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REVIEW ARTICLE

State-of-the-Art Constitutive Modelling of Frozen Soils

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Abstract

In recent decades, the constitutive modelling for frozen soils has attracted remarkable attention from scholars and engineers due to the continuously growing constructions in cold regions. Frozen soils exhibit substantial differences in mechanical behaviours compared to unfrozen soils, due to the presence of ice and the complexity of phase changes. Accordingly, it is more difficult to establish constitutive models to reasonably capture the mechanical behaviours of frozen soils than unfrozen soils. This study attempts to present a comprehensive review of the state of the art of constitutive models for frozen soils, which is a focal topic in geotechnical engineering. Various constitutive models of frozen soils under static and dynamic loads are summarised based on their underlying theories. The advantages and limitations of the models are thoroughly discussed. On this basis, the challenges and potential future research possibilities in frozen soil modelling are outlined, including the development of open databases and unified constitutive models with the aid of advanced techniques. It is hoped that the review could facilitate research on describing the mechanical behaviours of frozen soils, and promote a deeper understanding of the thermo-hydro-mechanical (THM) coupled process occurring in cold regions.

1 Introduction

Frozen soils are frequently encountered in various engineering activities. On the one hand, permafrost covers 24% of the earth land surface, accounting for up to 35,760,000 km^2 [31]. Construction and maintenance of railways, highways as well as pipelines in permafrost regions are inevitably involved with frozen soils. On the other hand, artificial ground freezing (AGF) techniques are widely applied in underground constructions [72, 82, 188], where frozen soils are artificially created to serve as temporary support and groundwater sealing through enhancing the soil modulus and minimising its hydraulic conductivity. Reasonably capturing the mechanical behaviours of frozen soils through robust constitutive models is an urgent task faced by scientists and engineers but remains challenging.

Due to the sensitivity of frozen soil to climate and environmental conditions [104, 142], variations in the mechanical properties of frozen soils can give rise to multiple engineering and environmental issues, such as ground subsidence floor risk, slope instability, infrastructure damage, and climate change impacts (e.g., [12, 26, 92, 103, 109, 137, 175, 187]). The occurrence of these disasters associated with frozen soil has not only garnered significant attention from researchers across various engineering fields, but also prompted corresponding decision-making and research initiatives by government departments. Understanding mechanical behaviours and properties, such as strength, deformation, and creep of frozen soils, plays a pivotal role in the investigation, design, construction, and operation of engineering projects in cold regions undergoing a gradual degradation caused by climate change and human activities [175, 187].

In addition, a large number of significant infrastructure and energy pipeline projects are being implemented in cold regions [175], such as the Qinghai-Tibet Railway (1142

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km, Golmud to Lhasa, China [14]), China-Russia Crude Oil Pipeline (953 km, from Skovorodina, Russia to Daqing, China [59]), Alaska Highway (2232 km, British Columbia, Canada to Alaska, USA [18]), and so on. These endeavours highlight the global scale and importance of construction initiatives in cold regions and emphasise the need for specialised engineering approaches and meticulous consideration of the unique and complex mechanical behaviours of frozen soils. Consequently, it is crucial to consider the mechanical behaviours of frozen soils when undertaking engineering activities in cold regions, particularly in light of the potential risks posed by deformation, to ensure safety and provide guidance for minimising these frozen soil engineering disasters and preventive measures for these crucial projects.

The mechanical behaviours of frozen soils are remarkably different from those of unfrozen soils owing to the existence of ice that can significantly alter the mechanical performance of frozen soils [30, 48, 65, 148]. Understanding the distinct mechanical responses of frozen soils is therefore essential for effectively addressing these engineering problems and ensuring the long-term performance and safety of infrastructure in cold regions. Accordingly, the constitutive model is an efficient and theoretical tool to quantitatively describe the complex behaviours of frozen soils. Up to now, tremendous constitutive models for unfrozen soils and other geomaterials have been established and applied in manifold engineering fields [15, 16, 144, 161, 185]. However, the constitutive models of frozen soils differ significantly from those applied to unfrozen soils due to the distinct features of frozen soils induced by ice, such as temperature dependency, time dependency, heterogeneity, and ice content. The unique mechanical behaviours of frozen soils are summarised as follows. (1) Frozen soils display a high level of temperature dependency; hence, the constitutive models for frozen soils should consider this temperature dependency, which is not usually necessary for unfrozen soils. (2) Frozen soils exhibit time-dependent behaviour (e.g., creep), which can substantially impact the deformation of frozen soils. (3) When water freezes into ice and expands, it induces volume change within the soil matrix, a phenomenon not commonly observed in unfrozen soils [107, 113]. Besides, the ice within frozen soils can significantly affect mechanical behaviours, which enhances the cohesive strength of soil solids so that the tensile strength of frozen soil is nonzero [53]. It has been reported that the enhanced strength of frozen soils stems from the intrinsic strength of the soil skeleton and the cementation of ice [9, 41, 55, 108]. (4) Frozen soils are prone to be highly heterogeneous and anisotropic owing to the impact of direction and connectivity of ice lenses [111].

Therefore, a reasonable constitutive model of frozen soils is critical for accurately describing and predicting the mechanical behaviours of frozen soils, which is capable of providing theoretical guidelines for the safety and stability of engineering projects in cold regions and is beneficial for the analysis of the thermo-hydro-mechanical (THM) coupled issues in geotechnical engineering. At present, numerous constitutive models have been proposed to capture and describe the complex mechanical behaviours of frozen soils based on different theories, but few of them can comprehensively account for these aforementioned unique characteristics of frozen soils. Given that numerous studies on constitutive models of frozen soils, herein, the distinguishing investigations in recent two decades have been checked according to the Web of Science (see Fig. 1, 13 articles before 2003). It can be noted that the publications on constitutive models of frozen soils are growing exponentially, while these constitutive models for frozen soils have not yet been adequately summarised.

This study comprehensively reviews the current investigations on the constitutive models of frozen soils under static and dynamic loadings. A systematic review of constitutive models of frozen soils upon static and dynamic loadings are respectively presented in Sect. 2 and Sect. 3, according to their underlying theories. Subsequently, the advantages and limitations of each type of constitutive model are discussed in Sect. 4. The last section highlights the critical findings and outlines future research trends in the field of constitutive models for frozen soils. This study contributes to a better understanding of the mechanisms governing the multiphysical field coupling of frozen soils and offers valuable references for engineering design and maintenance.

2 Constitutive Models for Static Loading Condition

Herein, the constitutive models for frozen models under static loadings are reviewed, which are divided into ten categories based on their underlying principles.

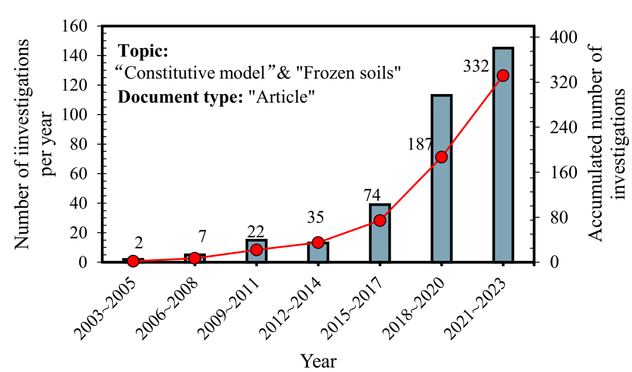


Fig. 1 Number of publications related to constitutive models of frozen soils in recent two decades (from 2003 to 2023)

Table 1 Typical investigations on empirical models of frozen soils under static loading

Tests	References	Equations	Soil types	Remarks
Uniaxial compression test	Vialov, 1959 [120]	$\sigma = \mathrm{A}\varepsilon^{\mathrm{m}}$	Frozen soil	–
	Liu and Peng, 2009 [78]	$\sigma = \begin{cases} E\varepsilon\varepsilon \le \varepsilon_y \\ \frac{E\varepsilon}{A+B\varepsilon+C\varepsilon^2+D\varepsilon^3}\varepsilon > \varepsilon_y \end{cases}$	Thawing soil	Considering the effects of initial water content, cooling temperature, and thawing temperature at the top of the sample
	Cai et al., 2017 [7]	$\sigma_1 = E(T)\varepsilon_1 + f(\varepsilon_1, \dot{\varepsilon}_1)g(\dot{\varepsilon}_1)h(T)$	Frozen loess	Considering the effects of loading rate and T
Creep test	Fish et al., 1984 [28]	$\varepsilon \approx \frac{\dot{\varepsilon}_0}{1-\eta}t^{1-\eta}\exp\left(\frac{t\eta}{t_{\mathrm{m}}}\right)$	Frozen soil	Only valid for extremely small values of t/t_{m}, and cannot satisfy the condition where the strain rate reaches a minimum at $t = t_{\mathrm{m}}$
	Gardner et al., 1984 [36]	$\frac{\varepsilon}{\varepsilon_{\mathrm{m}}-\varepsilon_0} = \left(\frac{t_{\mathrm{c}}}{t_{\mathrm{m}}}\right)^c \exp\left[\left(\sqrt{c}-c\right)\left(\frac{t}{t_{\mathrm{m}}}-1\right)\right]$	Frozen soil	Reliable estimation for the entire creep curve, exhibiting considerable fitness at the inflexion point
	Arenson and Springman, 2005 [3]	$\varepsilon = A\sigma_e^{3\omega_i}; \sigma_e = \frac{\sqrt{2}}{2}q$ $\log A = \frac{2}{1+\lvert T\rvert} + \log\left(5 + 10^{-11}\exp\left(-10.2\omega_i\right)\right)$	Ice-rich frozen soils	For frozen soils at temperatures close to 0 °C
	Yin et al., 2013 [162]	$\dot{\varepsilon} = A(T)(1-\exp(-bt))\left(3J_2\right)^{\frac{m-1}{2}}\sigma$	Frozen clay	Considering the impact of temperature

σ is stress; σ_{f} is long-term strength; σ_{e} is stress-invariant; σ_1 is axial strain; ε is strain; ε_1 is axial strain; ε_{y} is yield strain; $\dot{\varepsilon}$ is strain rate; $\dot{\varepsilon}_0$ is a parameter calculated at $t = t^* = 1$ min; t_{m} is the inflexion point of each creep curve where the strain rate reaches a temporary minimum; η is a dimensional parameter; T is temperature; A, B, C, D, m are material parameters; α, β and r are empirical parameters related to soils; ω_{i} is ice content; c is dimensional parameter describing creep curve shape; J_2 is second invariant of the deviatoric stress tensor

2.1 Empirical Models

Empirical constitutive models are primarily derived by fitting experimental data of frozen soils at specific conditions, which are easy to compute and often used by engineers [124]. Early scholars proposed empirical constitutive models of frozen soils by fitting the experimental data from uniaxial compression and creep tests under static loadings (see Table 1). Vialov [120] established a power function relation between stress and strain based on uniaxial compression tests. However, Zhu et al. [192] indicated that the power function is not suitable for describing the constitutive relations for frozen soils since some factors (e.g., temperature and water content) could significantly affect the relations. Accordingly, Cai et al. [7] proposed a novel constitutive model for frozen soil by considering the impacts of temperature and loading rate. Besides, Liu and Peng [78] also established an empirical model to describe the stress–strain relation for thawing soils involving the effects of water content, cooling and thawing temperature.

In addition, to describe the creep deformation of frozen soils, several empirical models were also developed. Fish [28] introduced an approximate equation to represent individual creep curves, illustrating the relationship between strain rate (ε) versus time (t). However, the applicability of this model was limited as it was only valid for extremely small values of t/t_m (where t_m represents the inflexion point of each creep curve, indicating the temporary minimum strain rate). Moreover, it failed to satisfy the condition where the strain rate reaches a minimum at $t=t_m$. To address these limitations, Gardner et al. [36] proposed a novel form of $\dot{\varepsilon}$-t. Arenson and Springman [3] derived a straightforward formula to describe the creep behaviours of ice-rich frozen soils near the melting temperatures of ice based on a series of laboratory investigations. Besides, Yin et al. [162] developed a negative exponential nonlinear equation to capture the creep deformation of frozen clay under triaxial stress.

Table 1 summarises typical investigations of the empirical constitutive models of frozen soils. It can be noted that involving less underlying physical mechanics of frozen soils limits the accuracy and applicability of empirical constitutive models to various soil types and conditions. Therefore, empirical models of frozen soils are efficient tools to evaluate the mechanical behaviours of frozen soils, but they should be employed with more caution and validated with experimental data. In addition, frozen soils often suffer complex stress conditions, while the empirical models fail to reflect the mechanical characteristics of frozen soils under generalised stress conditions.

2.2 Elastoplastic Modelling

Elastoplastic models of frozen soils have been proposed and developed in recent years since they can capture frozen soils' elastic (reversible deformation) and plastic behaviours (irreversible deformation). The advancements of elastoplastic constitutive models of frozen soils are analysed herein.

The plastic relation involves three essential contents: yield function, flow rule, and hardening law of yield surface. The yield function represents the initiation of plastic deformation, while the flow rule, in conjunction with the plastic potential function, governs dilatancy, defining the evolution and direction of plastic deformation. The hardening law describes the magnitude of plastic deformation [83]. Three variables (i.e., shear strain, volumetric strain and plastic work) are often utilised as hardening parameters in the elastoplastic constitutive models of soil [46]. Since the plastic work involves two plasticity variables (volumetric strain and shear strain), adopting the plastic work as a hardening parameter is likely to reduce the number of parameters that need to be determined. Based on the classical elastoplastic theory, the total strain increment ($d\varepsilon$) is the summation of elastic strain increment ($d\varepsilon^e$) and plastic strain increment ($d\varepsilon^p$).

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \tag{1}$$

Fish [29] initially determined a plastic yield surface as the critical state line (CSL) based on experimental results. However, this model did not consider the hardening and softening behaviours of frozen soils, leading to an inability to accurately predict volumetric strain in frozen soils. To solve this limitation, Lai et al. [53] proposed an elastoplastic constitutive model of frozen sandy soil by adopting plastic work as the hardening parameter where the plastic volumetric strain and shear strain were considered. In the work of Lai et al. [53], the *K-G* model was used to compute the elastic part of strain, and an ellipse yield function was proposed to describe the plastic potential loci based on Drucker's postulate. However, the associated flow rule employed in the model of Lai et al. [53] posed challenges in accurately capturing the volumetric deformation of frozen soil. To address this issue, Yang et al. [158] proposed an elastoplastic model for frozen silt soil that utilised a nonassociated follow rule to describe the stress–strain relationships under high confining pressure. An elastoplastic constitutive model of frozen silt was also developed by Lai et al. [54], which incorporates an elliptical yield surface for the compressive mechanism along with a nonassociated flow rule and two parabolic yield surfaces for the shear mechanism with nonassociated flow rules.

Table 2 Summary of existing elastoplastic models for frozen soils under static loading

References	Soil types	Tests	Influencing factors	Plastic parts			Features
				Yield function	Flow rule	Hardening parameter	
Nishimura et al., 2009 [96]	Frozen soils	–	–	Ellipse (1)	A	ε_v^p(1)	Analogy with the modified Cam-Clay model; Involves two stress variables
Lai et al. 2009 [53]	Frozen sandy soil (– 6°C)	T	σ_3 (1 ~ 18 MPa)	Ellipse (1)	A	W_p (1)	Describing the hardening and softening; associated flow rule results in difficulty in predicting the plastic volumetric deformation
Yang et al. 2010b [158]	Frozen silt (– 6°C)	T	σ_3 (4 ~ 14 MPa)	#Ellipse & parabolic (2)	N	W_p (1)	Describing the mechanical response under high σ_3
Lai et al., 2010 [53]	Frozen silt (– 6°C)	T	σ_3 (1 ~ 14 MPa)	# Ellipse & parabolic (2)	N	$f(\varepsilon_v^p), f(\varepsilon_s^p)$ (2)	Describing the mechanical response under low and high σ_3
Rotta Loria et al., 2017 [83]	Frozen silt (– 6°C)	T*1	σ_3 (1 ~ 14 MPa)	#Ellipse & parabolic (2)	A	$f(\varepsilon_v^p), f(\varepsilon_s^p)$ (2)	With fewer material parameters than Lai et al. [54]
Xu et al., 2017 [153]	Frozen Helin loess	UC	$\dot{\varepsilon}$(5E-5/s ~ E-2/s), T (− 2 ~ − 7°C)	Mises (1)	A	$f(T, \varepsilon_s^p)$(1)	The hardening parameter is an empirical function of temperature and plastic shear strain;
Chang et al., 2019 [10]	Frozen saline coarse sandy soil (−6°C)	T	σ_3 (1 ~ 16 MPa), f_{salt} (0 ~ 3.5%)	(3)	N	$f(\varepsilon_v^p), f(\varepsilon_s^p)$ (2)	Three yield surfaces consider considering the breakage, compression and shear mechanism
Shastri et al., 2021 [110]	Undisturbed natural frozen soil	T	T (− 6 ~ − 26°C), σ3 (0 ~ 54.6 MPa)	(1)	A	$f(d\varepsilon_v^p)$ (1)	Named Hiss sub-loading model; overcoming the difficulty in describing the shear behaviours of natural frozen soils using the elliptical yield surface by modifying the constitutive model by Nishimura et al. [96]
Sun and Zhou, 2021 [118]	Frozen soil (silt sand, sand, saline sandy soil)	T*2	(a) T (− 4, − 6°C) and σ_3 (0.3 ~ 1.0 MPa); (b) T (− 6°C) and σ_3 (3 ~ 7 MPa); (c) T (− 6°C) and σ_3 (1 ~ 6 MPa); (d) T (− 1 ~ − 10°C) and σ_3 (1 MPa)	(3)	A	$f(d\varepsilon_v^p), f(H), f(d\varepsilon_v^{mp}, d\varepsilon_v^{sp})$ (3)	Three yield surfaces include reference yield surface, subloading yield surface, and cryogenic suction yield surface

Table 2 (continued)

References	Soil types	Tests	Influencing factors	Plastic parts			Features
				Yield function	Flow rule	Hardening parameter	
Yu et al., 2022 [165]	Frozen and unfrozen soils	T, FT*3	T (−4°C and −6°C) σ_3 (0.3, 0.6, 0.8, and 1.0 MPa)	$F(p, q, p_c, \theta)$ (1)	N	$f(p, q, p_c, \theta)$ (1)	Establishing a unified model for frost heave and thaw consolidation, which can smoothly transit between frozen and unfrozen/thaw soils, based on the CASM model for unfrozen soil [164]

T is triaxial test; UC is uniaxial compression test; σ_3 is confining pressure; $\dot{\varepsilon}$ is strain rate; T is temperature; f_{salt} is salt content; A is associated flow rule; N is nonassociated flow rule; ε_v^p is plastic volumetric strain; ε_s^p is plastic shear strain; $\mathrm{d}\varepsilon_v^{mp}$ and $\mathrm{d}\varepsilon_v^{sp}$ are respectively the plastic strain induced by the mechanical process and cryogenic suction; W_{p} is the plastic work; p is the hydrostatic pressure; q is deviatoric stress; p_{c} the preconsolidation pressure of frozen soil; θ is temperature related state variable; F is yield function; *1 represent the triaxial tests conducted by [54]; *2a~c indicates the triaxial tests from Xu [149], Lai et al. [54], Lai et al. [57] and Xu et al. [151]; *3 indicates triaxial tests conducted by Xu [149] and freeze–thaw tests of Athabasca silty clay conducted by Nixon and Morgenstern [97]; # indicates the ellipse yield surface stands for compression mechanism and parabolic yield surface stands for shear mechanism and dilatation at the strain-softening stage. The numbers in brackets in "Yield function" and "Hardening parameter" respectively represent the number of yield functions and hardening parameters

Based on experimental data from triaxial tests conducted by Lai et al. [54], Rotta Loria et al. [83] proposed a non-linear constitutive model for frozen silt by modifying the elastoplastic constitutive model from [54] and adopting the associated flow rule, whose model reduces the number of material parameters and facilitates the practical application of the constitutive model. Xu et al. [153] proposed an elastoplastic model to describe the strength and deformation behaviours of Helin loss under varying temperature and strain rates based on a series of uniaxial compression tests, and the material parameters were obtained by fitting the experimental data. To consider the effect of breakage, an elastoplastic constitutive model for frozen saline soil was established by Chang et al. [10]. Based on a series of triaxial tests with varying temperatures and confine pressures, Shastri et al. [110] derived a Hiss sub-loading model to address the mechanical behaviours of undistributed frozen soils. Yu et al. [165] extended the one-dimensional (1D) elastoplastic model (i.e., Clay and Sand Model, CASM [164]) to describe the mechanical behaviours of both unfrozen soils and frozen soils by considering the ice-bonding impact and using the conception of residual stress. Sun and Zhou [118] built an elastoplastic model for frozen soils by considering three yield surfaces (i.e., reference yield surface, subloading yield surface, and cryogenic suction yield surface). In their model, cryogenic suction and solid-phase stress were employed to reveal the impacts of confining pressure and temperature on the mechanical behaviours of frozen soil, and a nonlinear relationship between cryogenic cohesion and cryogenic suction was established to account for the phenomena of the strength increase with decreasing temperature. Furthermore, the validation of this model was implemented by using four triaxial tests on different frozen soils from other scholars.

Table 2 lists the typical elastoplastic models for frozen soils, where the primary features are also summarised. In these studies, the *K-G* model is widely used to calculate the elastic part of strain. Some scholars [53, 83] assumed that the bulk modulus (i.e., K) is a power function of σ_3, while G is the parabola function of σ_3. Yang et al. [158] employed the exponential function of σ_3 to calculate K and a parabola function of σ_3 to determine G. Chang et al. [10] adopted exponential functions to describe the relationships between K/G and σ_3. In contrast, Xu et al. [153] derived the empirical relationships between K/G and strain rate by fitting data from uniaxial compression tests. It can be noted from Table 2 that existing elastoplastic models for frozen soils are mainly obtained based on triaxial tests of artificial frozen soils and involve the impact of confining pressure, temperature and strain rate, in addition to the model of Shastri et al. [110] who proposed elastoplastic model for undisturbed natural frozen soil. Besides, the material parameters included in these models are generally obtained by fitting experimental

Table 3 Summary of viscoplastic modelling for frozen soils under static loading

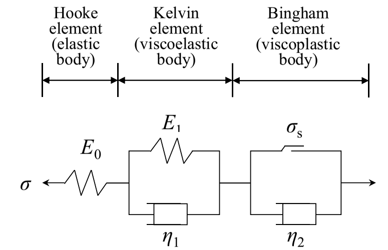
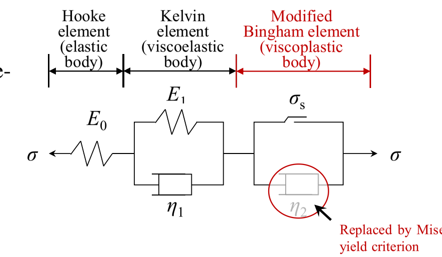
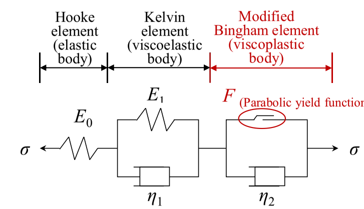
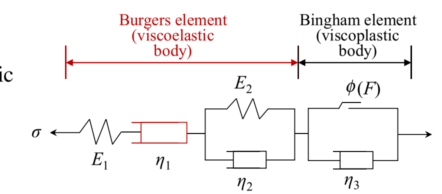
References	Soil types	Tests	External conditions	Remarks	Schematic diagrams
Nishihara, 1958 [95]	–	–	–	It is not suitable to describe the accelerating creep and failure process of frozen soils, since its elements are linear	
Wang et al., 2006 [123]	–	–	–	Modified Nashihara model by placing the Newtonian body with Mises yield criterion; Solved the issue that the Nishihara model is not capable of describing the creep at three stages under high-stress conditions	
Li et al., 2011 [60]	Clay	Triaxial creep test	T = − 5, − 10, − 15, − 20, − 25°C, σ_3 = 0, 7, 8, 9 MPa	Modified Nishihara's model by applying a parabolic yield function to define a viscoplastic body	
Wang et al., 2014 [124]	Fine sand	Direct shear creep test	T = − 2, − 5, − 10, − 25°C	Combining Burgers model (viscoelastic part) and Bingham body (viscoplastic part)	 The associated flow rule was employed where the viscoplastic potential function possesses the same form as the yield function

Table 3 (continued)

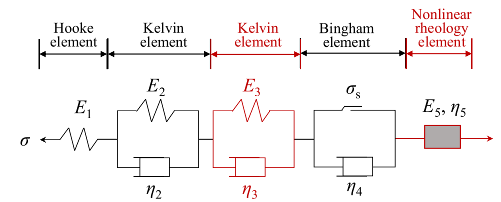
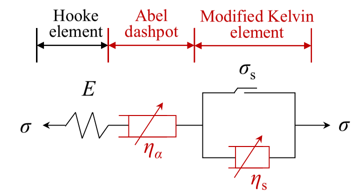
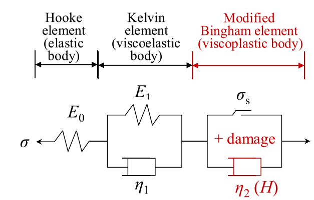
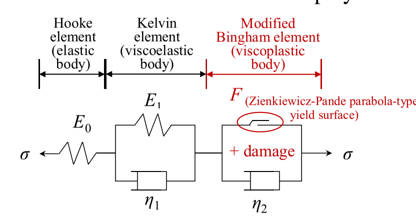
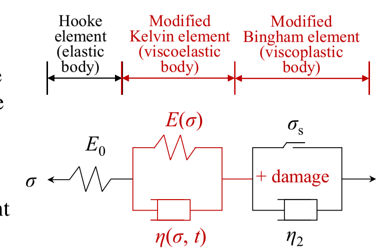
References	Soil types	Tests	External conditions	Remarks	Schematic diagrams
Yang et al., 2014 [159]	Silt	Triaxial creep test	$T = -8°C$, $\sigma_3 = 1, 2, 3$ MPa	Modified Nishihara's model by adding a Kelvin body and a nonlinear rheology element in series	
Hou et al., 2016 [43]	Silty clay mixed with quartz sand	Triaxial creep test	$T = -10°C$, $\sigma_3 = 0.3, 1.4, 6$ MPa	Modified Nishihara's model where Newton dashpot was replaced by Abel dashpot; A fractional constitutive model that considers the strengthing and weakening effect	 The associated flow rule was employed
Hou et al., 2018 [44]	Silty clay mixed with coarse grains	Triaxial creep test	$f_c = 0, 0.2, 0.4, 0.6$	Modified Nishihara's model where the hardening and damageeffects were incorporated into Bingham's model	 The associated flow rule was employed
Li et al., 2018 [62]	-	-	-	Modified Nishihara's model where the Saint–Venant body model was replaced by Zienkiewicz-Pande parabola-type yield surface	
Zhu et al., 2019 [196]	Sand	Triaxial creep test	$T = -0.5, -0.9, -1.2, -1.5, -2°C$, $\sigma_3 = 0.1$;*3	Modified Nishihara's model where E in the viscoelastic element was modified to be a function of stress, η in the viscoelastic element was revised by considering the effect of time and stress, and the viscoplastic element was improved by adding a damage variable	

Table 3 (continued)

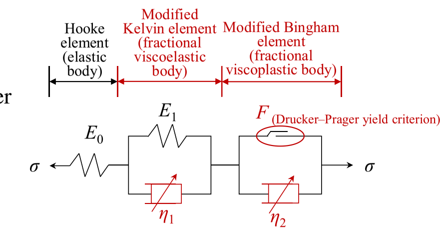
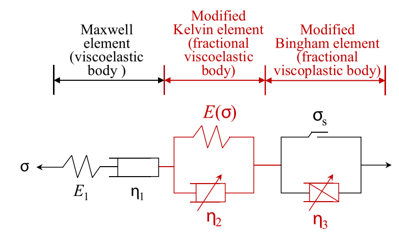
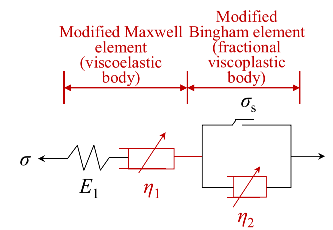
References	Soil types	Tests	External conditions	Remarks	Schematic diagrams
Li et al., 2020a [64]	Clay	Triaxial creep test	$T = -5, -10°C$, $\sigma_3 = 0 \sim 7$ MPa	Modified Nishihara's model where Newton dashpot was replaced by Abeel dashpot and Drucker-Prager yield criterion was used to describe creep yield	
Zhang et al., 2022 [176]	silt with sand	Triaxial creep test	$T = -5, -7, -10, -15°C$, $\sigma_3 = 0.5, 1, 1.5$ MPa	Modified Nishihara's model where the viscous body in the viscoelastic part and viscoplastic part were respectively replaced by the fractional viscous body and its improved version	
He et al., 2023a [40]	clay	Triaxial creep test	$T = -1°C$, $\sigma_3 = 1$ MPa	Consisting of a modified Maxwell body (a Hooke body and a fractional viscoelastic body) and a modified fractional Bingham body; Considering the microcosmic breakage mechanism	
Cai et al., 1990 [6]	Silty soil	Constant stress creep test	$T = -1, -3, -5°C$, stress = 0.448, 0.686, 1.127, 1.764 MPa)*[1]	Strain rate was divided into two parts: attenuation and nonattenuation creep Assumed that the frozen soils obey the Mises yield criterion and ignored the impact of hydrostatic pressure	The creep deformation was divided into attenuation creep and nonattenuation creep; Assumed that the frozen soils obey the Mises yield criterion and ignored the influence of hydrostatic pressure; Employed associated flow rule and isotropic assumption
Ghoreishian Amiri et a., 2016 [37]	Sand	Uniaxial compression tests	$T = -5, -15°C$, constant stress tests (stress = $1 \sim 8$ MPa)*[2]	Being developed within the framework of overstress theory that involves two stress-state variables; Dividing the total stress into fluid pressure, solid phase stress, and cryogenic suction	Based on the unsaturated soil modelling framework (i.e., Barcelona Basic Model, BBM); Involved two independent stress-state variables (i.e., cryogenic suction and solid phase stress); The associated flow rule was employed

Table 3 (continued)

References	Soil types	Tests	External conditions	Remarks	Schematic diagrams
Wang et al., 2020b [128]	Loess	Triaxial creep test	$T = -15°C$, $\sigma_3 = 0.3$ MPa, shear stress = 4.25 ~ 9.57 MPa	The creep model was derived by considering the internal creep deformation mechanism from a micro perspective	The viscoplastic properties of ice particles and soil matrix were considered; The ice was assumed to be associated with the flow rule

Notes: σ_3 is confining pressure; T is temperature; f_c is content of coarse grains; E is elastic modulus; η is coefficient of viscosity; σ is applied stress; $*^1$ and $*^2$ indicate the tests conducted by Zhu and Carbee [190] and Eckardt [24], respectively. * indicates that the frozen soils measured by Zhu et al. [196] with different particle distribution and dry density (i.e., content of particles with 0.075–1 mm size were 0.54, 0.6, 0.69; dry density was 1.75 and 1.92 g/cm^3). Table cells shaded in grey colour represent that the flow rule was incorporated into the viscoplastic model

data. When attempting to describe the mechanical behaviours of frozen soils under different external conditions, these constitutive models with complex mathematical formulations relying on many material parameters are adopted, which leads to the challenging and time-consuming analyses of practical engineering. Therefore, developing constitutive models with fewer material parameters considering various factors such as temperature, ice content, and salinity is more attractive, as they can capture the mechanical behaviours of frozen soils under complex conditions.

2.3 Viscoplastic Modelling

Frozen soils exhibit significant creep deformation under complex external loadings due to the viscosity of ice and unfrozen water [35, 160]. Therefore, it is essential to incorporate viscous behaviour into constitutive models for frozen soils to accurately describe the relationship between stress, strain, and time.

The rheological model, as a common type of creep model, has been established and applied to predict the creep deformation of geomaterials, which consists of several rheologic elements, e.g., Hooke elastic body (spring), Newton viscous body (dashpot) and Saint–Venant rigid plastic body (slider). A compilation of typical rheological elements and their combinations is provided in Appendix Table A. Furthermore, Table 3 summarises typical creep models of frozen soils based on viscoplastic theory and their primary features. As a classical model, the Nishihara model [95] consists of the Hooke element, Kelvin and Bingham model in series, which has been widely used to describe the transient and steady creep process of frozen soils. However, the linear constitutive elements of Nishihara model are inadequate for capturing nonlinear rheological characteristics, such as accelerating creep and failure processes. To address these limitations, as shown in Table 3, researchers have improved Nishihara model [95] by adding rheological bodies or modifying specific elements (e.g., [43, 44, 60, 62, 64, 141, 159, 176, 196]), as well as combining typical rheological models (e.g., [40, 124, 193]). For example, Li et al. [60] modified Nishihara's model by applying a parabolic yield function to define a viscoplastic body. Yang et al. [159] established a generalised Burgers model by combining Kelvin's body and Nishihara's model to address models that fail to describe the accelerated creep stage. Hou et al. [43] developed a novel creep model involving a hardening parameter and a damage variable to account for the strengthening and weakening effects. Zhu et al. [196] developed a nonlinear creep model highlighting the impacts of temperature, dry density and grain size distribution. Specifically, they improved the elastic modulus of viscoelastic elements by considering the stress dependence, revised the coefficient of viscosity in the viscoelastic element to account for time and stress effects, and enhanced the viscoplastic element by incorporating a damage variable.

Overall, these rheological models have relatively straightforward physical interpretations and are prone to be easily programmed. However, it is worth noting that combined elements were initially proposed to capture the rheological characteristics of other materials (e.g., metals and rubbers) at temperatures close to their melting points. Besides, when modelling the mechanical behaviours of frozen soils, the presence of heterogeneity and disordered internal structure poses challenges in determining the most appropriate combinations of elements. As a result, caution should be exercised when employing this approach. Simple combined models may struggle to accurately depict the actual mechanical behaviours of frozen soil, while complex combined models could introduce significant mathematical complexities.

Notably, the rheological elementary models derived from viscoplastic theory are primarily 1D models; therefore, the flow rule should be incorporated to describe frozen soils' mechanical behaviours under three-dimensional (3D) stress conditions. Accordingly, some scholars extended their proposed rheological elementary models into 3D models by employing the associated flow rules to reflect 3D stress state of frozen soils (e.g., [43, 44, 196]) or established novel models for frozen soils by combing viscoplastic theory and other principles. Specifically, Cai et al. [6] established a model to simulate the creep deformation of frozen soils under monotonic and cyclic loadings. In their model, the creep deformation was divided into attenuation creep (described by the linear viscoelastic model) and nonattenuation creep (represented by the overstress viscoplastic model). However, since this model assumed that the frozen soils obey the Mises yield criterion and ignored the influence of hydrostatic pressure, their predicted results are inconsistent with most of the experiment data and cannot describe the plastic volumetric strain of frozen soils. Ghoreishian Amiri et al. [37] developed a model based on the unsaturated soil modelling framework (i.e., Barcelona Basic Model, BBM), whose model involved two independent stress-state variables (i.e., cryogenic suction and solid phase stress). The calculated results from their model were compared with experimental data from triaxial, uniaxial, and creep tests, demonstrating its ability to describe mechanical behaviours from the frozen state to the unfrozen state using a set of parameters. However, this model has some limitations in describing the curves of σ-ε and volumetric changes under the combined impacts of temperature and confining pressure. For insurance, it would produce overestimations for the critical and peak strength of frozen soils due to the linear relations between cryogenic cohesion and suction [118].

In the aforementioned investigations, it is evident that the majority of studies on the viscoplastic model of frozen soils have treated frozen soil as homogeneous solid material and primarily focused on the macroscopic deformation and evolution of damage. However, the creep models based on the micromechanics of creep deformation for the frozen soils are limited, particularly regarding the effect of ice content and temperature. Frozen soil is a typical multiphase material, and its macro responses stem from the collective contribution of each component and its individual properties [63]. On the one hand, the viscosity of ice, unfrozen water and soil fabric enhance the creep properties of frozen soils. On the other hand, the pore ice in the frozen soils is likely to be melted and crushed caused by stress concentration, which in turn results in the weakening of ice-cement and increment of unfrozen water content, facilitating the readjustment of solid particle displacement. In light of this, Wang et al. [130] established a macro–micro creep model for frozen soils based on the equivalent method for viscoelastic and viscoplastic models and discussed the influences of temperature and stress level, whose model was validated to be capable of simulating three stages of non-attenuation creep.

2.4 Hypoplastic Modelling

Hypoplastic models have been developed by many scholars (e.g., [49, 138, 139]) to describe the mechanical responses of granular materials and applied to manifold geotechnical issues [25, 99, 106] since it is an efficient approach in elucidating soil behaviours (e.g., nonlinear mechanical response, dependency of stress path and soil dilatation). Based on continuum mechanics, the hypoplastic models are formulated based on the representation theorem of isotropic tensor

Table 4 Typical investigations on hypoplastic modelling for frozen soils under static loading

References	Soil types	Tests	Influencing factors
Xu et al., 2016a [151]	Frozen sand	Triaxial tests	T (−1, −2, −5, −10°C) and $\sigma_3 = 1$ MPa; $T = -4$°C and $\sigma_3 = 0.3, 0.6, 0.8, 1$ MPa
Xu et al., 2016b [152]	Frozen Fairbanks silt*[1]; Karlsruhe medium sand*[2]	Uniaxial compression tests*[1]; uniaxial creep tests*[2]	$\dot{\varepsilon}$ (1.1E-6 ~ 5.5E-2) and T (−0.5 ~ −10°C) *[1]; σ (1 ~ 14 MPa) and T (−2 ~ −20°C) *[2]
Xu et al., 2022 [154]	Frozen sandy soil*[2]	Triaxial test*[2]	T (−5°C, −10°C); σ_3 (3.5 ~ 6.5 MPa, 1 ~ 10 MPa)

σ_3 is confining pressure; σ is stress level; T is temperature; $\dot{\varepsilon}$ is strain rate; *[1] represents the triaxial tests conducted by Zhu and Carbee [190]; *[2] represents the uniaxial creep tests conducted by Orth [98]

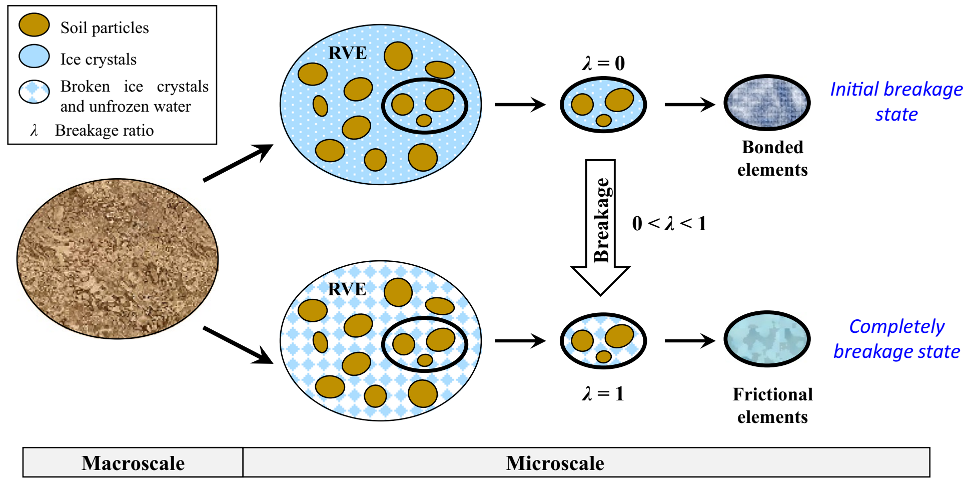


Fig. 2 Schematic diagram of bonded elements and frictional elements in binary element model

functions. They describe the relationship between stress rate, stress, and strain rate, which become hypoelastic when the nonlinear terms are omitted [149]. Table 4 lists the corresponding models based on the hypoplasticity theory.

Xu et al. [151] developed an extended hypoplastic model for granular materials, building upon the model proposed by Wu [139]. This extended model incorporates a backpressure term that accounts for the cohesion of frozen soils related to temperature, as well as a deformation-related function to capture strain softening. In the model of Xu et al. [151], eight parameters are included, including four parameters in the original model of Wu and Bauer [140], and four novel parameters where two parameters can be determined from the curves of cohesion-temperature and the other two parameters are relevant to the stress–strain curves that can be obtained by fitting the triaxial tests. The validation of this model was verified by two series of triaxial tests (i.e., varying temperature and constant confining pressure, varying confining pressure and constant temperature) where the simulated results are basically in accord with the experimental data. Nonetheless, it is worth noting that a distinct limitation in the extended hypoplastic model proposed by Xu et al. [151], i.e., the absence of consideration for the rheological properties (e.g., creep) of frozen soils. Accordingly, Xu et al. [152] presented a hypoplastic model to describe the three-stage creep deformation of frozen soils in a unified manner by coupling the extended hypoplastic model by Xu et al. [151] with the high-order model developed by Wu [141]. The model was verified by comparing the experimental data from uniaxial compression tests at different loading rates by Zhu and Carbee [190] as well as creep tests at different stress levels by Orth [98], which demonstrated that the model could efficiently capture the viscous characteristics of frozen soils, including the rate dependence of compressive strength, creep stain and creep rate of frozen soils. Xu et al. [154] also proposed a triaxial constitutive model with the framework of the hypoplastic model to characterise creep strains and strain rates in both axial and radial directions of frozen soil under various stress levels and temperatures. Unlike specialised models that only apply to specific stages of the creep process (e.g., [50, 121]), the proposed model can describe the entire three-stage creep process. Additionally, the proposed model eliminates the need to differentiate between high-stress and low-stress creep scenarios, where the specimen either fails or not.

2.5 Binary Element Models Based on Homogenization Theory

Homogenization theory is an efficient tool to establish the connection between microscale structure and macroscale behaviours of composite materials [66, 69, 76]. The concept of representative volume element (RVE) is often combined with homogenization theory [63]. The size of RVE should be carefully determined to ensure it is sufficiently large to encompass numerous internal particles at the microscale, while it should also be small enough to accurately represent the entirety of the composite material [69]. Based on the conception of RVE, the stress ($\boldsymbol{\sigma}_{ij}$) at any point within volume V for the macroscale RVE can be expressed as:

$$\sigma_{ij} = \frac{\int_V \sigma_{ij}^{local} dV}{V} \tag{2}$$

where σ_{ij}^{local} is the local stress within the RVE at microscale.

The frozen soil is considered a quasi-continuous medium comprising bonded blocks and weakened bands, respectively characterised as bonded and frictional elements. As shown in Fig. 2, the saturated frozen soil specimen is regarded as RVE at the macroscale, while it is composed of the bonded and frictional elements at the microscale that jointly withdraw the external loads. As for saturated frozen soils, the soil particles and ice crystals together constitute the bonded elements. In contrast, soil particles, broken ice crystals and unfrozen water are the frictional elements. The bonded elements can undergo irreversible breaking, transitioning into frictional elements when the external stress attains a certain strength threshold. The breakage ratio is an important parameter of binary materials such as frozen soils, which quantitatively describes the breakage degree of bonded elements transferring to the frictional elements and ranges from 0 to 1 with increasing external loads.

Therefore, within the concept of RVE and binary medium, the RVE, bonded element and frictional element volumes are defined as V, V_b and V_f ($V = V_b + V_f$). The conversion rate of the bonded element into the frictional element is expressed by the breakage rate (λ) that is equal to the ratio of V_f to V. At the macroscale level, the bonded and frictional elements are homogenous in their individual volume. Therefore, their average stress ($\sigma_{ij}^{average}$) can be computed as

$$\sigma_{ij}^{average} = \frac{\int_{V_b+V_f} \sigma_{ij}^{local} dV}{V} = \frac{\frac{1}{V_b}\int_{V_b} \sigma_{ij}^{b} dV + \frac{1}{V_f}\int_{V_f} \sigma_{ij}^{f} dV}{V} \tag{3}$$

where σ_{ij}^{b} and σ_{ij}^{f} are local stresses of the bonded element and frictional element. Based on the definition of breakage ratio, Eq. (3) can be rewritten as

$$\sigma_{ij}^{average} = (1-\lambda)\sigma_{ij}^{b} + \lambda\sigma_{ij}^{f} \tag{4}$$

Table 5 lists the existing binary element models based on homogenization theory for frozen soils. Chang et al. [10] developed a meso-macro constitutive model for frozen saline sandy soil based on homogenization theory, whose model can duplicate the strain softening/hardening, high dilatancy and pressure melting phenomenon well. In their mode, the saline frozen sandy soil was assumed to be composed of inclusion and matrix. The equivalent inclusion phase was described by an elastic-brittle model, while the sandy soil matrix was described by a critical state elastoplastic mode, and their expressions are derived from the three-phase sphere and Mori–Tanaka theory. Subsequently, Zhang and coworkers [167, 168, 170] proposed binary-medium constitutive models for frozen soils where the boned element and frictional elements were regarded as a matrix with elastic-brittle properties and inclusions with elastoplastic behaviours based on different constitutive relations for the bonded elements and frictional elements (see Table 5)

Besides, Wang and his coworkers (i.e., [125, 127, 130, 132, 134, 135]) have been devoted to developing the binary medium constitutive models for frozen soils according to homogenization theory in recent years. For example, given that the previous studies regarded the frozen soils as a whole solid material that cannot reflect the progressive breakage of ice-cement during the creep process, Wang et al. [125] initially analysed the breakage mechanism of creep deformation of frozen soils and developed a binary medium creep model to capture the creep deformation. By considering the effects of temperature and stress on the breakage evolution law, an exponential function for the breakage ratio was derived. To further reveal the mechanism of creep deformation of frozen soils under constant/varying temperatures, Wang et al. [132] proposed a unified macro-meso creep model to describe the creep process based on the creep tests of saturated frozen soils. In this model, the breakage ratio was derived by an empirical function of the plastic shear strain by referring to the determined methods of damage factors or plastic hardening parameters. Furthermore, some novel binary medium models to describe the stress–strain relations of frozen soils are also developed by Wang and his coworkers. Based on the experiment results from loading–unloading-loading tests, Wang et al. [127] observed that the elastic and plastic deformation would not generate simultaneously during the shear process. The mechanisms of these two types of deformation were analysed by a newly proposed binary medium model where the frozen soils were regarded as a binary medium consisting of bonded and frictional elements. Specifically, the boned elements have linear elastic properties described by generalised Hooke law, while the frictional elements exhibit plastic properties and are represented by the double-yield surface plasticity model. Since the breakage evolution is often empirically derived, reflecting the meso information within the constitutive models for frozen soils is difficult. Wang et al. [130] formulated a multi-scale incremental constitutive model to reflect this internal mechanical mechanism and introduced a new approach for determining the breakage ratio based on the thermodynamic theory that is more rational than previous empirical methods. Wang et al. [134] also established a binary-medium constitutive model combining the breakage mechanics and the homogenization method to describe the deformation mechanism of frozen soil with different coarse-grained contents. To address the limitation of the existing binary-medium-based models that are difficult to simulate the rate-dependent mechanical behaviours of frozen soils, Wang et al. [135] proposed a novel binary medium-based rate-dependent model by considering local bond breakage and the composition characteristics within the framework of breakage mechanics for geomaterials. Since the breaking process can be characterised as an irreversible thermodynamic process, a quantitative equation to describe the breaking ratio was also derived in the work of Wang et al. [135] by using a concept similar to that of plastic work. A traditional triaxial compression test

Table 5 Summary of existing binary element model based on homogenization theory for frozen soil under static loading

References	Breakage ratio	Bonded elements	Frictional elements	Soil types	Tests	External conditions	Remarks
Wang et al., 2019 [125]	An exponential function of temperature, time and stress	Burger model with elastic and viscoelastic properties	Quadratic yield function with associated flow rule	Frozen loess	Triaxial creep test	T = −15°C, σ_3 = 0.3 MPa	Creep
Zhang et al. 2019a [167]	An exponential function of the volumetric strain and shear strain	Structural blocks with elastic-brittle properties	Nonlinear Duncan-Chang hyperbolic model	Frozen silt soils	Triaxial compression test	T = −6°C, σ_3 = 0.3, 0.5, 0.8, 1.2, 1.5, 1.8 MPa	Stress–strain relationship
Zhang et al. 2019b [168]	Including volume and shear breakage ratio (exponential functions of volumetric strain and shear strain)	Elastic-brittle constitutive model	Elastoplastic model (a single yield surface with two hardening parameters with non-associated flow rule)	Frozen sand	Triaxial compression test	T = −6°C, σ_3 = 0.3, 0.5, 0.8, 1.2, 1.5, 1.8 MPa	Stress–strain relationship
Zhang et al., 2020a [170]	Weibull functions of volumetric stain and shear strain	Elastic-brittle constitutive model	A double hardening elastoplastic model with a non-associated flow rule	Frozen sand	Triaxial compression test	T = −6°C, σ_3 = 0.3, 0.5, 0.8, 1.2, 1.5, 1.8 MPa	Stress–strain relationship
Wang et al. 2020a [127]	Including volumetric and shear breakage ratio (exponential functions of mean stress and shear stress)	Generalisation Hooke law with linear elastic properties	Double yield surface plasticity model with plastic properties	Frozen loess	Loading–unloading-reloading test	T = −1.5°C, σ_3 = 0.3, 0.5, 1.4, 2 MPa, loading–unloading cyclic = 7	Stress–strain relationship
Wang et al., 2021b [130]	Derived from the first and second laws of thermodynamics	Generalisation Hooke law with linear elastic properties solved by the Mori–Tanaka method	Elastoplastic properties solved by Eshelby tensor	Frozen soil	Triaxial compression test	T = −4°C, σ_3 = 0.3, 3, 5, 7 MPa	Stress–strain relationship
Wang et al., 2022a [132]	An exponential function of plastic shear strain	Viscoelastic properties	Viscoplastic properties; an overstress model with D-P criterion (associated flow)	Frozen silty clay	Triaxial creep test	T = − 0.5, − 1.5, − 4°C, σ_3 = 0.3, 1.2, 3 MPa; T = − 4°C to − 1.5°C to − 0.5°C, σ_3 = 0.3, 1.2, 3 MPa	Creep
Chen et al., 2023b [13]	A function of volumetric strain and the shear strain	Elastic-brittle constitutive model described by generalised Hooke's law	A double hardening model with a non-associated flow rule	Frozen salinity sandy soil	Triaxial test	σ_3 = 4, 5, 6, 8 MPa	Stress–strain relationship
He et al., 2023a [40]	A function of creep time	Consisting of a Hooke body and a fractional viscoelastic body	Modified fractional Bingham body with Viscoplastic properties	Frozen clay	Triaxial creep test	T = − 1°C, σ_3 = 1 MPa	Creep

Table 5 (continued)

References	Breakage ratio	Bonded elements	Frictional elements	Soil types	Tests	External conditions	Remarks
Wang et al., 2023a [134]	A Weibull function of generalised shear strain	Eshelby model with elastic properties	Drucker-Prager yield criterion and a non-associate flow rule to describe the elastoplastic properties	Frozen soils with different coarse-grained contents	Triaxial compression test	$T = -10°C$, $\sigma_3 = 0.3$, 1.4 MPa, Freeze–thaw cycle = 0, 5, 10, $f_c = 0$, 0.3	Stress–strain relationship
Wang et al., 2023b [135]	An exponential function of axial strain and axial strain rate	Viscoelastic properties	Viscoplastic properties	Frozen soils	Triaxial compression test	$T = -1.5$, $-4°C$, $\sigma_3 = 1.4$ MPa, $\dot{\varepsilon} = 4.144E\text{-}3$, 8.64E-3, 4.32E-2/min	Stress–strain relationship

σ_3 is confining pressure; T is temperature; $\dot{\varepsilon}$ is strain rate; f_c is course-grained content

of frozen soils under different strain rates was simulated by this model, whose results demonstrated that the model could well predict the nonlinearity, dilatancy and strain softening/hardening characteristics.

Based on the homogenization method, Chen et al. [13] also developed a meso-macro constitutive model for frozen saline sandy soil by considering the influence of salt concentration. In their model, the bonded elements exhibit elastic-brittle mechanical properties, while the frictional elements possess elastic–plastic properties. He et al. [40] proposed a creep constitutive model according to the binary medium concept and homogenization theory to describe the creep deformation of frozen soils, including decaying creep and non-decaying creep. In their model, the constitutive relations of the bonded and frictional elements were formulated by using the creep breakage mechanism and the theory of fractional calculus, and the breakage ratio was defined as a function of creep time. Specifically, the bonded elements comprise a Hooke body and a fractional viscoelastic body, while the frictional elements are simulated by modified fractional Bingham body.

These binary element models based on homogenization theory have been verified to duplicate the primary characteristics of frozen soils, which offers a novel perspective and tool to capture and describe the mechanical behaviours of frozen soils within a multi-scale framework.

2.6 Hyperplastic Modelling

The mechanical and strength of frozen soils are highly influenced by negative temperature, ice-water phase change, and moisture migration. The deformation process is an intricate thermodynamic process [56]. Hence, incorporating the thermodynamic theory into constitutive modelling is capable of making the coupled thermal–mechanical model more reasonable and systematic. The core of the framework is that the constitutive relations of materials (i.e., elastic law, yield function, flow rule and hardening rule relating to the traditional elastoplastic theory) are entirely represented by two thermodynamic functions (i.e., Gibbs free energy function and dissipation function), which breaks the theoretical framework of conventional plastic mechanics [182].

Based on the irreversible thermodynamic theory, Ziegler [200] introduced a framework for establishing constitutive models, along with its foundation known as Ziegler's orthogonality principle [45]. Nevertheless, this principle has encountered significant criticism from various perspectives. Srinivasa [115] contended that Ziegler's orthogonality principle was based on an erroneous assumption. Nonetheless, Collins and Houlsby [17] argued that the principle is a weak assumption and can be employed to define a broad range of constitutive equations. Furthermore, using the convex analysis, Srinivasa [115] substantiated that the same orthogonality principle can be deduced from the maximum rate of dissipation criterion in both the rate-dependent and rate-independent scenarios (with the prerequisite of first-order homogeneity of the dissipation function). This method of constitutive modelling based on thermodynamics is also referred to as "hyperplasticity" [102], hereinafter referred to as "hyperplastic modelling". Hyperplasticity contains two types, i.e., single and multiple internal variable hyperplasticity that correspond to the single yield surface model and multiple yield surface model.

Establishing hyperplastic modelling possesses many advantages. Firstly, it ensures automatic adherence to thermodynamics laws. The second is the formulation exhibits a concise nature. It is crucial to establish competing constitutive models within a unified theoretical framework, facilitating convenient comparisons among different models and aiding in the selection of the most promising one. Thirdly, the constitutive modelling within the frame of hyperplasticity theory can seamlessly integrate with other differential equations, which makes the THM coupling model maintain a high level of consistency and systematicity [56].

Recently, some researchers have employed the hyperplasticity theory to describe the mechanical behaviours of frozen soils, as listed in Table 6. Fish [28] developed a hyperplastic model to describe the entire creep process (including primary, secondary, and tertiary creep) and failure for both constant stress tests and constant strain rate tests in the form of a unified constitutive equation and failure criteria. According to the second principle of thermodynamics and damage mechanics, He et al. [42] proposed a constitutive model for analysing the stress–strain relationship of frozen silt under complex stress conditions, whose model also explored damage development and the failure process of the internal structure of frozen soil. Sun et al. [116] established an elastoplastic anisotropic damage constitutive model by involving a plastic potential function considering the effect of volumetric deformation and anisotropic damage evolution. Lai et al. [56] adopted the orthogonality principle proposed by Srinicasa [115] to model Lanzhou frozen loess's mechanical behaviours. In their model, the specific expressions of the Gibbs free energy function and dissipation function for frozen loess were derived from the test results and the hyperplasticity theory. It is worth noting that the multiple internal variable hyperplasticity was employed since the test results showed that the direction of plastic strain increment was affected by the stress path. Based on the principles of hyperplasticity theory and using the non-associated flow rule, Lai et al. [57] established a double-yield surface model for frozen saline sandy soil, incorporating rotational hardening as well as initial anisotropy and loaded anisotropy to account for both plastic compression mechanisms and plastic shear mechanisms. The proposed model effectively captures the phenomena of pressure melting, crushing, and dilatancy in frozen saline sand. Furthermore, it accurately

Table 6 Summary of hyperplastic modelling for frozen soils under static loading

References	Soil types	Tests	External conditions	Remarks
Fish, 1984 [28]	Frozen Fairbanks silt	Unconfined compression creep tests under constant stress*[1] and constant strain rate*[2]	$T = -6$°C, constant stress tests ($\sigma_3 = 2 \sim 4$ MPa)*[1], and constant strain rate(1.1E-6 ~ 6.5E-2)*[2]	Creep deformation and stress–strain relationship
He et al., 2000 [42]	Frozen silt	Triaxial creep	$T = -5$°C, $\sigma_3 = 2 \sim 18$ MPa	Creep
Sun et al., 2005 [116]	Frozen soils	Triaxial test	$T = -10$°C, $\sigma_3 = 0.6 \sim 1.0$ MPa	Stress–strain relationship
Lai et al., 2014 [56]	Frozen loess	Constant-confining pressure triaxial compression test and constant-slope stress path test	$T = -6$°C, $\sigma_3 = 0.5 \sim 17.0$ MPa; constant-stress path test of $dq/dp = 3$, axial loading rate = 0.006 KN/s, confining pressure loading rate = 0.001 MPa/s	Stress–strain relationship
Lai et al., 2016 [57]	Frozen saline sandy soil	Triaxial test	$T = -6$°C, $\sigma_3 = 1 \sim 16$ MPa	Stress–strain relationship
Zhou et al., 2016 [186]	Frozen loess	Uniaxial, triaxial and hydrostatic test	$T = -3, -6, -9$°C	Creep deformation and stress–strain relationship
Chang et al., 2019 [10]	Frozen saline coarse sandy soil	Triaxial compression test	$f_{salt} = 0 \sim 3.5\%$, $T = -6$°C, $\sigma_3 = 1 \sim 16$ MPa	Stress–strain relationship
Sun et al., 2021 [117]	Frozen sand	Uniaxial compression creep test*[3]	$T = -5, -15$°C, $\sigma_3 = 1 \sim 8$ MPa*[3]	Stress–strain relationship

σ_3 is confining pressure; T is temperature; f_{salt} is saltcontent; p is mean stress; q is deviatoric stress; *[1~3] indicates the tests conducted by Zhu and Carbee [191], Zhu and Carbee [189] and Eckardt [24] accounts for the influence of salt content on stress–strain curves. Based on a series of different types of tests (i.e., uniaxial, triaxial and hydrostatic tests) on frozen loess, Zhou et al. [186] developed a rate-dependent constitutive model to characterise the multiaxial creep deformation by using the hyperplasticity theory and incorporated multiple internal variables. Chang et al. [10] established a constitutive model considering particle breakage, compression and shear mechanisms. In their model, the yield surface functions and plastic potential functions of three mechanisms are derived based on data from a series of triaxial compression tests for frozen saline coarse sandy soils with varying salt content under a wide range of confining pressure and stress paths. Sun et al. [117] presented a comprehensive framework that integrates poromechanics, thermodynamics, and continuum damage mechanics to account for visco-elasto-plastic creep and anisotropic damage in saturated frozen soils. To derive this constitutive model, the soil skeleton's free energy and the dissipation potential associated with the applied stress level are required. The decomposition of the first term includes four components: reversible free energy, strain energy stored in the soil skeleton, energy in the form of heat, and interfacial energy.

It can be noted that conventional constitutive models for frozen soils relied on Drucker's stability postulate or plastic potential theory. In these models, yield functions, flow rules, and hardening laws are introduced independently and occasionally contradict each other, which increases the likelihood of violating the hyperplasticity theory. In contrast, hyperplastic modelling has recently been developed and applied for geotechnical engineering, which guarantees adherence to the principles of hyperplastic by incorporating dissipative incremental and free energy functions.

2.7 Thermo-Poromechanics-Based Constitutive Modelling

Previous studies on modelling the mechanical behaviours of frozen soils regarded the frozen soils as continuous solid materials, which failed to reflect the interaction of different components within frozen soils. Besides, since soil is a typical pore medium [71], the constitutive modelling of frozen soils based on thermo-poromechanics theory is more closely aligned with its fundamental nature. Coussy and his colleagues [19, 20] have conducted investigations on frozen porous materials from a poromechanics perspective. In recent years, some scholars have applied thermo-poromechanics to the research of frozen soils, as summarised in Table 7.

Based on thermo-poromechanics theory, Liu et al. [79] formulated a constitutive model for poro-elastoplastic frozen soils to simulate the cryogenic triaxial compression of frozen soils by considering the Lagrangian saturation and solid–fluid interface interactions. In their model, the frozen soil was regarded as an open system, and both Lagrangian and Eulerian formulations presented the associated governing equations. Compared to other models, this model exhibits a more rigorous structure and derivation process, which is crucial for establishing a constitutive model for frozen soils. Subsequently, given the fact that the existing poromechanics-based models ignored the impact of rate dependence on the mechanical behaviours of frozen soils, Liu and Lai [81] established a framework to consider the time-dependent influence of frozen soils based on the extending of the poro-elastoplasticity to poro-elasto-viscoplasticity, whose model can be applied for the frost heave of freezing ground in the cold region. However, the impact of the water–ice phase change was not considered in their model. Therefore, as the temperature is close to the temperature of the water–ice phase change, the phase change between unfrozen water and ice crystals should be taken into account, especially for warm frozen soil. Accordingly, Ma et al. [88] proposed a

Table 7 Summary of thermo-poromechanics-based modelling for frozen soils under static loading

References	Soil type	Tests	Influencing factors	Remarks
Liu et al., 2018 [79]	Frozen sand soil*[1]; Frozen saline sands*[2]	Triaxial compression test	$T = -1, -5, -10$°C, $\sigma_3 = 1$ MPa*[1]; $T = -6$°C, $\sigma_3 = 1 \sim 10$ MPa*[2]	Stress–strain relationship
Liu and Lai, 2020 [81]	Frozen loess*[3–4]	Triaxial creep test	$T = -6$°C, $\sigma_3 = 6$ MPa*[3]; $T = -6$°C, $\sigma_3 = 1$ MPa, $\dot{\varepsilon} = 1.44$E-4/s, 7.2E-4, 1.44E-3/s)*[4]	Stress–strain relationship and creep
Ma et al., 2022 [88]	Warm frozen soil (Lanzhou silt and silty clay)	Triaxial test	$T = -1.5$°C, $\sigma_3 = 1, 2, 6$ MPa; $T = -1.5$°C, $\sigma_3 = 0.3, 3, 7$ MPa	Stress–strain relationship (especially for strain softening and volumetric dilatancy behaviours)

σ_3 is confining pressure; T is temperature; $\dot{\varepsilon}$ is strain rate; *[1~4] refer to tests by Xu et al. [151]; Lai et al. [57]; Zhou et al. [186]; Hou et al. [44]

poromechanics-based constitutive model for warm frozen soil by considering the interaction between soil particles, ice crystals and unfrozen water caused by pressure melting, which can predict the unfrozen water and ice pressure during the loading process. Furthermore, a comparison between experimental data and theoretical results has demonstrated that the model proposed by Ma et al. [88] can accurately simulate the mechanical behaviours of warm frozen soil, particularly the volumetric dilatancy behaviours and strain softening observed during both isotropic loading and axial stress loading stages. The aforementioned three thermo-poromechanics-based models introduce a novel way to construct frozen soil's constitutive model.

2.8 Constitutive Models Considering Damage Mechanics

The theory of damage mechanics describes the damage initiation, propagation, and eventual fracture of materials. Over the past few decades, damage mechanics theory has been widely utilised to develop constitutive models for unfrozen soils and rocks (e.g., [61, 80, 129]). More recently, damage mechanics has also been employed to develop constitutive models for frozen soils. The deterioration of macroscopic mechanical performance and the progressive failure of microscopic structures are the core of damage theory. By incorporating a "damage variable", the variation of mechanical responses caused by micro-defects can be quantified. This damage variable can be subsequently added to the constitutive model, also known as the damage model, to account for the impact of damage on the material's mechanical behaviour.

To construct a damage model for frozen soils, three aspects relating to the damage behaviour should be determined. The first is the definition of the damage variable, which is crucial in measuring the changes in macroscopic response resulting from the development of damage. Various variables have been adopted as the damage variable, such as particle orientation and area reduction [89], the deterioration level of elastic modulus [150], strength [52, 58], creep time [77] and so on. The second is the damage threshold value, which serves as a damage criterion to determine whether damage has occurred, given that the process is irreversible and often accompanied by irreversible deformation. The third aspect is the damage evolution law, which quantitatively describes the change in damage with irreversible deformation. Once these three aspects are established, a damage model can be used to describe the frozen soil's damage behaviours. Therefore, investigating these aspects is crucial in understanding the damage characteristics of and developing reliable damage models for frozen soils. Accordingly, the damage evolution laws of several important mechanical parameters (i.e., strength, stiffness, and viscosity) have received considerable attention. Four typical damage functions are employed to serve as damage evolution laws, including statistical functions, empirical functions, functions based on the energy dissipation principle, and functions based on elasticity theory.

(1) In the statistical damage model, some statistical distribution functions are applied to quantify the damage degree. Many different parameters are adopted to describe the damage variables. Lai et al. [52] indicated that the Weibull distribution is more suitable for describing the strength distribution law than normal and lognormal distribution. However, the axial strain was considered a random variable in the work of Lai et al. [52], and the frozen soil would be damaged when the axial strain reached a certain threshold. Accordingly, Li et al. [58] regarded the strength of soil elements as a random variable and improved the original statistical damage model of Lai et al. [52] for warm frozen clay and warm ice-rich frozen clay.

$$D = 1 - \frac{N_t}{N} = 1 - \exp\left(-\left(\frac{\varepsilon}{\eta}\right)^{\beta}\right) \tag{5}$$

where D is the damage variable; N_t is damage element numbers; N represents all the elements; ε is strain; β is the shape parameter; η is scale parameter.

(2) As for the empirical damage theory, the damage degree obeys some functional relationships with the damage variables. For instance, Liao et al. [77] established an exponential damage function for ice-rich frozen soil during the creep process.

$$\mathrm{D} = 1 - \exp(-\theta \mathrm{t}) \tag{6}$$

where D is the damage variable; θ is a parameter relating the damage evolution with time, which varies with the stress level; t is creep time. Besides, Hou et al. [44] defined a damage variable as

$$D(\sigma, t) = \begin{cases} 0 \; \sigma < \sigma_\infty \\ 1 - \exp(-Ct) \; \sigma \geq \sigma_\infty \end{cases} \tag{7}$$

where C is parameter reflecting the impact of damage; σ_∞ is long-term strength.

(3) Liu and Lai [81] regarded that the damage originated from the gradual breaking of ice bonding under external loading and determined the evolution of damage based on the assumption that the dissipation energy caused by damage is only dependent on the damage variable.

(4) According to Sidoroff's postulation [112], the form of the complementary elastic energy for a damaged

material is identical to that of an undamaged material. Accordingly, Lai et al. [53] defined a cross-anisotropic damage variable as follows.

$$\begin{aligned} D_1 &= 1 - \sqrt{\frac{\tilde{E}}{E}} \\ D_2 &= D_3 = 1 - \frac{v}{\tilde{v}}\left(1 - D_1\right) \end{aligned} \tag{8}$$

where D_1, D_2 and D_3 are the damage variables in the corresponding principal axes ($D_1 \neq D_2 = D_3$); E is elastic modulus; v is Poisson's ratio; $\tilde{E}$ and $\tilde{v}$ are the effective elastic modulus and Poisson ratio of damaged materials.

Table 8 summarises typical investigation on modelling the mechanical behaviours of frozen soils by damage theory. It is noted that the damage within frozen soil is 3D due to the anisotropic characteristics of ice crystals under loading. Therefore, assuming an isotropic hypothesis would not provide an accurate representation of the damage process in frozen soil. In this regard, Lai et al. [53] proposed cross-anisotropic damage variables and incorporated these into a novel elastoplastic constitutive model. The advantage of constitutive models based on damage theory is that they can elucidate the macroscopic behaviours of frozen soils from a microscopic perspective. However, describing the damage evolution for frozen soils in a reliable and efficient manner remains a significant challenge.

2.9 DEM-Based Constitutive Modelling

The microstructure of frozen soil plays a pivotal role in influencing the macro-mechanical characteristics of the material. However, limited by the nondestructive testing and monitoring techniques, most existing studies were conducted to explore the macroscopic mechanical behaviours of frozen soils. Some researchers have performed scanning technologies (e.g., computed tomography, CT) on the microstructures of frozen soils, and they noticed that the structures of frozen soils were continuously reorganised throughout the loading and deformation process, and the triaxial tests exerted simultaneous effects of structural reinforcement and weakening. Additionally, the failure of frozen soils was attributed to the slippage occurring between soil particles and bonded ice

Table 8 Summary of four typical damage functions used in static constitutive models for frozen soils

Damage functions		References	Soil types	Constitutive models	Remarks
Types	Specific descriptions				
Statistical functions	A Weibull function of strength	Lai et al. 2008 [52]	Warm frozen clay and warm ice-rich frozen clay	Stochastic damage constitutive model	Stress–strain behaviour
		Li et al., 2009 [58]	Warm frozen clay	An improved statistical damage constitutive model based on the Mohr–Coulomb failure criterion	Stress–strain behaviour
	A Weibull function of the random distribution of inner flaws (e.g., cracks)	Yang et al., 2010a [157]	Warm ice-rich frozen sand	A statistical damage constitutive model by modifying Nishihara's model	Creep behaviour
Empirical functions	An exponential function of creep time	Liao et al., 2017 [77]	Warm frozen silt	A fractional order creep constitutive model	Creep behaviour
	The elastic modulus is formulated by an exponential decay equation	Hou et al., 2018 [44]	Frozen soils with different contents of coarse grains	An improved Nishihara model	Creep behaviour
Functions based on energy dissipation principle	The energy dissipated by damage is only related to the damage variable	Liu and Lai, 2020 [81]	Saturated frozen soils	A thermo-poro-mechanics-based viscoplastic damage constitutive model	Stress–strain and creep behaviour
Functions based on elasticity theory	The complementary elastic energy of damaged materials is the same as that of undamaged materials	Shi et al., 2023 [111]	Frozen sandy soil	An elastoplastic damage model	Stress–strain behaviour

Table 9 Summary of DEM-based modelling for frozen soils under static loading

References	Soil type	Tests	DEM	Contact model	External conditions	Sample shape	Sample size (mm)	Particles		Porosity
								Radius (mm)	Numbers	
Zhou and Lai, 2010 [184]	Frozen sand clay	Triaxial test*[1]	PFC2D	Contact-bond model and slip model	σ_3 = 0.3, 0.6, 0.8 MPa, T = –2, –4, –6°C	2D plane	125 (H) × 61.8 (D)	0.25 ~ 0.5	–	0.35
Yin et al., 2016 [163]	Frozen Clay	Triaxial test	–	Linear contact bond model	σ_3 = 0.8 MPa, T = –0.5 ~ –6°C; σ_3 = 3 ~ 16 MPa, T = –1 ~ –4 °C	3D cylinder	125 (H) × 61.8 (D)	0.0001 ~ 2	–	0.35
An et al., 2018 [2]	Ice-Rich Frozen Sand	Triaxial compression*[2]	PFC2D	Nonlinear contact model with rolling resistance	σ_3 = 0.2 MPa, T = –10°C	2D plane	80 (H) × 39.1 (W)	0.075 ~ 2	–	–
Wang and Calvetti, 2020 [126]	Frozen granular soils with high ice content	Triaxial compression	PFC3D	Bonded model	σ_3 = 0.1 ~ 0.8 MPa	3D cylinder	100 (H) × 50 (D)	1.4 ~ 2.8 (S) 0.7 ~ 1.4 (I)	2500 ~ 17225	0.38 ~ 0.4
Wang and Calvetti, 2022 [131]	Frozen granular soils	Triaxial compression	PFC3D	Bonded model	σ_3 = 0.1 ~ 0.8 MPa	3D cylinder	100 (H) × 50 (D)	1.4 ~ 5.6 (S) 0.7 ~ 1.87 (I)	4793 ~ 112302 (S), 609 ~ 1837 (I)	0.36
Chang et al., 2023 [11]	Frozen sandy soil	Direct shear test	PFC3D	Linear parallel bond model	Shear rate = 6.67E-5	3D cylinder	500 (H) × 500 (D)	1.6 ~ 19.968#	100621	0.31
Sun et al., 2023 [119]	Frozen clayey sand	Triaxial compression	PFC3D	Ice bond contact model	σ_3 = 0.1, 0.2, 0.3 MPa, T = –5 ~ –15°C	3D cube	4.6 (H) × 2.3 (W, L)	0.05 ~ 0.25	20060 (S), 57611 (I)	-
Xu and Wang, 2023 [155]	Cemented methane hydrate-bearing sand	Triaxial creep*[3]	PFC2D	Parallel-bar model	σ_3 = 1 MPa, q = 3.9 ~ 4.8 MPa	2D plane	160 (H) × 80 (W)	0.3–0.5	17,918	0.2
Zhang et al., 2023c [179]	Frozen soils	Triaxial creep*[4]	PFC3D	Kelvin bond contact model	σ_3 = 1.4 MPa, q = 3.41 ~ 7.67 MPa	3D cylinder	125 (H) × 61.8 (D)	1 ~ 3	6113	0.3

σ_3 is confining pressure; T is temperature; q is deviatoric stress; S and I represent soil particles and ice particles; H, D, W and L represent height, diameter, width and length; *[1~4] refer to tests by Lai et al. [51], An et al. [1], Miyazaki et al. [90], Song et al. [114]; #For computation efficiency, the particle size in DEM model was expanded by 30 times that of actual sandy particles. Although these microscopic studies can observe the development of the structure of frozen soils, it is still a challenge to quantify the variations of microscopic information (e.g., article contact force and cementation of ice) within frozen soils under external loadings. Consequently, a noticeable research gap persists in understanding the underlying micromechanics governing the failure of frozen soils.

Initially proposed by Cundall and Strack [21], the discrete element method (DEM) has gradually become an alternative approach to explore the mechanical behaviours of geomaterials, especially the unfrozen soils (e.g., [5, 93, 100, 133, 156]). However, only a few studies have employed DEM to simulate the mechanical behaviours of frozen soils (see Table 9). Zhou and Lai [184] adopted a two-dimensional (2D) DEM method to simulate the triaxial test of frozen sandy soil where the contact bond model was used to replicate the cementation of ice within the frozen sand clay and a slip model to account for friction. Yin et al. [63] established a 3D model to simulate a series of triaxial tests on frozen clay and compared the numerical results with laboratory results. Wang and Calvetti [126, 131] presented a DEM model to assess the influence of ice content on the behaviours of the mixture of ice and granular soils in triaxial compression and shear tests. Xu and Wang [155] used a 2D DEM method to explore the creep deformation of cemented methane hydrate-bearing sand where the methane hydrate was simulated as contact bonds between particles, and proposed a parallel bar creep model to describe the evolution of damage with time. Their simulations showed that the methane hydrate-bearing sand displays considerable creep, similar to frozen sand. Considering the size effect in the conventional small-size direct shear test, Chang et al. [11] established a DEM model to explore the microscopic mechanical properties of frozen sand in direct shear tests, whose simulation results demonstrated that the DEM model was capable of elucidating the microscopic mechanisms underlying the macroscopic strength and deformation behaviour of frozen sandy soils. Zhang et al. [179] proposed a creep contact model for frozen soils to explore the bond effect on the mechanical behaviours of frozen soils by incorporating shear strength and tensile strength into the generalised Kelvin body. Based on this model, a series of triaxial creep DEM simulations for frozen soils were conducted to investigate macro- and meso-mechanical behaviours of frozen soils.

Nevertheless, the contact models utilised in these previous investigations neglect the impact of temperature on the ice's strength and stiffness. Accordingly, Sun et al. [119] extended an existing 3D contact model by involving the effect of temperature on ice cementation's strength and stiffness. By utilising this novel model based on DEM, they investigated the bond breakage mechanism during the loading process, which is not readily observable in conventional laboratory tests. They also qualitatively examined the influences of temperature on the macro–micro mechanical responses of frozen soils. These aforementioned investigations (as summarised in Table 9) offer a novel research avenue for comprehending the mechanical behaviours of frozen soils from a microscopic perspective and present opportunities for exploring other crucial properties of frozen soils in future studies, such as anisotropic and rheological behaviours, as well as the nonlinearity of strength. It is noted that modelling mechanical behaviours of frozen soils based on DEM is an efficient tool to explicitly simulate the interaction between ice and soil particles rather than the constitutive model based on a continuum approach for reproducing the macroscopic behaviours of frozen soils.

2.9.1 Machine Learning-Based Modelling

With the advancements in computation methods, some scholars have utilised machine learning (ML) methods to tackle geotechnical engineering issues [136, 173]. Machine learning methods have proven to be effective in predicting soil properties [67, 68, 73] and behaviours [74, 174, 180], which are helpful since direct measurements are relatively time-consuming and costly.

Therefore, some researchers have been devoted to modelling the mechanical behaviours of frozen soils in recent years (see Table 10). Nassr et al. [91] first presented the evolutionary polynomial regression (EPR) to model the thermos-mechanical behaviours of frozen soils with the consideration of the effects of strain rate, confining pressure and temperature. A comprehensive dataset of unconsolidated undrained triaxial tests on frozen soils from Iran was compiled and used to train the EPR model. This data-mining method was validated by comparing their predictions and experimental data. Pham et al. [101] also modelled the mechanical responses of frozen sand under varying confining pressure and temperature using the Bayesian Neural Network. However, these two models only focused on the evolution of deviatoric stress of frozen soils under external loadings and the volumetric variation was not considered. Accordingly, Li et al. [74] proposed a novel data-driven method (i.e., long short-term memory, LSTM) to capture and describe stress–strain relationships. Given the fact that uncertainty in modelling is one of the crucial sources of uncertainty in geotechnical engineering, their model combined with the Monte Carlo dropout scheme to provide reliability evaluations for the deviatoric stress and volumetric strain of frozen soils. Besides, Ren et al. [47] proposed a prediction model for unfrozen water based on a genetic algorithm-back propagation (GA-BP) neural network and analysed the mechanical behaviours of frozen soils under extremely-low-temperature conditions based on a series of laboratory experiments on frozen clay. The work of Ren et al. [47] indicated that the compressive strength of frozen

Table 10 Summary of machine learning-based modelling for frozen soils under static loading

References	Soil type	Targets	Tests	Methods	Influencing factors	Inputs	Outputs
Nassr et al., 2018 [91]	Frozen sand	Modelling mechanical behaviours (ε_a-q)	Unconsolidated undrained triaxial test	EPR	T (−0.5 ~ −11°C); σ_3 (0 ~ 0.8 MPa); $\dot{\varepsilon}$ (0.1 ~ 2 mm/min)	T, σ_3, $\dot{\varepsilon}$, ε_a, dε_a, q	$q_{(i+1)}$
Pham et al., 2021 [101]	Frozen silica sand	Modelling mechanical behaviours (ε_a-q)	Triaxial test	BNN	T (−1 ~ −20°C); σ_3 (0.3 ~ 2 MPa)	T, σ_3, ε_a	$q_{(i+1)}$
Li et al. 2023d [74]	Frozen sandy soil	Modelling mechanical behaviours (ε_a-q and ε_a-ε_v)	Triaxial test	LSTM	T (−4°C, −6°C); σ_3 (0.3, 0.6, 0.8, 1 MPa)	T, σ_3, ε_a, ε_v, q	$q_{(i+1)}$, $\varepsilon_{v(i+1)}$
Ren et al., 2023 [47]	Frozen clay	Predicting compressive strength	Uniaxial compression test	GA-BP	T (0 ~ −80°C); f_u (17%; 20%; 23%)	T, f_u	Compressive strength

σ is stress; σ_f is long-term strength; ε is strain; ε_a is axial strain; dε_a is axial strain increment; $\dot{\varepsilon}$ is strain rate; ε_v is volumetric strain; T is temperature; f_u is the content of unfrozen water; q is deviatoric stress; q_{i+1} is deviatoric stress for next increment; $\varepsilon_{v(i+1)}$ is volumetric strain for next increment; BNN is Bayesian neural network; EPR is evolutionary polynomial regression; GA-BP is genetic algorithm-back propagation; LSTM is long short-term memory

soil exhibits a direct proportionality with the absolute value of temperature, while the unfrozen water content follows a power function relationship with compressive strength.

It can be noted from Table 10 that the applicability of the ML method in modelling the mechanical behaviours of frozen soils, which exhibits considerable performance in capturing and forecasting the stress–strain relations of frozen soils as well as the strain softening/hardening phenomenon. Notably, these ML-based models require fewer parameters compared to conventional rheological models since these models based on the ML method are implemented on the available database from laboratory measurements (e.g., triaxial tests).

3 Constitutive Models for Dynamic Loading Condition

As summarised in Sect. 2, extensive research efforts have been dedicated to developing constitutive models for frozen soils under static loading conditions. However, it is essential to note that construction projects in cold regions are often subjected to dynamic loading rather than quasi-static loading conditions. The mechanical behaviours of frozen soils under dynamic loads are more sophisticated than those under static conditions. For example, owing to the short duration of impact loads, the constitutive model developed for quasi-static loading conditions is unsuitable for analysing frozen soils' mechanical responses at impact loading scenarios. Accordingly, investigations into dynamic constitutive models for frozen soils have achieved significant progress in recent decades. This section provides a brief overview and discussion of these dynamic models. Furthermore, Table 11 presents a compilation of constitutive models for frozen soils under dynamic loads, which is categorised into eight distinct types based on their underlying fundamental principles.

3.1 Empirical Models

Empirical dynamic constitutive models for frozen soil are mathematical formulations that aim to capture the dynamic behaviour of frozen soil through empirical functions and experimental observations. These models are developed based on experimental data obtained from dynamic tests of frozen soils, which often incorporate various parameters (i.e., strain rate, temperature, and stress) to describe the mechanical response of frozen soils under dynamic loading conditions. These empirical models provide an efficient approach to numerically predict the mechanical behaviour of frozen soils and are easily applied for engineering applications.

Zhang et al. [166] conducted the uniaxial stress and nearly uniaxial strain conditions to explore the dynamic behaviours of frozen soils and formulated a model to describe the temperature and strain-rate-dependent behaviour of frozen soils. However, their model exhibits limitations in capturing stress–strain curves when the strain approaches the final strain, and it fails to simulate the dynamic behaviours of frozen soils under general conditions, such as 3D scenarios. Since it is difficult to establish

Table 11 Summary of existing dynamic constitutive models for frozen soils

Model types	References	Soil types	Test conditions	Influencing factors	Details	Remarks
1 Empirical models	Zhang et al., 2013 [166]	Artificial frozen clay	uniaxial strain and uniaxial stress test	T, $\dot{\varepsilon}$	$\sigma = \begin{cases} \left(Y + \alpha\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)^m\right)\frac{\varepsilon}{\varepsilon_y}\left(\frac{T-T_r}{T_m-T_r}\right)^l, \varepsilon \le \varepsilon_y \\ \left(Y\left(\frac{\varepsilon}{\varepsilon_y}\right)^n + \alpha\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)^m\right)\left(\frac{T-T_r}{T_m-T_r}\right)^l, \varepsilon_y < \varepsilon < \varepsilon_d \end{cases}$	$\dot{\varepsilon}_0$ is reference strain rate; ε_y is the yield strain; ε_d is yield strain and the densification strain for the frozen soil m, n, and l are rate sensitivity, strain hardening exponents, and the thermal exponent α is a fitting parameter, Y is the yield strength under the quasi-static condition; T_r is room temperature (19.85°C), T_m is melting point of frozen soil (−0.15°C)
2 Elastoplastic modelling	Ning et al., 2014 [94]	Frozen sand	DUC	T, $\dot{\varepsilon}$	An elastoplastic model with modified Drucker-Prager yield function by considering the effect of strain rate and isotropic hardening rule	-
	Xie et al., 2020 [147]	Frozen soil	DUC	T, $\dot{\varepsilon}$	An elastoplastic model where the modified Drucker–Prager (DP) yield criterion was used to simulate the stress–strain relations before the peak stress and the interfacial debonding damage was considered to describe the strain softening	The deboning damage of ice particles obeys the Weibull distribution; Suitable for: $T < -2$°C and $\omega > 20\%$
	Qiao et al. 2022 [105]	Unsaturated frozen clay	DUC	$\dot{\varepsilon}$	A rate-dependent yield surface constitutive model was proposed based on plastic flow criterion and damage, where the matrix suction was considered by the modified Barcelona Basic Model (BBM)	Damage variable obeys Weibull distribution
3 Viscoelastic modelling	Wang et al., 2003 [122]	Frozen soils	DUC	$\dot{\varepsilon}$	$\sigma = E_0\varepsilon + \alpha\varepsilon^2 + \beta\varepsilon^3 + E_1\int_0^t \dot{\varepsilon}\exp\left(-\frac{t-\tau}{\theta_1}\right)d\tau$ $+E_2\int_0^t \dot{\varepsilon}\exp\left(-\frac{t-\tau}{\theta_2}\right)d\tau$	A nonlinear viscoelastic model was derived based on dynamic impact tests on high-polymer materials; its structure and corresponding formula are shown in Fig. 3

Table 11 (continued)

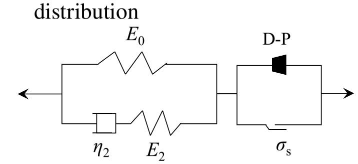
Model types	References	Soil types	Test conditions	Influencing factors	Details	Remarks
4 Viscoplastic modelling	Zhang et al., 2021a [172]	Frozen sand	DUC	T, $\dot{\varepsilon}$	A unified viscoplastic model involving the damage caused by adiabatic heating to describe the non-linear strain softening	-
	Zhang et al., 2023a [177]	Frozen Clay	DUC	Particle size	A dynamic model with a multiplicative hardening law was proposed based on Drucker–Prager viscoplastic theory	The soil particle size dependence wasinvolved by the equivalent strain gradient of the soil cell element
	Zhu et al., 2020 [197]	Frozen clay	DUC	T, ε	A visco-elastoplastic constitutive model was proposed based on Chaboche unified viscoplastic constitutive theory	Combining Drucker-Prager yield condition and associated flow rule; Rate-dependent damage evolution law was related to the activation energy to describe the thermal softening caused by adiabatic temperature rise
	Li et al., 2022d [70]	Frozen sand	DUC with F-T loading	F-T cycles	Viscoelastic part: modified ZWT model by ignoring the first item; Plastic part: D-P yield criterion (associated flow rule, strain rate term was added to the hardening criterion)	Considering two damage variables under impact loading and F-T cycles, F-T damage is defined by wave impedance and impact damage evolution obeying the Weibull distribution 
5 Energy-based models	Xie et al., 2014 [145]	Frozen sand	DUC	$\dot{\varepsilon}$	The absorbed energy was formulated by a GussAmp peak function	Considering energy absorption
	Ma et al., 2021b [87]	Frozen clay	DUC	Pre-existing cracks (positions, numbers, lengths, obliquities)		

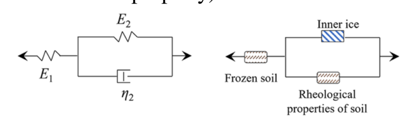
Table 11 (continued)

Model types	References	Soil types	Test conditions	Influencing factors	Details	Remarks
6 Two surface plasticity models	Zhao et al., 2020 [181]	Frozen saline silty clay	Cyclic DTC	Initial static stress, loading cycle, and salt content (0, 0.5, 1.5, 2.5%)	An anisotropic bounding surface elastoplastic constitutive model that can describe the effects of confining pressures and salt contents, considering the pressure melting and crushing of coarse particles and salt particles	(1) A linear critical stress ratio and bounding surface shape parameters were incorporated into the plastic potential surface function and bounding surface function, relating to σ_3; (2) Nonassociated flow rule; (3) A mixed hardening rule was provided to describe the changes in boundary surface shape and account for induced anisotropy caused by cyclic loading; (4) A radial mapping rule with a moveable centre to determine the image stress point on the boundary surface
7 Micromechanics-based models	Zhu et al., 2020 [197]	Frozen clay	multiaxial loading (DUC and DTC)	$\dot{\varepsilon}$	A dynamic model was developed based on micromechanics (i.e., Mori–Tanaka's concept of average stress)	Combining with damage theory (deboning damage of ice particles obeys the Weibull distribution, deriving based on thermal activation theory)
	Fu et al., 2021 [34]	Unsaturated frozen sand	DUC	T, $\dot{\varepsilon}$	A dynamic model was developed based on the micromechanics (i.e., Mori–Tanaka and Eshelby equivalent inclusion theories),	Combining damage theory where the crack propagation law and Sidoroff residual energy equivalent principle were incorporated to account for the strength degradation induced by microcrack damage
	Wang et al., 2021a [201]	Frozen silty soils with medium silica gravel*	Cyclic DTC	σ_3, coarse particle contents, dynamic deviatoric stress amplitudes	A dynamic model was developed based on micromechanics (i.e., homogenization and breakage theory), considering the effect of ice cementation failure and slippage of soil particles	Also named a binary medium model, assuming that frozen soils consist of bonded elements (undamaged part with elastic behaviours) and frictional elements (damaged part and simulated by double hardening constitutive model) Breakage: a powerful tool for describing mesoscale response of frozen soils, such as strain softening, volumetric contraction, and dilatancy
7 Micromechanics-based models	Zhang et al., 2023b [178]	Frozen silty clay	DUC	T, $\dot{\varepsilon}$	A dynamic model was developed based on the micromechanics (i.e., mixed theory), modified Drucker–Prager yielding criterion and damage criterion that considers the mutual effect of frictional sliding of local cracks and damage evolution for damage energy release rate	-

Table 11 (continued)

Model types	References	Soil types	Test conditions	Influencing factors	Details	Remarks
8 Constitutive models considering damage mechanics	Zhu et al., 2010 [193]	Frozen sand	DUC	T	The damage variable is a Weibull distribution function of strain	Combining with an equivalent inclusion method
	Xie et al., 2016 [146]	Frozen clay	DUC	T		Combining with micromechanisms theory (The frozen soil was assumed to be a viscoelastic medium and modelled by two parallel Maxwell bodies (linear elastic elements))
	Zhu et al., 2016 [194]	Frozen sand	DUC	T, $\dot{\varepsilon}$		Combining with an equivalent inclusion method
	Ma et al., 2017 [84]	Frozen sandy clay	DUC	T, $\dot{\varepsilon}$, σ_3		Modifying the ZWT model by considering the effect of T, $\dot{\varepsilon}$, and σ_3
	Fu et al., 2019b [33]	Frozen sand	DUC	T	The damage variable is a Weibull distribution function of strain	Modifying the ZWT model by considering the effect of temperature damage
	Ma et al., 2021b [86]	Frozen silty soil	DUC	$\dot{\varepsilon}$, prefabricated crack number		Modifying the ZWT model by considering the microdamage of frozen soils
	Ma et al., 2019 [85]	Frozen silty clay	DTC, DUC, and CSDL	$\dot{\varepsilon}$, loading conditions (coupled static and dynamic loads)		Modifying the ZWT model by considering the effect of axial pre-compressive stress on coupled static and dynamic strength and damage

Table 11 (continued)

Model types	References	Soil types	Test conditions	Influencing factors	Details	Remarks
8 Constitutive models considering damage mechanics	Zhang et al., 2020b [171]	Frozen clay	DUC	T, $\dot{\varepsilon}$	A damage variable was represented by the longitudinal wave velocity, which can measured via ultrasonic nondestructive testing techniques	Modifying the ZWT model by considering the generation and expansion of micro-cracks, where the Maxwell element was represented by low-frequency parameters instead of a simple spring
	Cao et al., 2018 [8]	Frozen soil	DUC	T, $\dot{\varepsilon}$	The damage evolution caused by the propagation of micro cracks and defects was derived by incorporating a strain rate term based on thermal activation theory	Considering the temperature softening and strain rate strengthening effects
	Zhu et al., 2021a [198]	Frozen sand	DUC	T, $\dot{\varepsilon}$	The damage evolution was derived from thermal activation energy	Proposed by modifying the nonlinear elastic model (i.e., Ottsen model) and thermal activation theory, considering the rate enhancement and damage softening effects under impact loadings
	Zhu et al., 2021b [199]	Frozen clay	DUC	T, $\dot{\varepsilon}$	A damage viscoelastic model was proposed to describe the stress–strain relations of frozen soils at a high strain rate by considering the effects of debonding (an exponential equation was derived based on thermal activation theory) and temperature softening (incorporating a temperature softening coefficient into the Johnson–Cook model)	The frozen soil was represented by an elastic element in series with a Kelvin element (a spring with elastic property and a dashpot with visco property) 
	Zhu et al., 2017 [195]	Frozen sand	DUC	T, $\dot{\varepsilon}$, particle size	Based on the Drucker-Prager yield rule and involves two damage mechanisms	Two damage mechanisms are considered, i.e., damage induced by microcrack defects and another caused by microvoid collapse
	Fu et al., 2019a [32]	Frozen soil	DUC	T, $\dot{\varepsilon}$	A modified Drucker-Prager yield function with two damage mechanisms to simulate the mechanical behaviours before peak stress and a cohesive crack model to describe the post-peak softening process	

D-P is Drucker-Prager (D-P) yield criterion;* represents that the experimental data are from Zhang et al. [169]; DUC is the dynamic uniaxial compression test (impact loading test); DTC is dynamic triaxial compression test; F-T is freeze–thaw cycle; T is temperature; $\dot{\varepsilon}$ is strain rate; σ_3 is confining pressure; ω is initial water content; E is elastic modulus; η is coefficient of viscosity

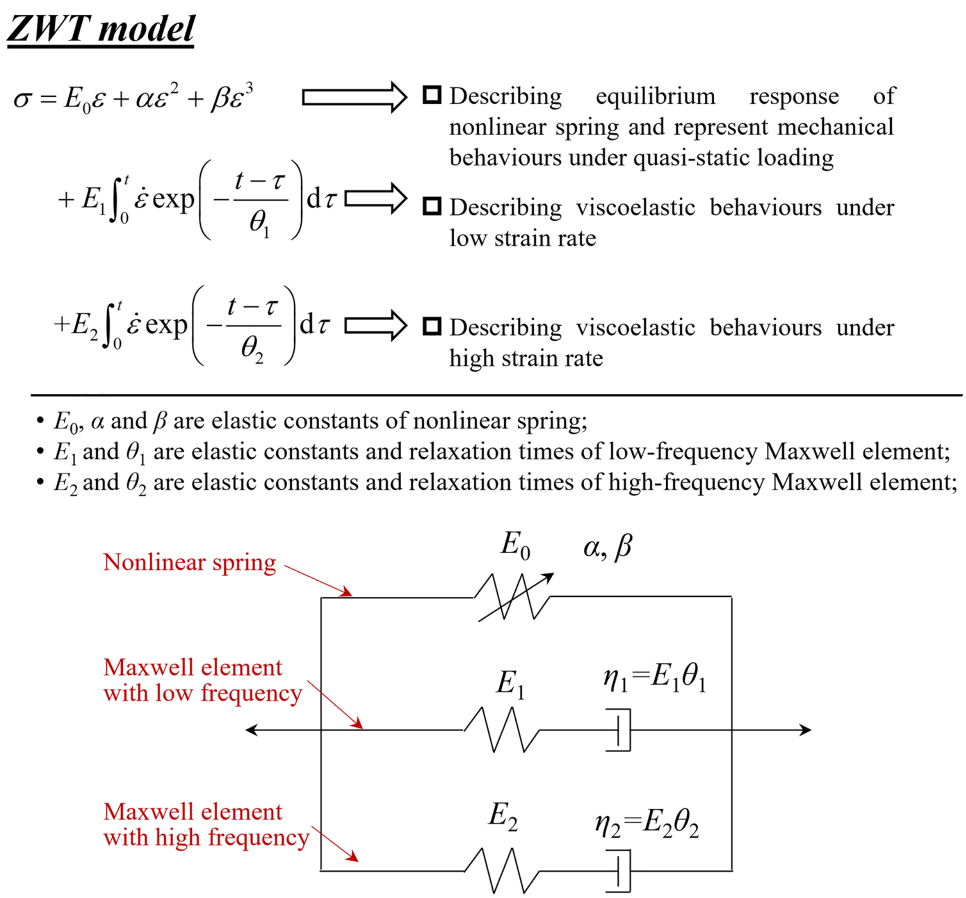


Fig. 3 Schematic diagram of ZWT nonlinear viscoelastic model and its formulation

a dynamic constitutive model for frozen soils with precise physical meanings based on experimental data, researchers have devised alternative approaches to modelling the dynamic behaviours of frozen soils.

3.2 Elastoplastic Modelling

Based on a series of dynamic experiments on frozen soils using the SHPB method, Ning et al. [94] proposed an isotropic hardening constitutive model with a modified Drucker-Prager yield function considering the effect of strain rate to describe the dynamic behaviours of frozen soils. Considering the adiabatic temperature increase and interfacial debonding, Xie et al. [147] developed an elastoplastic model for frozen soils where a modified D-P criterion was used to describe the strain hardening before peak stress and deboning damage of ice particles obeying the Weibull distribution was consider for reproduce the phenomenon of softening. Qiao et al. [105] proposed a rate-dependent yield surface model within the framework of elastoplastic theory by considering the matrix suction impact to describe the dynamic responses of unsaturated frozen soils, where the damage variable also obeyed the Weibull distribution.

3.3 Viscoelastic Modelling

The Zhu Wang Tang (ZWT) model is a typical nonlinear viscoelastic model that was derived based on dynamic impact tests on high-polymer materials [122], which includes two parts, i.e., a nonlinear transient response component independent of time and a linear transient response component closely associated with time. Figure 3 plots the schematic diagram of the ZWT model.

The ZWT model has been widely used for the dynamic behaviours of frozen soils. Since frozen soils are rate-dependent materials under impact loadings, Zhang et al. [171] noted that the low-frequency Maxwell element no longer affects the dynamical responses of frozen soils. Some scholars have devoted themselves to improving the ZWT model to better capture and describe frozen soils' dynamic behaviours.

3.4 Viscoplastic Modelling

Under impact loadings, the damage evolution within frozen soils can be regarded as a rate-dependent viscoplastic deformation process [8]. Some scholars have established viscoplastic models to replicate the mechanical behaviours of frozen soils under impact loading conditions. Zhu et al. [197] established a viscoplastic model to depict the dynamic mechanical behaviours, whose model was developed by integrating the Chaboche unified viscoplastic constitutive theory, the Drucker–Prager yield criterion and associated flow rule. Zhang et al. [172] derived a damage dynamic model for frozen soil based on the unified viscoplastic theory. This model incorporates an inelastic multiplier equation that accounts for the characteristic thermal softening observed during impact loading, while the phenomenon of adiabatic heating induces damage. Based on a series of dynamic tests of frozen soil under F-T loading and impact loadings, Li et al. [70] derived a viscoplastic model by an improved ZWT model and D-P yield criterion. Besides, their model also considered two damage mechanisms, i.e., F-T damage and impact damage, which were described by wave impedance and Weibull distribution function, respectively. Recognizing the influence of soil particle size on the dynamic responses of frozen soils, Zhang et al. [177] proposed a dynamic viscoplastic constitutive model by incorporating the average strain gradient of the soil cell model into a multiplicative hardening law that includes a work hardening function, a hardening deterioration function and a strain rate-temperature sensitive function.

3.5 Energy-Based Models

Energy-based methods are often adopted to indirectly establish the dynamic constitutive models of frozen soils. For instance, based on a series of impact compressive loading tests, Xie et al. [145] defined the energy absorbed per unit volume of frozen soil as the integral of the stress–strain curve and used a GuassAmp peak function to describe the absorbed energy. Their energy-based dynamical model can reasonably reflect the positive strain-rate sensitivity and negative temperature dependence of the artificial frozen soil, but it fails to simulate the failure process of frozen soils owing to the path independence of the energy-based method. Similarly, Ma et al. [87] also employed the GuassAmp peak function to describe the energy absorption curves and derived a dynamic model to reflect the strength and deformation of frozen clay with pre-existing cracks under impact loading conditions. The comparison between theoretical results and measurements showed that the predictive error could be constrained to 1.35 MPa.

3.6 Two Surface Plasticity Models

To improve the capacity of model for cyclic loadings, the concept of two surface plasticity theory is often introduced. Two surface plastic models, i.e., the bounding surface model (initially proposed by Dafalias and Popov [22, 23]) and the subloading surface model (firstly developed by Hashiguchi and Ueno [39]), have been acknowledged as powerful tools for simulating the deformation characteristics of soils under cyclic loading, owing to their clear concepts and straightforward numerical implementations. The boundary surface plasticity theory defines the loading surface and bounding surface (referring to the maximum yield surface that corresponds to the maximum loading stress experienced during the loading process). In addition to the bounding surface and plastic potential surface, the boundary surface plasticity theory also contains the hardening rule, mapping origin and radial mapping rule, as well as plastic modulus. The subloading surface model possesses a similar structure as the bounding surface model, consisting of a normal yield surface and a subloading yield surface.

In practice, the frozen soils are often subjected to cyclic loading conditions (e.g., traffic load and seismic action). The dynamic deformation of frozen soils under reciprocating dynamic loadings exhibits nonlinearity, hysteresis and plastic strain accumulation, which are more complex than unfrozen soils since the ice crystals considerably influence the soil cohesion. Therefore, it is not appropriate to directly apply the two surface plasticity model developed for unfrozen soils to frozen soils. To reproduce the dynamic behaviours of frozen saline silty clay under cyclic loading, Zhao et al. [181] formulated an anisotropic boundary surface elastoplastic constitutive model based on boundary surface plasticity theory. In their model, the rate dependence and visco impact were neglected. The comparison between experimental data and predicted deformation from the proposed model illustrates that the dynamic model of Zhao et al. [181] can reasonably simulate the deviator strain of saline frozen soils while some differences between measured volumetric strain and corresponding predictions can be observed. Their dynamic model comprehensively accounts for the impact of salt content by incorporating appropriate model parameters, which can accurately simulate the intricate deformation characteristics of frozen saline soil and unfrozen/frozen salt-free soils.

3.7 Micromechanics-Based Models

The frozen soil is a typical porous, multiphase and heterogeneous medium, and its macroscopic behaviours are affected by the internal microstructure. Therefore, it is reasonable and beneficial to construct the dynamic constitutive models for frozen soils based on micromechanics due to its unique features, such as the ability to shed light on the underlying mechanisms behind the effects of influencing factors (e.g., strain rate and temperature), elucidate the softening impact of adiabatic temperature rise that is initially reported by Zhu et al. [194] and model the microscopic constitutive relationships within frozen soils. For example, based on the representative volume element (RVE) concept, Xie et al. [146] assumed that the frozen soil could be represented as a binary medium with ice particles serving as spherical inclusions and soil particles acting as the matrix. Zhu et al. [193, 194] developed dynamic constitutive models for frozen soils by combining damage theory and equivalent inclusion methods. Using the Mori–Tanaka method, Zhu et al. [197] developed a dynamic constitutive model for frozen soils under a multiaxial state, whose model incorporates the debonding damage of ice particles and the damage evolution of the soil matrix based on thermal activation theory. Fu et al. [34] developed a dynamic damage constitutive model for frozen soil using Mori–Tanaka and Eshelby's equivalent theory, which can simulate the dynamical responses of frozen soils, such as rate-dependence, temperature sensitivity and dynamic deformation. Besides, in their model, the strength degradation caused by microcrack damage evolution was considered by the complementary strain energy equivalence principle. Wang et al. [201] formulated a dynamic binary medium model based on homogenization and breakage theory, whose model can reflect the influences of coarse-grain contents, axial deviatoric stress amplitude, and confining pressures on the dynamic behaviours of frozen soils under cyclic loading conditions. Zhang et al. [178] developed a microscopic dynamic model for unsaturated frozen soils to elucidate the underlying physical mechanism of local crack frictional sliding and damage evolution by incorporating a Drucker–Prager type yielding criterion and a damage criterion based on the damage energy release rate. The comparison between experimental data and model computation results showed that this model could capture crucial aspects such as the effect of strain rate and temperature as well as the nonlinear hardening and softening behaviour of unsaturated frozen soil.

3.8 Constitutive Models Considering Damage Mechanics

Based on the microcosmic mechanics mixing theory and damage theory, Zhu et al. [193] proposed an elastic constitutive model assuming that the damage variable was a Weibull distribution function of strain. By considering frozen soil as a particle-reinforced composite material consisting of clay soil (matrix) and ice particles, Xie et al. [146] formulated a micromechanical constitutive mode to characterise the dynamic compressive deformation of frozen soil. Based on the debonding damage theory, this model assumes that the damage variable follows a Weibull distribution of strain and accounts for strain rate and temperature's influence on frozen soil's dynamic compressive behaviour. However, the model of Xie et al. [146]) is unsuitable for warm frozen soils since it neglects the existence of unfrozen water. Zhu et al. [194] initially introduced the concept of temperature damage (i.e., strength reduction caused by adiabatic temperature increase), and developed a constitutive model for frozen soils to describe the heat damage and softening behaviour under dynamic loading conditions based on the equivalent inclusion method and damage theory.

The previously developed ZWT model [122] has demonstrated its capability to effectively capture and characterise the viscoelastic behaviour of solid materials, while the low-frequency Maxwell element in the ZWT model would lose its effectiveness due to the microcrack propagation. Accordingly, many scholars have improved the ZWT model by considering the damage stemming from macro- and microdamages. Considering the impacts of confining pressure, strain rate and temperature, Ma et al. [84] modified the ZWT model by considering the influences of these three factors on the dynamic behaviours of frozen sandy clay, whose model included a damage variable that also obeyed the Weibull distribution. Fu et al. [33] introduced the damage evolution and temperature damage into the original ZWT model to formulate a novel viscoelastic constitutive model to reflect the intricate interplay of strain rate effect, temperature effect, strain convergence effect, damage softening impact on dynamic behaviours of frozen soils. Ma et al. [86] improved the ZWT model by considering the damage of frozen soils where the Weibull distribution was used to describe the damage evolution within frozen silty soil, whose model can effectively capture the strain rate enhancement and the weakening influence of prefabricated crack number on the dynamic strength of frozen soils. Besides, Zhang et al. [171] employed the longitudinal wave velocity to represent the damage variable and developed a modified ZWT model by introducing this damage variable. They also investigated the macroscopic physical quantity and mesoscopic parameters to explain the changes in damage in frozen soil caused by internal microcracks.

Previous research studies have neglected the process of damage evolution in frozen soil and relied on the assumption that the strength of microcells within frozen soil follows a

Table 12 Typical characteristics of constitutive models for frozen soils under static loads

No	Methods	Advantages	Limitations
1	Empirical model	(1) Relatively simple to implement (2) It can provide reasonable predictions for certain frozen soils (3) It does not require detailed knowledge of the mechanical properties of frozen soils or complex mathematical formulations	(1) It lacks a fundamental theoretical basis; (2) Its applicability is limited beyond the specific conditions for which it was calibrated; (3) It might fail to capture the complex behaviours of frozen soils
2	Elastoplastic modelling	(1) It can accurately capture the nonlinear behaviour of frozen soils, especially for plastic deformation; (2) It is widely used in engineering applications	(1) It possesses relatively complex mathematical formulations and calibration procedures; (2) Its accuracy might be constrained for frozen soils exhibiting intricate behaviour or subjected to extreme loading conditions
3	Viscoplastic modelling	It accounts for time-dependent behaviours and viscoelastic effects, which is suitable for describing creep or time-dependent deformation	(1) The strain and its rate should be decomposed into elastic and plastic parts; (2) It often involves more parameters and requires more complex modelling techniques, which can increase computational costs and calibration efforts
4	Hypoplastic modelling	(1) It does not have an explicit elastic range, enabling it to effectively capture the nonlinear deformation observed during the initial stages of loading; (2) Failure criterion can be derived from the hypoplastic model, reducing the necessity for duplicated experimental tests; (3) Including the plastic potential concept is unnecessary, as the flow rule can be directly obtained from hypoplastic models; (4) The need to establish loading–unloading criteria is eliminated, enhancing the efficiency of numerical simulations; (5) It offers a concise mathematical formulation with few parameters, thereby reducing the need for extensive experiments to determine the parameters	(1) It is formulated within the theoretical framework of hypoplasticity, which is grounded in the principles of continuum mechanics. It does not consider the influence of internal structure (e.g., ice distribution) of frozen soil; (2) Rheological properties of frozen soil, such as creep and relaxation, are not considered
5	Binary element model based on homogenization theory	(1) It effectively captures the mechanical response of frozen soil by conceptualizing it as a binary element medium that consists of bonded elements and frictional elements; (2) It bridges the microscale structure and macroscale behaviours of frozen soils	It requires detailed information on each constituent's microstructure and corresponding mechanical properties to accurately reflect the local grain debonding process and non-uniformity
6	Hyperplastic modelling	(1) It provides a thermodynamically consistent framework for modelling mechanical behaviours of frozen soils, ensuring energy balance and stability in simulations (2) Once the free energy function and dissipation function are determined, all the components within the elastoplastic constitutive model (i.e., elastic stress–strain relations, yield functions, flow law, and hardening law) can be determined. Consequently, many arbitrary assumptions, such as Drucker's postulate and plastic potential functions, can be eliminated (3) It is capable of integrating with other differential equations, thereby ensuring a high degree of consistency and systematicity within the THM coupling model	It might be complex to implement and require significant computational resources, especially for frozen soils exhibiting highly nonlinear or rate-dependent behaviour
7	Thermo-poromechanics-based constitutive modelling	It considers the coupled thermo-mechanical and porosity-related effects in frozen soils, which is suitable for applications involving heat transfer, fluid flow, and deformation	It involves complex multi-physics formulations and requires specialised knowledge in both mechanics and thermodynamics

Table 12 (continued)

No	Methods	Advantages	Limitations
8	Constitutive model considering damage mechanics	(1) It can elucidate the macroscopic behaviours of frozen soils from a microscopic perspective, which links the progressive failure of microscopic structures and deterioration of macroscopic mechanical performance; (2) Incorporating a damage variable allows for quantifying the variations in mechanical responses induced by internal damage	It is based on the assumption that the damage within frozen soil is 3D isotropic, which is inconsistent with the anisotropic characteristics of ice crystals under loading in reality. Therefore, it would not accurately represent the damage process in frozen soil
9	DEM-based constitutive modelling	It is well-suited for modelling frozen granular soils, which can provide detailed insights into the micro-scale behaviour, facilitating a comprehensive understanding of the frozen soils' behaviours at the microscale	It is computationally expensive, especially for large-scale simulations, and requires careful calibration of parameters based on experimental or empirical data
10	Machine learning-based model	(1) It has the potential to capture complex material behaviour without relying on explicit mathematical formulations; (2) it can learn from available datasets to enhance prediction accuracy and performance	(1) It requires a large amount of high-quality training data and can be sensitive to the quality and representativeness of the training dataset (2) The model's ability to provide meaningfulexplanations for its predictions and accurately generalise for unseen data is intricate and requires careful consideration

probability distribution (i.e., Weibull distribution). Although this approach effectively reflects damage state at the macroscopic level, it fails to account for the temporal evolution of damage. To enhance the accuracy of describing the constitutive relation of frozen soil, it becomes imperative to explore the damage evolution law of frozen soils under impact loading. Cao et al. [8] incorporated the strain rate item into the damage evolution formula based on thermal activation theory [4] and derived a dynamic constitutive model with rate and temperature effects. Zhu et al. [199] modified the Ottsen model, a nonlinear elastic model, and introduced damage based on thermal activation theory to formulate a dynamic model. Zhu et al. [199] proposed a damage viscoelastic model by considering the debonding damage and temperature softening to describe the dynamic behaviours of frozen soils under high strain rates. Specifically, an exponential equation was derived for the debonding effect based on thermal activation theory, and a temperature softening coefficient was incorporated into the Johnson–Cook model to reflect the softening phenomenon.

Based on the Drucker–Prager failure criterion and coupled damage-plasticity, Zhu et al. [195] developed a new constitutive model to describe the dynamic mechanical behaviours of frozen soil with different particle sizes. Their equations involved two damage mechanisms (microcrack-induced damage and microvoid collapse-included damage). Fu et al. [32] also proposed a dynamic model where the Drucker–Prager yield function was modified by adding the strain rate enhancement impact to capture the mechanical behaviours of frozen soils before reaching the peak stress. Besides, since the impact loading process exhibits a pronounced strain-rate effect that becomes more unstable after the peak stress, a cohesive crack model based on fracture mechanics theory was established by Fu et al. [32] to describe this nonlinear process after peak stress.

Frozen soils are often subjected to static stresses resulting from geostatic forces, frozen-heave effects, and dynamic loads induced by impact stress waves. Therefore, frozen soil's mechanical behaviour differs significantly from materials subjected individually to static stress or dynamic loading. Referring to the test results for rocks (e.g., [27, 38]) that axial pre-stress significantly impacts materials' dynamic strength, failure mode, and energy dissipation features of

Table 13 Typical characteristics of constitutive models for frozen soils under dynamic loadings

No	Methods	Advantages	Limitations
1	Empirical model	As summarised in Table 12	As summarised in Table 12
2	Elastoplastic modelling	As summarised in Table 12	As summarised in Table 12
3	Viscoelastic modelling	It can effectively capture and characterise the viscoelastic behaviour of frozen soils	(1) It fails to reflect the viscoplastic properties of frozen soils; (2) It often requires the calibration of many material parameters (3) It might be computationally demanding, particularly for large-scale simulations or complex loading scenarios (4) It has limitations in generalising its predictions beyond the range of the calibration data or specific frozen soil conditions
4	Viscoplastic modelling	As summarised in Table 12	As summarised in Table 12
5	Energy-based model	It can ensure energy balance and stability in dynamic simulations, leading to physically meaningful results	It might be computationally expensive, particularly for large-scale simulations
6	Two surface plasticity model	It can efficiently simulate the deformation characteristics of soils under cyclic loading due to its clear concept and straightforward numerical implementation	(1) It requires calibration of material parameters; (2) It has complex mathematical formulations; (3) it might not accurately capture all aspects of dynamic behaviours
7	Micromechanics-based models	(1) It provides insights into the microscale behaviour and mechanisms (2) It can explore the influence of microstructural structures	(1) It requires detailed knowledge of material microstructure and associated properties; (2) It has complex mathematical formulations and computational implementation; (3) Calibration of material parameters can be challenging
8	Constitutive model considering damage	As summarised in Table 12	As summarised in Table 12

geomaterials, Ma et al. [85] conducted a series of laboratory tests on artificial frozen silty clay (i.e., static uniaxial compression test, the dynamic uniaxial compression test, and the coupled static and dynamic loading test), and established a dynamic constitutive model for artificial frozen silty clay under coupled static and dynamic loads based on modified ZWT model and damage theory.

4 Discussion

The aforementioned studies are the latest developments in the constitutive model for frozen soils, presenting significant advancements in exploring the mechanical behaviours of frozen soils under static and dynamic loadings.

Various types of constitutive models of frozen soils under static loads have been proposed, which can be divided into ten categories: empirical models, elastoplastic modelling, viscoplastic modelling, hypoplastic modelling, binary element models based on homogenization theory, hyperplastic modelling, thermo-poromechanics-based constitutive modelling, constitutive models considering damage mechanics, DEM-based constitutive modelling, machine learning-based modelling. As for constitutive models under dynamic loads, eight types of models have also been developed. The primary advantages and limitations of these constitutive models under static and dynamic loadings are summarised in Tables 12 ans 13 to provide references for engineers and researchers.

It can be seen that empirical models, viscoplastic models, constitutive models considering damage mechanics, energy-based models, and micromechanics-based models have been developed for both static and dynamic analyses of frozen soils. Among the static models, the hypoplastic model exhibits distinct advantages compared to empirical models and conventional elastoplastic models, which is a powerful tool to capture and describe the mechanical behaviours of frozen soils. Besides, models based on damage mechanics and homogenization theory can explore the macro mechanical behaviours of frozen soils from a microscopic perspective, which can shed light on the impact of heterogeneous microstructures and provide a comprehensive understanding of internal damage evolution law. Incorporating thermodynamic theory or thermo-poromechanics into constitutive models for frozen soils can make the coupled thermo-mechanical model more rational and systematic, which enhances the model's capability to represent the mechanical behaviour of frozen soils accurately. In contrast, constitutive models for frozen soils based on DEM or ML method are appealing in describing the mechanical behaviours of frozen soils, although research in these areas is relatively scarce. DEM-based models offer a granular-level understanding of frozen soil behaviour by simulating the interactions between individual particles, representing micro-scale phenomena such as ice particle rearrangement, contact forces, and particle breakage. In addition, ML-based models display the potential to capture complex relationships and patterns in frozen soil behaviour using data-driven approaches, which enables the development of robust predictive models for frozen soil properties and behaviours, even in cases where traditional constitutive models may be limited. Both DEM and ML methods provide advantages in capturing the intricate nature of frozen soil behaviours and offer new avenues for enhanced understanding and predictive capabilities. However, it is essential to note that these methods also present their challenges, including the need for accurate calibration, rigorous validation, and substantial computational resources in the case of DEM. Careful consideration and extensive verification of these models are necessary before their widespread application in engineering practice.

The dynamic models for frozen soils under dynamic loadings are similar to the static constitutive models, including empirical models, viscoplastic models, and constitutive models considering damage mechanics. In addition, inspired by the behaviour of other composite materials, some researchers have utilised the conventional viscoelastic model, such as ZWT model, to capture the time-dependent characteristics of frozen soils. However, it is important to note that this viscoelastic model loses its capability to accurately predict nonlinearity and align with experimental data at lower strain rates. Energy-based models, incorporating the Gaussian peak function to describe absorbed energy, have been proposed to capture the dynamic behaviours of frozen soils subjected to uniaxial compressive impact loadings. Additionally, two surface plasticity model has gained popularity in modelling the deformation characteristics of soils under cyclic dynamic loading due to its clear conceptual framework and ease of numerical implementation. It is crucial to acknowledge that developing dynamic constitutive models for frozen soils is an ongoing area of research, and different models have been established based on various principles. Therefore, careful consideration should be given to the specific requirements of the intended application and the availability of reliable data for calibration and validation purposes.

5 Conclusions and Prospects

Developing a reliable and accurate constitutive model for frozen soil is crucial for understanding and predicting its mechanical behaviour, which plays a significant role in foundation design and engineering damage prevention. However, accurately capturing the mechanical behaviours of frozen soil is challenging due to its unique properties and complex nature. In this study, an extensive review was undertaken to examine the existing constitutive models for frozen soils under both static and dynamic loadings according to their theoretical foundations.

The static models were divided into ten categories, while the dynamic models consisted of eight types, and their primary advantages and limitations are also summarised. Up to now, significant advancements have been achieved in the development of constitutive models for frozen soil, which are verified to be efficient in describing the mechanical responses of frozen soils and providing theoretical and engineering references for construction projects associated with frozen soils. However, due to the complexity and randomness of frozen soils, as well as the variability of external conditions, some issues need to be addressed.

(1) Previous constitutive models for frozen soils have been derived from different theories and verified using certain soil types and test conditions. These models possess complex mathematical formulations with model parameters obtained through fitting experimental data, typically considering individual factors while neglecting their combined effects. However, existing models have insufficiently addressed the random distribution

of defects within frozen soils, which plays a crucial role in the formation of ice lenses and contributes to the heterogeneous and anisotropic nature of frozen soils. Additionally, uncertainties in the mechanical behaviours of frozen soils associated with these defects have not been adequately discussed. Therefore, it is necessary to establish comprehensive constitutive models that consider the joint influences of factors and account for the unique characteristics of frozen soils (i.e., temperature/time dependency, heterogeneity, and ice lenses). Moreover, these models should involve as few parameters as possible to facilitate their practical applications and numerical implementation. Besides, conducting reliability evaluations based on advanced constitutive models is beneficial to provide valuable guidelines for practical engineering.

(2) Existing models often treat the strength criterion and creep model separately, leading to a fragmented representation of the stress–strain relationship and rate-dependent behaviours. In addition, creep deformation is often divided into two separate types (i.e., attenuation creep and nonattenuation creep), which hampers the overall applicability of these models. Therefore, there is an urgent need to establish a unified model that comprehensively captures and represents the diverse aspects of mechanical behaviours of frozen soils under complex conditions.

(3) The validation of constitutive models for frozen soil should not be limited to specific soil types. Future investigations should aim to validate the models using a wider range of soil samples and conditions to ensure their reliability and generalisability. Moreover, the determination of model parameters should not solely rely on fitting experimental data but also incorporate intelligent algorithms to enhance their accuracy and robustness. To facilitate this validation process and promote further research in the field, it would be beneficial to establish a comprehensive database that consolidates information on the mechanical behaviours of frozen soils. This database can be constructed by incorporating data from scientific publications and government reports, thereby providing a valuable resource for the constitutive modelling of frozen soils.

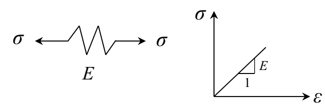
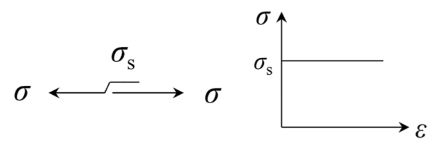
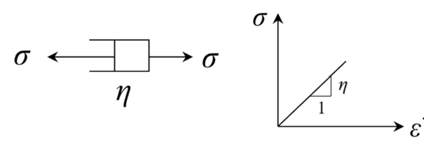
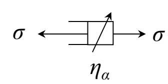
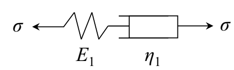
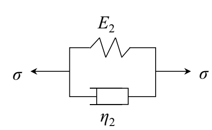
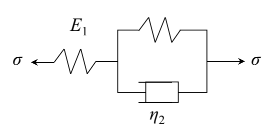
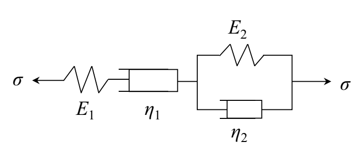
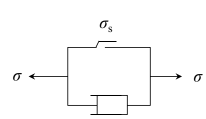
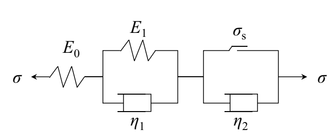
(4) The advancements in discrete element method (DEM) and machine learning (ML) techniques provide a promising approach to expanding the capabilities of constitutive modelling for frozen soils. These two powerful tools can be leveraged to explore the micro-mechanisms underlying the behaviour of frozen soils and enhance the capacity for data mining and analysis. With the ability to investigate the micromechanical properties and strong capacity in data mining, incorporating DEM or ML methods into constitutive modelling can provide a more comprehensive understanding of the complex interactions within frozen soils and unlock new possibilities for predictive modelling and analysis.

Overall, the development of an advanced and comprehensive constitutive model for frozen soil remains an ongoing research endeavour, with the potential to significantly enhance engineering practices in cold regions and improve the safety of infrastructure built on frozen ground. Further research and in-depth investigations should be conducted to advance the development of constitutive models for frozen soils, particularly those recently developed based on machine learning and DEM, and seamlessly integrate these models with computer simulations (such as the finite element method) to tackle the intractable thermo-hydro-mechanical problems associated with frozen soils.

Appendix

See Table 14.

Table 14 Summary of typical rheological elements and combinations

No	Elements	Properties		Schematic diagrams
Rheological element				
1	Hooke element (spring)	Elastic		
2	Saint–Venant element (slider)	Plastic		
3	Newton element (dashpot)	Viscous		
4	Abel element (dashpot)	Viscous; it is a fractional derivation of Newton element and a combination of Newton dashpot for an ideal Newton fluid and Hooke spring for an ideal elastic solid		
Combination of rheological elements				
1	Maxwell model	Viscoelastic	Hooke spring and Newton dashpot are connected in series;	
2	Kelvin model		Hooke spring and Newton dashpot are connected in parallel;	
3	Generalised Kelvin model		Hooke spring and Maxwell model are connected in parallel;	
4	Burgers model		Maxwell model and Kelvin model are connected in series;	
5	Bingham model	Viscoplastic	Saint–Venant slide and Newton dashpot are connected in parallel;	
6	Nishihara model	Visco-elasto-plastic	Hooke spring, Kelvin model and Bingham model are connected in series	

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Data Availability All data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors declare there are no competing interests.

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