x]_R} \frac{\mu_B^-(y) - \lambda}{\lambda}, \bigwedge_{y \in [x]_R} \frac{\mu_B^+(y) - \lambda}{1-\lambda} \right) \\ &= \underline{R}_{bv}(\Phi_f^{-1}(B))(x), \end{aligned}$$

i.e., $\Phi_f^{-1}(\underline{R}_{iv}(B)) = \underline{R}_{bv}(\Phi_f^{-1}(B))$. Similarly, $\Phi_f^{-1}(\bar{R}_{iv}(B)) = \bar{R}_{bv}(\Phi_f^{-1}(B))$. ■

It follows from Theorem 4.3 that $\underline{R}_{bv}(A)$ and $\bar{R}_{bv}(A)$ are equivalent to $\underline{R}_{iv}(\Phi_f(A))$ and $\bar{R}_{iv}(\Phi_f(A))$, respectively.

This means all properties on the lower approximation and the upper approximation of the bipolar-valued fuzzy set can be induced from the corresponding ones of interval-valued fuzzy set.

Figure 4.11 shows the relationships between the lower approximation and the upper approximation of the bipolar-valued fuzzy sets, and the lower approximation and the upper approximation of intersected interval-valued fuzzy sets.

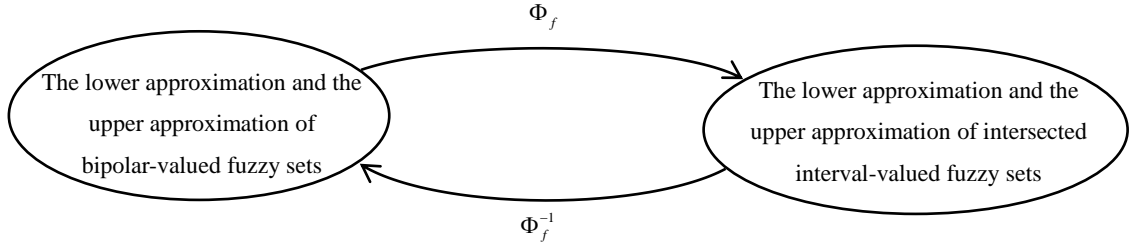


Figure 4.11. The relationships between the lower approximation and the upper approximation of bipolar-valued fuzzy sets, and the lower approximation and the upper approximation of intersected interval-valued fuzzy sets

Example 4.4. Theorem 2 1) in [18], $\underline{R}_{bv}(A) \subseteq_{bv} A \subseteq_{bv} \bar{R}_{bv}(A)$ holds from the corresponding property of interval-valued fuzzy set $\Phi_f(A)$, $\underline{R}_{iv}(\Phi_f(A)) \subseteq_{iv} \Phi_f(A) \subseteq_{iv} \bar{R}_{iv}(\Phi_f(A))$, and order isomorphism mapping Φ_f in Theorem 3.1. The following Table 4.5 shows corresponding equivalence relations on Theorem 2 in [18] and Theorem 3.2 in [16] if interval values are limited in $I_\lambda^{(2)}$.

Table 4.5

The list of corresponding equivalence relations on some properties between [18] and [16] (limited in $I_\lambda^{(2)}$)

Literature [18]	Literature [16] limited in $I_\lambda^{(2)}$
$\underline{R}_{bv}(A) \subseteq_{bv} A \subseteq_{bv} \bar{R}_{bv}(A)$	$\underline{R}_{iv}(\Phi_f(A)) \subseteq_{iv} \Phi_f(A) \subseteq_{iv} \bar{R}_{iv}(\Phi_f(A))$
$\underline{R}_{bv}(A^c) = (\bar{R}_{bv}(A))^c$	$\underline{R}_{iv}(\Phi_f(A)^c) = (\bar{R}_{iv}(\Phi_f(A)))^c$
$\bar{R}_{bv}(A^c) = (\underline{R}_{bv}(A))^c$	$\bar{R}_{iv}(\Phi_f(A)^c) = (\underline{R}_{iv}(\Phi_f(A)))^c$
$\underline{R}_{bv}(A \cap_{bv} B) = \underline{R}_{bv}(A) \cap_{bv} \underline{R}_{bv}(B)$	$\underline{R}_{iv}(\Phi_f(A) \cap_{iv} \Phi_f(B)) = \underline{R}_{iv}(\Phi_f(A)) \cap_{iv} \underline{R}_{iv}(\Phi_f(B))$
$\bar{R}_{bv}(A \cup_{bv} B) = \bar{R}_{bv}(A) \cup_{bv} \bar{R}_{bv}(B)$	$\bar{R}_{iv}(\Phi_f(A) \cup_{iv} \Phi_f(B)) = \bar{R}_{iv}(\Phi_f(A)) \cup_{iv} \bar{R}_{iv}(\Phi_f(B))$
$\bar{R}_{bv}(A \cap_{bv} B) \subseteq_{bv} \bar{R}_{bv}(A) \cap_{bv} \bar{R}_{bv}(B)$	$\bar{R}_{iv}(\Phi_f(A) \cap_{iv} \Phi_f(B)) \subseteq_{iv} \bar{R}_{iv}(\Phi_f(A)) \cap_{iv} \bar{R}_{iv}(\Phi_f(B))$
$\underline{R}_{bv}(A \cup_{bv} B) \subseteq_{bv} \underline{R}_{bv}(A) \cup_{bv} \underline{R}_{bv}(B)$	$\underline{R}_{iv}(\Phi_f(A) \cup_{iv} \Phi_f(B)) \subseteq_{iv} \underline{R}_{iv}(\Phi_f(A)) \cup_{iv} \underline{R}_{iv}(\Phi_f(B))$

The properties on bipolar-valued rough fuzzy sets not discussed in [18] can also be derived from the corresponding properties of interval-valued rough fuzzy sets in [16]. The following theorem is an example.

Theorem 4.4. Let (X, R) be a crisp approximation space and $A = (\varphi_A^-, \varphi_A^+) \in BVFS(X)$. If $(\underline{R}_{bv}(A), \bar{R}_{bv}(A))$ is a bipolar-valued rough fuzzy set of A , then

$$\begin{aligned} \underline{R}_{bv}(\underline{R}_{bv}(A)) &= \bar{R}_{bv}(\underline{R}_{bv}(A)) = \underline{R}_{bv}(A), \\ \bar{R}_{bv}(\bar{R}_{bv}(A)) &= \underline{R}_{bv}(\bar{R}_{bv}(A)) = \bar{R}_{bv}(A). \end{aligned}$$

Proof. Only prove $\underline{R}_{bv}(\underline{R}_{bv}(A)) = \underline{R}_{bv}(A)$ and others are similar.

It follows from Theorem 4.3 that

$$\begin{aligned} \underline{R}_{bv}(A) &= \Phi_f^{-1}(\underline{R}_{iv}(\Phi_f(A))) \\ &= \Phi_f^{-1}(\underline{R}_{iv}(\underline{R}_{iv}(\Phi_f(A)))) \quad (\text{by Theorem 3.2. (5) in [15]}) \\ &= \Phi_f^{-1}(\underline{R}_{iv}(\Phi_f(\underline{R}_{bv}(A)))) \\ &= \Phi_f^{-1}(\Phi_f(\underline{R}_{bv}(\underline{R}_{bv}(A)))) \\ &= \underline{R}_{bv}(\underline{R}_{bv}(A)). \quad \blacksquare \end{aligned}$$

Example 4.5. Consider Example 1 in [18]. Given a crisp approximation space (U, R) , where

$$U = \{x_0, x_1, \dots, x_{10}\} \text{ and}$$

$$U / R = \{\{x_0, x_2, x_3\}, \{x_1, x_4, x_5\}, \{x_6, x_8, x_{10}\}, \{x_7, x_9\}\},$$

a bipolar-valued fuzzy evaluation about each house based on the size is defined in Table 4.6.

Table 4.7 shows the lower approximation and upper approximation of the bipolar-valued fuzzy set H about (U, R) . The corresponding interval-valued fuzzy set H and its lower approximation and upper approximation about (U, R) are shown in Table 4.8 and Table 4.9, respectively.

Table 4.6

Bipolar-valued fuzzy evaluations

U	x_0	x_1	x_2	x_3	x_4	x_5
H	(0.0,1.0)	(-0.2,0.7)	(-0.1,0.8)	(-0.5,0.5)	(-0.7,0.4)	(-1.0,0.0)
U	x_6	x_7	x_8	x_9	x_{10}	
H	(-1.0,0.0)	(-1.0,0.0)	(-1.0,0.0)	(-1.0,0.0)	(-1.0,0.0)	

Table 4.7

Approximations of the bipolar-valued fuzzy set H on (U, R)

	$\{x_0, x_2, x_3\}$	$\{x_1, x_4, x_5\}$	$\{x_6, x_8, x_{10}\}$	$\{x_7, x_9\}$
$\underline{R}_{bv}(H)$	(-0.5,0.5)	(-1.0,0.0)	(-1.0,0.0)	(-1.0,0.0)
$\overline{R}_{bv}(H)$	(0.0,1.0)	(-0.2,0.7)	(-1.0,0.0)	(-1.0,0.0)

Table 4.8

Interval-valued fuzzy evaluation H (consider $\lambda = 0.5$)

U	x_0	x_1	x_2	x_3	x_4	x_5
H	[0.5,1.0]	[0.4,0.85]	[0.45,0.9]	[0.25,0.75]	[0.15,0.7]	[0.0,0.5]
U	x_6	x_7	x_8	x_9	x_{10}	
H	[0.0,0.5]	[0.0,0.5]	[0.0,0.5]	[0.0,0.5]	[0.0,0.5]	

Table 4.9

Approximations of the interval-valued fuzzy set H on (U, R) (consider $\lambda = 0.5$)

	$\{x_0, x_2, x_3\}$	$\{x_1, x_4, x_5\}$	$\{x_6, x_8, x_{10}\}$	$\{x_7, x_9\}$
$\underline{R}_n(H)$	[0.25,0.75]	[0.0,0.5]	[0.0,0.5]	[0.0,0.5]
$\overline{R}_n(H)$	[0.5,1.0]	[0.4,0.85]	[0.0,0.5]	[0.0,0.5]

4.3 Bipolar-valued t -norm and t -conorm

In [49], Zhang introduced YinYang bipolar t -norm and t -conorm as follows, which is named bipolar-valued t -norm and t -conorm in the following, respectively.

Definition 4.7. (See Zhang, [49]) A bipolar-valued t -norm is a function $T^{B_F}: B_F \times B_F \rightarrow B_F$ satisfying the following

conditions, $\forall (x, y), (u, v), (a, b), (c, d) \in B_F$,

Commutativity: $\mathbb{T}^{B_F}((x, y), (u, v)) = \mathbb{T}^{B_F}((u, v), (x, y))$;

Monotonicity: $\mathbb{T}^{B_F}((x, y), (u, v)) \leq_{ibv} \mathbb{T}^{B_F}((a, b), (c, d))$ if $(x, y) \leq_{ibv} (a, b)$ and $(u, v) \leq_{ibv} (c, d)$;

Associativity: $\mathbb{T}^{B_F}((x, y), \mathbb{T}^{B_F}((u, v), (a, b))) = \mathbb{T}^{B_F}(\mathbb{T}^{B_F}((x, y), (u, v)), (a, b))$;

Identity law: $\mathbb{T}^{B_F}((a, b), (-1, 1)) = (a, b)$.

A bipolar-valued t-conorm is a function $\perp^{B_F}: B_F \times B_F \rightarrow B_F$ satisfying the following conditions, $\forall (x, y), (u, v), (a, b), (c, d) \in B_F$,

Commutativity: $\perp^{B_F}((x, y), (u, v)) = \perp^{B_F}((u, v), (x, y))$;

Monotonicity: $\perp^{B_F}((x, y), (u, v)) \leq_{ibv} \perp^{B_F}((a, b), (c, d))$ if $(x, y) \leq_{ibv} (a, b)$, $(u, v) \leq_{ibv} (c, d)$;

Associativity: $\perp^{B_F}((x, y), \perp^{B_F}((u, v), (a, b))) = \perp^{B_F}(\perp^{B_F}((x, y), (u, v)), (a, b))$;

Identity law: $\perp^{B_F}((a, b), (0, 0)) = (a, b)$.

It is worth that Null law referred in [49], i.e., $\mathbb{T}^{B_F}((a, b), (0, 0)) = (0, 0)$ (resp. $\perp^{B_F}((a, b), (-1, 1)) = (-1, 1)$), can be implied from Monotonicity and Identity law.

Interval-valued t-norm \mathbb{T} and interval-valued t-conorm \perp over $I^{(2)}$ can be restricted to $I_{0.5}^{(2)}$.

Definition 4.8. A t-norm \mathbb{T} over $[0, 1]$ (A t-conorm \perp over $[0, 1]$) is a commutative, associative mapping $\mathbb{T}: [0, 1]^2 \rightarrow [0, 1]$ (resp. $\perp: [0, 1]^2 \rightarrow [0, 1]$) which is increasing in both arguments and for which $\mathbb{T}(a, 1) = a$ (resp. $\perp(a, 0) = a$), $\forall a \in [0, 1]$.

The following is immediate from Definition 10 and Definition 11.

Theorem 4.5. If \mathbb{T}^- and \mathbb{T}^+ is two t-norms on $[0, 1]$ and for $(x, y), (u, v) \in B_F$, we define

$$\mathbb{T}^{B_F}((x, y), (u, v)) = (-\mathbb{T}^-(x, u), \mathbb{T}^+(y, v))$$

then \mathbb{T}^{B_F} is a bipolar-valued t-norm on (B_F, \leq_{ibv}) .

Proof. (i) The commutativity of \mathbb{T}^{B_F} on (B_F, \leq_{ibv}) is known directly from that of \mathbb{T} on $[0, 1]$.

(ii) $\forall (x, y), (u, v), (w, z) \in B_F$,

$$\begin{aligned} \mathbb{T}^{B_F}(\mathbb{T}^{B_F}((x, y), (u, v)), (w, z)) &= \mathbb{T}^{B_F}((-\mathbb{T}^-(x, u), \mathbb{T}^+(y, v)), (w, z)) \\ &= (-\mathbb{T}^-(\mathbb{T}^-(x, u), -w), \mathbb{T}^+(\mathbb{T}^+(y, v), z)) \\ &= (-\mathbb{T}^-(x, \mathbb{T}^-(u, -w)), \mathbb{T}^+(y, \mathbb{T}^+(v, z))) \\ &= \mathbb{T}^{B_F}((x, y), (-\mathbb{T}^-(u, -w), \mathbb{T}^+(v, z))) \\ &= \mathbb{T}^{B_F}((x, y), \mathbb{T}^{B_F}((u, v), (w, z))). \end{aligned}$$

Namely, \mathbb{T}^{B_F} has associativity on (B_F, \leq_{ibv}) .

(iii) Let $(x, y), (u, v) \in B_F$ and $(x, y) \leq_{ibv} (u, v)$, i.e., $x \geq u, y \leq v$. Then

$$\mathbb{T}^{B_F}((x, y), (w, z)) = (-\mathbb{T}^-(x, -w), \mathbb{T}^+(y, z)) \leq_{ibv} (-\mathbb{T}^-(u, -w), \mathbb{T}^+(v, z)) = \mathbb{T}^{B_F}((u, v), (w, z)).$$

(iv) $\forall (u, v) \in B_F$, $\mathbb{T}^{B_F}((u, v), (-1, 1)) = (-\mathbb{T}^-(u, 1), \mathbb{T}^+(v, 1)) = (u, v)$. ■

It follows similar theorem on t-conorm from Theorem 4.5.

Theorem 4.6. If \perp^- and \perp^+ is two t-conorms on $[0, 1]$ and for $(x, y), (u, v) \in B_F$, we define

$$\perp^{B_F}((x, y), (u, v)) = (-\perp^-(x, u), \perp^+(y, v))$$

then \perp^{B_F} is a bipolar-valued t-conorm on (B_F, \leq_{ibv}) .

Theorem 4.5 and theorem 4.6 tell us that bipolar-valued t-norm and bipolar-valued t-conorm on (B_F, \leq_{ibv}) can be induced from t-norm and t-conorm over $[0,1]$, respectively.

Example 4.6. In [49], Zhang presented some examples of bipolar-valued t-norm (t-conorm) on (B_F, \leq_{ibv}) . Table 4.10 shows relationship between bipolar-valued t-norm (t-conorm) on (B_F, \leq_{ibv}) and t-norm (t-conorm) on $[0,1]$, where for $x, y \in [0,1]$,

(1) t-norm and t-conorm of Zadeh, written as T_M and \perp_M , respectively:

$$x \wedge y = \min(x, y), \quad x \vee y = \max(x, y).$$

(2) Probability t-norm and t-conorm, written as T_p and \perp_p , respectively:

$$x \hat{\wedge} y = xy, \quad x \hat{\vee} y = x + y - xy.$$

(3) t-norm and t-conorm of Lukasiewitch, written as T_L and \perp_L , respectively:

$$x \odot y = \max(0, x + y - 1), \quad x \oplus y = \min(x + y, 1).$$

Table 4.10

Zhang's bipolar-valued t-norms (t-conorms) on (B_F, \leq_{ibv}) induced from t-norms (t-conorms) on $[0,1]$, where $(x, y), (u, v) \in B_F$

t-norm (t-conorm) on $[0,1]$	bipolar-valued t-norm (t-conorm) on (B_F, \leq_{ibv})
T_M	$(-(x \wedge u), y \wedge v)$ (10a) in [99]
T_p	$(-(x \times u), y \times v)$ (10b) in [99]
T_L	$(-(x \odot u), y \odot v)$ (10c) in [99]
\perp_M	$(-(x \vee u), y \vee v)$ (11a) in [99]
\perp_p	$(-(x \hat{\vee} u), y \hat{\vee} v)$ (11b) in [99]
\perp_L	$(-(x \oplus u), y \oplus v)$ (11c) in [99]

From above Table 4.10, bipolar-valued t-norms (t-conorms) on (B_F, \leq_{ibv}) can be induced from interval-valued t-norms (t-conorms) on $[0,1]$. Moreover, from Theorems 4.5 and 4.6, our induced construction of bipolar-valued t-norm and bipolar-valued t-conorm are richer. Table 4.11 lists out induced bipolar-valued t-norm and bipolar-valued t-conorm which are different from Zhang's in Table 4.10.

Table 4.11

This paper's bipolar-valued t-norms (t-conorms) on (B_F, \leq_{ibv}) induced from t-norms (t-conorms) on $[0,1]$, where $(x, y), (u, v) \in B_F$

t-norm (t-conorm) on $[0,1]$	bipolar-valued t-norm (t-conorm) on (B_F, \leq_{ibv})
$\mathbb{T}^- = \mathbb{T}_M, \mathbb{T}^+ = \mathbb{T}_P$	$(-(x \wedge u), y \times v)$
$\mathbb{T}^- = \mathbb{T}_M, \mathbb{T}^+ = \mathbb{T}_L$	$(-(x \wedge u), y \odot v)$
$\mathbb{T}^- = \mathbb{T}_P, \mathbb{T}^+ = \mathbb{T}_M$	$(-(x \times u), y \wedge v)$
$\mathbb{T}^- = \mathbb{T}_P, \mathbb{T}^+ = \mathbb{T}_L$	$(-(x \times u), y \odot v)$
$\mathbb{T}^- = \mathbb{T}_L, \mathbb{T}^+ = \mathbb{T}_M$	$(-(x \odot u), y \wedge v)$
$\mathbb{T}^- = \mathbb{T}_L, \mathbb{T}^+ = \mathbb{T}_P$	$(-(x \odot u), y \times v)$
$\perp^- = \perp_M, \perp^+ = \perp_P$	$(-(x \vee u), y \hat{+} v)$
$\perp^- = \perp_M, \perp^+ = \perp_L$	$(-(x \vee u), y \oplus v)$
$\perp^- = \perp_P, \perp^+ = \perp_M$	$(-(x \hat{+} u), y \vee v)$
$\perp^- = \perp_P, \perp^+ = \perp_L$	$(-(x \hat{+} u), y \oplus v)$
$\perp^- = \perp_L, \perp^+ = \perp_M$	$(-(x \oplus u), y \vee v)$
$\perp^- = \perp_L, \perp^+ = \perp_P$	$(-(x \oplus u), y \hat{+} v)$

4.4 Others

Besides, in 2000, Lee [24] used the name of bipolar-valued fuzzy sets to represent the following sets.

Definition 4.9. A bipolar-valued fuzzy set in the sense of Lee on X is a mapping

$$A: X \rightarrow [-1, 1].$$

Since power of $[-a, b]$ ($a \geq 0, b \geq 0$ and $|a| + |b| \neq 0$) is the same as one of $[0, 1]$, this means that bipolar-valued fuzzy sets in the sense of Lee and Zadeh's fuzzy sets are identical. Otherwise, for the given positive number n , we can define a kind fuzzy set by a mapping $A: X \rightarrow [-n, n]$, even $A: X \rightarrow [-a, b]$ ($a \geq 0, b \geq 0$ and $|a| + |b| \neq 0$).

Greco and Rindone [17] adopted the bipolar scale $[-1, 1]$ to present their results on fuzzy integrals in the Section 4 "bipolar-valued fuzzy integrals on the scale $[-1, 1]$ " in [17].

In 2013, Mesiarová-Zemánková and Ahmad [30] adopted the bipolar scale $[-1, 1]$ to present aggregation operator.

Definition 4.10. (i) A mapping $A: \bigcup_{n \in \mathbb{N}} [0, 1]^n \rightarrow [0, 1]$ is called an aggregation operator if

- (A1) A is non-decreasing, i.e., for any $n \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in [0, 1]^n$, $\mathbf{x} \leq \mathbf{y}$ it holds $A(\mathbf{x}) \leq A(\mathbf{y})$;
- (A2) $0, 1$ are idempotent elements of A , i.e., for any $n \in \mathbb{N}$, $A(0, 0, \dots, 0) = 0$, and $A(1, 1, \dots, 1) = 1$;
- (A3) for $n = 1$, $A(x) = x$ for all $x \in [0, 1]$.

(ii) A mapping $B: \bigcup_{n \in \mathbb{N}} [-1, 1]^n \rightarrow [-1, 1]$ is called a bipolar aggregation operator if

- (B1) B is non-decreasing, i.e., for any $n \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in [-1, 1]^n$, $\mathbf{x} \leq \mathbf{y}$ it holds $B(\mathbf{x}) \leq B(\mathbf{y})$;
- (B2) $0, 1, -1$ are idempotent elements of B , i.e., for any $n \in \mathbb{N}$, $B(0, 0, \dots, 0) = 0$, $B(1, 1, \dots, 1) = 1$ and $B(-1, -1, \dots, -1) = -1$;
- (B3) for $n = 1$, $B(x) = x$ for all $x \in [-1, 1]$.

Since bipolar-valued fuzzy sets in the sense of Lee and Zadeh's fuzzy sets are identical, condition $B(0, 0, \dots, 0) = 0$ of bipolar aggregation operator B is identical with condition $A(0.5, 0.5, \dots, 0.5) = 0.5$ of aggregation

operator A . In aggregation operator A , the condition $A(0.5, 0.5, \dots, 0.5) = 0.5$ is superfluous, so do the condition $B(0, 0, \dots, 0) = 0$ in bipolar aggregation operator B .

Mesiarová-Zemánková and Ahmad extended polar aggregation operator to m -polar aggregation operator and multi-polar aggregation operator, and proved a 1-polar aggregation operator and 2-polar aggregation operator are aggregation operators [30].

In 2004, zhang and Zhang [50] introduced the following concept which was discussed by Mesiarová-Zemánková and Ahmad [31] in 2014.

Definition 4.11. Given the ordered sets $B_n^- = \{-n, \dots, -1, 0\}$ and $B_n^+ = \{0, 1, \dots, n\}$, $1 \leq n < \infty$, the Cartesian product $B_n = B_n^- \times B_n^+ = \{(a_i, b_j)\}$ is a strict bipolar (crisp truth) space.

As discussed above, $B_n = B_n^- \times B_n^+$ is a discrete situation of $[-n, 0] \times [0, n]$, which is a generalization of $[-1, 0] \times [0, 1]$ used by bipolar-valued fuzzy sets.

5. Conclusions

In this paper, we mainly investigate the relationships between two concepts, bipolar-valued fuzzy sets and interval-valued fuzzy sets, and the following are main conclusions of this paper.

(1) A bipolar-valued fuzzy set is a special interval-valued fuzzy set, which is equivalent to an intersected interval-valued fuzzy set proposed in this paper. Figure 5.1 shows the relationship among fuzzy sets, bipolar-valued fuzzy sets, intersected interval-valued fuzzy sets, interval-valued fuzzy sets and intuitionistic fuzzy sets.

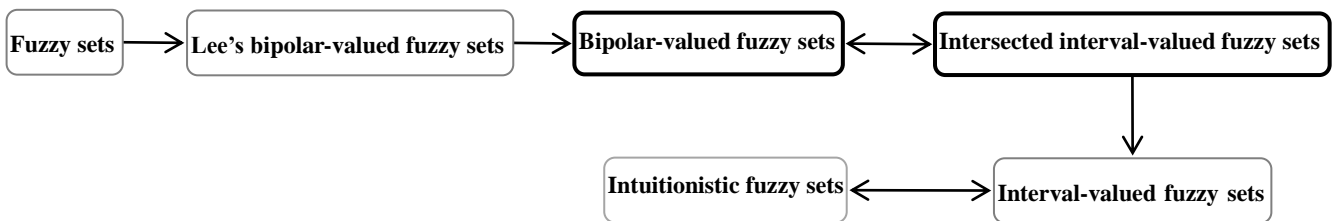


Figure 5.1. The relationship between bipolar-valued fuzzy sets and interval-valued fuzzy sets, where $\boxed{A} \rightarrow \boxed{B}$ means that A is B and $\boxed{A} \leftrightarrow \boxed{B}$ means that A is equivalent to B. For example, $\boxed{\text{Fuzzy sets}} \rightarrow \boxed{\text{Bipolar-valued fuzzy sets}}$ means that a fuzzy set is a bipolar-valued fuzzy set, $\boxed{\text{Bipolar-valued fuzzy sets}} \leftrightarrow \boxed{\text{Intersected interval-valued fuzzy sets}}$ means that a bipolar-valued fuzzy set is equivalent to an intersected interval-valued fuzzy set.

(2) All operations and properties of bipolar-valued fuzzy sets can be induced from corresponding those of interval-valued fuzzy sets and fuzzy sets.

(3) Some existing concepts and conclusion on bipolar-valued fuzzy sets can be induced from the corresponding concepts on interval-valued fuzzy sets. bipolar-valued fuzzy graphs and bipolar-valued rough fuzzy sets on bipolar-valued fuzzy sets can be induced from interval-valued fuzzy graphs and interval-valued rough fuzzy sets on interval-valued fuzzy sets, respectively. bipolar-valued t-norm (t-conorm) can be induced from t-norm (t-conorm) on I , but not from interval-valued t-norm (t-conorm).

(4) The work of bipolar-valued fuzzy sets should focus on semantic studies, intrinsic advantage and distinctive applications.

(5) In applications, do they not have, in the case of ranking decision objects, the rank-reversal feature that appears after adding a new object to the original set of compared objects? This is an issue to be explored in future work.

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Declarations

Conflict of interest The author declares that they have no conflict of interest.

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