



A Power Method for Computing the Dominant Eigenvalue of a Dual Quaternion Hermitian Matrix

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Received: 1 May 2023 / Revised: 17 March 2024 / Accepted: 30 April 2024
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Abstract

In this paper, we first study the projections onto the set of unit dual quaternions, and the set of dual quaternion vectors with unit norms. Then we propose a power method for computing the dominant eigenvalue of a dual quaternion Hermitian matrix. For a strict dominant eigenvalue, we show the sequence generated by the power method converges to the dominant eigenvalue and its corresponding eigenvector linearly. For a general dominant eigenvalue, we establish linear convergence of the standard part of the dominant eigenvalue. Based upon these, we reformulate the simultaneous localization and mapping problem as a rank-one dual quaternion completion problem. A two-block coordinate descent method is proposed to solve this problem. One block has a closed-form solution and the other block is the best rank-one approximation problem of a dual quaternion Hermitian matrix, which can be computed by the power method. Numerical experiments are presented to show the efficiency of our proposed power method.

Keywords Dual quaternion Hermitian matrix · Dominant eigenvalue · Power method · Simultaneous localization and mapping

Mathematics Subject Classification 15A66 · 15A18 · 65F15

1 Introduction

Dual quaternion numbers and dual quaternion matrices are important in robotic research, i.e., the hand-eye calibration problem [7], the simultaneous localization and mapping (SLAM)

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problem [1–3, 5, 23, 25], and the kinematic modeling and control [20]. In [19], Qi and Luo studied right and left eigenvalues of square dual quaternion matrices. If a right eigenvalue is a dual number, then it is also a left eigenvalue. In this case, this dual number is called an eigenvalue of that dual quaternion matrix. They showed that the right eigenvalues of a dual quaternion Hermitian matrix are dual numbers. Thus, they are eigenvalues. An n -by- n dual quaternion Hermitian matrix was shown to have exactly n eigenvalues. It is positive semidefinite, or positive definite, if and only if all of its eigenvalues are nonnegative, or positive and appreciable, dual numbers, respectively. A unitary decomposition for a dual quaternion Hermitian matrix was also proposed. In [20], it was shown that the eigenvalue theory of dual quaternion Hermitian matrices plays an important role in the multi-agent formation control. However, the numerical methods for computing the eigenvalues of a dual quaternion Hermitian matrix is blank.

The power method is one of the state of the art numerical approaches for computing eigenvalues, such as matrix eigenvalues [11], matrix sparse eigenvalues [27], tensor eigenvalues [12], nonnegative tensor eigenvalues [17], and quaternion matrix eigenvalues [15]. In this paper, we propose a power method for computing the dominant eigenvalue of a dual quaternion Hermitian matrix. We first study the projections onto the set of unit dual quaternions and the set of dual quaternion vectors with unit norms. Then we propose a power method for computing the dual quaternion Hermitian matrix eigenvalues. This fills the blank of dual quaternion matrix computation. We also define the convergence of a dual number sequence and study the convergence properties of the power method we proposed.

Dual quaternion is a powerful tool for solving the SLAM problem. The SLAM problem aims to build a map of the environment and to simultaneously localize within this map, which is an essential skill for mobile robots navigating in unknown environments in absence of external referencing systems such as GPS. An intuitive way to solve the SLAM problem is via graph-based formulation [10]. One line of work reformulates SLAM as a nonlinear least square and solves it by the Gauss-Newton method [13]. However, as this problem is nonconvex and Gauss-Newton may be trapped in local minimal points, a good initialization is important [5]. Several methods have been proposed to address the issue of global convergence, including the majorization minimization [9] and the Lagrangian duality [4]. In this paper, we reformulate SLAM as a rank-one dual quaternion completion problem and propose a two-block coordinate descent method to solve it. One block subproblem has a closed form solution and the other block subproblem can be solved by the dominant eigenpair. This connects the dual quaternion matrix theory with the SLAM problem.

The distribution of the remainder of this paper is as follows. In the next section, we review some basic properties of dual quaternions and dual quaternion matrices. We also define the convergence of a dual number sequence there. In Sect. 3, we show the projections onto the set of unit dual quaternions, and the projection onto the set of dual quaternion vectors with unit norms, can be obtained by normalization for appreciable numbers and vectors, respectively. In Sect. 4, we show that the sequence generated by the power method converges linearly to a strict dominant eigenvalue and eigenvector of a dual quaternion Hermitian matrix. For a general dominant eigenvalue, we show the convergence and the linear convergence rate of the standard part of the sequence to the standard part of the dominant eigenvalue. In Sect. 5, we present numerical experiment results on computing the eigenvalues of Laplacian matrices. In Sect. 6, we reformulate SLAM as a rank-one approximation model and present a block coordinate descent method for solving it. Some final remarks are made in Sect. 7.

2 Dual Quaternions and Dual Quaternion Matrices

2.1 Dual Quaternions

The sets of real numbers, dual numbers, quaternion, unit quaternion, dual quaternion, and unit dual quaternion are denoted as $\mathbb{R}, \mathbb{D}, \mathbb{Q}, \mathbb{U}, \hat{\mathbb{Q}},$ and $\hat{\mathbb{U}},$ respectively. Denote \mathbb{R}^3 as $\mathbb{V}.$

A quaternion $\tilde{q} = [q_0, q_1, q_2, q_3]$ is a real four-dimensional vector. We use a tilde symbol to distinguish a quaternion. We may also write $\tilde{q} = [q_0, \vec{q}],$ where $\vec{q} = [q_1, q_2, q_3] \in \mathbb{V}.$ See [6, 7, 22]. If $q_0 = 0,$ then $\tilde{q} = [0, \vec{q}]$ is called a vector quaternion. Suppose that we have two quaternions

$\tilde{p} = [p_0, \vec{p}], \tilde{q} = [q_0, \vec{q}] \in \mathbb{Q},$ where $p = [p_1, p_2, p_3], q = [q_1, q_2, q_3] \in \mathbb{V}.$ The addition of \tilde{p} and \tilde{q} is defined as $\tilde{p} + \tilde{q} = [p_0 + q_0, \vec{p} + \vec{q}].$ Denote the zero element of \mathbb{Q} as $\vec{0} := [0, 0, 0, 0] \in \mathbb{Q}.$ The multiplication of \tilde{p} and \tilde{q} is defined by

$$\tilde{p}\tilde{q} = [p_0q_0 - \vec{p} \cdot \vec{q}, p_0\vec{q} + q_0\vec{p} + \vec{p} \times \vec{q}],$$

where $\vec{p} \cdot \vec{q}$ is the dot product, i.e., the inner product of \vec{p} and $\vec{q},$ with

$$\vec{p} \cdot \vec{q} \equiv \vec{p}^T \vec{q} = p_1q_1 + p_2q_2 + p_3q_3,$$

and $\vec{p} \times \vec{q}$ is the cross product of \vec{p} and $\vec{q},$ with

$$\vec{p} \times \vec{q} = [p_2q_3 - p_3q_2, -p_1q_3 + p_3q_1, p_1q_2 - p_2q_1] = -\vec{q} \times \vec{p}.$$

Thus, in general, $\tilde{p}\tilde{q} \neq \tilde{q}\tilde{p},$ and we have $\tilde{p}\tilde{q} = \tilde{q}\tilde{p}$ if and only if $\vec{p} \times \vec{q} = \vec{0},$ i.e., either $\vec{p} = \vec{0}$ or $\vec{q} = \vec{0},$ or $\vec{p} = \alpha\vec{q}$ for some real number $\alpha.$ We can also represent the product of \tilde{p} and \tilde{q} as the product of a matrix and a vector, i.e.,

$$\mathcal{R}\tilde{p}\tilde{q} = \mathcal{M}(\tilde{p})\mathcal{R}(\tilde{q}), \text{ where } \mathcal{M}(\tilde{p}) = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix}, \mathcal{R}(\tilde{q}) = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

Furthermore, let $\tilde{\mathbf{p}} = [\tilde{p}_1, \dots, \tilde{p}_n]^T \in \mathbb{Q}^n$ and $\tilde{\mathbf{q}} = [\tilde{q}_1, \dots, \tilde{q}_n]^T \in \mathbb{Q}^n$ be two quaternion vectors, and $\tilde{P} = [\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_n] \in \mathbb{Q}^{n \times n}$ be a quaternion matrix. Then there is

$$\mathcal{R}(\tilde{\mathbf{p}}^T \tilde{\mathbf{q}}) = \mathcal{M}(\tilde{\mathbf{p}}^T)\mathcal{R}(\tilde{\mathbf{q}}) \text{ and } \mathcal{R}\tilde{P}\tilde{\mathbf{q}} = \mathcal{M}(\tilde{P})\mathcal{R}(\tilde{\mathbf{q}}), \tag{1}$$

where $\mathcal{M}(\tilde{\mathbf{p}}^T) = [\mathcal{M}(\tilde{p}_1), \dots, \mathcal{M}(\tilde{p}_n)] \in \mathbb{R}^{4 \times 4n}, \mathcal{R}(\tilde{\mathbf{p}}) = [\mathcal{R}(\tilde{p}_1)^T, \dots, \mathcal{R}(\tilde{p}_n)^T]^T \in \mathbb{R}^{4n \times 1},$ and $\mathcal{M}(\tilde{P}) = (\mathcal{M}(\tilde{p}_{ij})) \in \mathbb{R}^{4n \times 4n}.$

The conjugate of a quaternion $\tilde{q} = [q_0, q_1, q_2, q_3] \in \mathbb{Q}$ is defined as $\tilde{q}^* = [q_0, -q_1, -q_2, -q_3].$ Let $\vec{1} := [1, 0, 0, 0] \in \mathbb{Q}.$ Then for any $\tilde{q} \in \mathbb{Q},$ we have $\tilde{q}\vec{1} = \vec{1}\tilde{q} = \tilde{q},$ i.e., $\vec{1}$ is the identity element of $\mathbb{Q}.$ For any $\tilde{p}, \tilde{q} \in \mathbb{Q},$ we have $(\tilde{p}\tilde{q})^* = \tilde{q}^*\tilde{p}^*.$ Suppose that $\tilde{p}, \tilde{q} \in \mathbb{Q}, \tilde{p}\tilde{q} = \tilde{q}\tilde{p} = \vec{1}.$ Then we say that \tilde{p} is invertible and its inverse is $\tilde{p}^{-1} = \tilde{q}.$

For a quaternion $\tilde{q} = [q_0, q_1, q_2, q_3] \in \mathbb{Q},$ its magnitude is defined by

$$|\tilde{q}| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

If $|\tilde{q}| = 1,$ then it is called a unit quaternion. If $\tilde{p}, \tilde{q} \in \mathbb{U},$ then $\tilde{p}\tilde{q} \in \mathbb{U}.$ For any $\tilde{q} \in \mathbb{U},$ we have $\tilde{q}\tilde{q}^* = \tilde{q}^*\tilde{q} = \vec{1},$ i.e., \tilde{q} is invertible and $\tilde{q}^{-1} = \tilde{q}^*.$ Furthermore, we have $|\tilde{q}| = \|\mathcal{R}(\tilde{q})\|.$ For a quaternion vector $\tilde{\mathbf{q}} = [\tilde{q}_1, \dots, \tilde{q}_n]^T \in \mathbb{Q}^n,$ we then have $\|\tilde{\mathbf{q}}\| = \|\mathcal{R}(\tilde{\mathbf{q}})\|.$

A dual number $a = a_{st} + a_{\mathcal{I}}\epsilon$ consists of the standard part $a_{st} \in \mathbb{R}$ and the dual part $a_{\mathcal{I}} \in \mathbb{R}.$ The symbol ϵ is the infinitesimal unit. It satisfies $\epsilon \neq 0$ and $\epsilon^2 = 0.$ If $a_{st} \neq 0,$ then a is appreciable, otherwise, a is infinitesimal [18].

For $a = a_{st} + a_{\mathcal{I}}\epsilon, b = b_{st} + b_{\mathcal{I}}\epsilon \in \mathbb{D}$, the addition of a and b is defined as

$$a + b = a_{st} + b_{st} + (a_{\mathcal{I}} + b_{\mathcal{I}})\epsilon,$$

the multiplication of a and b is defined as

$$ab = a_{st}b_{st} + (a_{st}b_{\mathcal{I}} + a_{\mathcal{I}}b_{st})\epsilon,$$

and the division of a and b , when $a_{st} \neq 0$, or $a_{st} = 0$ and $b_{st} = 0$ is defined as

$$\frac{b_{st} + b_{\mathcal{I}}\epsilon}{a_{st} + a_{\mathcal{I}}\epsilon} = \begin{cases} \frac{b_{st}}{a_{st}} + \left(\frac{b_{\mathcal{I}}}{a_{st}} - \frac{b_{st} a_{\mathcal{I}}}{a_{st}^2}\right)\epsilon, & \text{if } a_{st} \neq 0, \\ \frac{b_{\mathcal{I}}}{a_{\mathcal{I}}} + c\epsilon, & \text{if } a_{st} = 0, b_{st} = 0, \end{cases}$$

where $c \in \mathbb{R}$ is an arbitrary real number. One may show that the division of dual numbers is the inverse operation of the multiplication of dual numbers.

The zero element of \mathbb{D} is $0_D := 0 + 0\epsilon$. The identity element of \mathbb{D} is $1_D = 1 + 0\epsilon$. The absolute value of $a \in \mathbb{D}$ is defined in [18] as

$$|a| = \begin{cases} |a_{st}| + \text{sgn}(a_{st})a_{\mathcal{I}}\epsilon, & \text{if } a_{st} \neq 0, \\ |a_{\mathcal{I}}|\epsilon, & \text{otherwise.} \end{cases}$$

When $a \neq 0_D$, we have $\frac{a}{|a|} = \text{sgn}(a_{st})$ if $a_{st} \neq 0$, and $\frac{a}{|a|} = \text{sgn}(a_{\mathcal{I}}) + c\epsilon$ for any $c \in \mathbb{R}$ if $a_{st} = 0$.

The total order of dual numbers was defined in [18]: we say that $a > b$ if $a_{st} > b_{st}$, or $a_{st} = b_{st}$ and $a_{\mathcal{I}} > b_{\mathcal{I}}$. Let $a > 0_D$ be a nonnegative dual number. Then the square root of a is defined by [18]

$$\sqrt{a} = \sqrt{a_{st}} + \frac{a_{\mathcal{I}}}{2\sqrt{a_{st}}}\epsilon.$$

Let $\{a_k = a_{k,st} + a_{k,\mathcal{I}}\epsilon : k = 1, 2, \dots\}$ be a dual number sequence. We say that this sequence is convergent and has a limit $a = a_{st} + a_{\mathcal{I}}\epsilon$ if both its standard part sequence $\{a_{k,st} : k = 1, 2, \dots\}$ and its dual part sequence $\{a_{k,\mathcal{I}} : k = 1, 2, \dots\}$ are convergent and have limits a_{st} and $a_{\mathcal{I}}$ respectively.

Given a dual number sequence $\{a_k : k = 1, 2, \dots\} \subset \mathbb{D}$ and a real number sequence $\{c_k : k = 1, 2, \dots\} \subset \mathbb{R}$, we introduce a big O_D notation as follows,

$$a_k = O_D(c_k) \text{ if } a_{k,st} = O(c_k) \text{ and } a_{k,\mathcal{I}} = O(c_k). \tag{2}$$

Clearly, in this case, if the real number sequence $\{c_k : k = 1, 2, \dots\}$ is convergent, then the dual number sequence $\{a_k : k = 1, 2, \dots\}$ is also convergent. Furthermore, if $c_k = O(c^k h(k))$ for a real positive number $c < 1$ and a polynomial $h(k)$, then we denote $c_k = \tilde{O}(c^k)$. Similarly, we denote

$$a_k = \tilde{O}_D(c_k) \text{ if } a_{k,st} = \tilde{O}(c_k) \text{ and } a_{k,\mathcal{I}} = \tilde{O}(c_k). \tag{3}$$

A dual quaternion $\hat{q} = \tilde{q}_{st} + \tilde{q}_{\mathcal{I}}\epsilon$ consists of two quaternions \tilde{q}_{st} , the standard part of \hat{q} , and $\tilde{q}_{\mathcal{I}}$, the dual part of \hat{q} . We use a hat symbol to distinguish a dual quaternion. If $\tilde{q}_{st} \neq \tilde{0}$, then \hat{q} is appreciable, otherwise, \hat{q} is infinitesimal [18].

Let $\hat{p} = \tilde{p}_{st} + \tilde{p}_{\mathcal{I}}\epsilon, \hat{q} = \tilde{q}_{st} + \tilde{q}_{\mathcal{I}}\epsilon \in \hat{\mathbb{Q}}$. Then the sum of \hat{p} and \hat{q} is

$$\hat{p} + \hat{q} = (\tilde{p}_{st} + \tilde{q}_{st}) + (\tilde{p}_{\mathcal{I}} + \tilde{q}_{\mathcal{I}})\epsilon$$

and the product of \hat{p} and \hat{q} is

$$\hat{p}\hat{q} = \tilde{p}_{st}\tilde{q}_{st} + (\tilde{p}_{st}\tilde{q}_{\mathcal{I}} + \tilde{p}_{\mathcal{I}}\tilde{q}_{st})\epsilon.$$

Again, in general, $\hat{p}\hat{q} \neq \hat{q}\hat{p}$. The conjugate of $\hat{p} = \tilde{p}_{st} + \tilde{p}_{\mathcal{I}}\epsilon$ is $\hat{p}^* = \tilde{p}_{st}^* + \tilde{p}_{\mathcal{I}}^*\epsilon$. Denote $\hat{0} = \tilde{0} + \tilde{0}\epsilon$ and $\hat{1} = \tilde{1} + \tilde{0}\epsilon$.

A dual quaternion $\hat{q} = \tilde{q}_{st} + \tilde{q}_{\mathcal{I}}\epsilon$ is called a unit dual quaternion if

$$|\tilde{q}_{st}| = 1 \text{ and } \tilde{q}_{st}\tilde{q}_{\mathcal{I}}^* + \tilde{q}_{\mathcal{I}}\tilde{q}_{st}^* = \tilde{0}.$$

The magnitude of a dual quaternion $\hat{q} \in \hat{\mathbb{Q}}$ is [18]

$$|\hat{q}| = \begin{cases} |\tilde{q}_{st}| + \frac{sc(\tilde{q}_{st}^*\tilde{q}_{\mathcal{I}})}{|\tilde{q}_{st}|}\epsilon, & \text{if } \tilde{q}_{st} \neq \tilde{0}, \\ |\tilde{q}_{\mathcal{I}}|\epsilon, & \text{otherwise.} \end{cases}$$

Here, $sc(\tilde{q}) = \frac{1}{2}(\tilde{q} + \tilde{q}^*)$ is the scalar part of \tilde{q} .

For any dual quaternion number $\hat{q} = \tilde{q}_{st} + \tilde{q}_{\mathcal{I}}\epsilon$ and dual number $a = a_{st} + a_{\mathcal{I}}\epsilon$ with $a_{st} \neq 0$, or $a_{st} = 0$ and $\tilde{q}_{st} = \tilde{0}$, there is

$$\frac{\tilde{q}_{st} + \tilde{q}_{\mathcal{I}}\epsilon}{a_{st} + a_{\mathcal{I}}\epsilon} = \begin{cases} \frac{\tilde{q}_{st}}{a_{st}} + \left(\frac{\tilde{q}_{\mathcal{I}}}{a_{st}} - \frac{\tilde{q}_{st} a_{\mathcal{I}}}{a_{st} a_{st}} \right) \epsilon, & \text{if } a_{st} \neq 0, \\ \frac{\tilde{q}_{\mathcal{I}}}{a_{\mathcal{I}}} + \tilde{c}\epsilon, & \text{if } a_{st} = 0, \tilde{q}_{st} = \tilde{0}, \end{cases}$$

where $\tilde{c} \in \mathbb{Q}$ is an arbitrary quaternion number.

2.2 Dual Quaternion Matrices

We denote the set of m -by- n real, quaternion, unit quaternion, dual quaternion, and unit dual quaternion matrices as $\mathbb{R}^{m \times n}$, $\mathbb{Q}^{m \times n}$, $\mathbb{U}^{m \times n}$, $\hat{\mathbb{Q}}^{m \times n}$, and $\hat{\mathbb{U}}^{m \times n}$, respectively. We use $\tilde{\mathbf{0}}_{m \times n}$ and $\hat{\mathbf{0}}_{m \times n}$ to denote the m -by- n zero quaternion and zero dual quaternion matrices, and $\tilde{\mathbf{I}}_{m \times m}$ and $\hat{\mathbf{I}}_{m \times m}$ to denote the m -by- m identity quaternion and identity dual quaternion matrices, respectively. Given $\hat{\mathbf{Q}} \in \hat{\mathbb{Q}}^{m \times n}$, if $\hat{\mathbf{Q}}_{st} \neq \tilde{\mathbf{0}}_{m \times n}$, then $\hat{\mathbf{Q}}$ is appreciable, otherwise, $\hat{\mathbf{Q}}$ is infinitesimal.

The 2-norm of a dual quaternion vector $\hat{\mathbf{x}} = (\hat{x}_i) \in \hat{\mathbb{Q}}^{n \times 1}$ is

$$\|\hat{\mathbf{x}}\|_2 = \begin{cases} \sqrt{\sum_{i=1}^n |\hat{x}_i|^2}, & \text{if } \tilde{\mathbf{x}}_{st} \neq \tilde{\mathbf{0}}_{n \times 1}, \\ \sqrt{\sum_{i=1}^n |\tilde{x}_{i,\mathcal{I}}|^2}\epsilon, & \text{if } \tilde{\mathbf{x}}_{st} = \tilde{\mathbf{0}}_{n \times 1} \text{ and } \hat{\mathbf{x}} = \tilde{\mathbf{x}}_{\mathcal{I}}\epsilon. \end{cases}$$

A vector $\hat{\mathbf{x}} = (\hat{x}_i) \in \hat{\mathbb{U}}^{n \times 1}$ is called a unit dual quaternion vector if each element in $\hat{\mathbf{x}}$ is a unit dual quaternion number. The set of $n \times 1$ quaternion vectors with unit 2-norms is denoted by $\mathbb{Q}_2^{n \times 1}$. A matrix $\hat{\mathbf{Q}} = (\hat{q}_{i,j}) \in \hat{\mathbb{U}}^{m \times n}$ is called a unit dual quaternion matrix if each element in $\hat{\mathbf{Q}}$ is a unit dual quaternion number.

Given a matrix $\hat{\mathbf{Q}} = (\hat{q}_{i,j}) \in \hat{\mathbb{Q}}^{m \times n}$, the transpose of $\hat{\mathbf{Q}}$ is $\hat{\mathbf{Q}}^T = (\hat{q}_{j,i}) \in \hat{\mathbb{Q}}^{n \times m}$, and the conjugate transpose of $\hat{\mathbf{Q}}$ is $\hat{\mathbf{Q}}^* = (\hat{q}_{j,i}^*) \in \hat{\mathbb{Q}}^{n \times m}$. If $\hat{\mathbf{Q}} = \hat{\mathbf{Q}}^*$, then it is a dual quaternion Hermitian matrix, and both its standard part and infinitesimal part are quaternion Hermitian matrices.

The F -norm of a dual quaternion matrix $\hat{\mathbf{Q}} = (\hat{Q}_{ij}) \in \hat{\mathbb{Q}}^{m \times n}$ is

$$\|\hat{\mathbf{Q}}\|_F = \begin{cases} \|\tilde{\mathbf{Q}}_{st}\|_F + \frac{sc(tr(\tilde{\mathbf{Q}}_{st}^*\tilde{\mathbf{Q}}_{\mathcal{I}}))}{\|\tilde{\mathbf{Q}}_{st}\|_F}\epsilon, & \text{if } \tilde{\mathbf{Q}}_{st} \neq \tilde{\mathbf{0}}_{m \times n}, \\ \|\tilde{\mathbf{Q}}_{\mathcal{I}}\|_F\epsilon, & \text{if } \tilde{\mathbf{Q}}_{st} = \tilde{\mathbf{0}}_{m \times n} \text{ and } \hat{\mathbf{Q}} = \tilde{\mathbf{Q}}_{\mathcal{I}}\epsilon, \end{cases}$$

and the F^* -norm is

$$\|\hat{\mathbf{Q}}\|_{F^*} = \begin{cases} \|\tilde{\mathbf{Q}}_{st}\|_F + \frac{\|\tilde{\mathbf{Q}}_{\mathcal{I}}\|_F^2}{2\|\tilde{\mathbf{Q}}_{st}\|_F} \epsilon, & \text{if } \tilde{\mathbf{Q}}_{st} \neq \tilde{\mathbf{O}}_{m \times n}, \\ \|\tilde{\mathbf{Q}}_{\mathcal{I}}\|_F \epsilon, & \text{if } \tilde{\mathbf{Q}}_{st} = \tilde{\mathbf{O}}_{m \times n} \text{ and } \hat{\mathbf{Q}} = \tilde{\mathbf{Q}}_{\mathcal{I}} \epsilon, \end{cases}$$

For convenience in the numerical experiments, we define the 2^R -norm of a dual quaternion vector $\hat{\mathbf{x}} = (\hat{x}_i) \in \hat{\mathbb{Q}}^{n \times 1}$ as

$$\|\hat{\mathbf{x}}\|_{2^R} = \sqrt{\|\tilde{\mathbf{x}}_{st}\|_2^2 + \|\tilde{\mathbf{x}}_{\mathcal{I}}\|_2^2},$$

and the F^R -norm of a dual quaternion matrix $\hat{\mathbf{Q}} = (\hat{q}_{ij}) \in \hat{\mathbb{Q}}^{m \times n}$ as

$$\|\hat{\mathbf{Q}}\|_{F^R} = \sqrt{\|\tilde{\mathbf{Q}}_{st}\|_F^2 + \|\tilde{\mathbf{Q}}_{\mathcal{I}}\|_F^2},$$

respectively. The 2^R -norm and F^R -norm here are not norms since they do not satisfy the scaling condition of norms. For instance, let $\alpha = 1 + \epsilon$, then $\|\alpha\|_{2^R} = \sqrt{2}$, and

$$\|\alpha \hat{\mathbf{x}}\|_{F^R} = \sqrt{\|\tilde{\mathbf{x}}_{st}\|_F^2 + \|\tilde{\mathbf{x}}_{st} + \tilde{\mathbf{x}}_{\mathcal{I}}\|_F^2} \neq \sqrt{2} \sqrt{\|\tilde{\mathbf{x}}_{st}\|_F^2 + \|\tilde{\mathbf{x}}_{\mathcal{I}}\|_F^2} = \|\alpha\|_{2^R} \|\hat{\mathbf{x}}\|_{F^R}.$$

We only use the 2^R -norm and F^R -norm to present the numerical results since they are real numbers, while the 2-norm and F -norm are dual numbers that are more complicated.

Let $A \in \hat{\mathbb{Q}}^{n \times n}$, $\mathbf{x} \in \hat{\mathbb{Q}}^n$ be appreciable, and $\lambda \in \hat{\mathbb{Q}}$. If

$$A\mathbf{x} = \mathbf{x}\lambda, \tag{4}$$

then λ is called a right eigenvalue of A , with \mathbf{x} as its corresponding right eigenvector. If

$$A\mathbf{x} = \lambda\mathbf{x}, \tag{5}$$

then λ is called a left eigenvalue of A , with a left eigenvector \mathbf{x} . It was proved in [19] that all the right eigenvalues of a dual quaternion Hermitian matrix A are dual numbers. As dual numbers are commutative with dual quaternions, they are also left eigenvalues. Thus, we may simply call them eigenvalues of A . Note that A may still have other left eigenvalues, which are not dual numbers. See an example of a quaternion matrix in [28].

3 Projection Onto the Set of Dual Quaternions with Unit Norms

Given a quaternion number $\tilde{q} \neq \tilde{0}$ and a quaternion vector $\tilde{\mathbf{q}} \neq \tilde{\mathbf{O}}_{n \times 1}$, we have

$$P_U(\tilde{q}) = \frac{\tilde{q}}{|\tilde{q}|} \text{ and } P_{\mathbb{Q}_2^{n \times 1}}(\tilde{\mathbf{q}}) = \frac{\tilde{\mathbf{q}}}{\|\tilde{\mathbf{q}}\|_2}.$$

Here, $\mathbb{Q}_2^{n \times 1}$ is the set of n dimensional quaternion vectors with unit 2-norms. In the following, we will show that for the dual quaternion number and the dual quaternion vector, the projection has a similar formulation.

Theorem 1 *Given a dual quaternion number $\hat{q} = \tilde{q}_{st} + \tilde{q}_{\mathcal{I}}\epsilon \in \hat{\mathbb{Q}}$, we have the following conclusions.*

(i) *If $\tilde{q}_{st} \neq \tilde{0}$, the normalization of \hat{q} is the projection of \hat{q} onto the unit dual quaternion set. Namely,*

$$\hat{u} = \frac{\hat{q}}{|\hat{q}|} \text{ with } \tilde{u}_{st} = \frac{\tilde{q}_{st}}{|\tilde{q}_{st}|}, \tilde{u}_{\mathcal{I}} = \frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{st}|} - \frac{\tilde{q}_{st}}{|\tilde{q}_{st}|} sc \left(\frac{\tilde{q}_{st}^*}{|\tilde{q}_{st}|} \frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{st}|} \right) \tag{6}$$

is a unit dual quaternion number and

$$\hat{u} \in \arg \min_{\hat{v} \in \hat{\mathbb{U}}} |\hat{v} - \hat{q}|^2. \tag{7}$$

(ii) If $\tilde{q}_{st} = \tilde{0}$ and $\tilde{q}_{\mathcal{I}} \neq \tilde{0}$, then the projection of \hat{q} onto the unit dual quaternion set is $\hat{u} = \tilde{u}_{st} + \tilde{u}_{\mathcal{I}}\epsilon$, where

$$\tilde{u}_{st} = \frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{\mathcal{I}}|}, \tilde{u}_{\mathcal{I}} \text{ is any quaternion number satisfying } sc(\tilde{q}_{\mathcal{I}}^* \tilde{u}_{\mathcal{I}}) = 0. \tag{8}$$

Proof (i) The normalization formula in equation (6) follows from

$$\frac{\hat{q}}{|\hat{q}|} = \frac{\tilde{q}_{st} + \tilde{q}_{\mathcal{I}}\epsilon}{|\tilde{q}_{st}| + \frac{sc(\tilde{q}_{st}^* \tilde{q}_{\mathcal{I}})}{|\tilde{q}_{st}|}\epsilon} = \frac{\tilde{q}_{st}}{|\tilde{q}_{st}|} + \left(\frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{st}|} - \frac{\tilde{q}_{st}}{|\tilde{q}_{st}|} sc \left(\frac{\tilde{q}_{st}^*}{|\tilde{q}_{st}|} \frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{st}|} \right) \right) \epsilon.$$

By direct computations, we have $\tilde{u}_{st}^* \tilde{u}_{st} = \tilde{1}$ and $\tilde{u}_{st}^* \tilde{u}_{\mathcal{I}} + \tilde{u}_{\mathcal{I}}^* \tilde{u}_{st} = 0$. Furthermore, by

$$|\hat{v} - \hat{q}|^2 = |\tilde{v}_{st} - \tilde{q}_{st}|^2 + 2sc((\tilde{v}_{st} - \tilde{q}_{st})^* (\tilde{v}_{\mathcal{I}} - \tilde{q}_{\mathcal{I}}))\epsilon$$

and the definition of the total order of dual numbers, we have

$$\tilde{u}_{st} = \arg \min_{\tilde{v}_{st} \in \mathbb{U}} |\tilde{v}_{st} - \tilde{q}_{st}|^2 = \frac{\tilde{q}_{st}}{|\tilde{q}_{st}|}. \tag{9}$$

Moreover, we have $sc((\tilde{u}_{st} - \tilde{q}_{st})^* (\tilde{v}_{\mathcal{I}} - \tilde{q}_{\mathcal{I}})) = -sc((\tilde{u}_{st} - \tilde{q}_{st})^* \tilde{q}_{\mathcal{I}})$ for all $\tilde{v}_{\mathcal{I}}$ satisfying $sc(\tilde{q}_{st}^* \tilde{v}_{\mathcal{I}}) = 0$. In other words, any \hat{u} with the standard part $\tilde{u}_{st} = \frac{\tilde{q}_{st}}{|\tilde{q}_{st}|}$ and the infinitesimal part satisfying $sc(\tilde{q}_{st}^* \tilde{u}_{\mathcal{I}}) = 0$ is an optimal solution. Hence, we conclude that \hat{u} in (6) is an optimal solution of (7).

(ii) When $\tilde{q}_{st} = 0$, we have

$$|\hat{v} - \hat{q}|^2 = |\tilde{v}_{st}|^2 + 2sc(\tilde{v}_{st}^* (\tilde{v}_{\mathcal{I}} - \tilde{q}_{\mathcal{I}}))\epsilon = 1 - 2sc(\tilde{v}_{st}^* \tilde{q}_{\mathcal{I}})\epsilon$$

for any $\hat{v} \in \hat{\mathbb{U}}$. By the definition of the total order of dual numbers, there is

$$\tilde{u}_{st} = \arg \min_{\tilde{v}_{st} \in \mathbb{U}} -sc(\tilde{v}_{st}^* \tilde{q}_{\mathcal{I}}).$$

Hence, $\tilde{u}_{st} = \frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{\mathcal{I}}|}$. As the objective function is independent of $\tilde{v}_{\mathcal{I}}$, the infinitesimal part of \hat{u} can be any quaternion number satisfying $sc(\tilde{q}_{st}^* \tilde{u}_{\mathcal{I}}) = 0$. This completes the proof. \square

For problem (7), any $\tilde{u}_{\mathcal{I}}$ satisfying $sc(\tilde{q}_{st}^* \tilde{u}_{\mathcal{I}}) = 0$ is an optimal solution. However, the choice of $\tilde{u}_{\mathcal{I}}$ in (6) is geometric meaningful, as shown in the following proposition.

Proposition 2 Suppose $\tilde{q}_{st} \neq \tilde{0}$. Then $\tilde{u}_{\mathcal{I}}$ defined by (6) is a solution to the following optimization problem,

$$\min_{\tilde{v} \in \mathbb{Q}} \left| \tilde{v} - \frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{\mathcal{I}}|} \right|^2 \text{ s.t. } sc(\tilde{v}^* \tilde{q}_{st}) = 0. \tag{10}$$

Furthermore, $\hat{u} = \frac{\hat{q}}{|\hat{q}|}$ is the optimal solution to the following problem

$$\hat{u} = \arg \min_{\hat{v} \in \hat{\mathbb{U}}} |\tilde{v}_{st} - \tilde{q}_{st}|^2 + \left| \tilde{v}_{\mathcal{I}} - \frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{\mathcal{I}}|} \right|^2 \epsilon.$$

Proof Problem (10) is a convex quaternion optimization problem [21] since the objective function is convex and the constraint is linear. Then it follows from the first-order optimality conditions given in Theorem 4.3 of [21], there is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\tilde{v} - \frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{\text{st}}|} + \lambda \tilde{q}_{\text{st}} = 0 \text{ and } sc(\tilde{v}^* \tilde{q}_{\text{st}}) = 0.$$

By multiplying \tilde{q}_{st}^* at the both sides of the first equation, there is $\lambda = \frac{sc(\tilde{q}_{\text{st}}^* \tilde{q}_{\mathcal{I}})}{|\tilde{q}_{\text{st}}|^3}$. Consequently, we have

$$\tilde{v} = \frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{\text{st}}|} - \lambda \tilde{q}_{\text{st}} = \frac{\tilde{q}_{\mathcal{I}}}{|\tilde{q}_{\text{st}}|} - \frac{sc(\tilde{q}_{\text{st}}^* \tilde{q}_{\mathcal{I}})}{|\tilde{q}_{\text{st}}|^3} \tilde{q}_{\text{st}}.$$

This completes the proof. □

Similarly, we can show the projection onto the set of dual quaternion vectors with unit norms also have closed-form solutions.

Theorem 3 Given a dual quaternion vector $\hat{\mathbf{q}} \in \hat{\mathbb{Q}}^{n \times 1}$, we have

(i) If $\tilde{\mathbf{q}}_{\text{st}} \neq \tilde{\mathbf{O}}_{n \times 1}$, the normalization of $\hat{\mathbf{q}}$ is the projection of $\hat{\mathbf{q}}$ onto the set of dual quaternion vectors with unit norms. Namely,

$$\hat{\mathbf{u}} = \frac{\hat{\mathbf{q}}}{\|\hat{\mathbf{q}}\|_2} \text{ with } \tilde{\mathbf{u}}_{\text{st}} = \frac{\tilde{\mathbf{q}}_{\text{st}}}{\|\tilde{\mathbf{q}}_{\text{st}}\|_2}, \tilde{\mathbf{u}}_{\mathcal{I}} = \frac{\tilde{\mathbf{q}}_{\mathcal{I}}}{\|\tilde{\mathbf{q}}_{\text{st}}\|_2} - \frac{\tilde{\mathbf{q}}_{\text{st}}}{\|\tilde{\mathbf{q}}_{\text{st}}\|_2} sc\left(\frac{\tilde{\mathbf{q}}_{\text{st}}^*}{\|\tilde{\mathbf{q}}_{\text{st}}\|_2} \frac{\tilde{\mathbf{q}}_{\mathcal{I}}}{\|\tilde{\mathbf{q}}_{\text{st}}\|_2}\right) \quad (11)$$

is a dual quaternion vector with unit norm and

$$\hat{\mathbf{u}} \in \arg \min_{\hat{\mathbf{v}} \in \hat{\mathbb{Q}}_2^{n \times 1}} \|\hat{\mathbf{v}} - \hat{\mathbf{q}}\|_2^2.$$

(ii) If $\tilde{\mathbf{q}}_{\text{st}} = \tilde{\mathbf{O}}_{n \times 1}$, the projection of $\hat{\mathbf{q}}$ onto the set of dual quaternion vectors with unit norms is $\hat{\mathbf{u}} = \tilde{\mathbf{u}}_{\text{st}} + \tilde{\mathbf{u}}_{\mathcal{I}} \epsilon$ satisfying

$$\tilde{\mathbf{u}}_{\text{st}} = \frac{\tilde{\mathbf{q}}_{\mathcal{I}}}{\|\tilde{\mathbf{q}}_{\mathcal{I}}\|_2}, \tilde{\mathbf{u}}_{\mathcal{I}} \text{ is any quaternion satisfying } sc(\tilde{\mathbf{q}}_{\mathcal{I}}^* \tilde{\mathbf{u}}_{\mathcal{I}}) = 0.$$

Proof The proof of this theorem is similar to the proof of Theorem 1. We do not repeat it here. □

Similar to Proposition 2, we have the following result.

Proposition 4 If $\tilde{\mathbf{q}}_{\text{st}} \neq \tilde{\mathbf{O}}_{n \times 1}$, $\tilde{\mathbf{u}}_{\mathcal{I}}$ in (11) is a solution to the following optimization problem,

$$\min_{\tilde{\mathbf{v}} \in \mathbb{Q}_2^{n \times 1}} \left\| \tilde{\mathbf{v}} - \frac{\tilde{\mathbf{q}}_{\mathcal{I}}}{\|\tilde{\mathbf{q}}_{\text{st}}\|_2} \right\|_2^2 \text{ s.t. } sc(\tilde{\mathbf{v}}^* \tilde{\mathbf{q}}_{\text{st}}) = 0.$$

Furthermore, $\hat{\mathbf{u}} = \frac{\hat{\mathbf{q}}}{\|\hat{\mathbf{q}}\|_2}$ is the optimal solution to the following problem

$$\hat{\mathbf{u}} = \arg \min_{\hat{\mathbf{v}} \in \hat{\mathbb{Q}}_2^{n \times 1}} \|\tilde{\mathbf{v}}_{\text{st}} - \tilde{\mathbf{q}}_{\text{st}}\|_2^2 + \left\| \tilde{\mathbf{v}}_{\mathcal{I}} - \frac{\tilde{\mathbf{q}}_{\mathcal{I}}}{\|\tilde{\mathbf{q}}_{\text{st}}\|_2} \right\|_2^2 \epsilon.$$

Proof The proof is similar to that of Proposition 2 and we omit it here. □

4 The Power Method for Computing the Dominant Eigenvalue of a Dual Quaternion Hermitian Matrix

For a quaternion matrix \tilde{Q} , the power method can return the eigenvalue with the maximum absolute value and its associated eigenvector [15]. We now study the power method for dual quaternion Hermitian matrices.

4.1 The Dominant Eigenvalues of Dual Quaternion Hermitian Matrices

Based on Theorem 4.2 in [19], a dual quaternion Hermitian matrix $\hat{Q} \in \hat{\mathbb{Q}}^{n \times n}$ has exactly n eigenvalues, which are dual numbers. It also has n eigenvectors $\hat{u}_1, \dots, \hat{u}_n$, which are orthonormal vectors, i.e.,

$$\hat{Q}\hat{u}_i = \hat{u}_i\lambda_i \text{ and } \hat{u}_i^*\hat{u}_j = \begin{cases} \hat{1}, & \text{if } i = j; \\ \hat{0} & \text{otherwise,} \end{cases} \quad \forall i, j, = 1, \dots, n.$$

Furthermore, $\{\hat{u}_i : i = 1, \dots, n\}$ forms an orthonormal basis of $\hat{\mathbb{Q}}^{n \times 1}$.

For any unit dual quaternion numbers $\hat{q}_i \in \hat{\mathbb{U}}, i = 1, \dots, n$, if we multiply \hat{q}_i on the right of \hat{u}_i , there is

$$\hat{Q}\hat{u}_i\hat{q}_i = \hat{u}_i\lambda_i\hat{q}_i = (\hat{u}_i\hat{q}_i)\lambda_i \text{ and } \hat{q}_i^*\hat{u}_i^*\hat{u}_j\hat{q}_j = \begin{cases} \hat{1}, & \text{if } i = j; \\ \hat{0} & \text{otherwise} \end{cases} \quad \forall i, j, = 1, \dots, n, \quad (12)$$

where the second equality follows from that a dual number is commutative with a dual quaternion matrix. Hence, for a dual quaternion Hermitian matrix,

the unit norm eigenvectors $\{\hat{u}_i\}_{i=1}^n$, which form an orthonormal basis of $\hat{\mathbb{Q}}^{n \times 1}$, are not unique.

Suppose that we have a dual quaternion Hermitian matrix $\hat{Q} \in \hat{\mathbb{Q}}^{n \times n}$. We say that an eigenvalue λ_1 of \hat{Q} is a dominant eigenvalue of \hat{Q} , and an eigenvector corresponding to λ_1 a dominant eigenvector, if for any eigenvalue λ_j of \hat{Q} , we have

$$|\lambda_{1,st}| \geq |\lambda_{j,st}|, \quad \forall j = 2, \dots, n.$$

We say that an eigenvalue λ_1 of \hat{Q} is a strict dominant eigenvalue of \hat{Q} , with multiplicity l , if \hat{Q} has eigenvalues λ_j for $j = 1, \dots, n$, and they satisfy

$$\lambda_1 = \dots = \lambda_l \text{ and } |\lambda_{1,st}| > |\lambda_{l+1,st}| \geq \dots \geq |\lambda_{n,st}| \geq 0. \quad (13)$$

Assumption A The standard parts of the dominant eigenvalues have the same sign.

If \hat{Q} has a strict dominant eigenvalue, then \hat{Q} satisfies Assumption A. On the other hand, if \hat{Q} does not satisfy Assumption A. Let $\hat{P} = \hat{Q} + \alpha\hat{1}_{n \times n}$, where α is a nonzero real number. Then \hat{P} must satisfy Assumption A, and λ is an eigenvalue of \hat{Q} if and only if $\lambda + \alpha$ is an eigenvalue of \hat{P} . Thus, it is adequate that we may assume that \hat{Q} satisfies Assumption A.

Given any initial dual quaternion vector $\hat{v}^{(0)}$ with unit norm, the power method computes

$$\hat{y}^{(k)} = \hat{Q}\hat{v}^{(k-1)}, \quad \lambda^{(k-1)} = (\hat{v}^{(k-1)})^*\hat{y}^{(k)}, \quad \hat{v}^{(k)} = \frac{\hat{y}^{(k)}}{\|\hat{y}^{(k)}\|_2} \quad (14)$$

iteratively. This process is repeated until convergent or the maximal iteration number is reached.

Algorithm 1 The power method for computing the dominant eigenvalues of a dual quaternion Hermitian matrix

Require: The Hermitian matrix $\hat{\mathbf{Q}}$, the initial point $\hat{\mathbf{v}}^{(0)}$, the maximal iteration number k_{\max} , and the tolerance δ .
for $k = 1, \dots, k_{\max}$ **do**
 Update $\hat{\mathbf{y}}^{(k)}$ by $\hat{\mathbf{y}}^{(k)} = \hat{\mathbf{Q}}\hat{\mathbf{v}}^{(k-1)}$.
 Update $\lambda^{(k-1)} = (\hat{\mathbf{y}}^{(k-1)})^* \hat{\mathbf{y}}^{(k)}$.
 if $\|\hat{\mathbf{y}}^{(k)} - \hat{\mathbf{v}}^{(k-1)}\lambda^{(k-1)}\|_{2R} \leq \delta \times \|\hat{\mathbf{Q}}\|_{FR}$ **then**
 Stop.
 end if
 Update $\hat{\mathbf{v}}^{(k)}$ by $\hat{\mathbf{v}}^{(k)} = \frac{\hat{\mathbf{y}}^{(k)}}{\|\hat{\mathbf{y}}^{(k)}\|_2}$.
end for
Output: $\hat{\mathbf{v}}^{(k-1)}$ and $\lambda^{(k)}$.

4.2 Computing the Strict Dominant Eigenvalues by the Power Method

Similar to the power method for the real matrices and quaternion matrices, we show $\hat{\mathbf{v}}^{(k)}$ converges to the eigenvector corresponding to a strict dominant eigenvalue linearly.

Theorem 5 Given a dual quaternion Hermitian matrix $\hat{\mathbf{Q}} \in \hat{\mathbb{Q}}^{n \times n}$ and a dual quaternion vector $\hat{\mathbf{v}}^{(0)} = \sum_{j=1}^n \hat{\mathbf{u}}_j \hat{\alpha}_j$ with unit norm, where $\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_n$ are orthonormal eigenvectors of $\hat{\mathbf{Q}}$. Suppose $\sum_{j=1}^l |\hat{\alpha}_{j,\text{st}}| \neq 0$, $\hat{\mathbf{Q}}_{\text{st}} \neq \tilde{\mathbf{O}}_{n \times n}$, and $\hat{\mathbf{Q}}$ has a strict dominant eigenvalue with multiplicity l that satisfies (13). Then the sequence generated by the power method (14) satisfies

$$\hat{\mathbf{v}}^{(k)} = s^k \sum_{j=1}^l \hat{\mathbf{u}}_j \hat{\gamma}_j \left(1_D + \tilde{O}_D \left(\left| \frac{\lambda_{l+1,\text{st}}}{\lambda_{1,\text{st}}} \right|^k \right) \right), \tag{15}$$

where $s = \text{sgn}(\lambda_{1,\text{st}})$, $\hat{\gamma}_j = \frac{\hat{\alpha}_j}{\sqrt{\sum_{i=1}^l |\hat{\alpha}_i|^2}}$, and

$$\lambda^{(k)} = (\hat{\mathbf{v}}^{(k)})^* \hat{\mathbf{y}}^{(k+1)} = \lambda_1 \left(1_D + \tilde{O}_D \left(\left| \frac{\lambda_{l+1,\text{st}}}{\lambda_{1,\text{st}}} \right|^{2k} \right) \right).$$

In other words, the sequence $\hat{\mathbf{v}}^{(k)} s^k$ converges to a strict dominant eigenvector $\sum_{j=1}^l \hat{\mathbf{u}}_j \hat{\gamma}_j$ at a rate of $\tilde{O}_D \left(\left| \frac{\lambda_{l+1,\text{st}}}{\lambda_{1,\text{st}}} \right|^k \right)$ and the sequence $\lambda^{(k)}$ converges to the strict dominant eigenvalue λ_1 at a rate of $\tilde{O}_D \left(\left| \frac{\lambda_{l+1,\text{st}}}{\lambda_{1,\text{st}}} \right|^{2k} \right)$.

Proof Let $\hat{\mathbf{v}}^{(0)}$ be a dual quaternion vector with unit norm and $\hat{\mathbf{v}}^{(0)} = \sum_{j=1}^n \hat{\mathbf{u}}_j \hat{\alpha}_j$, where $\hat{\alpha}_j, j = 1, \dots, n$ are dual quaternion numbers, $\sum_{j=1}^n \hat{\alpha}_j^* \hat{\alpha}_j = \hat{1}$, and $\sum_{j=1}^l |\hat{\alpha}_{j,\text{st}}| \neq \tilde{0}$. Combing this with (14), we have

$$\hat{\mathbf{v}}^{(k)} = \frac{\sum_{j=1}^n \hat{\mathbf{u}}_j \lambda_j^k \hat{\alpha}_j}{\sqrt{\sum_{j=1}^n |\lambda_j^k \hat{\alpha}_j|^2}} \text{ and } \hat{\mathbf{y}}^{(k+1)} = \frac{\sum_{j=1}^n \hat{\mathbf{u}}_j \lambda_j^{k+1} \hat{\alpha}_j}{\sqrt{\sum_{j=1}^n |\lambda_j^{k+1} \hat{\alpha}_j|^2}}. \tag{16}$$

By direct derivations, we have $|\lambda_j^k \hat{\alpha}_j| = |\lambda_j^k| |\hat{\alpha}_j|$.

Furthermore, there is

$$\lambda_j^k = \lambda_{j,\text{st}}^k + k\lambda_{j,\text{st}}^{k-1}\lambda_{j,\mathcal{I}}\epsilon = \lambda_{j,\text{st}}^k(1 + k\beta_j\epsilon) \text{ and } |\lambda_j^k| = |\lambda_{j,\text{st}}|^k(1 + k\beta_j\epsilon),$$

where $\beta_j = \lambda_{j,\text{st}}^{-1}\lambda_{j,\mathcal{I}}$.

When k goes to infinity, we have

$$\begin{aligned} & \sqrt{\sum_{j=1}^n |\lambda_j^k \hat{\alpha}_j|^2 - |\lambda_1|^k} \sqrt{\sum_{j=1}^l |\hat{\alpha}_j|^2} \\ &= \frac{\sum_{j=l+1}^n |\lambda_j^k \hat{\alpha}_j|^2}{\sqrt{\sum_{j=1}^n |\lambda_j^k \hat{\alpha}_j|^2 + |\lambda_1|^k} \sqrt{\sum_{j=1}^l |\hat{\alpha}_j|^2}} \\ &= \frac{\sum_{j=l+1}^n |\lambda_{j,\text{st}}|^{2k} (1 + 2k\beta_j\epsilon) |\hat{\alpha}_j|^2}{\left(\sum_{j=1}^n |\lambda_{j,\text{st}}|^{2k} (1 + 2k\beta_j\epsilon) |\hat{\alpha}_j|^2\right)^{1/2} + |\lambda_{1,\text{st}}|^k (1 + k\beta_1\epsilon) \sqrt{\sum_{j=1}^l |\hat{\alpha}_j|^2}} \\ &= O_D(|\lambda_{l+1,\text{st}}|^k), \end{aligned}$$

where $O_D(\cdot)$ is defined by (2). Hence,

$$\frac{\lambda_j^k \hat{\alpha}_j}{\sqrt{\sum_{j=1}^n |\lambda_j^k \hat{\alpha}_j|^2}} = \frac{\lambda_j^k \hat{\alpha}_j}{|\lambda_1|^k \sqrt{\sum_{j=1}^l |\hat{\alpha}_j|^2} + O_D(|\lambda_{l+1,\text{st}}|^k)}.$$

If $j \in \{1, \dots, l\}$, it holds that

$$\begin{aligned} \frac{\lambda_1^k \hat{\alpha}_j}{\sqrt{\sum_{j=1}^n |\lambda_j^k \hat{\alpha}_j|^2}} &= \frac{\lambda_1^k \hat{\alpha}_j}{|\lambda_1|^k \sqrt{\sum_{i=1}^l |\hat{\alpha}_i|^2}} + \tilde{O}_D\left(\left|\frac{\lambda_{l+1,\text{st}}}{\lambda_{1,\text{st}}}\right|^k\right) \\ &= s^k \hat{\gamma}_j + \tilde{O}_D\left(\left|\frac{\lambda_{l+1,\text{st}}}{\lambda_{1,\text{st}}}\right|^k\right). \end{aligned}$$

Here, $\hat{\gamma}_j = \frac{\hat{\alpha}_j}{\sqrt{\sum_{i=1}^l |\hat{\alpha}_i|^2}}$,

and the second equality follows from $\frac{\lambda_1^k}{|\lambda_1|^k} = s^k$. Furthermore, if $j \geq l + 1$, we have

$$\frac{|\lambda_j^k \hat{\alpha}_j|}{\sqrt{\sum_{j=1}^n |\lambda_j^k \hat{\alpha}_j|^2}} = \frac{|\lambda_j|^k |\hat{\alpha}_j|}{|\lambda_1|^k \sqrt{\sum_{j=1}^l |\hat{\alpha}_j|^2} + O_D(|\lambda_{l+1,\text{st}}|^k)} = \tilde{O}_D\left(\left|\frac{\lambda_{j,\text{st}}}{\lambda_{1,\text{st}}}\right|^k\right).$$

Consequently, (15) holds true and $\hat{\mathbf{v}}^{(k)} s^k$ converges to $\sum_{j=1}^l \hat{\mathbf{u}}_j \hat{\gamma}_j$. Since

$$\hat{\mathbf{Q}} \sum_{j=1}^l \hat{\mathbf{u}}_j \hat{\gamma}_j = \sum_{j=1}^l \lambda_j \hat{\mathbf{u}}_j \hat{\gamma}_j = \lambda_1 \sum_{j=1}^l \hat{\mathbf{u}}_j \hat{\gamma}_j,$$

and

$$\left\| \sum_{j=1}^l \hat{\mathbf{u}}_j \hat{\gamma}_j \right\|_F^2 = \left(\sum_{j=1}^l \hat{\mathbf{u}}_j \hat{\gamma}_j \right)^* \left(\sum_{j=1}^l \hat{\mathbf{u}}_j \hat{\gamma}_j \right) = \sum_{j=1}^l \hat{\gamma}_j^* \hat{\gamma}_j = 1_D,$$

we have that $\sum_{j=1}^l \hat{\mathbf{u}}_j \hat{\gamma}_j$ is also an eigenvector corresponding to λ_1 . Furthermore,

$$\lambda^{(k)} = (\hat{\mathbf{v}}^{(k)})^* \hat{\mathbf{y}}^{(k+1)} = \frac{\sum_{j=1}^n \lambda_j^{2k+1} |\hat{\alpha}_j|^2}{\sum_{j=1}^n |\lambda_j^k \hat{\alpha}_j|^2} = \lambda_1 \left(1_D + \tilde{O}_D \left(\left| \frac{\lambda_{l+1, \text{st}}}{\lambda_{1, \text{st}}} \right|^{2k} \right) \right).$$

This completes the proof. □

4.3 General Dominant Eigenvalues

In general, we may assume that $\hat{\mathbf{Q}}$ satisfies Assumption A.

Then we show the convergence of the standard part of the dominant eigenvalue.

Lemma 6 *Given a dual quaternion Hermitian matrix $\hat{\mathbf{Q}} \in \hat{\mathbb{Q}}^{n \times n}$ and a dual quaternion vector $\hat{\mathbf{v}}^{(0)} = \sum_{j=1}^n \hat{\mathbf{u}}_j \hat{\alpha}_j$ with unit norm, where $\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_n$ are orthonormal eigenvectors of $\hat{\mathbf{Q}}$. Suppose $\hat{\mathbf{Q}}_{\text{st}} \neq \hat{\mathbf{O}}_{n \times n}$, $\sum_{j=1}^l |\tilde{\alpha}_{j, \text{st}}|^2 \neq 0$, and (12) holds.*

Then for the sequence generated by the power method (14), we have

$$\tilde{\mathbf{v}}_{\text{st}}^{(k)} = s^k \sum_{j=1}^l \tilde{\mathbf{u}}_{j, \text{st}} \tilde{\gamma}_{j, \text{st}} \left(1 + O \left(\left| \frac{\lambda_{l+1, \text{st}}}{\lambda_{1, \text{st}}} \right|^k \right) \right), \tag{17}$$

where $s = \text{sgn}(\lambda_{1, \text{st}})$, $\tilde{\gamma}_{j, \text{st}} = \frac{\tilde{\alpha}_{j, \text{st}}}{\sqrt{\sum_{i=1}^l |\tilde{\alpha}_{i, \text{st}}|^2}}$, and

$$\lambda_{\text{st}}^{(k)} = (\tilde{\mathbf{v}}_{\text{st}}^{(k)})^* \tilde{\mathbf{y}}_{\text{st}}^{(k+1)} = \lambda_{1, \text{st}} \left(1 + O \left(\left| \frac{\lambda_{l+1, \text{st}}}{\lambda_{1, \text{st}}} \right|^{2k} \right) \right).$$

In other words, the sequence $\tilde{\mathbf{v}}_{\text{st}}^{(k)} s^k$ converges to $\sum_{j=1}^l \tilde{\mathbf{u}}_{j, \text{st}} \tilde{\gamma}_{j, \text{st}}$ at a rate of $O \left(\left| \frac{\lambda_{l+1, \text{st}}}{\lambda_{1, \text{st}}} \right|^k \right)$, and the sequence $\lambda_{\text{st}}^{(k)}$ converges to $\lambda_{1, \text{st}}$ at a rate of $O \left(\left| \frac{\lambda_{l+1, \text{st}}}{\lambda_{1, \text{st}}} \right|^{2k} \right)$.

Proof For the standard part of the coefficient of $\hat{\mathbf{u}}_j$ in the sequence $\hat{\mathbf{v}}^{(k)}$ presented in (16), we have

$$\frac{\lambda_{j, \text{st}}^k \tilde{\alpha}_{j, \text{st}}}{\sqrt{\sum_{j=1}^n |\lambda_{j, \text{st}}^k \tilde{\alpha}_{j, \text{st}}|^2}} = \frac{\lambda_{j, \text{st}}^k \tilde{\alpha}_{j, \text{st}}}{|\lambda_{1, \text{st}}|^k \sqrt{\sum_{j=1}^l |\tilde{\alpha}_{j, \text{st}}|^2 + O(|\lambda_{l+1, \text{st}}|^k)}}.$$

If $j \in \{1, \dots, l\}$, it holds that

$$\begin{aligned} \frac{\lambda_{j, \text{st}}^k \tilde{\alpha}_{j, \text{st}}}{\sqrt{\sum_{j=1}^n |\lambda_{j, \text{st}}^k \tilde{\alpha}_{j, \text{st}}|^2}} &= \frac{\lambda_{1, \text{st}}^k \tilde{\alpha}_{j, \text{st}}}{|\lambda_{1, \text{st}}|^k \sqrt{\sum_{j=1}^l |\tilde{\alpha}_{j, \text{st}}|^2}} + O \left(\left| \frac{\lambda_{l+1, \text{st}}}{\lambda_{1, \text{st}}} \right|^k \right) \\ &= s^k \tilde{\gamma}_{j, \text{st}} + O \left(\left| \frac{\lambda_{l+1, \text{st}}}{\lambda_{1, \text{st}}} \right|^k \right). \end{aligned}$$

Here, $\tilde{\gamma}_{j, \text{st}} = \frac{\tilde{\alpha}_{j, \text{st}}}{\sqrt{\sum_{i=1}^l |\tilde{\alpha}_{i, \text{st}}|^2}}$. If $j \geq l + 1$, we have

$$\frac{\lambda_{j, \text{st}}^k \tilde{\alpha}_{j, \text{st}}}{\sqrt{\sum_{j=1}^n |\lambda_{j, \text{st}}^k \tilde{\alpha}_{j, \text{st}}|^2}} = \frac{\lambda_{j, \text{st}}^k \tilde{\alpha}_{j, \text{st}}}{|\lambda_{1, \text{st}}|^k \sqrt{\sum_{j=1}^l |\tilde{\alpha}_{j, \text{st}}|^2 + O(|\lambda_{l+1, \text{st}}|^k)}} = O \left(\left| \frac{\lambda_{j, \text{st}}}{\lambda_{1, \text{st}}} \right|^k \right).$$

Consequently, (17) holds true and $\tilde{\mathbf{v}}_{st}^{(k)}$ converges to $\sum_{j=1}^l \tilde{\mathbf{u}}_{j,st} \tilde{\gamma}_{j,st}$. Since

$$\tilde{\mathbf{Q}}_{st} \sum_{j=1}^l \tilde{\mathbf{u}}_{j,st} \tilde{\gamma}_{j,st} = \sum_{j=1}^l \lambda_{j,st} \tilde{\mathbf{u}}_{j,st} \tilde{\gamma}_{j,st} = \lambda_{1,st} \sum_{j=1}^l \tilde{\mathbf{u}}_{j,st} \tilde{\gamma}_{j,st},$$

and

$$\left\| \sum_{j=1}^l \tilde{\mathbf{u}}_{j,st} \tilde{\gamma}_{j,st} \right\|_F^2 = \left(\sum_{j=1}^l \tilde{\mathbf{u}}_{j,st} \tilde{\gamma}_{j,st} \right)^* \left(\sum_{j=1}^l \tilde{\mathbf{u}}_{j,st} \tilde{\gamma}_{j,st} \right) = \sum_{j=1}^l \tilde{\gamma}_{j,st}^* \tilde{\gamma}_{j,st} = 1,$$

we have $\sum_{j=1}^l \tilde{\mathbf{u}}_{j,st} \tilde{\gamma}_{j,st}$ is also an eigenvector of $\tilde{\mathbf{Q}}_{st}$ corresponding to $\lambda_{1,st}$. Furthermore, $\lambda_{st}^{(k)} = (\tilde{\mathbf{v}}_{st}^{(k)})^* \tilde{\mathbf{Q}}_{st} \tilde{\mathbf{v}}_{st}^{(k)}$ converges to $\lambda_{1,st}$. This completes the proof. \square

In the general case, the dual parts of $\hat{\mathbf{v}}^{(k)}$ may not converge. We now give an example to illustrate this phenomenon.

Example 4.1 Let $\hat{A} = A_{st} + A_{\mathcal{I}} \epsilon$, where

$$A_{st} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A_{\mathcal{I}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{18}$$

Then \hat{A} is a dual number Hermitian matrix.

By the power method, if $\hat{\mathbf{v}}^{(0)} = [1, 0]^\top$, then there is $\hat{\mathbf{v}}^{(k)} = [1, k\epsilon]^\top$, $\lambda^{(k)} = 1$; if $\hat{\mathbf{v}}^{(0)} = [0, 1]^\top$, then there is $\hat{\mathbf{v}}^{(k)} = [k\epsilon, 1]^\top$, $\lambda^{(k)} = 1$. In the both cases, $\tilde{\mathbf{v}}_{\mathcal{I}}^{(k)}$ does not converge.

If the dual part of the sequence generated by the power method does not converge, Algorithm 1 can still return the standard part of the dominant eigenvalue λ_{st} successfully. Then we may compute $\lambda_{\mathcal{I}}$, $\tilde{\mathbf{u}}_{st}$, and $\tilde{\mathbf{u}}_{\mathcal{I}}$ via the definition [19], i.e.,

$$\begin{aligned} (\tilde{\mathbf{Q}}_{st} - \lambda_{st} \tilde{I}) \tilde{\mathbf{u}}_{st} &= \mathbf{0}, \\ (\tilde{\mathbf{Q}}_{st} - \lambda_{st} \tilde{I}) \tilde{\mathbf{u}}_{\mathcal{I}} + \tilde{\mathbf{Q}}_{\mathcal{I}} \tilde{\mathbf{u}}_{st} &= \lambda_{\mathcal{I}} \tilde{\mathbf{u}}_{st}, \\ \tilde{\mathbf{u}}_{st}^* \tilde{\mathbf{u}}_{st} &= 1. \end{aligned} \tag{19}$$

Here, the last equation is applied to guarantee that $\hat{\mathbf{u}}$ is appreciable. This is a quaternion nonlinear system, and may not be easy to solve. Fortunately, $\lambda_{\mathcal{I}}$ is a real number. By (1), we can reformulate (19) into

$$\begin{aligned} \mathcal{M}(\tilde{\mathbf{Q}}_{st} - \lambda_{st} \tilde{I}) \mathcal{R}(\tilde{\mathbf{u}}_{st}) &= \mathbf{0}, \\ \mathcal{M}(\tilde{\mathbf{Q}}_{st} - \lambda_{st} \tilde{I}) \mathcal{R}(\tilde{\mathbf{u}}_{\mathcal{I}}) + \mathcal{M}(\tilde{\mathbf{Q}}_{\mathcal{I}}) \mathcal{R}(\tilde{\mathbf{u}}_{st}) &= \lambda_{\mathcal{I}} \mathcal{R}(\tilde{\mathbf{u}}_{st}), \\ \mathcal{R}(\tilde{\mathbf{u}}_{st})^\top \mathcal{R}(\tilde{\mathbf{u}}_{st}) &= 1. \end{aligned} \tag{20}$$

Equation (20) is a real nonlinear system, and may be solved by many numerical algorithms such as the Levenberg–Marquardt (LM) method [8]. It should be noted that λ_{st} is already known as the standard part of the dominant eigenvalue by our power method, hence the LM method may return a dominant eigenpair of $\tilde{\mathbf{Q}}$. Otherwise, the LM method may return any eigenpair of $\hat{\mathbf{Q}}$.

For Example 4.1, by the LM method, we may obtain the eigenpairs $\lambda_1 = 1 + \epsilon$, $\hat{\mathbf{u}}_1 = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right]^\top + \tilde{\mathbf{u}}_{1,\mathcal{I}} \epsilon$, and $\lambda_2 = 1 - \epsilon$, $\hat{\mathbf{u}}_2 = \left[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right]^\top + \tilde{\mathbf{u}}_{2,\mathcal{I}} \epsilon$, where $\tilde{\mathbf{u}}_{1,\mathcal{I}}$ and $\tilde{\mathbf{u}}_{2,\mathcal{I}}$ are arbitrary quaternion vectors.

Algorithm 2 Computing all appreciable eigenvalues of a dual quaternion Hermitian matrix

Require: $\hat{\mathbf{Q}}$, the dimension n , and the tolerance γ .

$\hat{\mathbf{Q}}_1 = \hat{\mathbf{Q}}$.

for $k = 1, \dots, n$ **do**

 Compute $\lambda_k, \hat{\mathbf{u}}_k$ as the dominant eigenpair of $\hat{\mathbf{Q}}_k$ by Algorithm 1.

 Update $\hat{\mathbf{Q}}_{k+1} = \hat{\mathbf{Q}}_k - \lambda_k \hat{\mathbf{u}}_k \hat{\mathbf{u}}_k^*$.

if $\|\hat{\mathbf{Q}}_{k+1, \text{st}}\|_F \leq \gamma$ **then**
 Stop.

end if

end for

Output: Eigenvalues $\{\lambda_i\}_{i=1}^k$ and eigenvectors $\{\hat{\mathbf{u}}_i\}_{i=1}^k$.

4.4 All Appreciable Eigenvalues of a Dual Quaternion Hermitian Matrix

By Theorem 7.1 in [16], the Eckart–Young-like theorem holds for dual quaternion matrices. If $\hat{\mathbf{Q}} \in \hat{\mathbb{Q}}^{n \times n}$ is a dual quaternion Hermitian matrix, then by Theorem 4.1 in [19], $\hat{\mathbf{Q}}$ can be rewritten as

$$\hat{\mathbf{Q}} = \hat{\mathbf{U}} \Sigma \hat{\mathbf{U}}^* = \sum_{i=1}^n \lambda_i \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^*, \tag{21}$$

where $\hat{\mathbf{U}} = [\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_n] \in \hat{\mathbb{Q}}^{n \times n}$ is a unitary matrix, $\Sigma = \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathbb{D}^{n \times n}$ is a diagonal dual matrix, and $\lambda_i, i = 1, \dots, n$ are in the descending order.

Denote $\hat{\mathbf{Q}}_k = \hat{\mathbf{Q}} - \sum_{i=1}^{k-1} \lambda_i \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^*$. Then $\lambda_k, \hat{\mathbf{u}}_k$ is a dominant eigenpair of $\hat{\mathbf{Q}}_k$. If $\lambda_{k, \text{st}} \neq 0$, $\lambda_k, \hat{\mathbf{u}}_k$ can be computed by implementing the power method on $\hat{\mathbf{Q}}_k$. By repeating this process from $k = 1$ to n , we get all appreciable eigenvalues and their corresponding eigenvectors.

The process is summarized in Algorithm 2.

Lemma 7 *Given a dual quaternion Hermitian matrix $\hat{\mathbf{Q}} \in \hat{\mathbb{Q}}^{n \times n}$. Suppose that all appreciable eigenvalues λ_i either equal each other or have distinct standard parts. Then Algorithm 2 can return all appreciable eigenvalues and their corresponding eigenvectors.*

Proof This lemma follows directly from (21) and Theorem 5. □

4.5 Relationship with the Best Rank-One Approximation

Lemma 8 *Given an appreciable dual quaternion Hermitian matrix $\hat{\mathbf{Q}} \in \hat{\mathbb{Q}}^{n \times n}$. The dominant eigenvalue and eigenvector formulate the best rank-one approximation of $\hat{\mathbf{Q}}$ under the square of the F -norm, i.e.,*

$$\lambda_1, \hat{\mathbf{u}}_1 \in \arg \min_{\lambda \in \mathbb{D}, \hat{\mathbf{u}} \in \hat{\mathbb{Q}}_2^n} \|\hat{\mathbf{Q}} - \lambda \hat{\mathbf{u}} \hat{\mathbf{u}}^*\|_F^2. \tag{22}$$

Proof By the ordering of the eigenvalues, there is

$$\lambda_{1, \text{st}}, \tilde{\mathbf{u}}_{1, \text{st}} \in \arg \min_{\lambda \in \mathbb{R}, \tilde{\mathbf{u}} \in \mathbb{Q}_2^{n \times 1}} \|\tilde{\mathbf{Q}}_{\text{st}} - \lambda \tilde{\mathbf{u}} \tilde{\mathbf{u}}^*\|_F^2.$$

Furthermore, by

$$\|\hat{\mathbf{Q}} - \lambda \hat{\mathbf{u}} \hat{\mathbf{u}}^*\|_F^2 = \|\tilde{\mathbf{Q}}_{\text{st}} - \lambda_{\text{st}} \tilde{\mathbf{u}}_{\text{st}} \tilde{\mathbf{u}}_{\text{st}}^*\|_F^2$$

$$+2sc \left(tr \left((\tilde{\mathbf{Q}}_{st} - \lambda_{st} \tilde{\mathbf{u}}_{st} \tilde{\mathbf{u}}_{st}^*)^* (\tilde{\mathbf{Q}}_{\mathcal{I}} - \lambda_{\mathcal{I}} \tilde{\mathbf{u}}_{st} \tilde{\mathbf{u}}_{st}^* - \lambda_{st} \tilde{\mathbf{u}}_{\mathcal{I}} \tilde{\mathbf{u}}_{st}^* - \lambda_{st} \tilde{\mathbf{u}}_{st} \tilde{\mathbf{u}}_{\mathcal{I}}^*) \right) \right) \epsilon,$$

$\tilde{\mathbf{Q}}_{st} \tilde{\mathbf{u}}_{1,st} = \lambda_{1,st} \tilde{\mathbf{u}}_{1,st}$, $\tilde{\mathbf{u}}_{1,st}^* \tilde{\mathbf{u}}_{1,st} = \tilde{\mathbf{I}}$, and $sc(\tilde{\mathbf{u}}_{1,st}^* \tilde{\mathbf{u}}_{\mathcal{I}}) = 0$, we have

$$\begin{aligned} &sc \left(tr \left((\tilde{\mathbf{Q}}_{st} - \lambda_{1,st} \tilde{\mathbf{u}}_{1,st} \tilde{\mathbf{u}}_{1,st}^*)^* (\tilde{\mathbf{Q}}_{\mathcal{I}} - \lambda_{\mathcal{I}} \tilde{\mathbf{u}}_{1,st} \tilde{\mathbf{u}}_{1,st}^* - \lambda_{1,st} \tilde{\mathbf{u}}_{\mathcal{I}} \tilde{\mathbf{u}}_{1,st}^* - \lambda_{1,st} \tilde{\mathbf{u}}_{1,st} \tilde{\mathbf{u}}_{\mathcal{I}}^*) \right) \right) \\ &= sc(tr(\tilde{\mathbf{Q}}_{st} \tilde{\mathbf{Q}}_{\mathcal{I}})) - \lambda_{1,st} \lambda_{\mathcal{I}} - \lambda_{1,st}^2 sc(\tilde{\mathbf{u}}_{1,st}^* \tilde{\mathbf{u}}_{\mathcal{I}} + \tilde{\mathbf{u}}_{\mathcal{I}}^* \tilde{\mathbf{u}}_{1,st}) \\ &\quad - \lambda_{1,st} \tilde{\mathbf{u}}_{1,st}^* \tilde{\mathbf{Q}}_{\mathcal{I}} \tilde{\mathbf{u}}_{1,st} + \lambda_{1,st} \lambda_{\mathcal{I}} + \lambda_{1,st}^2 sc(\tilde{\mathbf{u}}_{1,st}^* \tilde{\mathbf{u}}_{\mathcal{I}} + \tilde{\mathbf{u}}_{\mathcal{I}}^* \tilde{\mathbf{u}}_{1,st}) \\ &= sc(tr(\tilde{\mathbf{Q}}_{st} \tilde{\mathbf{Q}}_{\mathcal{I}})) - \lambda_{1,st} \tilde{\mathbf{u}}_{1,st}^* \tilde{\mathbf{Q}}_{\mathcal{I}} \tilde{\mathbf{u}}_{1,st}, \end{aligned}$$

which is invariant of $\tilde{\mathbf{u}}_{\mathcal{I}}$.

Hence, λ_1 and $\hat{\mathbf{u}}_1$ is an optimal solution of (22). This completes the proof. □

5 Numerical Experiments for Computing Eigenvalues

In this section, we first show the numerical experiments for computing all appreciable eigenvalues of the Laplacian matrices of circles and random graphs.

5.1 Eigenvalues of Laplacian Matrices of Circles

In the multi-agent formation control, the Laplacian matrix of the mutual visibility graph plays a key role [20]. Given a vector $\hat{\mathbf{q}} \in \hat{\mathbb{U}}^{m \times 1}$ and a graph $G = (V, E)$ with n vertices and m edges, the Laplacian matrix of G is defined as follows,

$$\hat{\mathbf{L}} = \mathbf{D} - \hat{\mathbf{A}}, \tag{23}$$

where \mathbf{D} is a diagonal real matrix where the i -th diagonal element is equal to the degree of the i -th vertex, and $\hat{\mathbf{A}} = (\hat{a}_{ij})$,

$$\hat{a}_{ij} = \begin{cases} \hat{q}_i^* \hat{q}_j, & \text{if } (i, j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

In the multi-agent formation control, $\hat{q}_i \in \hat{\mathbb{U}}$ is the configurations of the i -th rigid body, $\hat{a}_{ij} \in \hat{\mathbb{U}}$ is the relative configuration of the rigid bodies i and j , and $\hat{\mathbf{A}}$ is the relative configuration adjacency matrix.

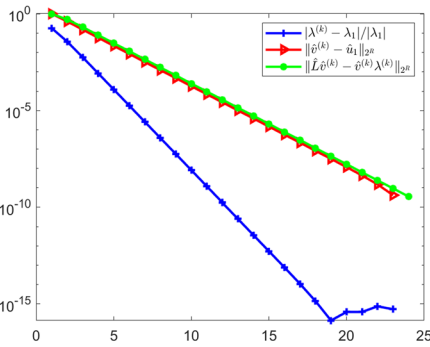
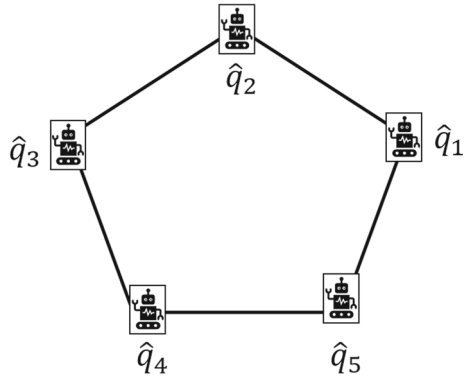
Consider a five-point circle as shown in Fig. 1. Suppose $\hat{\mathbf{q}}$ is a random unit dual quaternion vector as follows,

$$\hat{\mathbf{q}} = \begin{bmatrix} -0.4568 & -0.0602 & -0.0555 & -0.8858 \\ 0.6545 & -0.5512 & 0.0023 & -0.5175 \\ 0.3086 & 0.7100 & -0.5253 & -0.3533 \\ -0.5730 & 0.1223 & -0.6611 & -0.4688 \\ -0.5851 & 0.0650 & -0.1431 & -0.7956 \end{bmatrix} + \begin{bmatrix} 0.4701 & -0.6467 & 0.7286 & -0.2441 \\ 0.1108 & -0.4486 & 0.6871 & 0.6210 \\ -0.6448 & -0.1852 & -0.2048 & -0.6311 \\ -0.8584 & -0.2427 & 1.2512 & -0.7785 \\ -0.2806 & 0.4410 & 0.1730 & 0.2113 \end{bmatrix} \epsilon.$$

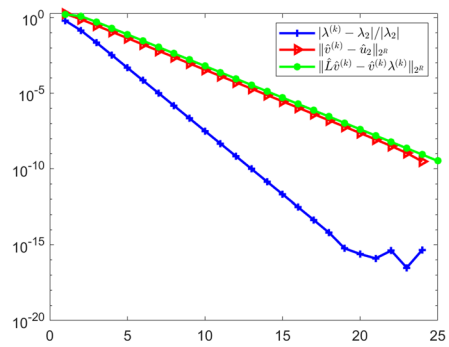
We now compute all appreciable eigenvalues of $\hat{\mathbf{L}}$ by Algorithm 2. After 23, 25, 1, and 1 iterations respectively, we get four eigenvalues 3.618, 3.618, 1.382, and 1.382. Then Algorithm 2 is stopped since the stopping criterion $\|\tilde{\mathbf{L}}_{5,\mathcal{I}}\|_F \leq 10^{-6} \times \|\tilde{\mathbf{L}}_{st}\|_F$ is satisfied and $\tilde{\mathbf{L}}_{5,\mathcal{I}}$ is also close to a zero matrix. Here, $\hat{\mathbf{L}}_5 = \hat{\mathbf{L}} - \sum_{k=1}^4 \lambda_k \hat{\mathbf{u}}_k \hat{\mathbf{u}}_k^*$. Hence, all five eigenvalues of $\hat{\mathbf{L}}$ are:

$$\{3.6180 + 0\epsilon, 3.6180 + 0\epsilon, 1.3820 + 0\epsilon, 1.3820 + 0\epsilon, 0 + 0\epsilon\}.$$

Fig. 1 The illustration of a five-point circle



(a) The first eigenpair for $n = 5$.



(b) The second eigenpair for $n = 5$.

Fig. 2 Convergence rate of power method for the Laplacian matrix of a five-point circle

Table 1 The eigenvalues (top row) for the Laplacian matrices of the circle and and the number of iterations (bottom row) of the power method

n	$[\lambda_1,$	$\lambda_2,$	$\lambda_3,$	$\lambda_4,$	$\lambda_5,$	$\lambda_6,$	$\lambda_7,$	$\lambda_8,$	$\lambda_9,$	$\lambda_{10}]$
3	[3.0000	3.0000	0]							
	[1	1	0]							
6	[4.0000	3.0000	3.0000	1.0000	1.0000	0]				
	[76	20	20	1	1	0]				
10	[4.0000	3.6180	3.6180	2.6180,	2.6180	1.3820	1.3820	0.3820	0.3820	0]
	[203	67	70	35	36	18	18	1	1	0]

The iterates of $\lambda^{(k)}$ and $\hat{v}^{(k)}$ for computing the first and second eigenpair are shown in Fig. 2. From this figure, we see that both the eigenvalue and the eigenvector converge linearly and the eigenvalue converges much faster than the eigenvector. Comparing the first and the second eigenpair, we see the convergence rate is the same. These conclusions are corresponding to our theory in Theorem 5 that $\hat{v}^{(k)}$ converges to a strict dominant eigenvector at a rate of $\tilde{O}_D((\frac{1.382}{3.618})^k)$ and $\lambda^{(k)}$ converges to a strict dominant eigenvalue at a rate of $\tilde{O}_D((\frac{1.382}{3.618})^{2k})$.

We further consider the Laplacian matrix of circles with 3, 6, and 10 points and list their eigenvalues and the number of iterations of the power method in Table 1. From this table,

we see that all appreciable eigenvalues are real numbers and the smallest eigenvalues are all zeros. In our numerical experiments, we also find that the eigenvalues are the same for any $\hat{\mathbf{q}}$ and are all nonnegative real numbers. The following theorem indicates that the observations are true.

Theorem 9 *Suppose $\hat{\mathbf{q}} \in \hat{\mathbb{U}}^n$ is a unit dual quaternion vector and*

$$\hat{L} = D - \hat{A}, \quad D = 2I, \quad \hat{A}_{ij} = \begin{cases} \hat{q}_i^* \hat{q}_j, & \text{if } |i - j| = 1 \text{ or } n - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Then the eigenvalues of \hat{L} have closed-form as follows,

$$\lambda_j = 2 - 2 \cos \frac{2\pi j}{n}, \quad j = 1, \dots, n, \tag{25}$$

and they are independent of the choice of $\hat{\mathbf{q}}$.

Proof Let $\hat{S} = \text{diag}(\hat{\mathbf{q}})$. Then \hat{S} is a unitary matrix, and

$$\hat{S} \hat{L} \hat{S}^* = \begin{bmatrix} 2 & -1 & \dots & -1 \\ -1 & 2 & -1 & \dots \\ & & \ddots & \ddots & \ddots \\ & & & \dots & 2 & -1 \\ -1 & & & \dots & -1 & 2 \end{bmatrix}.$$

Hence, \hat{L} is similar with a circulant matrix that admits eigenvalues as (25) up to reordering. Since the eigenvalues are invariant under unitary transformation, (25) are also the eigenvalues of \hat{L} . This completes the proof. \square

5.2 Eigenvalues of Laplacian Matrices of Random Graphs

We continue to consider the Laplacian matrix of a random graph $G = (V, E)$. Assume that G is a sparse undirected graph and E is symmetric. The sparsity of G is $s = m/n^2$, where $m = |E|$ is the number of edges in E . In practice, we randomly generate $\lceil \frac{m}{2} \rceil$ edges and let $E = \{(i, j), (j, i) : \text{if } (i, j) \text{ is sampled}\}$. All results are repeated ten times with different choices of $\hat{\mathbf{q}}$, different E , and different initial values in the power method. The average results are reported. We show the results in Table 2. In this table,

$$e_\lambda = \frac{1}{n_0} \sum_{i=1}^{n_0} \|\hat{\mathbf{L}} \hat{\mathbf{u}}_i - \lambda_i \hat{\mathbf{u}}_i\|_{2R} \quad \text{and} \quad e_L = \frac{1}{\|\hat{\mathbf{L}}\|_{FR}} \|\hat{\mathbf{L}} - \sum_{i=1}^{n_0} \lambda_i \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^*\|_{FR},$$

where λ_i and $\hat{\mathbf{u}}_i$ are the limit points obtained by Algorithm 1 and n_0 is the total number of eigenvalues obtained by Algorithm 2.

We also denote ‘ n_{iter} ’ as the average number of iterations and ‘Time (s)’ as the average CPU time in seconds for computing one eigenvalue.

From Table 2, we see that when $n = 10$, e_λ is less than 10^{-7} and e_L is less than 10^{-14} . Similarly, when $n = 100$, the error in eigenvalues and the residue of the result matrices is less than 1.1×10^{-3} and 10^{-7} , respectively. All results are obtained in less than 1.1 s. This shows the efficiency of the power method in computing all appreciable eigenvalues.

Table 2 Numerical results of power method for computing all appreciable eigenvalues of Laplacian matrices of random graphs with different sparsity values

s (%)	e_λ	e_L	n_{iter}	Time (s)	s (%)	e_λ	e_L	n_{iter}	Time (s)
$n = 10$									
10	2.27e-10	1.93e-16	13.72	0.0088	20	4.25e-10	4.03e-16	45.25	0.0288
30	1.31e-9	4.24e-16	88.98	0.0560	40	9.31e-10	2.40e-16	100.2	0.0631
50	1.72e-9	2.91e-16	120.63	0.0763	60	4.43e-8	4.98e-15	159.12	0.1005
$n = 100$									
5	1.42e-4	6.27e-8	707.85	0.7880	8	2.26e-4	2.38e-8	797.20	0.8857
10	2.99e-4	2.80e-8	840.98	0.9520	15	7.92e-4	2.18e-8	892.15	1.078
18	8.86e-4	1.45e-8	913.18	1.075	20	1.10e-3	3.91e-8	921.19	1.071

6 The Dominant Eigenvalue Method for SLAM

In this section, we first reformulate the SLAM problem as a rank-one dual quaternion matrix recovery problem. Then we present a two-block coordinate approach for solving this model.

At last, we present several numerical experiments to show the efficiency of our proposed method.

Given a directed graph $G = (V, E)$ with n vertices and m edges, where each vertex $i \in V$ corresponds to a robot pose $\hat{q}_i \in \hat{U}$ for $i = 1, \dots, n$, and each directed edge (arc) $(i, j) \in E$ corresponds to a relative measurement $\hat{q}_{ij} \in \hat{U}$. The aim of the SLAM problem is to find the best \hat{q}_i for $i = 1, \dots, n$, to satisfy $\hat{q}_{ij} = (\hat{q}_i)^* \hat{q}_j$ for $(i, j) \in E$ [5].

6.1 Reformulation

Denote $\hat{x} = [\hat{q}_1, \dots, \hat{q}_n]^* \in \hat{U}^n$ as a unit dual quaternion vector, and $\hat{Q} = (\hat{q}_{ij}) \in \hat{U}^{n \times n}$ as a dual quaternion matrix.

The aim of the SLAM problem is to find \hat{x} such that

$$(\hat{Q})_E = (\hat{x}\hat{x}^*)_E. \tag{26}$$

Here, $(\hat{Q})_E$ denotes the set $\{\hat{q}_{ij} : (i, j) \in E\}$. The variable \hat{x} can be computed by the least square model

$$\min_{\hat{x} \in \hat{U}^{n \times 1}} \|P_E(\hat{x}\hat{x}^* - \hat{Q})\|_F^2. \tag{27}$$

The solution of (27) is not unique because $\hat{x}\hat{x}^* = (\hat{x}\hat{q})(\hat{x}\hat{q})^*$ for any $\hat{q} \in \hat{U}$.

Instead of solving (27) directly, we introduce the auxiliary variables \hat{X} to solve the following problem

$$\begin{aligned} \min_{\hat{X} \in \hat{Q}^{n \times n}} & \frac{1}{2} \|P_E(\hat{X} - \hat{Q})\|_F^2 \\ \text{s.t.} & \hat{X} \in C_1 \cap C_2. \end{aligned} \tag{28}$$

Here,

$$C_1 = \{\hat{X} \in \hat{Q}^{n \times n} : \hat{X} = \hat{X}^*, \hat{x}_{ij} \in \hat{U}, \text{ and } \hat{x}_{ii} = \hat{1}\}, \tag{29}$$

$$C_2 = \{\hat{\mathbf{X}} \in \hat{\mathbb{Q}}^{n \times n} : \text{rank}(\hat{\mathbf{X}}) = 1\}. \tag{30}$$

In the following, we show the equivalence between problems (27) and (28).

Theorem 10 *The optimization problems (27) and (28) are equivalent in the following senses:*

- (i) *If $\hat{\mathbf{x}}$ is an optimal solution of (27), then $\hat{\mathbf{X}} = \hat{\mathbf{x}}\hat{\mathbf{x}}^*$ is an optimal solution of (28). Furthermore, the optimal value of problems (27) and (28) are the same.*
- (ii) *Let $\hat{\mathbf{X}}$ be the optimal solution for (28). Then its rank-one decomposition $\hat{\mathbf{X}} = \lambda\hat{\mathbf{u}}\hat{\mathbf{u}}^*$ satisfies $\lambda > 0_D$. Furthermore, $\hat{\mathbf{x}} = \sqrt{\lambda}\hat{\mathbf{u}}$ is an optimal solution of (27), and the optimal value of problems (27) and (28) are the same.*

Proof We first show that there is a one-to-one correspondence between $\hat{\mathbf{x}} \in \hat{\mathbb{U}}^{n \times 1}$ and $\hat{\mathbf{X}} \in C_1 \cap C_2$. On one hand, suppose that $\hat{\mathbf{x}} \in \hat{\mathbb{U}}^{n \times 1}$. Then it follows from direct derivations that

$$\hat{\mathbf{X}} = \hat{\mathbf{x}}\hat{\mathbf{x}}^* \in C_1 \cap C_2. \tag{31}$$

On the other hand, suppose that $\hat{\mathbf{X}} \in C_1 \cap C_2$. Then it follows from (21) that there exists $\lambda \in \mathbb{D}$ and $\hat{\mathbf{u}}^*\hat{\mathbf{u}} = \hat{\mathbf{I}}$ such that $\hat{\mathbf{X}} = \lambda\hat{\mathbf{u}}\hat{\mathbf{u}}^*$. By $x_{ii} = \lambda\hat{u}_i\hat{u}_i^* = \hat{\mathbf{I}}$, we have $\lambda_{st}\hat{u}_{i,st}\hat{u}_{i,st}^* = \hat{\mathbf{I}}$. Hence, $\lambda_{st} > 0$ and $\lambda > 0_D$. By [18], we have $\sqrt{\lambda} = \sqrt{\lambda_{st}} + \frac{\lambda_{\mathcal{T}}}{2\sqrt{\lambda_{st}}}\epsilon$. Let $\hat{\mathbf{x}} = \sqrt{\lambda}\hat{\mathbf{u}}$. Then $\hat{x}_i^*\hat{x}_i = \hat{\mathbf{I}}$. Hence, $\hat{\mathbf{x}} \in \hat{\mathbb{U}}^{n \times 1}$.

(i) From (31) we have $\hat{\mathbf{X}} = \hat{\mathbf{x}}\hat{\mathbf{x}}^*$ is a feasible solution of (28). Suppose that $\hat{\mathbf{X}}$ is not an optimal solution. Denote the optimal solution as $\hat{\mathbf{X}}'$. Then $\|P_E(\hat{\mathbf{X}}' - \hat{\mathbf{Q}})\|_F^2 < \|P_E(\hat{\mathbf{X}} - \hat{\mathbf{Q}})\|_F^2$. It follows from the above discussion, there also exists $\hat{\mathbf{x}}' \in \hat{\mathbb{U}}^{n \times 1}$ such that $\|P_E(\hat{\mathbf{x}}'(\hat{\mathbf{x}}')^* - \hat{\mathbf{Q}})\|_F^2 < \|P_E(\hat{\mathbf{x}}\hat{\mathbf{x}}^* - \hat{\mathbf{Q}})\|_F^2$. This contradicts the optimality of $\hat{\mathbf{x}}$.

(ii) Denote $\hat{\mathbf{x}} = \sqrt{\lambda}\hat{\mathbf{u}}$. Then it is a feasible solution of (27). The optimality of $\hat{\mathbf{x}}$ can be obtained by the contradiction method similarly.

This completes the proof. □

6.2 A Block Coordinate Descent Method

We introduce the auxiliary variables $\hat{\mathbf{X}}_1$ and $\hat{\mathbf{X}}_2$ to solve the following problem

$$\begin{aligned} \min_{\{\hat{\mathbf{X}}_i\}_{i=1}^2 \in \hat{\mathbb{Q}}^{n \times n}} \quad & \frac{1}{2} \|P_E(\hat{\mathbf{X}}_1 - \hat{\mathbf{Q}})\|_F^2 \\ \text{s.t.} \quad & \hat{\mathbf{X}}_1 \in C_1, \hat{\mathbf{X}}_2 \in C_2, \hat{\mathbf{X}}_1 = \hat{\mathbf{X}}_2. \end{aligned} \tag{32}$$

The quadratic penalty approach is applied to reformulate (32) as

$$\begin{aligned} \min_{\{\hat{\mathbf{X}}_i\}_{i=1}^2 \in \hat{\mathbb{Q}}^{n \times n}} \quad & \frac{1}{2} \|P_E(\hat{\mathbf{X}}_1 - \hat{\mathbf{Q}})\|_F^2 + \frac{\rho}{2} \|\hat{\mathbf{X}}_1 - \hat{\mathbf{X}}_2\|_F^2, \\ \text{s.t.} \quad & \hat{\mathbf{X}}_1 \in C_1, \hat{\mathbf{X}}_2 \in C_2. \end{aligned} \tag{33}$$

Then we compute $\hat{\mathbf{X}}_1$ and $\hat{\mathbf{X}}_2$ alternatively by the block coordinate descent method [26].

Specifically, at the k -th iteration, given $\rho^{(k-1)}$, $\hat{\mathbf{X}}_1^{(k-1)}$ and $\hat{\mathbf{X}}_2^{(k-1)}$, we compute $\hat{\mathbf{X}}_1^{(k)}$ as follows,

$$\hat{\mathbf{X}}_1^{(k)} = \arg \min_{\hat{\mathbf{X}}_1 \in C_1} \frac{1}{2} \|P_E(\hat{\mathbf{X}}_1 - \hat{\mathbf{Q}})\|_F^2 + \frac{\rho^{(k-1)}}{2} \|\hat{\mathbf{X}}_1 - \hat{\mathbf{X}}_2^{(k-1)}\|_F^2,$$

Algorithm 3 A two-block coordinate descent algorithm for SLAM

Require: $(\hat{\mathbf{Q}})_E$, the initial values $\rho^{(0)}$, $\hat{\mathbf{X}}_1^{(0)}$, $\hat{\mathbf{X}}_2^{(0)}$, the maximal iteration k_{\max} , the parameter ρ_1 , and the tolerance β .
for $k = 1, \dots, k_{\max}$ **do**
 Update $\hat{\mathbf{X}}_1^{(k)}$ by (34).
 Update $\hat{\mathbf{X}}_2^{(k)}$ by (36).
 Update $\rho^{(k)}$ by (37).
 if $\|\hat{\mathbf{X}}_1^{(k)} - \hat{\mathbf{X}}_2^{(k)}\|_{FR} \leq \beta$ **then**
 Stop.
 end if
end for
Output: $\hat{\mathbf{X}}_1^{(k)}$ and $\hat{\mathbf{X}}_2^{(k)}$.

which has an explicit solution as $\hat{x}_{1,ii}^{(k)} = 1$ and

$$\hat{x}_{1,ij}^{(k)} = P_{\hat{\mathbb{U}}} \left(c_{ij} \left(\delta_{E,ij} \hat{q}_{ij} + \delta_{E,ji} \hat{q}_{ji}^* + \rho^{(k-1)} (\hat{x}_{2,ij}^{(k-1)} + (\hat{x}_{2,ji}^{(k-1)})^*) \right) \right) \tag{34}$$

for all $i \neq j$. Here, $c_{ij} = \frac{1}{\delta_{E,ij} + \delta_{E,ji} + 2\rho^{(k-1)}}$, $\delta_{E,ij}$ is equal to one if $(i, j) \in E$ and zero otherwise, and $P_{\hat{\mathbb{U}}}(\cdot)$ is the projection onto the set of unit dual quaternion numbers defined by (6).

Then we compute $\hat{\mathbf{X}}_2^{(k)}$ as follows,

$$\hat{\mathbf{X}}_2^{(k)} = \arg \min_{\hat{\mathbf{X}}_2 \in C_2} \frac{\rho^{(k-1)}}{2} \|\hat{\mathbf{X}}_2 - \hat{\mathbf{X}}_1^{(k)}\|_F^2. \tag{35}$$

By (22), this problem admits an explicit solution as

$$\hat{\mathbf{X}}_2^{(k)} = \lambda \hat{\mathbf{u}} \hat{\mathbf{u}}^*, \tag{36}$$

where λ and $\hat{\mathbf{u}}$ are the dominant eigenvalue and eigenvector of $\hat{\mathbf{X}}_1^{(k)}$, respectively, which can be obtained by the power method (14).

We update the penalty parameter by

$$\rho^{(k)} = \rho_1 \rho^{(k-1)}, \tag{37}$$

where $\rho_1 > 1$ is a constant.

We summarize the block coordinate descent method for computing the SLAM problem in Algorithm 3.

6.3 Numerical Results for SLAM

We now consider solving the SLAM problem by our proposed model (28).

Denote $\hat{\mathbf{Q}}_0 = \hat{\mathbf{x}} \hat{\mathbf{x}}^* \in \hat{\mathbb{U}}^{n \times n}$, the noise matrix as $\hat{\mathbf{N}}$, and

$$\hat{\mathbf{Q}} = \hat{\mathbf{Q}}_0 + \hat{\mathbf{N}}, \tag{38}$$

The parameters of Algorithm 3 are given as follows. The penalty parameters $\rho^{(0)} = 0.01$ and $\rho_1 = 1.1$. The maximal iteration is set to be 1000, and the parameter in the stopping criterion is $\beta = 10^{-5}$. All experiments are repeated ten times with different choices of $\hat{\mathbf{q}}$, different noises, and different initial values in Algorithm 3. We only report the average performance.

Table 3 Numerical results for a five-point circle SLAM problem with different noise levels

l_{noise} (%)	e_x	e_Q	n_{iter}	Time (s)	l_{noise} %	e_x	e_Q	n_{iter}	time (s)
0	3.85e-6	4.72e-6	44.7	0.1872	5	1.68e-2	2.20e-2	144.3	0.5525
10	3.34e-2	4.40e-2	151.6	0.5958	15	4.49e-2	6.58e-2	156.1	0.6489
20	6.63e-2	8.76e-2	159.0	0.6531	25	8.25e-2	1.09e-1	161.7	0.6949

Table 4 Numerical results for random graph SLAM problems with different observation ratios

s (%)	e_x	e_Q	n_{iter}	Time (s)	s (%)	e_x	e_Q	n_{iter}	Time (s)
$n = 10$									
10	6.62e-1	4.63e-1	168.3	0.8036	20	1.50e-1	1.37e-1	149.0	0.6856
30	4.29e-3	4.24e-3	122.3	0.5492	40	5.74e-5	5.48e-5	86.3	0.3928
50	2.35e-6	1.55e-6	44.0	0.2137	60	1.28e-6	1.05e-6	33.7	0.1784
$n = 100$									
5	3.56e-1	1.81e-1	1000	14.14	8	3.95e-2	3.31e-2	1000	14.58
10	1.13e-2	1.15e-2	987.3	14.28	20	8.07e-5	7.83e-5	497.0	7.171
30	7.78e-7	8.45e-7	104.7	1.613	40	1.24e-7	1.40e-7	39.0	0.646

Firstly, consider the five-point circle SLAM problem with noisy observation (38). The results for different noise levels are given in Table 3. Here, ‘ l_{noise} ’ is the relative noise level defined by $l_{\text{noise}} = \frac{\|\hat{Q} - \hat{Q}_0\|_{FR}}{\|\hat{Q}_0\|_{FR}}$, ‘ e_x ’ and ‘ e_Q ’ are the error measurements defined by $e_x = \frac{\|\hat{y}_r - \hat{x}_r\|_{2R}}{\|\hat{x}_r\|_{2R}}$ and $e_Q = \frac{\|\hat{Q}_0 - \lambda \hat{u} \hat{u}^*\|_{FR}}{\|\hat{Q}_0\|_{FR}}$, where $\hat{x}_r = \hat{x}_{|\hat{x}_i|}$, $i = \arg \max_{i=1, \dots, n} |x_i|$, $\hat{y} = \hat{u} \sqrt{\lambda}$, and $\hat{y}_r = \hat{y}_{|\hat{y}_i|}$. ‘ n_{iter} ’ and ‘time (s)’ are the number of iterations and the total CPU time in seconds for implementing Algorithm 3 respectively. From Table 3, we can see that Algorithm 3 can return a solution with a reasonable error in one minute.

We continue to consider the SLAM problem of random graphs. Suppose the number of vertices is n and the observation rate is $s = \frac{m}{n^2}$. The numerical results are shown in Table 4. When $n = 10$, all results can be obtained in less than one second. As the observation ratio increases, the error in \hat{x} and \hat{Q} and the number of iterations are all decreasing.

When $n = 100$, we have similar observations.

7 Final Remarks

In this paper, we proposed a power method for computing the dominant eigenvalue of a dual quaternion Hermitian matrix and showed the convergence and convergence rate. We used the Laplacian matrices from circles and random graphs to demonstrate the efficiency of the power method. Then we further studied the SLAM problem by a two blocks coordinate descent approach, where one block has a closed form solution and the other block can be obtained by the power method. Our results fill the blank of dual quaternion matrix computation and build relationship between the dual quaternion matrix theory and the SLAM problem. Adding the relation between the dual quaternion matrix theory and the formation control

problem, established in [20], we see that the dual quaternion matrix has a solid application background and is worth being further studied.

Funding Open access funding provided by The Hong Kong Polytechnic University This work was supported by the R&D project of Pazhou Lab (Huangpu) (Grant no. 2023K0603), the Natural Science Foundation of China (No. 12131004), and the Fundamental Research Funds for the Central Universities (Grant No. YWF-22-T-204).

Data Availability Statement The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

Declarations

Conflict of interest The author declares no conflict of interest.

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