SIAM J. OPTIM.
Vol. 33, No. 3, pp. 2021–2040




# RELATIVE LIPSCHITZ-LIKE PROPERTY OF PARAMETRIC SYSTEMS VIA PROJECTIONAL CODERIVATIVES*

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**Abstract.** This paper concerns upper estimates of the projectional coderivative of implicit mappings and corresponding applications on analyzing the relative Lipschitz-like property. Under different constraint qualifications, we provide upper estimates of the projectional coderivative for solution mappings of parametric systems. For the solution mapping of affine variational inequalities, a generalized critical face condition is obtained for sufficiency of its Lipschitz-like property relative to a polyhedral set within its domain under a constraint qualification. The necessity is also obtainable under some regularity or when the polyhedral set is further the domain of the solution mapping. We further discuss possible conditions for the necessity and consider the solution mapping of a linear complementarity problem with a $Q_0$ matrix as an example.

**Key words.** parametric systems, relative Lipschitz-like property, generalized Mordukhovich criterion, affine variational inequality, linear complementarity problem

**MSC codes.** 49J40, 49J53, 49K40, 90C31, 90C33

**DOI.** 10.1137/22M151296X

## 1. Introduction.

In this paper, we focus on the relative Lipschitz-like property of the solution mapping of the following fully parametric system:

$$S(w) := \{x \in \mathbb{R}^n \mid 0 \in G(w,x) + M(w,x)\}, \tag{1.1}$$

where $G : \mathbb{R}^{m+n} \to \mathbb{R}^d$ is a $\mathcal{C}^1$ mapping, and $M : \mathbb{R}^{m+n} \rightrightarrows \mathbb{R}^d$ is a multifunction with closed graph. The Lipschitz-like property, originated from [1], plays a central role in stability analysis. The Mordukhovich criterion provides a complete characterization of a Lipschitz-like property by using the coderivative (see [26, 27, 31]).

An upper estimate, as well as the attainable equality, of the coderivative of the mapping $S$ in (1.1) were obtained in [23] under some constraint qualifications and applied to obtain the Lipschitz-like property for the stationary point multifunction of a parametric optimization problem. On the other hand, a critical face condition was developed in [8] (see also [9]) as sufficient and necessary conditions for the Lipschitz-like property of the solution mapping of an affine variational inequality (AVI) problem over a polyhedral set.

Sufficient conditions for a Lipschitz-like property of quasi-variational inequalities were obtained in [29] by developing a new coderivative calculus for special compositions. Sufficient conditions for the Lipschitz-like property of implicit multifunctions (that is, $G$ in (1.1) vanishes) were established in [11] by using a directional limiting coderivative, which was introduced in [10] (with a slightly different form in [13]). A formula was also derived for computing the directional limiting coderivative of the normal-cone map with a polyhedral set, which matches the well-known critical face condition framework in [8]. Determinantal conditions for the existence of a single-

*Received by the editors August 1, 2022; accepted for publication (in revised form) March 14, 2023; published electronically August 4, 2023.
https://doi.org/10.1137/22M151296X
**Funding:** The second author was partly supported by a project from the Research Grants Council of Hong Kong (15209921).

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valued, Lipschitz continuous, piecewise-affine solution trajectory to an AVI subject to both a canonical parameter and a perturbed normal cone of a polyhedral convex cone were obtained in [24]. Further extension of a critical face condition to the Lipschitz-like property of solution mapping to a generalized equation can also be found in [12] under the framework of [29]. Characterizations of a Lipschitz-like property for the solution mapping of linear semi-infinite and infinite systems were obtained in [6] by developing a Mordukhovich criterion in an arbitrary Banach space based on generalized differentiation. For more investigation on parametric optimization problems, see the monographs [4, 9, 18, 19].

The coderivative of a normalcone mapping was computed and applied to give some sufficient condition for the Lipschitz-like property of the stationary point multifunction of minimizing a quadratic function with a ball constraint in [21]. In [16, 17], the condition for coderivative equality in [23] was applied to study the Lipschitz-like property and the Robinson metric regularity of a parametric affine constraint system with a closed set under full perturbations. See also a recent monograph [20] on the study of stability of parametric quadratic programs and AVIs.

On the application side of stability theory, the calmness of the solution mapping of the Lasso relative to the positive half-line was established in [3, 5] (see also [15] for the relative $q$-order calmness for the $\ell_q$-regularization problem $(0 < q \leq 1)$.) As noted in [10], in order to guarantee some stationarity conditions, one may only need a regular behavior of the constraint systems with respect to one single critical direction, not on the whole space. These applications of stability properties are concerned with the case where the reference point may lie on the boundary of the domain or a set under consideration. In the case of the Lipschitz-like property, the well-known Mordukhovich criterion is not applicable. Recently, by virtue of a directional limiting coderivative of the normal-cone mapping and a variant of a critical face condition, sufficient conditions were obtained in [2] for the Lipschitz-like property relative to a closed set for the solution map of a class of parameterized variational systems. By employing the projection of the normal cone of the restricted graph of a multifunction on the product of the tangent cone of the concerned closed set and the solution space, a projectional coderivative was introduced and applied in [25] to derive a generalized Mordukhovich criterion for the Lipschitz-like property relative to a closed and convex set.

In this paper we begin by giving a comparison between the sufficient conditions for a relative Lipschitz-like property in [2, 25]. While an example was given to illustrate that the sufficient condition of the generalized Mordukhovich criterion in [25] holds but the sufficient condition in [2] does not hold, we further show that the sufficient condition of the relative Lipschitz-like property for an explicit multifunction in [2] implies the one in [25] when the relative set is closed and convex. We then focus on the investigation of a projectional coderivative and Lipschitz-like property of fully parametric systems. We first obtain an upper estimate of the projectional coderivative of the solution mapping (1.1) by posing different constraint qualifications. Following [23] and using the definition of the projectional coderivative, we calculate the outer limit of the projection of the normal cone of the restricted graph to the tangent cone of the relative set. The upper estimate becomes tight under a regularity condition and the relative set being a manifold. We also obtain some upper estimates of the projectional coderivative of (1.1) for the two special cases in which (i) $G(w,x)$ vanishes and (ii) $M(w,x)=M(x)$ and the relative set is equal to the dom $S$.

The obtained upper estimates of the projectional coderivative of $S(x)$ are applied to an AVI, where $G(q,x)=q+Mx$ and $M(x)=N_C(x)$ with $M$ being an $n\times n$ matrix and $C$ a polyhedron. Some upper estimates of the projectional coderivative of the solution mapping of the AVI are obtained under different constraint qualifications and regularity conditions. The upper estimate is tight if either the graph of the normal cone $N_C$ and the relative set are both regular or the relative set is equal to the domain of $S$. For studying the relative Lipschitz-like characterization of AVI, we consider that the relative set is a polyhedral subset of the domain. We obtain a sufficient condition for $S$ to be relative Lipschitz-like under a constraint qualification and represent this condition in the form of critical face condition as in [8]. This sufficient condition becomes necessary if the graph of the normal cone $N_C$ is regular or the relative set is further the dom $S$ along with convexity. Further, we simplify the condition when necessity is possible, where equivalent descriptions can be given. For the regularity of gph $N_C$, we show that it involves the property of the reference point and that instead of checking the condition with all combinations of the faces, the condition reduces to the critical cone only. For AVI on polyhedral cones, via equivalent description of dom $S$, we are able to give explicit expression of $N_{\text{dom } S}(\bar{q})$. We then illustrate this condition with an example: a linear complementarity problem (LCP) with a $Q_0$ matrix. It is known that the domain of LCP with a $Q_0$ matrix is a polyhedron (see [7]). We thus consider this domain as the relative set. Therefore the estimate of the projectional coderivative is tight and we are able to represent this condition in terms of a normal cone of the complementarity conditions of the LCP.

The organization of the paper is as follows. Section 2 introduces the standard notation and tools. Section 3 presents upper estimates of projectional coderivatives of parametric systems. Section 4 characterizes the Lipschitz-like property of the solution mapping of AVIs relative to a polyhedral set. Section 5 discusses the necessity of the generalized critical face condition and the corresponding simplification, along with an example on LCP relative to its domain.

**2. Preliminaries.** In this section, we review some notations and preliminary results that will be used in the following sections. The notations adopted are standard in variational analysis. Most of them can be found in monographs [28] and [31].

The norm and scalar product of an Euclidean space $\mathbb{R}^n$ are denoted as $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. The symbol $\mathbb{B}_r(x)$ stands for the closed unit ball with radius $r>0$ centered at $x$ and $\mathbb{B}:=\mathbb{B}_1(0)$. By $A^*$ we denote the transpose of matrix $A$ and also the adjoint operator of linear operator $A$, and by $A^{-1}$ we denote the inverse mapping of $A$.

For a vector $v\in\mathbb{R}^n$ we denote $[v]:=\{\lambda v\mid\lambda\in\mathbb{R}\}$ the linear subspace generated by $v$. For a nonempty set $C\subseteq\mathbb{R}^n$, the interior, the convex hull, and the positive hull of $C$ are denoted, respectively, by int $C$, conv $C$, and pos $C$. The orthogonal complement $C^\perp$ and the polar cone $C^*$ are defined, respectively, by

$$C^\perp := \{v\in\mathbb{R}^n \mid \langle v,x\rangle = 0\ \forall x\in C\},$$
$$C^* := \{v\in\mathbb{R}^n \mid \langle v,x\rangle \le 0\ \forall x\in C\}.$$

The distance from $x$ to $C$ is defined by $d(x,C) := \inf_{y\in C}||y-x||$. The projection mapping $\text{proj}_C$ is defined by $\text{proj}_C(x) := \{y\in C \mid d(x,y)=d(x,C)\}$. For a set $X\subset\mathbb{R}^n$, we denote the projection of $X$ onto $C$ by

$$\text{proj}_C X := \{y\in C \mid \exists x\in X \text{ with } d(x,y)=d(x,C)\}.$$

If $C=\emptyset$, by convention we set that $d(x,C):=+\infty$, $\text{proj}_C(x):=\emptyset$, and $\text{proj}_C X:=\emptyset$.

Let $x \in C$. We use $T_C(x)$ to denote the *tangent/contingent cone* to $C$ at $x$, i.e., $w \in T_C(x)$ if there exist sequences $t_k \searrow 0$ and $\{w_k\} \subset \mathbb{R}^n$ with $w_k \to w$ and $x + t_k w_k \in C$ $\forall k$. The *regular/Fréchet normal cone,* $\widehat{N}_C(x)$, is the polar cone of $T_C(x)$, which is equivalent to

$$\widehat{N}_C(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \limsup_{\substack{x' \xrightarrow{C} x \\ x' \neq x}} \frac{\langle v, x' - x\rangle}{\|x' - x\|} \leq 0 \right\}.$$

The *(basic/limiting/Mordukhovich) normal cone* to $C$ at $x$, $N_C(x)$, is defined via the outer limit of $\widehat{N}_C$ as

$$N_C(x) := \left\{ v \in \mathbb{R}^n \mid \exists \text{ sequences } x_k \xrightarrow{C} x,\ v_k \to v,\ v_k \in \widehat{N}_C(x_k)\, \forall k \right\}.$$

We say that $C$ is *locally closed* at a point $x \in C$ if $C \cap U$ is closed for some closed neighborhood $U \in \mathcal{N}(x)$. $C$ is said to be *regular at* $x$ in the sense of Clarke if it is locally closed at $x$ and $\widehat{N}_C(x) = N_C(x)$. $C$ is *regular around* $x$ if there exists a neighborhood of $x$ such that $C$ is regular at every point in it. For any $x \notin C$, we set by convention $T_C(x) = \emptyset,\ N_C(x) = \emptyset,\ \widehat{N}_C(x) = \emptyset$.

Let $C \subset \mathbb{R}^n$ be a nonempty convex set. A *face* of $C$ is a convex subset $C'$ of $C$ such that every closed line segment in $C$ with a relative interior point in $C'$ has both endpoints in $C'$. See the book [30] for more details. Moreover, we say $C$ is a *polyhedral* set if it can be expressed as the intersection of a finite number of closed half-spaces of hyperplanes (and is therefore convex).

For a multifunction $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we denote by gph $S := \{(x, u) \mid u \in S(x)\}$ and dom $S := \{x \mid S(x) \neq \emptyset\}$ the *graph* and the *domain* of $S$, respectively.

For a set $X \subset \mathbb{R}^n$, we denote by

$$S|_X(x) := S(x) \text{ if } x \in X;\ \emptyset \text{ if } x \notin X$$

the *restricted mapping* of $S$ on $X$. It is clear to see that gph $S|_X =$ gph $S \cap (X \times \mathbb{R}^m)$, dom $S|_X = X \cap \text{dom } S$ and also

$$\limsup_{x \xrightarrow{X} \bar{x}} S(x) = \limsup_{x \to \bar{x}} S|_X(x).$$

Definition 2.1 (outer semicontinuity [31, Definition 5.4]). *A multifunction $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous (osc) at $\bar{x}$ if* $\limsup_{x \to \bar{x}} S(x) = S(\bar{x})$.

Definition 2.2 (local boundedness relative to a set [31, p. 162]). *For a multifunction $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a closed set $X \subset \mathbb{R}^n$, and a given point $\bar{x} \in X$, if for some neighborhood $V \in \mathcal{N}(\bar{x})$, $S(V \cap X)$ is bounded, we say $S$ is locally bounded relative to $X$ at $\bar{x}$. Such a definition is equivalent to the local boundedness of $S|_X$ at $\bar{x}$.*

Next we present the definition of the relative Lipschitz-like property of a multifunction.

Definition 2.3 (Lipschitz-like property relative to a set [31, Definition 9.36]). *A multifunction $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is Lipschitz-like relative to $X$ at $\bar{x}$ for $\bar{u}$, where $\bar{x} \in X$ and $\bar{u} \in S(\bar{x})$, if* gph $S$ *is locally closed at $(\bar{x}, \bar{u})$ and there are neighborhoods $V \in \mathcal{N}(\bar{x})$ and $W \in \mathcal{N}(\bar{u})$ and a constant $\kappa \in \mathbb{R}_+$ such that*

$$S(x') \cap W \subset S(x) + \kappa \|x' - x\| \mathbb{B} \quad \forall x, x' \in X \cap V. \tag{2.1}$$

*The graphical modulus of $S$ relative to $X$ at $\bar{x}$ for $\bar{u}$ is defined as*

$$\operatorname{lip}_X S(\bar{x}\,|\,\bar{u}) := \inf\{\kappa \geq 0 \,|\, \exists V \in \mathcal{N}(\bar{x}), W \in \mathcal{N}(\bar{u}), \text{ such that}$$
$$S(x') \cap W \subset S(x) + \kappa\|x' - x\|\mathbb{B} \ \ \forall x, x' \in X \cap V\}.$$

*The property with $V$ in place of $X \cap V$ in* (2.1) *is the Lipschitz-like property along with the graphical modulus* $\operatorname{lip} S(\bar{x}\,|\,\bar{u})$.

Another notion of Lipschitz continuity mentioning $\bar{u}$ is the *local Lipschitz continuity around* $(\bar{x}, \bar{u})$, with the sets $S(x)$, $X \cap V$ being replaced by $S(x) \cap W$ and $X \cap V$, respectively (see [8]). It is also known as *truncated Lipschitz continuity* (see [9, p. 165]) and is generally stronger than the Lipschitz-like property.

Now we recall a projectional coderivative and a complete characterization of a Lipschitz-like property relative to a closed and convex set.

Definition 2.4 (see [25, Definition 2.2]). *The projectional coderivative $D^*_X S(\bar{x}\,|\,\bar{u}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of multifunction $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x} \in X \subseteq \mathbb{R}^n$ for any $\bar{u} \in S(\bar{x})$ with respect to $X$ is defined as*

$$t^* \in D^*_X S(\bar{x}\,|\,\bar{u})(u^*) \Longleftrightarrow (t^*, -u^*) \in \limsup_{(x,u) \xrightarrow{\text{gph } S|_X} (\bar{x},\bar{u})} \operatorname{proj}_{T_X(x) \times \mathbb{R}^m} N_{\text{gph } S|_X}(x, u).$$

A connection between projectional coderivative and coderivative is shown in the following corollary.

Corollary 2.5. *For a multifunction $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, and a closed set $X \subseteq \mathbb{R}^n$, for any $(\bar{x}, \bar{u}) \in \text{gph } S|_X$,*

$$D^* S|_X(\bar{x}\,|\,\bar{u})^{-1}(0) \subseteq D^*_X S(\bar{x}\,|\,\bar{u})^{-1}(0).$$

*Proof.* For $u^* \in D^* S|_X(\bar{x}\,|\,\bar{u})^{-1}(0)$, it is equivalent that $(0, -u^*) \in N_{\text{gph } S|_X}(\bar{x}, \bar{u})$. As $\operatorname{proj}_{T_X(\bar{x})}(0) = 0$, we have $(0, -u^*) \in \operatorname{proj}_{T_X(\bar{x}) \times \mathbb{R}^m} N_{\text{gph } S|_X}(\bar{x}, \bar{u})$. Then by the definition of projectional coderivatives, $u^* \in D^*_X S(\bar{x}\,|\,\bar{u})^{-1}(0)$. ◻

Theorem 2.6 (generalized Mordukhovich criterion [25, Theorem 2.4]). *Consider a multifunction $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $(\bar{x}, \bar{u}) \in$ gph $S|_X$. Suppose that* gph $S$ *is locally closed at $(\bar{x}, \bar{u})$ and that $X$ is closed and convex. Then $S$ has the Lipschitz-like property relative to $X$ at $\bar{x}$ for $\bar{u}$ if and only if $D^*_X S(\bar{x}\,|\,\bar{u})(0) = \{0\}$.*

When $\bar{x} \in \text{int } X$, the projectional coderivative mapping $D^*_X S(\bar{x}\,|\,\bar{u})$ reduces to the coderivative $D^* S(\bar{x}\,|\,\bar{u})$ and accordingly, the generalized Mordukhovich criterion reduces to the Mordukhovich criterion (see [26, 31]).

**3. Projectional coderivative and parametric systems.** In this section, we give a complete comparison between sufficient conditions for a relative Lipschitz-like property in [2, 25] and derive some upper estimates of a projectional coderivative for a fully parametric system (1.1).

First we introduce the definitions of the directional limiting normal cone and the directional limiting coderivative.

Definition 3.1 (see [13, Definition 2.3]). *For a closed set $\Omega \subset \mathbb{R}^n$ with $\bar{x} \in \Omega$ and a direction $u \in \mathbb{R}^n$, the directional limiting normal cone to $\Omega$ in direction $u$ at $\bar{x}$ is defined by*

$$N_\Omega(\bar{x}; u) := \limsup_{t \downarrow 0,\, u' \to u} \widehat{N}_\Omega(\bar{x} + tu'), \tag{3.1}$$

*while for a set-valued mapping* $S:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *having a locally closed graph around* $(\bar{w},\bar{x}) \in \mathrm{gph}\ S$ *and a pair of directions* $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$*, the set-valued mapping* $D^*S((\bar{w},\bar{x});(u,v)) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$*, defined by, for all* $v^* \in \mathbb{R}^m$*,*

$$D^*S((\bar{w},\bar{x});(u,v))(v^*) := \{u^* \in \mathbb{R}^n \mid (u^*,-v^*) \in N_{\mathrm{gph}\ S}((\bar{w},\bar{x});(u,v))\}, \tag{3.2}$$

*is called the directional limiting coderivative of* $S$ *in the direction* $(u,v)$ *at* $(\bar{w},\bar{x})$*.*

In the next theorem, we show that when we are referring to the set $X := \mathrm{dom}\ S$, the sufficient condition in [2, Theorem 3.5] implies the generalized Mordukhovich criterion when dom $S$ is closed and convex. For direct comparison, we adopt the explicit form introduced in [25, Theorem 2.5].

Theorem 3.2. *Consider* $S:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$*,* $\bar{x} \in \mathrm{dom}\ S \subset \mathbb{R}^n$*, and* $\bar{u} \in S(\bar{x})$*. Assume that* gph $S$ *is locally closed at* $(\bar{x},\bar{u})$ *and that* dom $S$ *is closed and convex around* $\bar{x}$*. Further assume that the following conditions are satisfied:*

(i) *For every* $x \in T_{\mathrm{dom}\ S}(\bar{x})$ *and every sequence* $t_k \downarrow 0$*, there exists some* $u \in \mathbb{R}^n$ *such that*

$$\liminf_{k\to\infty} \frac{d((\bar{x}+t_k x, \bar{u}+t_k u), \mathrm{gph}\ S)}{t_k} = 0.$$

(ii) *The equality*

$$D^*S((\bar{x},\bar{u});(x,u))(0) = \{0\}$$

*holds for all* $x \in T_{\mathrm{dom}\ S}(\bar{x})$ *and* $(x,u) \in T_{\mathrm{gph}\ S}(\bar{x},\bar{u})$ *with* $(x,u) \neq (0,0)$*.*

*Then* $D^*_{\mathrm{dom}\ S}S(\bar{x}\mid\bar{u})(0) = \{0\}$ *and* $S$ *has the Lipschitz-like property relative to* dom $S$ *at* $\bar{x}$ *for* $\bar{u}$*.*

*Proof.* Let $t^* \in D^*_{\mathrm{dom}\ S}S(\bar{x}\mid\bar{u})(0)$. By the definition of the projectional coderivative, there exist sequences

$$(x_k,u_k) \xrightarrow{\mathrm{gph}\ S} (\bar{x},\bar{u}),\ (x_k^*,-u_k^*) \in N_{\mathrm{gph}\ S}(x_k,u_k),\ t_k^* = \mathrm{proj}_{T_{\mathrm{dom}\ S}(x_k)}(x_k^*)$$

such that $t_k^* \to t^*$, $u_k^* \to 0$. Then by the definition of normal cones, we know that there exist sequences

$$(x_{kt},u_{kt}) \xrightarrow{\mathrm{gph}\ S} (x_k,u_k), (x_{kt}^*,-u_{kt}^*) \in \hat{N}_{\mathrm{gph}\ S}(x_{kt},u_{kt}), \text{ such that } (x_{kt}^*,u_{kt}^*) \to (x_k^*,u_k^*).$$

If $(x_k,u_k) \neq (\bar{x},\bar{u})$, let

$$\tau_{kt} := \|(x_{kt}-\bar{x},u_{kt}-\bar{u})\|,\ (x'_{kt},u'_{kt}) := \tfrac{(x_{kt}-\bar{x},u_{kt}-\bar{u})}{\|(x_{kt}-\bar{x},u_{kt}-\bar{u})\|},\ (x'_k,u'_k) := \tfrac{(x_k-\bar{x},u_k-\bar{u})}{\|(x_k-\bar{x},u_k-\bar{u})\|}. \tag{3.3}$$

Then we have

$$(x'_{kt},u'_{kt}) \to (x'_k,u'_k),\ \tau_{kt} \searrow 0 \text{ and } (x_{kt}^*,-u_{kt}^*) \in \hat{N}_{\mathrm{gph}\ S}((\bar{x},\bar{u})+\tau_{kt}(x'_{kt},u'_{kt})).$$

By the definition of a directional normal cone we have $(x_k^*,-u_k^*) \in N_{\mathrm{gph}\ S}((\bar{x},\bar{u}); (x'_k,u'_k))$. Besides, by the definition of tangent cones, $(x'_k,u'_k) \in T_{\mathrm{gph}\ S}(\bar{x},\bar{u}) \cap \mathbb{S}$ (where $\mathbb{S}$ denotes the unit sphere) and $x'_k \in T_{\mathrm{dom}\ S}(\bar{q})$. By condition (ii), we have that $u_k^* \to 0$ indicates $x_k^* \to 0$. Therefore

$$t_k^* = \mathrm{proj}_{T_{domS}(x_k)}(x_k^*) \to 0 = t^*.$$

If $(x_k,u_k)=(\bar x,\bar u)$, then we have

$$(x^*,-u^*)\in N_{\text{gph } S}(\bar x,\bar u) \text{ and } t^*=\text{proj}_{T_{\text{dom } S}(\bar x)}(x^*),\ u^*=0.$$

Without loss of generality, we can further assume that $(x^*,-u^*)\in \hat N_{\text{gph } S}(\bar x,\bar u)$, as otherwise by the construction in (3.3) we can always find some $(x'_k,u'_k)\in T_{\text{gph } S}(\bar x,\bar u)\cap \mathbb{S}$ and the argument is similar to the above. Given $t^*\in T_{\text{dom } S}(\bar x)$, by condition (i), we can find accordingly $u'$ such that $(t^*,u')\in T_{\text{gph } S}(\bar x,\bar u)$. By the polar relation between tangent cones and regular normal cones, we have

$$\langle (t^*,u'),(x^*,-u^*)\rangle = \langle t^*,x^*\rangle \le 0.$$

Given the convexity of $T_{\text{dom } S}(\bar x)$, the decomposition of $x^*=t^*+y^*$ is unique with $t^*=\text{proj}_{T_{\text{dom } S}(\bar x)}(x^*)$ and $y^*=\text{proj}_{N_{\text{dom } S}(\bar x)}(x^*)$, and $t^*\perp y^*$ by [31, Exercise 12.22]. Therefore we have $t^*=0$ and $D^*_{\text{dom } S}S(\bar x\mid\bar u)(0)=\{0\}$. By Theorem 2.6 $S$ is Lipschitz-like relative to dom $S$ at $\bar x$ for $\bar u$. ∎

Next we give an example where a fixed-point expression of the projectional coderivative can be given when the set $X$ is a smooth manifold around the point $\bar x$ (see [31, Example 6.8] for the definition of smooth manifold).

Lemma 3.3 (projectional coderivatives of a set-valued mapping restricted on a smooth manifold). *Consider $S:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ and $\bar u\in S(\bar x)$. Suppose that* gph $S$ *is locally closed at $(\bar x,\bar u)$ and $X$ is a smooth manifold around $\bar x$. Then we have*

$$D^*_X S(\bar x\mid\bar u)(u^*)=\text{proj}_{T_X(\bar x)}D^*S|_X(\bar x\mid\bar u)(u^*) \quad \forall u^*.$$

*Proof.* By Definition 2.4, it suffices to show

$$D^*_X S(\bar x\mid\bar u)(u^*)\subseteq\text{proj}_{T_X(\bar x)}D^*S|_X(\bar x\mid\bar u)(u^*) \quad \forall u^*.$$

Let $y^*\in D^*_X S(\bar x\mid\bar u)(u^*)$. Then there exist some sequences $(x_k,u_k)\xrightarrow{\text{gph } S|_X}(\bar x,\bar u)$ and $x^*_k\in D^*S|_X(x_k\mid u_k)(u^*_k)$ such that $u^*_k\to u^*$ and $y^*_k:=\text{proj}_{T_X(x_k)}(x^*_k)\to y^*$. By a representation of a tangent cone of the smooth manifold [31, Example 6.8], $T_X(\cdot)$ is continuous at $\bar x$ relative to $X$. Together with [31, Exercise 5.35], we have that $y^*=\text{proj}_{T_X(\bar x)}(x^*)$. This completes the proof. ∎

We now investigate upper estimates of a projectional coderivative of the mapping $S$ defined in (1.1). Recall that upper estimates of coderivatives of $S$ have been given in [23, Theorem 2.1]. We also discuss the upper estimates under two special cases: (i) $G(w,x)=0$, (ii) $M(w,x)=M(x)$ and $X=\text{dom } S$.

Theorem 3.4. *Consider the implicit mapping $S:\mathbb{R}^m\rightrightarrows\mathbb{R}^n$ of the form* (1.1) *with $G:\mathbb{R}^{m+n}\to\mathbb{R}^d$ a $\mathcal{C}^1$ mapping, and $M:\mathbb{R}^{m+n}\rightrightarrows\mathbb{R}^d$ a multifunction with closed graph. Consider a pair $(\bar w,\bar x)\in$ gph $S|_W$, where $W\subseteq$ dom $S$ is a closed set. Let $\mathcal{M}(w,x,y):=\nabla G(w,x)^*y+D^*M|_{W\times\mathbb{R}^n}((w,x)|-G(w,x).)(y)$. If the following basic constraint qualification holds,*

$$(0,0)\in\mathcal{M}(\bar w,\bar x,y)\Longrightarrow y=0, \tag{3.4}$$

*then we have*

$$D^*_W S(\bar w\mid\bar x)(r)\subseteq \limsup_{\substack{(w,x)\xrightarrow{\text{gph } S|_W}(\bar w,\bar x)\\ r'\to r}} \bigcup_{y\in\mathbb{R}^d}\left\{\text{proj}_{T_W(w)}(v)\ \mid (v,-r')\in\mathcal{M}(w,x,y)\right\}. \tag{3.5}$$

*If the following strong constraint qualification is satisfied,*

$$(0,0) \in \limsup_{\substack{(w,x)\xrightarrow{\text{gph } S|_W}(\bar{w},\bar{x}) \\ y'\to y}} \operatorname{proj}_{T_W(w)\times\mathbb{R}^n}\mathcal{M}(w,x,y') \Longrightarrow y=0, \tag{3.6}$$

*then the limsup in* (3.5) *can be equivalently put into the bracket as*

$$\begin{aligned} D^*_W S(\bar{w}\mid\bar{x})(r) \subseteq \Bigg\{ t\in\mathbb{R}^m \,\Bigg|\, & \exists y\in\mathbb{R}^d \text{ with } (t,-r) \\ & \in \limsup_{\substack{(w,x,-G(w,x))\xrightarrow{\text{gph } M|_{W\times\mathbb{R}^n}}(\bar{w},\bar{x},-G(\bar{w},\bar{x})) \\ y'\to y}} \operatorname{proj}_{T_W(w)\times\mathbb{R}^n}\mathcal{M}(w,x,y')\Bigg\}. \end{aligned} \tag{3.7}$$

*If in addition,* $M|_{W\times\mathbb{R}^n}$ *is graphically regular at* $(\bar{w},\bar{x},-G(\bar{w},\bar{x}))$ *and* $W$ *is a smooth manifold around* $\bar{w}$*, then*

$$D^*_W S(\bar{w}\mid\bar{x})(r) = \Bigg\{ t\in\mathbb{R}^m \,\Bigg|\, \exists y\in\mathbb{R}^d \text{ with } (t,-r)\in \operatorname{proj}_{T_W(\bar{w})\times\mathbb{R}^n}\mathcal{M}(\bar{w},\bar{x},y)\Bigg\}. \tag{3.8}$$

*Proof.* By a simple statement of contradiction with the osc of $D^*M|_{W\times\mathbb{R}^n}$, (3.4) also indicates that

$$(0,0)\in\mathcal{M}(w,x,y)\Longrightarrow y=0$$

for any $(w,x)\in \text{gph } S|_W$ sufficiently close to $(\bar{w},\bar{x})$. According to [23, Theorem 2.1], for any pair $(w,x)\in\text{gph } S|_W$ sufficiently near $(\bar{w},\bar{x})$, we have

$$N_{\text{gph } S|_W}(w,x) \subseteq \bigcup_{y\in\mathbb{R}^d}\mathcal{M}(w,x,y). \tag{3.9}$$

Therefore we have

$$\begin{aligned} &\limsup_{(w,x)\xrightarrow{\text{gph } S|_W}(\bar{w},\bar{x})} \operatorname{proj}_{T_W(w)\times\mathbb{R}^n} N_{\text{gph } S|_W}(w,x) \\ &\qquad \subseteq \limsup_{(w,x)\xrightarrow{\text{gph } S|_W}(\bar{w},\bar{x})} \bigcup_{y\in\mathbb{R}^d} \operatorname{proj}_{T_W(w)\times\mathbb{R}^n}\mathcal{M}(w,x,y) \end{aligned}$$

and accordingly the inclusion (3.5) holds.

Now, assume that the strong constraint qualification (3.6) holds. By the nature of projection and the outer limit, we can also see that (3.6) indicates (3.4). For $t$ belonging to the right-hand side of (3.5), there exist sequences $(w_k,x_k)\xrightarrow{\text{gph } S|_W}(\bar{w},\bar{x})$, $y_k\in\mathbb{R}^d$, and $(v_k,-r_k)\in\mathcal{M}(w_k,x_k,y_k)$, such that $t_k\in\operatorname{proj}_{T_W(w_k)}(v_k)\to t$ and $r_k\to r$. Taking a subsequence if necessary, we have either $y_k\to y\in\mathbb{R}^d$ or $\lambda_k y_k\to y\in\mathbb{R}^d$ with $\lambda_k\searrow 0$. For the first case, we directly have that $t$ belongs to the right-hand side of (3.7). For the second case, without loss of generality we assume $\|y\|=1$. With the conic structure we have $\lambda_k(v_k,-r_k)\in\mathcal{M}(w_k,x_k,\lambda_k y_k)$ and accordingly $\lambda_k t_k\in\lambda_k\operatorname{proj}_{T_W(w_k)}(v_k)\to 0$, $\lambda_k r_k\to 0$, which contradicts the constraint qualification (3.6) with $\|y\|=1$. Thus the second case is not possible. Given that $(w,x)\xrightarrow{\text{gph } S|_W}(\bar{w},\bar{x})$ is equivalent to $(w,x,-G(w,x))\xrightarrow{\text{gph } M|_{W\times\mathbb{R}^n}}(\bar{w},\bar{x},-G(\bar{w},\bar{x}))$ we have that $t$ also belongs to the set on the right-hand side of (3.7). Note that in general the right-hand side of (3.7) is included by that of (3.5) and therefore, with constraint qualification (3.6) being satisfied, these two sets are identical.

If furthermore $M|_{W\times\mathbb{R}^n}$ is graphically regular at $(\bar{w},\bar{x},-G(\bar{w},\bar{x}))$, again by [23, Theorem 2.1], we have (3.9) as an equation at the reference point $(\bar{w},\bar{x})$ and therefore

$$\begin{aligned}&\mathrm{proj}_{T_W(\bar{w})}D^*S|_W(\bar{w}\,|\,\bar{x})(r)\\ &=\left\{t\in\mathbb{R}^m\,\middle|\,\exists y\in\mathbb{R}^d \text{ with } (t,-r)\in\mathrm{proj}_{T_W(\bar{w})\times\mathbb{R}^n}\mathcal{M}(\bar{w},\bar{x},y)\right\}\\ &\subseteq D^*_W S(\bar{w}\,|\,\bar{x})(r).\end{aligned}\tag{3.10}$$

Besides, when $W$ is a smooth manifold at $\bar{w}$, by Lemma 3.3 and (3.9),

$$\begin{aligned}D^*_W S(\bar{w}\,|\,\bar{x})(r) &= \mathrm{proj}_{T_W(\bar{w})}D^*S|_W(\bar{w}\,|\,\bar{x})(r)\\ &\subseteq\left\{t\in\mathbb{R}^m\,\middle|\,\exists y\in\mathbb{R}^d \text{ with } (t,-r)\in\mathrm{proj}_{T_W(\bar{w})\times\mathbb{R}^n}\mathcal{M}(\bar{w},\bar{x},y)\right\}.\end{aligned}\tag{3.11}$$

Combining the conditions that $M|_{W\times\mathbb{R}^n}$ is graphically regular at $(\bar{w},\bar{x},-G(\bar{w},\bar{x}))$ and that $W$ is a smooth manifold at $\bar{w}$, (3.10) and (3.11) turn into (3.8). ◻

Next we use a simple example to illustrate how the strong constraint qualification (3.6) can be applied in calculating the projectional coderivative (3.8).

*Example* 3.5. For $S(w):=\{x\in\mathbb{R}^n\mid Ax+w\in K\}$, where $K\subseteq\mathbb{R}^m$ is a closed set, let $G(w,x)=-Ax-w$ and $M(w,x)=K$. For $W\subseteq\mathrm{dom}\,S$ we can write $\mathrm{gph}\,M|_{W\times\mathbb{R}^n}=W\times\mathbb{R}^n\times K$ and accordingly

$$D^*M|_{W\times\mathbb{R}^n}((w,x)\,|\,u)(y)=\begin{cases}N_W(w)\times\{0\} & \text{if } y\in -N_K(u),\\ \emptyset & \text{if } y\notin -N_K(u),\end{cases}$$
$$\nabla G(w,x)^*y=(-y,-A^*y).$$

Let $n=m=2$, $K=\mathbb{R}\times\{0\}\cup\{0\}\times\mathbb{R}$, $A=\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. Then $\mathrm{dom}\,S=K+\mathrm{range}A=\mathbb{R}^2$. Consider the particular pair $(\bar{w},\bar{x})\in\mathrm{gph}\,S|_W$, where $\bar{w}=(0,1)^\top$, $\bar{x}=(0,0)^\top$, and a smooth manifold $W=\mathbb{R}\times\{1\}\subseteq\mathrm{dom}\,S$. Then the constraint qualification (3.6), by Lemma 3.3, becomes

$$\begin{aligned}(0,0)&\in\mathrm{proj}_{T_W(\bar{w})\times\mathbb{R}^n}\mathcal{M}(\bar{w},\bar{x},y)\\ &=\left\{(\mathrm{proj}_{T_W(\bar{w})}(v-y),-A^*y)\mid v\in N_W(\bar{w}),y\in -N_K(A\bar{x}+\bar{w})\right\}\Longrightarrow y=0.\end{aligned}$$

As $-G(\bar{w},\bar{x})=A\bar{x}+\bar{w}=(0,1)^\top$, $K$ is regular at $-G(\bar{w},\bar{x})$ and $N_K(-G(\bar{w},\bar{x}))=N_K((0,1)^\top)=\mathbb{R}\times\{0\}$. Thus $M|_{W\times\mathbb{R}^n}$ is graphically regular at $(\bar{w},\bar{x},-G(\bar{w},\bar{x}))$. Besides, in view of the facts that $T_W(\bar{w})=\mathbb{R}\times\{0\}$ and $y\in -N_K(A\bar{x}+\bar{w})=\mathbb{R}\times\{0\}$ and by the polar relation between $N_W(\bar{w})$ and $T_W(\bar{w})$,

$$0=\mathrm{proj}_{T_W(\bar{w})}(v-y)=\mathrm{proj}_{T_W(\bar{w})}(-y)=-y\Longrightarrow y=0.$$

Thus the constraint qualification (3.6) is satisfied. Applying (3.8), we obtain

$$D^*_W S(\bar{w},\bar{x})(r)=\{y\mid y\in N_K(A\bar{w}+\bar{x}) \text{ with } A^*y=r\}=\begin{cases}\mathbb{R}\times\{0\} & \text{if } r=(0,0)^\top,\\ \emptyset & \text{if } r\neq(0,0)^\top.\end{cases}$$

Thus, $D^*_W S(\bar{w},\bar{x})((0,0)^\top)=\mathbb{R}\times\{0\}\neq\{(0,0)^\top\}$. As $W$ is also a convex set, $S$ does not enjoy the Lipschitz-like property relative to $W$ at $\bar{w}$ for $\bar{x}$ according to the generalized Mordukhovich criterion (Theorem 2.6).

By observing the right-hand side of the expression (3.7), we can see that $(t, -r)$ actually belongs to something that is very close to the projectional coderivative of the multifunction $G(w,x) + M(w,x)$ relative to the set $W \times \mathbb{R}^n$ at $(\bar{w}, \bar{x})$ for 0. Next we present a simpler model by taking $G(w,x) = 0$ so that the relation of projectional coderivatives between $S$ and $M$ can be revealed more clearly.

Corollary 3.6. *For an implicit mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ as*

$$S(w) = \{x \mid 0 \in M(w,x)\},$$

*where $M : \mathbb{R}^m \times \mathbb{R}^n \rightrightarrows \mathbb{R}^d$ has closed graph, let a closed set $W \subseteq \mathbb{R}^m$ be given and let $\bar{x} \in S|_W(\bar{w})$. If the basic constraint qualification holds,*

$$(0,0) \in D^* M|_{W\times\mathbb{R}^n}\left((\bar{w},\bar{x}) \mid 0\right)(y) \Longrightarrow y = 0, \tag{3.12}$$

*then*

$$\begin{aligned} &D^*_W S\left(\bar{w} \mid \bar{x}\right)(r) \\ &\quad\subseteq \limsup_{\substack{(w,x)\xrightarrow{\text{gph } S|_W}(\bar{w},\bar{x}) \\ r'\to r}} \bigcup_{y\in\mathbb{R}^d} \left\{ \text{proj}_{T_W(w)}(v) \,\middle|\, (v, -r') \in D^* M|_{W\times\mathbb{R}^n}\left((w,x) \mid 0\right)(y) \right\}. \end{aligned} \tag{3.13}$$

*If the strong constraint qualification holds,*

$$(0,0) \in D^*_{W\times\mathbb{R}^n} M\left((\bar{w},\bar{x}) \mid 0\right)(y) \Longrightarrow y = 0, \tag{3.14}$$

*then we have*

$$D^*_W S(\bar{w} \mid \bar{x})(r) \subseteq \left\{ t \mid \exists y \text{ s.t. } (t, -r) \in D^*_{W\times\mathbb{R}^n} M((\bar{w},\bar{x}) \mid 0)(y) \right\}. \tag{3.15}$$

*When in addition $M|_{W\times\mathbb{R}^n}$ is graphically regular at $(\bar{w}, \bar{x}, 0)$ and $W$ is a smooth manifold at $\bar{w}$, the inclusions* (3.13) *and* (3.15) *are identical and become equations.*

*Proof.* This corollary comes from direct application of Theorem 3.4 by taking $G(w,x) = 0$. As $(w,x) \xrightarrow{\text{gph } S|_W} (\bar{w}, \bar{x})$ is equivalent to $(w,x,0) \xrightarrow{\text{gph } M|_{W\times\mathbb{R}^n}} (\bar{w}, \bar{x}, 0)$, therefore

$$\limsup_{\substack{(w,x)\xrightarrow{\text{gph } S|_W}(\bar{w},\bar{x}) \\ y'\to y}} \text{proj}_{T_W(w)\times\mathbb{R}^n} D^* M|_{W\times\mathbb{R}^n}\left((w,x)|0\right)(y') \subseteq D^*_{W\times\mathbb{R}^n} M\left((\bar{w},\bar{x})|0\right)(y).$$

Then we can rewrite the inclusions in Theorem 3.4 as (3.13) and (3.15), respectively. ∎

In Theorem 3.4, two different constraint qualifications are mentioned. We can see that the basic one (3.4) is ensuring the upper estimate in two ways: (i) restricting $S$ to $W$; (ii) expressing the normal cone of gph $S$ via those of gph $G$ and gph $M$. The strong one, as shown in (3.14), aims at presenting the projectional coderivative of $S$ via that of $M$ when $G$ vanishes. In the next theorem, we give a setting where the basic constraint qualification (3.4) is automatically satisfied when we consider the largest possible $W$ as dom $S$.

THEOREM 3.7. *For $S$ defined in* (1.1), *if $M(w,x)=M(x)$ and $\nabla_w G(w,x)$ has full rank around $(\bar w,\bar x)\in \operatorname{gph} S$, then*

$$D^*_{\mathrm{dom}S} S(\bar w \mid \bar x)(r) = \limsup_{\substack{(w,x)\xrightarrow{\mathrm{gph}\ S}(\bar w,\bar x)\\ r'\to r}} \bigcup_{y\in\mathbb{R}^d} \Big\{ \mathrm{proj}_{T_{domS}(w)}(\nabla_w G(w,x)^* y) \tag{3.16}$$
$$\Big| -r' \in \nabla_x G(w,x)^* y + D^* M(x \mid -G(w,x))(y) \Big\}.$$

*Proof.* When the set $W := \operatorname{dom} S$, $S|_W = S$. By condition (b) in [23, Theorem 2.1], we have for any $(w,x)\in \operatorname{gph} S$,

$$N_{\mathrm{gph}\ S}(w,x) = \bigcup_{y\in\mathbb{R}^d} \left(\nabla G(w,x)^* y + \{0\}\times D^* M(x \mid -G(w,x))(y)\right).$$

Therefore we have

$$\begin{aligned} &\limsup_{(w,x)\xrightarrow{\mathrm{gph}\ S|_W}(\bar w,\bar x)} \mathrm{proj}_{T_W(w)\times\mathbb{R}^n} N_{\mathrm{gph}\ S|_W}(w,x) \\ =& \limsup_{(w,x)\xrightarrow{\mathrm{gph}\ S}(\bar w,\bar x)} \mathrm{proj}_{T_{\mathrm{dom}\ S}(w)\times\mathbb{R}^n} N_{\mathrm{gph}\ S}(w,x) \\ =& \limsup_{(w,x)\xrightarrow{\mathrm{gph}\ S}(\bar w,\bar x)} \bigcup_{y\in\mathbb{R}^d} \mathrm{proj}_{T_{\mathrm{dom}\ S}(w)\times\mathbb{R}^n} \left(\nabla G(w,x)^* y + \{0\}\times D^* M(x \mid -G(w,x))(y)\right) \end{aligned}$$

and thus the equality (3.16). ∎

## 4. Affine variational inequalities.

In this section, we consider the following AVI:

$$0 \in q + Mx + N_C(x),$$

where $C\subset\mathbb{R}^n$ is a polyhedral set and $M$ is an $n\times n$ matrix. Here we consider that $q$ is a parameter. The solution mapping of AVI is written as

$$S(q) = \{x \mid 0 \in q + Mx + N_C(x)\}. \tag{4.1}$$

Note that for AVI, gph $S$ is always a union of finitely many polyhedral sets as it is a linear transformation of gph $N_C$ (see [31, Example 12.31] and [8]). For a closed subset $Q\subseteq \operatorname{dom} S$, the graph of the multifunction $S$ restricted on $Q$ is

$$\operatorname{gph} S|_Q = \operatorname{gph} S \cap (Q\times\mathbb{R}^n) = \{(q,x)\in Q\times\mathbb{R}^n \mid 0\in q+Mx+N_C(x)\}. \tag{4.2}$$

We first obtain a upper estimate of the projectional coderivative of $S$, when $Q$ is a union of polyhedral sets. In this way, we can skip the limsup appearing in, e.g., (3.5), and express the upper estimate of $D^*_Q S$ in the form of a union within the range of a ball centered at $(\bar q,\bar x)$.

PROPOSITION 4.1. *For the solution mapping $S$* (4.1) *of AVI, consider a union of polyhedral sets $Q\subseteq \operatorname{dom} S$. Let $(\bar q,\bar x)\in \operatorname{gph} S|_Q$. If the following basic constraint qualification holds,*

$$u^*\in N_Q(\bar q),\ (M^*u^*,u^*)\in N_{\mathrm{gph}\ N_C}(\bar x,-M\bar x-\bar q) \Longrightarrow u^*=0, \tag{4.3}$$

*then*

$$D^*_Q S(\bar{q}\,|\,\bar{x})(y^*) \subseteq \bigcup_{(q,x)\in \mathrm{gph}\ S|_Q\cap \mathbb{B}_\varepsilon(\bar{q},\bar{x})} \Big\{ \mathrm{proj}_{T_Q(q)}(-u^*+w^*) \;\Big|\; w^* \in N_Q(q) \quad \exists\, u^* \text{ s.t. } (M^*u^*-y^*,u^*) \in N_{\mathrm{gph}\ N_C}(x,-Mx-q) \Big\} \tag{4.4}$$

*for sufficiently small* $\varepsilon > 0$. *If one of the following conditions is satisfied,*

(a) gph $N_C$ *and* $Q$ *are regular around* $(\bar{x}, -M\bar{x}-\bar{q})$ *and* $\bar{q}$, *respectively,*

(b) $Q = \mathrm{dom}\ S$ *(in this case the constraint qualification* (4.3) *can be avoided), then the inclusion* (4.4) *becomes an equality.*

*Proof.* For any $q, x \in \mathbb{R}^n$, let

$$\Gamma|_{Q\times\mathbb{R}^n}(q,x) := N_C(x) \text{ if } q\in Q; \quad \emptyset \text{ if } q \notin Q.$$

For any $q\in Q, x\in\mathbb{R}^n,$ and $v\in N_C(x)$, we have

$$D^*\Gamma|_{Q\times\mathbb{R}^n}((q,x)\,|\,v) = N_Q(q)\times D^*N_C(x\,|\,v).$$

Then the mapping $S$ (4.1) is rewritten as

$$S|_Q(q) = \{x \mid 0 \in q + Mx + \Gamma|_{Q\times\mathbb{R}^n}(q,x)\}. \tag{4.5}$$

Note that the constraint qualification (3.4) becomes

$$(0,0) = (u^*, M^*u^*) + (w^*, v^*) \text{ with } w^*\in N_Q(\bar{q}),\ v^*\in D^*N_C(\bar{x}\,|\,-M\bar{x}-\bar{q})(u^*) \Longrightarrow u^* = 0,$$

which is equivalent to

$$-u^*\in N_Q(\bar{q}),\ -M^*u^* \in D^*N_C(\bar{x}\,|\,-M\bar{x}-\bar{q})(u^*) \Longrightarrow u^* = 0.$$

By tuning the direction of $u^*$, we arrive at (4.3).

And the upper estimate (3.5) can be put as

$$\begin{aligned}
D^*_Q S(\bar{q}\,|\,\bar{x})(y^*) &\subseteq \limsup_{\substack{(q,x)\xrightarrow{\mathrm{gph}\ S|_Q}(\bar{q},\bar{x})\\ y'^*\to y^*}} \bigcup_{u^*\in\mathbb{R}^n} \Big\{ \mathrm{proj}_{T_Q(q)}(t^*) \;\Big|\; (t^*, -y'^*) \in (u^*, M^*u^*) \\
&\qquad + N_Q(q)\times D^*N_C(x\,|\,-Mx-q)(u^*) \Big\} \\
&= \limsup_{\substack{(q,x)\xrightarrow{\mathrm{gph}\ S|_Q}(\bar{q},\bar{x})\\ y'^*\to y^*}} \bigcup_{u^*\in\mathbb{R}^n} \Big\{ \mathrm{proj}_{T_Q(q)}(u^*+w^*) \;\Big|\; y'^* = -M^*u^*-v^*, \\
&\qquad w^*\in N_Q(q), v^*\in D^*N_C(x\,|\,-Mx-q)(u^*) \Big\} \\
&= \bigcup_{(q,x)\in\mathrm{gph}\ S|_Q\cap\mathbb{B}_\varepsilon(\bar{q},\bar{x})} \bigcup_{u^*\in\mathbb{R}^n} \Big\{ \mathrm{proj}_{T_Q(q)}(u^*+w^*) \Big| y^* = -M^*u^*-v^*, \\
&\qquad w^*\in N_Q(q), v^*\in D^*N_C(x\,|\,-Mx-q)(u^*) \Big\}
\end{aligned}$$

for sufficiently small $\varepsilon > 0$. Here the second equality comes from the polyhedrality of both gph $S|_Q$ and $Q$ as only finitely many possible statuses are considered. By adjusting the expression of $y^*$ and direction of $u^*$, we finally arrive at (4.4). Under condition (a), we have $N_{\text{gph } S|_Q}(q,x) = N_Q(q)\times\{0\} + N_{\text{gph } S}(q,x)$ for $(q,x)$ near $(\bar{q},\bar{x})$ by [31, Theorem 6.14]. Accordingly the inclusion in (4.4) becomes an equality. Under condition (b), the equality in (4.4) is derived as an application of Theorem 3.7. ∎

Given the upper estimate (4.4), we can give a sufficient condition for the Lipschitz-like property of $S$ relative to $Q$ when $Q$ is further convex, that is, $Q$ is a polyhedral set. Moreover, based on the critical face condition introduced in [8], the sufficient condition can be simplified concerning the given point only. As such we derive a "generalized critical face condition." To do this, let us review two notations introduced in [8]. For a polyhedral set $C \subset \mathbb{R}^n$ and $(x,v) \in \text{gph } N_C$, the critical cone $K(x,v)$ is defined by

$$K(x,v) = T_C(x) \cap [v]^{\perp}.$$

Let $\mathcal{F}(K)$ be the collection of all the closed faces of polyhedral cone $K$ in the form of $F = K \cap [v^*]^{\perp}$ with $v^* \in K^*$. It is clear that such an $F$ is also a polyhedral cone.

The following lemma plays an important role in the development of a critical face condition in [8], which will be useful here as well.

Lemma 4.2 (see [8, Reduction Lemma]). *For any $(x,v) \in \text{gph } N_C$, there is a neighborhood $U$ of $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that, for $(x',v') \in U$, one has*

$$v + v' \in N_C(x+x') \Longleftrightarrow v' \in N_{K(x,v)}(x').$$

*In particular, $T_{\text{gph } N_C}(x,v) = \text{gph } N_{K(x,v)}$.*

With the Reduction Lemma, the local geometry of gph $N_C$ around $(\bar{x},\bar{v})$ can be observed via that of gph $N_{K(\bar{x},\bar{v})}$ and allows us to express $N_{\text{gph } N_C}(x,v)$ via the faces of the critical cone $K(x,v)$. From the proof of [8, Theorem 2], for any pair $(x,v) \in \text{gph } N_C$, we have

$$N_{\text{gph } N_C}(x,v) = \left\{ (F_1 - F_2)^* \times (F_1 - F_2) \mid F_2 \subset F_1 \in \mathcal{F}(K(x,v)) \right\}. \tag{4.6}$$

In [11], the directional limiting normal cone of gph $N_C$ is expressed with critical faces. In light of this expression, we have the following result.

Lemma 4.3. *For a given pair $(x,v) \in \text{gph } N_C$ and $(x',v') \in T_{\text{gph } N_C}(x,v)$ sufficiently near to $(0,0)$, we have*

$$\begin{aligned} N_{\text{gph } N_C}(x+x', v+v') = \big\{ (F_1 - F_2)^* \times (F_1 - F_2) \mid\ & x' \in F_2 \subset F_1 \subset [v']^{\perp}, \\ & F_1, F_2 \in \mathcal{F}(K(x,v)) \big\}. \end{aligned} \tag{4.7}$$

*Proof.* As $(x',v') \in T_{\text{gph } N_C}(x,v)$ is sufficiently near to $(0,0)$, we have

$$\begin{aligned} N_{\text{gph } N_C}(x+x', v+v') &= \limsup_{(x'',v'') \to (x',v')} \widehat{N}_{\text{gph } N_C}\left((x,v) + (x'',v'')\right) \\ &= \limsup_{\substack{t \searrow 0 \\ (\tilde{x},\tilde{v}) \to (x',v')}} \widehat{N}_{\text{gph } N_C}\left((x,v) + t(\tilde{x},\tilde{v})\right) \\ &= N_{\text{gph } N_C}\left((x,v);(x',v')\right). \end{aligned}$$

The last equality comes from the definition of directional limiting normal cone (3.1). Then (4.7) is obtained by [11, Theorem 2.12]. ∎

In the following theorem, we present the "generalized critical face condition" as an inclusion, which reduces to the one in [8] when $\bar{q} \in \text{int } Q$.

THEOREM 4.4. *For* $(\bar{q},\bar{x}) \in \mathrm{gph}\, S|_Q$ *and* $Q$ *being a polyhedral set, suppose the following constraint qualification holds:*

$$N_Q(\bar{q}) \cap (F_1 - F_2) \cap (M(F_1 - F_2))^* = \{0\}, \tag{4.8}$$

*where* $F_1, F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))$ *are closed faces with* $F_2 \subset F_1$ *and* $\bar{v} = -M\bar{x} - \bar{q}$*. If, for any such* $F_1, F_2$*, the following inclusion is satisfied,*

$$(F_1 - F_2) \cap (M(F_1 - F_2))^* \subseteq -N_Q(\bar{q}), \tag{4.9}$$

*then* $S$ *has Lipschitz-like property relative to* $Q$ *at* $\bar{q}$ *for* $\bar{x}$*. Furthermore, we have the following:*

(i) *If* gph $N_C$ *is regular around* $(\bar{x},\bar{v})$*, then the sufficient condition* (4.9) *becomes also necessary.*

(ii) *If the set* $Q = \mathrm{dom}\, S$ *(in this case,* dom $S$ *is polyhedral), then the condition* (4.9) *is sufficient and necessary regardless of the satisfaction of* (4.8).

*Proof.* Noting that gph $S|_Q$ is a union of finitely many polyhedral sets, for any $(q,x) \in \mathrm{gph}\, S|_Q$ sufficiently close to $(\bar{q},\bar{x})$, $(q',x') := (q-\bar{q}, x-\bar{x}) \in T_{\mathrm{gph}\, S|_Q}(\bar{q},\bar{x})$ is sufficiently close to $(0,0)$. By (4.2) and [31, Exercise 6.7, Theorem 6.42], we have

$$\begin{aligned} T_{\mathrm{gph}\, S|_Q}(\bar{q},\bar{x}) &\subseteq (T_Q(\bar{q}) \times \mathbb{R}^n) \cap T_{\mathrm{gph}\, S}(\bar{q},\bar{x}) \\ &= \{(\tilde{q},\tilde{x}) \mid \tilde{q} \in T_Q(\bar{q}),\ (\tilde{x}, -M\tilde{x} - \tilde{q}) \in T_{\mathrm{gph}\, N_C}(\bar{x},\bar{v})\}. \end{aligned} \tag{4.10}$$

Therefore $q' \in T_Q(\bar{q})$ and $(x', -Mx' - q') \in T_{\mathrm{gph}\, N_C}(\bar{x},\bar{v})$. By polyhedrality of $Q$, it holds that

$$N_Q(\bar{q} + q') = N_Q(\bar{q}) \cap [q']^{\perp}. \tag{4.11}$$

As $(\bar{q},\bar{x}) \in \mathrm{gph}\, S|_Q$, we have $(\bar{x}, -M\bar{x} - \bar{q}) \in \mathrm{gph}\, N_C$. Thus by (4.6), we can derive the constraint qualification (4.3) as

$$u^* \in N_Q(\bar{q}),\ (M^*u^*, u^*) \in (F_1 - F_2)^* \times (F_1 - F_2) \Longrightarrow u^* = 0 \tag{4.12}$$

for some $F_1, F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))$ with $F_2 \subset F_1$.

By using the upper estimate in Proposition 4.1 and the generalized Mordukhovich criterion in Theorem 2.6, it would be sufficient to examine the Lipschitz-like property of $S$ relative to $Q$ at $(\bar{q},\bar{x})$ by checking

$$(M^*u^*, u^*) \in N_{\mathrm{gph}\, N_C}(x, -Mx - q),\ w^* \in N_Q(q) \Longrightarrow \mathrm{proj}_{T_Q(q)}(-u^* + w^*) = 0, \tag{4.13}$$

where $(q,x) \in \mathrm{gph}\, S|_Q$ is sufficiently close to $(\bar{q},\bar{x})$. By [31, Exercise 12.22] and the polar relation between $T_Q(q)$ and $N_Q(q)$, the following equivalence holds:

$$\mathrm{proj}_{T_Q(q)}(-u^* + w^*) = 0 \Longleftrightarrow -u^* + w^* \in N_Q(q) \quad \text{for any } w^* \in N_Q(q).$$

Noting $q = \bar{q} + q'$, it follows from (4.11) and the convexity and conic structure of $N_Q(q)$ that the above equivalence reduces to

$$\mathrm{proj}_{T_Q(q)}(-u^* + w^*) = 0 \text{ for any } w^* \in N_Q(q) \Longleftrightarrow -u^* \in N_Q(q) = N_Q(\bar{q}) \cap [q']^{\perp}. \tag{4.14}$$

As $(q',x') \in T_{\text{gph } S|_Q}(\bar{q},\bar{x})$, by (4.10), we have $(x',-Mx'-q') \in T_{\text{gph } N_C}(x,v)$. Note that $\bar{v} = -M\bar{x}-\bar{q}$. Let $v' = -Mx'-q'$. By Lemma 4.3, we have

$$N_{\text{gph } N_C}(x,-Mx-q) = \{ (F_1-F_2)^* \times (F_1-F_2) \mid x' \in F_2 \subset F_1 \subset [v']^\perp, \; F_1,F_2 \in \mathcal{F}(K(\bar{x},\bar{v})) \}, \tag{4.15}$$

where $(q,x) \in \text{gph } S|_Q$ is sufficiently close to $(\bar{q},\bar{x})$. Then by (4.13), (4.14), and (4.15), a sufficient condition can be derived as

$$\forall (M^*u^*,u^*) \in (F_1-F_2)^* \times (F_1-F_2) \Longrightarrow u^* \in -N_Q(\bar{q}) \cap [q']^\perp \tag{4.16}$$

for all closed faces $F_1,F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))$ with $x' \in F_2 \subset F_1 \subset [-Mx'-q']^\perp$.

It remains to show that the sufficient condition (4.16) can be equivalently replaced by the following condition:

$$\forall (M^*u^*,u^*) \in (F_1-F_2)^* \times (F_1-F_2) \Longrightarrow u^* \in -N_Q(\bar{q}) \tag{4.17}$$

for all closed faces $F_1,F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))$ with $F_2 \subset F_1$. It is obvious that (4.16) implies (4.17) by taking $x' = q' = 0$. Now we prove that (4.17) implies (4.16). For $F_1,F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))$ with $x' \in F_2 \subset F_1 \subset [-Mx'-q']^\perp$, we have

$$tx' \in F_2 \subset F_1 \;\; \forall t \geq 0$$

by the conic structure of $F_2$ and $F_1$. Thus we have

$$[x'] \subset F_1 - F_2 \subset [-Mx'-q']^\perp.$$

Note that $([-Mx'-q']^\perp)^* = [-Mx'-q']$ and $[x']^* = [x']^\perp$. Besides, $F_1 - F_2$ is still a convex polyhedral cone and by the polar relation, we also have

$$[-Mx'-q'] \subset (F_1-F_2)^* \subset [x']^\perp.$$

Therefore the following inclusions hold:

$$(M^*u^*,u^*) \in (F_1-F_2)^* \times (F_1-F_2) \subset [x']^\perp \times [-Mx'-q']^\perp.$$

From $M^*u^* \in (F_1-F_2)^* \subset [x']^\perp$, we have $\langle u^*, Mx'\rangle = \langle M^*u^*, x'\rangle = 0$. From $u^* \in F_1 - F_2 \subset [-Mx'-q']^\perp$, we have $\langle u^*, -q'\rangle = \langle u^*, -Mx'-q'\rangle = 0$. Thus, $u^* \in [q']^\perp$ holds. Therefore (4.16) holds when (4.17) is satisfied.

For any possible combinations of $F_1,F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))$ with $F_2 \subset F_1$, $F_1$ and $F_2$ are closed and convex cones and so is $F_1 - F_2$. By [30, Corollary 16.3.2], $M^*u^* \in (F_1-F_2)^*$ is equivalent to $u^* \in (M(F_1-F_2))^*$. Then (4.8) and (4.9) can be derived from (4.12) and (4.17), respectively.

When gph $N_C$ is regular at $(\bar{x},\bar{v})$, $S$ is regular at $(\bar{q},\bar{x})$. Besides, $Q$ is a polyhedral set. By the first equality in (4.2), [31, Theorem 6.42], and the regularity of gph $N_C$ at $(\bar{x},\bar{v})$, we have

$$N_Q(\bar{q}) + D^*S(\bar{q}\,|\,\bar{x})(y^*) \subseteq \widehat{D}^*S|_Q(\bar{q}\,|\,\bar{x})(y^*). \tag{4.18}$$

By [25, Theorem 2.1], when $S$ has the Lipschitz-like property relative to $Q$ at $\bar{q}$ for $\bar{x}$, given (4.18), there exists $\kappa \in \mathbb{R}_+$ such that

$$\max_{t \in T_Q(\bar{q}) \cap \mathbb{S}} \langle u^* + w^*, t\rangle \leq \kappa \|y^*\| \;\; \forall u^* \in D^*S(\bar{q}\,|\,\bar{x})(y^*), \; \forall w^* \in N_Q(\bar{q}).$$

Further, by polar relative between $T_Q(\bar{q})$ and $N_Q(\bar{q})$, the above inequality becomes

$$\|\mathrm{proj}_{T_Q(\bar{q})}(u^*)\| = \max\left\{\max_{t\in T_Q(\bar{q})\cap\mathbb{S}}\langle u^*, t\rangle, 0\right\} \le \kappa\|y^*\| \ \forall u^* \in D^*S(\bar{q}\,|\,\bar{x})(y^*),$$

which is equivalent to

$$D^*S(\bar{q}\,|\,\bar{x})(0) \subseteq N_Q(\bar{q}).$$

By calculation in [8]

$$D^*S(\bar{q}\,|\,\bar{x})(0) = \{-u^* \mid (M^*u^*, u^*) \in N_{\mathrm{gph}\ N_C}(\bar{x}, \bar{v})\}.$$

Then again by (4.15) we arrive at the necessity of the condition (4.9). Therefore (i) holds.

Besides, by Proposition 4.1(b), the upper estimate in (4.4) becomes exact when $Q = \mathrm{dom}\ S$. In this case, the sufficient condition becomes necessary without the satisfaction of the constraint qualification (4.8). Thus (ii) holds. ∎

**5. A discussion on the necessity of the generalized critical face condition.** For some AVI with specific structures, it is possible to further simplify the generalized critical face condition (4.9). In this section, we first revisit Theorem 4.4(i) and (ii) and further simplify the generalized critical face condition (4.9). We then discuss a LCP with a $Q_0$ matrix as an example of Proposition 5.3.

**5.1. Regularity of $N_{\mathbf{gph}\ N_C}$.** First we give some equivalent characterizations for the regularity of the set gph $N_C$.

LEMMA 5.1. *For the polyhedral set $C$ and the pair $(x, v) \in \mathrm{gph}\ N_C$, the following statements are equivalent:*

(i) gph $N_C$ *is regular at* $(x, v)$.

(ii) *The critical cone* $K(x, v) = T_C(x) \cap [v]^\perp$ *is a subspace.*

(iii) $v \in \mathrm{rint}\ N_C(x)$.

*Proof.* For $v \in N_C(x)$, from the calculation in [8]

$$\widehat{N}_{\mathrm{gph}\ N_C}(x, v) = K(x, v)^* \times K(x, v).$$

By comparing the expression of $N_{\mathrm{gph}\ N_C}(x, v)$ in (4.6) we obtain the equivalence between (i) and (ii) as any closed face of a subspace remains a subspace. Given that $C$ is a polyhedral set, $T_C(x)$ and $N_C(x)$ are two polyhedral cones polar to each other. For $v \in N_C(x)$, we have

$$K(x, v) = T_C(x) \cap [v]^\perp = N_{N_C(x)}(v).$$

By [22, Proposition 2.2], $N_{N_C(x)}(v)$ being a subspace is equivalent to $v \in \mathrm{rint}\ N_C(x)$. Thus the equivalence between (ii) and (iii) is derived. ∎

When the critical cone $K(\bar{x}, \bar{v})$ is also a subspace, we show in the next corollary that the left-hand side of the condition (4.9) is also a subspace. And instead of checking the condition on each possible combination of $F_1 - F_2$, we can directly check the condition with the set $K(\bar{x}, \bar{v})$.

PROPOSITION 5.2. *For $(\bar{q}, \bar{x}) \in \mathrm{gph}\ S|_Q$, where $S$ is defined as in* (4.1) *and $Q$ is a polyhedral set, when $\bar{v} = -M\bar{x} - \bar{q} \in \mathrm{rint}\ N_C(\bar{x})$, $K(\bar{x}, \bar{v})$ is a subspace and the necessary condition* (4.9) *becomes*

$$K(\bar{x}, \bar{v}) \cap (MK(\bar{x}, \bar{v}))^\perp \subseteq N_Q(\bar{q}). \tag{5.1}$$

*Proof.* From Lemma 5.1 we know that when $\bar{v} \in \operatorname{rint} N_C(\bar{x})$, it is equivalent that gph $N_C$ is regular at $(\bar{x}, \bar{v})$ and therefore

$$K(\bar{x},\bar{v}) = \{F_1 - F_2 \mid F_2 \subseteq F_1, F_1, F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))\}$$

is a subspace, and so is $MK(\bar{x},\bar{v})$. Then the condition (4.9) can be put as (5.1). ∎

**5.2. *C* being a polyhedral cone.** Next we consider the second case where $C$ in (4.1) is a polyhedral cone. In this case we can acquire the expression of dom $S$ when it is also convex. Then the condition for the Lipschitz-like property of $S$ relative to its domain becomes sufficient and necessary. In particular, when $C = \mathbb{R}^n_+$, we show that the condition can be exploited with the structure explicitly.

PROPOSITION 5.3. *For* $(\bar{q}, \bar{x}) \in \operatorname{gph} S$*, where* $S$ *is defined as in* (4.1)*, suppose* $C$ *is a polyhedral cone and* dom $S$ *is convex. In this case,* $S$ *is Lipschitz-like relative to* dom $S$ *at* $\bar{q}$ *for* $\bar{x}$ *if and only if*

$$(F_1 - F_2) \cap (M(F_1 - F_2))^* \subseteq C \cap (MC)^* \cap [\bar{q}]^\perp, \tag{5.2}$$

*where* $F_1, F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))$ *are closed faces with* $F_2 \subset F_1$ *and* $\bar{v} = -M\bar{x} - \bar{q}$.

*Proof.* When $C$ is a polyhedral cone, by [14, Proposition 2], dom $S$ is convex (and therefore a polyhedral cone as well) if and only if dom $S = -C^* - MC$. Then we have

$$-N_{\operatorname{dom} S}(\bar{q}) = -(\operatorname{dom} S)^* \cap [\bar{q}]^\perp = C \cap (MC)^* \cap [\bar{q}]^\perp.$$

Therefore the sufficient and necessary condition (4.9) can be put as (5.2). ∎

To end this section, we show as an example that necessary and sufficient condition (5.2) can be further exploited for an LCP with a $Q_0$ matrix.

Consider the following LCP:

$$x \geq 0,\ Mx + q \geq 0,\ x^\top (Mx + q) = 0, \tag{5.3}$$

where $q, x \in \mathbb{R}^n$ and $M$ is an $n \times n$ matrix. This is also a type of AVI with the polyhedral set $C$ in (4.1) being $\mathbb{R}^n_+$. We denote the solution mapping of (5.3) with $q$ being the parameter as $S$.

To avoid abuse of notation, we specify the unique index combination decided by $(\bar{q}, \bar{x}) \in \operatorname{gph} S$ as

$$\begin{aligned} I_1 &:= \{i \in I | \bar{x}_i = 0, (M\bar{x} + \bar{q})_i > 0\}, \\ I_2 &:= \{i \in I | \bar{x}_i > 0, (M\bar{x} + \bar{q})_i = 0\}, \\ I_3 &:= \{i \in I | \bar{x}_i = 0, (M\bar{x} + \bar{q})_i = 0\}, \end{aligned} \tag{5.4}$$

where $I := \{1, \dots, n\}$. To better illustrate the structure of $N_{\operatorname{gph} S}(\bar{q}, \bar{x})$, we introduce a set defined by index combinations:

$$W(I_1, I_2, I_3) := \left\{ (u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n \,\middle|\, \begin{array}{ll} (u_i^*, v_i^*) \in \{0\} \times \mathbb{R} & \text{if } i \in I_1 \\ (u_i^*, v_i^*) \in \mathbb{R} \times \{0\} & \text{if } i \in I_2 \\ (u_i^*, v_i^*) \in \Omega & \text{if } i \in I_3 \end{array} \right\}, \tag{5.5}$$

where $\Omega := (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \cup \mathbb{R}^2_-$. Note that for $(\bar{q}, \bar{x}) \in \operatorname{gph} S$,

$$(u^*, v^*) \in W(I_1, I_2, I_3) \Longleftrightarrow (v^*, -u^*) \in N_{\operatorname{gph} N_{\mathbb{R}^n_+}}(\bar{x}, -M\bar{x} - \bar{q}). \tag{5.6}$$

By calculation in [16] we have

$$N_{\mathrm{gph}\ S}(\bar{q},\bar{x}) = \{(u^*,\ M^*u^* + v^*) \mid (u^*, v^*) \in W(I_1, I_2, I_3)\}\,. \tag{5.7}$$

As the assumption of the generalized Mordukhovich criterion requires the relative set to be closed and convex, it is natural to ask under what condition dom $S$ is closed and convex. The following proposition provides the rationality behind such an assumption.

LEMMA 5.4 (see [7, Proposition 3.2.1]). *For an* LCP$(q, M)$ *defined as in* (5.3), *the following statements are equivalent:*

(i) $M$ *is a* $Q_0$ *matrix.*
(ii) dom $S$ *is a polyhedral cone in* $\mathbb{R}^n$*.*
(iii) dom $S = $ conv pos $(E, -M) = \mathbb{R}^n_+ - M\mathbb{R}^n_+$.

*Here a* $Q_0$ *matrix means the type of matrices with LCP* (5.3) *being solvable whenever feasible.*

Following Theorem 4.4, in the next theorem we prove that under some specific setting we can use only the information at the given point to obtain a sufficient and necessary condition for the relative Lipschitz-like property. Before presenting the condition, we introduce another set defined by an index combination similar to $W(I_1, I_2, I_3)$:

$$W'(I_1,I_2,I_3) := \left\{ (u^*,v^*) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; \begin{array}{ll} (u_i^*, v_i^*) \in \{0\} \times \mathbb{R}_- & \text{if } i \in I_1 \\ (u_i^*, v_i^*) \in \mathbb{R}_- \times \{0\} & \text{if } i \in I_2 \\ (u_i^*, v_i^*) \in \mathbb{R}^2_- & \text{if } i \in I_3 \end{array} \right\}. \tag{5.8}$$

Note that this set is generated by replacing all $\mathbb{R}$ with $\mathbb{R}_-$ in $W(I_1, I_2, I_3)$.

PROPOSITION 5.5. *For LCP* (5.3) *with* $M$ *being a* $Q_0$ *matrix, let* $(\bar{q},\bar{x}) \in \mathrm{gph}\ S$*. The solution mapping* $S$ *has the Lipschitz-like property relative to its domain at* $\bar{q}$ *for* $\bar{x}$ *if and only if*

$$\forall (-u^*, M^*u^*) \in W(I_1, I_2, I_3) \Longrightarrow (-u^*, M^*u^*) \in W'(I_1, I_2, I_3). \tag{5.9}$$

*Proof.* Given Lemma 5.4 and $M$ being a $Q_0$ matrix, dom $S$ is a polyhedral cone. By Proposition 5.3, the sufficient and necessary condition for the Lipschitz-like property of $S$ relative to its domain at $\bar{q}$ for $\bar{x}$ writes

$$(F_1 - F_2) \cap (M(F_1 - F_2))^* \subseteq \mathbb{R}^n_+ \cap (M\mathbb{R}^n_+)^* \cap [\bar{q}]^\perp, \tag{5.10}$$

where $F_1, F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))$ are closed faces with $F_2 \subset F_1$ and $\bar{v} = -M\bar{x} - \bar{q}$. From the representations of $N_{\mathrm{gph}\ S}(\bar{q},\bar{x})$ in (4.6), (5.7) and the definition of $W(I_1, I_2, I_3)$ (5.5) we can see that

$$(M^*u^*, u^*) \in \bigcup_{\substack{F_2 \subseteq F_1 \\ F_1, F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))}} \Big( (F_1 - F_2)^* \times (F_1 - F_2) \Big)$$
$$\Longleftrightarrow (-u^*, M^*u^*) \in W(I_1, I_2, I_3).$$

For $u^* \in \mathbb{R}^n_+ \cap (M\mathbb{R}^n_+)^* \cap [\bar{q}]^\perp$, we know that

$$\langle u^*, \bar{q}\rangle = \langle u^*, -\bar{v}\rangle - \langle u^*, M\bar{x}\rangle = \langle u^*, -\bar{v}\rangle - \langle M^*u^*, \bar{x}\rangle = 0.$$

Given that $u^* \in \mathbb{R}^n_+$, $-\bar{v} \in \mathbb{R}^n+$, $M^*u^* \in \mathbb{R}^n_-$, and $\bar{x} \in \mathbb{R}^n_+$, we have

$$\langle u^*, -\bar{v}\rangle = \langle M^*u^*, \bar{x}\rangle = 0.$$

Therefore, along with the index category in (5.4), we have that

$$u^* \in \mathbb{R}^n_+ \cap (M\mathbb{R}^n_+)^* \cap [\bar{q}]^\perp \Longleftrightarrow (-u^*, M^*u^*) \in W'(I_1, I_2, I_3).$$

The sufficient and necessary condition (5.9) is proved. ∎

*Remark* 5.6. When $\bar{q} \in \operatorname{int} \operatorname{dom} S$, $N_{\operatorname{dom} S}(\bar{q}) = \{0\}$ and the criterion (5.9) reduces to

$$(-u^*, M^*u^*) \in W\,(I_1, I_2, I_3) \Longrightarrow u^* = 0,$$

which is equivalent to the sufficient and necessary condition for the Lipschitz-like property of $S$ in [8, Theorem 4].

**6. Conclusions.** In this paper, we presented several upper estimates of projectional coderivative of the solution mapping for parametric systems. We compared two sufficient conditions that were given by a projectional coderivative and a directional limiting coderivative, respectively. We developed a sufficient condition of the relative Lipschitz-like property of the solution mapping for AVIs as a generalized critical face condition. For AVIs with specific structures, we showed that this sufficient condition can also be necessary. We considered a linear complementarity problem with a $Q_0$ matrix as an example and simplified the condition concerning the reference point information only.

**Acknowledgments.** The authors are grateful to the anonymous reviewers for their careful reading and valuable suggestions. In particular, the remarks of the reviewers on necessity of the critical face condition led us to a significant improvement of this paper. The authors would like to thank Dr. Li Minghua and Dr. Meng Kaiwen for their useful discussions and comments during the development of this project.

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