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Conjugate gradient method for Phase retrieval based on Wirtinger derivative

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A conjugate gradient-Wirtinger flow (CG-WF) algorithm for phase retrieval is proposed in this paper. It is shown that, compared with recently reported Wirtinger flow (WF) and its modified methods, the proposed CG-WF algorithm is able to dramatically accelerate the convergence rate, while keeping the dominant computational cost of each iteration unchanged. We numerically illustrate the effectiveness of our method in recovering 1D Gaussian signals and 2D natural color images under both Gaussian and coded diffraction pattern (CDP) models.

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1. INTRODUCTION

Phase retrieval (PR), which aims to reconstruct an unknown signal or image from the phaseless measurements, plays an important role in science and engineering such as astronomy [1], crystallography [2], microscopy [3], and optics [4, 5]. For example, in optical encryption systems, the PR technique is widely used to reconstruct phase only masks or original images [6, 7]. Recently, intense attentions have also been paid on an active field called coherent diffractive imaging (CDI) which combines X-ray diffraction, oversampling, and phase retrieval technique [8–10].

The most well-known algorithm for phase retrieval would be the alternating projection method proposed by Gerchberg and Saxton (GS) [11], which starts from a random initial guess and projects alternatively between frequency and time domain to correct the current estimate. Fienup has proposed a hybrid input-output method (HIO) [12] based on GS algorithm, and is widely used in various fields to extract phase information from diffraction intensity by applying certain constraints. Over the years, various modified GS algorithms have been proposed and demonstrated to be effective in phase retrieval such as difference map [13], guided HIO [14], and oversampling smoothness [15]. The major drawback of GS algorithm and its modified methods would be that it is very difficult to ensure the convergence theoretically due to the use of projections onto nonconvex constraint set [16, 17].

Recently, more works in PR are focused on theoretically convergent algorithms. A PhaseLift algorithm is proposed to convert the non-convex problem into a convex one by using the lift technique of semi-definite programming (SDP) [18]. It is shown that PhaseLift can reconstruct the signal with large probability when the measurements are random Gaussian. However, this approach requires a lift of matrix and thus highly increases the computational cost, which can hardly be applied for high dimensional signals such as 2D image. A gradient decent scheme, i.e., Wirtinger flow (WF) algorithm, has been reported recently and is demonstrated to allow exact recovery of the phase from magnitude measurements [19]. The WF algorithm is based on spectral initialization procedure and Wirtinger derivative. An empirical choice of stepsize is suggested in the WF method, but this heuristic stepsize selection is not an optimal one. In [20], a modified Wirtinger flow (MWF) method with optimal stepsize is further proposed to increase the convergence rate of WF method. In this paper, we propose a conjugate gradient Wirtinger flow (CG-WF) algorithm by using Polak-Ribière-Polyak (PRP) direction instead of Wirtinger derivative. Also, the process of calculating optimal stepsize in [20] is modified to decrease the computational cost at each iteration. We demonstrate the proposed CG-WF method by various numerical examples using both Gaussian and coded diffraction pattern (CDP) models. It is shown that compared with recently reported Wirtinger flow (WF) and its modified methods, the proposed CG-WF algorithm is able to dramatically accelerate the convergence rate, while keeping the dominant computational cost of each iteration unchanged.

2. THEORETICAL ANALYSIS

The phase retrieval problems are related with quadratic equations of the form

$$y_i = |\mathbf{a}_i^H \mathbf{z}|^2, \quad i = 1, 2, ..., m, \tag{1}$$

where the superscript H denotes the Hermitian operator. $\mathbf{z}$ is a complex vector to be recovered, $\mathbf{a_i}$ is the ith measurement vector, and y_i is the ith magnitude squared observation of the linear measurement of any complex vector $\mathbf{z}$. Usually, the following least-squared equation is used to solve $\mathbf{z}$:

$$min\ f(\mathbf{z}) = \frac{1}{2m}\sum_{i=1}^{m}\left(|\mathbf{a}_i^H\mathbf{z}|^2 - y_i\right)^2 \quad \textbf{(2)}$$

A. Wirtinger flow and modified Wirtinger flow methods

The Wirtinger flow (WF) method starts with an initialization $\mathbf{z_0}$ via a spectral method, i.e., $\mathbf{z_0}$ is calculated as the leading eigenvector of the positive semidefinite Hermitian matrix $\sum_i y_i\mathbf{a_i}\mathbf{a_i^H}$ by power iteration method [19, 21]. For $k = 0,\ 1,\ 2,\ ...$, WF method iteratively updates $\mathbf{z_{k+1}}$ through the following equation:

$$\mathbf{z}_{k+1} = \mathbf{z}_k - \alpha_k\mathbf{d}_k \quad \textbf{(3)}$$

where the search direction $\mathbf{d}_k$ is calculated as Wirtinger derivative of the objective function:

$$\mathbf{d}_k = \nabla f(\mathbf{z}_k) = \frac{1}{m}\sum_{i=1}^{m}(|\mathbf{a}_i^H\mathbf{z_k}|^2 - y_i)(\mathbf{a}_i\mathbf{a}_i^H)\mathbf{z_k} \quad \textbf{(4)}$$

The stepsize α_k is empirically chosen as

$$\alpha_k = min(1 - e^{(-k/k_0)}, \alpha_{max}) \quad \textbf{(5)}$$

with k_0 and α_{max} around 330 and 0.4, respectively. As is stated in [20], this heuristic stepsize selection rule is not optimal. A more appropriate stepsize is chosen in modified Wirtinger flow (MWF) method based on the fact that the objective function $f(\mathbf{z}_k - \alpha_k\mathbf{d}_k)$ is a quartic function of α_k. The minimum value of $f(\mathbf{z}_k - \alpha_k\mathbf{d}_k)$ occurs at the point satisfying the following optimal condition [20]:

$$df(\mathbf{z}_k - \alpha_k\mathbf{d}_k)/d\alpha_k = 0 \quad \textbf{(6)}$$

Eq. (6) is a univariate cubic equation of α_k given by:

$$a_c\alpha_k^3 + b_c\alpha_k^2 + c_c\alpha_k + d_c = 0 \quad \textbf{(7)}$$

where the constant coefficients $a_c = \sum_{i=1}^{m}|h_i|^4$, $b_c = -3\sum_{i=1}^{m}\mu_i|h_i|^2$, $c_c = \sum_{i=1}^{m} r_i|h_i|^2 + 2\mu_i^2$, and $d_c = -\sum_{i=1}^{m} u_i r_i$ with $h_i = \mathbf{a}_i^H\mathbf{d}_k$, $\mu_i = \mathrm{Re}(y_i^* h_i)$, and $r_i = |\mathbf{a}_i^H\mathbf{z_k}|^2 - y_i$ [20]. The optimal stepsize is one of the roots in Eq. (7), and a closed form solution of such cubic equation can be easily found. There are two cases for the property of the roots in Eq. (7): the first case has only one real root, which is just the optimal stepsize. The second case has three real roots, and the optimal stepsize is chosen as the real root associated with the minimum objective value.

B. Conjugate gradient-Wirtinger flow method

In this paper, we propose a conjugate gradient-Wirtinger flow (CG-WF) method for solving phase retrieval problem. In CG-WF, the calculation of initial value is the same as that in WF method. In the optimization process, instead of Wirtinger derivative, Polak-Ribière-Polyak (PRP) direction [22] is used as the direction of updating z_{k+1} in Eq. (3). Specifically, $\mathbf{d}_k$ in Eq. (3) is replaced with $-\mathbf{v}_k$, where $\mathbf{v}_k$ is calculated as follows: If $k = 0$, $\mathbf{v}_0 = -\nabla f(\mathbf{z}_0)$. Otherwise,

$$\mathbf{v}_k = -l_k + (\mathrm{Re}[l_k^H \cdot (l_k - l_{k-1})]/||l_{k-1}||^2)\mathbf{v}_{k-1} \quad \textbf{(8)}$$

with $||\cdot||$ be Euclidean norm and $l_k = \nabla f(\mathbf{z}_k)$, which can be calculated by Eq. (4).

The stepsize in CG-WF method is the same as that in MWF with replacing $\mathbf{d}_k$ in Eq. (7) by $-\mathbf{v}_k$. Whereas, the process of calculating the optimal size in CG-WF is slightly different with that in MWF. As is mentioned previously, when there are three real roots in Eq. (7), MWF chooses the real root associated with the minimum objective value. Practically, the computational cost associated with calculating objective value three times at each iteration by Eq. (2) is relatively high for large scale problems. Thus, in the proposed method, we first estimate the sign of the second derivative of $f(\mathbf{z}_k - \alpha_k\mathbf{d}_k)$, which is equivalent to decide the sign of

$$s(o_i) = 3a_c{o_i}^2 + 2b_c o_i + c_c,\ \ i = 1,2,3. \quad \textbf{(9)}$$

with o_i are the three real roots of Eq. (7). Since the optimal stepsize is corresponding to the case when $s(o_i)$ is positive, we only need to calculate the objective function at most twice. The complete implementation procedures of proposed CG-WF method are as follows:

- Step 1) Initial step, $k = 0$; Calculate the initial value $\mathbf{z}_0$ based on the power iteration method.
- Step 2) Determine the search direction: Calculate Wirtinger derivative $\nabla f(\mathbf{z}_k)$ according to Eq. (4). Then determine the Polak-Ribière-Polyak (PRP) direction $\mathbf{v}_k$ according to Eq. (8), and the search direction $\mathbf{d}_k = -\mathbf{v}_k$.
- Step 3) Determine the search length α_k according to the roots in Eq. (7). For the case of three roots in Eq. (7), only real roots that yield a positive sign of $s(o_i)$ in Eq. (9) are eligible.
- Step 4) Update $\mathbf{z}_{k+1} = \mathbf{z}_k - \alpha_k\mathbf{d}_k$
- Step 5) If termination condition such as reaching a maximum iterations is satisfied, stop iteration. Otherwise, let $k = k + 1$, and go to step 2).

3. NUMERICAL RESULTS

In this section, we present the simulation results to compare WF, MWF and CG-WF methods using both Gaussian and coded diffraction pattern (CDP) models [19, 23], where both Gaussian signals (1D and 2D) and natural images are considered. For the Gaussian model, $\mathbf{a_i} \sim N(0, \mathbf{I}/2) + iN(0, \mathbf{I}/2)$. For CDP models, we collect the data as the form of [19, 23]:

$$y_{l,q} = |\sum_{t=0}^{n-1} x[t]\bar{d}_l(t)e^{-i2\pi qt/n}|^2 \quad \textbf{(10)}$$

with $1 \le l \le L$ and $0 \le q \le n-1$. The CDP models generate the information about the spectrum of $x[t]$ modulated by the code $d_l[t]$. In this paper, we consider the case where d_l are i.i.d. distributed and a total number of L patterns are collected by changing the code d_l. In this section, we are interested in the behavior of normalized mean square error (NMSE) as a function of number of iterations. The NMSE is calculated as:

$$NMSE = ||\mathbf{z}^k - \mathbf{z}||_F^2/||\mathbf{z}||_F^2 \quad \textbf{(11)}$$

with $||\cdot||_F$ denotes Frobenius norm. $\mathbf{z}$ is the true signal, and $\mathbf{z}^k$ is the signal recovered at the kth iterations. It should be noted that $\mathbf{z}^k$ is multiplied by a constant value to get the rid of effect of constant phase shift. For a fair comparison, we make all the initial values and collected data the same among three methods (including the randomly distributed signals).

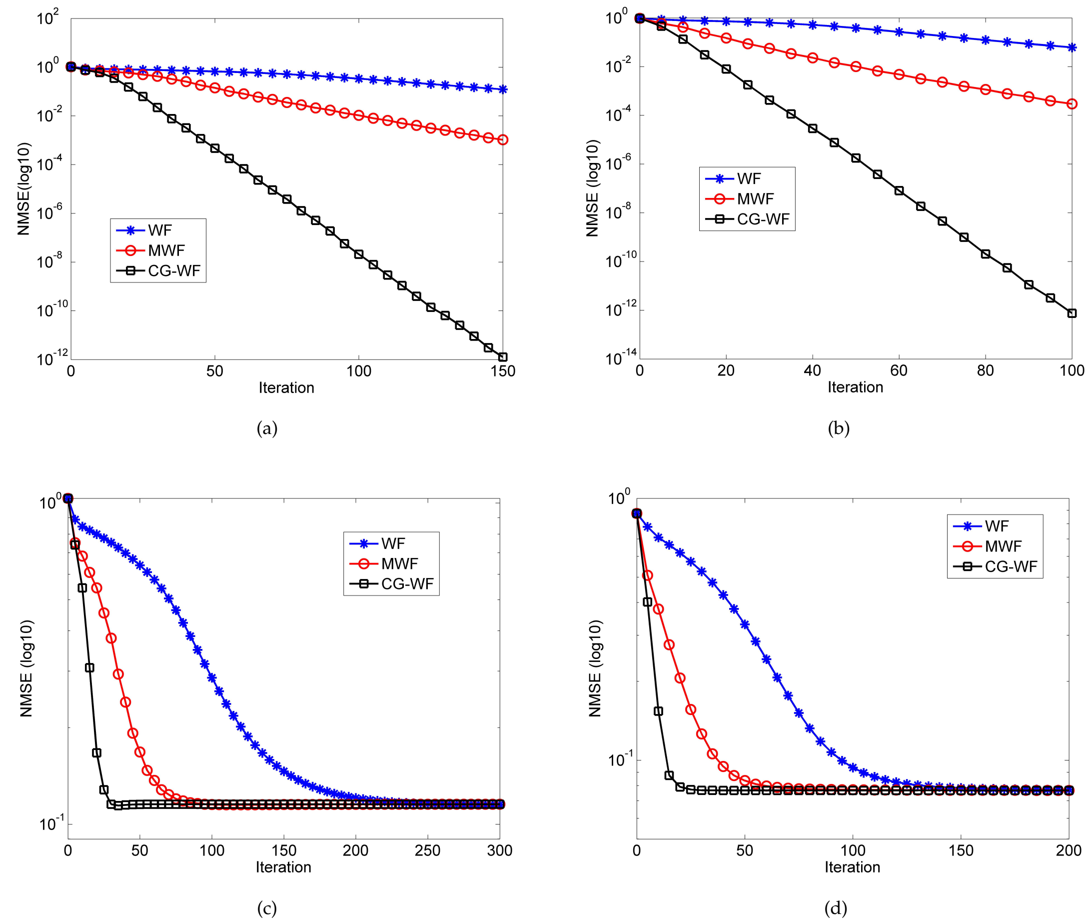

Fig. 1. NMSE of recovering 1D Gaussian signal in (a) Gaussian model, (b) CDP model, (c) Gaussian model with 10% noise, and (d) CDP model with 10% noise.

A. Recovery of 1D signal

We first consider the recovery of 1D Gaussian signals with the size of 128×1 under both Gaussian and CDP models. For the CDP models, a number of 6 masks are used. Fig. 1(a) and Fig. 1(b) present NMSEs of recovering 1D Gaussian signals in Gaussian model and CDP model, respectively, where NMSEs are plotted using a base 10 logarithmic scale. It is noted that, when NMSE reaches 0.01 for Gaussian model, it takes around 30, 100, and more than 150 iterations for CG-WF, MWF, and WF algorithms, respectively. Similar results can be obtained for recovering 1D Gaussian signals in CDP model. It is concluded that the proposed CG-WF method outperforms the recently reported WF and MWF methods since it accelerates the convergence speed dramatically. It is worthwhile to discuss the reasons of the results in Fig. 1. In WF and MWF method, a gradient decent direction is used, which makes the methods easily take steps in the same direction as earlier steps and zigzag move to the solution. Whereas, in CG-WF method, through a linear combination of the Wirtinger derivative and previous search direction, we are able to avoid this problem and thus move to the solution with much fewer steps [24].

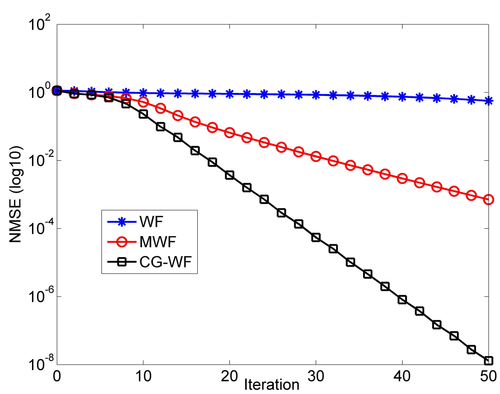

Fig. 2. NMSE of recovering 2D Gaussian signal in CDP model.

In the second part of numerical experiments, recovering signals from noisy received magnitudes are considered. In these examples, additive white Gaussian noise (AWGN) $\mathbf{n}$ is added to the received magnitude vector $\mathbf{y}$, and is quantified by $(||\mathbf{n}||_F/||\mathbf{y}||_F) \times 100\%$. Fig. 1(c) and Fig. 1(d) present NMSEs of recovering 1D Gaussian signals in Gaussian model and CDP model with 10% Gaussian noise, respectively. It suggests that the proposed CG-WF converges much faster to noise level than WF and MWF method. Specifically, for Gaussian model, it takes only about 30 iterations for CG-WF method to converge to the

(a)

(b)

(c)

Fig. 3. Performance of (a) WF, (b) MWF, and (c) CG-WF on a natural color image after 30 iterations.

noise level, but it takes around 100 and 250 iterations for MWF and WF methods, respectively.

B. Recovery of 2D signal

In the third part of numerical experiments, we first consider the recovering of 2D Gaussian signals with the size of 128×256 under the CDP model. Fig. 2 presents the NMSE varying with iteration number for recovering 2D Gaussian signals. Numerical results show that CG-WF method converges much faster than WF and MWF methods.

Numerical experiments have also been done on natural color image to compare the performance of different methods, as are done in [16], where a Stanford main quad color image with the size of 320×1280 pixels is considered. $L = 21$ random patterns are generated to gather the coded diffraction patterns for each color band, and the diffraction patterns are then used to recover the original images with WF, MWF, and CG-WF methods. Fig. 3(a), Fig. 3(b), and Fig. 3(c) show the recovered images after 30 iteration, and NMSEs for WF, MWF, and CG-WF are 1.2, 0.3, and 0.001, respectively. It is seen that the performance of recovered image by CG-WF is much better than those of WF and MWF methods, and for both WF and MWF methods, they need more iterations before obtaining a satisfying recovery of natural images.

4. CONCLUSIONS

In this paper, we have proposed a CG-WF algorithm for phase retrieval, which significantly improves the convergence rate compared with recently reported WF and MWF methods. Instead of using Wirtinger derivative as the search direction, Polak-Ribière-Polyak (PRP) direction is used. Numerical experiments have been done on both Gaussian and CDP models to recover both 1D Gaussian signals and 2D natural images. All the results show that the proposed phase retrieval method converges much faster than WF and MWF methods. In addition, the proposed CG-WF method can also be directly applied to other modified Wirtinger flow methods such as truncated Wirtinger flow (TWF) [25] to accelerate the convergence speed.

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