Self-complementary (Pseudo-)Split Graphs

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Abstract

We are concerned with split graphs and pseudo-split graphs whose complements are isomorphic to themselves. These special subclasses of self-complementary graphs are actually the core of self-complementary graphs. Indeed, we show that all self-complementary graphs with forcibly self-complementary degree sequences are pseudo-split graphs. We also give formulas to calculate the number of self-complementary (pseudo-)split graphs of a given order, and show that a stronger version of Trotignon's conjecture holds for all self-complementary split graphs.

1 Introduction

The *complement* of a graph G is a graph defined on the same vertex set of G, where a pair of distinct vertices are adjacent if and only if they are not adjacent in G. In this paper, we study the graph that is isomorphic to its complement, hence called *self-complementary*. The graph of order one is trivially self-complementary. There are one self-complementary graph of order four and two self-complementary graphs of order five. Figure 1 lists all self-complementary graphs with eight vertices. A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. The first three of Figure 1 are split graphs, and their renditions in Figures 2 highlight the partition.

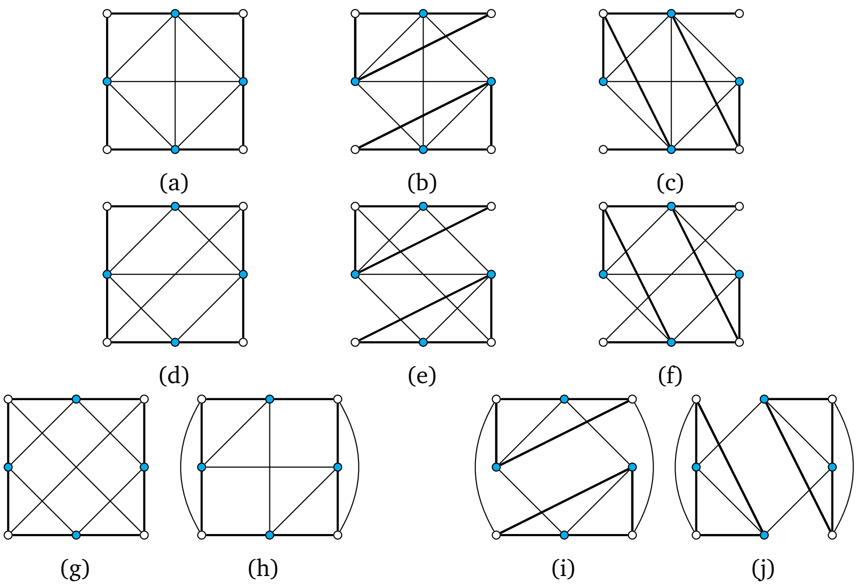

Figure 1: All self-complementary graphs on eight vertices. In each graph, the four vertices with lower degrees are represented as empty nodes, and others filled nodes.

These two families of graphs are connected by the following observation. An elementary counting argument convinces us that the order of a nontrivial self-complementary graph is either $4k$ or $4k+1$ for

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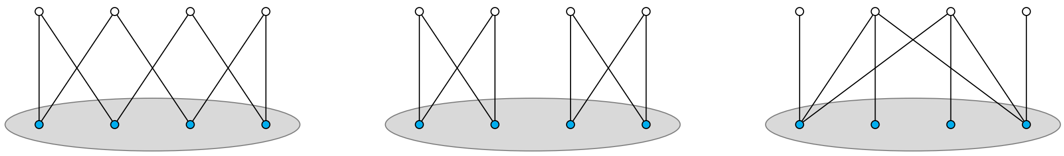


Figure 2: Self-complementary split graphs with eight vertices. Vertices in I are represented by empty nodes on the top, while vertices in K are represented by filled nodes on the bottom. For clarity, edges among vertices in K are omitted. Their degree sequences are (a) $(5^4, 2^4)$, (b) $(5^4, 2^4)$, and (c) $(6^2, 4^2, 3^2, 1^2)$.

some positive integer k. Consider a self-complementary graph G of order $4k$, where L (resp., H) represents the set of $2k$ vertices with lower (resp., higher) degrees. Note that $d(x) \leqslant 2k - 1 < 2k \leqslant d(y)$ for every pair of vertices $x \in L$ and $y \in H$. Xu and Wong [19] observed that the subgraphs of G induced by L and H are complementary to each other. More importantly, the bipartite graph spanned by the edges between L and H is closed under *bipartite complementation*, i.e., reversing edges in between but keeping both L and H independent. See the thick edges in Figure 1. When studying the connection between L and H, it is more convenient to add all the missing edges among H and remove all the edges among L, thereby turning G into a self-complementary split graph. In this sense, every self-complementary graph of order $4k$ can be constructed from a self-complementary split graph of the same order and a graph of order $2k$. For a self-complementary graph of an odd order, the self-complementary split graph is replaced by a self-complementary pseudo-split graph. A pseudo-split graph is either a split graph or a split graph plus a five-cycle such that every vertex on the cycle is adjacent to every vertex in the clique of the split graph and is nonadjacent to any vertex in the independent set of the split graph.

The decomposition theorem of Xu and Wong [19] was for the construction of self-complementary graphs, another ingredient of which is the degree sequences of these graphs (the non-increasing sequence of its vertex degrees). Clapham and Kleitman [5, 3] present a necessary condition for a degree sequence to be that of a self-complementary graph. However, a realization of such a degree sequence may or may not be self-complementary. A natural question is to ask about the degree sequences all of whose realizations are necessarily self-complementary, called *forcibly self-complementary*. All the degree sequences for self-complementary graphs up to order five, (0), $(2, 2, 1, 1)$, $(2, 2, 2, 2, 2)$, and $(3, 3, 2, 1, 1)$, are forcibly self-complementary. Of the four degree sequences for the self-complementary graphs of order eight, only $(5, 5, 5, 5, 2, 2, 2, 2)$ and $(6, 6, 4, 4, 3, 3, 1, 1)$ are focibly self-complementary. All the realizations of these forcibly self-complementary degree sequences turn out to be pseudo-split graphs. As we will see, this is not incidental.

We take p graphs S_1, S_2, …, S_p, each being either a four-path or one of the first two graphs in Figure 2. Note that $S_i, i = 1, \ldots, p$, admits a unique decomposition into a clique K_i and an independent set I_i. For any pair of i, j with $1 \leqslant i < j \leqslant p$, we add all possible edges between K_i and $K_j \cup I_j$. It is easy to verify that the resulting graph is self-complementary, and can be partitioned into a clique $\bigcup_{i=1}^{p} K_i$ and an independent set $\bigcup_{i=1}^{p} I_i$. By an *elementary self-complementary pseudo-split graph* we mean such a graph, or one obtained from it by adding a single vertex or a five-cycle and making them complete to $\bigcup_{i=1}^{p} K_i$. For example, we end with the graph in Figure 1(c) with $p = 2$ and both S_1 and S_2 being four-paths. It is a routine exercise to verify that the degree sequence of an elementary self-complementary pseudo-split graph is forcibly self-complementary. We show that the other direction holds as well, thereby fully characterizing forcibly self-complementary degree sequences.

Theorem 1.1. *A degree sequence is forcibly self-complementary if and only if every realization of it is an elementary self-complementary pseudo-split graph.*

Our result also bridges a longstanding gap in the literature on self-complementary graphs. Rao [12] has proposed another characterization for forcibly self-complementary degree sequences (we leave the statement, which is too technical, to Section 3). As far as we can check, he never published a proof of his characterization. It follows immediately from Theorem 1.1.

All self-complementary graphs up to order five are pseudo-split graphs, while only three out of the ten self-complementary graphs of order eight are. By examining the list of small self-complementary graphs, Ali [1] counted self-complementary split graphs up to 17 vertices. Whether a graph is a split graph can be determined solely by its degree sequence. However, this approach needs the list of all self-complementary graphs, and hence cannot be generalized to large graphs. Answering a question of Harary [9], Read [13] presented a formula for the number of self-complementary graphs with a specific number of vertices. Clapham [4] simplified Read's formula by studying the isomorphisms between a self-complementary graph and its complement. We take an approach similar to Clapham's for self-complementary split graphs with an even order, which leads to a formula for the number of such graphs. For other self-complementary pseudo-split graphs, we establish a one-to-one correspondence between self-complementary split graphs on $4k$ vertices and those on $4k+1$ vertices, and a one-to-one correspondence between self-complementary pseudo-split graphs of order $4k+1$ that are not split graphs and self-complementary split graphs on $4k-4$ vertices.

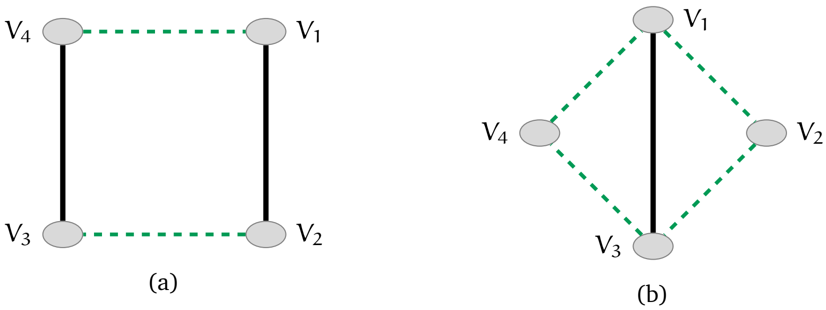


Figure 3: (a) The rectangle partition and (b) the diamond partition. Each node represents one part of the partition. A solid line indicates that all the edges between the two parts are present, a missing line indicates that there is no edge between the two parts, while a dashed line imposes no restrictions on the adjacency between the two parts.

We also study a conjecture of Trotignon [18], which asserts that if a self-complementary graph G does not contain a five-cycle, then its vertex set can be partitioned into four nonempty sets with the adjacency patterns of a rectangle or a diamond, as described in Figure 3. He managed to prove that certain special graphs satisfy this conjecture. The study of rectangle partitions in self-complementary graphs enabled Trotignon to present a new proof of Gibbs' theorem [8, Theorem 4]. We prove Trotignon's conjecture on self-complementary split graphs, with a stronger statement. We say that a partition of $V(G)$ is *self-complementary* if it forms the same partition in the complement of G, illustrated in Figure 4. Every self-complementary split graph of an even order admits a diamond partition that is self-complementary. Moreover, for each positive integer k, there is a single graph of order $4k$ that admits a rectangle partition. Note that in general, there are graphs that admit a partition, but do not admit any partition that is self-complementary [2].

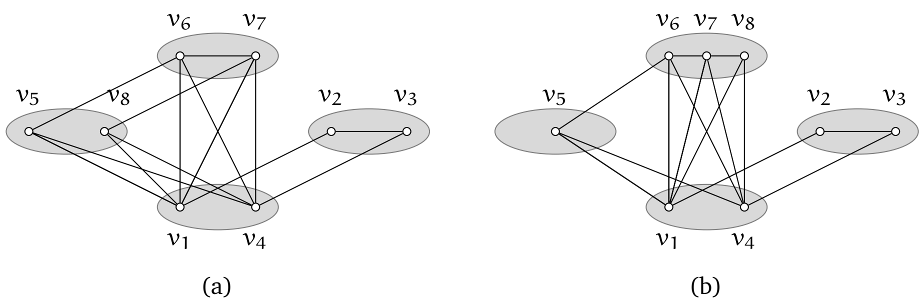


Figure 4: Two diamond partitions of a self-complementary graph; only the first is self-complementary.

Before closing this section, let us mention related work. There is another natural motivation to study self-complementary split graphs. Sridharan and Balaji [17] tried to understand self-complementary graphs that are chordal. They are self-complementary precisely split graphs [7]. The class of split graphs is *closed under complementation*.[1] We may study self-complementary graphs in other graph classes. Again,

[1]Some authors call such graph classes "self-complementary," e.g., the influential "Information System on Graph Classes and their

for this purpose, it suffices to focus on those closed under complementation. In the simplest case, we can define such a class by forbidding a graph F as well as its complement. It is not interesting when F consists of two or three vertices, or when it is the four-path. When F is the four-cycle, we end with the class of pseudo-split graphs, which is the simplest in this sense. A more important class closed under complementation is perfect graphs. We leave it open to characterize self-complementary perfect graphs. Another open problem is the recognition of self-complementary (pseudo)-split graphs. It is well known that the isomorphism test of both self-complementary graphs and (pseudo)-split graphs are GI-complete [6, 10].

2 Preliminaries

All the graphs discussed in this paper are finite and simple. The vertex set and edge set of a graph G are denoted by, respectively, $V(G)$ and $E(G)$. The two ends of an edge are *neighbors* of each other, and the number of neighbors of a vertex v, denoted by $d_G(v)$, is its *degree*. We may drop the subscript G if the graph is clear from the context. For a subset $U \subseteq V(G)$, let $G[U]$ denote the subgraph of G induced by U, whose vertex set is U and whose edge set comprises all the edges with both ends in U, and let $G - U = G[V(G) \setminus U]$, which is simplified to $G - u$ if U comprises a single vertex u. A *clique* is a set of pairwise adjacent vertices, and an *independent set* is a set of vertices that are pairwise nonadjacent. For $\ell \geqslant 1$, we use P_ℓ and K_ℓ to denote the path graph and the complete graph, respectively, on ℓ vertices. For $\ell \geqslant 3$, we use C_ℓ to denote the ℓ-cycle. We say that two sets of vertices are *complete* or *nonadjacent* to each other if there are all possible edges or if there is no edge between them, respectively.

An *isomorphism* between two graphs G_1 and G_2 is a bijection between their vertex sets, i.e., $\sigma\colon V(G_1) \to V(G_2)$, such that two vertices u and v are adjacent in G_1 if and only if $\sigma(u)$ and $\sigma(v)$ are adjacent in G_2. Two graphs with an isomorphism are *isomorphic* to each other. A graph is *self-complementary* if it is isomorphic to its *complement* $\overline{G}$, the graph defined on the same vertex set of G, where a pair of distinct vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. An isomorphism σ between G and $\overline{G}$ is a permutation of $V(G)$, called an *antimorphism*. We may abuse notation to use $\sigma(X) = Y$, where $X, Y \subseteq V(G)$ to denote that $Y = \bigcup_{x \in X}\{\sigma(x)\}$. We represent an antimorphism as the product of disjoint cycles

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_p,$$

where $\sigma_i = (v_{i1} v_{i2} \cdots)$ for all $i = 1, \ldots, p$. Sachs and Ringel [16, 14] independently showed that there can be at most one vertex v fixed by an antimorphism σ, i.e., $\sigma(v) = v$. For any other vertex u, the smallest number k satisfying $\sigma^k(u) = u$ has to be a multiplier of four. Gibbs [8] observed that $d(v) + d(\sigma(v)) = n - 1$, where n is the order of G. Hence, if v is fixed by σ, then $d(v) = (n-1)/2$. Furthermore, the vertices in every cycle of σ with a length of more than one alternate in degrees d and $n - 1 - d$ for some d.

Lemma 2.1 ([16, 14])**.** *If σ is an antimorphism of a self-complementary graph, then the length of each cycle in σ is either one or a multiplier of four. Moreover, there is a unique cycle of length one if and only if the order of the graph is odd.*

For any subset of cycles in σ, the vertices within those cycles induce a subgraph that is self-complementary. Indeed, the selected cycles themselves act as an antimorphism of the subgraph.

Proposition 2.2 ([8])**.** *Let G be a self-complementary graph and σ an antimorphism of G. For any subset of cycles in σ, the vertices within those cycles induce a self-complementary graph.*

A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. We use $K \uplus I$, where K being a clique and I an independent set, to denote a *split partition* of a split graph. The following is straightforward. Since it is not used in the present paper, we omit the proof.

Proposition 2.3. *Let G be a graph of an order $4k$, and let H and L be the $2k$ vertices of the higher and lower degrees, respectively. If G is self-complementary, then it remains self-complementary after H replaced by a clique and L an independent set.*

A split graph may have more than one split partition; e.g., a complete graph on n vertices has $n + 1$ different split partitions.

Inclusions" (https://www.graphclasses.org).

Lemma 2.4. *A self-complementary split graph on* $4k$ *vertices has a unique split partition and it is*

$$\{v \mid d(v) \geqslant 2k\} \uplus \{v \mid d(v) < 2k\}. \tag{1}$$

Proof. Let G be a self-complementary split graph with $4k$ vertices, and σ an antimorphism of G. By definition, for any vertex $v \in V(G)$, we have $d(v) + d(\sigma(v)) = 4k - 1$. Thus,

$$\min(d(v), d(\sigma(v))) \leqslant 2k - 1 < 2k \leqslant \max(d(v), d(\sigma(v))).$$

As a result, G does not contain any clique or independent set of order $2k + 1$. Suppose for contradiction that there exists a split partition $K \uplus I$ of G different from (1). There must be a vertex $x \in I$ with $d(x) \geqslant 2k$. We must have $d(x) = 2k$ and $N(x) \subseteq K$. But then there are at least $|N[x]| = 2k + 1$ vertices having degree at least $2k$, a contradiction. $\square$

The following observation correlates self-complementary split graphs having even and odd orders.

Proposition 2.5. *Let* G *be a split graph on* $4k + 1$ *vertices. If* G *is self-complementary, then* G *has exactly one vertex* v *of degree* $2k$*, and* $G - v$ *is also self-complementary.*

Proof. Let σ be an antimorphism of G. By Lemma 2.1, there exists a cycle of length one in σ. We take v to be the vertex in this cycle. We can write $\sigma = \sigma_1 \dots \sigma_p(v)$. By definition, for each $i = 1, \dots, p$, a vertex $x \in \sigma_i$ is adjacent to v if and only if $\sigma_i(x)$ is not adjacent to v. Thus, v is adjacent to half of vertices in σ_i, and $d(v) = 2k$. By Proposition 2.2, $\sigma_1 \dots \sigma_p$ is an antimorphism of $G - v$, which is thus self-complementary. Since $G - v$ is an induced subgraph of a split graph, it is a self-complementary split graph. It remains to show that no other vertex in G has a degree of $2k$. By Lemma 2.4, $G - v$ has a unique split partition; let it be $K \uplus I$. Note that $N(v) = K$, since G is a split graph and $d(v) = 2k$. Since a vertex $x \in K$ cannot be moved to I to make another split partition, it has at least one neighbor in I. Thus, $d(x) > 2k$. In a similar way, we can conclude that $d(x) < 2k$ for all $x \in I$. $\square$

A partition of $V(G)$ into four nonempty subsets $\{V_1, V_2, V_3, V_4\}$ is a *rectangle partition* if V_1 is complete to V_2 and nonadjacent to V_3, while V_4 is complete to V_3 and nonadjacent to V_2, or a *diamond partition* if V_1 is complete to V_3 while V_2 is nonadjacent to V_4. See Figure 3. It is obvious that every self-complementary split graph admits a diamond partition. We prove a stronger statement. We say that a rectangle or diamond partition of a graph G is *self-complementary* if the four parts form the same type of partition in the complement of G.

Lemma 2.6. *Every self-complementary split graph* G *admits a diamond partition. If* G *has an even order, then it admits a diamond partition that is self-complementary.*

Proof. Let $K \uplus I$ be a split partition of G. For any proper and nonempty subset $K' \subseteq K$ and proper and nonempty subset $I' \subseteq I$, the partition

$$K', I', K \setminus K', I \setminus I'$$

is a diamond partition.

Now suppose that the order of G is $4k$. We fix an arbitrary antimorphism $\sigma = \sigma_1 \sigma_2 \cdots \sigma_p$ of G. Since $\sigma(K) = I$, every cycle has vertices from both K and I. We may assume without loss of generality that for all $i = 1, \dots, p$, the first vertex in σ_i is in K. For $j = 1, \dots, |\sigma_i|$, we assign the jth vertex of σ_i to $V_{j \pmod 4}$. As a result, $\sigma(V_1) = V_2$ and $\sigma(V_3) = V_4$. Moreover, $V_1 \cup V_3 = K$ and $V_2 \cup V_4 = I$. Thus, $\{V_1, V_2, V_3, V_4\}$ is a self-complementary diamond partition of G. $\square$

For a positive integer k, let Z_k denote the graph obtained from a P_4 as follows. We substitute each degree-one vertex with an independent set of k vertices, and each degree-two vertex with a clique of k vertices. For example, P_4 itself is Z_1 and depicted in Figure 2(b) is Z_2.

Lemma 2.7. *A self-complementary split graph admits a rectangle partition if and only if it is isomorphic to* Z_k.

Proof. The sufficiency is trivial, and we consider the necessity. Suppose that G is a self-complementary split graph and it has a rectangle partition $\{V_1, V_2, V_3, V_4\}$. Let $K \uplus I$ be a split partition of G. There are at least one edge and at least one missing edge between any three parts. Thus, vertices in K are assigned to precisely two parts in the partition. By the definition of rectangle partition, K is either $V_2 \cup V_3$ or $V_1 \cup V_4$. Assume without loss of generality that $K = V_2 \cup V_3$. Since V_2 is complete to V_1 and nonadjacent to V_4, any antimorphism of G maps V_2 to either V_1 or V_4. If $|V_2| \neq |V_3|$, then the numbers of edges between K and I in G and $\overline{G}$ are different. This is impossible. It further implies $|V_1| = |V_4|$, and hence G is precisely $Z_{|V_1|}$. ☐

A *pseudo-split graph* is either a split graph, or a graph whose vertex set can be partitioned into a clique K, an independent set I, and a set C that (1) induces a C_5; (2) is complete to K; and (3) is nonadjacent to I. We say that $K \uplus I \uplus C$ is a *pseudo-split partition* of the graph, where C may or may not be empty. If C is empty, then $K \uplus I$ is a split partition of the graph. Otherwise, the graph has a unique pseudo-split partition. Similar to split graphs, the complement of a pseudo-split graph remains a pseudo-split graph.

Proposition 2.8. *Let G be a self-complementary pseudo-split graph with a pseudo-split partition $K \uplus I \uplus C$. If $C \neq \emptyset$, then $G - C$ is a self-complementary split graph of an even order.*

Proof. Let σ be an antimorphism of G. In both G and its complement, the only C_5 is induced by C. Thus, $\sigma(C) = C$. Since C is complete to K and nonadjacent to I, it follows that $\sigma(K) = I$ and $\sigma(I) = K$. Thus, $G - C$ is a self-complementary graph. It is clearly a split graph and has an even order. ☐

3 Forcibly self-complementary degree sequences

The *degree sequence* of a graph G is the sequence of degrees of all vertices, listed in non-increasing order, and G is a *realization* of this degree sequence. For our purpose, it is more convenient to use a compact form of degree sequences where the same degrees are grouped:

$$(d_i^{n_i})_{i=1}^{\ell} = (d_1^{n_1}, \dots, d_\ell^{n_\ell}) = \Bigg(\underbrace{d_1, \dots, d_1}_{n_1}, \underbrace{d_2, \dots, d_2}_{n_2}, \dots, \underbrace{d_\ell, \dots, d_\ell}_{n_\ell}\Bigg).$$

Note that we always have $d_1 > d_2 > \cdots > d_\ell$. For example, the degree sequences of the first two graphs in Figure 2 are both

$$(5^4, 2^4) = (5, 5, 5, 5, 2, 2, 2, 2).$$

These two graphs are not isomorphic; thus, a degree sequence may have non-isomorphic realizations.

For four vertices v_1, v_2, v_3, and v_4 such that v_1 is adjacent to v_2 but not to v_3 while v_4 is adjacent to v_3 but not to v_2, the operation of replacing v_1v_2 and v_3v_4 with v_1v_3 and v_2v_4 is a *2-switch*, denoted as

$$(v_1v_2, v_3v_4) \to (v_1v_3, v_2v_4).$$

See Figure 5. It is easy to check that this operation does not change the degree of any vertex. Indeed, it is well known that any two graphs of the same degree sequence can be transformed into each other by 2-switches [15].

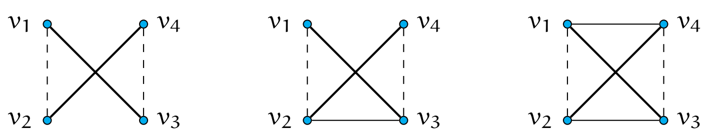


Figure 5: Illustrations for 2-switches: replacing the dashed edges with thick edges.

Lemma 3.1 ([15])**.** *Two graphs have the same degree sequence if and only if they can be transformed into each other by a series of 2-switches.*

The subgraph induced by the four vertices involved in a 2-switch operation must be isomorphic to a $2K_2$, P_4, or C_4. Moreover, after the operation, the four vertices induce an isomorphic subgraph. Since a split graph G cannot contain any $2K_2$ or C_4 [7], a 2-switch must be done on a P_4. In any split partition $K \uplus I$ of G, the two degree-one vertices of P_4 are from I, while the others from K. The graph remains a split graph after this operation. Thus, if a degree sequence has a realization that is a split graph, then all its realizations are split graphs [7]. A similar statement holds for pseudo-split graphs [11].

We do not have a similar claim on degree sequences of self-complementary graphs. Clapham and Kleitman [5, 3] have fully characterized all such degree sequences, called *potentially self-complementary degree sequences*. For simplicity, we only need a simpler statement on even-order graphs.

Theorem 3.2 ([5, 3])**.** *A degree sequence* $(d_i^{n_i})_{i=1}^{\ell}$ *of even order* n *is potentially self-complementary if and only if* ℓ *is even, and for all* $i = 1, \ldots, \ell/2$*,*

- $d_i + d_{\ell+1-i} = n - 1$*, and*
- $n_i = n_{\ell+1-i}$ *is even.*

Moreover, for all $p = 1, \ldots, \ell/2$

$$\sum_{i=1}^{p} n_i d_i \leqslant \left(\sum_{i=1}^{p} n_i\right)\left(n - 1 - \sum_{i=1}^{p} \frac{n_i}{2}\right).$$

A degree sequence is *forcibly self-complementary* if all of its realizations are self-complementary. We refer to the graph in Figure 1(a) as a *trampoline graph*.

Proposition 3.3. *The following degree sequences are all forcibly self-complementary:* $(2^2, 1^2)$*,* (2^5)*, and* $(5^4, 2^4)$*.*

Proof. Applying a 2-switch operation to a realization of $(2^2, 1^2)$ or (2^5) leads to an isomorphic graph. A 2-switch operation transforms a Z_2 into a trampoline, and vice versa. Thus, the statement follows from Lemma 3.1. □

We have seen that degree sequences (2^5), $(2^2, 1^2)$, and $(5^4, 2^4)$ are forcibly self-complementary. They are the only ones of these forms. In general, it is quite challenging to verify that a degree sequence is indeed forcibly self-complementary. On the other hand, to show that a degree sequence is not forcibly self-complementary, it suffices to construct a realization that is not self-complementary. The proof of the following proposition is deferred to the appendix.

Proposition 3.4. *The following degree sequences are not forcibly self-complementary.*

i) $((2k)^{4k+1})$*, where* $k \geqslant 2$*.*

ii) $(d^{2k}, (4k-1-d)^{2k})$*, where* $k \geqslant 2$ *and* $d \neq 5$*.*

iii) $(d^{2k_1}, (d-1)^{2k_2}, (4k-d)^{2k_2}, (4k-1-d)^{2k_1})$*, where* $k_1, k_2 > 0$ *and* $k = k_1 + k_2$*.*

We take p vertex-disjoint graphs S_1, S_2, ..., S_p, each of which is isomorphic to P_4, Z_2, or trampoline. For $i = 1, \ldots, p$, let $H_i \uplus L_i$ denote the unique split partition of S_i (Lemma 2.4). Let C be another set of 0, 1, or 5 vertices. We add all possible edges among $\bigcup_{i=1}^{p} H_i$ to make it a clique, and for each $i = 1, \ldots, p$, add all possible edges between H_i and $\bigcup_{j=i+1}^{p} L_j$.[2] Finally, we add all possible edges between C and $\bigcup_{i=1}^{p} H_i$, and add edges to make C a cycle if $|C| = 5$. Let $\mathcal{E}$ denote the set of graphs that can be constructed as above.

Lemma 3.5. *All graphs in* $\mathcal{E}$ *are self-complementary pseudo-split graphs, and their degree sequences are forcibly self-complementary.*

Proof. Let G be an arbitrary graph in $\mathcal{E}$. It has a split partition $(\bigcup_{i=1}^{p} H_i \cup C) \uplus (\bigcup_{i=1}^{p} L_i)$ when $|C| \leqslant 1$, and a pseudo-split partition $(\bigcup_{i=1}^{p} H_i) \uplus (\bigcup_{i=1}^{p} L_i) \uplus C$ otherwise. To show that G is self-complementary, we construct an antimorphism σ for it. For each $i = 1, \ldots, p$, we take an antimorphism σ_i of S_i, and set $\sigma(x) = \sigma_i(x)$ for all $x \in V(S_i)$. If C consists of a single vertex v, we set $\sigma(v) = v$. If $|C| = 5$, we take an antimorphism σ_{p+1} of C_5 and set $\sigma(x) = \sigma_{p+1}(x)$ for all $x \in C$. It is easy to verify that a pair of vertices u and v are adjacent in G if and only if $\sigma(u)$ and $\sigma(v)$ are adjacent in $\overline{G}$.

[2] The reader familiar with threshold graphs may note its use here. If we contract H_i and L_i into two vertices, we end with a threshold graph. Threshold graphs have a stronger characterization by degree sequences. Since a threshold graph free of $2K_2$, P_4, and C_4, no 2-switch is possible on it. Thus, the degree sequence of a threshold graph has a unique realization.

For the second assertion, we show that applying a 2-switch to G in $\mathcal{E}$ leads to another graph in $\mathcal{E}$. Since G is a split graph, a 2-switch can only be applied to a P_4. For two vertices $v_1 \in H_i$ and $v_2 \in H_j$ with $i < j$, we have $N[v_2] \subseteq N[v_1]$. Thus, there cannot be any P_4 involving both v_1 and v_2. A similar argument applies to two vertices in L_i and L_j with $i \neq j$. Therefore, a 2-switch can be applied either *inside* C or *inside* S_i for some $i \in \{1, \ldots, p\}$. By Proposition 3.3, the resulting graph is in $\mathcal{E}$, hence self-complementary. Thus, the degree sequence of G is forcibly self-complementary by Lemma 3.1. □

We refer to graphs in $\mathcal{E}$ as *elementary self-complementary pseudo-split graphs*. The rest of this section is devoted to showing that all realizations of forcibly self-complementary degree sequences are elementary self-complementary pseudo-split graphs. We fix a forcibly self-complementary degree sequence $\tau = (d_1^{n_1}, \ldots, d_\ell^{n_\ell})$ and a realization G of τ. For each $i = 1, \ldots, \ell$, let

$$V_i = \{v \in V(G) \mid d(v) = d_i\}, \quad V_i^+ = V_i \cup V_{\ell+1-i},$$

and we define the i*th slice* of G as the induced subgraph $G[V_i^+]$. Note that $V_i = V_{\ell+1-i}$ and $V_i^+ = V_i$ when ℓ is odd and $i = (\ell+1)/2$.

Each slice must be self-complementary, and more importantly, its degree sequence is forcibly self-complementary.

Lemma 3.6. *For all $i = 1, \ldots, \ell$, the degree sequence of the subgraph $G[V_i^+]$ is forcibly self-complementary.*

Proof. Let σ be an antimorphism of G. Since $d_1 > d_2 > \cdots > d_\ell$, we have $\sigma(V_i) = V_{\ell+1-i}$ and $\sigma(V_{\ell+1-i}) = V_i$ (note that V_i and $V_{\ell+1-i}$ are either identical or disjoint). Therefore, $n_i = n_{\ell+1-i}$. By Proposition 2.2, the cycles of σ consisting of vertices from V_i^+ is an antimorphism of $G[V_i^+]$, and $G[V_i^+]$ is self-complementary. To show that the degree sequence of $G[V_i^+]$ is forcibly self-complementary, let S be any other realization of the same degree sequence. By Lemma 3.1, we can transform $G[V_i^+]$ to S by a sequence of 2-switches applied on vertices in V_i^+. We can apply the same sequence of 2-switches to G, and denote by G' the resulting graph. By Lemma 3.1, the degree sequence of G' is also τ, and S is the ith slice of G'. By the first assertion, S is self-complementary. □

Lemma 3.6 imposes limitations on possible 2-switches applicable to G.

Corollary 3.7. *For all $i = 1, \ldots, \ell$, the number of edges in $G[V_i^+]$ or between V_i and $V_{\ell+1-i}$ cannot be changed by any sequence of 2-switches.*

Proof. Let G' be the graph obtained from G by a sequence of 2-switches. By the definition of 2-switches, every vertex has the same degree in G and G'. Since G' is a realization of τ, the subgraph $G'[V_i^+]$ is self-complementary. Thus, the number of edges in $G'[V_i^+]$ is the same as in $G[V_i^+]$. Since there are an antimorphism σ of G and an antimorphism σ' of G' such that $\sigma(V_i) = \sigma'(V_i) = V_{\ell+1-i}$, the number of edges between V_i and $V_{\ell+1-i}$ are the same. □

All the vertices in V_i share the same degree *in the ith slice*. In other words, the ith slice has at most two distinct degrees.

Lemma 3.8. *For each $i \in \{1, \ldots, \ell\}$, the vertices in V_i have the same degree in $G[V_i^+]$.*

Proof. Suppose for contradiction that vertices in V_i have different degrees in $G[V_i^+]$.

Case 1, there are two vertices v_1 and v_2 in V_i such that

$$d = d_{G[V_i^+]}(v_1) > d_{G[V_i^+]}(v_2) + 1.$$

There exists a vertex $x_1 \in V_i^+$ adjacent to v_1 but not to v_2. On the other hand, since $d_G(v_1) = d_G(v_2)$, there must be a vertex

$$x_2 \in N(v_2) \setminus \left(N(v_1) \cup V_i^+\right).$$

We apply the 2-switch $(x_1v_1, x_2v_2) \to (x_1v_2, x_2v_1)$ to G and denote by G' the resulting graph. By Lemma 3.6, $G[V_i^+]$ is self-complementary, and hence there are an even number of vertices with degree d in $G[V_i^+]$ by Theorem 3.2. The degree of a vertex x in $G'[V_i^+]$ is

$$\begin{cases} d_{G[V_i^+]}(x) - 1 & x = v_1, \\ d_{G[V_i^+]}(x) + 1 & x = v_2, \\ d_{G[V_i^+]}(x) & \text{otherwise.} \end{cases}$$

Thus, the number of vertices with degree d in $G'[V_i^+]$ is odd. Hence, $G'[V_i^+]$ is not self-complementary by Theorem 3.2. By Lemma 3.1, G' is also a realization of τ, and hence $G'[V_i^+]$ is self-complementary by Lemma 3.6. We end with a contradiction.

Case 2, the degree of vertices in V_i is either d or $d-1$ for some d in $G[V_i^+]$. By Lemma 3.6, the degree sequence of $G[V_i^+]$ is forcibly self-complementary. It cannot be of the form $(d^{2k_1}, (d-1)^{2k_2}, (n-d)^{2k_2}, (n-1-d)^{2k_1})$ by Proposition 3.4(iii). Thus, the degree sequence of $G[V_i^+]$ must be $(d^{2k}, (d-1)^{2k})$ for some k. By Proposition 3.4(ii), $k = 1$ and $d = 2$. Let $v_1v_2v_3v_4$ denote the path induced by V_i^+. By the applicability of the 2-switch $(v_1v_2, v_3v_4) \to (v_1v_3, v_2v_4)$ and Corollary 3.7, we must have $i = \ell + 1 - i$. Also note that $\ell > 1$ because vertices in V_i have different degrees in $G[V_i]$. Let σ be an antimorphism of G. In every cycle disjoint from V_i, the neighbors of v_1 and v_2 differ by an even number. Thus, $d_G(v_1) \neq d_G(v_2)$, a contradiction. □

We can now settle the interval structure of each slice.

Lemma 3.9. *For all* $i = 1, \ldots, \lfloor \frac{\ell}{2} \rfloor$,

i) the slice $G[V_i^+]$ *is isomorphic to either a* P_4*, a* Z_2*, or a trampoline, and*

ii) $V_i \uplus V_{\ell+1-i}$ *is a split partition of* $G[V_i^+]$.

Moreover, if ℓ *is odd, the slice* $G[V_{(\ell+1)/2}]$ *is either a* C_5 *or consists of a single vertex.*

Proof. For all $i = 1, \ldots, \ell$, the induced subgraph $G[V_i^+]$ of G is self-complementary by Lemma 3.6. Furthermore, $G[V_i^+]$ is either a regular graph or has two different degrees (Lemma 3.8). For all $i = 1, \ldots, \lfloor \frac{\ell}{2} \rfloor$, the sets V_i and $V_{\ell+1-i}$ are disjoint. Hence, $|V_i^+|$ is $4k$ for some positive k, and the degree sequence of $G[V_i^+]$ is of the form $(d^{2k}, (4k-1-d)^{2k})$. By Lemma 3.6 and Proposition 3.4(ii), $k = 1$ or $d = 5$. Thus, the degree sequence of $G[V_i^+]$ is either $(2^2, 1^2)$ or $(5^4, 2^4)$, whose realizations are either a P_4, a Z_2, or a trampoline. Let $H_i \uplus L_i$ be the unique split partition of $G[V_i^+]$. Suppose to the contradiction of (ii) that there is a vertex $v_1 \in V_i \cap L_i$. We can find a vertex $v_2 \in H_i \setminus N(v_1)$ and a vertex $x_2 \in N(v_2) \cap L_i$. Note that x_2 is not adjacent to v_1. Since $d_G(v_1) > d_G(v_2)$ while $d_{G[V_i^+]}(v_1) < d_{G[V_i^+]}(v_2)$, we can find a vertex x_1 in $V(G) \setminus V_i^+$ that is adjacent to v_1 but not v_2. The applicability of the 2-switch $(x_1v_1, x_2v_2) \to (x_1v_2, x_2v_1)$ violates Corollary 3.7.

If ℓ is odd, then $G[V_{(\ell+1)/2}]$ is a regular graph by Lemma 3.9. Hence, the degree sequence of $G[V_{(\ell+1)/2}]$ is $((2k)^{4k+1})$, where $k = (|V_{(\ell+1)/2}| - 1)/4$. By Lemma 3.6 and Proposition 3.4(i), $k \leqslant 1$. The statement follows. □

The next is on edges between different slices.

Lemma 3.10. *For every* $i \in \{1, 2, \ldots, \lfloor \ell/2 \rfloor\}$*, if a vertex in* $V(G) \setminus V_i$ *has a neighbor in* $V_{\ell+1-i}$*, then it is adjacent to all the vertices in* V_i^+.

Proof. Let $x_1 \in V(G) \setminus V_i$ be adjacent to $v_1 \in V_{\ell+1-i}$. Since $V_{\ell+1-i}$ is an independent set, it does not contain x_1. Suppose for contradiction that $V_i^+ \not\subseteq N(x_1)$, and let v_2 be a vertex in $V_i^+ \setminus N(x_1)$. If $v_2 \in V_i$, we can find a vertex $v_3 \in V_i \setminus N(v_1)$ by Lemma 3.9. The applicability of the 2-switch $(x_1v_1, v_2v_3) \to (x_1v_2, v_1v_3)$ violates Corollary 3.7. In the rest, $v_2 \in V_{\ell+1-i}$.

If there exists a vertex $x_2 \in V_i \cap N(v_2) \setminus N(v_1)$, then we can conduct the 2-switch $(x_1v_1, x_2v_2) \to (x_1v_2, x_2v_1)$, but the ith slice of the resulting graph cannot be isomorphic to P_4, Z_2, or trampoline, contradicting Lemma 3.9(i). Therefore, $V_i \cap N(v_2) \subseteq N(v_1)$, and $G[V_i^+]$ must be isomorphic to Z_2. We can find a vertex x_3 in $V_i \setminus N(v_1)$ and a vertex v_3 in $V_{\ell+1-i} \cap N(x_3)$. Note that neither x_2v_3 nor x_3v_1 is an edge. We may either conduct the 2-switch $(x_1v_3, x_2v_2) \to (x_1v_2, x_2v_3)$ or $(x_1v_1, x_3v_3) \to (x_1v_3, x_3v_1)$ to G, depending on whether x_1 is adjacent to v_3. In either case, the ith slice of the resulting graph contradicts Lemma 3.9(i). These contradictions conclude the proof. □

We are now ready to prove the main lemma.

Lemma 3.11. *The graph* G *is an elementary self-complementary pseudo-split graph.*

Proof. Let σ be an antimorphism of G. For each $i \in \{1, 2, \ldots, \lfloor \ell/2 \rfloor\}$, we denote $H_i = V_i$ and $L_i = V_{\ell+1-i}$. By Lemma 3.9, $H_i \uplus L_i$ is a split partition of $G[V_i^+]$. Let i, j be two distinct indices in $\{1, 2, \ldots, \lfloor \ell/2 \rfloor\}$. We argue that there cannot be any edge between H_i and L_j if $i > j$. Suppose for contradiction that there exists

$x \in H_i$ that is adjacent to $y \in L_j$ for some $i > j$. By Lemma 3.10, x is adjacent to all the vertices in $G[V_j^+]$. Consequently, $\sigma(x)$ is in L_i and has no neighbor in $G[V_j^+]$. Let v_1 be a vertex in H_j. Since v_1 is not adjacent to $\sigma(x)$, it has no neighbor in L_i by Lemma 3.10. Note that $G[V_i^+]$ is either a P_4, a Z_2, or a trampoline, and so does $G[V_j^+]$. If we focus on the graph induced by $V_i^+ \cup V_j^+$, we can observe that

$$d_{G[V_i^+ \cup V_j^+]}(v_1) < d_{G[V_i^+ \cup V_j^+]}(x).$$

Since $d_G(v_1) > d_G(x)$, we can find a vertex x_1 in $V(G) \setminus (V_i^+ \cup V_j^+)$ that is adjacent to v_1 but not x. Let v_2 be a neighbor of x in L_i. Note that v_2 is not adjacent to v_1. We can conduct the 2-switch $(x_1v_1, xv_2) \to (x_1x, v_1v_2)$, violating Corollary 3.7. Therefore, L_i is nonadjacent to $\bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} H_p$ for all $i = 1, \ldots, \lfloor \ell/2 \rfloor$. Since $\sigma(L_i) = H_i$ and $\sigma(\bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} H_p) = \bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} L_p$, we can obtain that H_i is complete to $\bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} L_p$. Moreover, H_i is complete to $\bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} H_p$ by Lemma 3.10, and hence L_i is nonadjacent to $\bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} L_p$.

We are done if ℓ is even. In the rest, we assume that ℓ is odd. By Lemma 3.9, the subgraph induced by $V_{(\ell+1)/2}$ is either a C_5 or contains exactly one vertex. It suffices to show that $V_{(\ell+1)/2}$ is complete to H_i and nonadjacent to L_i for every $i \in \{1, 2, \ldots, \lfloor \ell/2 \rfloor\}$. Suppose $\sigma(v) = v$. When $V_{(\ell+1)/2} = \{v\}$, the claim follows from Lemma 3.10 and that $\sigma(v) = v$ and $\sigma(V_i) = V_{\ell+1-i}$. Now $|V_{(\ell+1)/2}| = 5$. Suppose for contradiction that there is a pair of adjacent vertices $v_1 \in V_{(\ell+1)/2}$ and $x \in L_i$. Let $v_2 = \sigma(v_1)$. By Lemmas 3.10, v_1 is adjacent to all the vertices in $G[V_i^+]$. Accordingly, v_2 has no neighbor in $G[V_i^+]$. Since $G[V_{(\ell+1)/2}]$ is a C_5 , we can find $v_3 \in V_{(\ell+1)/2}$ that is adjacent to v_2 but not v_1. We can conduct the 2-switch $(xv_1, v_2v_3) \to (xv_2, v_1v_3)$ and denote by G' as the resulting graph. It can be seen that $G'[V_{(\ell+1)/2}]$ is not a C_5, contradicting Lemma 3.9. □

Lemmas 3.5 and 3.11 imply Theorem 1.1 and Rao's characterization of forcibly self-complementary degree sequences [12].

Theorem 3.12 ([12]). *A degree sequence $(d_i^{n_i})_{i=1}^{\ell}$ is forcibly self-complementary if and only if for all $i = 1, \ldots, \lfloor \ell/2 \rfloor$,*

$$n_{\ell+1-i} = n_i \in \{2, 4\}, \tag{2}$$

$$d_{\ell+1-i} = n - 1 - d_i = \sum_{j=1}^{i} n_j - \frac{1}{2} n_i, \tag{3}$$

and $n_{(\ell+1)/2} \in \{1, 5\}$ and $d_{(\ell+1)/2} = \frac{1}{2}(n-1)$ when ℓ is odd.

Proof. The sufficiency follows from Lemma 3.5: note that an elementary self-complementary pseudo-split graph in which $G[V_i^+]$ has $2n_i$ vertices satisfies the conditions. The necessity follows from Lemma 3.11. □

4 Enumeration

In this section, we consider the enumeration of self-complementary (pseudo-)split graphs. The following corollary of Propositions 2.5 and 2.8 focuses us on self-complementary split graphs of even orders. Let λ_n and λ'_n denote the number of split graphs and pseudo-split graphs, respectively, of order n that are self-complementary. For convenience, we set $\lambda_0 = 1$.

Corollary 4.1. *For each $k \geqslant 1$, it holds $\lambda_{4k+1} = \lambda_{4k}$. For each $n > 0$,*

$$\lambda'_n = \begin{cases} \lambda_n & n \equiv 0 \pmod 4, \\ \lambda_{n-1} + \lambda_{n-5} & n \equiv 1 \pmod 4. \end{cases}$$

Proof. Proposition 2.5 implies that there exists a one-to-one correspondence between self-complementary split graphs with $4k$ vertices and those with $4k + 1$ vertices. If a self-complementary pseudo-split graph is not a split graph, then it contains a five cycle and the removal of this five cycle from the graph resulting a self-complementary split graph of an even order by Proposition 2.8. □

Let $\sigma = \sigma_1 \dots \sigma_p$ be an antimorphism of a self-complementary graph of $4k$ vertices. We find the number of ways in which edges can be introduced so that the result is a self-complementary split graph with σ as an antimorphism. We need to consider adjacencies among vertices in the same cycle and the adjacencies between vertices from different cycles of σ. For the second part, we further separate into two cases depending on whether the cycles have the same length. We use G to denote a resulting graph and denote by G_i the graph induced by the vertices in the ith cycle, for $i = 1, \dots, p$. By Lemma 2.4, G has a unique split partition and we refer to it as $K \uplus I$.

(i) The subgraph G_i is determined if it has been decided whether v_{i1} is to be adjacent or not adjacent to each of the following $\frac{|\sigma_i|}{2}$ vertices in σ_i. Among those $\frac{|\sigma_i|}{2}$ vertices, half of them are odd-numbered in σ_i. Therefore, v_{i1} is either adjacent to all of them or adjacent to none of them by Lemma 2.4. The number of adjacencies to be decided is $\frac{|\sigma_i|}{4} + 1$.

(ii) The adjacencies between two subgraphs G_i and G_j of the same order are determined if it has been decided whether v_{i1} is to be adjacent or not adjacent to each of the vertices in G_j. By Lemma 2.4, the vertex v_{i1} and half of vertices of G_j are decided in K or in I after (i). The number of adjacencies to be decided is $\frac{|\sigma_j|}{2}$.

(iii) We now consider the adjacencies between two subgraphs G_i and G_j of different orders. We use $\gcd(x, y)$ to denote the greatest common factor of two integers x and y. The adjacencies between G_i and G_j are determined if it has been decided whether v_{i1} is to be adjacent or not adjacent to each of the first $\gcd(|\sigma_i|, |\sigma_j|)$ vertices of G_j. Among those $\gcd(|\sigma_i|, |\sigma_j|)$ vertices of G_j, half of them are decided in the same part of $K \uplus I$ as v_{i1} after (i). The number of adjacencies to be decided is $\frac{1}{2}\gcd(|\sigma_i|, |\sigma_j|)$.

By Lemma 2.1, $|\sigma_i| \equiv 0 \pmod 4$ for every $i = 1, \dots, p$. Let c be the cycle structure of σ. We use c_q to denote the number of cycles in c with length $4q$ for every $q = 1, 2, \dots, k$. The total number of adjacencies to be determined is

$$\begin{aligned} P &= \sum_{q=1}^{k} \left(c_q(q+1) + \frac{1}{2}c_q(c_q - 1) \cdot 2q\right) + \sum_{1 \leqslant r < s \leqslant k} c_r c_s \cdot \frac{1}{2}\gcd(4r, 4s) \\ &= \sum_{q=1}^{k} (q c_q^2 + c_q) + 2 \sum_{1 \leqslant r < s \leqslant k} c_r c_s \gcd(r, s)\,. \end{aligned}$$

For each adjacency, there are two choices. Therefore, the number of labeled self-complementary split graphs with this σ as an antimorphism is 2^P.

The number of distinct permutations of the cycle structure c consisting of c_q cycles of length $4q$ for every $q = 1, 2, \dots, k$ is

$$\frac{(4k)!}{\prod_{q=1}^{k} (4q)^{c_q} \cdot c_q!}\,,$$

and it is the number of possible choices for σ [4]. Let C_{4k} be the set that contains all cycle structures c that satisfy $\sum_{q=1}^{k} c_q \cdot 4q = 4k$. Then the number of antimorphisms with all possible labeled self-complementary split graphs with $4k$ vertices corresponding to each is

$$\sum_{c \in C_{4k}} \frac{(4k!)}{\prod_{q=1}^{k} (4q)^{c_q} \cdot c_q!} 2^P\,. \tag{4}$$

For a graph G with $4k$ vertices, let A_G be the set of automorphisms of G. Then, the number of different labelings of G is $(4k)!/|A_G|$. If G is self-complementary, then the number of antimorphisms of G is equal to the number of automorphisms of G. Let S be the set of all non-isomorphic self-complementary split graphs with $4k$ vertices and let $\lambda_{4k} = |S|$. The number of labeled self-complementary split graphs with all possible antimorphisms corresponding to each is equal to

$$\sum_{G \in S} |A_G| \frac{(4k)!}{|A_G|} = \lambda_{4k}\,(4k)!. \tag{5}$$

Let Equation (4) equal to Equation (5) and we solve for λ_{4k}:

$$\lambda_{4k} = \sum_{c \in C_{4k}} \frac{2^P}{\prod_{q=1}^{k} (4q)^{c_q} \cdot c_q!}\,.$$

In Table 1, we list the number of self-complementary (pseudo-)split graphs on up to 21 vertices.

Table 1: The number of self-complementary (pseudo)-split graphs on n vertices.

n	4	5	8	9	12	13	16	17	20	21
split graphs	1	1	3	3	16	16	218	218	9608	9608
pseudo-split graphs	1	2	3	4	16	19	218	234	9608	9826
all	1	2	10	36	720	5600	703760	11220000	9168331776	293293716992

A Appendix: Proof of Proposition 3.4

We start with a simple observation on potentially self-complementary degree sequences with two different degrees. It can be derived from the characterization of Clapham and Kleitman [5, 3]. Here we provide a direct and simple proof.

Proposition A.1. *If the degree sequence* $(d^{2k}, (4k-1-d)^{2k})$ *is potentially self-complementary, then* $2k \leqslant d \leqslant 3k-1$.

Proof. By the definition of degree sequences, $d > 4k-1-d$. Therefore, $d \geqslant 2k$. Let H be the set of vertices of degree d and L the set of vertices of degree $4k-1-d$. Each vertex in H has at most $|H|-1 = 2k-1$ neighbors in H. Thus, the number of edges between H and L is at least $2k(d-2k+1)$. On the other hand, the number of edges between H and L is at most $2k(4k-1-d)$. Thus, $4k-1-d \geqslant d-2k+1$, and $d \leqslant 3k-1$. □

Indeed, for any pair of positive integers d and k such that $2k \leqslant d \leqslant 3k-1$, we can construct a self-complementary graph $G(d,k)$ with a one-cycle antimorphism

$$(v_1 v_2 \cdots v_{4k}).$$

We set $N(v_1)$ to be

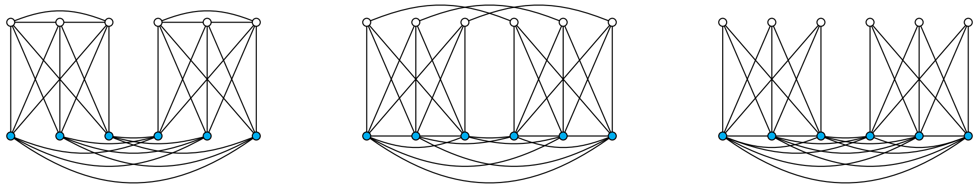
$$\{v_2, v_6, \ldots, v_{4k-2}\} \cup \begin{cases} \{v_3, v_5, \ldots, v_{d-k}\} \cup \{v_{2k+1}\} \cup \{v_{5k-d+2}, v_{5k-d+4}, \ldots, v_{4k-1}\}, & d \not\equiv k \pmod 2 \\ \{v_3, v_5, \ldots, v_{d-k+1}\} \cup \{v_{5k-d+1}, v_{5k-d+3}, \ldots, v_{4k-1}\}, & d \equiv k \pmod 2. \end{cases} \tag{6}$$

Since $v_i v_j \in E(G)$ if and only if $\sigma(v_i)\sigma(v_j) = v_{i+1 \pmod{|V(G)|}} v_{j+1 \pmod{|V(G)|}} \notin E(G)$, all the other adjacencies can be derived. We leave it to the reader to verify that $G(d,k)$ is a self-complementary graph. See Figure 6 for the constructions for $k = 3$ and all possible values of d. Note that (c) is Z_3, while the roles of cliques and independent sets are switched in (a). Let H denote the set of odd-numbered vertices, whose degrees are d, and L the set of even-numbered vertices, whose degrees are $4k-1-d$.

Proposition A.2. *The graph* $G(d,k)$ *is self-complementary in which*

i) $\{v_1, v_5, \ldots, v_{4k-3}\}$ *is complete to* $\{v_2, v_6, \ldots, v_{4k-2}\}$*, and*

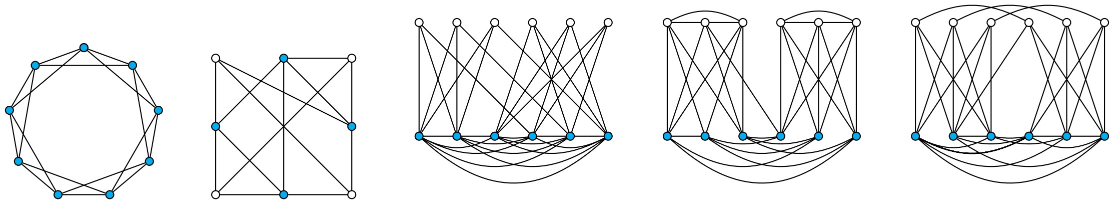
ii) $\{v_3, v_7, \ldots, v_{4k-1}\}$ *is complete to* $\{v_4, v_8, \ldots, v_{4k}\}$.



(a) $d = 6$ (b) $d = 7$ (c) $d = 8$

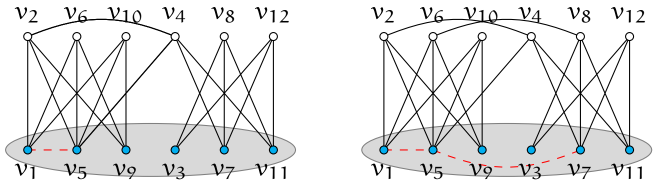
Figure 6: Self-complementary graphs constructed by (6) with $k = 3$.

We now prove Proposition 3.4 by constructions adapted from $G(d,k)$.



(a) (4^9) (b) $(4^4, 3^4)$ (c) $(8^6, 3^6)$ (d) $(6^6, 5^6)$ (e) $(7^6, 4^6)$

Figure 7: Constructions for (i) and (ii) of Proposition 3.4.



(a) $k_1 = 2, k_2 = 1$ (b) $k_1 = 1, k_2 = 2$

Figure 8: Constructions for degree sequences $(8^{2k_1}, 7^{2k_2}, 4^{2k_2}, 3^{2k_1})$. Except for the dashed lines, edges among solid nodes are all present.

Proof of Proposition 3.4. The statement holds vacuously if the degree sequence is not potentially self-complementary. Henceforth, we assume that they are.

(i) We start from a cycle graph on $4k+1$ vertices, and add an edge between every pair of vertices with distance at most k on this cycle. The resulting graph is denoted as C^k_{4k+1}. As an example, C^2_9, the graph for $k = 2$, is in Figure 7a. To see that the graph C^k_{4k+1} is not self-complementary, note that for any vertex v, there are $3k(k-1)/2$ edges among $N(v)$ and $k(k-1)/2$ missing edges among $V(G) \setminus N[v]$.

(ii) By Proposition A.1, we have $2k \leqslant d \leqslant 3k-1$. The graph in Figure 7b has degree sequence $(4^4, 3^4)$ and is not self-complementary. In the rest, $k \geqslant 3$.

Case 1: $d = 3k-1$. We take a Z_k, whose degree sequence is $(d^{2k}, (4k-1-d)^{2k}) = (d^{2k}, k^{2k})$. Let $\{u_1, \ldots, u_{2k}\} \uplus \{v_1, \ldots, v_{2k}\}$ be the split partition. For $i = 1, \ldots, k$, we conduct $(u_k v_i, u_{k+i} v_{k+i}) \to (u_k v_{k+i}, u_{k+i} v_i)$, and denote by G the resulting graph. Then

$$N[u_i] = \begin{cases} K \cup \{v_1, \ldots, v_k\}, & i = 1, \ldots, k-1, \\ K \cup \{v_{k+1}, \ldots, v_{2k}\}, & i = k, \\ K \cup \{v_{i-k}, v_{k+1}, \ldots, v_{2k}\} \setminus \{v_i\}, & i = k+1, \ldots, 2k. \end{cases}$$

See Figure 7c for the construction of $k = 3$. An antimorphism of G, if one exists, must map u_1 and u_2 to two vertices with the same open neighborhood. But $N(v_i) \neq N(v_i)$ for all i, j with $1 \leqslant i < j \leqslant k$. Thus, G is not self-complementary.

Case 2: $d < 3k-1$ and $d+k$ is odd. We start from $G(d, k)$. Note that v_1 is adjacent to v_{2k+1} but not to v_{2k-1}: from $d < 3k-1$ it follows that $d-k+1 < 2k$ and $5k-d+1 > 2k+2$. From $v_1 v_2 \in E(G)$ we can conclude that $v_{2k-1} v_{2k} \in E(G)$ and $v_{2k} v_{2k+1} \notin E(G)$. We conduct the 2-switch $(v_1 v_{2k+1}, v_{2k-1} v_{2k}) \to (v_1 v_{2k-1}, v_{2k} v_{2k+1})$, and denote by G' the resulting graph. See Figure 7d for the example with $d = 6$ and $k = 3$. Suppose that G' is a self-complementary graph. Any antimorphism σ of G' must map vertices v_{2k-1} and v_{2k+1} to L. Since they have different numbers of neighbors in L, they must be mapped to vertices in L with different numbers of neighbors in H. There are however no such two vertices. We end with a contradiction.

Case 3: $d < 3k-1$ and $d+k$ is even. We start from $G(d, k)$. Since $d-k+1 < 2k+1 < 5k-d+1$, the vertex v_1 is not adjacent to v_{2k+1}. From our construction, we know that $d-k$ is even and v_1 is adjacent to v_{d-k+1} and not adjacent to v_{d-k+3}. The fact that v_1 is adjacent to v_2 and not adjacent to v_4 implies v_{d-k+3} is adjacent to v_{d-k+4} and v_{d-k+1} is not adjacent to v_{d-k+4}. By conducting the 2-switch $(v_1 v_{d-k+1}, v_{d-k+3} v_{d-k+4}) \to (v_1 v_{d-k+3}, v_{d-k+1} v_{d-k+4})$. See Figure 7e for the example with $d = 7$ and $k = 3$. Note that vertices v_{d-k+1} and v_{d-k+3} have different numbers of neighbors in L. The argument is similar to the previous case.

(iii) We use τ to denote the degree sequence $(d^{2k_1}, (d-1)^{2k_2}, (4k-d)^{2k_2}, (4k-1-d)^{2k_1})$. By Theorem 3.2,

$$k_1 d + k_2(d-1) \leqslant (k_1+k_2)(4k-1-(k_1+k_2)) = k(3k-1).$$

Therefore,

$$d \leqslant 3k - 1 + \frac{k_2}{k} < 3k - 1.$$

We can use (6) to construct a realization G of $(d^{2k}, (4k-1-d)^{2k})$. Since $d > d-1 > 4k-d > 4k-1-d$, we have $d-(4k-1-d) \geqslant 3$. Thus, $d \geqslant 2k+2$ and $k \geqslant 3$ (by Proposition A.1). By construction, $v_1v_5 \in E(G)$. Note that for all $i = 0, 1, 2, \ldots, k-1$ and for all $x \in N_G(v_1)$, we have

$$\sigma^{4i}(v_1)\sigma^{4i}(x) \in E(G), \quad \sigma^{4i+1}(v_1)\sigma^{4i+1}(x) \notin E(G).$$

Let G' denote the graph obtained from G by removing all the edges $\sigma^{4i}(v_1)\sigma^{4i}(v_3)$ and adding edges $\sigma^{4i+1}(v_1)\sigma^{4i+1}(v_3)$ for all $i = 0, 1, 2, \ldots, k_2 - 1$. The degree of a vertex v_i in G' is

$$\begin{cases} d & i = 4k_2+1, 4k_2+3, \ldots, 4k-1, \\ d-1 & i = 1, 3, \ldots, 4k_2-1, \\ 4k-d & i = 2, 4, \ldots, 4k_2, \\ 4k-1-d & i = 4k_2+2, 4k_2+4, \ldots, 4k. \end{cases}$$

Thus, G' is a realization of the degree sequence τ. In G', the vertex v_1 is adjacent to v_5 and not to v_3, while v_4 is adjacent to v_3 and not to v_5. We conduct the 2-switch $(v_1v_5, v_3v_4) \to (v_1v_3, v_4v_5)$ and denote by G'' the resulting graph. The degree sequence of G'' is also τ by Lemma 3.1. See Figure 8 for illustrations.

We argue that in all the three graphs, each vertex in L has at least k neighbors in H in G. By Proposition A.2, each vertex $x \in L$ has at least k neighbors in H in G. This remains true in G' because the modifications from G to G' involve only edges in H and in L. During the modification from G' to G'', the vertex v_4 loses a neighbor in H and gets a new neighbor in H.

If G'' is a self-complementary graph, any antimorphism σ' of G'' maps H to L and vice versa. Thus, each vertex in H must have at least k non-neighbors in L. However, $\{v_4\} \cup \{v_2, v_6, \ldots, v_{4k-2}\} \subseteq N(v_5) \cap L$, a contradiction. $\square$

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