# Asymptotic scaling of optimal cost and asymptotic optimality of base-stock policy in several multi-dimensional inventory systems

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We consider three classes of inventory systems under long-run average cost: (i) periodic-review system with lost sales, positive lead times and a non-stationary demand process, (ii) periodic-review system for a perishable product with partial backorders and a non-stationary demand process, and (iii) continuous-review system with fixed lead times, Poisson demand process and lost sales. The state spaces for these systems are multi-dimensional and computations of their optimal control policies/costs are intractable. Since the unit shortage penalty cost is typically much higher than the unit holding cost, we analyze these systems in the regime of large unit penalty cost. When the lead-time demand is unbounded, we establish the asymptotic optimality of the best (modified) base-stock policy and obtain an explicit form solution for the optimal cost rate in each of these systems. This explicit form solution is given in terms of a simple fractile solution of lead-time demand distribution. We also characterize the asymptotic scaling of the optimal cost in the first two systems when the lead-time demand is bounded.

*Key words*: optimal cost, asymptotic scaling, large unit penalty cost, asymptotic optimality

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## 1. Introduction

In this paper, we study three inventory systems:

(i) Periodic-review lost-sales inventory system with fixed lead times and a non-stationary demand process.

(ii) Periodic-review system for a perishable product with fixed lifetime, partial backorders, and a non-stationary demand process.

(iii) Continuous-review system for a non-perishable product with fixed lead times and lost sales.

For systems (i) and (ii), the firm first reviews the system state in each period $t \geq 1$ and then makes a replenishment decision. The system state for system (i) includes the on-hand and pipeline inventories, and for system (ii) it includes the on-hand inventories with different ages and backlogged demand (if any). The order arrives after fixed lead time in system (i) and immediately in system (ii). Next, random demand in period $t$ is realized and satisfied as much as possible by the on-hand inventories (and following a first-in-first-out issuance policy in system (ii)). For system (ii), a random fraction of unmet demand is backlogged and the remaining fraction is lost. The demand in each period is a non-negative period-dependent deterministic part plus an *i.i.d.* non-negative *r.v.*. At the end of period $t$, the leftover inventories are carried to the next period (or outdated when reaching the end of their lifetime in system (ii)). For system (iii), demands arrive according to a Poisson process. At any time $t \geq 0$, the firm first reviews the system state, which includes the on-hand and pipeline inventories. If demand arrives at time $t$, then it is satisfied by the on-hand inventory and unmet demand is lost. After that, the firm makes a replenishment decision, and the order (if any) will arrive after fixed lead times. The objective for all three systems is to minimize the long-run average holding, backlogging/lost-sales penalty, and outdating (for system (ii)) cost.

The studies of the three systems described when demands are *i.i.d.* and unmet demand is fully lost/backlogged have a long history which dates back to 1950's (see, e.g., Veinott 1960, Karlin and Scarf 1958, Karush 1957). Since these systems have multi-dimensional state space, their optimal policies and costs are in general intractable due to the curse of dimensionality. Thus, the dominant approach in the literature is to develop and analyze simple heuristic policies for these systems, such as base-stock policy (Reiman 2004, Huh et al. 2009, Bijvank et al. 2014, Zhang et al. 2020, Bu et al. 2023), constant-order policy (Reiman 2004, Xin and Goldberg 2016), and approximation algorithm (Levi et al. 2008, Chao et al. 2015, Chao et al. 2018, Zhang et al. 2023). In practice, it is well known that the unit penalty cost is usually much larger than the unit holding cost. Thus many of these studies perform asymptotic analysis for the heuristic policies in the regime of large unit penalty cost (see, e.g., Reiman 2004, Huh et al. 2009, van Jaarsveld and Arts 2024, Bu et al. 2023). We refer to Goldberg et al. (2021) for a detailed review. However, almost no study characterizes the asymptotic behavior of the optimal cost in these systems as the unit penalty cost grows large. To our knowledge, Arts et al. (2015), combined with the results in Huh et al. (2009), provide the only such result for system (i) under a class of integer-valued demand distributions and *i.i.d.* demands. They do not consider system (i) under continuous demands or non-stationary demand process, and there is no such study for system (ii), or system (iii). In addition, the literature has shown that the base-stock policy is asymptotically optimal with large unit penalty cost for systems (i) and (ii) under *i.i.d.* demands, and pure backorder or lost-sales models ( Huh et al. 2009, Bijvank et al. 2014, Bu et al. 2023). However, for systems (i) and (ii) under the more general setting of this paper, and for the classical continuous-review system (iii), no existing paper studies asymptotically optimal policies with large unit penalty cost.

In this paper, we study the three inventory systems in the regime of large unit penalty cost. As a foundation of our analysis, we prove that the optimal cost of each system is bounded from below by some newsvendor cost. For systems (i) and (ii), we propose a class of modified base-stock policies that raises the inventory level/position to some *time-dependent* levels and prove that the long-run average cost of the best modified base-stock policy is bounded from above by another newsvendor cost. Thus, to understand the optimal cost of these systems, we first characterize the asymptotic scaling of the newsvendor cost with large unit penalty cost. We summarize our approach as follows. First, when demand is unbounded and its mean-residual life (MRL) is sub-linear, we show that the ratio between the newsvendor cost and a simple fractile solution converges to a constant as the unit penalty cost goes to infinity. Under a weaker condition that the MRL of the demand distribution is upper bounded by a linear function, we derive the asymptotic bounds on the newsvendor cost. Second, we characterize the exact rate of the newsvendor cost under six classes of demand distributions, and establish asymptotic bounds on the newsvendor cost under finite $k$-th order moment, sub-exponential, and sub-Gaussian demand distributions. Finally, we consider the case of bounded demand and characterize the rate at which the newsvendor cost converges to its finite limit under two classes of bounded demand distributions.

We then apply the above results to the three inventory systems described earlier. For systems (i) and (ii), when the lead-time demand is unbounded and its MRL is sub-linear (respectively, upper bounded by a linear function), we characterize the asymptotic scaling (respectively, asymptotic bounds) of the optimal cost in both systems as the unit penalty cost grows using a simple fractile solution. Importantly, this enables us to establish the asymptotic optimality of the best modified base-stock policy with large unit penalty cost in both systems (for system (ii), the sub-linear condition of MRL can be removed for the asymptotic optimality result to hold.) When demand is bounded in system (i) with non-stationary demand process and in system (ii) with *i.i.d.* demand process and lost sales, we also characterize the rate at which the optimal cost converges to its finite limit under different demand distributions. For system (iii), we obtain the exact rate of the optimal cost as the unit penalty cost grows and establish the asymptotic optimality of the best base-stock policy in this regime. To our knowledge, this is the first asymptotic optimality result in the continuous-review lost-sales inventory system. This paper contributes to the literature with the understanding of the optimal cost for three classical but notoriously complex inventory systems in the regime of large unit penalty costs. It also extends the asymptotic optimality result of simple base-stock policies (for both perishable and non-perishable inventory systems) to systems with *a non-stationary demand process*, and establishes asymptotic optimality result of simple base-stock policies for continuous-review lost-sales inventory system with Poisson demand. We expect our analysis and results to be useful in studying other complex inventory systems.

The rest of this paper is organized as follows. We characterize the asymptotic scaling of the newsvendor cost in §2 and that of the optimal cost of the three inventory systems in §3 to §5, respectively. We conclude the paper in §6 with a few remarks. Throughout this paper, for positive functions $f(x)$ and $g(x)$, we apply the notation $f(x) = o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = 0$, $f(x) = \mathcal{O}(g(x))$ if $\limsup_{x\to\infty} f(x)/g(x) < \infty$, $f(x) = \Theta(g(x))$ if $f(x) = \mathcal{O}(g(x))$ and $g(x) = \mathcal{O}(f(x))$, and $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. For random variable (*r.v.*) $X$, let $F_X(x)$ denote its cumulative distribution function (*c.d.f.*) and $f_X(x)$ denote its probability mass/density function (*p.m.f.* or *p.d.f.*). Let $\bar{F}_X(x) \triangleq 1 - F_X(x)$, and for any $0 < \alpha < 1$, $F_X^{-1}(\alpha) \triangleq \inf\{x \geq 0 : F_X(x) \geq \alpha\}$, where "$\triangleq$" stands for "defined as". Let $\Phi(\cdot)$ denote the *c.d.f.* of a standard normal *r.v.*.

## 2. Asymptotic Scaling of Newsvendor Cost

In this section, we study the asymptotic scaling of the newsvendor cost with large unit penalty cost. The results established in this section lay the foundation for characterizing the asymptotic scaling of the optimal cost of the infinite time horizon systems (i) to (iii) in the subsequent sections. The proofs of all the technical results in this section are provided in Appendix A.

In the newsvendor problem, the seller decides an order quantity $q \geq 0$ to satisfy non-negative random demand $D$ with known *c.d.f.* $F_D(\cdot)$ and *p.d.f.* or *p.m.f.* $f_D(\cdot)$. We assume that $\mathbb{E}[D] < \infty$ and denote $\bar{D} \triangleq \sup\{x : F_D(x) < 1\} \leq \infty$. Leftover inventory incurs a unit holding cost $h$ and unsatisfied demand incurs a unit penalty cost $p$. Let $C^{\mathrm{NV}}(h,p,F_D)$ denote the *newsvendor cost* with unit holding cost $h$, unit penalty cost $p$, and demand distribution $F_D$, i.e.,

$$C^{\mathrm{NV}}(h,p,F_D) = \min_{q\geq 0} \left\{ h\mathbb{E}[(q-D)^+] + p\mathbb{E}[(D-q)^+] \right\}.$$

It is well-known that the optimal order quantity $q^*$, referred to as the *newsvendor solution*, is the fractile solution $F_D^{-1}(\frac{p}{p+h})$, i.e., $q^* = \inf\{q \geq 0 : F_D(q) \geq \frac{p}{p+h}\}$.

THEOREM 1. *Suppose $D$ is unbounded. Then the following results hold:*

*(a) If $\mathbb{E}[D - x | D > x] = o(x)$, then*

$$\lim_{p\to\infty} \frac{C^{\mathrm{NV}}(h,p,F_D)}{F_D^{-1}(\frac{p}{p+1})} = h. \tag{1}$$

*(b) If $\mathbb{E}[D - x | D > x] = \mathcal{O}(x)$, then $C^{\mathrm{NV}}(h,p,F_D) = \Theta(F_D^{-1}(\frac{p}{p+1}))$.*

Equation (1) shows that the newsvendor cost asymptotically scales up as $h$ times the fractile solution $F_D^{-1}(p/(p+1))$ as $p \to \infty$ under condition $\mathbb{E}[D - x|D > x] = o(x)$. A similar condition appears in Huh et al. (2009), Bijvank et al. (2014) and van Jaarsveld and Arts (2024). As shown in Theorem 1 of Huh et al. (2009), many commonly used demand distributions satisfy this condition, including any distribution with an increasing failure rate (IFR). For any $\nu > 0$, Equation (1) implies that $\lim_{p\to\infty} C^{\mathrm{NV}}(h,p/\nu,F_D)/C^{\mathrm{NV}}(h,p,F_D)=1$, which has two implications. First, it implies (1) also holds when replacing $F_D^{-1}(p/(p+1))$ with $F_D^{-1}(p/(p+\nu))$ for any $\nu>0$. Second, it reveals that the newsvendor cost is a *slowly varying* function of $p$. A measurable function $f(\cdot):\mathbb{R}^+\to\mathbb{R}$ is called slowly varying if $\lim_{x\to\infty} f(\alpha x)/f(x)=1$ for any $\alpha>0$. It is well-known that any slowly varying function $f(x)$ satisfies $\lim_{x\to\infty} f(x)/x^\alpha=0$ for any $\alpha>0$ (see, e.g., Korevaar et al. 1949). Thus, when $D$ is unbounded and $\mathbb{E}[D-x|D>x]=o(x)$, the newsvendor cost grows slower than $p^\alpha$ for any $\alpha>0$, i.e., $\lim_{p\to\infty} C^{\mathrm{NV}}(h,p,F_D)/p^\alpha=0$. Part (b) shows a weaker result under the (weaker) condition $\mathbb{E}[D-x|D>x]=\mathcal{O}(x)$, that the newsvendor cost is asymptotically bounded from above and from below by $F_D^{-1}(p/(p+1))$ multiplied by certain constants as $p\to\infty$. Examples of distributions satisfying $\mathbb{E}[D-x|D>x]=\mathcal{O}(x)$ but not $\mathbb{E}[D-x|D>x]=o(x)$ include fat-tailed distributions such as Pareto distribution (see, e.g., Proposition 1 in Huh et al. 2009).

With Theorem 1, we can characterize the exact rate of the newsvendor cost as a function of $p$ by studying the growth rate of $F_D^{-1}(p/(p+1))$. The following proposition shows the results for six classes of commonly used demand distributions.

PROPOSITION 1. *Suppose $D$ is unbounded. Then the following results hold:*

*(a) If $D$ is a continuous r.v. and there exists $k\in(-1,\infty)$ such that $\lim_{x\to\infty} r_D(x)/x^k=\gamma\in(0,\infty)$, where $r_D(x)=f_D(x)/\bar{F}_D(x)$, then $C^{\mathrm{NV}}(h,p,F_D)\sim h(\frac{k+1}{\gamma})^{\frac{1}{k+1}}(\ln p)^{\frac{1}{k+1}}$.*

*(b) If $D$ is a Gumbel min r.v., i.e., $F_D(x)=1-e^{1-e^x}$ for $x\ge 0$, then $C^{\mathrm{NV}}(h,p,F_D)\sim h\ln\ln p$.*

*(c) If $D$ is a log-normal r.v., i.e., $F_D(x)=\Phi(\frac{\ln x-\mu}{\sigma})$ for $x>0$ with parameters $\mu\in\mathbb{R}$ and $\sigma>0$, then $C^{\mathrm{NV}}(h,p,F_D)\sim he^{\mu+\sigma\sqrt{2\ln p}}$.*

*(d) If $D$ is a continuous fat-tailed r.v., i.e., $\bar{F}_D(x)\sim\gamma x^{-\alpha}$ for $\gamma>0$ and $\alpha>1$, then $C^{\mathrm{NV}}(h,p,F_D)\sim\alpha(\alpha-1)^{-1}\gamma^{1/\alpha}h^{(\alpha-1)/\alpha}p^{1/\alpha}$.*

*(e) If $D$ is an integer-valued r.v. with $\lim_{n\to\infty} r_D(n)=\gamma\in(0,1)$, where $r_D(n)=\mathbb{P}(D=n|D\ge n)$ for $n\in\mathbb{N}$, then $C^{\mathrm{NV}}(h,p,F_D)\sim h(\ln\frac{1}{1-\gamma})^{-1}\ln p$.*

*(f) If $D$ is a Poisson r.v., then $C^{\mathrm{NV}}(h,p,F_D)\sim hg^{-1}(p)$, where $g^{-1}(\cdot)$ is the inverse function of $g(x)=x^x$ for $x>0$.*

Table 1 summarizes many commonly used distributions that belong to the six classes in parts (a)-(f) of Proposition 1, where the support, *p.d.f.* or *p.m.f.* and *c.d.f.* of $D$, and the exact rate of the newsvendor cost (scaled by $1/h$) are listed. For many distributions, the newsvendor cost scales up at the rate of a polynomial of $\ln p$. In particular, the failure rate of a geometric Poisson distribution converges to a constant in (0, 1) (see the proof in Appendix A.3), and the newsvendor cost under this distribution scales up at the rate of $\ln p$ from part (e) in Proposition 1. Pareto distribution with parameters $x_{\min}>0$ and $\alpha>1$ and Burr distribution with parameters $c>0$ and $k>1/c$ are

**Table 1** Exact Rate of Newsvendor Cost (Scaled by $1/h$) under Different Demand Distributions

| Distribution | Support | $f_D(x)$ | $F_D(x)$ | Exact rate of $C^{\mathrm{NV}}(h,p,F_D)/h$ |
|---|---|---|---|---|
| Exponential | $(0,\infty)$ | $\lambda e^{-\lambda x}$ | $1-e^{-\lambda x}$ | $\lambda^{-1}\ln p$ |
| Laplace | $(-\infty,\infty)$ | $\frac{1}{2b}e^{-\frac{\lvert x-\mu\rvert}{b}}$ | $1-\frac{1}{2}e^{-\frac{x-\mu}{b}}, \forall x\geq\mu$ | $b\ln p$ |
| $\chi^2$ | $(0,\infty)$ | $\frac{x^{\frac{\alpha}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})}$ | * | $2\ln p$ |
| Gamma | $(0,\infty)$ | $\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ | * | $\beta^{-1}\ln p$ |
| Logistic | $(-\infty,\infty)$ | $\frac{e^{-\frac{x-\mu}{s}}}{s(1+e^{-\frac{x-\mu}{s}})^2}$ | $\frac{1}{1+e^{-\frac{x-\mu}{s}}}$ | $s\ln p$ |
| Gumbel max | $(0,\infty)$ | $\frac{e^{-x}e^{-e^{-x}}}{1-e^{-1}}$ | $\frac{e^{-e^{-x}}-e^{-1}}{1-e^{-1}}$ | $\ln p$ |
| Inverse Gaussian | $(0,\infty)$ | $\sqrt{\frac{\lambda}{2\pi x^3}}e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}$ | * | $2\mu^2\lambda^{-1}\ln p$ |
| Gaussian | $(-\infty,\infty)$ | $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ | $\Phi(\frac{x-\mu}{\sigma})$ | $\sqrt{2}\sigma\sqrt{\ln p}$ |
| Weibull | $(0,\infty)$ | $\frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}e^{-(\frac{x}{\alpha})^\beta}$ | $1-e^{-(\frac{x}{\alpha})^\beta}$ | $\alpha(\ln p)^{\frac{1}{\beta}}$ |
| Gumbel min | $(0,\infty)$ | $e^{1+x-e^x}$ | $1-e^{1-e^x}$ | $\ln\ln p$ |
| Log-normal | $(0,\infty)$ | $\frac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}$ | $\Phi(\frac{\ln x-\mu}{\sigma})$ | $e^{\mu+\sigma\sqrt{2\ln p}}$ |
| Pareto | $(x_{\min},\infty)$ | $\alpha x_{\min}^{\alpha}x^{-\alpha-1}$ | $1-x_{\min}^{\alpha}x^{-\alpha}$ | $\frac{\alpha x_{\min}}{\alpha-1}(p/h)^{\frac{1}{\alpha}}$ |
| Burr | $(0,\infty)$ | $\frac{ckx^{c-1}}{(1+x^c)^{k+1}}$ | $1-(1+x^c)^{-k}$ | $\frac{ck}{ck-1}(p/h)^{\frac{1}{ck}}$ |
| Geometric | $\{0,1,2,\ldots\}$ | $(1-q)^x q$ | $1-(1-q)^{x+1}$ | $(\ln\frac{1}{1-q})^{-1}\ln p$ |
| Negative binomial | $\{r,r+1,r+2,\ldots\}$ | $C_{x-1}^{r-1}(1-q)^{x-r}q^r$ | * | $(\ln\frac{1}{1-q})^{-1}\ln p$ |
| Geometric Poisson | $\{0,1,2,\ldots\}$ | $\sum_{k=1}^{n}e^{-\lambda}\frac{\lambda^k}{k!}(1-\gamma)^{n-k}\gamma^k C_{n-1}^{k-1}$ | * | $(\ln\frac{1}{1-\gamma})^{-1}\ln p$ |
| Poisson | $\{0,1,2,\ldots\}$ | $e^{-\lambda}\lambda^x/x!$ | * | $g^{-1}(p)^\dagger$ |

$^\dagger$ $g^{-1}(\cdot)$ is the inverse function of $g(x)=x^x$, $\forall x>0$.

both fat-tailed distributions, whose resulting newsvendor cost grows polynomially in $p$ according to part (d) of Proposition 1.

The following proposition presents weaker asymptotic bounds on the newsvendor cost.

PROPOSITION 2. *Suppose $D$ is unbounded and $\mathbb{E}[D-x|D>x]=\mathcal{O}(x)$. The following results hold:*

*(a) If $\mathbb{E}[D^k]<\infty$ but $\mathbb{E}[D^{k+1}]=\infty$ for some $k>1$, then $C^{\mathrm{NV}}(h,p,F_D)=o(p^{1/k})$.*

*(b) If $D$ is sub-exponential, i.e., $\bar{F}_D(x)=\mathcal{O}(e^{-cx})$ for some $c>0$, then $C^{\mathrm{NV}}(h,p,F_D)=\mathcal{O}(\ln p)$.*

*(c) If $D$ is sub-Gaussian, i.e., $\bar{F}_D(x)=\mathcal{O}(e^{-cx^2})$ for some $c>0$, then $C^{\mathrm{NV}}(h,p,F_D)=\mathcal{O}(\sqrt{\ln p})$.*

Parts (a)-(c) reveal that the asymptotic bound of the newsvendor cost depends on the rate at which the tail function $\bar{F}_D(x)$ converges to zero. For any distribution with up to $k$-th order moment being finite, the newsvendor cost grows slower than $p^{1/k}$ does because in this case, it can be shown that $\bar{F}_D(x)$ decays to zero in $o(x^{-k})$. For sub-exponential and sub-Gaussian distributions, $\bar{F}_D(x)$ decays no faster than the tail functions of exponential and Gaussian distributions respectively. Thus, the newsvendor cost asymptotically scales up no faster than $\ln p$ and $\sqrt{\ln p}$ as the exact rate of the newsvendor cost under these two distributions respectively.

For completeness, we end this section by presenting two results for the case with bounded $D$ (i.e., $\bar{D} < \infty$). In this case, $\lim_{p\to\infty} C^{\rm NV}(h,p,F_D) = h(\bar{D} - \mathbb{E}[D])$, and the following proposition characterizes the rate at which the newsvendor cost converges to its finite limit.

PROPOSITION 3. *(a) If $D$ is a continuous bounded r.v. and $\lim_{x\uparrow\bar{D}} \bar{F}_D(x)/(\bar{D}-x)^k = \gamma$ for some $k>0$ and $\gamma>0$, then $h(\bar{D}-\mathbb{E}[D]) - C^{\rm NV}(h,p,F_D) \sim \frac{k}{k+1} h^{1+\frac{1}{k}} \gamma^{-\frac{1}{k}} \cdot p^{-\frac{1}{k}}$.*

*(b) If $D$ is an integer-valued bounded r.v., then $C^{\rm NV}(h,p,F_D) = h(\bar{D} - \mathbb{E}[D])$ for any $p \geq h\big((\mathbb{P}(D=\bar{D}))^{-1} - 1\big)$.*

We note that many bounded continuous distributions satisfy the condition in part (a), such as uniform distribution (with $k=1$), triangular distribution (with $k=2$), and Beta distribution with parameters $\alpha>0$ and $\beta>0$ (with $k=\beta$).

## 3. Lost-Sales System with Positive Lead Times and Non-Stationary Demand

In this section, we consider a periodic-review lost-sales inventory system with positive replenishment lead times of $L \in \mathbb{N}^+$ periods and a non-stationary demand process. When demand distributions are stationary and unmet demand is backlogged, it is well-known that the optimal policy is a base-stock policy that keeps a constant inventory position. When unmet demand is lost, even under *i.i.d.* demands, the optimal policy is very complex and intractable in computation. Many heuristic policies have been proposed in the literature (see, e.g., Zipkin 2008, Huh et al. 2009, Bijvank et al. 2014, Arts et al. 2015, Xin and Goldberg 2016, Xin 2021, van Jaarsveld and Arts 2024). We refer to Bijvank et al. (2023) for a recent review on this stream of literature. In this section, we consider a slightly more general system where the demands follow a particular non-stationary process.

We first introduce the non-stationary demand process considered in this section. For each $t \geq 1$, we assume that the random demand $D_t$ in period $t$ takes the following form:

$$D_t = d_t + W_t, \tag{2}$$

where $\{d_t : t \geq 1\}$ is a sequence of non-negative deterministic numbers and $\{W_t : t \geq 1\}$ is a sequence of *i.i.d.* non-negative *r.v.'s* with the same distribution as *r.v.* $W$, and $0 < \mathbb{E}[W] < \infty$. The non-negativity of $W_t$ is without loss of generality. To see this, we first note that since $D_t$ is a non-negative *r.v.*, $W_t$ has to be bounded from below by some constant $w_{\min} \in \mathbb{R}$ satisfying $d_t + w_{\min} \geq 0$. Then we can express $D_t$ as $(d_t + w_{\min}) + (W_t - w_{\min})$, with both $d_t + w_{\min}$ and $W_t - w_{\min}$ being non-negative and satisfying our assumption. The special form of non-stationary demand (2) arises when the first part is the non-stationary average demand whereas the second part is *i.i.d.* random noise. For example, $\{d_t : t \geq 1\}$ captures the known seasonal effect of demand, and $\{W_t : t \geq 1\}$ captures the *i.i.d.* random noise in the demand. We leave the study for a more general non-stationary demand process with non i.i.d. noise as an important future research direction.

The sequence of events in each period $t \geq 1$ is as follows. First, the firm receives the order placed $L$ periods ago. Then, it reviews the system state $\mathbf{x}_t \triangleq (I_t, x_{t,1}, \ldots, x_{t,L-1})$, where $I_t$ is the on-hand inventory level, and $x_{t,i}$ is the order quantity placed in period $t+i-L$. Second, the firm decides the order quantity $q_t \geq 0$, which will arrive at the beginning of period $t+L$. We denote $x_{t,L} \triangleq q_t$. Third, $D_t$ is realized and satisfied by the on-hand inventory as much as possible. At the end of period $t$, unmet demand is lost, incurring a unit lost-sales penalty cost $p$. Leftover inventory is carried to the next period, incurring unit holding cost $h$. For simplicity, we assume that the system is initially empty, i.e., $\mathbf{x}_1 = \mathbf{0}$.

An admissible replenishment policy is described by a sequence of measurable functions $\{\varphi_t(\cdot) : t \geq 1\}$, where each $\varphi_t(\cdot)$ maps the system state $\mathbf{x}_t$ to a non-negative order quantity. Given an admissible policy $\pi$, the total cost in period $t$ is $C_t^\pi \triangleq h(I_t^\pi - D_t)^+ + p(D_t - I_t^\pi)^+$. The long-run average cost of policy $\pi$, denoted by $C^\pi$, is defined as $C^\pi \triangleq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^T \mathbb{E}[C_t^\pi]$. and the optimal cost, denoted by OPT, is defined as $\mathrm{OPT} \triangleq \inf_\pi C^\pi$.

We then introduce a class of heuristic replenishment policies for the system described above. When demands are stationary, the class of base-stock policies, which orders to raise the *inventory position* (i.e., the sum of on-hand and pipeline inventories) to a constant level as much as possible, has been studied extensively (e.g., Zipkin 2008, Huh et al. 2009). When demands are non-stationary, we introduce a new class of policies, called modified base-stock policies, that carefully takes the demand non-stationarity into account. The modified base-stock policy with level $S \geq 0$, denoted by $\pi_S$, places the following order quantity in each period $t \geq 1$:

$$q_t^{\pi_S} = d_{t+L} + \left(S + \sum_{i=t}^{t+L-1} d_i - \left(I_t^{\pi_S} + \sum_{i=1}^{L-1} x_{t,i}^{\pi_S}\right)\right)^+ \cdot \mathbb{I}\{t \geq L+1\}. \tag{3}$$

That is, in the first $L$ periods, $\pi_S$ orders $d_{1+L}, d_{2+L}, \ldots, d_{2L}$ units sequentially, and subsequently, it raises the inventory position in each period $t \geq L+1$ to $\max\{S + \sum_{i=t}^{t+L} d_i, I_t^{\pi_S} + \sum_{i=1}^{L-1} x_{t,i}^{\pi_S} + d_{t+L}\}$. Therefore, $\pi_S$ seeks to maintain a *time-dependent* order-up-to level $S + \sum_{i=t}^{t+L} d_i$ in each period $t$ by ordering at least $d_{t+L}$ units to make sure that the deterministic part of the demand in period $t+L$ is always fulfilled. Then a crucial question is whether the post-ordering inventory position always attains the desired level $S + \sum_{i=t}^{t+L} d_i$ in each period $t$ under the modified base-stock policy $\pi_S$. To be seen from the proof of Lemma 1 later, we prove that this is true for each $t \geq L+1$ using a sample-path inductive argument. This attainability result will be crucial for establishing a newsvendor-type upper bound on the long-run average cost of modified base-stock policies. The long-run average cost of the modified base-stock policy $\pi_S$ is denoted by $C(S)$ and the best base-stock level is denoted by $S^*$, i.e., $S^* = \arg\min_{S\geq 0} C(S)$. For OPT, $C(S)$ and $S^*$, we add superscript "(I)", meaning system (I), to differentiate them from those for other systems to be introduced in the subsequent sections.

The following lemma establishes a lower bound on the optimal cost and an upper bound on the long-run average cost of the modified base-stock policy. These bounds extend those in Theorem 5 of Janakiraman et al. (2007) and Lemma 5 of Huh et al. (2009), respectively, to the setting with a non-stationary demand process. In particular, to construct the upper bound, we first prove the aforementioned attainability result for the order-up-to level $S+\sum_{i=t}^{t+L} d_i$ in each period $t \geq L+1$. The proof for both the upper and lower bounds is based on a sample-path argument and is provided in Appendix B.1. We denote $\mathscr{W}_{L+1} =^d \sum_{i=1}^{L+1} W_i$.

Lemma 1. *The following inequalities hold:*

$$\mathrm{OPT}^{(\mathrm{I})} \geq C^{\mathrm{NV}}\Big(h, \frac{p}{1+L}, F_{\mathscr{W}_{L+1}}\Big), \tag{4}$$

$$C^{(\mathrm{I})}(S) \leq h\mathbb{E}[(S-\mathscr{W}_{L+1})^+] + (p+Lh)\mathbb{E}[(\mathscr{W}_{L+1}-S)^+], \quad \forall S \geq 0. \tag{5}$$

Lemma 1 enables us to characterize the asymptotic scaling of the optimal cost and establish the asymptotic optimality of the best modified base-stock policy. The following theorem follows from Lemma 1, Theorem 1 and part (d) of Proposition 1 (see Appendix B.2 for a proof). To highlight the dependency on the unit penalty cost, we add a subscript "$p$" to relevant quantities.

Theorem 2. *Suppose $W$ is unbounded. Then the following results hold:*

*(a) If $\mathbb{E}[\mathscr{W}_{L+1}-x|\mathscr{W}_{L+1}>x] = o(x)$, then*

$$\lim_{p\to\infty} \frac{\mathrm{OPT}_p^{(\mathrm{I})}}{F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1})} = \lim_{p\to\infty} \frac{C_p^{(\mathrm{I})}(S_p^{(\mathrm{I}),*})}{F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1})} = h. \tag{6}$$

*(b) If $\mathbb{E}[\mathscr{W}_{L+1}-x|\mathscr{W}_{L+1}>x] = \mathcal{O}(x)$, then $\mathrm{OPT}_p^{(\mathrm{I})} = \Theta(F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1}))$ and $C_p^{(\mathrm{I})}(S_p^{(\mathrm{I}),*}) = \Theta(F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1}))$. Moreover, if $\mathscr{W}_{L+1}$ follows a continuous fat-tailed distribution, i.e., $\bar{F}_{\mathscr{W}_{L+1}}(x) \sim \gamma x^{-\alpha}$ for $\alpha > 1$ and $\gamma > 0$, then*

$$\frac{\alpha\gamma^{1/\alpha}h^{(\alpha-1)/\alpha}}{(\alpha-1)(1+L)^{1/\alpha}} \leq \liminf_{p\to\infty} \frac{\mathrm{OPT}_p^{(\mathrm{I})}}{p^{1/\alpha}} \leq \limsup_{p\to\infty} \frac{C_p^{(\mathrm{I})}(S_p^{(\mathrm{I}),*})}{p^{1/\alpha}} \leq \frac{\alpha\gamma^{1/\alpha}h^{(\alpha-1)/\alpha}}{\alpha-1}. \tag{7}$$

Equation (6) implies the asymptotic optimality of the best modified base-stock policy with large unit penalty costs under condition $\mathbb{E}[\mathscr{W}_{L+1}-x|\mathscr{W}_{L+1}>x] = o(x)$, which extends the result in Huh et al. (2009) to the setting with a non-stationary demand process. Similar to Bijvank et al. (2014), we can show that these asymptotic results also hold for a class of easy-to-compute heuristic base-stock policies, and we refer the reader to Appendix E for details. We remark that under the weaker condition $\mathbb{E}[\mathscr{W}_{L+1}-x|\mathscr{W}_{L+1}>x] = \mathcal{O}(x)$, it remains unknown whether the best modified base-stock policy is asymptotically optimal in the system studied in this section (including the classical lost-sales inventory system with *i.i.d.* demands). Our current approach for proving the asymptotic optimality is based on the newsvendor-type upper and lower bounds on the optimal cost in inequalities (4) and (5). These bounds are asymptotically identical when $\mathbb{E}[\mathscr{W}_{L+1}-x|\mathscr{W}_{L+1}>x]=o(x)$, but they are generally not asymptotically identical when this condition is violated. For example, under fat-tailed distributions, part (b) in Theorem 2 shows that their ratio asymptotically equals $(1+L)^{1/\alpha}$, and due to this, we are only able to establish $\limsup_{p\to\infty} C_p^{(\mathrm{I})}(S_p^{(\mathrm{I}),*})/\mathrm{OPT}_p^{(\mathrm{I})}\le (1+L)^{1/\alpha}$. We leave the asymptotic optimality of the best modified base-stock policy under the weaker condition $\mathbb{E}[\mathscr{W}_{L+1}-x|\mathscr{W}_{L+1}>x]=\mathcal{O}(x)$ as future research.

As discussed earlier, the condition $\mathbb{E}[\mathscr{W}_{L+1}-x|\mathscr{W}_{L+1}>x]=o(x)$ is satisfied by many demand distributions. When $\mathscr{W}_{L+1}$ belongs to one of the six classes in Proposition 1, the optimal cost rate for this lost-sales inventory system with lead times and non-stationary demand process can be characterized accordingly. Below we provide some examples:

(i) If $W$ is a continuous IFR *r.v.* with $\lim_{x\to\infty} r_W(x)=\gamma\in(0,\infty)$, then $\lim_{x\to\infty} r_{\mathscr{W}_{L+1}}(x)=\gamma$ (see Appendix B.3 for a proof) and $\mathrm{OPT}_p^{(\mathrm{I})}\sim h(\ln p)/\gamma$ by Proposition 1-(a).

(ii) If $W$ is an integer-valued *r.v.* with $\lim_{n\to\infty} r_W(n)=\gamma\in(0,1)$, then $\lim_{n\to\infty} r_{\mathscr{W}_{L+1}}(n)=\gamma$ by Proposition 1 in Arts et al. (2017) and $\mathrm{OPT}_p^{(\mathrm{I})}\sim h(\ln p)/(-\ln(1-\gamma))$ by Proposition 1-(e).

(iii) If $W$ is a Poisson *r.v.*, then so does $\mathscr{W}_{L+1}$ and $\mathrm{OPT}_p^{(\mathrm{I})}\sim hg^{-1}(p)$ by Proposition 1-(f).

When $\mathscr{W}_{L+1}$ follows any distribution in Proposition 2, the asymptotic bounds of the optimal cost can be further characterized. In particular, if $W$ follows a distribution in parts (a)-(c) of Proposition 2, it is easy to verify that $\mathscr{W}_{L+1}$ also follows a distribution in parts (a)-(c), respectively. Thus, the asymptotic bounds on the optimal cost can be characterized accordingly under these distributions of $W$.

When $W$ is bounded, one can easily verify from Lemma 1 that $\lim_{p\to\infty}\mathrm{OPT}_p^{(\mathrm{I})}=h(L+1)(\bar{W}-\mathbb{E}[W])$, where $\bar{W}\triangleq\sup\{x:F_W(x)<1\}$. The following proposition characterizes the rate at which the optimal cost converges to its finite limit. We omit its proof for brevity.

Proposition 4. *(a) If $W$ is a continuous bounded r.v. and $\lim_{x\uparrow(L+1)\bar{W}}\bar{F}_{\mathscr{W}_{L+1}}(x)/((L+1)\bar{W}-x)^k=\gamma\in(0,\infty)$ for $k>0$, then $h(L+1)(\bar{W}-\mathbb{E}[W])-\mathrm{OPT}_p^{(\mathrm{I})}=\Theta(p^{-1/k})$.*

*(b) If $W$ is an integer-valued bounded r.v., then $\mathrm{OPT}_p^{(\mathrm{I})}=h(L+1)(\bar{W}-\mathbb{E}[W])$ when $p$ is sufficiently large, in particular when $p\ge h(1+L)\big((\mathbb{P}(W=\bar{W}))^{-1}-1\big)$.*

## 4. Perishable Inventory System with Fixed Lifetime and Partial Backorders

In this section, we study a periodic-review inventory system for a perishable product with fixed lifetime of $m\in\mathbb{N}^+$ periods, partial backorders, zero replenishment lead time and the same non-stationary demand process in §3. When demands are stationary and unmet demand is fully backlogged or lost, the system has been studied extensively in the literature (e.g., Nahmias 1976, Chazan and Gal 1977, Cooper 2001, Li et al. 2016, Chao et al. 2015, Zhang et al. 2023, Bu et al. 2023). We refer to Karaesmen et al. (2011) and Nahmias (2011) for detailed reviews and Li and Yu (2023) for a more recent review. In this section, we consider a more general system where a random fraction of unmet demand in each period is backlogged whereas the rest is lost, and the demand also follows a non-stationary process. Clearly, when the random fraction is always equal to one, it reduces to the lost-sales model, and when the random fraction is always equal to 0, it reduces to the backlogging model. Thus, the model includes perishable inventory system with lost sales and perishable inventory system with backlogs as special cases.

The demand process $\{D_t : t \geq 1\}$ is assumed to follow (2), with an extra mild assumption that $\liminf_{T\to\infty} \sum_{t=1}^{T} d_t/T < \infty$. The sequence of events in each period $t \geq 1$ is described as follows. First, the firm reviews the system state $\mathbf{x}_t \triangleq (x_{t,1}, x_{t,2}, \ldots, x_{t,m-1}, b_{t-1})$, where each $x_{t,i}$ denotes the amount of on-hand inventory whose remaining lifetime is no more than $i$ periods and $b_{t-1}$ is the amount of backorders at the beginning of period $t$. Second, the firm places an order of $q_t$ units and receives it immediately. If $b_{t-1} > 0$, then the backorders are satisfied by the new order to the maximum extent. Let $x_{t,m} \triangleq x_{t,m-1} - b_{t-1} + q_t$. Third, demand $D_t$ is realized and satisfied by the on-hand inventory as much as possible following the first-in-first-out (FIFO) issuance policy. At the end of period $t$, a random fraction $\zeta_t \in [0,1]$ of unsatisfied demands is backlogged, incurring a unit backlogging cost $b$, whereas the remaining $(1-\zeta_t)$-fraction is lost, incurring a unit lost-sales penalty cost $p$. Assume that $\{\zeta_t : t \geq 1\}$ is a sequence of *i.i.d. r.v.'s* with the same distribution as *r.v.* $\zeta$, and it is independent of the amounts of unsatisfied demands. Let $\vartheta \triangleq \mathbb{E}[\zeta]$. Any leftover inventory incurs a unit holding cost $h$. Finally, leftover inventories reaching the end of the lifetime are outdated, incurring a unit outdating cost $\theta \geq 0$. The amount of outdated inventory in period $t$ is denoted by $o_t \triangleq (x_{t,1} - D_t)^+$. The remaining inventories are carried to the next period, with their remaining lifetimes deducted by one period. Same as §3, we assume that the system is initially empty. Given an admissible policy $\pi$, the total cost in period $t$ is

$$C_t^{\pi} \triangleq h(x_{t,m}^{\pi} - D_t)^+ + \big(b\zeta_t + p(1-\zeta_t)\big)(D_t - x_{t,m}^{\pi})^+ + \theta o_t^{\pi}.$$

The long-run average cost of policy $\pi$ and the optimal cost are defined in the same way as those in §3.

We now introduce a class of modified base-stock policies. The modified base-stock policy with level $S \geq 0$, denoted by $\pi_S$, places the order quantity $q_t^{\pi_S} = d_t + \big(S - (x_{t,m-1}^{\pi_S} - b_{t-1}^{\pi_S})\big)^+$ in each period $t \geq 1$. That is, $\pi_S$ orders in each period $t \geq 1$ to maintain a *time-dependent* order-up-to level $S + d_t$ by ordering at least $d_t$ units to make sure the deterministic part of the demand in period $t$ is always fulfilled. One can easily verify that in each period $t \geq 1$, the order up-to level $S + d_t$ can always be attained after ordering (see the proof of Lemma 2 for details). Similar to §3, for any $S \geq 0$, let $C(S)$ denote the long-run average cost under modified base-stock policy $\pi_S$. Also denote

$S^* \triangleq \min_{S\geq 0} C(S)$. Similar to §3, we add superscript "(II)" to OPT, $C(S)$ and $S^*$ in this section, indicating that these measures are for system (II).

The following lemma establishes a lower bound on the optimal cost and an upper bound on the long-run average cost of the modified base-stock policy, which extend Proposition 2 and Lemma 3 in Bu et al. (2023) respectively to the setting with partial backorder and a non-stationary demand process. Its proof is also based on a sample-path argument and deferred to Appendix C.1.

LEMMA 2. *The following inequalities hold:*

$$\mathrm{OPT}^{(\mathrm{II})} \geq C^{\mathrm{NV}}\Big(h+\frac{\theta}{m}, \vartheta b+(1-\vartheta)p, F_W\Big) - \frac{m-1}{m}\theta\Big(\mathbb{E}[W]+\liminf_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} d_t\Big), \tag{8}$$

$$C^{(\mathrm{II})}(S) \leq \Big(h+\frac{\theta}{m}\Big)\mathbb{E}[(S-W)^+] + \big(\vartheta b+(1-\vartheta)p\big)\mathbb{E}[(W-S)^+], \quad \forall S\geq 0. \tag{9}$$

Lemma 2 implies that when demand is unbounded, the best modified base-stock policy is asymptotically optimal when $p\to\infty$ (suppose $\vartheta<1$) and/or $b\to\infty$ (suppose $\vartheta>0$), i.e., $(C^{(\mathrm{II})}(S^{(\mathrm{II}),*}) - \mathrm{OPT}^{(\mathrm{II})})/\mathrm{OPT}^{(\mathrm{II})}$ goes to zero in the asymptotic regime mentioned. Lemma 2, together with Theorem 1, also enables us to characterize the asymptotic scaling of the optimal cost. The asymptotic regime we consider is large unit backlogging and lost-sales penalty costs while keeping their ratio constant. That is, we consider $p\to\infty$ while keeping $b=\kappa_0 p$ for some constant $\kappa_0>0$. Denote $\nu \triangleq \vartheta(\kappa_0-1)+1$. The following theorem summarizes our main results in this section, whose proof is similar to that of Theorem 2 and thus omitted.

THEOREM 3. *Suppose $W$ is unbounded. Then the following results hold:*

*(a) If $\mathbb{E}[W-x|W>x]=o(x)$, then*

$$\lim_{p\to\infty}\frac{\mathrm{OPT}_p^{(\mathrm{II})}}{F_W^{-1}(\frac{p}{p+1})} = \lim_{p\to\infty}\frac{C_p^{(\mathrm{II})}(S_p^{(\mathrm{II}),*})}{F_W^{-1}(\frac{p}{p+1})} = h+\frac{\theta}{m}. \tag{10}$$

*(b) If $\mathbb{E}[W-x|W>x]=\mathcal{O}(x)$, then $\mathrm{OPT}_p^{(\mathrm{II})}=\Theta(F_W^{-1}(\frac{p}{p+1}))$ and $C_p^{(\mathrm{II})}(S_p^{(\mathrm{II}),*})=\Theta(F_W^{-1}(\frac{p}{p+1}))$. Moreover, if $W$ follows a continuous fat-tailed distribution, i.e., $\bar{F}_W(x)\sim\gamma x^{-\alpha}$ for $\alpha>1$ and $\gamma>0$, then*

$$\lim_{p\to\infty}\frac{\mathrm{OPT}_p^{(\mathrm{II})}}{p^{1/\alpha}} = \lim_{p\to\infty}\frac{C_p^{(\mathrm{II})}(S_p^{(\mathrm{II}),*})}{p^{1/\alpha}} = \alpha(\alpha-1)^{-1}(\gamma\nu)^{1/\alpha}(h+\theta/m)^{(\alpha-1)/\alpha}. \tag{11}$$

Part (a) states that under the condition $\mathbb{E}[W-x|W>x]=o(x)$, when the unit backlogging and lost-sales penalty costs are scaled up proportionally, the optimal cost and the long-run average cost of the best modified base-stock policy asymptotically scale up as $F_W^{-1}(p/(p+1))$. Under the weaker condition $\mathbb{E}[W-x|W>x]=\mathcal{O}(x)$, part (b) shows that the long-run average costs of the optimal policy and the best modified base-stock policy are asymptotically bounded by $F_W^{-1}(p/(p+1))$. For a fat-tailed distribution, it also gives an exact characterization on the cost rates of the optimal policy and the best modified base-stock policy.

We can also apply Propositions 1 and 2 to characterize the asymptotic rates or bounds of the two costs under different demand distributions. In particular, all the results in Table 1 of §2 for various probability distributions hold true as the optimal cost rate of the perishable inventory system of this section. Same as the non-perishable inventory system studied in §3, we can show that these asymptotic results in Theorem 3 also hold for a class of heuristic base-stock policies. See Appendix E for details.

When demand $D$ is bounded, the bounds in Lemma 2 are not tight enough to characterize the asymptotic scaling of the optimal cost as $p \to \infty$. When unmet demand is fully lost and demands are stationary (i.e., $d_t$ is a constant for all $t \geq 1$), by applying two tighter bounds on the optimal cost developed in Bu et al. (2023), we have the following result (whose proof can be found in Appendix C.2). Denote $\mathrm{OPT}_{\infty}^{(\mathrm{II})} \triangleq h(\bar{D} - \mathbb{E}[D]) + \theta \mathbb{E}[O_{\infty}(\bar{D})]$, where $\mathbb{E}[O_{\infty}(\bar{D})]$ denotes the long-run average outdates under base-stock policy $\pi_{\bar{D}}$. For the existence of random variable $O_{\infty}(\bar{D})$, we refer to Lemma 1 in Bu et al. (2023) for details.

Proposition 5. *Suppose $\{D_t : t \geq 1\}$ is a sequence of i.i.d. r.v.'s, and unmet demand is lost (i.e., $\zeta \equiv 0$). Then, $\lim_{p \to \infty} \mathrm{OPT}_p^{(\mathrm{II})} = \mathrm{OPT}_{\infty}^{(\mathrm{II})}$ and the following results hold:*

*(a) If $D$ is a continuous r.v. and $\lim_{x \uparrow \bar{D}} \bar{F}_D(x)/(\bar{D} - x)^k = \gamma \in (0, \infty)$ for $k > 0$, then $\mathrm{OPT}_{\infty}^{(\mathrm{II})} - \mathrm{OPT}_p^{(\mathrm{II})} = \mathcal{O}(p^{-1/k})$.*

*(b) If $D$ is an integer-valued r.v., then $\mathrm{OPT}_p^{(\mathrm{II})} = \mathrm{OPT}_{\infty}^{(\mathrm{II})}$ when $p$ is sufficiently large, in particular when $p \geq (h + \theta)/\mathbb{P}(D = \bar{D}) - h$.*

## 5. Continuous-review Inventory System with Lead Times and Lost Sales

In this section, we consider a classical continuous-review inventory system with fixed lead times and lost sales, which has been studied extensively in inventory literature (e.g., Karush 1957, Reiman 2004, Xin 2022). In particular, Reiman (2004) characterizes the asymptotic scaling of the long-run average cost of the best base-stock policy when the lead time and the unit penalty cost grow proportionally. In the same asymptotic regime, Xin (2022) proves that the long-run average cost of the best capped base-stock policy is asymptotically at most 1.79 times of the optimal cost. In this section, we characterize the asymptotic scaling of the optimal cost in the regime of large unit lost-sales penalty cost (with fixed lead times). Our result also implies that the best base-stock policy is asymptotically optimal in this asymptotic regime.

We briefly describe the system as follows. Demand arrivals follow a Poisson process with rate $\lambda > 0$. The replenishment lead times are fixed and equal to $L > 0$. Let $D(t)$ be the cumulative demand during $[0, t]$. The demand occurred at time $t$ is given by $d(t) = D(t) - D(t^-)$, where

$D(t^-) \triangleq \lim_{s\uparrow t} D(s)$. The firm's control policy is given by a family of functions $\{Q(t): t\geq 0\}$, where $Q(t)$ denotes the cumulative orders placed during $[0,t]$. Thus, the order placed at time $t$, denoted as $q(t)$, equals $Q(t)-Q(t^-)$, where $Q(t^-)\triangleq \lim_{s\uparrow t} Q(s)$.

The sequence of events is as follows. At time $t$, the firm first reviews the on-hand inventory level $I(t^-)$ and receives the order placed $L$ units of time ago, i.e., $q(t-L)$. If a demand arrives at time $t$, then the firm uses the on-hand inventory to satisfy it. Unmet demand is lost. The on-hand inventory level $I(t)$ and the lost-sales quantity $l(t)$ at time $t$ are given by $I(t)=(I(t^-)+q(t-L)-d(t))^+$ and $l(t)=(d(t)-I(t^-)-q(t-L))^+$, respectively. Next, the firm places an order of size $q(t)$, which will arrive at time $t+L$. Let $h$ be the cost rate of holding one unit of inventory and $p$ be the cost of losing one unit of demand. The long-run average cost of an admissible policy $\pi$ is defined as $C^\pi = \limsup_{T\to\infty}\frac{1}{T}\{h\int_0^T \mathbb{E}[I^\pi(t)]dt + p\mathbb{E}[A^\pi(T)]\}$, where $A^\pi(T)$ denotes the cumulative lost-sales quantity during $[0,T]$ under policy $\pi$. The firm's objective is to minimize the long-run average cost. Let $\mathrm{OPT}^{(\mathrm{III})}$ be the optimal long-run average cost.

We next introduce the class of base-stock policies. At any time $t\geq 0$, base-stock policy $\pi_S$ with level $S\geq 0$ keeps the inventory position $\mathrm{IP}(t)$ (i.e., the on-hand inventory plus all the orders in transit) at the constant level $S$. Under base-stock policy $\pi_S$, an order is triggered whenever the inventory position drops below $S$, or equivalently, a new demand arrives. As shown in Karush (1957), the system under a base-stock policy $\pi_S$ is equivalent to an Erlang B system with $S$ servers, Poisson customer arrival rate $\lambda$ and service time $L$. Moreover, the long-run average cost of base-stock policy $\pi_S$, denoted by $C^{(\mathrm{III})}(S)$, is given by

$$C^{(\mathrm{III})}(S) = h(S-\lambda L) + \lambda(p+hL)B(S,\lambda L), \quad \forall S\geq 0, \tag{12}$$

where the function $B(S,a)\triangleq \frac{a^S/S!}{\sum_{n=0}^S a^n/n!}$ for any $S\in\mathbb{N}$ and $a>0$ is known as the Erlang loss function. Karush (1957) shows that $B(S,a)$ is decreasing and convex in $S$ on $\mathbb{N}$. Thus, $C^{(\mathrm{III})}(S)$ is convex in $S$ on $\mathbb{N}$ and the best base-stock level, denoted by $S^{(\mathrm{III}),*}\triangleq \arg\min_{S\in\mathbb{N}} C^{(\mathrm{III})}(S)$, satisfies the following inequalities (see, e.g., Smith 1977):

$$B(S^{(\mathrm{III}),*},\lambda L) - B(S^{(\mathrm{III}),*}+1,\lambda L) \leq \frac{h}{p+hL} \leq B(S^{(\mathrm{III}),*}-1,\lambda L) - B(S^{(\mathrm{III}),*},\lambda L). \tag{13}$$

The following lemma provides a lower bound on the optimal cost. This result has been established in Lemma 5 of Xin (2022) based on a comparison to a counterpart backlogging system. In Appendix D.1, we provide a different proof based on a sample-path argument.

Lemma 3. $\mathrm{OPT}^{(\mathrm{III})} \geq C^{\mathrm{NV}}(h, p/L, F_{D(L)})$.

The following theorem characterizes the asymptotic scaling of the optimal cost and the long-run average cost of the best base-stock policy. It implies that the best base-stock policy is asymptotically optimal with large unit penalty cost. To our knowledge, this is the first asymptotic optimality result established for continuous-review lost-sales inventory systems. The proof of Theorem 4 is built upon equation (12), inequality (13) and Lemma 3, which is deferred to Appendix D.2.

THEOREM 4. $\lim_{p\to\infty} \mathrm{OPT}_p^{(\mathrm{III})}/g^{-1}(p) = \lim_{p\to\infty} C_p^{(\mathrm{III})}(S_p^{(\mathrm{III}),*})/g^{-1}(p) = h$, *where* $g^{-1}(\cdot)$ *is the inverse function of* $g(x) = x^x$ *for* $x > 0$.

## 6. Final Remarks

In this paper, we study the asymptotic scaling of the newsvendor cost and apply it to analyze the asymptotic scaling of the optimal cost and establish asymptotic optimality of the best (modified) base-stock policy for three multi-dimensional inventory systems. We expect our analysis to be useful in studying other complex inventory systems (e.g., perishable inventory systems with positive lead times, serial inventory systems with lost sales, etc.). To this end, one needs to construct tight newsvendor-type upper and lower bounds on the optimal cost, and then apply similar analysis to Theorem 2 to obtain the desired results. In this paper, the upper bounds we construct on the optimal costs come from those on the long-run average costs of the (modified) base-stock policy. One may also establish newsvendor-type upper bounds on the long-run average costs of other classes of heuristic policies, e.g., the projected inventory level (PIL) policy proposed in van Jaarsveld and Arts (2024). Finally, since the demand process in the real world is usually non-stationary, the results in §3 and §4 for inventory systems with non-stationary demand are of great practical interest. However, we are able to prove the results only for the case when the uncertainty (or noise) for various periods are *i.i.d.* It will be particularly interesting to explore the direction of general non-stationary demand process for practically important complex inventory systems.

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# Online Appendix for "Asymptotic scaling of optimal cost and asymptotic optimality of base-stock policy in several multi-dimensional inventory systems"

By Jinzhi Bu, Xiting Gong, and Xiuli Chao

## Appendix A: Proofs of Statements in Section 2

### A.1. Proof of Theorem 1

**Proof of Part (a)**. We first note the following equation:

$$\frac{C^{\text{NV}}(h,p,F_D)}{F_D^{-1}(\frac{p}{p+1})}=\frac{C^{\text{NV}}(h,p,F_D)}{F_D^{-1}(\frac{p}{p+h})}\times\frac{F_D^{-1}(\frac{p}{p+h})}{F_D^{-1}(\frac{p}{p+1})}.$$

Therefore, it suffices to prove the following two equations:

$$\lim_{p\to\infty}\frac{F_D^{-1}(\frac{p}{p+h})}{F_D^{-1}(\frac{p}{p+1})}=1, \tag{EC.1}$$

$$\lim_{p\to\infty}\frac{C^{\text{NV}}(h,p,F_D)}{F_D^{-1}(\frac{p}{p+h})}=h. \tag{EC.2}$$

We first prove Equation (EC.1) for $0<h\le 1$. If $h>1$, we notice that

$$\lim_{p\to\infty}\frac{F_D^{-1}(\frac{p}{p+h})}{F_D^{-1}(\frac{p}{p+1})}=\lim_{p\to\infty}\frac{F_D^{-1}(\frac{p/h}{p/h+1})}{F_D^{-1}(\frac{p/h}{p/h+1/h})}=\lim_{p\to\infty}\frac{F_D^{-1}(\frac{p}{p+1})}{F_D^{-1}(\frac{p}{p+1/h})}.$$

When $h>1$, we have $1/h<1$ and if Equation (EC.1) can be shown for any $0<h\le 1$, it also holds for any $h>1$ from the above equation. When $0<h\le 1$, we have $F_D^{-1}(p/(p+h))\ge F_D^{-1}(p/(p+1))$. In addition, we also have the following inequalities:

$$\begin{aligned}\frac{F_D^{-1}(\frac{p}{p+h})}{F_D^{-1}(\frac{p}{p+1})}&\le\frac{F_D^{-1}(\frac{p}{p+h})}{C^{\text{NV}}(1,\frac{p}{h},F_D)}\times\frac{\mathbb{E}[(F_D^{-1}(\frac{p}{p+1})-D)^+]+\frac{p}{h}\mathbb{E}[(D-F_D^{-1}(\frac{p}{p+1}))^+]}{F_D^{-1}(\frac{p}{p+1})}\\&\le\frac{F_D^{-1}(\frac{p}{p+h})}{\mathbb{E}[(F_D^{-1}(\frac{p}{p+h})-D)^+]}\times\left(1+\frac{p\mathbb{E}[(D-F_D^{-1}(\frac{p}{p+1}))^+]}{hF_D^{-1}(\frac{p}{p+1})}\right),\end{aligned} \tag{EC.3}$$

where the first inequality holds since $F_D^{-1}(p/(p+h))=\arg\min_{S\ge 0}\{\mathbb{E}[(S-D)^+]+p/h\mathbb{E}[(D-S)^+]\}$. For the first term in the RHS of (EC.3), it is easy to see from $\mathbb{E}[D]<\infty$ and $\bar{D}=\infty$ that

$$\lim_{p\to\infty}\frac{F_D^{-1}(\frac{p}{p+h})}{\mathbb{E}[(F_D^{-1}(\frac{p}{p+h})-D)^+]}=1. \tag{EC.4}$$

For the second term, we note that

$$\begin{aligned}\lim_{p\to\infty}\frac{p\mathbb{E}[(D-F_D^{-1}(\frac{p}{p+1}))^+]}{F_D^{-1}(\frac{p}{p+1})}&=\lim_{p\to\infty}\frac{p\mathbb{E}\big[D-F_D^{-1}(\frac{p}{p+1})\big|D>F_D^{-1}(\frac{p}{p+1})\big]\mathbb{P}(D>F_D^{-1}(\frac{p}{p+1}))}{F_D^{-1}(\frac{p}{p+1})}\\&\le\lim_{p\to\infty}\frac{p}{p+1}\frac{\mathbb{E}\big[D-F_D^{-1}(\frac{p}{p+1})\big|D>F_D^{-1}(\frac{p}{p+1})\big]}{F_D^{-1}(\frac{p}{p+1})}=0,\end{aligned} \tag{EC.5}$$

where the inequality follows from the definition of $F_D^{-1}(p/(p+1))$ and the last equality follows from $\mathbb{E}[D-x|D>x]=o(x)$ and $\lim_{p\to\infty} F_D^{-1}(p/(p+1))=\infty$ (due to $\bar{D}=\infty$). Combining (EC.4) and (EC.5), the right-hand-side of (EC.3) converges to 1 as $p\to\infty$ and Equation (EC.1) holds.

To see Equation (EC.2), similar to (EC.4) and (EC.5), we have the following two equations respectively:

$$\lim_{p\to\infty}\frac{\mathbb{E}[(F^{-1}(\frac{p}{p+h})-D)^+]}{F^{-1}(\frac{p}{p+h})}=1,\quad \lim_{p\to\infty}\frac{p\mathbb{E}[(D-F^{-1}(\frac{p}{p+h}))^+]}{F^{-1}(\frac{p}{p+h})}=0,$$

which, together with the definition of $C^{\mathrm{NV}}(h,p,F_D)$, imply Equation (EC.2).

**Proof of Part (b).** From the definition of $C_p^{NV}(h,p,F_D)$, we have the following inequalities:

$$\frac{h\mathbb{E}[(F_D^{-1}(\frac{p}{p+h})-D)^+]}{F_D^{-1}(\frac{p}{p+h})}\le\frac{C_p^{NV}(h,p,F_D)}{F_D^{-1}(\frac{p}{p+h})}\le h+\frac{p\mathbb{E}[(D-F_D^{-1}(\frac{p}{p+h}))^+]}{F_D^{-1}(\frac{p}{p+h})}. \tag{EC.6}$$

Since $\mathbb{E}[D-x|D>x]=\mathcal{O}(x)$, we have

$$\begin{aligned}\frac{p\mathbb{E}[(D-F_D^{-1}(\frac{p}{p+h}))^+]}{F_D^{-1}(\frac{p}{p+h})}&=\frac{p\mathbb{E}[D-F_D^{-1}(\frac{p}{p+h})|D>F_D^{-1}(\frac{p}{p+h})]\cdot\mathbb{P}(D>F_D^{-1}(\frac{p}{p+h}))}{F_D^{-1}(\frac{p}{p+h})}\\&\le\frac{ph}{p+h}\cdot\frac{\mathbb{E}[D-F_D^{-1}(\frac{p}{p+h})|D>F_D^{-1}(\frac{p}{p+h})]}{F_D^{-1}(\frac{p}{p+h})}=\mathcal{O}(1).\end{aligned} \tag{EC.7}$$

When $p$ is sufficiently large, it follows from $\lim_{p\to\infty}F_D^{-1}(\frac{p}{p+h})=\infty$ that

$$h\mathbb{E}[(F_D^{-1}(\frac{p}{p+h})-D)^+]/F_D^{-1}(\frac{p}{p+h})\ge h(1-\mathbb{E}[D]/F_D^{-1}(\frac{p}{p+h}))\ge h/2.$$

Combining this with inequalities (EC.6) and (EC.7), we obtain

$$C^{NV}\Big(h,p,F_D^{-1}\Big(\frac{p}{p+h}\Big)\Big)=\Theta\Big(F_D^{-1}\Big(\frac{p}{p+h}\Big)\Big).$$

We next prove that $F_D^{-1}(\frac{p}{p+h})=\Theta(F_D^{-1}(\frac{p}{p+1}))$. If $h<1$, $F_D^{-1}(\frac{p}{p+h})\ge F_D^{-1}(\frac{p}{p+1})$ and it suffices to prove $F_D^{-1}(\frac{p}{p+h})=\mathcal{O}(F_D^{-1}(\frac{p}{p+1}))$. Note that inequality (EC.3) and Equation (EC.4) continue to hold. It then suffices to show

$$\frac{p\mathbb{E}[(D-F_D^{-1}(\frac{p}{p+1}))^+]}{F_D^{-1}(\frac{p}{p+1})}=\mathcal{O}(1),$$

which follows from similar arguments to inequality (EC.7) and the assumption $\mathbb{E}[D-x|D>x]=\mathcal{O}(x)$. If $h\ge1$, the proof is similar and omitted for brevity. Therefore, we have $C^{NV}(h,p,F_D)=\Theta(F_D^{-1}(\frac{p}{p+1}))$. Q.E.D.

### A.2. Proof of Proposition 1

We first show the results in parts (a)-(c) and (e)-(f). We first verify that all the demand distributions in parts (a)-(c) and (e)-(f) in Proposition 1 satisfy $\mathbb{E}[D-x|D>x]=o(x)$. From the result in part (c) of Theorem 1 in Huh et al. (2009), it suffices to show the following two properties for these distributions: $\mathbb{E}[D^2]<\infty$ and $\lim_{x\to\infty} x\cdot r_D(x)=\infty$.

**(a)** For demand distributions in part (a), since $\lim_{x\to\infty} r_D(x)/x^k=\gamma$ for some $k>-1$, we directly have $\lim_{x\to\infty} x\cdot r_D(x)=\infty$. Moreover, when $t$ is sufficiently large, we have $r_D(t)\geq \frac{1}{2}\gamma t^k$. Noting that $\bar{F}_D(x)=\exp\left(-\int_0^x r_D(t)dt\right)$, we then have for sufficiently large $x$,

$$\bar{F}_D(x)\leq \exp\left(-\frac{1}{2}\gamma\int_{\frac{1}{2}x}^{x} t^k dt\right)=\exp\left(-\frac{\gamma}{2(k+1)}(1-\frac{1}{2^{k+1}})x^{k+1}\right).$$

This implies that

$$\mathbb{E}[D^2]=\int_0^\infty \mathbb{P}(D^2>x)dx=\int_0^\infty \bar{F}_D(\sqrt{x})dx=2\int_0^\infty t\bar{F}_D(t)dt<\infty,$$

where the last equality follows from the change of variable letting $t=\sqrt{x}$.

**(b)** For Gumbel min distribution in part (b), the two properties $\mathbb{E}[D^2]<\infty$ and $\lim_{x\to\infty} x\cdot r_D(x)=\infty$ can be easily verified from $\bar{F}_D(x)=e^{1-e^x}$ and $r_D(x)=e^x$.

**(c)** For log-normal distribution in part (c), it is easy to verify that $\mathbb{E}[D^2]=e^{2(\mu+\sigma^2)}$. Moreover,

$$\lim_{x\to\infty} x\cdot r_D(x)=\lim_{x\to\infty}\frac{x\cdot\frac{1}{\sigma x}\phi(\frac{\ln x-\mu}{\sigma})}{\bar{\Phi}(\frac{\ln x-\mu}{\sigma})}=\frac{1}{\sigma}\cdot\lim_{t\to\infty}\frac{\phi(t)}{\bar{\Phi}(t)}=\frac{1}{\sigma}\cdot\lim_{t\to\infty}\frac{-t\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}}{-\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}}=\infty,$$

where the third equation follows from L'Hospital's rule.

**(e) & (f)** For any integer-valued *r.v.* satisfying $\lim_{n\to\infty} r_D(n)=\gamma\in(0,1]$, by applying similar arguments to inequality (62) in Arts et al. (2015), we can easily verify that for any $\epsilon\in(0,\gamma)$, $\mathbb{E}[D-x|D>x]\leq\frac{1}{\gamma-\epsilon}$ when $x$ is sufficiently large. Thus, $\mathbb{E}[D-x|D>x]=o(x)$. For Poisson distribution in part (f), Proposition 5.3 in Arts et al. (2015) shows that $\lim_{n\to\infty} r_D(n)=1$. Thus, all the distributions in parts (e) and (f) satisfy $\mathbb{E}[D-x|D>x]=o(x)$.

Based on Equation (1), it only remains to establish the following five equations for parts (a)-(c) and (e)-(f) respectively:

$$\text{For part (a):}\quad \lim_{p\to\infty} F_D^{-1}\left(\frac{p}{p+1}\right)/\left((k+1)\gamma^{-1}\ln p\right)^{\frac{1}{k+1}}=1; \tag{EC.8}$$

$$\text{For part (b):}\quad \lim_{p\to\infty} F_D^{-1}\left(\frac{p}{p+1}\right)/\ln\ln p=1; \tag{EC.9}$$

$$\text{For part (c):}\quad \lim_{p\to\infty} F_D^{-1}\left(\frac{p}{p+1}\right)/e^{\mu+\sigma\sqrt{2\ln p}}=1; \tag{EC.10}$$

$$\text{For part (e):}\quad \lim_{p\to\infty} F_D^{-1}\left(\frac{p}{p+1}\right)/\left((\ln\frac{1}{1-\gamma})^{-1}\ln p\right)=1; \tag{EC.11}$$

$$\text{For part (f):}\quad \lim_{p\to\infty} F_D^{-1}\left(\frac{p}{p+1}\right)/g^{-1}(p)=1. \tag{EC.12}$$

A similar result to Equation (EC.11) has been shown in Lemma A.2 of Arts et al. (2015). Thus, we omit its proof and refer to Arts et al. (2015). We next prove Equations (EC.8), (EC.9), (EC.10) and (EC.12) respectively. For convenience, we denote $S_p^\dagger \triangleq F_D^{-1}(\frac{p}{p+1})$.

**Proof of Equation (EC.8).** We first note that

$$\lim_{p\to\infty} \frac{(S_p^\dagger)^{k+1}}{\ln p} = \lim_{p\to\infty} \frac{(k+1)\cdot (S_p^\dagger)'\cdot (S_p^\dagger)^k}{p^{-1}} = (k+1)\lim_{p\to\infty} p\cdot (S_p^\dagger)'\cdot (S_p^\dagger)^k,$$

where the first identity follows from L'Hospital's rule. Then, it remains to prove that

$$\lim_{p\to\infty} p\cdot (S_p^\dagger)'\cdot (S_p^\dagger)^k = \frac{1}{\gamma}.$$

Since $\bar{F}_D(S_p^\dagger) = \frac{1}{p+1}$, by taking the derivative with respect to $p$ on both sides, we obtain

$$(S_p^\dagger)' f_D(S_p^\dagger) = \frac{1}{(p+1)^2} = \frac{1}{p+1}\bar{F}_D(S_p^\dagger),$$

which implies that

$$(S_p^\dagger)' = (p+1)^{-1}\bar{F}_D(S_p^\dagger)(f_D(S_p^\dagger))^{-1} = ((p+1)r_D(S_p^\dagger))^{-1}.$$

Thus,

$$\lim_{p\to\infty} p\cdot (S_p^\dagger)'\cdot (S_p^\dagger)^k = \lim_{p\to\infty} \frac{p\cdot (S_p^\dagger)^k}{(p+1)\cdot r_D(S_p^\dagger)} = \lim_{p\to\infty} \frac{(S_p^\dagger)^k}{r_D(S_p^\dagger)} = \frac{1}{\gamma},$$

where the last identity holds since $\lim_{x\to\infty} r_D(x)/x^k = \gamma$ and $\lim_{p\to\infty} S_p^\dagger = \infty$.

**Proof of Equation (EC.9).** For Gumbel min distribution, since $\bar{F}_D(x) = \exp(1-e^x)$, we can solve from

$$\bar{F}_D(S_p^\dagger) = \exp(1-e^{S_p^\dagger}) = \frac{1}{p+1}$$

that $S_p^\dagger = \ln(1+\ln(1+p))$. By applying L'Hospital's rule, we obtain

$$\lim_{p\to\infty} \frac{S_p^\dagger}{\ln\ln p} = \lim_{p\to\infty} \frac{\ln(1+\ln(1+p))}{\ln\ln p} = \lim_{p\to\infty} \frac{\frac{1}{1+p}\cdot\frac{1}{1+\ln(1+p)}}{\frac{1}{p}\cdot\frac{1}{\ln p}} = \lim_{p\to\infty} \frac{\ln p}{1+\ln(1+p)} = 1.$$

**Proof of Equation (EC.10).** We first prove Equation (EC.10) for $\mu = 0$ and $\sigma = 1$. We start with the following bounds on the tail function of standard Gaussian distribution (see, e.g., Gordon 1941):

$$\frac{1}{\sqrt{2\pi}}\frac{x}{x^2+1}e^{-\frac{x^2}{2}} \le 1-\Phi(x) \le \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}}. \tag{EC.13}$$

From (EC.13), when $p$ is sufficiently large, the following inequality holds:

$$\frac{1}{2\sqrt{2\pi}\ln S_p^\dagger}e^{-\frac{(\ln S_p^\dagger)^2}{2}} \le \frac{1}{p+1} \le \frac{1}{\sqrt{2\pi}\ln S_p^\dagger}e^{-\frac{(\ln S_p^\dagger)^2}{2}}. \tag{EC.14}$$

Taking logarithm on each side of inequality (EC.14) and after simple algebra, we get

$$-2\ln(2\sqrt{2\pi}) - 2\ln(\ln S_p^\dagger) \le (\ln S_p^\dagger)^2 - 2\ln(p+1) \le -2\ln(\sqrt{2\pi}) - 2\ln(\ln S_p^\dagger). \tag{EC.15}$$

By dividing $\ln S_p^\dagger + \sqrt{2\ln(p+1)}$ on each side of inequality (EC.15), we further have

$$\frac{-2\ln(2\sqrt{2\pi}) - 2\ln(\ln S_p^\dagger)}{\ln S_p^\dagger + \sqrt{2\ln(p+1)}} \le \ln S_p^\dagger - \sqrt{2\ln(p+1)} \le \frac{-2\ln(\sqrt{2\pi}) - 2\ln(\ln S_p^\dagger)}{\ln S_p^\dagger + \sqrt{2\ln(p+1)}}.$$

Since $\lim_{p\to\infty} S_p^\dagger = \infty$, the left-most side and the right-most side of the above inequality both converge to zero as $p\to\infty$. Thus, we have $\lim_{p\to\infty}(\ln S_p^\dagger - \sqrt{2\ln(p+1)}) = 0$. Since

$$\lim_{p\to\infty}\left(\sqrt{2\ln(p+1)} - \sqrt{2\ln p}\right) = \lim_{p\to\infty}\frac{2\ln(p+1) - 2\ln p}{\sqrt{2\ln(p+1)} + \sqrt{2\ln p}} = 0,$$

we obtain $\lim_{p\to\infty}(\ln S_p^\dagger - \sqrt{2\ln p}) = 0$. Thus,

$$\lim_{p\to\infty}\frac{S_p^\dagger}{e^{\sqrt{2\ln p}}} = \lim_{p\to\infty} e^{\ln S_p^\dagger - \sqrt{2\ln p}} = 1,$$

which completes the proof of the equation in part (c) for $\mu = 0$ and $\sigma = 1$.

We next prove Equation (EC.10) for general $\mu \in \mathbb{R}$ and $\sigma > 0$. To facilitate discussion, we highlight the dependency of $S_p^\dagger$ on $\mu$ and $\sigma$ by writing it as $S_p^\dagger(\mu,\sigma)$. We can easily verify the following equations:

$$\ln S_p^\dagger(\mu,\sigma) = \mu + \sigma\cdot\Phi^{-1}\left(\frac{p}{p+1}\right) = \mu + \sigma\cdot\ln S_p^\dagger(0,1).$$

Then, $S_p^\dagger(\mu,\sigma) = e^{\mu}(S_p^\dagger(0,1))^{\sigma}$. Since

$$\lim_{p\to\infty} S_p^\dagger(0,1)/e^{\sqrt{2\ln p}} = 1,$$

we conclude

$$\lim_{p\to\infty} S_p^\dagger(\mu,\sigma)/e^{\mu+\sigma\sqrt{2\ln p}} = 1.$$

**Proof of Equation (EC.12).** We first show that Equation (EC.12) is implied from the following equation:

$$\lim_{p\to\infty}\frac{S_p^\dagger \ln S_p^\dagger}{g^{-1}(p)\ln(g^{-1}(p))} = 1. \tag{EC.16}$$

Suppose to the contrary, Equation (EC.12) does not hold and $\limsup_{p\to\infty} S_p^\dagger/g^{-1}(p) > 1$. Then there exists some $\epsilon_0 > 0$ and a sequence $p_n \to \infty$ as $n\to\infty$ such that $S_{p_n}^\dagger \ge (1+\epsilon_0)g^{-1}(p_n)$ and $\ln S_{p_n}^\dagger \ge \ln(1+\epsilon_0) + \ln(g^{-1}(p_n))$, which imply

$$\frac{S_{p_n}^\dagger \ln S_{p_n}^\dagger}{g^{-1}(p_n)\ln(g^{-1}(p_n))} \ge (1+\epsilon_0)\left(\frac{\ln(1+\epsilon_0)}{\ln g^{-1}(p_n)} + 1\right),$$

leading to contradiction with (EC.16). Similarly, $\liminf_{p\to\infty} S_p^\dagger/g^{-1}(p) < 1$ will also lead to contradiction.

Now we prove Equation (EC.16). From the definition of $S_p^\dagger$, we have

$$F_D(S_p^\dagger - 1) < \frac{p}{p+1} \le F_D(S_p^\dagger). \tag{EC.17}$$

As shown in Theorem 2 of Short (2013), when $D$ is a Poisson $r.v.$, the following bounds hold for its cumulative distribution function: for any $n \in \mathbb{N}^+$,

$$\Phi\Big(1_{\{n-\mu>0\}} \cdot \sqrt{2H(\mu,n)}\Big) < \mathbb{P}(D \le n) < \Phi\Big(1_{\{n+1-\mu>0\}} \cdot \sqrt{2H(\mu,n+1)}\Big), \tag{EC.18}$$

where $H(x,y) = x - y(1+\ln x) + y\ln y$ and $\mu \triangleq \mathbb{E}[D]$. Applying (EC.18) to (EC.17), we have for sufficiently large $p$,

$$\Phi\Big(\sqrt{2H(\mu,S_p^\dagger - 1)}\Big) < \frac{p}{p+1} < \Phi\Big(\sqrt{2H(\mu,S_p^\dagger + 1)}\Big). \tag{EC.19}$$

After substituting the expressions of $H(\mu,S_p^\dagger-1)$ and $H(\mu,S_p^\dagger+1)$ into the above inequalities, and dividing $\ln p$ on each side of the above inequalities, we have

$$\frac{(S_p^\dagger - 1)\big(\ln(S_p^\dagger - 1) - 1 - \ln\mu\big)}{\ln p} \le \frac{1}{2\ln p}\Big(\Phi^{-1}\Big(\frac{p}{p+1}\Big)\Big)^2 - \frac{\mu}{\ln p}, \tag{EC.20}$$

and

$$\frac{(S_p^\dagger + 1)\big(\ln(S_p^\dagger + 1) - 1 - \ln\mu\big)}{\ln p} \ge \frac{1}{2\ln p}\Big(\Phi^{-1}\Big(\frac{p}{p+1}\Big)\Big)^2 - \frac{\mu}{\ln p}. \tag{EC.21}$$

For standard Gaussian distribution, we have

$$\lim_{x\to\infty}\frac{r_D(x)}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}e^{-\frac{x^2}{2}}}{\int_x^\infty e^{-\frac{t^2}{2}}dt} = \lim_{x\to\infty}\frac{-\frac{1}{x^2}e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}}}{-e^{-\frac{x^2}{2}}} = 1.$$

By applying the result in part (a) to normal distribution, we have

$$\lim_{p\to\infty}\frac{\Phi^{-1}\big(\frac{p}{p+1}\big)}{\sqrt{2\ln p}} = 1. \tag{EC.22}$$

Combining Equation (EC.22) with inequality (EC.20), we have

$$\limsup_{p\to\infty}\frac{(S_p^\dagger - 1)(\ln(S_p^\dagger - 1) - 1 - \ln\mu)}{\ln p} \le 1.$$

Since $\lim_{p\to\infty} S_p^\dagger = \infty$, it then follows from the property of lim sup that

$$\limsup_{p\to\infty}\frac{S_p^\dagger \ln S_p^\dagger}{\ln p} = \limsup_{p\to\infty}\frac{(S_p^\dagger - 1)(\ln(S_p^\dagger - 1) - 1 - \ln\mu)}{\ln p} \cdot \lim_{p\to\infty}\frac{S_p^\dagger \ln S_p^\dagger}{(S_p^\dagger - 1)(\ln(S_p^\dagger - 1) - 1 - \ln\mu)} \le 1.$$

Similarly, combining Equation (EC.22) and inequality (EC.20) with $\lim_{p\to\infty} S_p^\dagger = \infty$ leads to $\liminf_{p\to\infty} \frac{S_p^\dagger \ln S_p^\dagger}{\ln p} \geq 1$. Therefore, $\lim_{p\to\infty} \frac{S_p^\dagger \ln S_p^\dagger}{\ln p} = 1$. By the definition of $g(\cdot)$, we have $\ln p = g^{-1}(p)\ln(g^{-1}(p))$. Thus, Equation (EC.16) holds.

Finally, we prove the result in part (d). Note that

$$\frac{C^{\mathrm{NV}}(h,p,F_D)}{p^{\frac{1}{\alpha}}} = \frac{h\mathbb{E}\Big[\Big(F_D^{-1}\big(\frac{p}{p+h}\big) - D\Big)^+\Big] + p\mathbb{E}\Big[\Big(D - F_D^{-1}\big(\frac{p}{p+h}\big)\Big)^+\Big]}{F_D^{-1}\big(\frac{p}{p+h}\big)} \cdot \frac{F_D^{-1}\big(\frac{p}{p+h}\big)}{p^{\frac{1}{\alpha}}},$$

Since $\lim_{p\to\infty} h\mathbb{E}[(F_D^{-1}(\frac{p}{p+h}) - D)^+]/F_D^{-1}(\frac{p}{p+h}) = h$ due to $\bar{D} = \infty$, it then suffices to establish the following two equations:

$$\lim_{p\to\infty} \frac{p\mathbb{E}\Big[\Big(D - F_D^{-1}\big(\frac{p}{p+h}\big)\Big)^+\Big]}{F_D^{-1}\big(\frac{p}{p+h}\big)} = \frac{h}{\alpha - 1}, \tag{EC.23}$$

$$\lim_{p\to\infty} \frac{F_D^{-1}\big(\frac{p}{p+h}\big)}{p^{\frac{1}{\alpha}}} = \gamma^{\frac{1}{\alpha}} \cdot h^{-\frac{1}{\alpha}}. \tag{EC.24}$$

We first show Equation (EC.23). Note that

$$\begin{aligned} \frac{p\mathbb{E}\Big[\Big(D - F_D^{-1}\big(\frac{p}{p+h}\big)\Big)^+\Big]}{F_D^{-1}\big(\frac{p}{p+h}\big)} &= \frac{p\mathbb{E}\Big[D - F_D^{-1}\big(\frac{p}{p+h}\big)\Big| D > F_D^{-1}\big(\frac{p}{p+h}\big)\Big] \cdot \mathbb{P}\Big(D > F_D^{-1}\big(\frac{p}{p+h}\big)\Big)}{F_D^{-1}\big(\frac{p}{p+h}\big)} \\ &= \frac{ph}{p+h} \cdot \frac{\mathbb{E}\Big[D - F_D^{-1}\big(\frac{p}{p+h}\big)\Big| D > F_D^{-1}\big(\frac{p}{p+h}\big)\Big]}{F_D^{-1}\big(\frac{p}{p+h}\big)}. \end{aligned}$$

Therefore, to prove Equation (EC.23), it suffices to prove the following equation:

$$\lim_{x\to\infty} \frac{\mathbb{E}[D - x | D > x]}{x} = \frac{1}{\alpha - 1}. \tag{EC.25}$$

To see Equation (EC.25), we first note from $\bar{F}_D(x) \sim \gamma x^{-\alpha}$ that, for any $\epsilon > 0$, when $x$ is sufficiently large, we have $(\gamma - \epsilon)x^{-\alpha} \leq \bar{F}_D(x) \leq (\gamma + \epsilon)x^{-\alpha}$. Since $\mathbb{E}[D - x|D > x]/x = \int_x^\infty \bar{F}_D(t)dt/\big(\bar{F}_D(x)x\big)$, by applying the previous inequality, we have the following inequality for sufficiently large $x$:

$$\frac{(\gamma - \epsilon)\int_x^\infty t^{-\alpha}dt}{(\gamma + \epsilon)\cdot x^{-\alpha}\cdot x} \leq \frac{\mathbb{E}[D - x|D > x]}{x} \leq \frac{(\gamma + \epsilon)\int_x^\infty t^{-\alpha}dt}{(\gamma - \epsilon)\cdot x^{-\alpha}\cdot x}.$$

After simple algebra and by letting $x \to \infty$, we have

$$\frac{\gamma - \epsilon}{\gamma + \epsilon} \cdot \frac{1}{\alpha - 1} \leq \liminf_{x\to\infty} \frac{\mathbb{E}[D - x|D > x]}{x} \leq \limsup_{x\to\infty} \frac{\mathbb{E}[D - x|D > x]}{x} \leq \frac{\gamma + \epsilon}{\gamma - \epsilon} \cdot \frac{1}{\alpha - 1}.$$

Letting $\epsilon \downarrow 0$ in each side of the above inequality, we obtain Equation (EC.25).

Equation (EC.24) is easily shown using the following argument. Let $x = F_D^{-1}(p/(p+h))$. Then we have $p = hF_D(x)/\bar{F}_D(x)$ and

$$\lim_{p\to\infty} \frac{F_D^{-1}\big(\frac{p}{p+h}\big)}{p^{\frac{1}{\alpha}}} = \lim_{x\to\infty} \frac{x\cdot\Big(\bar{F}_D(x)\Big)^{\frac{1}{\alpha}}}{h^{1/\alpha}\cdot\Big(F_D(x)\Big)^{\frac{1}{\alpha}}} = h^{-1/\alpha}\cdot\lim_{x\to\infty}\left(\frac{\bar{F}_D(x)}{x^{-\frac{1}{\alpha}}}\right)^{\frac{1}{\alpha}} = \gamma^{\frac{1}{\alpha}}\cdot h^{-\frac{1}{\alpha}},$$

which shows Equation (EC.24). Q.E.D.

### A.3. Limiting Failure Rate for Geometric Poisson Distribution

Let $D$ be a geometric Poisson *r.v.*, with the rate of Poisson being $\lambda$ and the success probability of the compounding geometric distribution being $\gamma \in (0,1)$. Then, the *p.m.f.* of $D$ is given by $f_D(0) = e^{-\lambda}$ and $f_D(n) = \sum_{k=1}^{n} e^{-\lambda} \frac{\lambda^k}{k!}(1-\gamma)^{n-k}\gamma^k C_{n-1}^{k-1}$ for any $n \geq 1$. From Theorem 1 in Özel and Inal (2010), the following equation holds for any $n \geq 2$:

$$f_D(n) = \frac{2n-2+z}{n}(1-\gamma) f_D(n-1) - \frac{n-2}{n}(1-\gamma)^2 f_D(n-2), \tag{EC.26}$$

where $z \triangleq \lambda\gamma/(1-\gamma)$. Dividing $\mathbb{P}(D \geq n-1)$ on both sides of Equation (EC.26), we have

$$r_D(n)\frac{\mathbb{P}(D \geq n)}{\mathbb{P}(D \geq n-1)} = \frac{2n-2+z}{n}(1-\gamma) r_D(n-1) - \frac{n-2}{n}(1-\gamma)^2 r_D(n-2)\frac{\mathbb{P}(D \geq n-2)}{\mathbb{P}(D \geq n-1)}. \tag{EC.27}$$

By the definition of $r_{D(L)}(k)$, for any $k \geq 2$, we have

$$\frac{\mathbb{P}(D \geq k)}{\mathbb{P}(D \geq k-1)} = \frac{\mathbb{P}(D \geq k-1) - \mathbb{P}(D = k-1)}{\mathbb{P}(D \geq k-1)} = 1 - r_D(k-1).$$

Plugging the above equation to (EC.27), we obtain

$$r_D(n)\big(1 - r_D(n-1)\big) = \frac{2n-2+z}{n}(1-\gamma) r_D(n-1) - \frac{n-2}{n}(1-\gamma)^2 \frac{r_D(n-2)}{1 - r_D(n-2)}. \tag{EC.28}$$

From Theorem 2.2 in Ninh and Prékopa (2013), $D$ has a log-concave *p.m.f.*. Therefore, the failure rate $r_D(n)$ increases in $n$ (see Barlow and Proschan 1965). Since $r_D(n) \leq 1$, the limit $r_\infty \triangleq \lim_{n\to\infty} r_D(n)$ exists. Letting $n \to \infty$ on both sides of (EC.28), we obtain the following equation:

$$r_\infty(1 - r_\infty) = 2(1-\gamma) r_\infty - (1-\gamma)^2 \frac{r_\infty}{1 - r_\infty}.$$

Since $r_\infty > 0$ from the increasing property of $r_D(n)$, dividing $r_\infty$ on both sides of the above equation and re-arranging the terms, we obtain $\lim_{n\to\infty} r_D(n) = r_\infty = \gamma$. Q.E.D.

### A.4. Proof of Proposition 2

**Proof of Part (a).** We note that

$$\mathbb{E}[D^k] = \int_0^\infty \mathbb{P}(D^k > x)dx = \int_0^\infty \bar{F}_D(x^{\frac{1}{k}})dx = k\int_0^\infty t^{k-1}\bar{F}_D(t)dt.$$

Since $\mathbb{E}[D^k] < \infty$, we then have

$$\lim_{x\to\infty}\int_x^\infty t^{k-1}\bar{F}_D(t)dt = \int_0^\infty t^{k-1}\bar{F}_D(t)dt - \lim_{x\to\infty}\int_0^x t^{k-1}\bar{F}_D(t)dt = 0. \tag{EC.29}$$

Note that for any $x > 0$, we have the following inequality:

$$\int_x^\infty t^{k-1}\bar{F}_D(t)dt \geq \int_x^{2x} t^{k-1}\bar{F}_D(t)dt \geq \int_x^{2x} x^{k-1}\bar{F}_D(2x)dt = x^k\bar{F}_D(2x) = \frac{1}{2^k}(2x)^k\bar{F}_D(2x), \tag{EC.30}$$

where the second inequality holds since $t^{k-1}$ increases in $t$ by the assumption that $k>1$ and $\bar{F}_D(t)$ decreases in $t$. Combining (EC.29) with (EC.30), we obtain $\lim_{x\to\infty} x^k \bar{F}_D(x)=0$. Thus, we have

$$\begin{aligned}\lim_{p\to\infty}\Big(F_D^{-1}\big(\frac{p}{p+1}\big)\Big)^k \frac{1}{p+1} &\le \lim_{p\to\infty}\Big(F_D^{-1}\big(\frac{p}{p+1}\big)\Big)^k \bar{F}_D\Big(\frac{1}{2}F_D^{-1}\big(\frac{p}{p+1}\big)\Big)\\ &= 2^k \lim_{x\to\infty} x^k \bar{F}_D(x)=0,\end{aligned} \tag{EC.31}$$

where the inequality holds since $\frac{1}{2}F_D^{-1}(p/(p+1))<F_D^{-1}(p/(p+1))$ implies $\bar{F}_D(\frac{1}{2}F_D^{-1}(p/(p+1)))>1/(p+1)$ from the definition of $F_D^{-1}(p/(p+1))$. Inequality (EC.31) shows that $F_D^{-1}(p/(p+1))=o(p^{1/k})$. It then follows from part (b) in Theorem 1 that $C^{\rm NV}(h,p,F_D)=o(p^{1/k})$.

**Proof of Part (b).** When $D$ follows a sub-exponential distribution, there exists positive constants $c'$ and $c$ such that $\bar{F}_D(x)\le c'\exp(-cx)$ when $x$ is sufficiently large. Therefore, when $p$ is sufficiently large,

$$\frac{1}{p+1}<\bar{F}_D\Big(\frac{1}{2}F_D^{-1}\big(\frac{p}{p+1}\big)\Big)\le c'\exp\Big(-\frac{c}{2}F_D^{-1}\big(\frac{p}{p+1}\big)\Big), \tag{EC.32}$$

where the first inequality holds due to the same reason to the inequality in (EC.31). Re-arranging the terms of the above inequality, we obtain

$$F_D^{-1}(p/(p+1))<\frac{2}{c}\ln(c'(p+1))=\mathcal{O}(\ln p).$$

It then follows from part (b) in Theorem 1 that $C^{\rm NV}(h,p,F_D)=\mathcal{O}(\ln p)$.

**Proof of Part (c).** When $D$ follows a sub-Gaussian distribution, similar to inequality (EC.32), there exist positive constants $c$ and $c'$ such that $\bar{F}_D(x)\le c'\exp(-cx^2)$ when $x$ is sufficiently large. Therefore, when $p$ is sufficiently large,

$$\frac{1}{p+1}<\bar{F}_D\Big(\frac{1}{2}F_D^{-1}\big(\frac{p}{p+1}\big)\Big)\le c'\exp\Big(-\frac{c}{4}\big(F_D^{-1}\big(\frac{p}{p+1}\big)\big)^2\Big),$$

which then implies

$$F_D^{-1}(p/(p+1))<\sqrt{\frac{4}{c}\ln(c'(p+1))}=\mathcal{O}(\sqrt{\ln p}).$$

It then follows from part (b) in Theorem 1 that $C^{\rm NV}(h,p,F_D)=\mathcal{O}(\sqrt{\ln p})$. Q.E.D.

### A.5. Proof of Proposition 3

**Proof of Part (a).** To show the equation in part (a), we first note the following identities:

$$\begin{aligned}h(\bar{D}-\mathbb{E}[D])-C^{\rm NV}(h,p,F_D) &= h\Big(\bar{D}-F_D^{-1}\big(\frac{p}{p+h}\big)\Big)-(h+p)\mathbb{E}\Big[\Big(D-F_D^{-1}\big(\frac{p}{p+h}\big)\Big)^+\Big]\\ &= h\Big(\bar{D}-F_D^{-1}\big(\frac{p}{p+h}\big)\Big)-(h+p)\int_{F_D^{-1}\left(\frac{p}{p+h}\right)}^{\bar{D}}\bar{F}_D(t)dt.\end{aligned}$$

Let $x = F_D^{-1}(\frac{p}{p+h})$. Then it is easy to verify that $p = hF_D(x)/\bar{F}_D(x)$ and $p+h = h/\bar{F}_D(x)$. Therefore, we have the following equations:

$$\frac{h\Big(\bar{D} - F_D^{-1}\big(\frac{p}{p+h}\big)\Big) - (h+p)\int_{F_D^{-1}\left(\frac{p}{p+h}\right)}^{\bar{D}} \bar{F}_D(t)dt}{p^{-\frac{1}{k}}} = \frac{h(\bar{D}-x) - \frac{h}{\bar{F}_D(x)}\int_x^{\bar{D}} \bar{F}_D(t)dt}{\left(\frac{h\cdot F_D(x)}{\bar{F}_D(x)}\right)^{-\frac{1}{k}}}$$

$$= h^{1+\frac{1}{k}} \cdot \big(F_D(x)\big)^{\frac{1}{k}} \cdot \left(\frac{\bar{F}_D(x)}{(\bar{D}-x)^k}\right)^{-\frac{1}{k}} \cdot \left(1 - \frac{\int_x^{\bar{D}} \bar{F}(t)dt}{(\bar{D}-x)\bar{F}_D(x)}\right).$$

Since $\lim_{x\uparrow\bar{D}} F_D(x) = 1$ and $\lim_{x\uparrow\bar{D}} \bar{F}_D(x)/(\bar{D}-x)^k = \gamma$, it suffices to prove the following equation:

$$\lim_{x\to\bar{D}} \frac{\int_x^{\bar{D}} \bar{F}_D(t)dt}{(\bar{D}-x)\bar{F}_D(x)} = \frac{1}{k+1}. \tag{EC.33}$$

Since $\lim_{x\uparrow\bar{D}} \bar{F}_D(x)/(\bar{D}-x)^k = \gamma$, for any $\varepsilon \in (0,\gamma)$, when $t$ is sufficiently close to $\bar{D}$, we have

$$(\gamma-\varepsilon)(\bar{D}-t)^k \le \bar{F}_D(t) \le (\gamma+\varepsilon)(\bar{D}-t)^k.$$

Then, we have

$$\frac{\int_x^{\bar{D}} (\gamma-\varepsilon)(\bar{D}-t)^k dt}{(\gamma+\varepsilon)(\bar{D}-x)^{k+1}} \le \frac{\int_x^{\bar{D}} \bar{F}_D(t)dt}{(\bar{D}-x)\bar{F}_D(x)} \le \frac{\int_x^{\bar{D}} (\gamma+\varepsilon)(\bar{D}-t)^k dt}{(\gamma-\varepsilon)(\bar{D}-x)^{k+1}},$$

which implies

$$\frac{\gamma-\varepsilon}{\gamma+\varepsilon}\cdot\frac{1}{k+1} \le \liminf_{x\uparrow\bar{D}} \frac{\int_x^{\bar{D}} \bar{F}_D(t)dt}{(\bar{D}-x)\bar{F}_D(x)} \le \limsup_{x\uparrow\bar{D}} \frac{\int_x^{\bar{D}} \bar{F}_D(t)dt}{(\bar{D}-x)\bar{F}_D(x)} \le \frac{\gamma+\varepsilon}{\gamma-\varepsilon}\cdot\frac{1}{k+1}.$$

Since the above inequalities hold for any $0 < \varepsilon < \gamma$, letting $\varepsilon \downarrow 0$, we obtain Equation (EC.33).

**Proof of Part (b).** It is easy to see that when $D$ is an integer-valued *r.v.* and

$$p > h\big((\mathbb{P}(D=\bar{D}))^{-1} - 1\big), \quad F_D^{-1}(\frac{p}{p+h}) = \bar{D},$$

and thus, $C^{\mathrm{NV}}(h,p,F_D) = h(\bar{D} - \mathbb{E}[D])$. Q.E.D.

## Appendix B: Proof of Statements in Section 3

### B.1. Proof of Lemma 1

**Proof of Inequality (4).** Let $l_t$ be the amount of lost-sales quantity in each period $t \ge 1$, i.e., $l_t \triangleq (D_t - I_t)^+$. For any admissible policy $\pi$, it is easy to verify the following equation from the system dynamics: for any $t \ge 1$,

$$I_{t+1}^\pi = I_t^\pi - (D_t - l_t^\pi) + x_{t,1}^\pi,$$

which then implies the following equation for any $t \geq 1$:

$$I_{t+L}^{\pi} = I_t^{\pi} - \sum_{i=t}^{t+L-1} (D_i - l_i^{\pi}) + \sum_{i=1}^{L} x_{t,i}^{\pi}. \tag{EC.34}$$

Therefore, we have

$$\big(I_{t+L}^{\pi} - D_{t+L}\big)^+ = \Big(I_t^{\pi} + \sum_{i=1}^{L} x_{t,i}^{\pi} - \sum_{i=t}^{t+L} D_i + \sum_{i=t}^{t+L-1} l_i^{\pi}\Big)^+ \geq \Big(I_t^{\pi} + \sum_{i=1}^{L} x_{t,i}^{\pi} - \sum_{i=t}^{t+L} D_i\Big)^+. \tag{EC.35}$$

In addition, we also have

$$\begin{aligned}\big(D_{t+L} - I_{t+L}^{\pi}\big)^+ &= \Big(\sum_{i=t}^{t+L} D_i - \sum_{i=t}^{t+L-1} l_i^{\pi} - I_t^{\pi} - \sum_{i=1}^{L} x_{t,i}^{\pi}\Big)^+ \\ &\geq \Big(\sum_{i=t}^{t+L} D_i - I_t^{\pi} - \sum_{i=1}^{L} x_{t,i}^{\pi}\Big)^+ - \sum_{i=t}^{t+L-1} l_i^{\pi}.\end{aligned}$$

which then implies

$$\sum_{i=t}^{t+L} (D_i - I_i^{\pi})^+ \geq \Big(\sum_{i=t}^{t+L} D_i - I_t^{\pi} - \sum_{i=1}^{L} x_{t,i}^{\pi}\Big)^+.$$

Taking the expectation on each side of the above inequality and summing over $t = 1, 2, \ldots, T$, we obtain the following inequality:

$$\sum_{t=1}^{T+L} \mathbb{E}\big[\big(D_t - I_t^{\pi}\big)^+\big] \geq \frac{1}{L+1} \sum_{t=1}^{T} \mathbb{E}\Big[\Big(\sum_{i=t}^{t+L} D_i^{\pi} - I_t^{\pi} - \sum_{i=1}^{L} x_{t,i}^{\pi}\Big)^+\Big]. \tag{EC.36}$$

Therefore, we have the following inequality for any $T \geq 1$:

$$\begin{aligned}&\sum_{t=1}^{T+L} \mathbb{E}[C_t^{\pi}] \\ &\geq h \sum_{t=1}^{T} \mathbb{E}[(I_{t+L}^{\pi} - D_{t+L})^+] + p \sum_{t=1}^{T+L} \mathbb{E}\big[\big(D_t - I_t^{\pi}\big)^+\big] \\ &\geq \sum_{t=1}^{T} \Bigg(h\mathbb{E}\Big[\Big(I_t^{\pi} + \sum_{i=1}^{L} x_{t,i}^{\pi} - \sum_{i=t}^{t+L} D_i\Big)^+\Big] + \frac{p}{L+1}\mathbb{E}\Big[\Big(\sum_{i=t}^{t+L} D_i^{\pi} - I_t^{\pi} - \sum_{i=1}^{L} x_{t,i}^{\pi}\Big)^+\Big]\Bigg) \\ &= \sum_{t=1}^{T} \Bigg(h\mathbb{E}\Big[\Big(I_t^{\pi} + \sum_{i=1}^{L} x_{t,i}^{\pi} - \sum_{i=t}^{t+L} d_i - \sum_{i=t}^{t+L} W_i\Big)^+\Big] + \frac{p}{L+1}\mathbb{E}\Big[\Big(\sum_{i=t}^{t+L} W_i - I_t^{\pi} - \sum_{i=1}^{L} x_{t,i}^{\pi} + \sum_{i=t}^{t+L} d_i\Big)^+\Big]\Bigg) \\ &\geq \sum_{t=1}^{T} \Bigg(h\mathbb{E}\Big[\Big(\mathbb{E}\Big[I_t^{\pi} + \sum_{i=1}^{L} x_{t,i}^{\pi} - \sum_{i=t}^{t+L} d_i\Big] - \sum_{i=t}^{t+L} W_i\Big)^+\Big] + \frac{p}{L+1}\mathbb{E}\Big[\Big(\sum_{i=t}^{t+L} W_i - \mathbb{E}\Big[I_t^{\pi} + \sum_{i=1}^{L} x_{t,i}^{\pi} - \sum_{i=t}^{t+L} d_i\Big]\Big)^+\Big]\Bigg) \\ &\geq \sum_{t=1}^{T} \min_{S \in \mathbb{R}} \Bigg\{h\mathbb{E}\Big[\Big(S - \sum_{i=t}^{t+L} W_i\Big)^+\Big] + \frac{p}{L+1}\mathbb{E}\Big[\Big(\sum_{i=t}^{t+L} W_i - S\Big)^+\Big]\Bigg\} \\ &= T \cdot \min_{S \geq 0} \Big\{h\mathbb{E}\Big[\Big(S - \mathscr{W}_{L+1}\Big)^+\Big] + \frac{p}{L+1}\mathbb{E}\Big[\Big(\mathscr{W}_{L+1} - S\Big)^+\Big]\Big\} \\ &= T \cdot C^{\mathrm{NV}}\Big(h, \frac{p}{L+1}, F_{\mathscr{W}_{L+1}}\Big),\end{aligned}$$

where the second inequality follows from (EC.35) and (EC.36), the third inequality follows from the conditional Jensen's inequality and the fact that $I_t^\pi + \sum_{i=1}^{L} x_{t,i}^\pi - \sum_{i=t}^{t+L} d_i$ is independent of $\sum_{i=t}^{t+L} W_i$, and the second identity holds since $\{W_t : t \geq 1\}$ is a sequence of non-negative *i.i.d. r.v.'s* and the minimum value is achieved at $S \in [0, \infty)$.

By dividing $T+L$ and letting $\limsup_{T\to\infty}$ on each side of the above inequality, we have $C^\pi \geq C^{\text{NV}}(h, \frac{p}{L+1}, F_{\mathscr{W}_{L+1}})$. Since this inequality holds for any admissible policy $\pi$, taking $\inf_\pi$ on both sides, we obtain inequality (4) from the definition of the optimal cost.

**Proof of Inequality (5).** We first show that under the modified base-stock policy $\pi_S$ with $S \geq 0$, the following inequality holds under any demand sample path and for each $t \geq L+1$:

$$I_t^{\pi_S} + \sum_{i=1}^{L-1} x_{t,i}^{\pi_S} \leq S + \sum_{i=t}^{t+L-1} d_i. \tag{EC.37}$$

When $t = L+1$, from the assumption of empty initial state, the system dynamics and equation (3), we know that

$$I_{L+1}^{\pi_S} + \sum_{i=1}^{L-1} x_{L+1,i}^{\pi_S} = d_{L+1} + \sum_{i=1}^{L-1} x_{L+1,i}^{\pi_S} = d_{L+1} + \sum_{i=L+2}^{2L} d_i \leq S + \sum_{i=L+1}^{2L} d_i.$$

Thus, inequality (EC.37) holds for period $t = L+1$. Suppose inequality (EC.37) holds for some period $t \geq L+1$, and we next prove it for period $t+1$. To this end, we notice the following inequality:

$$\begin{aligned}
I_{t+1}^{\pi_S} + \sum_{i=1}^{L-1} x_{t+1,i}^{\pi_S} &= I_t^{\pi_S} - D_t + l_t^{\pi_S} + x_{t,1}^{\pi_S} + \sum_{i=2}^{L} x_{t,i}^{\pi_S} \\
&= I_t^{\pi_S} + \sum_{i=1}^{L} x_{t,i}^{\pi_S} - d_t - W_t + \Big(W_t - \big(I_t^{\pi_S} - d_t\big)\Big)^+ \\
&= S + \sum_{i=t+1}^{t+L} d_i - W_t + \Big(W_t - \big(I_t^{\pi_S} - d_t\big)\Big)^+ \\
&\leq S + \sum_{i=t+1}^{t+L} d_i,
\end{aligned}$$

where the first identity follows from the system dynamics, the second identity follows from $D_t = d_t + W_t$ and $l_t = (D_t - I_t)^+$, the third identity holds since when $t \geq L+1$, the inductive assumption and equation (3) imply that $I_t^{\pi_S} + \sum_{i=1}^{L} x_{t,i}^{\pi_S} = S + \sum_{i=t}^{t+L} d_i$, and the inequality holds because $I_t^{\pi_S} = (I_{t-1}^{\pi_S} - D_{t-1})^+ + q_{t-L}^{\pi_S} \geq q_{t-L}^{\pi_S} \geq d_t$ for each $t \geq L+1$. Thus, inequality (EC.37) also holds for period $t+1$, completing the inductive argument.

Now we are ready to prove inequality (5). We notice from the identity in (EC.35) and inequality $(x+y)^+ \leq x^+ + y^+$ that for any admissible policy $\pi$ and $t \geq 1$,

$$\big(I_{t+L}^\pi - D_{t+L}\big)^+ \leq \Big(I_t^\pi + \sum_{i=1}^{L} x_{t,i}^\pi - \sum_{i=t}^{t+L} D_i\Big)^+ + \sum_{i=t}^{t+L-1} l_i^\pi$$

$$= \Big(I_t^\pi + \sum_{i=1}^{L} x_{t,i}^\pi - \sum_{i=t}^{t+L} D_i\Big)^+ + \sum_{i=t}^{t+L-1} (D_i - I_i^\pi)^+, \tag{EC.38}$$

and

$$\big(D_{t+L} - I_{t+L}^\pi\big)^+ = \Big(\sum_{i=t}^{t+L} D_i - \sum_{i=t}^{t+L-1} l_i^\pi - I_t^\pi - \sum_{i=1}^{L} x_{t,i}^\pi\Big)^+ \le \Big(\sum_{i=t}^{t+L} D_i - I_t^\pi - \sum_{i=1}^{L} x_{t,i}^\pi\Big)^+. \tag{EC.39}$$

Thus, we have the following inequality for the modified base-stock policy $\pi_S$:

$$\sum_{t=L+1}^{T} \mathbb{E}[C_{t+L}^{\pi_S}] \le h \sum_{t=L+1}^{T} \mathbb{E}\Big[\Big(I_t^{\pi_S} + \sum_{i=1}^{L} x_{t,i}^{\pi_S} - \sum_{i=t}^{t+L} D_i\Big)^+\Big] + h \sum_{t=L+1}^{T} \sum_{i=t}^{t+L-1} \mathbb{E}\Big[\Big(\sum_{j=i-L}^{i} D_j - I_{i-L}^{\pi_S} - \sum_{j=1}^{L} x_{i-L,j}^{\pi_S}\Big)^+\Big]$$
$$+ p \sum_{t=L+1}^{T} \mathbb{E}\Big[\Big(\sum_{i=t}^{t+L} D_i - I_t^{\pi_S} - \sum_{i=1}^{L} x_{t,i}^{\pi_S}\Big)^+\Big]. \tag{EC.40}$$

From our definition of the order quantity under the modified base-stock policy in Equation (3) and inequality (EC.37), for each period $t \ge L+1$, we have

$$I_t^{\pi_S} + \sum_{i=1}^{L} x_{t,i}^{\pi_S} - \sum_{i=t}^{t+L} D_i = S + \sum_{i=t}^{t+L} d_i - \Big(\sum_{i=t}^{t+L} d_i + \sum_{i=t}^{t+L} W_i\Big) = S - \sum_{i=t}^{t+L} W_i.$$

By plugging the above equation into the RHS of inequality (EC.40), we obtain the following inequality:

$$\sum_{t=L+1}^{T} \mathbb{E}[C_{t+L}^{\pi_S}] \le h \sum_{t=L+1}^{T} \mathbb{E}\Big[\Big(S - \sum_{i=t}^{t+L} W_i\Big)^+\Big] + h \sum_{t=L+1}^{T} \sum_{i=t}^{t+L-1} \mathbb{E}\Big[\Big(\sum_{j=i-L}^{i} W_j - S\Big)^+\Big]$$
$$+ p \sum_{t=L+1}^{T} \mathbb{E}\Big[\Big(\sum_{i=t}^{t+L} W_i - S\Big)^+\Big]$$
$$= (T-L)\Big(h\mathbb{E}\big[(S - \mathscr{W}_{L+1})^+\big] + (p + Lh)\mathbb{E}\big[(\mathscr{W}_{L+1} - S)^+\big]\Big).$$

By dividing $T$ on each side of the above inequality and taking $\limsup_{T\to\infty}$, we obtain inequality (5). Q.E.D.

### B.2. Proof of Theorem 2

From Lemma 1, we have the following inequality:

$$\frac{1}{L+1} \cdot \frac{C^{\mathrm{NV}}\Big(h(L+1), p, F_{\mathscr{W}_{L+1}}\Big)}{F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1})} \le \frac{\mathrm{OPT}_p^{(\mathrm{I})}}{F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1})} \le \frac{C_p^{(\mathrm{I})}(S_p^{(\mathrm{I}),*})}{F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1})} \le \frac{C^{\mathrm{NV}}(h, p, F_{\mathscr{W}_{L+1}}) + Lh\mathbb{E}[\mathscr{W}_{L+1}]}{F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1})}.$$

Since $W$ is unbounded, we have $\lim_{p\to\infty} Lh\mathbb{E}[\mathscr{W}_{L+1}]/F^{-1}_{\mathscr{W}_{L+1}}(p/(p+1)) = 0$. By applying Theorem 1 to the left-hand side and right-hand side of the above inequality, we obtain part (a) and the asymptotic bound $\Theta(F^{-1}_{\mathscr{W}_{L+1}}(p/(p+1))$ in part (b). Inequality (7) is also easily obtained by combining Lemma 1 and part (d) in Proposition 1, whose details are omitted for brevity. Q.E.D.

### B.3. Limiting Failure Rate for Convolution under IFR Distributions

In this appendix, we prove the following statement claimed in Section 3: if $W$ is a continuous *r.v.* with an increasing failure rate and $\lim_{x\to\infty} r_W(x)=\gamma\in(0,\infty)$, then $\lim_{x\to\infty} r_{\mathscr{W}_{L+1}}(x)=\gamma$.

We first show that for any unbounded non-negative continuous *r.v.* $X$ with p.d.f. $f_X(\cdot)$, c.d.f. $F_X(\cdot)$ and an increasing failure rate $r_X(\cdot)$, $\lim_{x\to\infty} r_X(x)=\lambda$ for some $0<\lambda<\infty$ if and only if $\lim_{x\to\infty}\frac{\bar{F}_X(x+y)}{\bar{F}_X(x)}=e^{-\lambda y}$ for any $y>0$. To see this, we note the following equation from the relationship $\bar{F}_X(x)=e^{-\int_0^x r_X(t)dt}$: for any $x>0$ and $y>0$,

$$\frac{\bar{F}_X(x+y)}{\bar{F}_X(x)}=e^{-\int_x^{x+y} r_X(t)dt}=e^{-\int_0^y r_X(x+t)dt}. \tag{EC.41}$$

If $\lim_{x\to\infty} r_X(x)=\lambda$ for some $0<\lambda<\infty$, it then follows from (EC.41) that

$$\lim_{x\to\infty}\frac{\bar{F}_X(x+y)}{\bar{F}_X(x)}=e^{-\lim_{x\to\infty}\int_0^y r_X(x+t)dt}=e^{-\lambda y}.$$

On the other hand, if $\lim_{x\to\infty}\bar{F}_X(x+y)/\bar{F}_X(x)=e^{-\lambda y}$, we have from (EC.41) that

$$\lim_{x\to\infty}\int_0^y r_X(x+t)dt=\lambda y.$$

This implies that $r_X(x)$ is bounded. Since $r_X(x)$ is also increasing by assumption, its limit $\lim_{x\to\infty} r_X(x)$ exists, and thus,

$$\lim_{x\to\infty}\int_0^y r_X(x+t)dt=y\lim_{x\to\infty} r_X(x),$$

which then implies $\lim_{x\to\infty} r_X(x)=\lambda$.

We now turn to proving $\lim_{x\to\infty} r_{\mathscr{W}_{L+1}}(x)=\gamma$. When $r_W(x)$ increases in $x$ and $\lim_{x\to\infty} r_W(x)=\gamma$, by applying the above property, we have $\lim_{x\to\infty}\bar{F}_W(x+y)/\bar{F}_W(x)=e^{-\lambda y}$ for any $y>0$. From Theorem 3-(b) in Embrechts and Goldie (1980), if two distributions with tail functions $\bar{F}_1$ and $\bar{F}_2$ satisfy $\lim_{x\to\infty}\bar{F}_i(x+y)/\bar{F}_i(x)=e^{-\lambda y}$, $i=1,2$ for some $\lambda\geq 0$ and any $y>0$, then the convolution of $F_1$ and $F_2$, with the tail function denoted by $\overline{G}$, also satisfies $\lim_{x\to\infty}\overline{G}(x+y)/\overline{G}(x)=e^{-\lambda y}$ for any $y>0$. Thus, by repeatedly applying this result, we obtain

$$\lim_{x\to\infty}\overline{F}_{\mathscr{W}_{L+1}}(x+y)/\overline{F}_{\mathscr{W}_{L+1}}(x)=e^{-\gamma y}. \tag{EC.42}$$

When $W$ has an IFR distribution, from the closure property of IFR distributions (see, e.g., Theorem 3.2 of Barlow et al. 1963), $\mathscr{W}_{L+1}$ also has an IFR distribution. This, combined with (EC.42) and the above property, implies $\lim_{x\to\infty} r_{\mathscr{W}_{L+1}}(x)=\gamma$. Q.E.D.

## Appendix C: Proofs of Statements in Section 4

### C.1. Proof of Lemma 2

**Proof of Inequality (8).** For any admissible policy $\pi$, we first note the following identities:

$$\begin{aligned}\sum_{t=1}^{T+m-1}\mathbb{E}[C_t^\pi] &= \sum_{t=1}^{T+m-1}\Big(h\mathbb{E}[(x_{t,m}^\pi - D_t)^+] + b\mathbb{E}[\zeta_t(D_t - x_{t,m}^\pi)^+] + p\mathbb{E}[(1-\zeta_t)(D_t - x_{t,m}^\pi)^+] + \theta\mathbb{E}[o_t^\pi]\Big)\\ &= \sum_{t=1}^{T+m-1}\Big(h\mathbb{E}[(x_{t,m}^\pi - D_t)^+] + b\vartheta\mathbb{E}[(D_t - x_{t,m}^\pi)^+] + p(1-\vartheta)\mathbb{E}[(D_t - x_{t,m}^\pi)^+] + \theta\mathbb{E}[o_t^\pi]\Big)\\ &= \sum_{t=1}^{T+m-1}\Big(h\mathbb{E}[(x_{t,m}^\pi - D_t)^+] + \big(\vartheta b + (1-\vartheta)p\big)\mathbb{E}[(D_t - x_{t,m}^\pi)^+] + \theta\mathbb{E}[o_t^\pi]\Big), \end{aligned} \tag{EC.43}$$

where the second identity follows from the assumption that $\zeta_t$ is independent of $D_t - x_{t,m}^\pi$ and $\mathbb{E}[\zeta_t] = \vartheta$ for each period $t \geq 1$. Due to a similar explanation to inequality (16) in the proof of Proposition 2 in Bu et al. (2023), the following inequality holds for any $T \geq 1$ and any given demand sample path:

$$\sum_{t=1}^{T+m-1} o_t^\pi \geq \frac{1}{m}\sum_{t=1}^{T}\Big(x_{t,m}^\pi - \sum_{i=t}^{t+m-1} D_i\Big)^+. \tag{EC.44}$$

For each $t \geq 1$, denote $\delta_t^\pi \triangleq x_{t,m}^\pi - d_t$. Applying (EC.44) to the RHS of (EC.43), we have

$$\begin{aligned} &\sum_{t=1}^{T+m-1}\mathbb{E}[C_t^\pi]\\ &\geq \sum_{t=1}^{T}\Big(h\mathbb{E}[(x_{t,m}^\pi - D_t)^+] + \big(\vartheta b + (1-\vartheta)p\big)\mathbb{E}[(D_t - x_{t,m}^\pi)^+] + \frac{\theta}{m}\mathbb{E}\Big[\Big(x_{t,m}^\pi - \sum_{i=t}^{t+m-1} D_i\Big)^+\Big]\Big)\\ &= \sum_{t=1}^{T}\Big(h\mathbb{E}\Big[\big((d_t+\delta_t^\pi) - (d_t + W_t)\big)^+\Big] + \big(\vartheta b + (1-\vartheta)p\big)\mathbb{E}\Big[\big((d_t+W_t) - (d_t+\delta_t^\pi)\big)^+\Big]\\ &\qquad + \frac{\theta}{m}\mathbb{E}\Big[\Big((d_t+\delta_t^\pi) - \sum_{i=t}^{t+m-1}(d_i + W_i)\Big)^+\Big]\Big)\\ &\geq \sum_{t=1}^{T}\Big(h\mathbb{E}\big[(\delta_t^\pi - W_t)^+\big] + (\vartheta b + (1-\vartheta)p)\mathbb{E}\big[(W_t - \delta_t^\pi)^+\big] + \frac{\theta}{m}\mathbb{E}[(\delta_t^\pi - W_t)^+] - \frac{\theta}{m}\sum_{i=t+1}^{t+m-1}(d_i + \mathbb{E}[W_i])\Big)\\ &\geq \sum_{t=1}^{T}\Big(\Big(h + \frac{\theta}{m}\Big)\mathbb{E}\big[(\mathbb{E}[\delta_t^\pi] - W_t)^+\big] + \big(\vartheta b + (1-\vartheta)p\big)\mathbb{E}\Big[(W_t - \mathbb{E}[\delta_t^\pi])^+\Big] - \frac{\theta}{m}\sum_{i=t+1}^{t+m-1}(d_i + \mathbb{E}[W_i])\Big)\\ &\geq \sum_{t=1}^{T}\min_{S\in\mathbb{R}}\Big\{\Big(h+\frac{\theta}{m}\Big)\mathbb{E}[(S-W)^+] + \big(\vartheta b + (1-\vartheta)p\big)\mathbb{E}[(W-S)^+]\Big\} - \frac{\theta(m-1)}{m}\sum_{t=1}^{T+m-1}(d_t + \mathbb{E}[W])\\ &= \sum_{t=1}^{T}\min_{S\geq 0}\Big\{\Big(h+\frac{\theta}{m}\Big)\mathbb{E}[(S-W)^+] + \big(\vartheta b + (1-\vartheta)p\big)\mathbb{E}[(W-S)^+]\Big\} - \frac{\theta(m-1)}{m}\sum_{t=1}^{T+m-1}(d_t + \mathbb{E}[W])\\ &= T\cdot C^{\mathrm{NV}}(h + \frac{\theta}{m}, \vartheta b + (1-\vartheta)p, F_W) - \frac{m-1}{m}\theta\sum_{t=1}^{T+m-1}(d_t + \mathbb{E}[W]), \end{aligned}$$

where the first identity follows from the definition of $\delta_t^\pi$ and $D_t = d_t + W_t$, the second inequality follows from $(x-y)^+ \geq x^+ - y^+$ and non-negativity of $d_i$ and $W_i$, the third inequality follows from the conditional Jensen's inequality and the fact that $\delta_t^\pi$ is independent of $W_t$, the fourth inequality follows from the assumption that $\{W_t : t \geq 1\}$ is a sequence of *i.i.d. r.v.'s*, and the second identity holds since the assumption that $W$ is non-negative implies that the minimum value is achieved at $S \in [0, \infty)$.

Dividing $T + m - 1$ on each side of the above inequalities and taking $\limsup_{T\to\infty}$, we obtain

$$C^\pi \geq C^{\mathrm{NV}}\big(h + \frac{\theta}{m}, \vartheta b + (1-\vartheta)p, F_W\big) - \frac{m-1}{m}\theta\Big(\mathbb{E}[W] + \liminf_{T\to\infty}\frac{1}{T}\sum_{t=1}^T d_t\Big).$$

Since the above inequality holds for any admissible policy $\pi$, taking $\inf_\pi$ on both sides, we obtain inequality (8) from the definition of the optimal cost.

**Proof of Inequality (9).** We first show that, under the modified base-stock policy $\pi_S$ with $S \geq 0$, $x_{t,m}^{\pi_S} = S + d_t$ holds for each period $t \geq 1$. To this end, we prove by induction that $x_{t,m-1}^{\pi_S} - b_{t-1}^{\pi_S} \leq S$ for each $t \geq 1$, which then implies from the definition of $x_{t,m}^{\pi_S}$ that $x_{t,m}^{\pi_S} = S + d_t$. When $t = 1$, we have $x_{t,m-1}^{\pi_S} - b_{t-1}^{\pi_S} = 0 \leq S$. Suppose $x_{t,m-1}^{\pi_S} - b_{t-1}^{\pi_S} \leq S$. Then we have

$$x_{t+1,m-1}^{\pi_S} - b_t^{\pi_S} = x_{t,m}^{\pi_S} - D_t + (1-\zeta_t)(D_t - x_{t,m}^{\pi_S})^+ - o_t^{\pi_S} = S - \big(W_t - (1-\zeta_t)\big(W_t - S\big)^+\big) - o_t^{\pi_S} \leq S,$$

where the first identity follows from the system dynamics, the second identity holds since the inductive assumption implies $x_{t,m}^{\pi_S} = S + d_t$, and the inequality holds since $(1-\zeta_t)\big(W_t - S\big)^+ \leq W_t$ and $o_t^{\pi_S} \geq 0$. This completes the inductive argument.

Now we are ready to prove inequality (9). Note that for any $t \geq 1$,

$$\sum_{i=t}^{t+m-1} o_i^{\pi_S} \leq (x_{t,m}^{\pi_S} - D_t)^+ = (S - W_t)^+, \tag{EC.45}$$

where the inequality holds since as explained in the proof of Lemma 3 in Bu et al. (2023), all the outdates in periods $t, t+1, \ldots, t+m-1$ come from the leftover inventory in period $t$ after satisfying demand, i.e., $(x_{t,m}^{\pi_S} - D_t)^+$, and the identity follows from $x_{t,m}^{\pi_S} = S + d_t$ and $D_t = d_t + W_t$. Thus, we have

$$\begin{aligned}
\sum_{t=1}^T \mathbb{E}[C_t^{\pi_S}] &= \sum_{t=1}^T \Big(h\mathbb{E}[(x_{t,m}^{\pi_S} - D_t)^+] + \big(\vartheta b + (1-\vartheta)p\big)\mathbb{E}[(D_t - x_{t,m}^{\pi_S})^+] + \theta\mathbb{E}[o_t^{\pi_S}]\Big) \\
&= \sum_{t=1}^T \Big(h\mathbb{E}[(S - W_t)^+] + \big(\vartheta b + (1-\vartheta)p\big)\mathbb{E}[(W_t - S)^+] + \theta\mathbb{E}[o_t^{\pi_S}]\Big) \\
&\leq T\Big(h\mathbb{E}[(S-W)^+] + \big(\vartheta b + (1-\vartheta)p\big)\mathbb{E}[(W-S)^+]\Big) + \Big\lceil \frac{T}{m}\Big\rceil \cdot \theta\mathbb{E}[(S-W)^+],
\end{aligned}$$

where the second identity follows from $x_{t,m}^{\pi_S} = S + d_t$ and $D_t = d_t + W_t$, and the inequality follows from (EC.45) and the assumption that $\{W_t : t \geq 1\}$ a sequence of *i.i.d. r.v.'s*. Dividing $T$ on each side of the above inequalities and taking $\limsup_{T\to\infty}$, we obtain inequality (9). Q.E.D.

### C.2. Proof of Proposition 5

We first prove

$$\mathrm{OPT}_{\infty}^{(\mathrm{II})} - \mathrm{OPT}_{p}^{(\mathrm{II})} = \mathcal{O}\Big(\bar{D} - F_D^{-1}\Big(\frac{p-\theta}{p+h}\Big)\Big). \tag{EC.46}$$

Then, since $\lim_{p\to\infty} F_D^{-1}\big(\frac{p-\theta}{p+h}\big) = \bar{D}$, it immediately leads to $\lim_{p\to\infty} \mathrm{OPT}_p^{(\mathrm{II})} = \mathrm{OPT}_\infty^{(\mathrm{II})}$. From Proposition 3 and Lemma 1 in Bu et al. (2023), when $D$ is bounded, we have the following inequalities for the lost-sales model:

$$h\mathbb{E}\Big[\Big(F_D^{-1}(\frac{p-\theta}{p+h}) - D\Big)^+\Big] + \theta\mathbb{E}\Big[O_\infty\Big(F_D^{-1}(\frac{p-\theta}{p+h})\Big)\Big] \le \mathrm{OPT}_p^{(\mathrm{II})} \le C_p^{(\mathrm{II})}(\bar{D}) = \mathrm{OPT}_\infty^{(\mathrm{II})}, \tag{EC.47}$$

where $\mathbb{E}[O_\infty(S)]$ denotes the long-run average outdates under base-stock policy $\pi_S$. Combining inequality (EC.47) with the definition of $\mathrm{OPT}_\infty^{(\mathrm{II})}$, we obtain

$$\begin{aligned}0 \le \mathrm{OPT}_{\infty}^{(\mathrm{II})} - \mathrm{OPT}_{p}^{(\mathrm{II})} &\le h(\bar{D}-\mu) + \theta\mathbb{E}[O_\infty(\bar{D})] - h\mathbb{E}\Big[\Big(F_D^{-1}(\frac{p-\theta}{p+h}) - D\Big)^+\Big] - \theta\mathbb{E}\Big[O_\infty\Big(F_D^{-1}(\frac{p-\theta}{p+h})\Big)\Big] \\ &\le (h+\theta)\Big(\bar{D} - F_D^{-1}(\frac{p-\theta}{p+h})\Big),\end{aligned} \tag{EC.48}$$

where the third inequality holds due to $x^+ - y^+ \le (x-y)^+$ for any $x, y \in \mathbb{R}$ and $\mathbb{E}[O_\infty(S_2)] - \mathbb{E}[O_\infty(S_1)] \le S_2 - S_1$ for any $0 \le S_1 \le S_2$ from the proof of Theorem 3 in Bu et al. (2023). Thus, we have (EC.46).

**(a)** To prove part (a), from the above analysis, it suffices to prove $\lim_{p\to\infty}(\bar{D} - F_D^{-1}(\frac{p-\theta}{p+h}))/p^{-\frac{1}{k}} \in (0,\infty)$. Let $x = F_D^{-1}(\frac{p-\theta}{p+h})$. It is easy to verify that $p = \frac{hF_D(x)+\theta}{\bar{F}_D(x)}$. Then we have

$$\lim_{p\to\infty} \frac{\bar{D} - F_D^{-1}(\frac{p-\theta}{p+h})}{p^{-\frac{1}{k}}} = \lim_{x\uparrow\bar{D}} \frac{\bar{D}-x}{\Big(\frac{hF_D(x)+\theta}{\bar{F}_D(x)}\Big)^{-\frac{1}{k}}} = (h+\theta)^{\frac{1}{k}} \cdot \lim_{x\uparrow\bar{D}} \frac{\bar{D}-x}{(\bar{F}_D(x))^{\frac{1}{k}}} = \Big(\frac{h+\theta}{\gamma}\Big)^{\frac{1}{k}},$$

where the last identity follows from the assumption that $\lim_{x\uparrow\bar{D}} \frac{\bar{F}_D(x)}{(\bar{D}-x)^k} = \gamma$. This completes the proof of part (a).

**(b)** It is straightforward to verify that when $p > (h+\theta)/\mathbb{P}(D=\bar{D}) - h$, $F_D^{-1}(\frac{p-\theta}{p+h}) = \bar{D}$ holds. Thus, when $p > (h+\theta)/\mathbb{P}(D=\bar{D}) - h$, from inequality (EC.48), we have $\mathrm{OPT}_p^{(\mathrm{II})} = \mathrm{OPT}_\infty^{(\mathrm{II})}$, completing the proof of part (b). Q.E.D.

## Appendix D: Proof of Statements in Section 5

### D.1. Proof of Lemma 3

For any admissible policy $\pi$ and any $t \ge L$, we note that the total demand during time $t-L$ to $t$ is $D(t) - D(t-L)$, and the maximum amount of sales from these demands is $\min\{\mathrm{IP}^\pi(t-L), D(t) -$

$D(t-L)\}$. Since all pipeline inventories from the inventory position $\mathrm{IP}^{\pi}(t-L)$ will arrive at or before time $t$, then the on-hand inventory level $I^{\pi}(t)$ at time $t$ has the following lower bound:

$$I^{\pi}(t) \geq \Big(\mathrm{IP}^{\pi}(t-L) - \big(D(t) - D(t-L)\big)\Big)^{+}, \tag{EC.49}$$

and the cumulative amount of lost-sales during time $t-L$ to $t$ has the following lower bound:

$$A^{\pi}(t) - A^{\pi}(t-L) \geq \Big(D(t) - D(t-L) - \mathrm{IP}^{\pi}(t-L)\Big)^{+}. \tag{EC.50}$$

Since $A^{\pi}(t)$ is an increasing function in $t$, we then have

$$\begin{aligned}
A^{\pi}(T) &\geq \frac{1}{L}\int_{T-L}^{T} A^{\pi}(t)dt \\
&\geq \frac{1}{L}\Big(\int_{L}^{T} A^{\pi}(t)dt - \int_{0}^{T-L} A^{\pi}(t)dt\Big) \\
&= \frac{1}{L}\int_{L}^{T} \big(A^{\pi}(t) - A^{\pi}(t-L)\big)dt \\
&\geq \frac{1}{L}\int_{L}^{T} \Big(D(t) - D(t-L) - \mathrm{IP}^{\pi}(t-L)\Big)^{+} dt,
\end{aligned} \tag{EC.51}$$

where the last inequality follows from (EC.50). Thus, we have the following inequalities:

$$\begin{aligned}
C^{\pi} &= \limsup_{T\to\infty} \frac{1}{T}\left(h\int_{L}^{T} \mathbb{E}[I^{\pi}(t)]dt + p\mathbb{E}[A^{\pi}(T)]\right) \\
&\geq \limsup_{T\to\infty} \frac{1}{T}\int_{L}^{T} \Big(h\mathbb{E}[(\mathrm{IP}^{\pi}(t-L) - (D(t) - D(t-L)))^{+}] + \frac{p}{L}\mathbb{E}[(D(t) - D(t-L) - \mathrm{IP}^{\pi}(t-L))^{+}]\Big)\, dt \\
&\geq \limsup_{T\to\infty} \frac{1}{T}\int_{L}^{T} \min_{S\geq 0}\Big\{h\mathbb{E}[(S - D(L))^{+}] + \frac{p}{L}\mathbb{E}[(D(L) - S)^{+}]\Big\}\, dt \\
&= C^{\mathrm{NV}}\Big(h, \frac{p}{L}, F_{D(L)}\Big),
\end{aligned}$$

where the first inequality follows from inequalities (EC.49) and (EC.51), and the second inequality follows from $\mathrm{IP}^{\pi}(t-L) \geq 0$ and $D(t) - D(t-L) =^{d} D(L)$ for any $t \geq L$ due to the assumption of Poisson arrival. Since the above inequality holds for any admissible policy $\pi$, taking $\inf_{\pi}$ on each side, we prove Lemma 3. Q.E.D.

### D.2. Proof of Theorem 4

Since $D(L)$ is a Poisson *r.v.*, part (f) of Proposition 1 shows $\lim_{p\to\infty} C^{\mathrm{NV}}(h, p/L, F_{D(L)})/g^{-1}(p) = h$. From Lemma 3, it suffices to prove $\lim_{p\to\infty} C_p^{(\mathrm{III})}(S_p^{(\mathrm{III}),*})/g^{-1}(p) = h$ or the following equations:

$$\lim_{p\to\infty} C_p^{(\mathrm{III})}(S_p^{(\mathrm{III}),*})/S_p^{(\mathrm{III}),*} = h, \tag{EC.52}$$

$$\lim_{p\to\infty} S_p^{(\mathrm{III}),*}/g^{-1}(p) = 1. \tag{EC.53}$$

We first show Equation (EC.52). By the definition of $C^{(\mathrm{III})}(S)$ in Equation (12) and noting that $\lim_{p\to\infty} S_p^{(\mathrm{III}),*}=\infty$, it suffices to prove that $(p+hL)B(S_p^{(\mathrm{III}),*},\lambda L)$ is bounded for all $p\geq 0$. To this end, we note the following inequality from the second inequality in (13):

$$\begin{aligned}\frac{h}{\lambda(p+Lh)} &\geq B\big(S_p^{(\mathrm{III}),*},\lambda L\big)-B\big(S_p^{(\mathrm{III}),*}+1,\lambda L\big)\\ &= B\big(S_p^{(\mathrm{III}),*},\lambda L\big)\left(1-\frac{\lambda L}{S_p^{(\mathrm{III}),*}+1}\cdot\frac{\sum_{n=0}^{S_p^{(\mathrm{III}),*}}\frac{(\lambda L)^n}{n!}}{\sum_{n=0}^{S_p^{(\mathrm{III}),*}+1}\frac{(\lambda L)^n}{n!}}\right).\end{aligned}$$

Since $\lim_{p\to\infty} S_p^{(\mathrm{III}),*}=\infty$ and $\lim_{S\to\infty}\sum_{n=0}^{S}\frac{(\lambda L)^n}{n!}=e^{\lambda L}$, the second term on the RHS of the above equality converges to 1 as $p\to\infty$. Thus, by multiplying $p+Lh$ on each side of the above inequality and letting $p\to\infty$, we obtain $\limsup_{p\to\infty}(p+Lh)B\big(S_p^{(\mathrm{III}),*},\lambda L\big)\leq h/\lambda$, showing that $(p+hL)B(S_p^{(\mathrm{III}),*},\lambda L)$ is bounded with respect to $p\geq 0$.

We next prove Equation (EC.53). From the proof of Equation (EC.12) in Appendix A.1, it suffices to prove $\lim_{p\to\infty} S_p^{(\mathrm{III}),*}\ln(S_p^{(\mathrm{III}),*})/\ln(p)=1$, or equivalently,

$$\liminf_{p\to\infty} S_p^{(\mathrm{III}),*}\ln(S_p^{(\mathrm{III}),*})/\ln(p)\geq 1; \tag{EC.54}$$

$$\limsup_{p\to\infty} S_p^{(\mathrm{III}),*}\ln(S_p^{(\mathrm{III}),*})/\ln(p)\leq 1. \tag{EC.55}$$

We prove inequality (EC.54) as follows. Note that $\sum_{n=0}^{S_p^{(\mathrm{III}),*}}\frac{(\lambda L)^n}{n!}<e^{\lambda L}$, and when $p$ is sufficiently large, $\sum_{n=0}^{S_p^{(\mathrm{III}),*}+1}\frac{(\lambda L)^n}{n!}\geq\frac{1}{2}e^{\lambda L}$. It then follows from the first inequality in (13) that when $p$ is sufficiently large,

$$\begin{aligned}\frac{h}{\lambda(p+hL)} &\geq B\big(S_p^{(\mathrm{III}),*},\lambda L\big)-B\big(S_p^{(\mathrm{III}),*}+1,\lambda L\big)\\ &\geq e^{-\lambda L}\frac{(\lambda L)^{S_p^{(\mathrm{III}),*}}}{(S_p^{(\mathrm{III}),*})!}-2e^{-\lambda L}\frac{(\lambda L)^{S_p^{(\mathrm{III}),*}+1}}{(S_p^{(\mathrm{III}),*}+1)!}=e^{-\lambda L}\frac{(\lambda L)^{S_p^{(\mathrm{III}),*}}}{(S_p^{(\mathrm{III}),*})!}\left(1-\frac{2\lambda L}{S_p^{(\mathrm{III}),*}+1}\right).\end{aligned}$$

By taking the logarithm on each side of the above inequality and after simple algebra, we obtain

$$\ln\big((S_p^{(\mathrm{III}),*})!\big)\geq S_p^{(\mathrm{III}),*}\ln(\lambda L)+\ln\left(\frac{\lambda}{e^{\lambda L}h}\left(1-\frac{2\lambda L}{S_p^{(\mathrm{III}),*}+1}\right)\right)+\ln(p+hL).$$

By applying the upper bound of Stirling's approximation $n!\leq n^{n+\frac{1}{2}}e^{1-n}$ for any $n\geq 1$, we have

$$1+\big(S_p^{(\mathrm{III}),*}+\tfrac{1}{2}\big)\ln\big(S_p^{(\mathrm{III}),*}\big)\geq S_p^{(\mathrm{III}),*}\big(1+\ln(\lambda L)\big)+\ln\left(\frac{\lambda}{e^{\lambda L}h}\left(1-\frac{2\lambda L}{S_p^{(\mathrm{III}),*}+1}\right)\right)+\ln(p+hL).$$

After dividing $S_p^{(\mathrm{III}),*}\ln S_p^{(\mathrm{III}),*}$ and taking $\limsup_{p\to\infty}$ on both sides of the above inequality, we have $\limsup_{p\to\infty}\ln p/\big(S_p^{(\mathrm{III}),*}\ln S_p^{(\mathrm{III}),*}\big)\leq 1$, which leads to inequality (EC.54).

We prove inequality (EC.55) as follows. Similar to the proof of inequality (EC.54), when $p$ is sufficiently large,

$$\begin{aligned}\frac{h}{\lambda(p+hL)} &< B\big(S_p^{(\mathrm{III}),*}-1,\lambda L\big) - B\big(S_p^{(\mathrm{III}),*},\lambda L\big)\\ &\le 2e^{-\lambda L}\frac{(\lambda L)^{S_p^{(\mathrm{III}),*}-1}}{\big(S_p^{(\mathrm{III}),*}-1\big)!} - e^{-\lambda L}\frac{(\lambda L)^{S_p^{(\mathrm{III}),*}}}{\big(S_p^{(\mathrm{III}),*}\big)!} = e^{-\lambda L}\frac{(\lambda L)^{S_p^{(\mathrm{III}),*}-1}}{\big(S_p^{(\mathrm{III}),*}-1\big)!}\left(2-\frac{\lambda L}{S_p^{(\mathrm{III}),*}}\right).\end{aligned}$$

By taking the logarithm on each side of the above inequality and after simple algebra, we obtain

$$\ln\Big(\big(S_p^{(\mathrm{III}),*}-1\big)!\Big) < \big(S_p^{(\mathrm{III}),*}-1\big)\ln(\lambda L) + \ln\left(\frac{\lambda}{e^{\lambda L}h}\left(2-\frac{\lambda L}{S_p^{(\mathrm{III}),*}}\right)\right) + \ln(p+hL).$$

By applying the lower bound of Stirling's approximation $n! \ge \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$ for any $n\ge 1$, we have

$$\begin{aligned}&\ln\big(\sqrt{2\pi}\big) + \big(S_p^{(\mathrm{III}),*}-\tfrac{1}{2}\big)\ln\big(S_p^{(\mathrm{III}),*}-1\big)\\ &< \big(S_p^{(\mathrm{III}),*}-1\big)\big(1+\ln(\lambda L)\big) + \ln\left(\frac{\lambda}{e^{\lambda L}h}\left(2-\frac{\lambda L}{S_p^{(\mathrm{III}),*}}\right)\right) + \ln(p+hL).\end{aligned}$$

After dividing $S_p^{(\mathrm{III}),*}\ln S_p^{(\mathrm{III}),*}$ and taking $\liminf_{p\to\infty}$ on both sides of the above inequality, we have $\liminf_{p\to\infty}\ln p/\big(S_p^{(\mathrm{III}),*}\ln S_p^{(\mathrm{III}),*}\big)\ge 1$, which leads to inequality (EC.55). Q.E.D.

## Appendix E: Asymptotic Scaling and Optimality for a Class of Heuristic Base-Stock Policies

In this appendix, we extend the results in Theorems 2 and 3 to a class of heuristic modified base-stock policies. For any modified base-stock policy $\pi_{\hat{S}}$, we introduce the following condition on its base-stock level $\hat{S}$, where $X$ denotes the lead-time demand and will be specified in the formal result later. Again, we add subscript "$p$" to highlight the dependency on the unit penalty cost.

**Condition 1** *There exists a triple $(p_0,\lambda_1,\lambda_2)$ where $0<p_0<\infty$ and $0<\lambda_2<\lambda_1<\infty$, such that*

$$F_X^{-1}\Big(\frac{p}{p+\lambda_1}\Big)\le \hat{S}_p \le F_X^{-1}\Big(\frac{p}{p+\lambda_2}\Big),\quad \forall p\ge p_0.$$

PROPOSITION EC.1. *Consider the modified base-stock policy $\pi_{\hat{S}_p}$, where $\hat{S}_p$ satisfies Condition 1 for $X =^d \mathscr{W}_{L+1}$ and $W$ in Sections 3 and 4, respectively, and suppose $X$ is unbounded. Then the following results hold:*

*(a) If $\mathbb{E}[X-x|X>x]=o(x)$, then $C_p^{\mathrm{I}}(\hat{S}_p)\sim hF_{\mathscr{W}_{L+1}}^{-1}(p/(p+1))$ and $C_p^{\mathrm{II}}(\hat{S}_p)\sim (h+\theta/m)F_W^{-1}(p/(p+1))$.*

*(b) If $\mathbb{E}[X-x|X>x]=\mathcal{O}(x)$, then $C_p^{\mathrm{I}}(\hat{S}_p)=\Theta(F_{\mathscr{W}_{L+1}}^{-1}(p/(p+1)))$ and $C_p^{\mathrm{II}}(\hat{S}_p)=\Theta(F_W^{-1}(p/(p+1)))$.*

Part (a) in Proposition EC.1 directly implies the asymptotic optimality of a class of modified base-stock policies satisfying Condition 1 for the two systems studied in Sections 3 and 4 under the assumption $\mathbb{E}[X-x|X>x]=o(x)$. We next prove Proposition EC.1.

**Proof of Proposition EC.1.** The proofs for the two systems are similar and we next provide a complete proof for the first system while omitting the details for the second one.

**(a)** Note that we have the following inequalities from $\hat{S}_p \le F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+\lambda_2})$:

$$\frac{\mathrm{OPT}_p^{(\mathrm{I})}}{F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1})} \le \frac{C_p^{(\mathrm{I})}(\hat{S}_p)}{F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1})} \le \frac{C_p^{(\mathrm{I})}(\hat{S}_p)}{\hat{S}_p} \cdot \frac{F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+\lambda_2})}{F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1})}. \tag{EC.56}$$

From inequality (5), we have

$$\begin{aligned}\frac{C_p^{(\mathrm{I})}(\hat{S}_p)}{\hat{S}_p} &\le h\frac{\mathbb{E}[(\hat{S}_p-\mathscr{W}_{L+1})^+]}{\hat{S}_p} + p\Pr(\mathscr{W}_{L+1}>\hat{S}_p)\times\frac{\mathbb{E}[\mathscr{W}_{L+1}-\hat{S}_p|\mathscr{W}_{L+1}>\hat{S}_p]}{\hat{S}_p} + \frac{Lh\mathbb{E}[\mathscr{W}_{L+1}]}{\hat{S}_p}\\ &\le h\frac{\mathbb{E}[(\hat{S}_p-\mathscr{W}_{L+1})^+]}{\hat{S}_p} + \frac{p\lambda_1}{p+\lambda_1}\times\frac{\mathbb{E}[\mathscr{W}_{L+1}-\hat{S}_p|\mathscr{W}_{L+1}>\hat{S}_p]}{\hat{S}_p} + \frac{Lh\mathbb{E}[\mathscr{W}_{L+1}]}{\hat{S}_p}. \end{aligned}\tag{EC.57}$$

Therefore, $\limsup_{p\to\infty} C_p^{(\mathrm{I})}(\hat{S}_p)/\hat{S}_p \le h$ from the assumption $\mathbb{E}[\mathscr{W}_{L+1}-x|\mathscr{W}_{L+1}>x]=o(x)$ and $\lim_{p\to\infty}\hat{S}_p=\infty$ due to unboundedness of $W$. On the other hand, from Equation (EC.1), we have $\lim_{p\to\infty} F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+\lambda_2})/F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1})=1$. Therefore, the $\limsup_{p\to\infty}$ of the most RHS in inequality (EC.56) is no more than $h$. From part (a) of Theorem 2, the most LHS of inequality (EC.56) converges to $h$ as $p\to\infty$. Thus, we have $C_p^{(\mathrm{I})}(\hat{S}_p)\sim hF^{-1}_{\mathscr{W}_{L+1}}(p/(p+1))$.

**(b)** From inequality (EC.57) and $\mathbb{E}[\mathscr{W}_{L+1}-x|\mathscr{W}_{L+1}>x]=\mathcal{O}(x)$, we know that $C_p^{(\mathrm{I})}(\hat{S}_p)=\mathcal{O}(\hat{S}_p)$. On the other hand, it is also easy to see the following inequality from inequality (4):

$$\frac{C_p^{(\mathrm{I})}(\hat{S}_p)}{\hat{S}_p} \ge \frac{h\mathbb{E}[(\hat{S}_p-\mathscr{W}_{L+1})^+]}{\hat{S}_p},$$

which then implies $C_p^{(\mathrm{I})}(\hat{S}_p)=\Theta(\hat{S}_p)$. Moreover, from the proof of part (b) in Theorem 1, we know that $F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+\lambda})=\Theta(F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1}))$ for any $\lambda>0$. Therefore, $C_p^{(\mathrm{I})}(\hat{S}_p)=\Theta(F^{-1}_{\mathscr{W}_{L+1}}(\frac{p}{p+1}))$. Q.E.D.