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A UNIFIED APPROACH TO LINEAR-QUADRATIC-GAUSSIAN MEAN-FIELD TEAM: HOMOGENEITY, HETEROGENEITY AND QUASI-EXCHANGEABILITY

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This paper aims to systematically solve stochastic team optimization of a large-scale system, in a linear-quadratic-Gaussian framework. Concretely, the underlying large-scale system involves considerable weakly coupled cooperative agents for which the individual admissible controls: (i) enter the diffusion terms, (ii) are constrained in some closed-convex subsets and (iii) subject to a general *partial decentralized information* structure. A more important but serious feature: (iv) all agents are heterogenous with *continuum* instead of *finite* diversity. Combination of (i)–(iv) yields a quite general modeling of stochastic team-optimization, but on the other hand, also fails current existing techniques of team analysis. In particular, classical team consistency with continuum heterogeneity collapses because of (i). As the resolution, a novel *unified approach* is proposed under which the intractable *continuum heterogeneity* can be converted to a more tractable *homogeneity*. As a trade-off, the underlying randomness is augmented, and all agents become (quasi) weakly exchangeable. Such an approach essentially involves a subtle balance between homogeneity v.s. heterogeneity, and left (prior-sampling)- v.s. right (posterior-sampling) information filtration. Subsequently, the consistency condition (CC) system takes a new type of forward-backward stochastic system with *double-projections* (due to (ii), (iii)), along with *spatial mean* on continuum heterogenous index (due to (iv)). Such a system is new in team literature and its well-posedness is also challenging. We address this issue under mild conditions. Related asymptotic optimality is also established.

1. Introduction. The starting point of the present work is the well-studied mean-field team (MFT). In its standard form, a MFT involves a large-scale system with considerable weakly interactive but *cooperative* agents $\{\mathcal{A}_i\}_{i=1}^N$. All agents are endowed with an individual (*principal*) state, cost functional and admissible decision set respectively in the following manner. The individual state dynamic of $\mathcal{A}_i$ is formulated by a controlled Itô-type linear stochastic differential equation (LSDE):

$$(1)\qquad \begin{cases} dx_i(t) = \big[A(t)x_i(t) + B(t)u_i(t) + F(t)x^{(N)}(t) + f_t\big]dt + \sigma_t\, dW_i(t), \\ x_i(0) = \xi \in \mathbb{R}^n, \quad 1 \le i \le N, \end{cases}$$

where $x^{(N)} := \frac{1}{N}\sum_{i=1}^N x_i$ is the weakly coupled state-average across all agents, W_i is a Brownian motion (BM) that might be vector-valued (e.g., with a common noise). For each $\mathcal{A}_i$, its *principal cost* $\mathcal{J}_i$ (while we may call $\{\mathcal{J}_j\}_{j\neq i}$ the *marginal costs* for $\mathcal{A}_i$) is measured by the

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following quadratic functional:

$$\mathcal{J}_i(\mathbf{u}(\cdot)) = \frac{1}{2}\mathbb{E}\int_0^T \big[\big\langle Q(t)\big(x_i(t) - H(t)x^{(N)}(t)\big), x_i(t) - H(t)x^{(N)}(t)\big\rangle + \big\langle R(t)u_i(t), u_i(t)\big\rangle\big]\,dt, \tag{2}$$

with admissible team strategy $\mathbf{u}(\cdot) = (u_1^\top(\cdot), \dots, u_N^\top(\cdot))^\top$. Note individual admissible $u_i(\cdot) \in \mathcal{U}_{i,\mathrm{op}}^{d,f} = L^2_{\mathbb{F}^i}(0, T; \mathbb{R}^m)$ with filtration $\mathbb{F}^i$ defined later, representing the decentralized open-loop information of $\mathcal{A}_i$.

A subtle point here is the distinction between centralized ($\mathcal{U}_i^{c,f}$), and decentralized ($\mathcal{U}_{i,\mathrm{op}}^{d,f}$, $\mathcal{U}_{i,\mathrm{cl}}^{d,f}$) but of full information. This makes team-optimization differing from classical vector optimization/control. Superscripts "*cl*", "*ol*" denote the closed-loop and open-loop and "*f*" the full-information. We will address this point in more detail in Section 2. Hereafter, we may exchange the usage of $\mathbf{u} = (u_1, \dots, u_N) \in \mathbb{R}^{m\times N}$, $\mathbf{u} = (u_1^\top, \dots, u_N^\top)^\top \in \mathbb{R}^{mN}$ and $\mathbf{u} = (u_i, u_{-i}) \in \mathbb{R}^{m\times N}$ with $u_{-i} = (u_1, \dots, u_{i-1}, u_{i+1}, u_N) \in \mathbb{R}^{m\times(N-1)}$ by noting all of them represent the team profile among all agents, but only differ in formations. For simplicity, we focus on *Lagrange problem* only, and no essential difficulty to the *Bolza problem* extension.

By mean-field "team" (MFT), we refer all weakly coupled agents $\{\mathcal{A}_i\}_{i=1}^N$ are cooperative, aiming to optimize the following social (or, team) cost functional (the related optimal functional is called *mean-field team*): $\mathcal{J}_{\mathrm{soc}}^{(N)}(\mathbf{u}(\cdot)) = \sum_{i=1}^N \mathcal{J}_i(\mathbf{u}(\cdot))$. Due to the new framework, MFT is different from mean-field control (MFC) problem and mean-field games (MFG).

MFT v.s. MFC. (i) MFT aims to analyze a complex large-scale system including many *cooperative* coupled agents, while MFC (e.g. [43]) only concerns a *single* agent with state distribution (or, mean) entering dynamics or cost. So, essentially, MFT is for multi-agent systems with *decentralized* information but MFC only for a single-agent with (of course) *centralized* information. Consequently, MFT seeks some (joint) strategy but without information compilation across team members; by contrast, MFC only involves a single agent so naturally seeks control by its own centralized information. (ii) Owning to the information distinction above, analysis of MFT and MFC also proceed very differently. For MFT, two crucial steps are *variational decomposition* and *duality procedure* to construct auxiliary problem for a *representative agent*. By comparison, MFC analysis is rather straightforward, no need to invoke variation and duality since it involves single-agent and central-information only. In addition, MFT essentially invokes some fixed-point arguments but this is not needed in MFC. (iii) Although in context of the homogeneous model (i.e., all agents are symmetric), there exists some connection between MFT and MFC (to partial content) in analysis. However, such connections will no longer be valid for the heterogenous model in the presence of nonsymmetric agents, especially with continuum heterogeneity.

MFT v.s. MFG. Furthermore, MFT is also quite different from MFG. (i) Concept difference. Although both are for large population systems, MFC is for cooperative agents towards a social-optima (Pareto) while MFG for noncooperative agents to an Nash equilibrium. (ii) Analysis difference. By (i), MFG and MFT analysis are very distinctive, especially for fixed-point arguments. In MFG, we can directly freeze state-average limit $\lim_{N\to+\infty} x^{(N)}$ to construct cost functional of auxiliary problem, and derive a consistency condition (CC) to complete the fixed point argument. However, for MFT, we cannot freeze $x^{(N)}$ directly as in MFG. Instead, MFT auxiliary functional must be specified in an *indirect* way. Roughly speaking, we should apply variational decomposition and weak duality, then auxiliary cost based on it, then fixed-point argument. Noting such pivotal *variational decomposition* is not needed at all for MFG because of its noncooperative nature. (iii) Moreover, verifications of the above MFT asymptotic social-optima and MFG asymptotic Nash equilibrium are also very distinctive. For example, due to the cooperative structure, all agents $\mathcal{A}_1, \dots, \mathcal{A}_N$ in MFT cooperate to minimize $\mathcal{J}_{\text{soc}}^{(N)}(\mathbf{u}(\cdot))$, and u_{-i} (i.e., all decisions except $\mathcal{A}_i$) cannot be viewed as *endogenous* terms as in MFG.

Please refer to [3, 5, 9, 11, 13, 30, 31] for some recent work on MFG and refer to [8, 10, 18, 29, 36] for the limit relation between MFG and noncooperate N-player games. The interested readers may refer to, for example, [27, 34, 39], for a detailed analysis comparison between MFG and MFT, and [38, 40] for some recent studies from various perspectives with different modeling variants. In particular, see [23] for MFT with volatility uncertainty; [25] for linear-quadratic-Gaussian (LQG) mean-field social optimization with a major player; [30] for MFG with optimal investment under relative performance criteria; [33] for LQG games with a major player and continuum-parametrized minor players; and [41] for mean-field team in LQG models with Markov jump parameters.

Our work distinguishes itself from all the above MFT literature by the following fairly (even not the most) general formulation, in LQG context. Unlike (1), the individual dynamic of agent $\mathcal{A}_i$ now takes

$$
\begin{cases} dx_i(t) = \big[A_{\Theta_i}(t)x_i(t) + B(t)u_i(t) + F(t)x^{(N)}(t)\big]\,dt \\ \qquad\qquad + \big[C(t)x_i(t) + D_{\Theta_i}(t)u_i(t) + \widetilde{F}(t)x^{(N)}(t)\big]\,dW_i(t), \\ x_i(0) = \xi \in \mathbb{R}^n, \quad 1 \le i \le N, \end{cases} \tag{3}
$$

where $\{\Theta_i\}_{i=1}^N$ is a sequence of independent random variables which are also independent of $\{W_i(s), s \ge 0\}_{i=1}^N$ to represent diversity. The range of $\{\Theta_i\}_{i=1}^N$ is a (possibly continuum) subset in $\mathbb{R}^k$, hence our framework includes both finite diversity and continuum diversity. Please refer to Section 2 for more information. The admissible strategy set for $\mathcal{A}_i$ is

$$
\mathcal{U}_i^{d,p} = \big\{u_i(\cdot) | u_i(\cdot) \in L^2_{\mathbb{G}^i}(0, T; \Gamma)\big\}, \tag{4}
$$

where $\mathbb{G}^i \subseteq \mathbb{F}^i$ or $\mathbb{G}^i \subseteq \mathbb{H}^i$ is a subfiltration representing the partial information; $\Gamma \subset \mathbb{R}^m$ is a nonempty closed convex set representing the input constraint.

There are four main modeling features in formulation (3), (4):

(i) *Weakly coupled controlled-diffusion*. It is remarkable that in (3), when $D_{\Theta_i} \neq 0$ so control process enters diffusion terms of LSDE, and when $\widetilde{F} \neq 0$ so all individual states are weakly coupled in diffusion terms also. In this case, we may call (3) to be *diffusion-controlled and weakly coupled*. This differs from [27] in modeling that is only *drift-controlled and weakly coupled*. Such modeling difference also brings considerable analysis distinctions, for example, on the relevant study of Hamiltonian systems, as well as consistency condition (CC) (see more comparison details in Section 3 and Section 6). Without loss of generality, no forcing terms such as f, σ involve in (3).

(ii) *Random diversity*. Recall that (1) is *homogenous* since all agents are endowed with identical parameters thus they become symmetric. Subsequently, the (decentralized) optimal strategy and states, still denoted as $\{u_i\}_{i=1}^N$ and $\{x_i\}_{i=1}^N$, should turn to be exchangeable. By contrast, in (3), a random index Θ_i is introduced in parameter A, D (also possible to be equipped on other parameters including the cost) to model the diversity across the underlying large-scale system. All agents thereby become heterogenous. Although the heterogenous large-scale system is already addressed in works such as [21, 25], we point out in these works, the heterogenous index is technically treated as some realization after random sampling, along with necessary *ordinal arrangements* within each subclass. Thus, essentially the index therein is some deterministic realization. This differs substantially from our random index treatment here along with related analysis, to be highlighted later. In addition, our index Θ_i can assume a *continuum* support that distinguishes from most heterogenous literature with only finite/discrete support (see, e.g., [21, 25]). Moreover, although continuum heterogeneity is also discussed in, for example, [33], but analysis therein heavily relies upon the

LQ structure with full input and resultant explicit representation. Such analysis collapses in the current formulation (3), due to the intrinsic *diffusion-controlled weakly coupled* feature introduced before, and an input constraint feature to be introduced below.

(iii) *Input constraint*. Note that a convex-closed set Γ is introduced in (4) denoting some pointwise constraint in control input. Recall that such pointwise input constraint is well documented in, for example, [14, 16, 22, 32]. A typical example is $\Gamma = \mathbb{R}^+$ representing the positive control, or *no-shorting* constraint in portfolio selection ([32]). Other examples may include subspace ([16]) or a general convex cone ([22]). We remark that pointwise input constraint is also studied in large-scale/large-population context such as [20] but in a competitive MFG setup, which differs from our cooperative MFT here.

(iv) *Partial information*. Last but not least, the admissible control set is confined on a partial information set $L^2_{\mathbb{G}^i}(0, T; \Gamma)$. LQG control with partial information is also well documented (e.g., [42]). Also, partial information for large population systems is also addressed recently (see [6, 7, 17, 24] for partial information/observation mean-field game). However, to our best knowledge, it is the first time addressing partial information in *mean-field team* context. Notice that the partial information setting differs from that of partial observation ([4]) for which some filtering method with innovation process should be invoked. We defer more detailed information structure to Section 2 after more rigorous formulation.

To a certain content, our aim in our current work is to solve the LQG MFT problem in a rather general setup, by combining the aforementioned features (i)–(iv) together. Although we admit various effective techniques have been already proposed to tackle these features *individually*, however their *combination* brings much more technical hurdles, and makes the associated analysis rather challenging. For example, the continuum heterogenous large-scale system is well studied by [33] in mean-field game setup. Nevertheless, its parallel analysis variant to MFT fails to work in the current formulation because of the following reasoning. Due to the controlled-diffusion feature (i), the related CC does not admit direct characterization because the adjoint process of some backward SDE should enter CC dynamics. Therefore, the direct augmented method in [40] fails to work here. Instead, some indirect embedding method [21, 38] becomes necessary in the presence of (i). Nevertheless, due to the continuum heterogenous feature (ii), the classical embedding CC in [21, 38] no longer works since we have to construct an infinite-dimensional Brownian motion-driven system (on continuum-valued space) to replicate the empirical distribution generated by the controlled large-scale system. Meanwhile, the method in [38] is also not infeasible since it mainly relies on some closed-form representation of optimal state/cost. This becomes unavailable because of the input constraint (iii) imposed above. In a nutshell, in case (i) or (iii) not combined together, we may still handle continuum heterogenous MFT with (ii) by modifying existing methods in, for example, [38]. However, the combination of (i), (ii), (iii) together makes all such existing methods no longer workable.

Other examples include the person-by-person procedure due to continuum heterogeneous (ii), and the tailor-made decentralized strategy in presence of both pointwise constraint (iii) and partial information constraint (iv). To circumvent these difficulties, we propose some novel analysis techniques such as weak duality and modified embedding representation, etc. More analysis details are illuminated in Section 3 and Section 4.3.

Our main contributions can be sketched as follows: (1) First, we devise a new framework to unify homogenous and heterogenous (discrete or continuum) setups in the large-scale system. In particular, it is enabled to transform the heterogenous setup into a homogenous one, with the tradeoff of an augmented randomness. (2) Second, under such new framework, we derive a modified embedding representation of the CC system (a crux in MFT analysis) to accommodate the continuum diversities. (3) Third, the input constraint and partial information constraint are both tackled, and a CC system with *double projection* operator is derived.

Specifically, the CC system takes a coupled mean-field type forward backward stochastic differential equations (FBSDEs) involving both projection mapping and conditional expectation. This seems quite novel in large-scale literature. (4) Last, the well-posedness of the CC system and asymptotic team optimality are established under mild conditions. Please refer to Section 6 for detailed literature comparisons and discussions on homogeneity and heterogeneity.

The remainder of this paper is organized as follows. In Section 2, we give the formulation of the LQG heterogeneous agents problem with input constraints and partial information pattern. In Section 3, we apply variational decomposition and weak duality to find the auxiliary control problem of the individual agent. The decentralized strategy and well-posedness of consistency condition is established in Section 4. Section 5 studies the asymptotic optimality of decentralized strategy. We give a synthetic analysis on homogeneity and heterogeneity and compare our framework with those in the current literature in Section 6.

2. Problem formulation. We first introduce some standard notations used throughout this paper. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the inner product denoted by $\langle\cdot,\cdot,\rangle$. $\mathbb{R}^{n\times m}$ is the space of all $(n\times m)$ matrices, endowed with the inner product $\langle M_1, M_2\rangle = \operatorname{tr}[M_1^\top M_2]$, where $x^\top$ denotes the transpose of a matrix (or vector) x and tr is the trace of a matrix. $M\in\mathbb{S}^n$ denotes the set of symmetric $n\times n$ matrices with real elements. $M>(\geq)0$ denotes that $M\in\mathbb{S}^n$ which is positive (semi)definite, while $M\gg 0$ denotes that, $M-\varepsilon I\geq 0$ for some $\varepsilon>0$.

Assume that $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space on which $\{W_i(t), 0\leq t\leq T\}_{i=1}^N$ is a N-fold Brownian motion (note here W_i might be vector-valued, say, including a common noise component W_0) and $\{\Theta_i\}_{i=1}^N$ is a sequence of independent random variables to represent diversity. In some sense, we may interpret $\{\Theta_i\}$ as some endogenous randomness, while $\{W_i\}$ some exogenous randomness for the generic agent $\mathcal{A}_i$. Moreover, we assume $\{\Theta_i\}_{i=1}^N$ are also independent of $\{W_i(s), s\geq 0\}_{i=1}^N$. Let $\{\mathcal{F}_t^W\}_{0\leq t\leq T}$ be the filtration generated by $\{W_i(s), 0\leq s\leq t\}_{i=1}^N$ and define $\mathcal{F}_t^{W,\Theta}=\sigma(\Theta_i, 1\leq i\leq N)\vee\mathcal{F}_t^W$. The set of null sets on Ω is defined by $\mathcal{N}_{\mathbb{P}}=\{M\in\Omega|\exists G\in\mathcal{F}_\infty^{W,\Theta}$ with $M\subset G$ and $\mathbb{P}(G)=0\}$. Consider the augmented filtration $\mathbb{F}=\{\mathcal{F}_t\}_{0\leq t\leq T}$ with $\mathcal{F}_t=\sigma(\mathcal{F}_t^{W,\Theta}\cup\mathcal{N}_{\mathbb{P}})$. Then $\mathbb{F}=\{\mathcal{F}_t\}_{0\leq t\leq T}$ represents the centralized information including all Brownian motions (BMs) and diversity index components across all agents (principal and marginals).

For any Euclidean space $\mathbb{V}$, $1\leq p<\infty$, and $T>0$, introduce the following spaces:

- $L^p_{\mathcal{F}_T}(\Omega;\mathbb{V}) := \{\eta:\Omega\to\mathbb{V}|\eta$ is $\mathcal{F}_T$-measurable such that $\mathbb{E}|\eta|^p<\infty\}$.
- $L^\infty(0,T;\mathbb{V}) := \{\varphi(\cdot):[0,T]\to\mathbb{V}$ such that $\operatorname{esssup}_{0\leq s\leq T}|\varphi(s)|<\infty\}$.
- $L^p_{\mathbb{F}}(0,T;\mathbb{V}) := \{\varphi(\cdot):\Omega\times[0,T]\to\mathbb{V}$ is progressively measurable such that $\mathbb{E}\int_0^T|\varphi(s)|^p\,ds<\infty\}$.

We consider a weakly coupled large population system of heterogeneous agents $\{\mathcal{A}_i : 1\leq i\leq N\}$ with the dynamics of the agents given in (3), and cost functional (2). For the sake of presentation, we restate them as follows:

$$
\begin{cases}
dx_i = \big[A_{\Theta_i}x_i + Bu_i + Fx^{(N)}\big]dt + \big[Cx_i + D_{\Theta_i}u_i + \widetilde{F}x^{(N)}\big]dW_i,\\
\quad x_i(0)=\xi\in\mathbb{R}^n,\\
\mathcal{J}_i(\mathbf{u}(\cdot)) = \dfrac{1}{2}\mathbb{E}\displaystyle\int_0^T\big[\langle Q(x_i-Hx^{(N)}), x_i-Hx^{(N)}\rangle + \langle Ru_i,u_i\rangle\big]dt,\\
\quad 1\leq i\leq N.
\end{cases}
\tag{5}
$$

As mentioned before, state (3) and functional (2) formulate a weakly coupled large-scale system with heterogeneous agents. The aggregate team functional of N agents is $\mathcal{J}_{\text{soc}}^{(N)}(\mathbf{u}(\cdot))=$

$\sum_{i=1}^{N}\mathcal{J}_i(\mathbf{u}(\cdot))$. In (5), $(A_{\Theta_i}(\cdot), B(\cdot), C(\cdot), D_{\Theta_i}(\cdot), F(\cdot), \widetilde{F}(\cdot))$ are called the state-coefficient datum, while $(Q(\cdot), H(\cdot), R(\cdot))$ the cost weight datum. We explain more details for the above datum. F, $\widetilde{F}$ are *weakly coupling coefficients* on state-drift and state-diffusion respectively; H is a weakly coupling coefficient on functional; C, D_{Θ_i} are diffusion state-dependence and diffusion control-dependence coefficients respectively. Note that $D_{\Theta_i}\neq 0$ represents the case when control enters diffusion alike the risky portfolio selection (e.g., [22, 32, 44]); $F, \widetilde{F}\neq 0$ denotes the agents are coupled in the dynamics such as the price formation problem (e.g., [19, 28]); $H\neq 0$ denotes the *relative performance* formulation (e.g., [16]).

Unlike state (1), we introduce $\{\Theta_i\}_{i=1}^{N}$ in (3) as some diversity index to characterize the possible heterogenous features among all agents. We point out that Θ_i may be vector-valued on a Cartesian *grid* space, say $[a_1, b_1]\times[a_2, b_2]$ or $[a_1, b_1]\times\{1,\dots,K\}$, to represent various feature dimensions, either in continuum space or discrete space, or in a hybrid manner.

For simplicity, we only assume that the coefficients A and D are dependent on Θ_i. Similar analysis can be generalized to the case when all other coefficients are also Θ_i-dependent. Besides, in what follows the time variable t will usually be suppressed if no confusion occurs. We now introduce the following assumption on distribution and coefficient datum set:

(A1) For $i=1,\dots,N$, $\Theta_i:\Omega\to\mathcal{S}$ are independent identically distributed (i.i.d) with the distribution function $\Phi(\theta)$, that is, $\int_{\mathcal{S}} d\Phi(\theta)=1$, where $\mathcal{S}$ is a (possibly continuum) subset in Cartesian space $\mathbb{R}^k$. Note that the discrete set, that is, finite diversity, is a special example.

(A2) For any $\theta\in\mathcal{S}$, $A_\theta(\cdot), F(\cdot), C(\cdot), \widetilde{F}\in L^\infty(0,T;\mathbb{R}^{n\times n})$, $B(\cdot)$, $D_\theta(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m})$, $Q(\cdot)\in L^\infty(0,T;\mathbb{S}^n)$, $H(\cdot)\in L^\infty(0,T;\mathbb{S}^n)$, $R(\cdot)\in L^\infty(0,T;\mathbb{S}^m)$.

(A3) $Q(\cdot)\geq 0$, $R(\cdot)\gg 0$.

REMARK 2.1. We remark that discrete- or finite-valued Θ_i might be transformed into continuum one by assigning uniform distribution on compact intervals along with given partitions. Indeed, this is equivalent to simulating a given discrete random variable using the quantile method by uniform distribution. Thus, hereafter we focus on vector-valued index Θ_i on $\mathbb{R}^k$. Similar analysis can be found in [33] for MFG with continuum-parametrized heterogeneous minor players and [35] for MFG with a continuum of agents.

Under assumptions (A1)–(A2), the state (3) admits a unique strong solution $x(\cdot)=(x_1(\cdot),\dots,x_N(\cdot))\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{N\times n})$, and the cost functional is well defined for each admissible control strategy $\mathbf{u}(\cdot)$ on appropriate admissible space, to be detailed soon. Moreover, under assumption (A3), the cost functional is uniform convex, that is, there exists some $\delta>0$ such that $\mathcal{J}_{\mathrm{soc}}^{(N)}(\mathbf{u})\geq\delta\mathbb{E}\int_0^T|\mathbf{u}(s)|^2\,ds$.

Given state (3) and functional (2), we can specify the associated information structures. Because of interactive coupling by state-average $x^{(N)}:=\frac{1}{N}\sum_{i=1}^{N}x_i$, $\mathcal{J}_i(u_i,u_{-i})$ depends on total team-decision $\mathbf{u}=(u_i,u_{-i})$. In this sense, (3) exhibits the so-called *weakly interactive coupling* in decision when $N\to+\infty$. Again, by such interactive coupling, the information structure of (3) becomes more involved:

- *Decentralized, open-loop information*: consider the filtration $\mathcal{F}_t^{W_i}=\sigma(W_i(s), 0\leq s\leq t)$, $\mathcal{F}_t^{W_i,\Theta_i}=\sigma(\Theta_i)\vee\mathcal{F}_t^{W_i}$, $0\leq t<\infty$, and the set of null sets $\mathcal{N}_{\mathbb{P}}^i=\{M\in\Omega|\exists G\in\mathcal{F}_\infty^{W_i,\Theta_i}$ with $M\subset G$ and $\mathbb{P}(G)=0\}$, and create the augmented filtration $\mathbb{F}^i=\{\mathcal{F}_t^i\}_{0\leq t\leq T}$ with $\mathcal{F}_t^i=\sigma(\mathcal{F}_t^{W_i,\Theta_i}\cup\mathcal{N}_{\mathbb{P}}^i)$. Note that $\{\mathcal{F}_t^i\}$ only depends on W_i and Θ_i instead of state x_i itself, thus we call it an open-loop (although it also differs from classical open-loop due to the mean-field nature) information since it depends directly on underlying randomness.

- *Decentralized, closed-loop information*: denote by $\{\mathcal{H}_t^i\}_{0\le t\le T}$ the filtration of individual state x_i augmented by $\mathcal{N}_{\mathbb{P}}^i$, that is, $\mathcal{H}_t^i = \sigma\{x_i(s), 0 \le s \le t\} \vee \mathcal{N}_{\mathbb{P}}^i$. Note that $\{\mathcal{H}_t^i\}$ only depends on the underlying principal state x_i itself, thus we call it closed-loop (although it also differs from classical closed-loop due to mean-field nature). We remark that x_i is not adapted to $\mathbb{F}^i$ due to weak coupling.
- *Decentralized, partial information*: Let $\mathcal{G}_t^i \subseteq \mathcal{F}_t^i$ be a sub-σ-field of $\mathcal{F}_t^i$ (or, $\mathcal{G}_t^i \subseteq \mathcal{H}_t^i$ be a sub-σ-field of $\mathcal{H}_t^i$), then $\mathbb{G}^i = \{\mathcal{G}_t^i\}_{0\le t\le T}$ represents the decentralized open-loop (or closed-loop) partial information available to $\mathcal{A}_i$.

REMARK 2.2. For decentralized, partial information pattern, $\mathcal{G}_t^i$ is a given filtration representing the information available to $\mathcal{A}_i$ at time t. For example, $\mathcal{G}_t^i = \mathcal{F}_{(t-\delta)+}^i$, or $\mathcal{G}_t^i = \mathcal{H}_{(t-\delta)+}^i$, $t \in [0, T]$, where $\delta > 0$ denotes the fixed delay of information. In this case, $\mathcal{G}_t^i$ represent the partial information in an open-loop or closed-loop sense, respectively. Another example is that $W_i = (\widetilde{W}_i, \widetilde{W}_0)$ takes vector-valued Brownian motion including a common noise component $\widetilde{W}_0$, then $\mathcal{G}_t^i = \sigma\{\widetilde{W}_i(s), \Theta_i, 0 \le s \le t\}$ denotes the partial information in an open-loop. Also, in case $\Theta_i = (\Theta_{i1}, \Theta_{i2})$, then $\mathcal{G}_t^i = \sigma\{W_i(s), \Theta_{i1}, 0 \le s \le t\}$ denotes the partial information to underlying diversity.

Therefore, $\mathcal{B}_t^i = \mathcal{F}_t^i \vee \mathcal{H}_t^i$ and $\mathbb{B}^i := \{\mathcal{B}_t^i\}_{0\le t\le T}$ represent (full) decentralized information. Then we have the following structure inclusion chart:

$$\begin{aligned}\mathbb{G}^i &\subset \{\mathbb{F}^i\,(\text{decentralized open-loop}), \mathbb{H}^i\,(\text{decentralized closed-loop})\} \\ &\subset \mathbb{B}^i\,(\text{decentralized}) \subset \mathbb{F}(\text{full}).\end{aligned}$$

Noticing due to state-average $x^{(N)}$, $x_i(t) \notin \mathcal{F}_t^i$, thus, NO inclusion relations between open-loop $\mathbb{F}^i = \{\mathcal{F}_t^i\}_{0\le t\le T}$ and closed-loop $\mathbb{H}^i = \{\mathcal{H}_t^i\}_{0\le t\le T}$. This is different from classical control where the open-loop information includes closed-loop information. Given the information structure, we are ready to formulate the relevant admissible control sets:

- Centralized full-information set: $\mathcal{U}_i^{c,f} = \{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathbb{F}}(0, T; \Gamma)\}$.
- Decentralized full-information open-loop set: $\mathcal{U}_{i,\mathrm{op}}^{d,f} = \{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathbb{F}^i}(0, T; \Gamma)\}$.
- Decentralized full-information closed-loop set: $\mathcal{U}_{i,\mathrm{cl}}^{d,f} = \{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathbb{H}^i}(0, T; \Gamma)\}$.
- Decentralized partial-information set: $\mathcal{U}_i^{d,p} = \{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathbb{G}^i}(0, T; \Gamma)\}$.

We point out here $\mathbb{G}^i$ is general to include both open-loop or closed-loop partial information. Now we propose the following optimization problem.

Problem LQG-MFT. Find a team strategy set $\bar{\mathbf{u}}(\cdot) = (\bar{u}_1(\cdot), \dots, \bar{u}_N(\cdot))$ where $\bar{u}_i(\cdot) \in \mathcal{U}_i^{c,f}$, $1 \le i \le N$, such that

$$\mathcal{J}_{\mathrm{soc}}^{(N)}(\bar{\mathbf{u}}(\cdot)) = \inf_{u_i \in \mathcal{U}_i^{c,f}, 1\le i\le N} \mathcal{J}_{\mathrm{soc}}^{(N)}(u_1(\cdot), \dots, u_i(\cdot), \dots, u_N(\cdot)).$$

Under some mild conditions on datum (Q, R) (e.g., (A3)), it is possible to ensure the existence and uniqueness of optimal mean-field team strategy in a centralized sense. This can proceed by classical vector-optimization or control method but in a high-dimension setting because of the existence of a large number of weakly coupled team agents. However, such a strategy, from a computational viewpoint, turns out to be intractable because of the information requirement to collect all agents' states simultaneously. Instead, it is more tractable to consider some decentralized strategy for which only the local (distributed) information for a given agent is needed. Moreover, considering the partial information pattern, we introduce the following definition on asymptotic social optimality.

DEFINITION 2.3. A strategy set $\widetilde{\mathbf{u}}(\cdot) = (\widetilde{u}_1(\cdot), \dots, \widetilde{u}_N(\cdot))$ with $\{\widetilde{u}_i \in \mathcal{U}_i^{d,p}\}_{i=1}^N$ is said to be ε-social optimal if there exists $\varepsilon = \varepsilon(N) > 0$, $\lim_{N\to+\infty}\varepsilon(N) = 0$ such that

$$\frac{1}{N}\Big(\mathcal{J}_{\mathrm{soc}}^{(N)}(\widetilde{\mathbf{u}}(\cdot)) - \inf_{u\in\mathcal{U}_i^{c,f}} \mathcal{J}_{\mathrm{soc}}^{(N)}(\mathbf{u}(\cdot))\Big) \le \varepsilon.$$

REMARK 2.4. In Remark 2.2, we emphasize W_i might be a vector-valued Brownian motion including a common noise component. For simplicity, in the following we assume that W_i, $i = 1, \dots, N$ are independent one-dimensional Brownian motions. Note that for the case $W_i = (\widetilde{W}_i, \widetilde{W}_0)$ takes vector-valued Brownian motion including a common noise component $\widetilde{W}_0$ and $\widetilde{W}_i$, $i = 1, \dots, N$ being independent one-dimensional Brownian motions, the procedures in Section 3 and Section 4 are still workable. However, in this case $\mathbb{E}\alpha$ in (26) should be the conditional expectation $\mathbb{E}[\alpha|\mathcal{F}_t^0]$ where $\{\mathcal{F}_t^0\}$ is the filtration generated by the common noise $\widetilde{W}_0$. For this kind of consistency system, please refer to [21] for more information.

As discussed before, centralized strategy based on classical vector optimization/control turns out to be ineffective to handle weakly coupled but highly complex LQG-MFT. Alternatively, it is more amenable to construct some decentralized strategy on distributional information only. Such strategy construction can be implemented using mean-field team analysis through the following steps:

Step 1 (Section 3.1): applying variational decomposition for a generic agent;

Step 2 (Section 3.2): applying weak duality to construct auxiliary control (AC) problem;

Step 3 (Section 4): solving AC to determine the limiting state-average by consistency condition (CC);

Step 4 (Section 5): verifying the asymptotic social optimality of the derived decentralized team strategy.

We now proceed step by step to construct the distributed LQG-MFT strategy.

3. Mean-field team analysis.

Our current work focuses on team optimization, so a variational analysis becomes unavoidable to calibrate a response of related componentwise Fréchet differentials for a generic agent, say $\mathcal{A}_i$. Such an analysis is not required in MFG as all agents there are noncooperative. Hence, unlike MFG, we need to quantify the *total* variation of social cost $\delta\mathcal{J}_{\mathrm{soc}}^{(N)}(\delta u_i)$ triggered by *individual* variation δu_i of $\mathcal{A}_i$.

3.1. *Variational decomposition.*

In order to quantify (total) variation $\delta\mathcal{J}_{\mathrm{soc}}^{(N)}(\delta u_i)$ owning to basic δu_i by a generic $\mathcal{A}_i$, we need to compute the variation of social cost when $\mathcal{A}_i$ adopts an alternative strategy while all others' decisions remain unchanged. Subsequently, in Step 1, we would like to figure out a variational decomposition for the original (5) around the centralized strategy (although we prefer to circumvent its direct but high-dimensional computation). The variational decomposition consists of three substeps, as detailed below.

3.1.1. *Decomposition of $\delta\mathcal{J}_{soc}^{(N)}(\delta u_i)$.*

First we will obtain $\delta\mathcal{J}_{\mathrm{soc}}^{(N)}(\delta u_i)$ when $\mathcal{A}_i$ uses an alternative strategy. Let $\{\bar{u}_i \in \mathcal{U}_i^{c,f}\}_{i=1}^N$ be the centralized optimal team strategy (its existence can be ensured under some mild convexity conditions. But, as discussed above, such strategies are intractable for real computation purposes because of the "curse of dimensionality"). Now consider the perturbation for a given benchmark agent, say, $\mathcal{A}_i$ to use the alternative strategy $u_i \in \mathcal{U}_i^{c,f}$ and all other agents still apply the strategy $\bar{u}_{-i} = (\bar{u}_1, \dots, \bar{u}_{i-1}, \bar{u}_{i+1}, \dots, \bar{u}_N)$. The realized state (3) corresponding to $(u_i, \bar{u}_{-i})$ and $(\bar{u}_i, \bar{u}_{-i})$ are denoted by $(x_1, \dots, x_N)$ and $(\bar{x}_1, \dots, \bar{x}_N)$, respectively. We denote the agent index set as

$\mathcal{I} = \{1, \dots, N\}$. To start the variational decomposition, it is helpful to present the following causal-relation flow-chart first:

$$\underbrace{\delta u_i = u_i - \bar{u}_i}_{\text{principal basic variation}}$$

$$\Longrightarrow \quad \underbrace{\delta x_i = x_i(u_i) - \bar{x}_i(\bar{u}_i)}_{\text{principal intermediate variation}} \quad \Longrightarrow \quad \underbrace{\delta x_{j,i} = x_j(x_i) - \bar{x}_j(\bar{x}_i)}_{\text{marginal variation}}$$

$$\Longrightarrow \quad \underbrace{\delta \mathcal{J}_{j,i}(\delta u_i)}_{\text{marginal cost variation}} = \mathcal{J}_j(u_i, \bar{u}_{-i}) - \mathcal{J}_j(\bar{u}_i, \bar{u}_{-i}), \quad j = 1, \dots, N,$$

$$\Longrightarrow \quad \underbrace{\delta \mathcal{J}_{\mathrm{soc}}^{(N)}(\delta u_i)}_{\text{total cost variation}} = \mathcal{J}_{\mathrm{soc}}^{(N)}(u_i, \bar{u}_{-i}) - \mathcal{J}_{\mathrm{soc}}^{(N)}(\bar{u}_i, \bar{u}_{-i}),$$

where δu_i is the most basic variation "block" for other variation structures; we write $x_i(u_i)$ to emphasize its dependence of x_i on u_i, and similarly for $\bar{x}_i(\bar{u}_i)$; we call δx_i the *principal intermediate* variation as it depends indirectly on basic δu_i via principal state; similarly, $\delta x_{j,i}$ *marginal variations* from the point of $\mathcal{A}_i$; also $x_j(x_i)$ depends on x_i via weak-coupling $x^{(N)}$, similar to $\bar{x}_j(\bar{x}_i)$. Note that the subscripts of $\delta x_{j,i}$ means that x_i is the principal state while x_j, $j \neq i$, are marginal ones, from the viewpoint of $\mathcal{A}_i$. Moreover, from the standpoint of $\mathcal{A}_i$, the variational equations for the principal state x_i, and marginal states $\{x_j\}_{j \neq i}$ satisfy

$$\begin{aligned} &d\delta x_i = \left[A_{\Theta_i}\delta x_i + B\delta u_i + F\delta x^{(N)}\right]dt + \left[C\delta x_i + D_{\Theta_i}\delta u_i + \widetilde{F}\delta x^{(N)}\right]dW_i, \\ &\delta x_i(0) = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &j \neq i, \quad d\delta x_{j,i} = \left[A_{\Theta_j}\delta x_{j,i} + F\delta x^{(N)}\right]dt + \left[C\delta x_{j,i} + \widetilde{F}\delta x^{(N)}\right]dW_j, \\ &\delta x_{j,i}(0) = 0. \end{aligned} \tag{7}$$

Denote $\delta x_{-i} = \sum_{j \neq i} \delta x_{j,i}$ the aggregate variation of marginal agents (benchmark to $\mathcal{A}_i$), so applying linear state-aggregation,

$$\begin{aligned} d\delta x_{-i} = &\left[\sum_{j \neq i} A_{\Theta_j}\delta x_{j,i} + (N-1)F\delta x^{(N)}\right]dt \\ &+ \sum_{j \neq i}\left[C\delta x_{j,i} + \widetilde{F}\delta x^{(N)}\right]dW_j, \delta x_{-i}(0) = 0. \end{aligned} \tag{8}$$

Similarly, we can obtain the variation of cost functionals as follows. For the principal cost of $\mathcal{A}_i$:

$$\delta \mathcal{J}_i(\delta u_i) = \mathbb{E}\int_0^T \left[\langle Q(\bar{x}_i - H\bar{x}^{(N)}), \delta x_i - H\delta x^{(N)}\rangle + \langle R\bar{u}_i, \delta u_i\rangle\right]dt.$$

For marginal costs of $\mathcal{A}_i$:

$$\delta \mathcal{J}_{j,i}(\delta u_i) = \mathbb{E}\int_0^T \langle Q(\bar{x}_j - H\bar{x}^{(N)}), \delta x_{j,i} - H\delta x^{(N)}\rangle dt, \quad j \neq i.$$

Therefore, the total variation of the social cost, from person-by-person variation of $\mathcal{A}_i$, becomes

$$\begin{aligned} &\delta \mathcal{J}_{\mathrm{soc}}^{(N)}(\delta u_i) \\ &\quad = \mathbb{E}\int_0^T \Bigg[\langle Q(\bar{x}_i - H\bar{x}^{(N)}), \delta x_i - H\delta x^{(N)}\rangle + \sum_{j \neq i}\langle Q(\bar{x}_j - H\bar{x}^{(N)}), \delta x_{j,i} - H\delta x^{(N)}\rangle \\ &\qquad + \langle R\bar{u}_i, \delta u_i\rangle\Bigg]dt \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\int_0^T \Big[\langle Q\bar{x}_i, \delta x_i\rangle - \langle QH\bar{x}^{(N)}, \delta x_i\rangle - \langle QN\bar{x}^{(N)}, H\delta x^{(N)}\rangle \\
&\quad + \langle QH\bar{x}^{(N)}, H\delta x^{(N)}\rangle + \sum_{j\neq i}\langle Q\bar{x}_j, \delta x_{j,i}\rangle - \sum_{j\neq i}\langle QH\bar{x}^{(N)}, \delta x_{j,i}\rangle \\
&\quad + (N-1)\langle QH\bar{x}^{(N)}, H\delta x^{(N)}\rangle + \langle R\bar{u}_i, \delta u_i\rangle \Big] dt \\
&= \mathbb{E}\int_0^T \Big[\langle Q\bar{x}_i, \delta x_i\rangle + \langle \mathcal{H}(H,Q)\bar{x}^{(N)}, \delta x_i\rangle + \Big\langle \mathcal{H}(H,Q)\bar{x}^{(N)}, \sum_{j\neq i}\delta x_{j,i}\Big\rangle \\
&\quad + \sum_{j\neq i}\langle Q\bar{x}_j, \delta x_{j,i}\rangle + \langle R\bar{u}_i, \delta u_i\rangle \Big] dt \\
&=: I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned}
\tag{9}
$$

where $\mathcal{H}(H,Q) := -QH - HQ + HQH$. There arise five decomposition terms in (9). Among them, I_5 depends directly on the *principal basic variation* δu_i, whereas I_1, I_2 depend on *principal intermediate variation* δx_i that further depends on the basic δu_i. Moreover, I_3, I_4 depend on the *marginal variations* $\{\delta x_{j,i}\}_{j\neq i}$ that further depends on the principal ones δx_i, δu_i. We denote $\|\delta x_i\|_{L^2} = (\mathbb{E}\int_0^T |\delta x_i|^2 ds)^{1/2}$. By a standard SDE estimation, $\|\delta x_i\|_{L^2} \le (K + O(N^{-\frac{1}{2}}))\|\delta u_i\|_{L^2}$ where K is independent on N, and only depends on coefficients of (3). Moreover, $\|\delta x_{j,i}\|_{L^2} = O(N^{-\frac{1}{2}})\|\delta u_i\|_{L^2}$ for $j \neq i$. Also, note that in general, it is not true that $\|\delta u_i\|_{L^2} = O(\|\delta x_i\|_{L^2})$.

Noting in (9), only I_1 and I_5 directly depend on (basic) principal variations δu_i, δx_i whereas I_2, I_3, I_4 are intermediate with indirect dependence ($\bar{x}^{(N)}$, $\delta x_{j,i}$). Thus, the reformulation below is invoked, in the spirit of mean-field approximation, to get rid of such implicit dependence in I_2, I_3, I_4 progressively.

3.1.2. *Reformulation of* I_2, I_3. For I_2, I_3, we need to approximate the empirical state-average $\bar{x}^{(N)}$ by its mean-field limit using heuristic reasoning. Therefore, replacing $\bar{x}^{(N)}$ of I_2, I_3 in (9) by state-average limit $\hat{x}$ (to be determined later in Step 3) will yield

$$
\begin{aligned}
\delta\mathcal{J}_{\mathrm{soc}}^{(N)}(\delta u_i) &= \mathbb{E}\int_0^T \Big[\langle Q\bar{x}_i, \delta x_i\rangle + \langle \mathcal{H}(H,Q)\hat{x}, \delta x_i\rangle + \langle \mathcal{H}(H,Q)\hat{x}, \delta x_{-i}\rangle \\
&\qquad + \frac{1}{N}\sum_{j\neq i}\langle Q\bar{x}_j, N\delta x_{j,i}\rangle + \langle R\bar{u}_i, \delta u_i\rangle \Big] dt + \varepsilon_1 \\
&=: I_1 + \widehat{I}_2 + \widehat{I}_3 + I_4 + I_5 + \varepsilon_1,
\end{aligned}
\tag{10}
$$

where

$$
\varepsilon_1 = -\mathbb{E}\int_0^T \langle \mathcal{H}(H,Q)(\hat{x} - \bar{x}^{(N)}), N\delta x^{(N)}\rangle dt.
$$

Note that terms I_1, $\widehat{I}_2$, I_5 in (10) already depend on the principal variations δu_i or δx_i. Thus, we only need to analyze the limiting behavior for the remaining term $\widehat{I}_3$ and I_4. It is remarkable that $\widehat{I}_3$, I_4 respectively involve components: δx_{-i} and $\frac{1}{N}\sum_{j\neq i}\langle Q\bar{x}_j, N\delta x_{j,i}\rangle$ that both depend on principal basic δu_i in a rather implicit manner.

3.1.3. *Reformulation of* $\widehat{I}_3$, I_4. Note that for $j \neq i$, $\|\delta x_{j,i}\|_{L^2} = O(N^{-\frac{1}{2}})\|\delta u_i\|_{L^2}$, so $\lim_{N\to+\infty}\|\delta x_{j,i}\|_{L^2} = 0$. Hence we need to introduce some limiting term x_j^* to replace the re-scaled $N\delta x_{j,i}$ in rate $\|x_j^* - N\delta x_{j,i}\| = O(N^{-\frac{1}{2}})\|\delta u_i\|_{L^2}$. This helps us deal with the variation of I_4. In addition, we introduce limiting term $x^{**} = \int_{\mathcal{S}} x_\theta^{**}\, d\Phi(\theta)$ to replace δx_{-i} in rate $\|x^{**} - \delta x_{-i}\| = O(N^{-\frac{1}{2}})\|\delta u_i\|_{L^2}$. This will help us deal with $\widehat{I}_3$. Moreover, by the independence between $\{\Theta_j\}$, $\{W_j\}$ and heuristic mean-field arguments, we construct the following coupled system:

$$
\begin{cases}
dx_j^* = \Big[A_{\Theta_j} x_j^* + F\delta x_i + F\displaystyle\int_{\mathcal{S}} x_\theta^{**}\, d\Phi(\theta)\Big]dt \\
\qquad\quad + \Big[C x_j^* + \widetilde{F}\delta x_i + \widetilde{F}\displaystyle\int_{\mathcal{S}} x_\theta^{**}\, d\Phi(\theta)\Big]dW_j, \\
dx_\theta^{**} = \big[A_\theta x_\theta^{**} + F\delta x_i + F x_\theta^{**}\big]dt, \quad x_\theta^{**}(0) = 0, \\
x_j^*(0) = 0, \quad j \neq i, \theta \in \mathcal{S}.
\end{cases}
\tag{11}
$$

Therefore,

$$
\begin{aligned}
\delta\mathcal{J}_{\mathrm{soc}}^{(N)}(\delta u_i) &= \mathbb{E}\int_0^T \Big[\langle Q\bar{x}_i, \delta x_i\rangle + \big\langle \mathcal{H}(H,Q)\hat{x}, \delta x_i\big\rangle + \big\langle \mathcal{H}(H,Q)\hat{x}, x^{**}\big\rangle \\
&\qquad + \frac{1}{N}\sum_{j\neq i}\big\langle Q\bar{x}_j, x_j^*\big\rangle + \langle R\bar{u}_i, \delta u_i\rangle\Big]dt + \sum_{l=1}^{3}\varepsilon_l \\
&=: I_1 + \widehat{I}_2 + \widetilde{I}_3 + \widetilde{I}_4 + I_5 + \sum_{l=1}^{3}\varepsilon_l,
\end{aligned}
\tag{12}
$$

where

$$
\begin{cases}
\varepsilon_2 = -\mathbb{E}\displaystyle\int_0^T \big\langle \mathcal{H}(H,Q)\hat{x}, x^{**} - \delta x_{-i}\big\rangle dt, \\
\varepsilon_3 = \mathbb{E}\displaystyle\int_0^T \frac{1}{N}\sum_{j\neq i}\big\langle Q\bar{x}_j, N\delta x_{j,i} - x_j^*\big\rangle dt.
\end{cases}
$$

Noting $\widetilde{I}_4$ of (12) connects to a sequence of exchangeable random variables $\{\int_0^T \langle Q\bar{x}_j, x_j^*\rangle\, dt\} \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$. By de Finetti theorem, they are *conditionally* independent identically distributed with respect to some tail sigma-algebra. Also, it is observable that such tail sigma-algebra should depend on δx_i in a rather implicit way. Then, we may apply the conditional law of large numbers to identify the related average. We present some weak duality as an approach to break away $\delta\mathcal{J}_{\mathrm{soc}}^{(N)}(\delta u_i)$ from dependence on x_j^* and x^{**}.

3.2. *Weak duality.* Although (12) is free of the marginal $\{\delta x_{j,i}\}_{j\neq i}$ but a trade-off is the raised limiting quantities x_j^* and x^{**} that are still intermediate. For auxiliary construction, we need to further eliminate them via some duality. Due to continuum heterogeneity and state-coupling dynamics, pertinent duality will take a fairly complex argument, and heavily depend on some weak equivalence in distributional rather than a (strong) pathwise sense. Thus, we term it as a "weak" duality procedure, as detailed below. More specifically, in order to break away $\delta\mathcal{J}_{\mathrm{soc}}^{(N)}(\delta u_i)$ of (12) from direct dependence on x_j^* and x^{**} (see $\widetilde{I}_3$, $\widetilde{I}_4$), we introduce the following adjoint equations $\{y_1^j\}_{j\neq i}$ and y_2^θ satisfying

$$
\begin{cases}
dy_1^j = \alpha_1^j\, dt + \beta_1^{jj}\, dW_j + \displaystyle\sum_{l=1, l\neq j}^{N} \beta_1^{jl}\, dW_l, \quad y_1^j(T) = 0, j \neq i, \\
dy_2^\theta = \alpha_2^\theta\, dt, \qquad y_2^\theta(T) = 0, \quad \theta \in \mathcal{S},
\end{cases}
\tag{13}
$$

where $\{W_l\}_{l\neq i}$ are some Brownian motion copies matching all marginal agents in a large-scaled system, from the benchmark point of $\mathcal{A}_i$. We remark that y_2^θ is parametrized by the diversity index in continuum support: $\theta \in \mathcal{S}$, while y_1^j is parametrized by the marginal agent index $j \in \mathcal{I}\backslash\{i\}$. Accordingly, the duality below should be in *distributional* and *agentwise* sense, respectively indexed by $\theta \in \mathcal{S}$ and $j \in \mathcal{I}$. To start, first apply Itô's formula to $\langle y_1^j, x_j^* \rangle$ for each marginal agent index $j \neq i$, and take expectation, by countable agentwise addition for all $j \in \mathcal{I}\backslash\{i\}$,

$$
\begin{aligned}
0 = \mathbb{E}\int_0^T \Bigg[& \frac{1}{N}\sum_{j\neq i}\langle \alpha_1^j + A_{\Theta_j}^\top y_1^j + C^\top \beta_1^{jj}, x_j^* \rangle + \frac{1}{N}\sum_{j\neq i}\langle F^\top y_1^j + \widetilde{F}^\top \beta_1^{jj}, x^{**}\rangle \\
& + \frac{1}{N}\sum_{j\neq i}\langle F^\top y_1^j + \widetilde{F}^\top \beta_1^{jj}, \delta x_i \rangle \Bigg] dt.
\end{aligned}
\tag{14}
$$

Similarly, by distributed integral on all $\theta \in \mathcal{S}$,

$$
0 = \int_0^T \left[\int_{\mathcal{S}} \langle \alpha_2^\theta + A_\theta^\top y_2^\theta + F^\top y_2^\theta, x_\theta^{**} \rangle d\Phi(\theta) + \int_{\mathcal{S}} \langle F^\top y_2^\theta, \delta x_i \rangle d\Phi(\theta) \right] dt. \tag{15}
$$

Combing (14) and (15) with (12)

$$
\begin{aligned}
\delta \mathcal{J}_{\mathrm{soc}}^{(N)}(\delta u_i) = \mathbb{E}\int_0^T \Bigg[& \langle Q\bar{x}_i, \delta x_i \rangle + \langle \mathcal{H}(H,Q)\hat{x}, \delta x_i \rangle - \frac{1}{N}\sum_{j\neq i}\langle F^\top y_1^j + \widetilde{F}^\top \beta_1^{jj}, \delta x_i \rangle \\
& - \int_{\mathcal{S}} \langle F^\top y_2^\theta, \delta x_i \rangle d\Phi(\theta) + \langle R\bar{u}_i, \delta u_i \rangle \Bigg] dt \\
& + \mathbb{E}\int_0^T \left[\frac{1}{N}\sum_{j\neq i}\langle Q\bar{x}_j - \alpha_1^j - A_{\Theta_j}^\top y_1^j - C^\top \beta_1^{jj}, x_j^* \rangle \right] dt \\
& - \mathbb{E}\int_0^T \int_{\mathcal{S}} \Bigg\langle -\mathcal{H}(H,Q)\hat{x} + \frac{1}{N}\sum_{j\neq i}(F^\top y_1^j + \widetilde{F}^\top \beta_1^{jj}) \\
& + \alpha_2^\theta + A_\theta^\top y_2^\theta + F^\top y_2^\theta, x_\theta^{**} \Bigg\rangle d\Phi(\theta)\, dt + \sum_{l=1}^3 \varepsilon_l.
\end{aligned}
\tag{16}
$$

Let

$$
\begin{cases}
\alpha_1^j = Q\bar{x}_j - A_{\Theta_j}^\top y_1^j - C^\top \beta_1^{jj}, \\
\alpha_2^\theta = \mathcal{H}(H,Q)\hat{x} - F^\top \mathbb{E} y_1^j - \widetilde{F}^\top \mathbb{E}\beta_1^{jj} - A_\theta^\top y_2^\theta - F^\top y_2^\theta,
\end{cases}
$$

hence we reach the following weak duality adjoint process:

$$
\begin{cases}
dy_1^j = (Q\bar{x}_j - A_{\Theta_j}^\top y_1^j - C^\top \beta_1^{jj})\,dt + \beta_1^{jj}\, dW_j + \displaystyle\sum_{l=1, l\neq j}^N \beta_1^{jl}\, dW_l, \\
dy_2^\theta = -(-\mathcal{H}(H,Q)\hat{x} - (F^\top \mathbb{E} y_1^j + \widetilde{F}^\top \mathbb{E}\beta_1^{jj}) - A_\theta^\top y_2^\theta - F^\top y_2^\theta)\,dt, \\
y_1^j(T) = 0, \quad j \neq i, \qquad y_2^\theta(T) = 0, \quad \theta \in \mathcal{S}.
\end{cases}
\tag{17}
$$

We point out that the above system can be rewritten as

$$
\begin{cases}
dy_1^j = (Q\bar{x}_j - A_{\Theta_j}^\top y_1^j - C^\top \beta_1^{jj})\,dt + \beta_1^{jj}\, dW_j + \displaystyle\sum_{l=1, l\neq j}^N \beta_1^{jl}\, dW_l, \\
dy_2^\Theta = -(-\mathcal{H}(H,Q)\hat{x} - (F^\top \mathbb{E} y_1^j + \widetilde{F}^\top \mathbb{E}\beta_1^{jj}) - A_\Theta^\top y_2^\Theta - F^\top y_2^\Theta)\,dt, \\
y_1^j(T) = 0, \quad j \neq i, \qquad y_2^\Theta(T) = 0.
\end{cases}
$$

We remark that y_2^Θ is a degenerate BSDE by noting $\Theta \in \mathcal{F}_0$. Also, it is not necessary to specify any dependence assumption between Θ_j and Θ since y_1^j and y_2^Θ get coupled only through the expectation operator. In other words, the coupling and associated consistency condition only concern their expectations. Still, we may term the resulting duality as weak duality. Substituting (17) into (16), we have

$$\delta\mathcal{J}_{\rm soc}^{(N)}(\delta u_i) = \mathbb{E}\int_0^T\Big[\langle Q\bar{x}_i, \delta x_i\rangle + \langle \mathcal{H}(H,Q)\hat{x}, \delta x_i\rangle - \frac{1}{N}\sum_{j\neq i}\langle F^\top y_1^j + \widetilde{F}^\top \beta_1^{jj}, \delta x_i\rangle$$
$$- \int_{\mathcal{S}}\langle F^\top y_2^\theta, \delta x_i\rangle d\Phi(\theta) + \langle R\bar{u}_i, \delta u_i\rangle\Big]dt + \sum_{l=1}^4 \varepsilon_l,$$

where

$$\varepsilon_4 = \mathbb{E}\int_0^T\Big\langle F^\top\Big(\mathbb{E}[y_1^j] - \frac{1}{N}\sum_{j\neq i} y_1^j\Big) + \widetilde{F}^\top\Big(\mathbb{E}[\beta_1^{jj}] - \frac{1}{N}\sum_{j\neq i}\beta_1^{jj}\Big), x^{**}\Big\rangle dt.$$

We observe that the initial terms such as $\langle Q\bar{x}_j, x_j^*\rangle$ in (12), are now reformulated with some inner product between principal intermediate variation δx_i and some quantities in terms by y_2^θ and y_1^j in an agentwise (i.e., $j\neq i$) manner. Then, we can identify the tail filtration for exchangeable $\{\int_0^T\langle Q\bar{x}_j, x_j^*\rangle\,dt\}_{j\neq i}$ based on δx_i with a degenerated filtration. So, applying the conditional law of large number, and noticing $\{y_1^j,\ j\neq i\}$ are identical distributed, we reach the following representation with expectation operator:

$$\delta\mathcal{J}_{\rm soc}^{(N)}(\delta u_i) = \mathbb{E}\int_0^T\Big[\langle Q\bar{x}_i, \delta x_i\rangle - \Big\langle -\mathcal{H}(H,Q)\hat{x} + F^\top\mathbb{E}[y_1] + \widetilde{F}^\top\mathbb{E}[\beta_1^1]$$
$$+ F^\top\int_{\mathcal{S}} y_2^\theta\, d\Phi(\theta), \delta x_i\Big\rangle + \langle R\bar{u}_i, \delta u_i\rangle\Big]dt + \sum_{l=1}^5\varepsilon_l, \tag{18}$$

where y_1 (depending on $\bar{x}_1$, that is, the optimized state for the generic agent) is some copy with the same distribution for generic y_1^j:

$$\begin{cases} dy_1 = [Q\bar{x}_1 - A_\Theta^\top y_1 - C^\top\beta_1^1]dt + \beta_1^1\,dW_1 + \displaystyle\sum_{l=1,l\neq 1}^N \beta_1^l\,dW_l, \\ dy_2^\theta = [\mathcal{H}(H,Q)\hat{x} - (F^\top\mathbb{E}y_1 + \widetilde{F}^\top\mathbb{E}\beta_1^1) - A_\theta^\top y_2^\theta - F^\top y_2^\theta]dt, \\ y_1(T) = 0, \qquad y_2^\theta(T) = 0, \quad \theta\in\mathcal{S}, \end{cases} \tag{19}$$

and

$$\varepsilon_5 = \mathbb{E}\int_0^T\Big\langle F^\top\Big(\mathbb{E}[y_1] - \frac{1}{N}\sum_{j\neq i} y_1^j\Big) + \widetilde{F}^\top\Big(\mathbb{E}[\beta_1^1] - \frac{1}{N}\sum_{j\neq i}\beta_1^{jj}\Big), \delta x_i\Big\rangle dt.$$

We remark that y_1 has the same distribution with generic y_1^j. This again explains why we term the above procedure as "weak" duality. We point out all variations terms in (18), are now directly depending only on principal (basic, or intermediate) variations. Thus, we now formulate a decentralized auxiliary cost differential $\delta J_i(\delta u_i)$:

$$\delta J_i(\delta u_i) = \mathbb{E}\int_0^T\Big[\langle Q\bar{x}_i, \delta x_i\rangle - \Big\langle -\mathcal{H}(H,Q)\hat{x} + F^\top\hat{y}_1 + \widetilde{F}^\top\hat{\beta}_1$$
$$+ F^\top\int_{\mathcal{S}} y_2^\theta\, d\Phi(\theta), \delta x_i\Big\rangle + \langle R\bar{u}_i, \delta u_i\rangle\Big]dt. \tag{20}$$

REMARK 3.1. There are four undetermined terms in (20) respectively: $\hat{x}$ by (10) is the state-average limit; $(\hat{y}_1 = \mathbb{E}[y_1], \hat{\beta}_1 = \mathbb{E}[\beta_1^1], y_2^\theta)$ is from (19) because of the weak duality procedure. All these terms, especially $\hat{x}$, will be determined by CC in Section 4.

REMARK 3.2. In (20), we introduce the first variation of auxiliary cost functional $\delta J_i(\delta u_i)$ and ignore the error term $\varepsilon_l, l = 1, \ldots, 5$. The convergence rate estimation of these terms and the rigorous proofs will be given in Section 5.

4. Auxiliary control problem and consistency condition. This section aims to complete Step 3 concerning the auxiliary problem, which has a double-fold role in its formulation and solvability. By weakly coupling of MFT, the centralized strategy is infeasible due to the curse of dimensionality. Alternatively, the decentralized one is more desirable that it can be derived by formulating an auxiliary cost with a frozen state-average limit. Counterpart formulation in MFG is quite straightforward because of the competitive feature. However, for MFT, such auxiliary formulation becomes more complicated because each agent must take into account social cost on others. Subsequently, formulation of the auxiliary problem, together with an earlier variational decomposition and weak duality, will jointly complete the above complex freezing procedure. Next, solvability of the auxiliary problem enables us to design decentralized MFT strategies with asymptotic optimality. Now, we present more role details.

4.1. *Auxiliary control with double-projection.* (20) contains only the principal terms δx_i and δu_i, thus it links to an optimal control problem using *local* information of $\mathcal{A}_i$ only. Now we can introduce the following auxiliary control (AC) problem for a generic $\mathcal{A}_i$:

$$\text{(AC)}: \quad \begin{cases} \text{Minimize } J_i(u_i(\cdot)) = \dfrac{1}{2}\mathbb{E}\displaystyle\int_0^T \big[\langle Qx_i, x_i\rangle - 2\langle \Xi, x_i\rangle + \langle Ru_i, u_i\rangle\big]\,dt, \\ \text{subject to } dx_i(t) = [A_{\Theta_i}x_i + Bu_i + F\hat{x}]\,dt + [Cx_i + D_{\Theta_i}u_i + \widetilde{F}\hat{x}]\,dW_i(t), \\ \quad x_i(0) = \xi, \end{cases}$$

with

$$\Xi(t; \hat{x}, y_2^\theta, \hat{y}_1, \hat{\beta}_1) = -\mathcal{H}(H, Q)\hat{x} + F^\top \hat{y}_1 + \widetilde{F}^\top \hat{\beta}_1 + F^\top \int_{\mathcal{S}} y_2^\theta \, d\Phi(\theta), \tag{21}$$

where $\hat{x}$ is the limiting state-average term introduced in (10); $(y_2^\theta, \hat{y}_1, \hat{\beta}_1)$ depends on $\hat{x}$ satisfying dynamics (19). Also, we remark that $\hat{y}_1$ depends on optimal state $\bar{x}_j$.

We will apply the stochastic maximum principle to study *Problem (AC)*. To this end, we introduce the following first-order adjoint equation:

$$dp_i(t) = -\big[A_{\Theta_i}^\top p_i + Qx_i - \Xi + C^\top q_i\big]\,dt + q_i\,dW_i(t), \quad p_i(T) = 0.$$

Let u_i^* be the optimal control and (x_i^*, p_i^*, q_i^*) the corresponding state and adjoint state. For any $u_i \in L^2_{\mathbb{G}^i}(0, T; \mathbb{R}^m)$ such that $u_i^* + u_i \in \mathcal{U}_{i,\mathrm{op}}^{d,p}$, we have $u_i^\epsilon := u_i^* + \epsilon u_i \in \mathcal{U}_{i,\mathrm{op}}^{d,p}$. The corresponding state and adjoint state with respect to u_i^ϵ are denoted by $(x_i^\epsilon, p_i^\epsilon, q_i^\epsilon)$. Introduce the following variational equation:

$$dy_i(t) = [A_{\Theta_i}y_i + Bu_i]\,dt + [Cy_i + D_{\Theta_i}u_i]\,dW_i(t), \quad y_i(0) = 0.$$

Applying Itô's formula to $\langle p_i, y_i\rangle$, by the optimality of u_i^* (i.e., $J_i(u_i^\epsilon) - J_i(u_i^*) \geq 0$), we have $\mathbb{E}\int_0^T \langle Ru_i^* + B^\top p_i + D_{\Theta_i}^\top q_i, u_i\rangle\,ds \geq 0$. For any $0 \leq t \leq T$ and $\mathcal{G}_t^i$-measurable random variable η_i, let

$$u_i^*(s) + u_i(s) = \begin{cases} u_i^*(s), & s \notin [t, t+\epsilon]; \\ \eta_i, & s \in [t, t+\epsilon]. \end{cases}$$

Therefore, $\frac{1}{\epsilon}\mathbb{E}\int_t^{t+\epsilon}\langle Ru_i^* + B^\top p_i + D_{\Theta_i}^\top q_i, \eta_i - u_i^*\rangle\,ds \geq 0$. Let $\epsilon \to 0$, we have $\mathbb{E}\langle R(t) \times u_i^*(t) + B^\top(t)p_i^*(t) + D_{\Theta_i}^\top(t)q_i^*(t), \eta_i - u_i^*(t)\rangle \geq 0$, $t \in [0, T]$. For any $v \in \Gamma$ and $A \in$

$\mathcal{G}_t^i$, define $\eta_i = vI_A + u_i^*(t)I_{A^c}$, we have $\mathbb{E}\langle R(t)u_i^*(t) + B^\top(t)p_i^*(t) + D_{\Theta_i}^\top(t)q_i^*(t), v - u_i^*(t)\rangle I_A \geq 0$, $t \in [0,T]$. Since $A \in \mathcal{G}_t^i$ is arbitrary, we have $\mathbb{E}[\langle R(t)u_i^*(t) + B^\top(t)p_i^*(t) + D_{\Theta_i}^\top(t)q_i^*(t), v - u_i^*(t)\rangle|\mathcal{G}_t^i] \geq 0$, $t \in [0,T]$, $\mathbb{P}$-a.s., that is,

$$\begin{aligned}&\langle -R(t)u_i^*(t) + \mathbb{E}[-B^\top(t)p_i^*(t) - D_{\Theta_i}^\top(t)q_i^*(t)|\mathcal{G}_t^i], v - u_i^*(t)\rangle \leq 0,\\ &\quad t \in [0,T], \mathbb{P}\text{-a.s.}\end{aligned} \tag{22}$$

Since $v \in \Gamma$ is arbitrary and Γ is a closed convex set, it follows from the well-known results of convex analysis that (22) is equivalent to

$$u_i^*(t) = \mathbf{P}_\Gamma[R^{-1}\mathbb{E}[-B^\top p_i^*(t) - D_{\Theta_i}^\top q_i^*(t)|\mathcal{G}_t^i]], \quad \text{a.e. } t \in [0,T], \mathbb{P}\text{-a.s.}, \tag{23}$$

where $\mathbf{P}_\Gamma[\cdot]$ is the projection mapping from $\mathbb{R}^m$ to its closed convex subset Γ under the norm $\|v\|_R^2 := \langle R^{\frac{1}{2}}v, R^{\frac{1}{2}}v\rangle$. Note that there involves two projections in (23), because of the input constraint and partial information constraint. This differs from [20, 21] which include only input constraint. Furthermore, the two projections are noncommutative due to above maximum principle arguments. In this case, the related Hamiltonian system for (AC) becomes

$$\begin{cases} dx_i^* = [A_{\Theta_i}x_i^* + B\mathbf{P}_\Gamma[R^{-1}\mathbb{E}[-B^\top p_i^*(t) - D_{\Theta_i}^\top q_i^*(t)|\mathcal{G}_t^i]] + F\hat{x}]\,dt \\ \qquad\quad + [Cx_i^* + D_{\Theta_i}\mathbf{P}_\Gamma[R^{-1}\mathbb{E}[-B^\top p_i^*(t) - D_{\Theta_i}^\top q_i^*(t)|\mathcal{G}_t^i]] + \widetilde{F}\hat{x}]\,dW_i(t), \\ dp_i^* = -[A_{\Theta_i}^\top p_i^* + Qx_i^* - \Xi + C^\top q_i^*]\,dt + q_i^*\,dW_i(t), \\ x_i^*(0) = \xi, \qquad p_i^*(T) = 0, \end{cases} \tag{24}$$

which is a fully coupled FBSDEs with double-projection: the mapping on the input convex-closed set, and the filtering for partial information (i.e., conditional expectation on subspace).

4.2. *Consistency condition.* Note that the optimal strategy for the auxiliary control problem involves some undetermined terms $(\hat{x}, \hat{y}_1, \hat{\beta}_1)$. In this section, we will characterize the undetermined processes, especially state-average limit $\hat{x}$, in (21) via some consistency matching scheme. Given the Hamiltonian system by (24), all agents should apply some exchangeable team decisions $\{u_i^*\}_{i=1}^N$ and the realized states should be as follows:

$$\begin{cases} dx_i^* = [A_{\Theta_i}x_i^* + B\mathbf{P}_\Gamma[R^{-1}\mathbb{E}[-B^\top p_i^*(t) - D_{\Theta_i}^\top q_i^*(t)|\mathcal{G}_t^i]] + Fx^{*,(N)}]\,dt \\ \qquad\quad + [Cx_i^* + D_{\Theta_i}\mathbf{P}_\Gamma[R^{-1}\mathbb{E}[-B^\top p_i^*(t) - D_{\Theta_i}^\top q_i^*(t)|\mathcal{G}_t^i]] + \widetilde{F}x^{*,(N)}]\,dW_i(t), \\ x_i^*(0) = \xi, \end{cases}$$

where $x^{*,(N)} = \frac{1}{N}\sum_{i=1}^N x_i^*$ and (p_i^*, q_i^*) is the solution of (24). Making all such exchangeable strategies aggregated, and applying the de Finetti theorem, we can obtain the limiting system by identifying $\hat{x} = \mathbb{E}x^*$,

$$\begin{cases} d\widetilde{x} = [A_\Theta\widetilde{x} + B\mathbf{P}_\Gamma[R^{-1}\mathbb{E}[-B^\top\widetilde{p}(t) - D_\Theta^\top\widetilde{q}(t)|\mathcal{G}_t]] + F\mathbb{E}\widetilde{x}]\,dt \\ \qquad\quad + [C\widetilde{x} + D_\Theta\mathbf{P}_\Gamma[R^{-1}\mathbb{E}[-B^\top\widetilde{p}(t) - D_\Theta^\top\widetilde{q}(t)|\mathcal{G}_t]] + \widetilde{F}\mathbb{E}\widetilde{x}]\,dW(t), \\ d\widetilde{p} = -\Big[A_\Theta^\top\widetilde{p} + Q\widetilde{x} + \mathcal{H}(H,Q)\mathbb{E}\widetilde{x} - F^\top\hat{y}_1 - \widetilde{F}^\top\hat{\beta}_1 - F^\top\int_{\mathcal{S}} y_2^\theta\,d\Phi(\theta) \\ \qquad\quad + C^\top\widetilde{q}\Big]\,dt + \widetilde{q}\,dW(t), \\ \widetilde{x}(0) = \xi, \qquad \widetilde{p}(T) = 0, \end{cases} \tag{25}$$

where Θ is a random variable with distribution defined in (A1), $W(t)$ is a generic Brownian motion independent of Θ, $\mathcal{G}$ is a subfiltration representing the partial information and $(\hat{y}_1 = \mathbb{E}[y_1], \hat{\beta}_1 = \mathbb{E}[\beta_1^1], y_2^\theta)$ is from (19). Note that we suppress subscript i in (25) as all agents are statistically identical in the distribution sense. Combining with (19), we will obtain consistency condition (CC) of the *Problem LQG-MFT*. For simplicity, define $\mathcal{E}_t[-B^\top\gamma - D_\Theta^\top\vartheta] = \mathbb{E}[-B^\top\gamma - D_\Theta^\top\vartheta|\mathcal{G}_t]$. Hence we have the following result.

PROPOSITION 4.1. *The undetermined parameters of* (21) *can be determined by*

$$(\hat{x}, \hat{y}_1, \hat{\beta}_1, y_2^\theta) = (\mathbb{E}\alpha, \mathbb{E}\check{y}_1, \mathbb{E}\check{\beta}_1, \check{y}_2^\theta),$$

where $(\alpha, \gamma, \vartheta, \check{y}_1, \check{\beta}_1, \check{y}_2^\theta)$ *is the solution of the consistency condition of* Problem LQG-MFT:

$$(26)\qquad \begin{cases} d\alpha = [A_\Theta \alpha + B\mathbf{P}_\Gamma[R^{-1}\mathcal{E}_t[-B^\top \gamma - D_\Theta^\top \vartheta]] + F\mathbb{E}\alpha]\,dt \\ \qquad\quad + [C\alpha + D_\Theta \mathbf{P}_\Gamma[R^{-1}\mathcal{E}_t[-B^\top \gamma - D_\Theta^\top \vartheta]] + \widetilde{F}\mathbb{E}\alpha]\,dW, \\ d\gamma = \Big[-Q\alpha - \mathcal{H}(H, Q)\mathbb{E}\alpha - A_\Theta^\top \gamma + F^\top \int_{\mathcal{S}} \check{y}_2^\theta \, d\Phi(\theta) \\ \qquad\quad + F^\top \mathbb{E}\check{y}_1 - C^\top \vartheta + \widetilde{F}^\top \mathbb{E}\check{\beta}_1\Big]\,dt + \vartheta\, dW, \\ d\check{y}_1 = [Q\alpha - A_\Theta^\top \check{y}_1 - C^\top \check{\beta}_1]\,dt + \check{\beta}_1\, dW, \\ d\check{y}_2^\theta = [\mathcal{H}(H, Q)\mathbb{E}\alpha - F^\top \mathbb{E}\check{y}_1 - \widetilde{F}^\top \mathbb{E}\check{\beta}_1 - A_\theta^\top \check{y}_2^\theta - F^\top \check{y}_2^\theta]\,dt, \\ \alpha(0) = \xi, \qquad \gamma(T) = 0, \qquad \check{y}_1(T) = 0, \qquad \check{y}_2^\theta(T) = 0, \quad \theta \in \mathcal{S}. \end{cases}$$

REMARK 4.2. (26) is a new type of fully coupled FBSDEs with double-projection (projection mapping on the convex-closed subset and partial-information subspace). Moreover, both temporal variable t and spatial variable θ appear in (26). Considering this, we can rewrite (26) in the following more compact form:

$$(27)\qquad \begin{cases} d\alpha = [A_\Theta \alpha + B\mathbf{P}_\Gamma[R^{-1}\mathcal{E}_t[-B^\top \gamma - D_\Theta^\top \vartheta]] + F\mathbb{E}\alpha]\,dt \\ \qquad\quad + [C\alpha + D_\Theta \mathbf{P}_\Gamma[R^{-1}\mathcal{E}_t[-B^\top \gamma - D_\Theta^\top \vartheta]] + \widetilde{F}\mathbb{E}\alpha]\,dW, \\ d\gamma = [-Q\alpha - \mathcal{H}(H, Q)\mathbb{E}\alpha - A_\Theta^\top \gamma + F^\top \mathbb{E}\check{y}_2^\Theta \\ \qquad\quad + F^\top \mathbb{E}\check{y}_1 - C^\top \vartheta + \widetilde{F}^\top \mathbb{E}\check{\beta}_1]\,dt + \vartheta\, dW(t), \\ d\check{y}_1 = [Q\alpha - A_\Theta^\top \check{y}_1 - C^\top \check{\beta}_1]\,dt + \check{\beta}_1\, dW, \\ d\check{y}_2^\Theta = [\mathcal{H}(H, Q)\mathbb{E}\alpha - F^\top \mathbb{E}\check{y}_1 - \widetilde{F}^\top \mathbb{E}\check{\beta}_1 - A_\Theta^\top \check{y}_2^\Theta - F^\top \check{y}_2^\Theta]\,dt, \\ \alpha(0) = \xi, \qquad \gamma(T) = 0, \qquad \check{y}_1(T) = 0, \qquad \check{y}_2^\Theta(T) = 0. \end{cases}$$

Note that by the independence between Θ and W, (27) can be viewed as defined on the product space $\Omega_1 \times \Omega_2 \to \mathcal{S} \times \mathbb{R}^n$. This is a general system which includes many frameworks in current literature as special cases. For more information, please refer to Section 6.1.

4.3. *Wellposedness of consistency condition.* This subsection continues to complete (Step 3) by establishing some well-posedness to CC derived in Section 4.2. Note that (26) is fully coupled FBSDEs involved with double projections whose well-posedness cannot be guaranteed by current literature. Moreover, as explained in Section 6.2, (26) is obtained by converting the system with continuum heterogeneity to a homogenous one but with augmented randomness ($\{\Theta_i, W_i\}_{i=1}^N$) as a trade-off. Based on this, we will apply the discounting method to study (26) which would provide some mild conditions to ensure the existence and uniqueness of fully coupled FBSDEs as (26). Define $X = \alpha$, $Y = (\gamma^\top, \check{y}_1^\top, (\check{y}_2^\theta)^\top)^\top$ and $Z = (\vartheta^\top, \check{\beta}_1^\top, 0)^\top$. For simplicity, let $\mathcal{E}_t[Y] = \mathbb{E}[Y|\mathcal{G}_t]$ and $\mathcal{E}_t[Z] = \mathbb{E}[Z|\mathcal{G}_t]$, $\widetilde{\mathbb{E}}[Y] = ((\int_{\mathcal{S}} \gamma \, d\Phi(\theta))^\top, (\int_{\mathcal{S}} \check{y}_1 \, d\Phi(\theta))^\top, (\int_{\mathcal{S}} \check{y}_2^\theta \, d\Phi(\theta))^\top)^\top$, then (26) takes the following form:

$$(28)\qquad \begin{cases} dX = [A_\Theta X + F\mathbb{E}[X] + \mathbb{B}_1(Y, Z)]\,dt + [CX + \widetilde{F}\mathbb{E}[X] + \mathbb{D}_\Theta(Y, Z)]\,dW, \\ dY = [\mathbb{A}_2 X + \bar{\mathbb{A}}_2 \mathbb{E}[X] + \mathbb{B}_2 Y + \bar{\mathbb{B}}_2 \mathbb{E}[Y] + \widetilde{\mathbb{B}}_2 \widetilde{\mathbb{E}}[Y] + \mathbb{C}_2 Z + \bar{\mathbb{C}}_2 \mathbb{E}[Z]]\,dt \\ \qquad\quad + Z\, dW, \\ X(0) = \xi, \qquad Y(T) = (0, \dots, 0)^\top, \end{cases}$$

where

$$\begin{cases} \mathbb{B}_1(Y,Z) = B\mathbf{P}_\Gamma\big[R^{-1}\mathcal{E}_t\big[-B^\top\gamma - D_\Theta^\top\vartheta\big]\big] \\ \qquad\qquad = B\mathbf{P}_\Gamma\big[R^{-1}\big((-B^\top,0,\dots,0)\mathcal{E}_t[Y] + (-D_\Theta^\top,0,\dots,0)\mathcal{E}_t[Z]\big)\big], \\ \mathbb{D}_\Theta(Y,Z) = D_\Theta\mathbf{P}_\Gamma\big[R^{-1}\mathcal{E}_t\big[-B^\top\gamma - D_\Theta^\top\vartheta\big]\big] \\ \qquad\qquad = D_\Theta\mathbf{P}_\Gamma\big[R^{-1}\big((-B^\top,0,\dots,0)\mathcal{E}_t[Y] + (-D_\Theta^\top,0,\dots,0)\mathcal{E}_t[Z]\big)\big], \\ \mathbb{A}_2 = \begin{pmatrix} -Q \\ Q \\ 0 \end{pmatrix}, \qquad \bar{\mathbb{A}}_2 = \begin{pmatrix} -\mathcal{H}(H,Q) \\ 0 \\ \mathcal{H}(H,Q) \end{pmatrix}, \\ \mathbb{B}_2 = \begin{pmatrix} -A_\Theta^\top & 0 & 0 \\ 0 & -A_\Theta^\top & 0 \\ 0 & 0 & -A_\theta^\top - F^\top \end{pmatrix}, \\ \bar{\mathbb{B}}_2 = \begin{pmatrix} 0 & F^\top & 0 \\ 0 & 0 & 0 \\ 0 & -F^\top & 0 \end{pmatrix}, \qquad \widetilde{\mathbb{B}}_2 = \begin{pmatrix} 0 & 0 & F^\top \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathbb{C}_2 = \begin{pmatrix} -C^\top & 0 & 0 \\ 0 & -C^\top & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \bar{\mathbb{C}}_2 = \begin{pmatrix} 0 & \widetilde{F}^\top & 0 \\ 0 & 0 & 0 \\ 0 & -\widetilde{F}^\top & 0 \end{pmatrix}, \end{cases}$$

and 0 denotes the zero vector or zero matrix with suitable dimensions. Note that in (28), $\widetilde{\mathbb{B}}_2\widetilde{\mathbb{E}}[Y] = \widetilde{\mathbb{E}}[\widetilde{\mathbb{B}}_2 Y] = \mathbb{E}[\widetilde{\mathbb{B}}_2 Y]$. To start, we first give some results for the general nonlinear mean-field forward-backward system with double projections:

$$\begin{cases} dX = b\big(t,X,\mathbb{E}[X],\mathcal{E}_t[Y],\mathcal{E}_t[Z]\big)\,dt \\ \qquad\quad + \sigma\big(t,X,\mathbb{E}[X],\mathcal{E}_t[Y],\mathcal{E}_t[Z]\big)\,dW, \quad X(0)=x, \\ dY(t) = -f\big(t,X,\mathbb{E}[X],Y,\mathbb{E}[Y],\widetilde{\mathbb{E}}[Y],Z,\mathbb{E}[Z]\big)\,dt + Z\,dW, \quad Y(T)=0, \end{cases} \tag{29}$$

where $\mathbb{E}[\widetilde{\mathbb{E}}[Y]] = \mathbb{E}[Y]$ and the coefficients satisfy the following conditions:

(H1) There exist $\rho_1,\rho_2\in\mathbb{R}$ and positive constants k_i, $i=1,\dots,8$ such that for all $t\in[0,T]$, $x,x_1,x_2,\bar{x},\bar{x}_1,\bar{x}_2\in\mathbb{R}^n$, $y,y_1,y_2,\bar{y},\bar{y}_1,\bar{y}_2,\hat{y},\hat{y}_1,\hat{y}_2,\widetilde{y},\widetilde{y}_1,\widetilde{y}_2\in\mathbb{R}^m$, $z,z_1,z_2,\bar{z},\bar{z}_1,\bar{z}_2,\widetilde{z},\widetilde{z}_1,\widetilde{z}_2\in\mathbb{R}^m$, a.s.,

$$\begin{aligned} &\big\langle b(t,x_1,\bar{x},y,\hat{y},z,\hat{z}) - b(t,x_2,\bar{x},y,\hat{y},z,\hat{z}), x_1-x_2\big\rangle \le \rho_1|x_1-x_2|^2, \\ &\big|b(t,x,\bar{x}_1,y_1,\hat{y}_1,z_1,\hat{z}_1) - b(t,x,\bar{x}_2,y_2,\hat{y}_2,z_1,\hat{z}_2)\big| \\ &\quad \le k_1|\bar{x}_1-\bar{x}_2| + k_2|y_1-y_2| + k_2|\hat{y}_1-\hat{y}_2| + k_2|z_1-z_2| + k_2|\hat{z}_1-\hat{z}_2|, \\ &\big\langle f(t,x,\bar{x},y_1,\bar{y},\widetilde{y},z,\bar{z}) - f(t,x,\bar{x},y_2,\bar{y},\widetilde{y},z,\bar{z}), y_1-y_2\big\rangle \le \rho_2|y_1-y_2|^2, \\ &\big|f(t,x_1,\bar{x}_1,y,\bar{y}_1,\widetilde{y}_1,z_1,\bar{z}_1) - f(t,x_2,\bar{x}_2,y,\bar{y}_2,\widetilde{y}_2,z_2,\bar{z}_2)\big| \\ &\quad \le k_2|x_1-x_2| + k_2|\bar{x}_1-\bar{x}_2| + k_3|\bar{y}_1-\bar{y}_2| + k_4|\widetilde{y}_1-\widetilde{y}_2| \\ &\qquad + k_5|z_1-z_2| + k_6|\bar{z}_1-\bar{z}_2|, \\ &\big|\sigma(t,x_1,\bar{x}_1,y_1,\hat{y}_1,z_1,\hat{z}_1) - \sigma(t,x_2,\bar{x}_2,y_2,\hat{y}_2,z_2,\hat{z}_2)\big|^2 \\ &\quad \le k_7^2|x_1-x_2|^2 + k_8^2|\bar{x}_1-\bar{x}_2|^2 + k_2^2|\hat{y}_1-\hat{y}_2|^2 + k_2^2|\hat{z}_1-\hat{z}_2|^2. \end{aligned}$$

(H2)

$$\mathbb{E}\int_0^T \big[\big|b(t,0,0,0,0)\big|^2 + \big|\sigma(t,0,0,0,0)\big|^2 + \big|f(t,0,0,0,0,0,0,0,0)\big|^2\big]\,dt < \infty.$$

Similar to [21] and [37], we have the following result of the solvability of (29). For the readers' convenience, we give the proof in the Appendix.

THEOREM 4.3. *Suppose* (H1) *and* (H2) *hold. There exists a constant* $\delta_1 > 0$ *depending on* $\rho_1, \rho_2, T, k_i, i = 1, 3, 4, 5, 6, 7, 8$ *such that if* $k_2 \in [0, \delta_1)$, *FBSDEs* (29) *admits a unique adapted solution* $(X, Y, Z) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$. *Furthermore, if* $2\rho_1 + 2\rho_2 < -2k_1 - 2k_3 - 2k_4 - k_5^2 - k_6^2 - k_7^2 - k_8^2$, *there exists a constant* $\delta_2 > 0$ *depending on* $\rho_1, \rho_2, k_i, i = 1, 3, 4, 5, 6, 7, 8$ *such that if* $k_2 \in [0, \delta_2)$, *FBSDEs* (29) *admits a unique adapted solution* $(X, Y, Z) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$.

Let $\rho_1^* = \operatorname{ess\,sup}_{0 \le s \le T} \operatorname{ess\,sup}_{\theta \in \mathcal{S}} \Lambda_{\max}(-\frac{1}{2}(A_\theta(s) + A_\theta(s)^\top))$ and $\rho_2^* = \operatorname{ess\,sup}_{0 \le s \le T} \Lambda_{\max}(-\frac{1}{2}(\mathbb{B}(s) + \mathbb{B}(s)^\top))$, where $\Lambda_{\max}(M)$ is the largest eigenvalue of the matrix M. For $M(\cdot) \in L^\infty_{\mathbb{F}}(0, T; \mathbb{R}^{n \times n})$, $\|M(\cdot)\| \triangleq \operatorname{ess\,sup}_{0 \le s \le T} \operatorname{ess\,sup}_{\omega \in \Omega} \|M(s)\|$. Comparing (29) with (28), we can check that the parameters of (H1) and (H2) can be chosen as follows:

$$
\begin{aligned}
&k_1 = \|F\|, \qquad k_3 = \|\bar{\mathbb{B}}_2\|, \qquad k_4 = \|\widetilde{\mathbb{B}}_2\|, \qquad k_5 = \|\mathbb{C}_2\|, \\
&k_6 = \|\bar{\mathbb{C}}_2\|, \qquad k_7 = \sqrt{3}\|C\|, \qquad k_8 = \sqrt{3}\|\widetilde{F}\|, \\
&k_2 = \max\{\|B\|^2\|R^{-1}\|, \|B\|\|R^{-1}\|\|D_\Theta\|, \|\mathbb{A}_2\|, \|\bar{\mathbb{A}}_2\|, \\
&\qquad \sqrt{6}\|D_\Theta\|\|R^{-1}\|\|B\|, \sqrt{6}\|D_\Theta\|\|R^{-1}\|\|D_\Theta\|\}.
\end{aligned}
$$

Now we introduce the following assumption:

(A4) $2\rho_1^* + 2\rho_2^* < -2k_1 - 2k_3 - 2k_4 - k_5^2 - k_6^2 - k_7^2 - k_8^2$.

It follows from Theorem 4.3 that:

PROPOSITION 4.4. *Under* (A4), *there exists a constant* $\delta_3 > 0$ *depending on* ρ_1^*, ρ_2^*, k_i, $i = 1, 3, 4, 5, 6, 7, 8$ *such that if* $k_2 \in [0, \delta_3)$, *FBSDEs* (28) *admits a unique adapted solution* $(X, Y, Z) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{3n}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{3n})$.

5. Asymptotic ε-optimality. This section aims to complete (Step 4) so as to verify the asymptotic optimality of the mean-field team strategy derived in Section 4. Here we proceed with our verification based on the assumption in Section 4.3, that is, (A4). Contrary to MFG entailing only one-side perturbation for a single agent to the asymptotic Nash equilibrium, MFT must take into account team (integrated) perturbations upon all agents. Meanwhile, (cooperative) social cost is more intertwined than the individual one of single agent, so a quadratic functional representation, as formalized below, will greatly facilitate our targeted analysis. For the sake of clear presentation, we divide the related analysis into four substeps in separated subsections.

5.1. *Quadratic representation of social cost.* We first give a quadratic representation of the team functional that gives a tractable fortiori formulation of Fréchet differentials of social cost. Rewrite the large-population system (3) as follows:

$$
d\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}u)\,dt + \sum_{i=1}^{N} (\mathbf{C}_i \mathbf{x} + \mathbf{D}_i u)\,dW_i, \quad \mathbf{x}(0) = \widetilde{\xi}, \tag{30}
$$

where

$$
\mathbf{A} = \begin{pmatrix} A_{\Theta_1} + \frac{F}{N} & \frac{F}{N} & \cdots & \frac{F}{N} \\ \frac{F}{N} & A_{\Theta_2} + \frac{F}{N} & \cdots & \frac{F}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F}{N} & \frac{F}{N} & \cdots & A_{\Theta_N} + \frac{F}{N} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix},
$$

$$
\mathbf{B} = \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix},
$$

$$
\mathbf{C}_i = \begin{array}{c} 1 \\ \vdots \\ i \\ \vdots \\ N \end{array} \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\widetilde{F}}{N} & \cdots & \frac{\widetilde{F}}{N} & \frac{\widetilde{F}}{N} + C & \frac{\widetilde{F}}{N} & \cdots & \frac{\widetilde{F}}{N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},
$$

$$
\mathbf{D}_i = \begin{array}{c} 1 \\ \vdots \\ i \\ \vdots \\ N \end{array} \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & D_{\Theta_i} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \widetilde{\xi} = \begin{pmatrix} \xi \\ \vdots \\ \xi \end{pmatrix}.
$$

Similarly, the social cost takes the following form:

$$
\mathcal{J}_{\mathrm{soc}}^{(N)}(u) = \frac{1}{2}\mathbb{E}\int_0^T \left[\langle \mathbf{Q}\mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{R}u, u \rangle \right] dt,
$$

where

$$
\mathbf{Q} = \begin{pmatrix} Q + \frac{1}{N}\mathcal{H}(H, Q) & \frac{1}{N}\mathcal{H}(H, Q) & \cdots & \frac{1}{N}\mathcal{H}(H, Q) \\ \frac{1}{N}\mathcal{H}(H, Q) & Q + \frac{1}{N}\mathcal{H}(H, Q) & \cdots & \frac{1}{N}\mathcal{H}(H, Q) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N}\mathcal{H}(H, Q) & \frac{1}{N}\mathcal{H}(H, Q) & \cdots & Q + \frac{1}{N}\mathcal{H}(H, Q) \end{pmatrix},
$$

$$
\mathbf{R} = \begin{pmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{pmatrix}.
$$

Next, by the variation of the constant formula, the strong solution of (30) becomes

$$\mathbf{x}(t)=\Phi(t)\widetilde{\xi}+\Phi(t)\int_0^t\Phi(s)^{-1}\Bigg[\Bigg(\mathbf{B}-\sum_{i=1}^N\mathbf{C}_i\mathbf{D}_i\Bigg)u(s)\Bigg]ds$$
$$+\sum_{i=1}^N\Phi(t)\int_0^t\Phi(s)^{-1}\mathbf{D}_iu(s)\,dW_i(s),$$

where $d\Phi(t)=\mathbf{A}\Phi(t)\,dt+\sum_{i=1}^N\mathbf{C}_i\Phi(t)\,dW_i(t)$, $\Phi(0)=I$. Define the following operators:

$$\begin{cases}\phi(u)(\cdot):=\Phi(\cdot)\Bigg\{\displaystyle\int_0^\cdot\Phi(s)^{-1}\Bigg[\Bigg(\mathbf{B}-\sum_{i=1}^N\mathbf{C}_i\mathbf{D}_i\Bigg)u(s)\Bigg]ds+\sum_{i=1}^N\int_0^\cdot\Phi(s)^{-1}\mathbf{D}_iu\,dW_i(s)\Bigg\},\\ \widetilde{\phi}(u):=\phi(u)(T),\qquad \mathcal{S}(y)(\cdot):=\Phi(\cdot)\Phi^{-1}(0)\widetilde{\xi},\qquad \widetilde{\mathcal{S}}(y):=\mathcal{S}(y)(T),\end{cases}$$

then for any admissible control u, we have $\mathbf{x}(\cdot)=\phi(u)(\cdot)+\mathcal{S}(y)(\cdot)$, $\mathbf{x}(T)=\widetilde{\phi}(u)+\widetilde{\mathcal{S}}(y)$. Note that $\phi(\cdot):(L^2_{\mathcal{F}}(0,T;\Gamma),\dots,L^2_{\mathcal{F}}(0,T;\Gamma))\to(L^2_{\mathcal{F}}(0,T;\mathbb{R}^n),\dots,L^2_{\mathcal{F}}(0,T;\mathbb{R}^n))$ is a bounded linear operator, thus there exists a unique bounded linear operator $\phi^*(\cdot):(L^2_{\mathcal{F}}(0,T;\mathbb{R}^n),\dots,L^2_{\mathcal{F}}(0,T;\mathbb{R}^n))\to(L^2_{\mathcal{F}}(0,T;\Gamma),\dots,L^2_{\mathcal{F}}(0,T;\Gamma))$ such that for any $u(\cdot)\in(L^2_{\mathcal{F}}(0,T;\Gamma),\dots,L^2_{\mathcal{F}}(0,T;\Gamma))$ and $\mathbf{x}(\cdot)\in(L^2_{\mathcal{F}}(0,T;\mathbb{R}^n),\dots,L^2_{\mathcal{F}}(0,T;\mathbb{R}^n))$,

$\mathbb{E}\int_0^T\langle\phi(u)(t),\mathbf{x}(t)\rangle\,dt=\mathbb{E}\int_0^T\langle u(t),\phi^*(\mathbf{x})(t)\rangle\,dt$. Hence, we can rewrite the cost functional as follows:

$$2\mathcal{J}^{(N)}_{\mathrm{soc}}(u)=\mathbb{E}\int_0^T\big[\langle(\phi^*\mathbf{Q}\phi+\mathbf{R})u,u\rangle+2\langle\phi^*\mathbf{Q}\mathcal{S}(y),u\rangle+\langle\mathbf{Q}\mathcal{S}(y),\mathcal{S}(y)\rangle\big]dt$$
$$:=\langle M_2(u)(\cdot),u(\cdot)\rangle+2\langle M_1,u(\cdot)\rangle+M_0,$$

where we have used $\langle\cdot,\cdot\rangle$ as inner products in different Hilbert spaces. Note that, $M_2(\cdot)$ is a bounded self-adjoint positive semidefinite linear operator.

5.2. *Agent $\mathcal{A}_i$ perturbation.* This subsection gives a perturbation for the single agent $\mathcal{A}_i$ that further triggers a team perturbation across the population, (see Section 5.4). Let $\widetilde{\mathbf{u}}=(\widetilde{u}_1,\dots,\widetilde{u}_N)$ be the decentralized strategy given by

$$\widetilde{u}_i(t)=\varphi_{\Theta_i}\big(p_i(t),q_i(t)\big):=\mathbf{P}_\Gamma\big[R(t)^{-1}\mathbb{E}\big[B(t)^\top p_i(t)+D_{\Theta_i}(t)^\top q_i(t)|\mathcal{G}^i_t\big]\big],\tag{31}$$

where (p_i,q_i) is the solution of

$$\begin{cases}dx_i=\big[A_{\Theta_i}x_i+B\varphi_{\Theta_i}(p_i,q_i)+F\mathbb{E}\alpha\big]dt+\big[Cx_i+D_{\Theta_i}\varphi_{\Theta_i}(p_i,q_i)+\widetilde{F}\mathbb{E}\alpha\big]dW_i(t),\\ dp_i=\Big[-Qx_i-\mathcal{H}(H,Q)\mathbb{E}\alpha-A_{\Theta_i}^\top p_i+F^\top\displaystyle\int_{\mathcal{S}}\check{y}_2^\theta\,d\Phi(\theta)+F^\top\mathbb{E}\check{y}_1\\ \qquad-C^\top q_i+\widetilde{F}^\top\mathbb{E}\check{\beta}_1\Big]dt+q_i\,dW_i(t),\\ x_i(0)=\xi,\qquad p_i(T)=0,\quad i=1,\dots,N.\end{cases}$$

Here, $(\alpha,\check{y}_1,\check{\beta}_1,\check{y}_2^\theta)$ is the solution of (26). Correspondingly, the realized decentralized states $(\widetilde{x}_1,\dots,\widetilde{x}_N)$ satisfy

$$\begin{cases}d\widetilde{x}_i=\big[A_{\Theta_i}\widetilde{x}_i+B\varphi_{\Theta_i}(p_i,q_i)+F\widetilde{x}^{(N)}\big]dt\\ \qquad+\big[C\widetilde{x}_i+D_{\Theta_i}\varphi_{\Theta_i}(p_i,q_i)+\widetilde{F}\widetilde{x}^{(N)}\big]dW_i,\\ \widetilde{x}_i(0)=\xi,\end{cases}\tag{32}$$

and $\widetilde{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \widetilde{x}_i$. Let us consider the case that the agent $\mathcal{A}_i$ (without loss of generality, assume $i > 1$) uses an alternative strategy $u_i \in \mathcal{U}_i^{c,f}$ while the other agents $\mathcal{A}_j$, $j \neq i$ use the strategy $\widetilde{u}_{-i}$. The realized state with the ith agent's perturbation is

$$\begin{cases} d\acute{x}_i = \big[A_{\Theta_i}\acute{x}_i + Bu_i + F\acute{x}^{(N)}\big]dt + \big[C\acute{x}_i + D_{\Theta_i}u_i + \widetilde{F}\acute{x}^{(N)}\big]dW_i, \\ d\acute{x}_j = \big[A_{\Theta_j}\acute{x}_j + B\varphi_{\Theta_j}(p_j,q_j) + F\acute{x}^{(N)}\big]dt + \big[C\acute{x}_j + D_{\Theta_j}\varphi_j(p_j,q_j) + \widetilde{F}\acute{x}^{(N)}\big]dW_j, \\ \acute{x}_i(0) = \xi, \quad \acute{x}_j(0) = \xi, \quad 1 \le j \le N, j \neq i, \end{cases}$$

where $\acute{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \acute{x}_i$. For $j = 1, \dots, N$, denote the perturbation $\delta u_i = u_i - \widetilde{u}_i$, $\delta x_{j,i} = \acute{x}_j - \widetilde{x}_j$, $\delta\mathcal{J}_{j,i} = \mathcal{J}_j(u_i, \widetilde{u}_{-i}) - \mathcal{J}_j(\widetilde{u}_i, \widetilde{u}_{-i})$. Introducing the following frozen states:

$$\begin{cases} d\widetilde{l}_j = \big[A_{\Theta_j}\widetilde{l}_j + B\varphi_{\Theta_j}(p_j,q_j) + F\mathbb{E}\alpha\big]dt \\ \qquad\quad + \big[C\widetilde{l}_j + D_{\Theta_j}\varphi_{\Theta_j}(p_j,q_j) + \widetilde{F}\mathbb{E}\alpha\big]dW_j, \\ \widetilde{l}_j(0) = \xi, \quad j = 1, \dots, N, \end{cases} \tag{33}$$

and

$$\begin{cases} d\acute{l}_i = [A_{\Theta_i}\acute{l}_i + Bu_i + F\mathbb{E}\alpha]dt + [C\acute{l}_i + D_{\Theta_i}u_i + \widetilde{F}\mathbb{E}\alpha]dW_i, \\ d\acute{l}_j = \big[A_{\Theta_j}\acute{l}_j + B\varphi_{\Theta_j}(p_j,q_j) + F\mathbb{E}\alpha\big]dt + \big[C\acute{l}_j + D_{\Theta_j}\varphi_j(p_j,q_j) + \widetilde{F}\mathbb{E}\alpha\big]dW_j, \\ \acute{l}_i(0) = \xi, \qquad \acute{l}_j(0) = \xi, \quad 1 \le j \le N, j \neq i. \end{cases}$$

Similar to the computations in Section 3.1, we have

$$\delta\mathcal{J}_{\text{soc}}^{(N)} = \mathbb{E}\int_0^T \big[\langle Q\widetilde{l}_i, \delta l_i\rangle - \langle \Xi, \delta l_i\rangle + \langle R\widetilde{u}_i, \delta u_i\rangle\big]dt + \sum_{l=1}^7 \epsilon_l, \tag{34}$$

where

$$\begin{cases} \epsilon_1 = -E\int_0^T \big\langle \mathcal{H}(H,Q)\big(\mathbb{E}\alpha - \widetilde{x}^{(N)}\big), N\delta x^{(N)}\big\rangle dt, \\ \epsilon_2 = -E\int_0^T \big\langle \mathcal{H}(H,Q)\mathbb{E}\alpha, x^{**} - \delta x_{-i}\big\rangle dt, \\ \epsilon_3 = E\int_0^T \frac{1}{N}\sum_{j\neq i}\big\langle Q\widetilde{x}_j, N\delta x_{j,i} - x_j^*\big\rangle dt, \\ \epsilon_4 = \mathbb{E}\int_0^T \Big\langle F^\top\Big(\mathbb{E}[y_1^1] - \frac{1}{N}\sum_{j\neq i} y_1^j\Big) + \widetilde{F}^\top\Big(\mathbb{E}[\beta_1^{11}] - \frac{1}{N}\sum_{j\neq i}\beta_1^{jj}\Big), \delta x_i\Big\rangle dt, \\ \epsilon_5 = \mathbb{E}\int_0^T \Big\langle F^\top\Big(\mathbb{E}[y_1^1] - \frac{1}{N}\sum_{j\neq i} y_1^j\Big) + \widetilde{F}^\top\Big(\mathbb{E}[\beta_1^{11}] - \frac{1}{N}\sum_{j\neq i}\beta_1^{jj}\Big), x^{**}\Big\rangle dt, \\ \epsilon_6 = \mathbb{E}\int_0^T \big[\langle \acute{l}_i - \acute{x}_i, \Xi\rangle + \langle \widetilde{l}_i - \widetilde{x}_i, \Xi\rangle\big]dt, \\ \epsilon_7 = \mathbb{E}\int_0^T \big[\big\langle Q(\widetilde{x}_i - \widetilde{l}_i), \delta x_i\big\rangle + \langle Q\widetilde{l}_i, \acute{x}_i - \acute{l}_i\rangle + \langle Q\widetilde{l}_i, \widetilde{x}_i - \widetilde{l}_i\rangle\big]dt. \end{cases}$$

5.3. *Preliminary estimations*. By (34), in order to establish asymptotic optimality of decentralized strategies, we need to rely on some estimates on $\epsilon_1, \dots, \epsilon_7$, based on structural estimations of variational equations (6), (7), (8) and mean-field approximations in Section 3.1. More elaborate estimates are thereby needed considering continuum heterogeneity. So this subsection will first study the properties of involved variational equations and mean-field approximations. Below, L denotes a generic constant whose value may change from line to line. Applying the same technique as in [21], Lemma 5.1, we have:

LEMMA 5.1. *There exists a constant L independent of N such that*

$$\mathbb{E}\sup_{0\le t\le T}\big[|\alpha|^2+|\gamma|^2+|\check{y}_1|^2+|\check{y}_2^\theta|^2\big]+\sum_{j=1}^N\mathbb{E}\sup_{0\le t\le T}\big[|x_j|^2+|p_j|^2\big]$$
$$+\mathbb{E}\int_0^T\big[|\vartheta|^2+|\check{\beta}_1|^2\big]dt+\sum_{j=1}^N\mathbb{E}\int_0^T\big[|q_j|^2+\big|\varphi_{\Theta_j}(p_j,q_j)\big|^2\big]dt\le L,$$

and

$$\sup_{1\le j\le N}\mathbb{E}\sup_{0\le t\le T}\big|\widetilde{x}_j(t)\big|^2+\sup_{1\le j\le N}\mathbb{E}\sup_{0\le t\le T}\big|\widetilde{l}_j(t)\big|^2\le L.$$

Next we give some estimations on variational equations (6), (7) and (8).

LEMMA 5.2. *There exists a constant L independent of N such that*

$$\mathbb{E}\sup_{0\le s\le t}\big|\delta x^{(N)}\big|^2+\sup_{1\le j\le N,j\ne i}\mathbb{E}\sup_{0\le t\le T}|\delta x_{j,i}|^2\le\frac{L}{N^2}.$$

PROOF. Recall the equations (6), (7) and (8), we have

$$\mathbb{E}\sup_{0\le s\le t}|\delta x_i|^2\le L+L\mathbb{E}\int_0^t|\delta x_i|^2\,ds+L\mathbb{E}\int_0^t\big|\delta x^{(N)}\big|^2\,ds,$$
$$\mathbb{E}\sup_{0\le s\le t}|\delta x_{j,i}|^2\le L\mathbb{E}\int_0^t|\delta x_{j,i}|^2\,ds+L\mathbb{E}\int_0^t\big|\delta x^{(N)}\big|^2\,ds,$$
$$\mathbb{E}\sup_{0\le s\le t}|\delta x_{-i}|^2\le L\mathbb{E}\int_0^t|\delta x_{-i}|^2\,ds+LN^2\mathbb{E}\int_0^t\big|\delta x^{(N)}\big|^2\,ds.$$

Note that $\delta x^{(N)}=\frac{1}{N}\delta x_i+\frac{1}{N}\delta x_{-i}$, we have

$$\mathbb{E}\sup_{0\le s\le t}|\delta x_i|^2\le L+L\mathbb{E}\int_0^t|\delta x_i|^2\,ds+\frac{L}{N^2}\mathbb{E}\int_0^t|\delta x_{-i}|^2\,ds,$$
$$\mathbb{E}\sup_{0\le s\le t}|\delta x_{-i}|^2\le L\mathbb{E}\int_0^t|\delta x_{-i}|^2\,ds+L\mathbb{E}\int_0^t|\delta x_i|^2\,ds.$$

Therefore, it follows from the Gronwall inequality that we have

$$\sup_{1\le j\le N,j\ne i}\mathbb{E}\sup_{0\le s\le t}|\delta x_{j,i}|^2\le\frac{L}{N^2}.\qquad\square$$

Now we study mean-field approximations: due to the continuum heterogeneous setting, some new estimates are thus required with their own interests. Specifically, Lemma 5.3 is not a standard SDE estimate, thus some specific techniques are invoked in its proof.

LEMMA 5.3. *There exists a constant L independent of N such that*

$$\sup_{0\le t\le T}\mathbb{E}\big|\widetilde{x}^{(N)}(t)-\mathbb{E}\alpha\big|^2\le\frac{L}{N}.$$

PROOF. First, for any $\theta \in \mathcal{S}$, let

$$\begin{cases} d\widetilde{x}_{\theta,j} = \big[A_\theta \widetilde{x}_{\theta,j} + B\varphi_\theta(p_j, q_j) + F\widetilde{x}_\theta^{(N)}\big]\,dt \\ \qquad\qquad + \big[C\widetilde{x}_{\theta,j} + D_\theta\varphi_\theta(p_j, q_j) + \widetilde{F}\widetilde{x}_\theta^{(N)}\big]\,dW_j(t), \\ \widetilde{x}_{\theta,j}(0) = \xi, \end{cases}$$

$$\begin{cases} d\widetilde{l}_{\theta,j} = \big[A_\theta \widetilde{l}_{\theta,j} + B\varphi_\theta(p_j, q_j) + F\mathbb{E}\alpha_\theta\big]\,dt \\ \qquad\qquad + \big[C\widetilde{l}_{\theta,j} + D_\theta\varphi_\theta(p_j, q_j) + \widetilde{F}\mathbb{E}\alpha_\theta\big]\,dW_j(t), \\ \widetilde{l}_{\theta,j}(0) = \xi, \end{cases}$$

where $\widetilde{x}_\theta^{(N)} = \frac{1}{N}\sum_{j=1}^N \widetilde{x}_{\theta,j}$ and α_θ is the solution of (26) corresponding to $\Theta \equiv \theta$. By the Cauchy–Schwarz inequality and the Burkholder–Davis–Gundy inequality, we have

$$\mathbb{E}\sup_{0\le s\le t}\big|\widetilde{x}_{\theta,j}(s) - \widetilde{l}_{\theta,j}(s)\big|^2 \le L\mathbb{E}\int_0^t \big[\big|\widetilde{x}_{\theta,j}(s) - \widetilde{l}_{\theta,j}(s)\big|^2 + \big|\widetilde{x}_\theta^{(N)}(s) - \mathbb{E}\alpha_\theta(s)\big|^2\big]\,ds.$$

By the Gronwall inequality, we have

$$\mathbb{E}\sup_{0\le s\le t}\big|\widetilde{x}_{\theta,j}(s) - \widetilde{l}_{\theta,j}(s)\big|^2 \le L\mathbb{E}\int_0^t \big|\widetilde{x}_\theta^{(N)}(s) - \mathbb{E}\alpha_\theta(s)\big|^2\,ds. \tag{35}$$

Next, recalling the state equations (32) and (33), similarly we have

$$\mathbb{E}\sup_{0\le s\le t}\big|\widetilde{x}_j(s) - \widetilde{l}_j(s)\big|^2 \le L\mathbb{E}\int_0^t \big|\widetilde{x}^{(N)}(s) - \mathbb{E}\alpha(s)\big|^2\,ds. \tag{36}$$

Note that for any $t \in [0, T]$,

$$\begin{aligned} &\mathbb{E}\big|\widetilde{x}^{(N)}(t) - \mathbb{E}\alpha(t)\big|^2 \\ &\quad \le 2\mathbb{E}\bigg|\frac{1}{N}\sum_{j=1}^N \widetilde{x}_j(t) - \frac{1}{N}\sum_{j=1}^N \int_{\mathcal{S}} \widetilde{x}_{\theta,j}(t)\,d\Phi(\theta)\bigg|^2 \\ &\qquad + 2\mathbb{E}\bigg|\frac{1}{N}\sum_{j=1}^N \int_{\mathcal{S}} \widetilde{x}_{\theta,j}(t)\,d\Phi(\theta) - \int_{\mathcal{S}} \mathbb{E}\big[\alpha(t)|\Theta=\theta\big]\,d\Phi(\theta)\bigg|^2 \\ &\quad \le \frac{6}{N}\sum_{j=1}^N \mathbb{E}\big|\widetilde{x}_j(t) - \widetilde{l}_j(t)\big|^2 + \frac{6}{N^2}\sum_{j=1}^N \mathbb{E}\bigg|\widetilde{l}_j(t) - \int_{\mathcal{S}} \widetilde{l}_{\theta,j}(t)\,d\Phi(\theta)\bigg|^2 \\ &\qquad + \frac{12}{N^2}\sum_{1\le j\ne k\le N}\bigg\langle \mathbb{E}\bigg(\widetilde{l}_j(t) - \int_{\mathcal{S}} \widetilde{l}_{\theta,j}(t)\,d\Phi(\theta)\bigg), \mathbb{E}\bigg(\widetilde{l}_k(t) - \int_{\mathcal{S}} \widetilde{l}_{\theta,k}(t)\,d\Phi(\theta)\bigg)\bigg\rangle \\ &\qquad + 6\mathbb{E}\bigg|\frac{1}{N}\sum_{j=1}^N \int_{\mathcal{S}} \widetilde{l}_{\theta,j}(t)\,d\Phi(\theta) - \frac{1}{N}\sum_{j=1}^N \int_{\mathcal{S}} \widetilde{x}_{\theta,j}(t)\,d\Phi(\theta)\bigg|^2 \\ &\qquad + 2\int_{\mathcal{S}} \mathbb{E}\bigg|\frac{1}{N}\sum_{j=1}^N \widetilde{x}_{\theta,j}(t)\,d\Phi(\theta) - \mathbb{E}\big[\alpha(t)|\Theta=\theta\big]\bigg|^2\,d\Phi(\theta). \end{aligned} \tag{37}$$

Similar to Lemma 5.1, there exists a constant L such that $\sup_{\theta\in\mathcal{S}}\sup_{1\le j\le N} \mathbb{E}\sup_{0\le t\le T}|\widetilde{x}_{\theta,j}(t)|^2 \le L$. Consequently,

$$\frac{6}{N^2}\sum_{j=1}^N \mathbb{E}\bigg|\widetilde{l}_j(t) - \int_{\mathcal{S}} \widetilde{l}_{\theta,j}(t)\,d\Phi(\theta)\bigg|^2 \le \frac{L}{N}. \tag{38}$$

From $\mathbb{E}\alpha = \int_{\mathcal{S}} \mathbb{E}\alpha_\theta \, d\Phi(\theta)$ and $\mathbb{E}(A_{\Theta_j}\widetilde{l}_j) = \int_{\mathcal{S}} \mathbb{E}(A_\theta \widetilde{l}_{\theta,j}) \, d\Phi(\theta)$, we have

$$\mathbb{E}\Big(\widetilde{l}_j(t) - \int_{\mathcal{S}} \widetilde{l}_{\theta,j}(t) \, d\Phi(\theta)\Big) = 0. \tag{39}$$

It is easy to see that

$$\begin{aligned} &\mathbb{E}\bigg|\frac{1}{N}\sum_{j=1}^{N}\int_{\mathcal{S}} \widetilde{l}_{\theta,j}(t) \, d\Phi(\theta) - \frac{1}{N}\sum_{j=1}^{N}\int_{\mathcal{S}} \widetilde{x}_{\theta,j}(t) \, d\Phi(\theta)\bigg|^2 \\ &\quad = \mathbb{E}\bigg|\frac{1}{N}\sum_{j=1}^{N}\int_{\mathcal{S}} \big(\widetilde{l}_{\theta,j}(t) - \widetilde{x}_{\theta,j}(t)\big) \, d\Phi(\theta)\bigg|^2 \\ &\quad \le \frac{1}{N}\sum_{j=1}^{N}\int_{\mathcal{S}} \mathbb{E}\big|\widetilde{l}_{\theta,j}(t) - \widetilde{x}_{\theta,j}(t)\big|^2 \, d\Phi(\theta). \end{aligned} \tag{40}$$

Substituting (35), (36), (38), (39), and (40) into (37), we have

$$\begin{aligned} \mathbb{E}\big|\widetilde{x}^{(N)}(t) - \mathbb{E}\alpha(t)\big|^2 \le{}& L\mathbb{E}\int_0^t \big|\widetilde{x}^{(N)}(s) - \mathbb{E}\alpha(s)\big|^2 \, ds + \frac{L}{N} \\ &+ \frac{L}{N}\sum_{j=1}^{N}\int_{\mathcal{S}} \mathbb{E}\int_0^t \big|\widetilde{x}_\theta^{(N)}(s) - \mathbb{E}\alpha_\theta(s)\big|^2 \, ds \, d\Phi(\theta) \\ &+ 2\int_{\mathcal{S}} \mathbb{E}\bigg|\frac{1}{N}\sum_{j=1}^{N}\widetilde{x}_{\theta,j}(t) - \mathbb{E}\big[\alpha(t)|\Theta = \theta\big]\bigg|^2 \, d\Phi(\theta). \end{aligned}$$

Applying similar method as the homogeneous case (e.g., [38], Lemma 6.3), we have $\mathbb{E}|\frac{1}{N}\sum_{j=1}^{N}\widetilde{x}_{\theta,j}(t) - \mathbb{E}[\alpha(t)|\Theta = \theta]|^2 \le \frac{L}{N}$, and $\mathbb{E}\int_0^t |\widetilde{x}_\theta^{(N)}(s) - \mathbb{E}\alpha_\theta(s)|^2 \, ds \le \frac{L}{N}$. Therefore, there exists a constant L independent of t such that $\mathbb{E}|\widetilde{x}^{(N)}(t) - \mathbb{E}\alpha|^2 \le L\mathbb{E}\int_0^t |\widetilde{x}^{(N)}(s) - \mathbb{E}\alpha(s)|^2 \, ds + \frac{L}{N}$. By the Gronwall inequality, we have

$$\mathbb{E}\big|\widetilde{x}^{(N)}(t) - \mathbb{E}\alpha\big|^2 \le \frac{L}{N}e^{Lt}. \qquad \square$$

LEMMA 5.4. *There exists some constant L independent of N such that*

$$\sup_{0\le t\le T} \mathbb{E}\big|x^{**} - \delta x_{-i}\big|^2 \le \frac{L}{N}, \tag{41}$$

$$\mathbb{E}\sup_{0\le t\le T} \big|N\delta x_{j,i} - x_j^*\big|^2 \le \frac{L}{N}, \quad j \ne i. \tag{42}$$

PROOF. Introduce the following equations:

$$\begin{cases} d\delta\check{x}_i = \Big[A_{\Theta_i}\delta\check{x}_i + B\delta u_i + \dfrac{F}{N}\delta x_i + \dfrac{F}{N}x^{**}\Big]dt \\ \qquad\quad + \Big[C\delta\check{x}_i + D_{\Theta_i}\delta u_i + \dfrac{\widetilde{F}}{N}\delta x_i + \dfrac{\widetilde{F}}{N}x^{**}\Big]dW_i, \\ d\delta\check{x}_j = \Big[A_{\Theta_j}\delta\check{x}_j + \dfrac{F}{N}\delta x_i + \dfrac{F}{N}x^{**}\Big]dt + \Big[C\delta\check{x}_j + \dfrac{\widetilde{F}}{N}\delta x_i + \dfrac{\widetilde{F}}{N}x^{**}\Big]dW_j, \quad j \ne i, \\ \delta\check{x}_i(0) = 0, \qquad \delta\check{x}_j(0) = 0. \end{cases}$$

Recalling (7), by the Cauchy–Schwarz inequality, the Burkholder–Davis–Gundy inequality, and the Gronwall inequality, we have

$$\mathbb{E}\sup_{0\le s\le t}\big|\delta x_{j,i}(s)-\delta\check{x}_j(s)\big|^2\le\frac{L}{N^2}\mathbb{E}\int_0^t\big|\delta x_{-i}(s)-x^{**}(s)\big|^2\,ds. \tag{43}$$

For any $\theta\in\mathcal{S}$, let

$$\begin{cases} d\delta x_{\theta,i}=\big[A_\theta\delta x_{\theta,i}+B\delta u_i+F\delta x_\theta^{(N)}\big]dt+\big[C\delta x_{\theta,i}+D_\theta\delta u_i+\widetilde{F}\delta x_\theta^{(N)}\big]dW_i,\\ \quad \delta x_{\theta,i}(0)=0,\\ j\ne i,\quad d\delta x_{\theta,j}=\big[A_\theta\delta x_{\theta,j}+F\delta x_\theta^{(N)}\big]dt+\big[C\delta x_{\theta,j}+\widetilde{F}\delta x_\theta^{(N)}\big]dW_j,\\ \quad \delta x_{\theta,j}(0)=0,\end{cases}$$

$$\begin{cases} d\delta\check{x}_{\theta,i}=\Big[A_\theta\delta\check{x}_{\theta,i}+B\delta u_i+\dfrac{F}{N}\delta x_{\theta,i}+\dfrac{F}{N}x_\theta^{**}\Big]dt\\ \qquad\qquad+\Big[C\delta x_{\theta,i}+D_\theta\delta u_i+\dfrac{\widetilde{F}}{N}\delta x_{\theta,i}+\dfrac{\widetilde{F}}{N}x_\theta^{**}\Big]dW_i,\\ d\delta\check{x}_{\theta,j}=\Big[A_\theta\delta\check{x}_{\theta,j}+\dfrac{F}{N}\delta x_{\theta,i}+\dfrac{F}{N}x_\theta^{**}\Big]dt+\Big[C\delta\check{x}_{\theta,j}+\dfrac{\widetilde{F}}{N}\delta x_{\theta,i}+\dfrac{\widetilde{F}}{N}x_\theta^{**}\Big]dW_j,\\ \delta\check{x}_{\theta,i}(0)=0,\qquad \delta\check{x}_{\theta,j}(0)=0,\quad j\ne i,\end{cases}$$

where $\delta x_\theta^{(N)}=\frac{1}{N}\sum_{j=1}^N\delta x_{\theta,j}$. Similarly,

$$\mathbb{E}\sup_{0\le s\le t}\big|\delta x_{\theta,j}(s)-\delta\check{x}_{\theta,j}(s)\big|^2\le\frac{L}{N^2}\mathbb{E}\int_0^t\Big|\sum_{j\ne i}\delta x_{\theta,j}(s)-x_\theta^{**}(s)\Big|^2\,ds. \tag{44}$$

For any $t\in[0,T]$,

$$\begin{aligned}&\mathbb{E}\big|x^{**}(t)-\delta x_{-i}(t)\big|^2\\ &\quad\le 6(N-1)\sum_{j\ne i}\mathbb{E}|\delta x_j-\delta\check{x}_j|^2+6\sum_{j\ne i}\mathbb{E}\Big|\delta\check{x}_j-\int_{\mathcal{S}}\delta\check{x}_{\theta,j}\,d\Phi(\theta)\Big|^2\\ &\qquad+12\sum_{1\le j\ne k\le N,j,k\ne i}\mathbb{E}\Big\langle\delta\check{x}_j-\int_{\mathcal{S}}\delta\check{x}_{\theta,j}\,d\Phi(\theta),\delta\check{x}_k-\int_{\mathcal{S}}\delta\check{x}_{\theta,k}\,d\Phi(\theta)\Big\rangle\\ &\qquad+6(N-1)\sum_{j\ne i}\int_{\mathcal{S}}\mathbb{E}|\delta\check{x}_{\theta,j}-\delta x_{\theta,j}|^2\,d\Phi(\theta)+2\int_{\mathcal{S}}\mathbb{E}\Big|\sum_{j\ne i}\delta x_{\theta,j}-x_\theta^{**}\Big|^2\,d\Phi(\theta).\end{aligned} \tag{45}$$

Similar to Lemma 5.3, we have

$$\begin{aligned}&\mathbb{E}\big|x^{**}(t)-\delta x_{-i}(t)\big|^2\\ &\quad\le L\mathbb{E}\int_0^t\big|\delta x_{-i}(s)-x^{**}(s)\big|^2\,ds+\frac{L}{N}+L\int_{\mathcal{S}}\mathbb{E}\int_0^t\Big|\sum_{j\ne i}\delta x_{\theta,j}(s)-x_\theta^{**}(s)\Big|^2\,ds\,d\Phi(\theta)\\ &\qquad+2\int_{\mathcal{S}}\mathbb{E}\Big|\sum_{j\ne i}\delta x_{\theta,j}-x_\theta^{**}\Big|^2\,d\Phi(\theta).\end{aligned}$$

Applying a similar technique as in the homogeneous case (e.g., page 29 in [38]), we have $\mathbb{E}\sup_{0\le s\le t}|\sum_{j\ne i}\delta x_{\theta,j}(s)-x_\theta^{**}|^2(s)\le\frac{L}{N}$. Therefore, there exists a constant L independent of t such that $\mathbb{E}|x^{**}(t)-\delta x_{-i}(t)|^2\le L\mathbb{E}\int_0^t|\delta x_{-i}(s)-x^{**}(s)|^2\,ds+\frac{L}{N}$. By the Gronwall inequality, we have $\mathbb{E}|x^{**}(t)-\delta x_{-i}(t)|^2 \le \frac{L}{N}e^{Lt}$. Hence (41) follows. Note that

$$\begin{cases} d(x_j^* - N\delta x_{j,i}) = [A_{\Theta_j}(x_j^* - N\delta x_{j,i}) + F(x^{**} - \delta x_{-i})]\,dt \\ \qquad\qquad\qquad + [C(x_j^* - N\delta x_{j,i}) + \widetilde{F}(x^{**} - \delta x_{-i})]\,dW_j, \\ (x_j^* - \delta x_{j,i})(0) = 0. \end{cases}$$

By (41), we have (42). □

The following result follows directly by Lemma 5.3 together with the common Cauchy–Schwarz inequality, the Burkholder–Davis–Gundy inequality and the Gronwall inequality.

LEMMA 5.5. *There exists a constant L independent of N such that*

$$\sup_{1\le j\le N} \mathbb{E} \sup_{0\le t\le T} |\widetilde{l}_j - \widetilde{x}_j|^2 \le \frac{L}{N}. \tag{46}$$

5.4. *Asymptotic optimality.* In view of Section 5.1–5.3, we are now ready to complete Step 4, that is, to establish the asymptotic optimality of $\widetilde{u} = (\widetilde{u}_1, \dots, \widetilde{u}_N)$. In order to prove asymptotic optimality, it suffices to consider the perturbations $u_i \in \mathcal{U}_i^c$ such that $\mathcal{J}_{\mathrm{soc}}^{(N)}(u_1, \dots, u_N) \le \mathcal{J}_{\mathrm{soc}}^{(N)}(\widetilde{u}_1, \dots, \widetilde{u}_N)$. It is easy to check that $\mathcal{J}_{\mathrm{soc}}^{(N)}(\widetilde{u}_1, \dots, \widetilde{u}_N) \le LN$, where L is a constant independent of N. Therefore, in the following we only consider perturbations $u_i \in \mathcal{U}_i^c$ satisfying $\sum_{i=1}^N \mathbb{E}\int_0^T |u_i|^2\,dt \le LN$. Therefore, similar to Lemma 5.3 and Lemma 5.5, we have the following lemma.

LEMMA 5.6. *There exist a constant L independent of N such that*

$$\mathbb{E} \sup_{0\le t\le T} |\acute{x}^{(N)}(t) - \mathbb{E}\alpha|^2 \le \frac{L}{N}, \qquad \sup_{1\le j\le N} \mathbb{E} \sup_{0\le t\le T} |\acute{l}_j - \acute{x}_j|^2 \le \frac{L}{N}.$$

Let $\delta u_i = u_i - \widetilde{u}_i$, and consider a perturbation $u = \widetilde{u} + (\delta u_1, \dots, \delta u_N) := \widetilde{u} + \delta u$. Then by Section 5.1, we have

$$\begin{aligned} 2\mathcal{J}_{\mathrm{soc}}^{(N)}(\widetilde{u} + \delta u) &= \langle M_2(\widetilde{u} + \delta u), \widetilde{u} + \delta u\rangle + 2\langle M_1, \widetilde{u} + \delta u\rangle + M_0 \\ &= 2\mathcal{J}_{\mathrm{soc}}^{(N)}(\widetilde{u}) + 2\sum_{i=1}^N \langle M_2(\widetilde{u}) + M_1, \delta u_i\rangle + \langle M_2(\delta u), \delta u\rangle, \end{aligned}$$

where $M_2(\widetilde{u}) + M_1$ is the Fréchet differential of $\mathcal{J}_{\mathrm{soc}}^{(N)}$ on $\widetilde{u}$.

THEOREM 5.7. *Under the assumptions* (A1)–(A5), $\widetilde{u} = (\widetilde{u}_1, \dots, \widetilde{u}_N)$ *defined in* (31) *is a* $(\frac{1}{\sqrt{N}})$*-social optimal strategy for the agents.*

PROOF. From Section 5.2, we have

$$\langle M_2(\widetilde{u}) + M_1, \delta u_i\rangle = \mathbb{E}\int_0^T [\langle Q\widetilde{l}_i, \delta l_i\rangle - \langle \Xi, \delta l_i\rangle + \langle R\widetilde{u}_i, \delta u_i\rangle]\,dt + \sum_{l=1}^7 \varepsilon_l.$$

From the optimality of $\widetilde{u}$, we have $\mathbb{E}\int_0^T [\langle Q\widetilde{l}_i, \delta l_i\rangle - \langle \Xi, \delta l_i\rangle + \langle R\widetilde{u}_i, \delta u_i\rangle]\,dt \ge 0$. Suppose this is not true, then for u_i such that $\widetilde{u}_i + u_i \in \mathcal{U}_i^{d,p}$, we have $\widetilde{u}_i + \rho u_i \in \mathcal{U}_i^{d,p}$, $0 < \rho < 1$, and $\lim_{\rho\to 0} \frac{J_i(\widetilde{u}_i + \rho u_i, \widetilde{u}_{-i}) - J_i(\widetilde{u}_i, \widetilde{u}_{-i})}{\rho} < 0$. Thus, $J_i(\widetilde{u}_i + \rho u_i, \widetilde{u}_{-i}) < J_i(\widetilde{u}_i, \widetilde{u}_{-i})$ for sufficiently small ρ, which is a contradiction with the optimality of $\widetilde{u}_i$. Moreover, combing Lemmas 5.3–5.6 with iteration analysis (e.g., [23]), we have $\sum_{l=1}^{7} \varepsilon_l = O(\frac{1}{\sqrt{N}})$. Therefore,

$$
\begin{aligned}
&\mathcal{J}_{\text{soc}}^{(N)}(\widetilde{u}+\delta u)\\
&\quad = \mathcal{J}_{\text{soc}}^{(N)}(\widetilde{u}) + \sum_{i=1}^{N} \mathbb{E}\int_0^T \big[\langle Q\widetilde{l}_i, \delta l_i\rangle - \langle \Xi, \delta l_i\rangle + \langle R\widetilde{u}_i, \delta u_i\rangle\big]\,dt\\
&\qquad + \sum_{i=1}^{N}\sum_{l=1}^{5} \varepsilon_l + \frac{1}{2}\langle M_2(\delta u), \delta u\rangle.
\end{aligned}
$$

Note that $\sum_{i=1}^{N} \mathbb{E}\int_0^T [\langle Q\widetilde{l}_i, \delta l_i\rangle - \langle \Xi, \delta l_i\rangle + \langle R\widetilde{u}_i, \delta u_i\rangle]\,dt + \frac{1}{2}\langle M_2(\delta u), \delta u\rangle \geq 0$, and $\sum_{i=1}^{N}\sum_{l=1}^{7} \varepsilon_l = O(\sqrt{N})$, there exists a constant L independent of N such that

$$
\frac{1}{N}\Big(\mathcal{J}_{\text{soc}}^{(N)}(\widetilde{u}) - \inf_{u\in\mathcal{U}_i^c} \mathcal{J}_{\text{soc}}^{(N)}(u)\Big) \leq \frac{L}{\sqrt{N}}. \qquad \square
$$

6. Synthetic analysis on homogeneity and heterogeneity.

6.1. *Literature comparison.* We now present comparisons to some relevant mean-field literature.

6.1.1. *Homogeneous case without diversity.* For the homogeneous case with $\mathcal{S} = \{s_1\}$ being a singleton set, we have $A_{\Theta_i} = A_{s_1} := A$ and $D_{\Theta_i} = D_{s_1} := D$ for $i = 1, \dots, N$. In this case, we do not need to introduce x_θ^{**} as in (11) when applying variational decomposition. We only need to introduce x^{**} to replace δx_{-i}. In fact, in the current case, x^{**} satisfies

$$
dx^{**} = \big[(A+F)x^{**} + F\delta x_i\big]\,dt, \quad x^{**}(0) = 0.
$$

Moreover, CC in the homogeneous case becomes

$$
\begin{cases}
d\alpha = \big[A\alpha + B\mathbf{P}_\Gamma\big[R^{-1}\mathcal{E}_t[-B^\top\gamma - D^\top\vartheta]\big] + F\mathbb{E}\alpha\big]\,dt\\
\qquad\; + \big[C\alpha + D\mathbf{P}_\Gamma\big[R^{-1}\mathcal{E}_t[-B^\top\gamma - D^\top\vartheta]\big] + \widetilde{F}\mathbb{E}\alpha\big]\,dW,\\
d\gamma = \big[-Q\alpha - \mathcal{H}(H,Q)\mathbb{E}\alpha - A^\top\gamma + F^\top\check{y}_2 + F^\top\mathbb{E}\check{y}_1 - C^\top\vartheta + \widetilde{F}^\top\mathbb{E}\check{\beta}_1\big]\,dt\\
\qquad\; + \vartheta\,dW(t),\\
d\check{y}_1 = \big[Q\alpha - A^\top\check{y}_1 - C^\top\check{\beta}_1\big]\,dt + \check{\beta}_1\,dW,\\
d\check{y}_2 = \big[\mathcal{H}(H,Q)\mathbb{E}\alpha - F^\top\mathbb{E}\check{y}_1 - \widetilde{F}^\top\mathbb{E}\check{\beta}_1 - A^\top\check{y}_2 - F^\top\check{y}_2\big]\,dt,\\
\alpha(0) = \xi, \qquad \gamma(T) = 0, \qquad \check{y}_1(T) = 0, \qquad \check{y}_2(T) = 0.
\end{cases} \tag{47}
$$

This is the special case of (26) with $\Phi(\theta)$ being a Dirac distribution. Subsequently, our framework covers the homogeneous case as its special case. Furthermore, in the case $C = D = F = \widetilde{F} = 0$, $\Gamma = \mathbb{R}^m$ and $\mathbb{G}^i = \mathbb{F}^i$, by taking expectation, $\bar{\alpha} = \mathbb{E}\alpha$ and $\bar{\gamma} = \mathbb{E}\gamma$ satisfy

$$
\begin{cases}
d\bar{\alpha} = \big[A\bar{\alpha} - BR^{-1}B^\top\bar{\gamma}\big]\,dt, & \bar{\alpha}(0) = \xi,\\
d\bar{\gamma} = \big[(-Q - \mathcal{H}(H,Q))\bar{\alpha} - A^\top\bar{\gamma}\big]\,dt, & \bar{\gamma}(T) = 0.
\end{cases} \tag{48}
$$

This is just the special case discussed in page 1742 of [27] (see (42),(43) therein). The only difference is that (48) is open-loop ($\bar{\gamma}$ is the adjoint process) while (42) and (43) in [27] are closed-loop ($\Pi\bar{x} + s$ is of feedback form).

6.1.2. *Heterogeneous case with finite diversities.* Specifically, we assume that Θ_i is deterministic (post-sampling) and assumes values in a finite discrete set $\mathcal{S} = \{1, 2, \ldots, K\}$. For $1 \le k \le K$, introduce $\mathcal{I}_k = \{i | \Theta_i = k, 1 \le i \le N\}$, $N_k = |\mathcal{I}_k|$, where N_k is the cardinality of index set $\mathcal{I}_k$ (i.e., cardinality of set of k-type agents). For $1 \le k \le K$, let $\pi_k^{(N)} = \frac{N_k}{N}$, then $\pi^{(N)} = (\pi_1^{(N)}, \ldots, \pi_K^{(N)})$ is a probability vector representing the empirical distribution of $\Theta_1, \ldots, \Theta_N$. Suppose there exists a probability mass vector $\pi = (\pi_1, \ldots, \pi_K)$ such that $\lim_{N\to+\infty} \pi^{(N)} = \pi$ and $\min_{1\le k\le K} \pi_k > 0$. Under these assumptions, the variational decomposition procedure still proceeds as in Section 3.1. Let $\delta x_{(k)} = \sum_{j\in\mathcal{I}_k, j\neq i} \delta x_{j,i}$. By exchangeability of agents within the same type, we only need to consider a representative agent in each type when using a limit to approximate $\delta x_{(k)}$. Therefore, for $k = 1, \ldots, K$, we should introduce the term x_k^{**} to replace $\delta x_{(k)}$, where x_k^{**} satisfies the following dynamics:

$$dx_k^{**} = \left[A_k x_k^{**} + F\pi_k \delta x_i + F\pi_k \sum_{l=1}^{K} x_l^{**} \right] dt, \quad x_k^{**}(0) = 0, k = 1, \ldots, K.$$

Furthermore, if $\mathbb{G}^i = \mathbb{F}^i$, CC of the heterogeneous case with finite diversities becomes:

$$\begin{cases} d\alpha_k = \left[A_k\alpha_k + B\mathbf{P}_\Gamma\big[R_k^{-1}\big(B^\top \gamma_k + D_k^\top \vartheta_k\big)\big] + F\sum_{l=1}^{K} \pi_l \mathbb{E}\alpha_l \right] dt \\ \qquad + \left[C\alpha_k + D_k\mathbf{P}_\Gamma\big[R_k^{-1}\big(B^\top \gamma_k + D_k^\top \vartheta_k\big)\big] + \widetilde{F}\sum_{l=1}^{K} \pi_l \mathbb{E}\alpha_l \right] dW_k(t), \\ d\gamma_k = \left[-Q\alpha_k - \mathcal{H}(H, Q)\sum_{l=1}^{K} \pi_l \mathbb{E}\alpha_l - A_k^\top \gamma_k + F^\top \sum_{l=1}^{K} \pi_l \check{y}_2^l + F^\top \sum_{l=1}^{K} \pi_l \mathbb{E}\check{y}_1^l \right. \\ \qquad \left. - C^\top \vartheta_k + \widetilde{F}^\top \sum_{l=1}^{K} \pi_l \mathbb{E}\check{\beta}_1^l \right] dt + \vartheta_k\, dW_k(t), \\ d\check{y}_1^k = \big[Q\alpha_k - A_k^\top \check{y}_1^k - C^\top \check{\beta}_1^k\big] dt + \check{\beta}_1^k\, dW_k, \\ d\check{y}_2^k = \left[\mathcal{H}(H, Q)\sum_{l=1}^{K} \pi_l \mathbb{E}\alpha_l - \sum_{l=1}^{K} \pi_l\big(F^\top \mathbb{E}\check{y}_1^l + \widetilde{F}^\top \mathbb{E}\check{\beta}_1^l\big) - A_k^\top \check{y}_2^k - F^\top \sum_{l=1}^{K} \pi_l \check{y}_2^l \right] dt, \\ \alpha_k(0) = \xi, \qquad \gamma_k(T) = 0, \qquad \check{y}_1^k(T) = 0, \qquad \check{y}_2^k(T) = 0, \quad k = 1, \ldots, K. \end{cases} \tag{49}$$

(49) is similar to the CC in [21] (see (2.15) therein). [21] deals with MFG with the heterogeneous case with finite diversities, hence the CC only involves the Hamiltonian system of the auxiliary control problem. While for LQG-MFT, besides the Hamilton system (25), CC also includes (19) by the weak duality procedure.

6.1.3. *Heterogeneous case with continuum diversities but without state-coupling.* When $F = \widetilde{F} = 0$, that is, there is no weakly coupling in state, by (7) we have $\delta x_{j,i} \equiv 0$, $j \neq i$, thus x_j^*, x_θ^{**} both vanish in (11). The resulting (12) takes a rather simpler form than (9),

$$\delta\mathcal{J}_{\text{soc}}^{(N)} = \mathbb{E}\int_0^T \big[\langle Q\bar{x}_i, \delta x_i\rangle + \langle \mathcal{H}(H, Q)\hat{x}, \delta x_i\rangle + \langle R\bar{u}_i, \delta u_i\rangle\big] dt + \varepsilon_1, \tag{50}$$

where

$$\varepsilon_1 = -E\int_0^T \big\langle \mathcal{H}(H, Q)\big(\hat{x} - \bar{x}^{(N)}\big), N\delta x^{(N)}\big\rangle dt.$$

From (50) we can obtain the auxiliary control problem directly, that is, it becomes unnecessary to introduce the limit terms (11) and adjoint processes (13). This is similar to the case in Section IV.A of [27]. Note that in [27], there is no pointwise constraint or partial information constraint on the admissible control, hence the main focus is to find the optimal closed-loop control for the auxiliary control problem (see (32) therein). While with the above two constraints, we will obtain the optimal open-loop control for the auxiliary control problem (see (23)). In this case, (26) reduces to

$$\begin{cases} d\alpha = [A_\Theta \alpha + B\mathbf{P}_\Gamma[R^{-1}\mathcal{E}_t[-B^\top \gamma - D_\Theta^\top \vartheta]]]\,dt \\ \qquad + [C\alpha + D_\Theta \mathbf{P}_\Gamma[R^{-1}\mathcal{E}_t[-B^\top \gamma - D_\Theta^\top \vartheta]]]\,dW, & \alpha(0) = \xi, \\ d\gamma = [-Q\alpha - \mathcal{H}(H, Q)\mathbb{E}\alpha - A_\Theta^\top \gamma - C^\top \vartheta]\,dt + \vartheta\,dW(t), & \gamma(T) = 0, \end{cases} \tag{51}$$

for which the well-posedness is much easier to establish. Furthermore, if $C = D_\Theta = 0$, $\Gamma = \mathbb{R}^m$ and $\mathbb{G}^i = \mathbb{F}^i$, by taking expectation to (51), the derived FBSDEs reduces to the case on page 1740 of [27].

By contrast, when $F, \widetilde{F} \neq 0$, the variation functional $\delta\mathcal{J}_{\text{soc}}^{(N)}(\delta u_i)$ of (12) becomes rather involved depending both on x_j^* and x^{**}. Those two terms are some *intermediate variation limits* related to basic variation term δx_i in an indirect manner. Thus, the current representation (12) cannot lead a direct construction to an auxiliary control. Some duality method are required to remove dependence on these intermediate variations.

6.1.4. *Other cases*. For the homogeneous case, [20] studies linear-quadratic mean-field games with control process constrained in a closed convex subset of full space $\mathbb{R}^m$; [24] studies backward mean-filed linear-quadratic games with partial information. When there involves only constraints on the control or only partial information, our framework is the extension of [20] and [24] for mean-field team case.

6.2. *Homogeneity and heterogeneity*: *A unified quasi-exchangeable approach*. Recall that the mean-field theory has been extensively applied to study the large-scale weakly coupled system along both (competitive) game and (cooperative) team directions see, for example, [5, 11, 12, 21, 25, 26, 31] for recent relevant studies for game; and [38, 40] for team. Essentially, such mean-field analysis is build on some *exchangeability* among all individual weakly coupled agents. It can be proved that any exchangeable sequences should be conditional independent with respect to some tail-sigma algebra. Thus, applying the de Finetti theorem, the original complex weakly coupling structure can be replaced by a deterministic- or common-noise-driven process as agent number N tends to infinity. By this, all agents thus become asymptotically decoupled along with chaos propagation. Subsequently, original game or team can be reduced to low-dimensional single agent optimization problems with some off-line quantities via a consistency condition that matches the above exchangeable reasoning. In this sense, mean-field analysis connects closely to exchangeable game/team in random context, and further to *symmetric* game/team [15] in deterministic context. We remark that all agents in the symmetric game are endowed with same underlying parameters and so become identical in analysis. So, the primal high-dimensional computation can be greatly reduced using "mirror" argument among all symmetric agents.

Regarding large-scale system, there exist three progressive levels of diversity relevant to the aforementioned exchangeability: homogeneous, heterogenous with finite/discrete diversity, and heterogenous with continuum diversity. Among them, the homogenous case is the most special but tractable one because all agents are statistically identical and the designed optimal team strategies should also be exchangeable. Consequently, the resulting optimized states are thus exchangeable. We refer to [38] for recent studies in such cases for team, and [20] for game.

Compared with the homogenous case, the heterogenous case with finite/discrete diversity is more realistic. Virtually, most systems in reality demonstrate some diversities in their random behaviors. In this case, all agents, from the whole system scale, are no longer identical because they are endowed with diversified parameters. However, all agents inside a subsystem with the same diversity index, are still exchangeable in small scale. Thus, we can treat the large-scale system as some mixed combination of *finite* exchangeable subsystems. The previous mean-field analysis to the homogenous can be suitably modified to tackle such cases, with some technical but straightforward arguments. We refer to [1, 2] for recent studies in such cases for team in discrete time setup, and [21, 25] for game, where a similar *partial exchangeability* is introduced.

The heterogenous case with continuum diversity, as discussed in [27, 33], should be the most realistic setup for a practical large-scale system. Indeed, it is less possible that the diversity of a real system can only be limited on a finite or discrete support set. Instead, considerable statistical diversity demonstrates its support on a continuum set such as the compact closed interval. On the other hand, such heterogenous cases should be most difficult to handle. One reason for the continuum heterogeneity to be analytically intractable, is that the subclass exchangeability featured in the finite heterogeneity case will shrink to zero mass along with the continuum diversity support. For this reason, the relevant results for continuum heterogeneity seem few compared with homogeneous- or finite-heterogenous-case.

We remark that [33] discussed mean-field analysis with continuum diversity in the game setup, and [27] in the team setup, using a direct state-aggregating method. However, the settings in both works are relatively simple, in particular, its weakly coupled dynamics is only drift-controlled. This corresponds to our model with $C = D = \widetilde{F} = 0$, and cannot cover various applications such as portfolio selection with relative performance. Our setup is more general (diffusion-controlled and -coupled) and the above aggregation method no longer works. Meanwhile, due to continuum diversity, we cannot apply the weak embedding representation method used in [20, 21, 38] when tackling diffusion controlled systems but of finite diversities only. Indeed, the analysis of [21] relies on a construction of K independent copies of optimized states with individual BMs, where K is the finite cardinality of diversity. This becomes impossible for the current case in the presence of continuum diversities.

As a resolution, this paper proposes some unified approaches to homogenous-, and heterogenous-cases using a quasi-exchangeable method. The main idea is as follows: first, note that

$$\begin{cases} dx_i = \big[A_{\Theta_i} x_i + Bu_i + F x^{(N)}\big]dt + \big[Cx_i + D_{\Theta_i} u_i + \widetilde{F} x^{(N)}\big]dW_i, \\ x_i(0) = \xi \in \mathbb{R}^n, \quad 1 \le i \le N, \end{cases}$$

can be reformulated as follows:

$$\begin{cases} dx_i = \big[A\big(z_i(t), t\big)x_i + Bu_i + F x^{(N)}\big]dt + \big[Cx_i + D\big(z_i(t), t\big)u_i + \widetilde{F} x^{(N)}\big]dW_i, \\ dz_i(t) \equiv 0, \\ x_i(0) = \xi \in \mathbb{R}^n; \qquad z_i(0) = \Theta, \quad 1 \le i \le N, \end{cases}$$

that can be further written with some augmented state as

$$d\mathbf{x}_i = \big[\mathbf{A}(\mathbf{x}_i)\mathbf{x}_i + Bu_i + \mathbf{F}\mathbf{x}^{(N)}\big]dt + \big[\mathbf{C}\mathbf{x}_i + \mathbf{D}(\mathbf{x}_i)u_i + \widetilde{\mathbf{F}}\mathbf{x}^{(N)}\big]dW_i, \quad \mathbf{x}_i(0) = \big(\xi_i^\top, \Theta^\top\big)^\top.$$

In other words, initial weakly coupled systems with continuum diversity can be viewed as some quasi-linear SDE with augmented state $\mathbf{x}_i = (x_i^\top, z_i^\top)^\top$ and random initial conditions $\mathbf{x}_i(0)$ (noting $\Theta \in \mathcal{F}_0$, although ξ might be deterministic).

To proceed, we introduce the following three systems. To ease notation, we are inclined to adopt symbols like $A(\mathbf{x})$ instead of $\mathbf{A}(\mathbf{x})$ when no confusion occurs. The first system is a McKean–Vlasov SDE with random initials:

$$\mathcal{P}_1: \quad d\mathbf{x} = \big[A(\mathbf{x})\mathbf{x} + Bu + F\mathbb{E}\mathbf{x}\big]dt + \big[C\mathbf{x} + D(\mathbf{x})u + \widetilde{F}\mathbb{E}\mathbf{x}\big]dW, \quad \mathbf{x}(0) = \big(\xi^\top, \Theta^\top\big)^\top.$$

For the sake of illustration, we set $\Theta \in \Lambda = \{\theta_1, \theta_2, \dots, \theta_K\}$ with the mass $m_1, \dots, m_K$ to admit finite K diversity classes. Later, we will illustrate its possible extension to infinite continuum diversities. The second system is a stochastic mixture: $\widetilde{\mathbf{x}} = \sum_{j=1}^K m_j \widetilde{\mathbf{x}}_j$ but driven by identical noise W:

$$\mathcal{P}_2: \quad d\widetilde{\mathbf{x}}_j = [A_{\theta_j}\widetilde{\mathbf{x}}_j + Bu + F\mathbb{E}\widetilde{\mathbf{x}}]\,dt + [C\widetilde{\mathbf{x}}_j + D_{\theta_j}u + \widetilde{F}\mathbb{E}\widetilde{\mathbf{x}}]\,dW, \quad \widetilde{\mathbf{x}}_j(0) = (\xi^\top, \theta_j^\top)^\top.$$

By contrast, the third system is also a stochastic mixture $\widehat{\mathbf{x}} = \sum_{j=1}^K m_j \widehat{\mathbf{x}}_j$ but driven by K i.i.d noises $\{W_j\}_{j=1}^K$:

$$\mathcal{P}_3: \quad d\widehat{\mathbf{x}}_j = [A_{\theta_j}\widehat{\mathbf{x}}_j + Bu + F\mathbb{E}\widehat{\mathbf{x}}]\,dt + [C\widehat{\mathbf{x}}_j + D_{\theta_j}u + \widehat{F}\mathbb{E}\widehat{\mathbf{x}}]\,dW_j, \quad \widehat{\mathbf{x}}_j(0) = (\xi^\top, \theta_j^\top)^\top.$$

It is obvious that above three systems: $\mathbf{x}$, $\widetilde{\mathbf{x}}$ and $\widehat{\mathbf{x}}$ are not of the same distributions. Actually, $\mathbf{x}$ has different initial distribution at $t = 0$ with $\widetilde{\mathbf{x}}$, $\widehat{\mathbf{x}}$, whereas $\widehat{\mathbf{x}}$ is driven by different noise with $\mathbf{x}$, $\widetilde{\mathbf{x}}$. However, they have same expectation dynamics, as verified using the *tower property* of conditional expectation, $\forall t \in [0, T] : \mathbb{E}\mathbf{x}(t) = \mathbb{E}(\mathbb{E}(\mathbf{x}(t)|\Theta)) = \sum_{j=1}^K m_j \mathbb{E}\widetilde{\mathbf{x}}_j(t) = \mathbb{E}\widetilde{\mathbf{x}}(t) = \sum_{j=1}^K m_j \mathbb{E}\widehat{\mathbf{x}}_j(t) = \mathbb{E}\widehat{\mathbf{x}}(t)$. Besides, all three systems have different second-moment function, and other finite-dimensional distributions. For example,

$$\mathbb{E}|\mathbf{x}(t)|^2 = \mathbb{E}(\mathbb{E}(|\mathbf{x}(t)|^2|\Theta)) = \sum_{j=1}^K m_j \mathbb{E}|\widetilde{\mathbf{x}}_j(t)|^2,$$

$$\mathbb{E}|\widetilde{\mathbf{x}}(t)|^2 = \sum_{j=1}^K m_j^2 \mathbb{E}|\widetilde{\mathbf{x}}_j(t)|^2 + \sum_{1\le j<l\le K} m_j m_l \mathbb{E}[\widetilde{\mathbf{x}}_j(t)\widetilde{\mathbf{x}}_l(t)],$$

$$\mathbb{E}|\widehat{\mathbf{x}}(t)|^2 = \sum_{j=1}^K m_j^2 \mathbb{E}|\widehat{\mathbf{x}}_j(t)|^2 + \sum_{1\le j<l\le K} m_j m_l \mathbb{E}[\widehat{\mathbf{x}}_j(t)\widehat{\mathbf{x}}_l(t)] = \sum_{j=1}^K m_j^2 \mathbb{E}|\widetilde{\mathbf{x}}_j(t)|^2.$$

Noticing the above expectation equivalence is a special degenerated version of the Jensen inequality, thanks to the underlying LQG context. Such a property cannot be extended to nonlinear moments hence $\mathbf{x}$, $\widetilde{\mathbf{x}}$ and $\widehat{\mathbf{x}}$ are with the same expectation but different distributions.

Corresponding to $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, we may construct three weakly coupled systems $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$:

$$\mathcal{M}_1: \quad d\mathbf{x}_i = [A(\mathbf{x}_i)\mathbf{x}_i + Bu_i + F\mathbf{x}^{(N)}]\,dt + [C\mathbf{x}_i + D(\mathbf{x}_i)u_i + \widetilde{F}\mathbf{x}^{(N)}]\,dW_i,$$
$$\mathbf{x}_i(0) = (\xi^\top, \Theta)^\top,$$

where $\mathbf{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \mathbf{x}_i$. Another is system $\mathcal{M}_2 : \{\widetilde{\mathbf{x}}_i\}_{i=1}^N$ with $\widetilde{\mathbf{x}}_i = \sum_{j=1}^K m_j \widetilde{\mathbf{x}}_{i,j}$,

$$\mathcal{M}_2: \quad d\widetilde{\mathbf{x}}_{i,j} = [A_{\theta_j}\widetilde{\mathbf{x}}_{i,j} + Bu_i + F\widetilde{\mathbf{x}}^{(N)}]\,dt + [C\widetilde{\mathbf{x}}_{i,j} + D_{\theta_j}u_i + \widetilde{F}\widetilde{\mathbf{x}}^{(N)}]\,dW_i,$$
$$\widetilde{\mathbf{x}}_{i,j}(0) = (\xi^\top, \theta_j^\top)^\top,$$

where $\widetilde{\mathbf{x}}^{(N)} = \frac{1}{N}\sum_{i=1}^N \widetilde{\mathbf{x}}_i$. For $1 \le j \le K$, we can introduce $\widehat{\mathcal{M}}_2^j : \{\widetilde{\mathbf{x}}_{i,j}\}_{i=1}^N$ that is a homogeneous weakly coupled system indexed by θ_j. Abusing notation, we may write informally that $\mathcal{M}_2 = \sum_{j=1}^K m_j \widehat{\mathcal{M}}_2^j$, in other words, $\mathcal{M}_2$ is a finite mixture of homogeneous systems $\{\widehat{\mathcal{M}}_2^j\}_{j=1}^K$. Noticing for $\widehat{\mathcal{M}}_2^j$, the driving BMs become $\{W_i\}_{i=1}^N$ which are the same as that of $\widehat{\mathcal{M}}_2^{j'}$ for $j \ne j'$. Thus, totally there involve N independent BMs for $\mathcal{M}_2$. Moreover, if we introduce a sampling sequence from $\{1, \dots, K\}$ with $\mathcal{I}_j = \{\theta_i = j, 1 \le i \le N\}$ and $\lim_{N\to+\infty} \frac{\mathrm{Card}\mathcal{I}_j}{N} = m_j$, $1 \le j \le K$. Then, $\mathcal{M}_2$ is equivalent in a weak sense to the stochastic K-heterogenous weakly coupled system introduced in [21, 25].

The third system is $\mathcal{M}_3 : \{\widehat{\mathbf{x}}_i\}_{i=1}^N$ with $\widehat{\mathbf{x}}_i = \sum_{j=1}^K m_j \widehat{\mathbf{x}}_{i,j}$,

$$\mathcal{M}_3 : \quad d\widehat{\mathbf{x}}_{i,j} = \big[A_{\theta_j}\widehat{\mathbf{x}}_{i,j} + Bu_i + F\widehat{\mathbf{x}}^{(N)}\big]dt + \big[C\widehat{\mathbf{x}}_{i,j} + D_{\theta_j}u_i + \widehat{F}\widehat{\mathbf{x}}^{(N)}\big]dW_{i,j},$$
$$\widehat{\mathbf{x}}_{i,j}(0) = \big(\xi^\top, \theta_j^\top\big)^\top,$$

where $\widehat{\mathbf{x}}^{(N)} = \frac{1}{N}\sum_{i=1}^N \widehat{\mathbf{x}}_i$. For $1 \le j \le K$, we can introduce $\widehat{\mathcal{M}}_3^j : \{\widehat{\mathbf{x}}_{i,j}\}_{i=1}^N$ that is, a homogeneous weakly coupled system indexed by θ_j. Noticing for $\widehat{\mathcal{M}}_3^j$, the driving BMs become $\{W_{i,j}\}_{i=1}^N$. So, totally there arise $N \times K$ independent BMs for $\mathcal{M}_3$, or re-scale to N BMs for each subsystem $\widehat{\mathcal{M}}_3^j$, $1 \le j \le K$. This is not problematic when K is finite. Again, $\mathcal{M}_3$ is a finite mixture of homogeneous system $\{\widehat{\mathcal{M}}_3^j\}_{j=1}^K$. We remark that $\widehat{\mathcal{M}}_3^j$ and $\widehat{\mathcal{M}}_2^j$ are driven by different BMs, but they are equivalently weak-coupled homogenous systems in weak sense. This is because they share the same state-average limit by law of large numbers, although they are driven by different BMs systems.

Moreover, we can introduce an augmented state $\mathbf{y}_i = (\widehat{\mathbf{x}}_{i,1}^\top, \dots, \widehat{\mathbf{x}}_{i,K}^\top)^\top$ and $\widehat{\mathbf{x}}^{(N)} = \frac{1}{N}\sum_{i=1}^N \widehat{\mathbf{x}}_i$, it follows that

$$d\mathbf{y}_i = \big[\widehat{A}\mathbf{y}_i + \widehat{B}\widehat{u}_i + \mathbf{F}\mathbf{y}^{(N)}\big]dt + \sum_{j=1}^K \big[\widehat{C}_j\mathbf{y}_i + \widehat{D}_j u_i + \widehat{\mathbf{F}}_j\mathbf{y}^{(N)}\big]dW_{i,j},$$
$$\mathbf{y}_i(0) = \big(\xi^\top, \theta_1^\top \cdots, \xi^\top, \theta_K^\top\big)^\top,$$

where

$$\widehat{A} = \begin{pmatrix} A_{\theta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{\theta_K} \end{pmatrix}_{(nK\times nK)}, \qquad \widehat{B} = \begin{pmatrix} B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B \end{pmatrix}_{(nK\times mK)},$$

$$\widehat{u}_i = \begin{pmatrix} u_i \\ \vdots \\ u_i \end{pmatrix}_{(mK\times 1)}, \qquad \mathbf{F} = \begin{pmatrix} Fm_1 & \cdots & Fm_K \\ \vdots & \vdots & \vdots \\ Fm_1 & \cdots & Fm_K \end{pmatrix}_{(nK\times nK)},$$

$$\widehat{C}_j = \begin{array}{c} 1 \\ \vdots \\ j \\ \vdots \\ K \end{array} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & C & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}_{(nK\times nK)},$$

$$\widehat{D}_j = \begin{array}{c} 1 \\ \vdots \\ j \\ \vdots \\ K \end{array} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & D_{\theta_j} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}_{(nK\times mK)}, \qquad \widehat{\mathbf{F}}_j = \begin{array}{c} 1 \\ \vdots \\ j \\ \vdots \\ K \end{array} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ \widehat{F}m_1 & \cdots & \widehat{F}m_K \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{(nK\times nK)}.$$

It follows that $\mathcal{M}_3 : \{\widehat{\mathbf{x}}_i\}_{i=1}^N$ satisfying $\widehat{\mathbf{x}}_i = \mathbf{m}\cdot\mathbf{y}_i$ with $\mathbf{m} = (m_1, \dots, m_K)$. Noticing that $\{\mathbf{y}_i\}_{i=1}^N$ is homogenous for $1 \le i \le N$ and so is the case for $\{\widehat{\mathbf{x}}_i\}_{i=1}^N$, thus $\mathcal{M}_3$ can be viewed as a homogenous system but with augmented state $\mathbf{y}_i$. Hence, $\mathcal{M}_3$ can be formulated either as a finite mixture of K-homogeneous system $\{\widehat{\mathcal{M}}_3^j\}_{j=1}^K$, or a single homogenous system but with augmented (mixed) state $\mathbf{y}_i$. Note that the later formulation on augmented $\mathbf{y}_i$ actually connects to the so-called direct method ([40]). In fact, by formulation on $\mathbf{y}_i$, we can apply the direct method proposed in ([40]) for the homogenous system only but now on a more intractable (finite) heterogenous system. As the trade-off, the associated Riccati or Hamiltonian system become augmented accordingly with coupled block structure due to K diversity.

The above three weakly coupled systems $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ have different distributions but always with the same asymptotic empirical state-average as $N \to +\infty$. In fact, they are generated from same underlying weakly coupled stochastic systems but differ in filtration on given timing point. To be precise, all agents in $\mathcal{M}_1$ are exchangeable in the quasi-sense (at filtration point $\mathcal{F}_0$) before the diversity sampling. In this case, $\mathbf{x}_i(t) = \mathbb{E}(\mathbf{x}_i(t)|\mathcal{F}_t) = \mathbb{E}(\mathbf{x}_i(t)|\Theta, W_i(s), 0 \le s \le t, 1 \le i \le N)$. On the other hand, $\mathcal{M}_2$ is the same system but conditional on the pre-sampled diversity index Θ_i. In this case, $\widetilde{\mathbf{x}}_i(t) = \mathbb{E}(\mathbb{E}(\mathbf{x}_i(t)|\Theta)|W_i(s), 0 \le s \le t, 1 \le i \le N)$. Last, $\mathcal{M}_3$ is the same weak-coupled system but after the sampling of diversity Θ and $\widehat{\mathcal{M}}_3^j$ is just the re-labeled system with the realization $\Theta = \theta_j$. In this sense, all three systems $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ characterize the same underlying dynamics but from a different temporal section. Thus, they are equivalent for mean-field analysis because they share the same state-average limit (in formulation, and Step 1 for decomposition) and expectation operator (in Step 3 for CC).

To recap, we present the following diagram where "$\Longleftrightarrow$" represents the equivalent expectation operator in the first line, while the asymptotic state-average operator is in the second line:

$$
\begin{cases}
\text{single-agent:} & \mathcal{P}_1 \Longleftrightarrow \mathcal{P}_2 \Longleftrightarrow \mathcal{P}_3, \\
\text{weakly coupled agents:} & \mathcal{M}_1 \Longleftrightarrow \mathcal{M}_2 \Longleftrightarrow \mathcal{M}_3 \Longleftrightarrow \mathcal{M} \\
 & \text{(stochastic } K\text{-heterogenous system)}, \\
\mathcal{M}_1: & \text{homogenous but with random diversity index } \Theta, \\
 & \text{augmented randomness, pre-sampling} \\
\mathcal{M}_2: & \text{mixture of } K \text{ homogenous system, pre-sampling} \\
\mathcal{M}_3: & \text{homogenous system with (augmented) mixture} \\
 & \text{of states, post-sampling} \\
\mathcal{M}: & K \text{ heterogenous system defined by relative frequency} \\
 & \text{of diversity sequence, post-sampling.}
\end{cases}
\tag{52}
$$

The above arguments in (52) are on the basis that Θ is finite-valued only. Now we present its generalization to the case when Θ has continuum diversity support. In this case, we have

$$
\begin{cases}
\mathcal{M}_1^c: & \text{homogenous but with random diversity index } \Theta, \text{ augmented randomness,} \\
 & \text{pre-sampling} \\
\mathcal{M}_2^c: & \text{mixture of continuum homogenous system, pre-sampling} \\
\mathcal{M}_3^c: & \text{homogenous system with (augmented) mixture of states, post-sampling} \\
\mathcal{M}^c: & \text{continuum heterogenous system defined by empirical distribution of diversity} \\
 & \text{sequence, post-sampling.}
\end{cases}
$$

$\mathcal{M}_1^c$ is still well defined and we have already proceeded with the analysis as in Section 3. On the other hand, $\mathcal{M}_3^c$ is no longer well defined since now we have to introduce continuum-valued BMs for $\widehat{\mathcal{M}}_3^{\theta,c}$ to model the diversity. By contrast, $\mathcal{M}_2^c$ is still well defined since we still need only to formulate countable BMs for each $\widehat{\mathcal{M}}_2^{\theta,c}, \theta \in \mathcal{S}$, and in total, only countable BMs are still invoked. In this case, we may further set $\widetilde{\mathbf{x}}_i = \int_{\mathcal{S}} \widetilde{\mathbf{x}}_{i,\theta} \, d\Phi(\theta)$ and proceed with the classical mean-field analysis as in [27]. However, classical mean-field analysis only works on $\mathcal{M}_2^c$ with $C = D = F = \widetilde{F} = 0$. In the general case with $F, \widetilde{F} \neq 0$, such classical analysis fails because its CC system should invoke an embedding representation (see, e.g., [23]), and a continuum-valued BMs system will be required to replicate the distribution for a generic agent who is still continuum-heterogenous (diversified). Moreover, in [26], the continuum heterogeneity is defined through some limiting empirical distribution by the Glivenko–Cantelli lemma. Note that the continuum set therein is required to be compact when using Glivenko–Cantelli arguments, while in our framework of $\mathcal{M}_1^c$, such compactness is not required. Consequently, this paper can deal with general continuum diversity based on $\mathcal{M}_1^c$, as summarized as follows.

First, we can verify that $\mathcal{M}_1^c$, $\mathcal{M}_2^c$ as well as $\mathcal{M}^c$ (note that $\mathcal{M}_3^c$ becomes infeasible to be defined) are still of the same asymptotic state-average limit. In this sense, the generic agents in $\mathcal{M}^c$ are quasi-exchangeable because although they are not exchangeable after diversity sapling, $\mathcal{M}^c$ shares the same expectation and asymptotic state-average limit with $\mathcal{M}_1^c$, $\mathcal{M}_2^c$, and all agents of $\mathcal{M}_1^c$ are exchangeable before the sampling. Second, given such quasi-exchangeable property, the original $\mathcal{M}^c$ or $\mathcal{M}_2^c$ system with continuum heterogeneity can be converted to $\mathcal{M}_1^c$ that is, a homogenous one but with augmented randomness ($\{\Theta_i, W_i\}_{i=1}^N$) as a trade-off. Third, as discussed in Section 3, some new type of variation-decomposition and auxiliary control problem can thus be constructed, and CC can be represented via the construction on continuum diversity support as in Proposition 4.1.

APPENDIX

First, for any given $(Y, Z) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$ and $0 \le t \le T$, the following SDE has a unique solution:

$$
\begin{aligned}
X(t) = x &+ \int_0^t b\big(s, X, \mathbb{E}[X], \mathcal{E}_t[Y], \mathcal{E}_t[Z]\big)\, ds \\
&+ \int_0^t \sigma\big(s, X, \mathbb{E}[X], \mathcal{E}_t[Y], \mathcal{E}_t[Z]\big)\, dW(s).
\end{aligned} \tag{53}
$$

Therefore, we can introduce a map $\mathcal{M}_1 : L^2_{\mathbb{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^m) \to L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$. Moreover, by the standard estimations of SDE, we have the following result:

LEMMA A.1. *Let X_i be the solution of (53) corresponding to (Y_i, Z_i), $i = 1, 2$ respectively. Then for all $\rho \in \mathbb{R}$ and some constants $l_1 > 0$, we have*

$$
\begin{aligned}
&\mathbb{E}e^{-\rho t}\big|\hat{X}(t)\big|^2 + \bar{\rho}_1 \mathbb{E}\int_0^t e^{-\rho s}\big|\hat{X}(s)\big|^2\, ds \\
&\quad \le \big(k_2 l_1 + k_2^2\big)\mathbb{E}\int_0^t e^{-\rho s}\big|\hat{Y}(s)\big|^2\, ds + \big(k_2 l_1 + k_2^2\big)\mathbb{E}\int_0^t e^{-\rho s}\big|\hat{Z}(s)\big|^2\, ds, \\
&\mathbb{E}e^{-\rho t}\big|\hat{X}(t)\big|^2 \le \big(k_2 l_1 + k_2^2\big)\mathbb{E}\int_0^t e^{-\bar{\rho}_1(t-s)-\rho s}\big[\big|\hat{Y}(s)\big|^2 + \big|\hat{Z}(s)\big|^2\big]\, ds,
\end{aligned}
$$

where $\bar{\rho}_1 = \rho - 2\rho_1 - 2k_1 - 2k_2 l_1^{-1} - k_7^2 - k_8^2$ and $\hat{\Phi} := \Phi_1 - \Phi_2$, $\Phi = X, Y, Z$. Moreover,

$$
\begin{aligned}
&\mathbb{E}\int_0^T e^{-\rho t}\big|\hat{X}(t)\big|^2\, dt \le \big(k_2 l_1 + k_2^2\big)\frac{1 - e^{-\bar{\rho}_1 T}}{\bar{\rho}_1}\mathbb{E}\int_0^T e^{-\rho s}\big[\big|\hat{Y}(s)\big|^2 + \big|\hat{Z}(s)\big|^2\big]\, ds, \\
&\qquad e^{-\rho T}\mathbb{E}\big|\hat{X}(T)\big|^2 \le \big(1 \vee e^{-\bar{\rho}_1 T}\big)\bigg\{\big(k_2 l_1 + k_2^2\big)\mathbb{E}\int_0^T e^{-\rho t}\big[\big|\hat{Y}(t)\big|^2 + \big|\hat{Z}(t)\big|^2\big]\, dt\bigg\}.
\end{aligned}
$$

Specially, if $\bar{\rho}_1 > 0$,

$$
e^{-\rho T}\mathbb{E}\big|\hat{X}(T)\big|^2 \le \big(k_2 l_1 + k_2^2\big)\mathbb{E}\int_0^T e^{-\rho t}\big|\hat{Y}(t)\big|^2\, dt + \big(k_2 l_1 + k_2^2\big)\mathbb{E}\int_0^T e^{-\rho t}\big|\hat{Z}(t)\big|^2\, dt.
$$

Next, for any given $X \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, consider the following BSDE:

$$Y(t) = \int_t^T f\big(s, X, \mathbb{E}[X], Y, \mathbb{E}[Y], \widetilde{\mathbb{E}}[Y], Z, \mathbb{E}[Z]\big)\,ds - \int_t^T Z(s)\,dW(s). \tag{54}$$

PROPOSITION A.2. (54) *admits a unique solution* $(Y,Z) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$.

PROOF. For any fixed $(y,z) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$,

$$Y(t) = \int_t^T f\big(s, X, \mathbb{E}[X], Y, \mathbb{E}[y], \widetilde{\mathbb{E}}[y], z, \mathbb{E}[z]\big)\,ds - \int_t^T Z(s)\,dW(s)$$

admits a unique solution $(Y,Z) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$. Hence we can introduce the mapping $\mathcal{N} : (y,z) \to (Y,Z)$. For any $(y,z), (y',z') \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, denote $(Y,Z) = \mathcal{N}(y,z)$ and $(Y',Z') = \mathcal{N}(y',z')$. Let $(\hat{y},\hat{z},\hat{Y},\hat{Z}) = (y-y', z-z', Y-Y', Z-Z')$. Applying Itô's formula to $e^{\delta x}|\hat{Y}(s)|^2$, we have

$$\begin{aligned}
&e^{\delta t}|\hat{Y}(t)|^2 + \int_t^T e^{\delta s}|\hat{Z}(s)|\,ds + \int_t^T \delta e^{\delta s}|\hat{Y}(s)|\,ds \\
&\quad\le \int_t^T e^{\delta s}(2\rho_2 + 4k_3^2 + 4k_4^2 + 4k_5^2 + 4k_6^2)|\hat{Y}(s)|^2\,ds \\
&\qquad + \frac{1}{4}\int_t^T e^{\delta s}\big(\mathbb{E}[|\hat{y}|^2] + \widetilde{\mathbb{E}}[|\hat{y}^2|] + |\hat{z}|^2 + \mathbb{E}[|\hat{z}|^2]\big)\,ds + 2\int_t^T e^{\delta s}\langle \hat{Y}(s), \hat{Z}(s)\,dW(s)\rangle.
\end{aligned}$$

Note that $\mathbb{E}[\widetilde{\mathbb{E}}[|\hat{y}^2|]] = \mathbb{E}[|\hat{y}^2|]$, letting $\delta = 2\rho_2 + 4k_3^2 + 4k_4^2 + 4k_5^2 + 4k_6^2$ and taking expectation, we have $\mathbb{E}\int_t^T e^{\delta s}(|\hat{Y}(s)|^2 + |\hat{Z}(s)|^2)\,ds \le \frac{1}{2}\mathbb{E}\int_t^T e^{\delta s}(|\hat{y}(s)|^2 + |\hat{z}(s)|^2)\,ds$, that is, $\mathcal{N}$ is a contraction mapping. Hence (54) admits a unique solution $(Y,Z) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$. $\square$

Thus, we can introduce another map $\mathcal{M}_2 : L^2_{\mathbb{F}}(0,T;\mathbb{R}^n) \to L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$. By the standard estimation of BSDE, we have the following result:

LEMMA A.3. *Let* (Y_i, Z_i) *be the solution of* (54) *corresponding to* X_i, $i = 1,2$, *respectively. Then for all* $\rho \in \mathbb{R}$ *and some constants* $l_1, l_2, l_3 > 0$, *we have*

$$\begin{aligned}
&\mathbb{E}e^{-\rho t}|\hat{Y}(t)|^2 + \bar{\rho}_2\mathbb{E}\int_t^T e^{-\rho s}|\hat{Y}(s)|^2\,ds + (1 - k_5 l_2 - k_6 l_3)\mathbb{E}\int_t^T e^{-\rho s}|\hat{Z}(s)|^2\,ds \\
&\quad\le 2k_2 l_1 \mathbb{E}\int_t^T e^{-\rho s}|\hat{X}(s)|^2\,ds, \\
&\mathbb{E}e^{-\rho t}|\hat{Y}(t)|^2 + (1 - k_2 l_1 - k_3 l_1)\mathbb{E}\int_t^T e^{-\rho s}|\hat{Z}(s)|^2\,ds \\
&\quad\le k_2 l_1 \mathbb{E}\int_t^T e^{-\bar{\rho}_2(s-t)-\rho s}|\hat{X}(s)|^2\,ds,
\end{aligned}$$

where $\bar{\rho}_2 = -\rho - 2\rho_2 - 2k_3 - 2k_4 - 2k_2 l_1^{-1} - k_5 l_2^{-1} - k_6 l_3^{-1}$, *and* $\hat{\Phi} := \Phi_1 - \Phi_2$, $\Phi = X, Y, Z$. *Moreover,*

$$\mathbb{E}\int_0^T e^{-\rho t}|\hat{Y}(t)|^2\,dt \le \frac{1 - e^{-\bar{\rho}_2 T}}{\bar{\rho}_2} 2k_2 l_1 \mathbb{E}\int_0^T e^{-\rho s}|\hat{X}(s)|^2\,ds,$$

$$\mathbb{E}\int_0^T e^{-\rho t}|\hat{Z}(t)|^2\,dt \le \frac{2k_2 l_1(1 \vee e^{-\bar{\rho}_2 T})}{(1 - k_5 l_2 - k_6 l_3)(1 \wedge e^{-\bar{\rho}_2 T})}\mathbb{E}\int_0^T e^{-\rho s}|\hat{X}(s)|^2\,ds.$$

Specially, if $\bar{\rho}_2 > 0$,

$$\mathbb{E}\int_0^T e^{-\rho t}\big|\hat{Z}(t)\big|^2\,dt \le \frac{2k_2l_1}{1-k_5l_2-k_6l_3}\mathbb{E}\int_0^T e^{-\rho s}\big|\hat{X}(s)\big|^2\,ds.$$

PROOF OF THEOREM 4.3. Define $\mathcal{M} := \mathcal{M}_2 \circ \mathcal{M}_1$, where $\mathcal{M}_1$ is defined by (53) and $\mathcal{M}_2$ is defined by (54). Thus $\mathcal{M}$ is a mapping from $L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ into itself. For $(U_i, V_i) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, let $X_i := \mathcal{M}_1(U_i, V_i)$ and $(Y_i, Z_i) := \mathcal{M}(U_i, V_i)$. Therefore,

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{-\rho t}\big|Y_1(t)-Y_2(t)\big|^2\,dt + \mathbb{E}\int_0^T e^{-\rho t}\big|Z_1(t)-Z_2(t)\big|^2\,dt\\
&\quad\le\left[\frac{1-e^{-\bar{\rho}_2 T}}{\bar{\rho}_2} + \frac{1\vee e^{-\bar{\rho}_2 T}}{(1-k_5l_2-k_6l_3)(1\wedge e^{-\bar{\rho}_2 T})}\right]2k_2l_1\frac{1-e^{-\bar{\rho}_1 T}}{\bar{\rho}_1}\\
&\qquad\times\bigg\{(k_2l_1+k_2^2)\mathbb{E}\int_0^T e^{-\rho t}\big|U_1(t)-U_2(t)\big|^2\,dt\\
&\qquad+(k_2l_1+k_2^2)\mathbb{E}\int_0^T e^{-\rho t}\big|V_1(t)-V_2(t)\big|^2\,dt\bigg\}.
\end{aligned}$$

Choosing suitable ρ, we get that $\mathcal{M}$ is a contraction mapping.

Furthermore, if $2\rho_1 + 2\rho_2 < -2k_1 - 2k_3 - 2k_4 - k_5^2 - k_6^2 - k_7^2 - k_8^2$, we can choose $\rho \in \mathbb{R}$, $0 < k_5l_2 < \frac{1}{2}$ and $0 < k_6l_3 < \frac{1}{2}$ and sufficient large l_1 such that $\bar{\rho}_1 > 0$, $\bar{\rho}_2 > 0$, $1 - k_5l_2 - k_6l_3 > 0$. Therefore,

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{-\rho t}\big|Y_1(t)-Y_2(t)\big|^2\,dt + \mathbb{E}\int_0^T e^{-\rho t}\big|Z_1(t)-Z_2(t)\big|^2\,dt\\
&\quad\le\left[\frac{1}{\bar{\rho}_2}+\frac{1}{1-k_5l_2-k_6l_3}\right]\frac{1}{\bar{\rho}_1}2k_2l_1(k_2l_1+k_2^2)\\
&\qquad\times\mathbb{E}\int_0^T e^{-\rho t}\big[\big|U_1(t)-U_2(t)\big|^2+\big|V_1(t)-V_2(t)\big|^2\big]\,dt.
\end{aligned}$$

Thus, the proof is complete. □

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