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Relative Regularity Conditions and Linear Regularity Properties for Split Feasibility Problems in Normed Linear Spaces

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Abstract: The bounded linear regularity property plays a key role in the study of the strong convergence and/or convergence rate of the CQ algorithm for solving split feasibility problems. To establish some sufficient conditions ensuring the bounded linear regularity property for split feasibility problems in normed linear spaces, we introduce the notion of a relative regularity condition and its associated relative regularity constant in spirit of the regularity condition used in Burke and Ferris (Math. Program. 71: 179-194, 1995). Based on convex analysis techniques, we explore equivalent characterizations of the relative regularity condition, which in particular extend the classical results in Burke and Ferris (Math. Program. 71: 179-194, 1995) from the Euclidean space to general normed linear spaces, and then establish some important and useful properties in terms of the related relative regularity constant. Consequently, we develop a new technique to establish some sufficient conditions ensuring the bounded linear regularity property for split feasibility problems in normed linear spaces. The sufficient conditions presented in this paper are in terms of the relative regularity constant, which seem completely new. Applied to the case of Hilbert spaces, our results extend and improve the corresponding ones in Wang et al. (Inverse Probl. 33: 055017, 2017) by relaxing the relevant assumptions.

Keywords: Relative regularity condition, Linear regularity, Split feasibility problem, Normed linear spaces

1 Introduction

Let X and Y be normed linear spaces, and let T be a bounded linear operator from X to Y. Consider the following split feasibility problem (SFP): find a point $x \in X$ such that

$$x \in C \quad \text{and} \quad Tx \in Q, \tag{1}$$

where C and Q are arbitrary nonempty closed and convex subsets of X and Y, respectively. Throughout the whole paper, we always assume that the solution set S of (1) is nonempty, that is,

$$S := C \cap T^{-1}Q \neq \emptyset.$$

In the special case when X and Y are Euclidean spaces or Hilbert spaces, the SFP was introduced by Censor and Elfving in [15], which provides a unified framework for many inverse problems and has been used in various areas such as signal processing [14], image reconstruction [33] and intensity-modulated radiation therapy [16, 17].

One of the most popular and practical algorithms to solve the SFP is the CQ algorithm, which is an application of the subgradient method introduced in [28] for solving the optimization problem:

$$\min_{x \in C} f(x) \tag{2}$$

where $f : X \to \mathbb{R}$ is given by $f(x) := \mathrm{d}^2(Tx, Q)$ for each $x \in X$, and can also be regarded as a special case of the project gradient method for solving (2) (cf. [22]). The CQ algorithm has also been studied in [13] for finite-dimensional spaces, and in [27, 36] for Hilbert spaces where it was shown to be weakly convergent in general. In [32], the authors considered the SFP in Banach spaces and generalized the CQ algorithm with Bregman projections to obtain an iterative algorithm in Banach spaces to solve the SFP. This generalized CQ algorithm has been modified in [34] to solve the multiple-sets SFP in Banach spaces, where the strong convergence with respect to Bregman distance of the modified algorithm was established. In [4], the authors constructed an iterative method to find the Bregman projection of a point onto a countable intersection of closed convex sets in a reflexive Banach space. The iterative method is different from the one given in [34], but the problem of finding the Bregman projection of a point onto a countable intersection of closed convex sets in a reflexive Banach space covers the multiple-sets SFP in Banach spaces. To study the strong convergence and/or convergence rate of the CQ algorithm in Hilbert spaces, Wang *et al.* [35] introduced a bounded linear regularity property for the SFP and then used this property to establish a linear convergence result for the CQ algorithm. Thus how to verify that the SFP has the bounded linear regularity property is an interesting problem. Some sufficient conditions based on the interior/polyhedron conditions ensuring the bounded linear regularity property for the SFP in Hilbert spaces were provided in [35, Proposition 2.5].

The main interests of the present paper are focused on the issue of the bounded linear regularity property for the SFP (1) in general normed linear spaces. In spirit of the regularity condition used in [12, Definition 3.1 and Lemma 3.2], we introduce the notion of a relative regularity condition, together with its related relative regularity constant. Based on convex analysis techniques, we explore equivalent characterizations of the relative regularity condition, which in particular extend the classical results in [12, Lemma 3.2] from the Euclidean space to general normed linear spaces, and then establish some important and useful properties in terms of the related relative regularity constant. Consequently, we develop a new technique, completely different from that of [35] and others, to derive some sufficient conditions ensuring the bounded linear regularity property for SFP (1). The sufficient conditions presented in this paper are in terms of the relative regularity constant, which seem completely new. Applied to the case when X and Y are Hilbert spaces, our results (see Theorem 4.3 and Corollary 4.4) extend and improve the corresponding ones in [35, Proposition 2.5]. In particular, the polyhedron C in (i) of [35, Proposition 2.5] is relaxed to a generalized polyhedron (see Theorem 4.3); while in the case when $R(T)$ (the range of T) is complete, the polyhedron Q in (iii) of [35, Proposition 2.5] is weakened to a generalized polyhedron (see Corollary 4.4).

The remainder of the paper is organized as follows. As usual, some notations and auxiliary results are presented in the next section, where some useful sufficient criteria for the bounded linear regularity for two closed and convex subsets are also presented. In Sec. 3, we introduce the definition of a relative regularity condition and its associated relative regularity constant, together with some related useful results. The main results about sufficient conditions ensuring the bounded linear regularity property for SFP (1) in normed linear spaces are provided in Sec. 4. Conclusions are provided in the last section.

2 Notations and Preliminaries

The notations used here are standard. As usual, we use $X, Y, Z, \cdots$ to stand for normed linear spaces. The topological dual of normed linear space X is denoted by X^*; while the dual pairing between X and X^* is denoted by $\langle \cdot, \cdot \rangle$. Let $\mathbf{B}(x, r)$ and $\mathbf{U}(x, r)$ denote respectively the closed and open ball in X with center at x and radius r. In particular, we use $\mathbf{B}$ and $\mathbf{B}^*$ to denote the unit ball in X and X^*, respectively. Let A be a subset of X. The interior (resp., closure, convex hull, closed convex hull, affine hull, closed affine hull, closed conical hull and boundary) of A is denoted by intA (resp., $\overline{A}$, coA, $\overline{\text{co}}$A, affA, $\overline{\text{aff}}$A, $\overline{\text{cone}}$A and bdA). The conical hull of A is denoted by cone$A$ and defined by cone$A := \{\lambda a : a \in A, \lambda \geq 0\}$. Similar notations will also be employed for subsets of X^* but with respect to the weak*-topology.

Let $H \subseteq X$. Recall from [37, P. 2] that the relative interior of A with respect to H is denoted by $\text{int}_H A$ and defined by

$$\text{int}_H A := \{x \in A : \text{ there exists } \delta > 0 \text{ such that } \mathbf{B}(x, \delta) \cap H \subseteq A\}.$$

Following [7, Definition 2.1], the relative interior of A is denoted by ri$A$ and defined by $\text{ri}A := \text{int}_{\overline{\text{aff}}A} A$. Note that a point $\bar{x} \in \text{ri}A$ if and only if aff$A$ is closed and $\bar{x} \in \text{int}_{\text{aff}A} A$. In fact, let $\bar{x} \in \text{ri}A$. Then there exists $r > 0$ such that $\mathbf{B}(\bar{x}, r) \cap \overline{\text{aff}}A \subseteq A$. Let $\bar{y} \in \overline{\text{aff}}A$. Thus, there exits $t \in (0, 1)$ such that $z = t\bar{x} + (1 - t)\bar{y} \in \mathbf{B}(\bar{x}, r)$ and so $z \in A$. Note that $\bar{y} = \frac{1}{1-t} z + \frac{-t}{1-t} \bar{x}$. Therefore, $\bar{y} \in \text{aff}A$ and so aff$A$ is closed.

Let A be a nonempty subset of X. The polar set and the dual cone of A are defined respectively by

$$A^\circ := \{x^* \in X^* : \langle x^*, x \rangle \leq 1 \text{ for each } x \in A\}$$

and

$$A^{\ominus} := \{x^* \in X^* : \langle x^*, x\rangle \le 0 \text{ for each } x \in A\}.$$

Let $x_0 \in X$. The normal cone of a convex subset A at x_0 is the set

$$N_A(x_0) := \{x^* \in X^* : \langle x^*, x - x_0\rangle \le 0 \text{ for each } x \in A\} \;\text{ if } x_0 \in A,$$

and $N_A(x_0) := \emptyset$ if $x_0 \notin A$. The indicator function δ_A, the support function σ_A and the distance funtion $\mathrm{d}(\cdot, A)$ are respectively defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & \text{otherwise}, \end{cases}$$

$$\sigma_A(x^*) := \sup_{x\in A}\langle x^*, x\rangle \quad \text{ for each } x^* \in X^*$$

and

$$\mathrm{d}(x, A) := \inf_{z\in A} \|x - z\| \quad \text{ for each } x \in X.$$

Note that $\mathrm{d}(x, A) = 0$ if and only if $x \in \overline{A}$. The distance from A to the origin is simply denoted by

$$\|A\| := \inf\{\|x\| : \; x \in A\}. \tag{3}$$

Recall that A is a polyhedron if there exist p pairs $(x_i^*, \alpha_i) \in X^* \times \mathbb{R}$ such that

$$A = \{x \in X : \langle x_i^*, x\rangle \le \alpha_i \;\text{ for each } i = 1, 2, \cdots, p\}.$$

Recall also that A is a generalized polyhedron if there exist a closed affine subspace $A_L \subseteq X$ and a polyhedron A_P such that

$$A = A_L \cap A_P$$

(see e.g., [6, P. 133]). Such a pair $\{A_L, A_P\}$ is called a polyhedron-linear space decomposition (P-L decomposition, for short) of the generalized polyhedron A.

The following lemma lists some useful facts about polars and interiors, where (i) and (ii) can be found in [24, Lemma 2.1], (iii) refers to [29, Lemma 1]; (iv) is a direct consequence of [1, Lemma 2.4].

Lemma 2.1. *Let $A, B \subset Y$. Then the following assertions hold:*

(i) *If $0 \in A \cap B$, then $\overline{\mathrm{co}}A \subseteq \overline{\mathrm{co}}B \Leftrightarrow B^\circ \subseteq A^\circ$.*

(ii) *If $0 \in A \cap B$ and A is a cone, then $(A + B)^\circ = A^\circ \cap B^\circ$.*

(iii) *If B is convex and $E \subseteq Y$ is bounded and $A + E \subseteq B + E$, then $A \subseteq \overline{B}$.*

(iv) *If A and B are convex, and $B \subseteq \overline{A}$, then $(1 - \varepsilon)B \subseteq A$ for each $\varepsilon \in (0, 1)$.*

Let $J : X \rightrightarrows Y$ be a multifunction. As usual, the domain, norm and the inverse of J are respectively denoted by $\mathrm{D}(J)$, $\|J\|$ and $J^{-1} : Y \rightrightarrows X$, and defined by

$$\mathrm{D}(J) := \{x \in X : J(x) \neq \emptyset\}, \quad \|J\| := \sup\{\|J(x)\| : x \in \mathrm{D}(J) \cap \mathbf{B}\}$$

(noting that $\|J(x)\| = \inf\{\|y\| : \; y \in J(x)\}$ by (3)), and

$$J^{-1}(y) := \{x \in X : y \in J(x)\} \quad \text{for each } y \in Y.$$

Let $H \subseteq X$. We use JH to denote the image of J on H, that is, $JH := \bigcup\{J(x) : x \in H\}$. In particular, the image of J on X is denoted by $\mathrm{R}(J)$ and defined by $\mathrm{R}(J) := JX$. Following Rockafellar [30,31], we say that J is normed if $\|J\| < +\infty$. Recall from [31, p.413] that J is called a convex process from X to Y if it satisfies that $0 \in J(0)$, $J(\lambda x) = \lambda J(x)$ for all $\lambda > 0, x \in X$, and $J(x + y) \supseteq J(x) + J(y)$ for all $x, y \in X$.

We use $\mathcal{L}(X, Y)$ to denote the space containing all the continuous linear operators from X to Y endowed with standard operator norm. The kernel of T is denoted by $\ker T$, and defined by $\ker T := \{x \in X : \; Tx = 0\}$. Let $T \in \mathcal{L}(X, Y)$ and let $K \subseteq X$ be a nonempty, closed and convex cone. We use $T_K : X \rightrightarrows Y$ to denote the convex process defined by

$$T_K(x) := Tx - K \quad \text{for each } x \in X. \tag{4}$$

Note that if $R(T_K)$ is a complete linear subspace, then T_K^{-1} is normed (cf. [25, Proposition 2.2 (ii)]). In particular, if $R(T)$ is complete, then T^{-1} is normed.

Let V be a subspace of X. For a map $h : X \to Y$, we use $h|_V : V \to Y$ to denote the restriction of h on V. Consider a proper convex function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ with its domain denoted by $\mathrm{dom}(f) := \{x \in X : f(x) < +\infty\}$. The subdifferential of f at $x_0 \in \mathrm{dom}\, f$ is defined by

$$\partial f(x_0) := \{x^* \in X^* : \langle x^*, x - x_0\rangle \leq f(x) - f(x_0) \text{ for each } x \in X\}.$$

Let A be a convex subset of X. We have that

$$\partial \delta_A(x) = N_A(x) \quad \text{for each } x \in A$$

and

$$\partial \mathrm{d}(x, A) = N_A(x) \cap \mathbf{B}^* \quad \text{for each } x \in \overline{A}. \tag{5}$$

We need the following chain rule of subdifferentials which is known in [19] for Banach space setting and a direct consequence of [37, Theorem 2.8.3 (iii)] for normed linear space setting.

Lemma 2.2. *Let $T \in \mathcal{L}(X, Y)$ and let $f : Y \to \overline{\mathbb{R}}$ be a proper convex function. Then for each $\bar{x} \in X$ such that f is locally Lipschitz continuous at $T\bar{x}$, one has*

$$\partial(f \circ T)(\bar{x}) = T^* \partial f(T\bar{x}). \tag{6}$$

The following useful proposition is about separation between a convex subset and the origin $\{0\}$ in normed linear space setting.

Proposition 2.1. *Let $D \subseteq X$ be a convex subset such that $0 \notin D$ and $\mathrm{ri}D \neq \emptyset$. Then there exists $y^* \in X^* \setminus \{0\}$ such that*

$$\langle y^*, y\rangle \leq 0 \quad \textit{for each } y \in D. \tag{7}$$

Proof. We first consider the case when $0 \in \overline{\mathrm{aff}}D$. Then $\{0\}, D \subseteq \overline{\mathrm{aff}D}$, $\overline{\mathrm{aff}D} = \overline{\mathrm{span}}D$ and $\mathrm{int}_{\overline{\mathrm{span}}D}D = \mathrm{ri}D \neq \emptyset$. We apply the separation theorem [37, Theorem 1.1.3] to $D, \{0\}, \overline{\mathrm{span}}D$ in place of A, B, X to get that there exists $x^* \in (\overline{\mathrm{span}}D)^* \setminus \{0\}$ such that

$$\langle x^*, y\rangle \leq 0 \quad \text{for each} \quad y \in D.$$

By Hahn-Banach Theorem (cf. [18, Corollary 6.5, page 79]), there exists $y^* \in X^*$ such that $y^*(x) = x^*(x)$ for each $x \in \overline{\mathrm{aff}}D$; hence (7) is seen to hold. Next we consider the case when $0 \notin \overline{\mathrm{aff}D}$. Then there exists $r_0 > 0$ such that $r_0\mathbf{B} \cap \overline{\mathrm{aff}D} = \emptyset$. By the separation theorem [37, Theorem 1.1.3] (applied to $r_0\mathbf{B}, \overline{\mathrm{aff}D}$ in place of A, B), there exists $y^* \in X^* \setminus \{0\}$ such that

$$\langle y^*, y\rangle \leq 0 \quad \text{for each} \quad y \in D.$$

The proof is completed. □

Next, we provide the the notion of linear regularity property for two closed and convex subsets, and its some relatively useful propositions. The linear regularity property plays a key role in various branches of convex optimization, such as convex feasibility problem, constrained approximation and systems of convex inequalities (see [2, 3, 5] and references therein). We recall from [2] the following definitions of the linear regularity and bounded linear regularity for two closed and convex subsets of X.

Definition 2.1. *Let E and F be closed and convex subsets of X such that $E \cap F \neq \emptyset$. The pair $\{E, F\}$ is said to have*

(a) *the linear regularity property if there exists $\alpha > 0$ such that*

$$\mathrm{d}(x, E \cap F) \leq \alpha \max\{\mathrm{d}(x, E), \mathrm{d}(x, F)\} \quad \textit{for each } x \in X;$$

(b) *the bounded linear regularity property if for any $r > 0$ there exists $\alpha_r > 0$ such that*

$$\mathrm{d}(x, E \cap F) \leq \alpha_r \max\{\mathrm{d}(x, E), \mathrm{d}(x, F)\} \quad \textit{for each } x \in r\mathbf{B}.$$

We will need the following propositions about the bounded linear regularity property. Proposition 2.2 is known in [38, Proposition 3.1, Theorem 3.1 and Proposition 3.6] for the Hilbert space setting. The arguments used there can be modified to work for the normed linear space setting, and so the proof of Proposition 2.2 is omitted here.

Proposition 2.2. *Let E and F be closed and convex subsets of a normed linear space X such that $E \cap F \neq \emptyset$. Then the pair $\{E, F\}$ has the bounded linear regularity property provided one of the following conditions holds:*

(a) *E and F are polyhedrons;*

(b) $E \cap \text{int}F \neq \emptyset$;

(c) $\text{ri}E \cap F \neq \emptyset$ *and* F *is a polyhedron;*

(d) $\text{ri}E \cap \text{ri}F \neq \emptyset$ *and* F *is finite codimensional;*

(e) *the pair* $\{E, F\}$ *has the bounded linear regularity property on* X_1 *which is a closed and linear subspace of* X.

Proposition 2.3. *Let* C *and* Q *be nonempty closed and convex subsets of* X *and* Y*, respectively. Let* $T \in \mathcal{L}(X, Y)$*. Suppose that* $T(\text{ri}C) \cap Q \neq \emptyset$ *and* $V = \overline{\text{span}}C$*. Then the pair* $\{T^{-1}Q \cap V, C\}$ *has the bounded linear regularity property.*

Proof. By the assumption, $\text{ri}C \cap T^{-1}Q \neq \emptyset$. Without loss of generality, we assume that $0 \in C$. Then, one has that $\text{aff}C = \text{span}C$ and so $\text{ri}C = \text{int}_V C$ (noting $V = \overline{\text{span}}C$). Therefore $T^{-1}Q \cap V \cap \text{int}_V C \neq \emptyset$, and it follows from Proposition 2.2(b) that the pair $\{T^{-1}Q \cap V, C\}$ has the bounded linear regularity property in space V. Then, in view of Proposition 2.2(e), it follows that the pair $\{T^{-1}Q \cap V, C\}$ has the bounded linear regularity property. □

3 Regularity Conditions

Recall that $C \subseteq X$, $Q \subseteq Y$ are closed and convex subsets and $T \in \mathcal{L}(X, Y)$. In spirit of the regularity condition used in [12, Definition 3.1 and Lemma 3.2], we introduce the relative regularity condition in the following definition.

Definition 3.1. *Let* $V \subset X$ *be a closed linear subspace. SFP* (1) *is said to satisfy*

(a) *the relative regularity condition at* z *on* V *if there exist* $\delta > 0$ *and* $\mu > 0$ *such that*

$$\delta \mathbf{B} \cap \overline{TV} \subseteq \mu T(\mathbf{B} \cap V) + Q - z; \tag{8}$$

(b) *the relative regularity condition at* z *if* (8) *holds with* X *in place of* V.

Define the relative regularity constant $\beta_T^V(z)$ at $z \in Y$ by

$$\beta_T^V(z) := \inf\{\frac{\mu}{\delta} : \delta \mathbf{B} \cap \overline{TV} \subseteq \mu T(\mathbf{B} \cap V) + Q - z\},$$

and in particular, in the case when $V = X$, we write for short that

$$\beta_T(z) := \beta_T^X(z) = \inf\{\frac{\mu}{\delta} : \delta \mathbf{B} \cap \overline{R(T)} \subseteq \mu T(\mathbf{B}) + Q - z\}. \tag{9}$$

Then it is clear that SFP (1) satisfies the relative regularity condition at z on V (resp. the relative regularity condition at z) if and only if $\beta_T^V(z) < +\infty$ (resp. $\beta_T(z) < +\infty$).

Throughout the whole paper, we always assume that Q is a closed and convex subset of Y. We recall from [12, p. 183] that the multifunction $\Gamma_Q : Y \rightrightarrows Y^*$ (where the setting is the Euclidean space) is defined by

$$\Gamma_Q(y) := [\text{cone}(Q - y)]^\circ = \{y^* \in Y^* : \langle y^*, y\rangle - \sigma_Q(y^*) \geq 0\}$$

for each $y \in Y$. Clearly, for each $y \in Y$, $\Gamma_Q(y)$ is a closed cone and coincides with the normal cone $N_Q(y)$ if $y \in Q$. Below, we proceed with a key proposition characterizing equivalent conditions for the relative regularity condition, which (applied to Y in place of H) in particular extends the classical results in [12, Lemma 3.2] from the Euclidean space to general normed linear spaces. Let $H \subset Y$ and let $H^\perp := \{y^* \in Y^* : \langle y^*, y\rangle = 0, \ \forall y \in H\}$.

Proposition 3.1. *Let $H \subset Y$ be a closed linear subspace and let $z \in Y$. Let $\delta > 0$ and $\mu > 0$. Consider the following equality/inclusions:*

$$\delta\mathbf{B} \cap H \subseteq \mu T\mathbf{B} + Q - z; \tag{10}$$

$$(T^*)^{-1}(\mathbf{B}^*) \cap \Gamma_Q(z) \subseteq \frac{\mu}{\delta}\mathbf{B}^* + H^\perp; \tag{11}$$

$$\delta\mathbf{B} \cap H \subseteq \mu T\mathbf{B} + \text{cone}(Q - z); \tag{12}$$

$$\overline{R(T) + \text{cone}(Q - z)} \supseteq H; \tag{13}$$

$$\ker T^* \cap \Gamma_Q(z) \subseteq H^\perp. \tag{14}$$

Then the following equivalences/implications hold:

$$(10) \Longrightarrow (11) \Longleftrightarrow (12) \Longrightarrow (13) \Longleftrightarrow (14). \tag{15}$$

Furthermore, if $\text{ri}((\mu_0 T\mathbf{B} + Q - z) \cap H) \neq \emptyset$ *for some* $\mu_0 > 0$ *and*

$$\overline{(R(T) + \text{cone}(Q - z)) \cap H} = \overline{R(T) + \text{cone}(Q - z)} \cap H, \tag{16}$$

then equality/inclusions (10)-(14) *are equivalent with possibly different* $\delta > 0$.

Proof. Note first by definition that

$$(\mu T\mathbf{B})^\circ = (T^*)^{-1}(\mu^{-1}\mathbf{B}^*), \ (R(T))^\circ = \ker T^*,$$

$$(\delta\mathbf{B} \cap H)^\circ = \frac{1}{\delta}\mathbf{B}^* + H^\perp, \ H^\circ = H^\perp \text{ and } (\text{cone}(Q - z))^\circ = \Gamma_Q(z).$$

Note further that $0 \in \mu T\mathbf{B} \cap \text{cone}(Q - z)$. Thus one applies Lemma 2.1(ii) to $\mu T\mathbf{B}, \text{cone}(Q - z)$ in place of A, B to check that

$$(\mu T\mathbf{B} + \text{cone}(Q - z))^\circ = (\mu T\mathbf{B})^\circ \cap (\text{cone}(Q - z))^\circ = (T^*)^{-1}(\mu^{-1}\mathbf{B}^*) \cap \Gamma_Q(z).$$

Similarly, one has that

$$(R(T) + \text{cone}(Q - z))^\circ = (R(T))^\circ \cap (\text{cone}(Q - z))^\circ = \ker T^* \cap \Gamma_Q(z). \tag{17}$$

Hence the equivalences (11) $\iff$ (12) and (13) $\iff$ (14) follow from Lemma 2.1(i) (applied to $\delta\mathbf{B}\cap H, \mu T\mathbf{B}+\mathrm{cone}(Q-z)$ and $H, R(T)+\mathrm{cone}(Q-z)$ in place of A, B, respectively). Since the implication (10) $\Longrightarrow$ (12) holds trivially, it remains to show the implication (12) $\Rightarrow$ (13). To do this, we assume that (12) holds and take the cone hull of both sides of (12) to deduce that $H\subseteq\overline{R(T)+\mathrm{cone}(Q-z)}$ and so (13) is seen to hold. The proof of (15) is completed.

Suppose that $\mathrm{ri}((\mu_0 T\mathbf{B}+Q-z)\cap H)\neq\emptyset$ for some $\mu_0>0$ and (16) holds. To complete the proof, it suffices to show that (13)$\Longrightarrow$(10). To do this, suppose that (13) holds and set

$$\Omega := \bigcup_{\mu>0}\mathrm{ri}((\mu T\mathbf{B}+Q-z)\cap H). \tag{18}$$

Note by definition that

$$\overline{\mathrm{aff}((\mu T\mathbf{B}+Q-z)\cap H)}=\overline{\mathrm{aff}\Omega}\quad \text{for each } \mu>0, \tag{19}$$

because for any $\mu',\mu''>0$, $\mathrm{aff}((\mu' T\mathbf{B}+Q-z)\cap H)=\mathrm{aff}((\mu'' T\mathbf{B}+Q-z)\cap H)$ (noting $0\in T\mathbf{B}$). We claim that $0\in\Omega$. Assuming this holds,

$$H\supseteq\overline{\mathrm{aff}\Omega}=\overline{\mathrm{span}\Omega}\supseteq\overline{(R(T)+\mathrm{cone}(Q-z))\cap H}\supseteq H \tag{20}$$

(due to (16) and (13)) and there exists $\mu>0$ such that $0\in\mathrm{ri}((\mu T\mathbf{B}+Q-z)\cap H)$, that is, there exists $\delta>0$ such that

$$\delta\mathbf{B}\cap\overline{\mathrm{aff}((\mu T\mathbf{B}+Q-z)\cap H)}\subset\mu T\mathbf{B}+Q-z;$$

hence (10) holds by (19) and (20). It remains to show that $0\in\Omega$. To this end, we first note by (19) and the nonempty assumption of $\mathrm{ri}((\mu_0 T\mathbf{B}+Q-z)\cap H)$ that $\mathrm{ri}\Omega\neq\emptyset$. Suppose on the contrary that $0\notin\Omega$. By Proposition 2.1 (applied to Ω, H in place of D, X), there exists $y^*\in H^*\setminus\{0\}$ such that

$$\langle y^*, y\rangle\leq 0\quad\text{for each}\quad y\in\Omega.$$

This implies that

$$\overline{\mathrm{cone}}\Omega\subseteq\{y\in H:\ \langle y^*, y\rangle\leq 0\}. \tag{21}$$

Since H is a closed linear subspace, it follows that

$$\overline{\mathrm{cone}}((\mu T\mathbf{B}+Q-z)\cap H)=\overline{\mathrm{cone}(\mu T\mathbf{B}+Q-z)\cap H}. \tag{22}$$

Note by basic techniques and definitions in convex analysis that for a convex subset $D\subseteq Y$ if $\mathrm{ri}D\neq\emptyset$, then $\overline{\mathrm{ri}D}=\overline{D}$ and so $\overline{\mathrm{cone}}D=\overline{\mathrm{cone}}(\mathrm{ri}D)$. Thus, by the nonempty assumption of $\mathrm{ri}((\mu_0 T\mathbf{B}+Q-z)\cap H)$, one has that

$$\overline{\mathrm{cone}}(\mathrm{ri}((\mu T\mathbf{B}+Q-z)\cap H))=\overline{\mathrm{cone}}((\mu T\mathbf{B}+Q-z)\cap H)\quad\text{for each } \mu>\mu_0.$$

Combining this with (22) yields that

$$\overline{\mathrm{cone}}(\mathrm{ri}((\mu T\mathbf{B}+Q-z)\cap H))=\overline{\mathrm{cone}(\mu T\mathbf{B}+Q-z)\cap H}\quad\text{for each } \mu>\mu_0.$$

Hence, it follows from (18) that

$$\overline{\mathrm{cone}}\Omega\supseteq\bigcup_{\mu>0}\overline{\mathrm{cone}(\mu T\mathbf{B}+Q-z)\cap H}\supseteq\overline{(R(T)+\mathrm{cone}(Q-z))\cap H}.$$

Thus, we have from (13), (16) and (21) that $H\subseteq\{y\in H:\ \langle y^*|_H, y\rangle\leq 0\}$. Noting that H is a linear subspace, one has that $y^*=0$ which contradicts with $y^*\in H^*\setminus\{0\}$. The proof is completed. □

Proposition 3.2. *Let $z \in Y$ and suppose that the pair (τ, K) satisfies that*

$$\begin{aligned} &\tau \geq 0,\ K \text{ is a nonempty closed and convex cone such that} \\ &-K \subseteq \tau T\mathbf{B} + Q - z. \end{aligned} \tag{23}$$

Then, the following two assertions hold:

(i) *If $\|T_K^{-1}\| < +\infty$ and $R(T_K)$ is closed (e.g., $R(T_K)$ is a complete linear subspace), then*

$$\beta_T(z) \leq \tau + \|T_K^{-1}\|. \tag{24}$$

(ii) *If $R(T_K)$ is a closed linear subspace and* $\mathrm{ri}((\mu_0 T\mathbf{B} + Q - z) \cap R(T_K)) \neq \emptyset$ *for some $\mu_0 > 0$, then $\beta_T(z) < +\infty$.*

Proof. (i) Assume that $\|T_K^{-1}\| < +\infty$ and $R(T_K)$ is closed. Now let $\epsilon > 0$ and $y \in \mathbf{B} \cap \mathrm{R}(T_K)$. Then, there exists $x \in T_K^{-1}y$ such that $\|x\| \leq \|T_K^{-1}y\| + \epsilon \leq \|T_K^{-1}\| + \epsilon$, and so

$$y \in T_K x \subseteq T_K[(\|T_K^{-1}\| + \epsilon)\mathbf{B}] = (\|T_K^{-1}\| + \epsilon)T_K\mathbf{B}.$$

Noting that $y \in \mathbf{B} \cap \mathrm{R}(T_K)$ is arbitrary, and letting $\epsilon \to 0$, we deduce that

$$\mathbf{B} \cap R(T_K) \subseteq \|T_K^{-1}\|T_K\mathbf{B} = \|T_K^{-1}\|T\mathbf{B} - K \tag{25}$$

(see the definition T_K in (4)). Note that $\overline{R(T)} \subseteq R(T_K)$ because $R(T_K)$ is closed by assumption. It follows from (23) and (25) that

$$\mathbf{B} \cap \overline{R(T)} \subseteq \|T_K^{-1}\|T\mathbf{B} - K \subseteq (\tau + \|T_K^{-1}\|)T\mathbf{B} + Q - z.$$

Thus, we arrive at (24) by (9), and complete the proof of assertion (i).

(ii) Assume that $R(T_K)$ is a closed linear subspace and $\mathrm{ri}((\mu_0 T\mathbf{B} + Q - z) \cap R(T_K)) \neq \emptyset$ for some $\mu_0 > 0$. Note by (23) that

$$R(T_K) = R(T) - K \subseteq R(T) + \tau T\mathbf{B} + Q - z \subseteq R(T) + \mathrm{cone}(Q - z).$$

We have that (16) and (13) hold automatically with $R(T_K)$ in place of H. Thus, thanks to the assumption that $\mathrm{ri}((\mu_0 T\mathbf{B} + Q - z) \cap R(T_K)) \neq \emptyset$ for some $\mu_0 > 0$, Proposition 3.1 is applicable (to $R(T_K)$ in place of H), and therefore equality/inclusions (10)-(14) are equivalent (with possibly different $\delta > 0$). In particular, there exist $\delta, \mu > 0$ such that (10) holds with $R(T_K)$ in place of H, that is, $\delta\mathbf{B} \cap R(T_K) \subseteq \mu T\mathbf{B} + Q - z$. Hence we have that $\delta\mathbf{B} \cap \overline{R(T)} \subseteq \mu T\mathbf{B} + Q - z$ as $\overline{R(T)} \subseteq \overline{R(T_K)}$ and $R(T_K)$ is closed. By the definition of $\beta_T(z)$ again, we have that $\beta_T(z) < +\infty$, and the proof is complete. □

Corollary 3.1. *Let $z \in R(T)$, and suppose that $R(T)$ is complete. Then, for each $\tau > \|T^{-1}\|\mathrm{d}(z, Q \cap R(T))$, the pair $(\tau, \{0\})$ satisfies (23), and so*

$$\beta_T(z) \leq \|T^{-1}\|(1 + \mathrm{d}(z, Q \cap R(T))).$$

Proof. Since $R(T)$ is complete, one has that $\|T^{-1}\| < \infty$. By Proposition 3.2(i) (applied to $\{0\}$) in place of K), it suffices to prove the first assertion. To do this, note that (25) holds for any pair (τ, K) with K being a nonempty closed and convex cone such that $\|T_K^{-1}\| < +\infty$. Thus, one has by (25) (applied to $\{0\}$ in place of K) that

$$R(T) \cap \mathbf{B} \subseteq \|T^{-1}\| T\mathbf{B}. \tag{26}$$

Let $\tau > \|T^{-1}\| \mathrm{d}(z, Q \cap R(T))$. Then we check by definition (noting that $z \in R(T)$) that

$$(z - Q \cap R(T)) \cap [\frac{\tau}{\|T^{-1}\|}\mathbf{B} \cap R(T)] \neq \emptyset.$$

This, together with (26), implies that

$$\{0\} \subseteq \tau T\mathbf{B} + Q \cap R(T) - z \subseteq \tau T\mathbf{B} + Q - z.$$

This means that the pair $(\tau, \{0\})$ satisfies (23). The proof is complete. □

Corollary 3.2. *Let $z \in Y$, and suppose that $R(T)$ is closed and* $\mathrm{ri}((\mu_0 T\mathbf{B} + Q - z) \cap R(T)) \neq \emptyset$ *for some $\mu_0 > 0$. Then, there exists $\tau > 0$ such that the pair $(\tau, \{0\})$ satisfies* (23)*, and so $\beta_T(z) < +\infty$.*

Proof. As in the proof of Corollary 3.1 (but using Proposition 3.2(ii) instead of Proposition 3.2(i)), we only need to show the first assertion. Note that $R(T) \subseteq R(T) + \mathrm{cone}(Q - z)$ is trivial. Thus, by the arguments for proving assertion (ii) of Proposition 3.2 (applied to $\{0\}$ in place of K, one checks that there exist $\delta, \tau > 0$ such that $\delta\mathbf{B} \cap \overline{R(T)} \subseteq \tau T\mathbf{B} + Q - z$. This clearly implies that the pair $(\tau, \{0\})$ satisfies (23), and the proof is complete. □

Proposition 3.3. *Let H be a closed linear subspace of Y such that $R(T) \subseteq H$. Let $\bar{z} \in H$, $\bar{\delta} > 0$ and $\bar{\mu} > 0$ be such that*

$$\bar{\delta}\mathbf{B} \cap H \subseteq \bar{\mu} T\mathbf{B} + Q - \bar{z}. \tag{27}$$

Let $\bar{R} > 0$ and $\bar{r} > 0$ be such that $\bar{\mu}\bar{R} + \bar{r} \leq \frac{\bar{\delta}}{2}$. Then for any $(\Delta, z) \in \mathcal{L}(X, Y) \times H$ with

$$\|\Delta\| \leq \bar{R}, \quad \|z - \bar{z}\| \leq \bar{r} \quad \textit{and} \quad R(\Delta) \subseteq H, \tag{28}$$

one has that

$$\beta_{T+\Delta}(z) \leq \frac{2\bar{\mu}}{\bar{\delta}}. \tag{29}$$

Proof. By the assumption that $R(T) \subseteq H$, one has that

$$(\bar{\mu} T\mathbf{B} + Q - \bar{z}) \cap H = \bar{\mu} T\mathbf{B} + Q \cap H - \bar{z}$$

and so it follows from (27) that

$$\bar{\delta}\mathbf{B} \cap H \subseteq \bar{\mu} T\mathbf{B} + Q \cap H - \bar{z}. \tag{30}$$

Now fix $(\Delta, z) \in \mathcal{L}(X, Y) \times H$ that satisfies (28). Then

$$\|\Delta\| \leq \bar{R}, \quad R(T + \Delta) \subseteq H \quad \text{and} \quad z - \bar{z} \in \bar{r}\mathbf{B} \cap H.$$

Using this, we check by definition that

$$T\mathbf{B} \subseteq (T + \Delta)\mathbf{B} + \|\Delta\|\mathbf{B} \cap H \subseteq (T + \Delta)\mathbf{B} + \bar{R}\mathbf{B} \cap H;$$

hence

$$\bar{\mu}T\mathbf{B} + Q \cap H - \bar{z} \subseteq \bar{\mu}(T + \Delta)\mathbf{B} + Q \cap H - z + (\bar{\mu}\bar{R} + \bar{r})\mathbf{B} \cap H. \tag{31}$$

Since $\bar{\mu}\bar{R} + \bar{r} \leq \frac{\bar{\delta}}{2}$ by the assumption, it follows from (30) and (31) that

$$\frac{\bar{\delta}}{2}\mathbf{B} \cap H + \frac{\bar{\delta}}{2}\mathbf{B} \cap H \subseteq \bar{\mu}(T + \Delta)\mathbf{B} + Q \cap H - z + \frac{\bar{\delta}}{2}\mathbf{B} \cap H$$

(noting that $\frac{\bar{\delta}}{2}\mathbf{B} \cap H + \frac{\bar{\delta}}{2}\mathbf{B} \cap H = \bar{\delta}\mathbf{B} \cap H$). Now applying Lemma 2.1(iii) (to $\frac{\bar{\delta}}{2}\mathbf{B} \cap H, \bar{\mu}(T + \Delta)\mathbf{B} + Q \cap H - z, \frac{\bar{\delta}}{2}\mathbf{B} \cap H$ in place of A, B, E), we deduce that

$$\frac{\bar{\delta}}{2}\mathbf{B} \cap H \subseteq \overline{\bar{\mu}(T + \Delta)\mathbf{B} + Q \cap H - z}.$$

Thus applying Lemma 2.1(iv) (with $\bar{\mu}(T + \Delta)\mathbf{B} + Q \cap H - z$, $\frac{\bar{\delta}}{2}\mathbf{B} \cap H$ in place of A,B), we deduce that

$$\frac{\bar{\delta}}{2}(1 - \varepsilon)\mathbf{B} \cap H \subseteq \bar{\mu}(T + \Delta)\mathbf{B} + Q \cap H - z \quad \forall \varepsilon \in (0, 1). \tag{32}$$

Noting that $\overline{R((T + \Delta))} \subseteq H$, one has from (32) that

$$\frac{\bar{\delta}}{2}(1 - \varepsilon)\mathbf{B} \cap \overline{R((T + \Delta))} \subseteq \bar{\mu}(T + \Delta)\mathbf{B} + Q \cap H - z \subseteq \bar{\mu}(T + \Delta)\mathbf{B} + Q - z \quad \forall \varepsilon \in (0, 1).$$

Then, applying (9) to $T + \Delta$ in place of T, one has that

$$\beta_{T+\Delta}(z) \leq \frac{2\bar{\mu}}{\bar{\delta}(1 - \varepsilon)} \quad \forall \varepsilon \in (0, 1).$$

Since $\varepsilon \in (0, 1)$ is arbitrary, (29) is seen to hold. The proof is completed. □

The following proposition establishes a useful relationship between $T^*(\Gamma_Q(z) \cap \mathbf{B}^*)$ and $T^*(\Gamma_Q(z)) \cap \mathbf{B}^*$.

Proposition 3.4. *Let $z \in Y$. Then*

(i)
$$T^*(\Gamma_Q(z) \cap \mathbf{B}^*) \subseteq T^*(\Gamma_Q(z)) \cap \|T\|\mathbf{B}^*; \tag{33}$$

(ii) *if $z \in \overline{R(T)}$ and $\beta_T(z) < \infty$, then*

$$(T^*\Gamma_{Q\cap\overline{R(T)}}(z)) \cap \mathbf{B}^* \subseteq T^*(\Gamma_{Q\cap\overline{R(T)}}(z) \cap \beta_T(z)\mathbf{B}^*).$$

Proof. (i) Inclusion (33) is trivial as $\|T^*y^*\| \le \|T^*\| = \|T\|$ for any $y^* \in \Gamma_Q(z) \cap \mathbf{B}^*$.

(ii) Since $\beta_T(z) < \infty$, for any $\epsilon > 0$, there exist $\delta > 0$ and $\mu > 0$ such that

$$\frac{\mu}{\delta} < \beta_T(z) + \epsilon \quad \text{and} \quad \delta\mathbf{B} \cap \overline{R(T)} \subset \mu T\mathbf{B} + Q \cap \overline{R(T)} - z. \tag{34}$$

Set $Y_0 := \overline{R(T)}$ for simplicity, and consider the operator $T_0 : X \to Y_0$ be defined by $T_0 x := Tx$ for each $x \in X$. Let $\Gamma_Q^{Y_0} : Y_0 \rightrightarrows Y_0^*$ be defined by

$$\Gamma_Q^{Y_0}(y) := [\text{cone}(Q - y)]^\circ = \{y^* \in Y_0^* : \langle y^*, y\rangle - \sigma_Q(y^*) \ge 0\}$$

for each $y \in Y_0$. Then, by (34), we apply the implication (10) $\Longrightarrow$ (11) in Proposition 3.1 (with $T_0, Y_0, Y_0, Q \cap Y_0$ in place of T, Y, H, Q) to conclude that

$$(T_0^*)^{-1}(\mathbf{B}^*) \cap \Gamma^{Y_0}_{Q\cap\overline{R(T)}}(z) \subseteq \frac{\mu}{\delta}\mathbf{B}^* + Y_0^\perp = \frac{\mu}{\delta}\mathbf{B}^*, \tag{35}$$

where the equality holds because $Y_0^\perp = \{0\}$ on Y_0. Noting that for any $(x^*, y^*) \in X^* \times Y^*$ with $T^*y^* = x^*$, one has that $x^* = T_0^*(y^*|_{Y_0})$ and

$$\Gamma_{Q\cap\overline{R(T)}}(z)|_{Y_0} := \{y^*|_{Y_0} \in Y_0^* : y^* \in \Gamma_{Q\cap\overline{R(T)}}(z)\} \subseteq \Gamma^{Y_0}_{Q\cap\overline{R(T)}}(z).$$

Hence, it follows that

$$\Big((T^*)^{-1}\mathbf{B}^* \cap \Gamma_{Q\cap\overline{R(T)}}(z)\Big)|_{Y_0} \subseteq (T_0^*)^{-1}(\mathbf{B}^*) \cap \Gamma^{Y_0}_{Q\cap\overline{R(T)}}(z).$$

This, together with (35), implies that

$$\Big((T^*)^{-1}\mathbf{B}^* \cap \Gamma_{Q\cap\overline{R(T)}}(z)\Big)|_{Y_0} \subseteq \frac{\mu}{\delta}\mathbf{B}^*. \tag{36}$$

Let $x^* \in T^*\Gamma_{Q\cap\overline{R(T)}}(z) \cap \mathbf{B}^*$. By definition, there exists $y^* \in (T^*)^{-1}\mathbf{B}^* \cap \Gamma_{Q\cap\overline{R(T)}}(z)$ such that $x^* = T^*y^*$. Hence, it follows from (36) that $y^*|_{Y_0} \in \frac{\mu}{\delta}\mathbf{B}^*$ and so $\|y^*|_{Y_0}\| \le \frac{\mu}{\delta}$. This, together with (34), implies that $\|y^*|_{Y_0}\| \le \beta_T(z) + \epsilon$. Since $\epsilon > 0$ is arbitrary, one has that $\|y^*|_{Y_0}\| \le \beta_T(z)$. By Hahn-Banach theorem (cf. [18, Corollary 6.5, page 79]), there exists $\hat{y}^* \in Y^*$ such that $\hat{y}^*|_{Y_0} = y^*|_{Y_0}$ and $\|\hat{y}^*\|| = \|y^*|_{Y_0}\|$. Consequently, one can check that $\hat{y}^* \in \Gamma_{Q\cap\overline{R(T)}}(z) \cap \beta_T(z)\mathbf{B}^*$ and $x^* = T^*\hat{y}^*$. The proof is completed. □

The notion of the global weak sharp minima for optimization problem introduced by Ferris in [21], as well as the notions of boundedly weak sharp minima and local weak sharp minima introduced by Burke and Deng in [9], has been extensively studied and widely applied in the sensitivity and convergence analysis of many optimization algorithms; see [10–12] and references therein. For the proofs of Proposition 3.5 below and Theorem 4.1 in the next section, we need the following lemma regarding some useful facts relative to the notion of weak sharp minima; see [26, Theorem 3.8 and Proposition 3.5]. Let $f : X \to \overline{\mathbb{R}}$ be a proper convex and lower semicontinuous function and we use $\bar{S}_f := \text{argmin}_{x\in X} f(x)$ to denote the set of all minimizers of f.

Lemma 3.1. *Let $\alpha > 0, r > 0$ and $x_0 \in \bar{S}_f$. Suppose that $f(x_0) = 0$ and consider the following inequalities/inclusion:*

$$\mathrm{d}(x, \bar{S}_f) \leq \alpha f(x) \quad \text{for each } x \in \mathbf{B}(x_0, r); \tag{37}$$

$$\mathbf{B}^* \cap N_{\bar{S}_f}(z) \subseteq \alpha \partial f(z) \quad \text{for each } z \in \bar{S}_f \cap \mathbf{B}(x_0, r); \tag{38}$$

$$\mathrm{d}(x, \bar{S}_f) \leq 2\alpha f(x) \quad \text{for each } x \in \mathbf{B}(x_0, \frac{r}{2}). \tag{39}$$

Then (37) $\Longrightarrow$ (38) $\Longrightarrow$ (39).

For convenience, let f be the function defined by

$$f(x) := \mathrm{d}(T(x), Q \cap \overline{R(T)}) \quad \text{for each } x \in X \tag{40}$$

(and so $f(x) = 0$ for each $x \in \bar{S}_f$). Then

$$\bar{S}_f = T^{-1}(Q \cap \overline{R(T)}) = T^{-1}Q \ \text{ and } \ \partial f(x) = T^*(N_{Q\cap\overline{R(T)}}(Tx) \cap \mathbf{B}^*) \tag{41}$$

for each $x \in T^{-1}Q$ (due to (6) and (5)).

Proposition 3.5. *Let $x \in T^{-1}Q$. Then*

$$T^*N_Q(Tx) \subseteq T^*N_{Q\cap\overline{R(T)}}(Tx) \subseteq \overline{T^*N_{Q\cap\overline{R(T)}}(Tx)} \subseteq N_{T^{-1}Q}(x). \tag{42}$$

Suppose further that there exists $\bar{z} \in R(T)$ such that $\beta_T(\bar{z}) < +\infty$. Then,

$$N_{T^{-1}Q}(x) = \overline{T^*N_{Q\cap\overline{R(T)}}(Tx)} = T^*N_{Q\cap\overline{R(T)}}(Tx). \tag{43}$$

Proof. To show (42), we only need to show the last inclusion as the others are trivial. Thus, it suffices to verify that $T^*N_{Q\cap\overline{R(T)}}(Tx) \subseteq N_{T^{-1}Q}(x)$ as $N_{T^{-1}Q}(x)$ is w*-closed. To do this, let $x^* := T^*y^* \in T^*N_{Q\cap\overline{R(T)}}(Tx)$ with $y^* \in N_{Q\cap\overline{R(T)}}(Tx)$. Then, for each $x' \in T^{-1}Q$, one has that

$$\langle x^*, x' - x\rangle = \langle T^*y^*, x' - x\rangle = \langle y^*, Tx' - Tx\rangle \leq 0$$

(as $Tx' \in Q \cap \overline{R(T)}$). This shows that $x^* \in N_{T^{-1}Q}(x)$ as desired to show, and completes the proof of (42).

Now assume that $\bar{z} \in R(T)$ is such that $\beta_T(\bar{z}) < +\infty$ and let $\bar{x} \in T^{-1}(\bar{z})$. Then there exist constants $\delta > 0$ and $\mu > 0$ such that

$$\delta\mathbf{B} \cap \overline{R(T)} \subseteq \mu T\mathbf{B} + Q \cap \overline{R(T)} - \bar{z}. \tag{44}$$

To show (43), it suffices to verify that

$$N_{T^{-1}Q}(x) \subseteq T^*N_{Q\cap\overline{R(T)}}(Tx). \tag{45}$$

To do this, let f be defined by (40), and consider the multifunction $F : X \rightrightarrows \overline{R(T)}$ defined by

$$F(v) := Q \cap \overline{R(T)} - Tv \quad \text{for each } v \in X.$$

Then

$$\mathrm{d}(0, F(\cdot)) = \mathrm{d}(T(\cdot), Q \cap \overline{R(T)}) = f(\cdot) \tag{46}$$

and

$$F^{-1}(0) = T^{-1}(Q \cap \overline{R(T)}) = T^{-1}Q = \bar{S}_f \neq \emptyset; \tag{47}$$

see (40) and (41). Furthermore, $F(v)$ is closed for each $v \in X$ and satisfies that

$$\lambda F(v_1) + (1-\lambda)F(v_2) \subseteq F(\lambda v_1 + (1-\lambda)v_2) \quad \text{for all } \lambda \in (0,1), v_1, v_2 \in X.$$

This means that F is a convex multifunction with closed values. Note further that $\mathrm{aff}(F(X)) \subseteq R(T)$ and $F(\bar{x} + \mu\mathbf{B}) = \mu T(\mathbf{B}) + Q \cap \overline{R(T)} - \bar{z}$ (due to $T(\mathbf{B}) = -T(\mathbf{B})$). Thus by (44), one sees that $\mathbf{B}(0, \delta) \cap \mathrm{aff}(F(X)) \subseteq F(\bar{x} + \mu\mathbf{B})$, and [39, Theorem 2.1] is applicable (to $\bar{x}$, 0, $\overline{R(T)}$ in place of x_0, y_0, Y) to concluding that

$$\mathrm{d}(v, F^{-1}(0)) \leq \frac{\mathrm{d}(0, F(v))}{\delta}(\mu + \|v - \bar{x}\|) \quad \text{for all } v \in X. \tag{48}$$

Set $\alpha := \frac{\mu+\delta+\|x-\bar{x}\|}{\delta}$. Then $\frac{\mu+\|v-\bar{x}\|}{\delta} \leq \alpha$ for all $v \in \mathbf{B}(x, \delta)$. Hence, it follows from (48), together with (46) and (47), that

$$\mathrm{d}(v, T^{-1}Q) \leq \alpha \mathrm{d}(Tv, Q \cap \overline{R(T)}) = \alpha f(v) \quad \text{for all } v \in \mathbf{B}(x, \delta).$$

This means that (37) holds with x, δ in place of x_0, r, and Lemma 3.1 is applicable to concluding that (38) holds, that is,

$$N_{T^{-1}Q}(v) \cap \mathbf{B}^* \subseteq \alpha \partial f(v) \quad \text{for all } v \in \mathbf{B}(x, \delta) \cap T^{-1}Q.$$

This, together with (41) , implies that

$$N_{T^{-1}Q}(x) \subseteq \mathrm{cone}(T^*(N_{Q\cap\overline{R(T)}}(Tx) \cap \mathbf{B}^*)) = T^* N_{Q\cap\overline{R(T)}}(Tx),$$

and (45) follows. The proof is completed. □

4 Linear Regularity Property for the SFP

Recall that $C \subseteq X$ and $Q \subseteq Y$ are nonempty closed and convex subsets and $T \in \mathcal{L}(X, Y)$. Linear regularity property have been widely used to analyze the convergence rates of many algorithms; see [12,35] and references therein. We begin with the following definition on the linear regularity property for SFP (1). In particular, item (b) was introduced in [35].

Definition 4.1. *Let $\bar{x} \in S$, and let $\Omega \subseteq X$ be such that $\Omega \cap S \neq \emptyset$. SFP* (1) *is said to have*

(a) *the linear regularity property on Ω if there exists $\gamma > 0$ such that*

$$\mathrm{d}(x, S) \leq \gamma \mathrm{d}(Tx, Q) \quad \textit{for each } x \in \Omega \cap C; \tag{49}$$

(b) *the bounded linear regularity property if, for any $r > 0$, there exists $\gamma_r > 0$ such that* (49) *holds with $\gamma := \gamma_r$ and $\Omega := \mathbf{B}(\bar{x}, r)$;*

(c) *the local linear regularity property at* $\bar{x}$ *if there exist* $r > 0$ *and* $\gamma > 0$ *such that* (49) *holds with* $\Omega := \mathbf{B}(\bar{x}, r)$.

Remark 4.1. *Let* $\bar{x} \in S$. *Consider the following property: for any* $r > 0$, *there exists* $\gamma_r > 0$ *satisfying*

$$\mathrm{d}(x, S) \leq \gamma_r \max\{\mathrm{d}(x, C), \mathrm{d}(Tx, Q)\} \quad \text{for each } x \in \mathbf{B}(\bar{x}, r).$$

Clearly, this property implies that SFP (1) *has the bounded linear regularity property. We claim that the converse is also true. In fact, let* $r > 0$ *and suppose that there exists* $\gamma_r > 0$ *such that*

$$\mathrm{d}(x, S) \leq \gamma_r \mathrm{d}(Tx, Q) \quad \text{for each } x \in \mathbf{B}(\bar{x}, r) \cap C. \tag{50}$$

Below, we show that

$$\mathrm{d}(x, S) \leq \beta_r \max\{\mathrm{d}(x, C), \mathrm{d}(Tx, Q)\} \quad \text{for each } x \in \mathbf{B}(\bar{x}, r), \tag{51}$$

where $\beta_r := 4\gamma_r(1 + \gamma_r\|T\|)$. *Since* $C \cap \mathbf{U}(\bar{x}, r) \neq \emptyset$, *by Proposition 2.2(b), one has that the pair* $\{C, \mathbf{B}(\bar{x}, r)\}$ *has the bounded linear regularity property. Hence*

$$\mathrm{d}(x, C \cap \mathbf{B}(\bar{x}, r)) \leq \gamma_r \mathrm{d}(x, C) \quad \text{for each } x \in \mathbf{B}(\bar{x}, r). \tag{52}$$

Let $x \in \mathbf{B}(\bar{x}, r)$. *Then, there exists* $x_1 \in C \cap \mathbf{B}(\bar{x}, r)$ *satisfying*

$$\mathrm{d}(x, x_1) \leq 2\mathrm{d}(x, C \cap \mathbf{B}(\bar{x}, r)).$$

This, together with (52), *implies that*

$$\mathrm{d}(x, x_1) \leq 2\gamma_r \mathrm{d}(x, C). \tag{53}$$

Since

$$\mathrm{d}(Tx_1, Q) \leq \mathrm{d}(Tx, Q) + \|T(x_1) - T(x)\| \leq \mathrm{d}(Tx, Q) + \|T\|\mathrm{d}(x, x_1),$$

it follows from (50) *that*

$$\mathrm{d}(x_1, S) \leq \gamma_r \mathrm{d}(Tx_1, Q) \leq \gamma_r(\mathrm{d}(Tx, Q) + \|T\|\mathrm{d}(x, x_1)).$$

Combining this with (53) *yields that*

$$\mathrm{d}(x, S) \leq \mathrm{d}(x, x_1) + \mathrm{d}(x_1, S) \leq 2\gamma_r(1 + \gamma_r\|T\|)(\mathrm{d}(x, C) + \mathrm{d}(Tx, Q)).$$

Hence, (51) *is seen to hold, and the claim stands.*

4.1 Linear Regularity Property for the Inclusion Problem

This subsection is devoted to the study of linear regularity property for the special case of the SPF (1) when $C = X$, that is, the inclusion problem:

$$Tx \in Q. \tag{54}$$

Theorem 4.1 below provides some sufficient conditions for ensuring that the inclusion problem (54) has the linear regularity property on $\mathbf{B}(\bar{x}, \frac{r}{2})$ for some $\bar{x} \in T^{-1}Q$ and $r > 0$.

Theorem 4.1. *Let $r, \alpha_r > 0$ and $\bar{x} \in T^{-1}Q$ be such that*

$$\beta_T(Tx) < \alpha_r \quad \text{for each } x \in \mathbf{B}(\bar{x}, r) \cap T^{-1}Q. \tag{55}$$

Suppose that there exists $\eta > 0$ such that

$$\mathrm{d}(y, Q \cap \overline{R(T)}) \leq \eta \max\{\mathrm{d}(y, Q), \mathrm{d}(y, \overline{R(T)})\} \quad \text{for each } y \in \mathbf{B}(T\bar{x}, \tfrac{r\|T\|}{2}). \tag{56}$$

Then

$$\mathrm{d}(x, T^{-1}Q) \leq 2\alpha_r \eta \mathrm{d}(Tx, Q) \quad \text{for each } x \in \mathbf{B}(\bar{x}, \frac{r}{2}).$$

Proof. By (56), it suffices to show that

$$\mathrm{d}(x, T^{-1}Q) \leq 2\alpha_r \mathrm{d}(Tx, Q \cap \overline{R(T)}) \quad \text{for each } x \in \mathbf{B}(\bar{x}, \frac{r}{2}). \tag{57}$$

To proceed, fix $x \in \mathbf{B}(\bar{x}, r) \cap T^{-1}Q$. By (55), we have that

$$\beta_T(Tx) \leq \alpha_r < +\infty.$$

Noting that $N_{Q\cap\overline{R(T)}}(Tx) = \Gamma_{Q\cap\overline{R(T)}}(Tx)$ (due to $Tx \in Q \cap \overline{R(T)}$), one apply Proposition 3.4(ii) to Tx in place of z to obtain that

$$(T^* N_{Q\cap\overline{R(T)}}(Tx)) \cap \mathbf{B}^* \subseteq T^*(N_{Q\cap\overline{R(T)}}(Tx) \cap \beta_T(Tx)\mathbf{B}^*). \tag{58}$$

As $\beta_T(Tx)\mathbf{B}^* \subseteq \alpha_r \mathbf{B}^*$ (due to (55)), we have that

$$T^*(N_{Q\cap\overline{R(T)}}(Tx) \cap \beta_T(Tx)\mathbf{B}^*) \subseteq \alpha_r T^*(N_{Q\cap\overline{R(T)}}(Tx) \cap \mathbf{B}^*).$$

Then, it follows from (58) that

$$(T^* N_{Q\cap\overline{R(T)}}(Tx)) \cap \mathbf{B}^* \subseteq \alpha_r T^*(N_{Q\cap\overline{R(T)}}(Tx) \cap \mathbf{B}^*).$$

Since $x \in T^{-1}Q$, we apply Proposition 3.5 to conclude that

$$N_{T^{-1}Q}(x) \cap \mathbf{B}^* = (T^* N_{Q\cap\overline{R(T)}}(Tx)) \cap \mathbf{B}^*.$$

Now let f be the function defined by (40). Then by (41) we see that

$$N_{T^{-1}Q}(x) \cap \mathbf{B}^* \subseteq \alpha_r \partial f(x). \tag{59}$$

Recalling that $\bar{S}_f = T^{-1}Q$ and noting that $x \in \mathbf{B}(\bar{x}, r) \cap T^{-1}Q$ is arbitrary, (59) implies that (38) holds with $\alpha_r, \bar{x}$ in place of α, x_0. Thus we apply Lemma 3.1 to conclude that

$$\mathrm{d}(x, T^{-1}Q) \leq 2\alpha_r f(x) \quad \text{for each } x \in \mathbf{B}(\bar{x}, \frac{r}{2}).$$

Hence, (57) is seen to hold. The proof is completed. □

The following proposition is on the boundedness of the relative regularity constants. To proceed, we need the notion of finite codimensional subsets (cf. [23, Definition 4.1]).

Definition 4.2. *Let A and B be nonempty convex subsets of Y. Let H be a closed linear subspace of Y such that $A \subseteq H$. A is said to be*

(a) *finite codimensional in H if the quotient space $H/\overline{\text{span}A}$ is finite dimensional;*

(b) *finite codimensional if A is finite codimensional in Y;*

(c) *finite codimensional in B if the closed subspace $\overline{\text{span}B} \cap \overline{\text{span}A}$ is finite codimensional in $\overline{\text{span}B}$.*

Obviously, in the case when B is finite dimensional, any nonempty convex subset A is finite codimensional in B.

Proposition 4.1. *Suppose that $R(T) \cap Q \neq \emptyset$. Then for any $r > 0$, there exists $\alpha_r > 0$ such that*

$$\beta_T(Tx) < \alpha_r \quad \textit{for each } x \in \mathbf{B}(0,r) \cap T^{-1}Q$$

provided one of the following conditions holds:

(a) *$R(T) \cap \text{int}_{\overline{R(T)}} Q \neq \emptyset$;*

(b) *$R(T) \cap \text{ri}Q \neq \emptyset$ and Q is finite codimensional;*

(c) *Q is a polyhedron.*

Proof. Let $r > 0$. It suffices to show that there exist $\eta > 0$ and $x_0 \in X$ such that, for each $\alpha \geq 1 + r + \|x_0\|$,

$$\eta\mathbf{B} \cap \overline{R(T)} \subseteq \alpha T\mathbf{B} + Q - Tx \quad \text{for each } x \in \mathbf{B}(0,r) \cap T^{-1}Q. \tag{60}$$

Suppose condition (a) holds. Then there exist $x_0 \in X$ and $\eta > 0$ such that $\eta\mathbf{B} \cap \overline{R(T)} \subseteq Q - Tx_0$. Let $x \in \mathbf{B}(0,r) \cap T^{-1}Q$. Since $-Tx_0 = \|x - x_0\| T(\frac{x-x_0}{\|x-x_0\|}) - Tx$, it follows that

$$\eta\mathbf{B} \cap \overline{R(T)} \subseteq Q - Tx_0 \subseteq \|x - x_0\| T\mathbf{B} + Q - Tx \subseteq (r + \|x_0\|)T\mathbf{B} + Q - Tx.$$

Thus, (60) holds for each $\alpha \geq 1 + r + \|x_0\|$.

Suppose condition (b) holds. Set $Z := \text{span}(Q - Q) \cap \overline{R(T)}$. By the assumption, $\text{span}(Q - Q)$ is finite codimensional and so Z is finite codimensional in $\overline{R(T)}$. Thus there exists a finite dimensional space Z_0 such that $\overline{R(T)} = Z \oplus Z_0$. Using the standard argument for proving the equivalence property of norms in finite dimensional spaces, one checks that

$$\lambda\mathbf{B} \cap \overline{R(T)} \subseteq \mathbf{B} \cap Z + \mathbf{B} \cap Z_0, \tag{61}$$

holds for some $\lambda > 0$. Moreover, by the assumption that $R(T) \cap \text{ri}Q \neq \emptyset$, there exist $x_0 \in X$ and $\eta > 0$ such that $\text{aff}Q \cap (Tx_0 + \eta\mathbf{B}) \subseteq Q$; hence

$$\eta\mathbf{B} \cap Z \subseteq \eta\mathbf{B} \cap (\text{aff}Q - Tx_0) \subseteq Q - Tx_0.$$

As done for (4.1), we have that

$$\eta\mathbf{B} \cap Z \subseteq (r + \|x_0\|)T\mathbf{B} + Q - Tx \quad \text{ for each } x \in \mathbf{B}(0, r) \cap T^{-1}Q.$$

Set $m := \dim Z_0$, and let $\{Tu_i : i = 1, ..., m\}$ be a basis of Z_0 with each $u_i \in \mathbf{B}$. Define

$$\|z\|_1 := \sum_{i=1}^{m} |\alpha_i| \quad \text{for each } z := \sum_{i=1}^{m} \alpha_i T u_i \in Z_0.$$

Then $\|\cdot\|_1$ is a norm on Z_0 and equivalent to the original norm $\|\cdot\|$ on Z_0 as Z_0 is a finite dimensional subspace; hence there exists $\eta_0 > 0$ such that $\eta_0\|\cdot\|_1 \leq \|\cdot\|$ on Z_0. Then,

$$\eta_0\mathbf{B} \cap Z_0 \subseteq T\mathbf{B} \tag{62}$$

because, for each $z := \sum_{i=1}^{m} \alpha_i T u_i \in \eta_0\mathbf{B} \cap Z_0$, one has that $z = T(\sum_{i=1}^{m} \alpha_i u_i) \in T\mathbf{B}$ and

$$\|\sum_{i=1}^{m} \alpha_i u_i\| \leq \sum_{i=1}^{m} |\alpha_i| = \|z\|_1 \leq \frac{\|z\|}{\eta_0} \leq 1.$$

Thus, using (61)-(62), we conclude that

$$\lambda \min\{\eta_0, \eta\}\mathbf{B} \cap \overline{R(T)} \subseteq \eta\mathbf{B} \cap Z + \eta_0\mathbf{B} \cap Z_0 \subseteq (1 + r + \|x_0\|)T\mathbf{B} + Q - Tx,$$

holds for all $x \in \mathbf{B}(0, r) \cap T^{-1}Q$. Ths means that, for each $\alpha \geq 1 + r + \|x_0\|$, (60) holds with $\lambda \min\{\eta_0, \eta\}$ in place of η.

Suppose condition (c) holds. Then $Q := \cap_{i=1}^{m} H_i$ where each H_i is a half space of Y given by

$$H_i := \{y \in Y : \langle y_i^*, y\rangle \leq b_i\},$$

with $\{(y_i^*, b_i)\} \subseteq Y^* \times \mathbb{R}$. Set

$$I_B := \{i : Q \subseteq \mathrm{bd} H_i\} \quad \text{and} \quad I_N := \{i : Q \cap \mathrm{int} H_i \neq \emptyset\},$$

and we adopt the convention that $\cap_{i\in\emptyset} H_i = Y$. Then one has by definition that $I_B \cup I_N = \{1, \cdots, m\}$,

$$Q = (\cap_{i\in I_B} \mathrm{bd} H_i) \cap (\cap_{i\in I_N} H_i) \quad \text{and} \quad \mathrm{ri} Q = (\cap_{i\in I_B} \mathrm{bd} H_i) \cap (\cap_{i\in I_N} \mathrm{int} H_i).$$

We claim that condition (b) is satisfied. To proceed, we assume, without loss of generality, that $I_N = \{1, \cdots, n\}$ with $n \leq m$. Then, for each $i \in I_N$, we have by the definition of I_N that there exists $x_i \in T^{-1}Q$ such that $Tx_i \in \mathrm{int} H_i$ and so $\langle y_i^*, Tx_i\rangle < b_i$. Set $x_0 := \frac{1}{n}\sum_{i\in I_N} x_i$. Since $Tx_i \in Q$ for each $i \in I_N$, one has that $Tx_0 \in \cap_{j\in I_B} \mathrm{bd} H_j$. For each $j \in I_N$, $\langle y_j^*, Tx_j\rangle < b_j$ and $\langle y_j^*, Tx_i\rangle \leq b_j$ for each $i \in I_N \setminus \{j\}$. This implies that $\langle y_j^*, Tx_0\rangle = \frac{1}{n}\sum_{i\in I_N}\langle y_j^*, Tx_i\rangle < b_j$. Then we arrive at that $Tx_0 \in \mathrm{int} H_j$ and so $Tx_0 \in \cap_{i\in I_N} \mathrm{int} H_i$ (noting that $j \in I_N$ is arbitrary). Therefore, it follows that $Tx_0 \in \mathrm{ri} Q$. Then condition (b) is checked as Q is finite codimensional by definition. The proof is completed. □

4.2 Linear Regularity Property for the SFP

Recall that $C \subseteq X$ and $Q \subseteq Y$ are nonempty closed and convex subsets and $T \in \mathcal{L}(X, Y)$. For simplicity, write

$$V := \overline{\text{span}} C \quad \text{ and } \quad \mathbf{B}_V := \mathbf{B} \cap V.$$

Recall that $T|_V : V \to Y$ is the restriction of T on V. Then we have by definition that

$$(T|_V)^{-1} Q = T^{-1} Q \cap V, \ \ T|_V \mathbf{B}_V = T\mathbf{B}_V, \|T|_V\| \le \|T\|.$$

Theorem 4.2. *Let $r, \alpha_r > 0$, $\bar{x} \in S$ and suppose that*

(a) $\beta_T^V(Tx) < \alpha_r$ *for each $x \in \mathbf{B}(\bar{x}, r) \cap T^{-1}Q \cap V$;*

(b) *there exists $\eta_r > 0$ such that*

$$\mathrm{d}(y, Q \cap \overline{TV}) \le \eta_r \max\{\mathrm{d}(y, Q), \mathrm{d}(y, \overline{TV})\} \quad \textit{for each } y \in \mathbf{B}(T\bar{x}, \frac{r\|T\|}{2});$$

(c) *there exists $\nu_r > 0$ such that*

$$\mathrm{d}(x, S) \le \nu_r \max\{\mathrm{d}(x, C), \mathrm{d}(x, T^{-1}Q \cap V)\} \quad \textit{for each } x \in \mathbf{B}(\bar{x}, \frac{r}{2}) \cap V. \tag{63}$$

Then there exists $\gamma_r > 0$ such that

$$\mathrm{d}(x, S) \le \gamma_r \mathrm{d}(Tx, Q) \quad \textit{for each } x \in \mathbf{B}(\bar{x}, \frac{r}{2}) \cap C. \tag{64}$$

Proof. By assumption, Theorem 4.1 is applicable with $T|_V, V$ in place of T, X to concluding that, for each $x \in \mathbf{B}(\bar{x}, \frac{r}{2}) \cap V$,

$$\mathrm{d}(x, (T|_V)^{-1}Q) \le 2\alpha_r\eta_r \mathrm{d}(T|_V x, Q),$$

that is,

$$\mathrm{d}(x, T^{-1}Q \cap V) \le 2\alpha_r\eta_r \mathrm{d}(Tx, Q).$$

This, together with (63), gives that, for each $x \in \mathbf{B}(\bar{x}, \frac{r}{2}) \cap V$,

$$\mathrm{d}(x, S) \le \nu_r \max\{\mathrm{d}(x, C), 2\alpha_r\eta_r \mathrm{d}(Tx, Q)\}. \tag{65}$$

Set $\gamma_r := \nu_r \max\{1, 2\alpha_r\eta_r\}$. Then, one has from (65) that

$$\mathrm{d}(x, S) \le \gamma_r \max\{\mathrm{d}(x, C), \mathrm{d}(Tx, Q)\} \quad \text{for each } x \in \mathbf{B}(\bar{x}, \frac{r}{2}) \cap V,$$

which implies that

$$\mathrm{d}(x, S) \le \gamma_r \mathrm{d}(Tx, Q) \quad \text{for each } x \in \mathbf{B}(\bar{x}, \frac{r}{2}) \cap C.$$

Thus, (64) is seen to hold. The proof is completed. □

The following corollary is directly from Theorem 4.2.

Corollary 4.1. *Let $\bar{x} \in S$ and suppose that*

(a) *for any $r > 0$, there exists $\alpha_r > 0$ such that*

$$\beta_T^V(Tx) < \alpha_r \quad \text{for each } x \in \mathbf{B}(\bar{x}, r) \cap T^{-1}Q \cap V;$$

(b) *the pair $\{\overline{TV}, Q\}$ has the bounded linear regularity property;*

(c) *the pair $\{T^{-1}Q \cap V, C\}$ has the bounded linear regularity property.*

Then SFP (1) *has the bounded linear regularity property.*

Corollary 4.2. *Let $\bar{x} \in S$. Suppose that $\beta_T^V(T\bar{x}) < +\infty$. Suppose further that there exist $r > 0$ and $\nu_r, \eta_r > 0$ such that assumptions* (b) *and* (c) *in Theorem 4.2 hold. Then SFP* (1) *has the local linear regularity property at $\bar{x}$.*

Proof. Since $\beta_T^V(T\bar{x}) < +\infty$, there exist $\bar{\delta} > 0$ and $\bar{\mu} > 0$ such that

$$\bar{\delta}\mathbf{B} \cap \overline{TV} \subseteq \bar{\mu}T(\mathbf{B} \cap V) + Q - T\bar{x}.$$

Let $0 < \bar{r} < \frac{\bar{\delta}}{2}$. Then, we apply Proposition 3.3 (with $\overline{TV}, T|_V, T\bar{x}, 0$ in place of $H, T, \bar{z}, \Delta$) to get that

$$\beta_{T|_V}(z) \le \frac{2\bar{\mu}}{\bar{\delta}} \quad \text{for each } z \in \mathbf{B}(T|_V\bar{x}, \bar{r})$$

and so

$$\beta_T^V(Tx) = \beta_{T|_V}(T|_V x) \le \frac{2\bar{\mu}}{\bar{\delta}} \quad \text{for each } x \in \mathbf{B}(\bar{x}, \frac{\bar{r}}{\|T|_V\|}) \cap T^{-1}Q \cap V.$$

Thus, assumption (a) in Theorem 4.2 is seen to hold and so Theorem 4.2 is applicable to concluding that SFP (1) has the local linear regularity property at $\bar{x}$. The proof is completed. □

The following theorem gives some sufficient conditions ensuring the bounded linear regularity property for SFP (1) with X and Y being normed linear spaces. Even in the case when X and Y are Hilbert spaces, our results extend and improve the corresponding ones in [35, Proposition 2.5]. In particular, the polyhedron C in (i) of [35, Proposition 2.5] is relaxed to be a generalized polyhedron.

Theorem 4.3. *SFP* (1) *has the bounded linear regularity property provided one of the following conditions is satisfied:*

(a) *Q is a polyhedron and C is a generalized polyhedron;*

(b) *$TC \cap \mathrm{int}_{\overline{TV}}Q \neq \emptyset$ and $Q \subseteq \overline{TV}$;*

(c) *$TC \cap \mathrm{int}Q \neq \emptyset$;*

(d) *$T(\mathrm{ri}C) \cap Q \neq \emptyset$ and Q is a polyhedron;*

(e) *$TC \cap \mathrm{ri}Q \neq \emptyset$, C is a polyhedron and Q is finite codimensional;*

(f) $T(\mathrm{ri}C) \cap \mathrm{ri}Q \neq \emptyset$ *and* Q *is finite codimensional.*

Proof. It suffices to show that any one of the conditions (a)-(f) guarantees assumptions (a)-(c) of Corollary 4.1 hold.

Assume condition (a) holds. We first prove the following implication:

$$[Q \text{ is a polyhedron}] \Rightarrow [\text{(a) and (b) of Corollary 4.1}]. \tag{66}$$

In fact, since Q is a polyhedron, we apply Proposition 4.1(c) to $T|_V$ in place of T to conclude that assumption (a) of Corollary 4.1 holds. Noting that $\overline{TV}$ is a closed linear subspace, one has that $\mathrm{ri}(\overline{TV}) = \overline{TV}$. Thus, $\mathrm{ri}(\overline{TV}) \cap Q = \overline{TV} \cap Q \neq \emptyset$ and so Proposition 2.2 (c) is applicable to concluding that the pair $\{\overline{TV}, Q\}$ has the bounded linear regularity property. Thus assumption (b) of Corollary 4.1 holds, showing (66). Hence, assumptions (a) and (b) of Corollary 4.1 hold. Since $T^{-1}Q$ is a polyhedron, we have that $T^{-1}Q \cap V$ is a generalized polyhedron on V. On the other hand, as C is a generalized polyhedron, there exist a closed affine subspace L and a polyhedron C_0 such that $C = L \cap C_0$. Since $V = \overline{\mathrm{span}}C = \overline{\mathrm{span}}(L \cap C_0) \subseteq L$, it follows that $C = C \cap V = L \cap C_0 \cap V = C_0 \cap V$ and so C is also a polyhedron on V. Hence, Proposition 2.2(a) is applicable with $T^{-1}Q \cap V, C, V$ in place of E, F, V to concluding that the pair $\{T^{-1}Q \cap V, C\}$ has the bounded linear regularity property on V. Then, in view of Proposition 2.2(e), we have that the pair $\{T^{-1}Q \cap V, C\}$ has the bounded linear regularity property on X and so assumption (c) of Corollary 4.1 holds.

Assume condition (b) holds. We first show the following implication:

$$[TC \cap \mathrm{int}_{\overline{TV}} Q \neq \emptyset] \Rightarrow [\text{(a) and (c) of Corollary 4.1}]. \tag{67}$$

To do this, since $TC \cap \mathrm{int}_{\overline{TV}} Q \neq \emptyset$, one has that $TV \cap \mathrm{int}_{\overline{TV}} Q \neq \emptyset$. Then, Proposition 4.1(a) is applicable with $T|_V$ in place of T to concluding that assumption (a) of Corollary 4.1 holds. Noting that $TC \cap \mathrm{int}_{\overline{TV}}(Q \cap \overline{TV}) \neq \emptyset$, there exists $x_0 \in C$ such that $Tx_0 \in \mathrm{int}_{\overline{TV}}(Q \cap \overline{TV})$ which implies that there is $\delta > 0$ such that $\mathbf{B}(Tx_0, \delta) \cap \overline{TV} \subseteq Q \cap \overline{TV}$. Since by definition that $\mathbf{B}(x_0, \frac{\delta}{\|T\|}) \cap V \subseteq T|_V^{-1}(\mathbf{B}(Tx_0, \delta) \cap \overline{TV})$, it follows that

$$\mathbf{B}(x_0, \frac{\delta}{\|T\|}) \cap V \subseteq T|_V^{-1}(Q \cap \overline{TV}) = T^{-1}Q \cap V.$$

Thus, we get that $C \cap \mathrm{int}_V(T^{-1}Q \cap V) \neq \emptyset$. Hence, it follows from Proposition 2.2(b) (with $C, T^{-1}Q \cap V, V$ in place of E, F, X) that the pair $\{C, T^{-1}Q \cap V\}$ has the bounded linear regularity property on V. Then, in view of Proposition 2.2(e), we have that the pair $\{C, T^{-1}Q \cap V\}$ has the bounded linear regularity property on X and so assumption (c) of Corollary 4.1 holds, showing implication (67). Hence, assumptions (a) and (c) of Corollary 4.1 hold. Since $Q \subseteq \overline{TV}$, by definition, the pair $\{\overline{TV}, Q\}$ has the bounded linear regularity property directly, that is, assumption (b) of Corollary 4.1 holds.

Assume condition (c) holds. Then by the assumption, $TC \cap \mathrm{int}_{\overline{TV}} Q \neq \emptyset$ and so it follows from (67) that assumptions (a) and (c) of Corollary 4.1 hold. Note that $\overline{TV} \cap \mathrm{int}Q \neq \emptyset$. In view of Proposition 2.2(b), we have that the pair $\{\overline{TV}, Q\}$ has the bounded linear regularity property, that is, assumption (b) of Corollary 4.1 hold.

Assume condition (d) holds. Since Q is a polyhedron, by (66), we have that assumptions (a) and (b) of Corollary 4.1 hold. Noting $T(\mathrm{ri}C) \cap Q \neq \emptyset$, one has from Proposition 2.3 that the pair $\{C, T^{-1}Q \cap V\}$ has the bounded linear regularity property, that is, assumption (c) of Corollary 4.1 holds.

Assume condition (e) holds. We first verify the following implication:

$$[TC \cap \mathrm{ri}Q \neq \emptyset \text{ and } Q \text{ is finite codimensional}] \Rightarrow [\text{(a) and (b) of Corollary 4.1}]. \tag{68}$$

In fact, since $TV \supseteq TC$, it follows that $TV \cap \mathrm{ri}Q \neq \emptyset$. Thus, we apply Proposition 4.1(b) to $T|_V$ in place of T to get that assumption (a) of Corollary 4.1 holds. Noting that $\overline{TV}$ is a closed linear subspace, one has that $\mathrm{ri}(\overline{TV}) = \overline{TV}$. Thus $\mathrm{ri}(\overline{TV}) \cap \mathrm{ri}Q = TV \cap \mathrm{ri}Q \neq \emptyset$. By this and the fact that Q is finite codimensional, we apply Proposition 2.2(d) to get that the pair $\{\overline{TV}, Q\}$ has the bounded linear regularity property, showing that assumption (b) of Corollary 4.1 holds. Hence, (68) holds and it follows that assumptions (a) and (b) of Corollary 4.1 hold. Below, we claim that

$$\mathrm{ri}(T^{-1}Q) \cap C \neq \emptyset. \tag{69}$$

Assuming this holds, as C is a polyhedron, it follows from Proposition 2.2(c) that the pair $\{T^{-1}Q, C\}$ has the bounded linear regularity property. This, together with the definition of the bounded linear regularity property, implies that the pair $\{T^{-1}Q \cap V, C\}$ has the bounded linear regularity property because $C \subset V$. Thus, assumption (c) of Corollary 4.1 holds. To proceed, by assumption, we can choose $x_0 \in C$ such that $Tx_0 \in \mathrm{ri}Q$. This means that there exists $\delta > 0$ such that $\mathbf{B}(Tx_0, \delta) \cap \mathrm{aff}Q \subseteq Q$. Consequently,

$$T^{-1}(\mathbf{B}(Tx_0, \delta)) \cap T^{-1}(\mathrm{aff}Q) = T^{-1}(\mathbf{B}(Tx_0, \delta) \cap \mathrm{aff}Q) \subseteq T^{-1}Q. \tag{70}$$

Since by definition we have $\mathbf{B}(x_0, \frac{\delta}{\|T\|}) \subseteq T^{-1}(\mathbf{B}(Tx_0, \delta))$ and $\mathrm{aff}(T^{-1}Q) \subseteq T^{-1}(\mathrm{aff}Q)$, it follows from (70) that

$$\mathbf{B}(x_0, \frac{\delta}{\|T\|}) \cap \mathrm{aff}(T^{-1}Q) \subseteq T^{-1}Q.$$

This implies that $x_0 \in \mathrm{ri}(T^{-1}Q)$ and so (69) is proved.

Assume condition (f) holds. Noting $TC \supseteq T(\mathrm{ri}C)$, it follows from (68) that assumptions (a) and (b) of Corollary 4.1 hold. Since $T(\mathrm{ri}C) \cap Q \neq \emptyset$ (noting $Q \supseteq \mathrm{ri}Q$), it follows from Proposition 2.3 that the pair $\{T^{-1}Q \cap V, C\}$ has the bounded linear regularity property, that is, assumption (c) of Corollary 4.1 holds. The proof is completed. □

Recall that $V = \overline{\mathrm{span}}C$. With the help of Proposition 3.2 (applied to V in place of X), together with Proposition 2.3 and [38, Proposition 3.7], one can apply Corollary 4.2 to get the following corollary, for which the argument of the proof is standard and so omitted.

Corollary 4.3. *Let $\bar{x} \in S$, and suppose that there exists a pair (τ, K) satisfying* (23) *with $T\bar{x}$ in place of z. Suppose also that*

(H) *$T(\mathrm{ri}C) \cap Q \neq \emptyset$ and Q is a generalized polyhedron with the P-L decomposition $\{Q_L, Q_P\}$ satisfying that $\overline{TV}$ is finite codimensional in Q_L or Q_L is finite codimensional in $\overline{TV}$ (e.g., Q_L or $\overline{TV}$ is finite dimensional or finite codimensional as noted after Definition 4.2).*

Then SFP (1) *has the local linear regularity property provided one of the following conditions hold:*

(a) $T_K V$ *is a closed linear subspace and* $\mathrm{ri}((\mu_0 T(\mathbf{B} \cap V) + Q - T\bar{x}) \cap T_K V) \neq \emptyset$ *for some* $\mu_0 > 0$ *(e.g.,* TV *is closed and* $\mathrm{ri}((\mu_0 T(\mathbf{B} \cap V) + Q - T\bar{x}) \cap TV) \neq \emptyset$ *for some* $\mu_0 > 0$*).*

(b) $T_K V$ *is a complete linear subspace (e.g.,* TV *is complete).*

Similarly, we have the following corollary, which in particular in the case when TV is complete, extends/improves partially [35, Proposition 2.5(iii)].

Corollary 4.4. *Suppose that assumption* (H) *in Corollary 4.3 holds and that* TV *is complete. Then SFP* (1) *has the bounded linear regularity property.*

Remark 4.2. *The SFP* (1) *can be reformulated as the following inequality problem:*

$$f(x) \leq 0, \qquad x \in C, \tag{71}$$

where $f : X \to \mathbb{R}$ *is given by* $f(x) := \mathrm{d}(Tx, Q)$ *for each* $x \in X$*. Then the bounded linear regularity property for* (1) *is equivalent to the bounded error bound for* (71)*: for each bounded subset* W *such that* $W \cap S \neq \emptyset$*, there exists* $\alpha_W > 0$ *such that*

$$\mathrm{d}(x, S) \leq \alpha_W f(x)_+ \quad \text{for each } x \in W \cap C,$$

where $\gamma_+ = \max\{0, \gamma\}$ *for* $\gamma \in \mathbb{R}$*. A powerful tool for studying the existence of the bounded error bound for the inequality* (71) *is the well known Slater qualification condition (cf. [8,20,28]): there exists a point* $\bar{x} \in C$ *such that* $f(\bar{x}) < 0$*. Obviously, in the present paper, the Slater qualification condition for* (71) *is not satisfied.*

5 Conclusions

To study the strong convergence and/or convergence rate of the CQ algorithm for the SFP in Hilbert spaces, Wang *et al.* [35] introduced a bounded linear regularity property for the SFP and then used this property to establish a linear convergence result for the CQ algorithm. Thus how to verify that the SFP has the bounded/local linear regularity property is an interesting problem. Some sufficient conditions based on the interior/polyhedron conditions ensuring the bounded linear regularity property for the SFP in Hilbert spaces are provided in [35, Proposition 2.5]. The present paper is focused on the issue of the bounded/local linear regularity property for the SFP in general normed linear spaces. To establish some sufficient conditions ensuring the bounded/local linear regularity property for the SFP in normed linear spaces, we introduced the notion of a relative regularity condition and its associated relative regularity constant in spirit of the regularity condition used in [12]. Based on convex analysis techniques, we explored equivalent characterizations of the relative regularity condition, which in particular extends the classical results in [12] from the Euclidean space to general normed linear spaces, and then established some important and useful properties in terms of the related relative regularity constant. Consequently, we developed a new technique to establish some sufficient conditions ensuring the bounded/local linear regularity property for the SFP in normed linear spaces. Applied to the case of Hilbert spaces, our results extend and improve the corresponding ones in [35].

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