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Alternate closed-form weibit-based model for assessing travel choice with an oddball alternative

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Abstract

Herein, a weibit-based model is proposed as an alternative to Recker's logit choice model with an "oddball" alternative, which explicitly focuses on a single alternative that has unique attributes to other conventional alternatives in the choice set [*Recker, W.W. (1995). Discrete choice with an oddball alternative. Transportation Research, 29B, 201–212*]. While retaining the closed-form probability expression, the proposed model handles the oddball alternative using a multiplicative random disutility function assuming Weibull distributed random components. The proposed model thus allows disutility-dependent perception variances of both the conventional and oddball alternatives and a flexible variance ratio between them, which are effective to reflect the way how individuals perceive varying magnitudes of travel disutility from different attributes. This gives the proposed model more flexibility to consider various heterogeneity issues, including the heterogeneous perceptions of conventional alternatives, heterogeneous perceptions of unique and common attributes of the oddball alternative, and heterogeneous features of conventional and oddball alternatives. The empirical application of the proposed model is explored to highlight its practical performance. The results demonstrate the superiority of the proposed model in the choice contexts where both oddball effect and heterogeneity issues need to be considered. The proposed model could further provide new behavioral insights into various decision-making scenarios of transportation networks, such as transportation mode choices in the current era of emerging technologies and destination choices in urban agglomerations.

Keywords: Oddball alternative; Closed-form choice model; Weibull distribution

1. Introduction

With significant progress in transportation technologies and rapid lifestyle changes, travelers are likely to consider innovative transportation alternatives that have different attributes to the conventional alternatives, which are thus labeled "*oddball*" in the choice set. For example, emerging technologies such as autonomous vehicles are set to be added to the multi-modal transportation network, which will lead to new travel modes with uncommon service features, such as automated navigation, that cannot be obtained from conventional travel modes. In another case, with development of urban agglomeration, more travelers now tend to choose destinations located in neighboring cities (Huang et al., 2020). Differing from the traditional destination choices within a city, choosing a destination in a neighboring city involves inter-city trips that may lead to unique opportunities for work, education, and entertainment.

The oddball alternative with unique attributes can be handled via alternative-specific random components in open-form choice models, such as the heteroscedastic extreme-value (HEV) model (Bhat, 1995). However, these models often require additional model parameters and lack a closed-form probability expression, which pose additional difficulties to model estimation, interpretation, and evaluation. Furthermore, the closed-form choice probability is valuable for the applications, where the stochastic choice behavior of travelers is embedded in higher-level optimization problems. For instance, facility location and assortment optimization problems often incorporate choice models to reproduce the probabilistic customer demand. Compared with the integration of closed-form logit-type models that renders linear reformulation and efficient solution algorithms (Haase and Müller, 2014; Davis et al., 2014; Méndez-Vogel et al., 2023), embedding open-form models can significantly increase the complexity of the optimization problem and often requires approximation algorithms for determining location solutions (Rusmevichientong et al., 2014; Strauss et al., 2018). In transportation studies, choice models are widely incorporated in equilibrium travel demand models to reproduce the multidimensional travel behavior with flow-dependent travel costs (Sheffi, 1985; Oppenheim, 1995; Zhou et al., 2009; Kitthamkesorn and Chen, 2017; Gu and Chen, 2023). As the travel costs are functions of travel demand in congested networks, the travel demand model can be viewed as a fixed-point problem where choice probabilities exist on both sides of the equation. Solving such models often requires an iterative procedure to update the demand pattern with choice probabilities evaluated at each iteration. Owing to the computationally burdensome evaluation of open-form choice probabilities, the equilibrium travel demand models become intractable when the choice set contains more than a handful of alternatives, particularly impractical for large-scale transportation networks with thousands or even millions of routes (Du et al., 2021; Li et al., 2023). The computational burden will become even heavier in the network design problem embedded with a network equilibrium model, which can be formulated as a bi-level program with a user equilibrium (UE) or a stochastic user equilibrium (SUE) traffic assignment model at the lower level, or a single-level mathematical programming with equilibrium constraints (Yang and Bell, 1998; Chen et al., 2011). In contrast, conventional closed-form travel choice models, such as the multinomial logit (MNL) model and generalized extreme value (GEV) models, can significantly reduce the computational requirement for choice probability evaluation and are frequently adopted in travel demand forecasting and network design studies. However, MNL and GEV models mainly assume identically Gumbel distributed total random errors and an identical variance across alternatives with distinct magnitudes of disutility, which are inadequate to capture the oddball effect (Ben-Akvia and Lerman, 1985; Prashker and Bekhor, 2004; Koppelman and Sethi, 2008). Therefore, it is imperative to develop a closed-form choice model that can explicitly assess travel choice behavior with the new oddball alternatives of modern transportation networks.

In a pioneering study, Recker (1995) proposed a multinomial logit model with an oddball alternative (hereafter referred to as the MNL-O model), which explicitly accounts for the random utility associated with the unique features of a single oddball alternative while retaining a closed-form probability expression, with the oddball alternative's unique attributes considered separately in random components. The closed-from probability expression allows a straightforward interpretation of the relationship between the observed variables and choice probabilities, enabling the efficient and precise maximum likelihood estimation approach to be applied, eliminating the computational burden of needing additional numerical or simulation approaches for the probability evaluation (Koppelman, 2008; Mondal and Bhat, 2021). With these advantages, the closed-form MNL-O model could be applied to distinguish the oddball alternative in econometric studies or be incorporated into transportation models that require efficient computation of many choice probabilities, such as the well-established network equilibrium models (Prashker and Bekhor, 2004).

However, the MNL-O model inherits some limitations from the MNL model based on the additive utility function. First, it assumes an identical perception variance for all conventional alternatives, which is inadequate for addressing the scale heterogeneity. Second, although a different perception variance is assumed for the oddball alternative, it is still fixed and independent of its utility, keeping the variance ratio between oddball and conventional alternatives fixed. This could lead to an unrealistic expectation that the common and unique features of the oddball alternative have equal and fixed contributions to the random utility of the oddball alternative with the service level disregarded. Finally, the choice probability expressions of the MNL-O model are based on the absolute differences in utility, which cannot address the distinct magnitudes of utility in the networks. These limitations make it difficult for the MNL-O model to reflect the heterogeneous service levels of various travel modes or the distinct trip lengths associated with different destinations, which could hinder its application to the complex decision-making scenarios of modern transportation networks.

This study proposes an alternate weibit-based model for assessing travel choice with an oddball alternative, addressing the inherent heterogeneity issues in the MNL-O model while retaining the closed-form probability expression. Based on the multiplicative disutility function with Weibull distributed random components, the proposed model allows disutility-dependent variances for all alternatives. Thus, the model can naturally consider various heterogeneity issues, including the heterogeneous perceptions of conventional alternatives, heterogeneous perceptions of unique and common attributes of the oddball alternative, and heterogeneous features of conventional and oddball alternatives. Also, the multiplicative error structure adopted in the proposed model coincides with the psychophysical laws on how individuals perceive different magnitudes of travel disutility, and thus can have better behavioral interpretations than the additive error structure in commonly used logit models (Chakroborty et al., 2021; Nirmale and Pinjari, 2023). Herein, derivations of closed-form choice probabilities of both conventional and oddball alternatives are proposed. The perception variances and elasticities with respect to both conventional and oddball alternatives are also analytically derived, illustrating the appealing theoretical properties of the proposed model.

The remainder of this paper is organized as follows. Section 2 briefly reviews the formulation, properties, and limitations of Recker's logit-based model with an oddball alternative. In Section 3, a weibit-based model with an oddball alternative is then developed, with detailed derivations of closed-form choice probabilities of both conventional and oddball alternatives provided. The theoretical properties and advantages of the proposed model are then thoroughly discussed via comparisons with some existing closed-form travel choice models that also focus on addressing the heterogeneity issues. Section 4 investigates the empirical performance of the proposed model. Finally, Section 5 concludes the paper, discusses potential applications of the proposed model, and provides some directions for future research.

2. Problem statement

To facilitate the presentation of the essential ideas, the notations used are listed in Section 2.1. Recker's (1995) logit choice model with an oddball alternative is then introduced in Section 2.2, together with a discussion on its properties and limitations.

2.1. Notations

Sets

A	Set of all travel alternatives
$A\text{-}r$	Set of conventional travel alternatives without the oddball alternative
I	Set of common attributes shared by all alternatives
J	Set of unique attributes of the oddball alternative
$\boldsymbol{\tau}_{\mathbf{k}}$	Set of attribute levels of alternative *k*

Parameters and variables

γ	Euler's constant
$P(k\|A)$	Choice probability of alternative k in choice set A
$R_{MNL\text{-}O}$	Variance ratio of the MNL-O model
$R_{MNW\text{-}O}$	Variance ratio of the MNW-O model
$E_{\tau_k^i}^{P(k\|A)}$	Direct elasticity of alternative k with respect to attribute i
$E_{\tau_l^i}^{P(k\|A)}$	Cross elasticity of alternative k with respect to attribute i of alternative l
V_k	Total perceived utility/disutility of alternative k
$\overline{V}_r$	Perceived utility/disutility of the common attributes of oddball alternative r
$\tilde{V}_r$	Perceived utility/disutility of the unique attributes of oddball alternative r
v_k	Total system utility/disutility of alternative k
$\overline{v}_r$	System utility/disutility of the common attributes of oddball alternative r
$\tilde{v}_r$	System utility/disutility of the unique attributes of oddball alternative r
ε_k	Random error associated with the common attributes of alternative k
ξ_r	Random error associated with the unique attributes of oddball alternative r
τ_k^i	Level of attribute i of alternative k
$\overline{\tau}_k^i$	Level of common attribute i of alternative k
$\tilde{\tau}_r^j$	Level of unique attribute j of oddball alternative r
ω^i	Coefficient of attribute i
λ	Location parameter of the Weibull distribution
α_k	Scale parameter of the Weibull distribution of alternative k
β	Shape parameter of the Weibull distribution
η_k	Location parameter of the Gumbel distribution of alternative k
θ	Scale parameter of the Gumbel distribution

2.2. Basis for Recker's logit choice model with an oddball alternative

2.2.1. Additive perceived utility of oddball alternative

A homogeneous choice set A is assumed in the conventional MNL model, in which each alternative shares a common set of attributes $\overline{\boldsymbol{\tau}} = \left\{\overline{\tau}^1, \overline{\tau}^2, \ldots, \overline{\tau}^m\right\}$. Let $\boldsymbol{\tau}_\mathbf{k} = \overline{\boldsymbol{\tau}}_\mathbf{k} = \left\{\overline{\tau}_k^1, \overline{\tau}_k^2, \ldots, \overline{\tau}_k^m\right\}$ denote the level of attributes of the k^{th} alternative. Then, a linear-in-parameters specification is assumed for the system utility of k: $v_k = \overline{v}_k = \sum_{i \in I} \omega^i \overline{\tau}_k^i$. The additive utility function is adopted in the MNL model:

$$V_k = v_k + \varepsilon_k, \forall k \in A, \tag{1}$$

where ε_k is the random error term. The random error is assumed to be independently and identically distributed (IID) Gumbel variable with the same scale parameter θ, i.e., $\varepsilon_k \sim G(\eta_k, \theta)$. Letting $\theta = 1$, the MNL choice probability can be expressed as

$$P_{MNL}(k|A) = \frac{e^{v_k+\eta_k}}{\sum_{l \in A} e^{v_l+\eta_l}}, \forall k \in A. \quad (2)$$

However, not all alternatives necessarily share the same set of attributes. Recker (1995) further considered a single oddball alternative, which additionally possessed a unique set of attributes $\tilde{\boldsymbol{\tau}} = \{\tilde{\tau}^{m+1}, \tilde{\tau}^{m+2}, \ldots, \tilde{\tau}^{n}\}$ in addition to the common attributes $\bar{\boldsymbol{\tau}}$. Suppose that the r^{th} alternative is the oddball in the choice set. Then, let $\bar{v}_r = \sum_{i \in I} \omega^i \bar{\tau}_r^i$ and ε_r denote the system utility and random error, respectively, that are associated with the common attributes of the oddball alternative, and let $\tilde{v}_r = \sum_{j \in J} \omega^j \tilde{\tau}_r^j$ and ξ_r denote the system utility and random error, respectively, that are associated with its unique attributes. Thus, the additive utility function of the oddball alternative is

$$V_r = \bar{V}_r + \tilde{V}_r = (\bar{v}_r + \varepsilon_r) + (\tilde{v}_r + \xi_r). \quad (3)$$

By assuming ε_r and ξ_r are IID Gumbel variables, i.e., $\varepsilon_r \sim G(\eta_{r1}, \theta)$ and $\xi_r \sim G(\eta_{r2}, \theta)$ (Recker, 1995), the oddball alternative differs from conventional alternatives in that its random error term is a summation of two random IID Gumbel variables, ε_r and ξ_r.

2.2.2. Formulation of logit choice model with an oddball alternative

Based on the additive utility function and assumptions given in Section 2.2.1, Recker (1995) developed the MNL-O model, which considers the oddball alternative to have unique features while retaining a closed-form choice probability expression. Letting $v_r = \bar{v}_r + \tilde{v}_r$ denote the total system utility and setting $\theta = 1$, the choice probabilities of the oddball and conventional alternatives can be derived based on the principle of utility maximization and the IID assumption as follows (Recker, 1995):

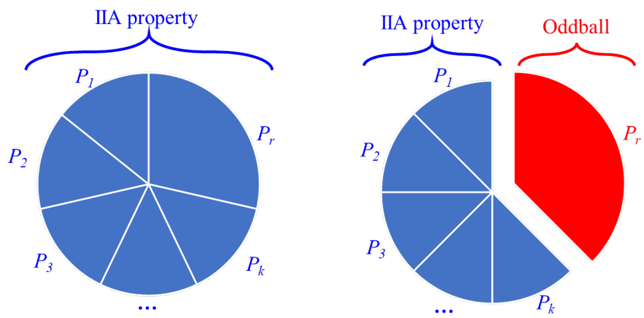
$$\begin{aligned} P_{MNL-O}(k|A) &= \frac{e^{v_k+\eta_k}}{\sum_{l \neq r \in A} e^{v_l+\eta_l}} \cdot \left[1 - \phi_r^L \cdot e^{\phi_r^L} \cdot E_1(\phi_r^L)\right] \\ &= P_{MNL}(k|A-r) \cdot \left[1 - \phi_r^L \cdot e^{\phi_r^L} \cdot E_1(\phi_r^L)\right], \forall k \neq r \in A \end{aligned}, \quad (4)$$

$$P_{MNL-O}(r|A) = \phi_r^L \cdot e^{\phi_r^L} \cdot E_1(\phi_r^L), \quad (5)$$

where $\phi_r^L = \frac{e^{v_r+\eta_{r1}+\eta_{r2}}}{\sum_{l\neq r\in A} e^{v_l+\eta_l}}$. $E_1(x)=\int_x^{+\infty}\frac{e^{-x}}{x}dx$ is the exponential integral, the value of which has been tabulated previously (Harris, 1957).

2.2.3. Properties and limitations of the logit-based approach

The MNL-O model is demonstrated in Figure 1 via a comparison with the basic MNL model. The MNL-O model tends to estimate a higher choice probability of the oddball alternative, while the MNL model always results in a higher choice probability for a conventional alternative (Recker, 1995). In addition, Eqs. (4) and (5) clearly show that the independence from irrelevant alternatives (IIA) property no longer holds for the oddball alternative in the MNL-O model. Instead, the IIA property only remains within the subset of conventional alternatives in the MNL-O model.

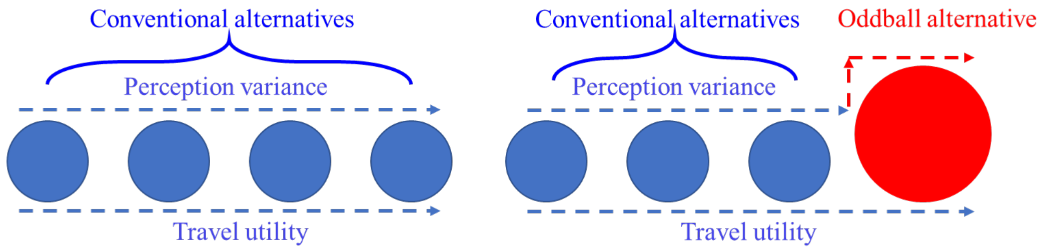


(a) MNL model (b) MNL-O model

Figure 1. Oddball choice probability derived from different models.

Figure 2 illustrates the different ways to consider an oddball alternative in the MNL and MNL-O models. As shown in Figure 2(a), the MNL model treats the oddball alternative in the same way as conventional alternatives. Because of the IID assumption and the properties of the Gumbel distribution, the MNL model assumes a fixed perception variance of $\pi^2/6\theta^2$ for all alternatives given the scale parameter θ. By comparison, the MNL-O model associates the oddball alternative with an additional random error term, which implies a higher uncertainty owing to the unique attributes that distinguishes the oddball alternative from conventional alternatives (Figure 2(b)). Because the random errors of common and unique attributes are assumed to be IID Gumbel variables (Recker, 1995), the variance of the oddball alternative can be derived as

$$D_{MNL-O}(V_r)=D(\bar{V}_r+\tilde{V}_r)=D(\bar{V}_r)+D(\tilde{V}_r)=\pi^2/6\theta^2+\pi^2/6\theta^2=\pi^2/3\theta^2 . \quad (6)$$



(a) MNL model (b) MNL-O model

Figure 2. Choice sets considered in different logit-based models.

Remark. In the MNL-O model, the assumption of identical variance remains within the subset of conventional alternatives. Although a different variance is assumed for the oddball alternative, it is still fixed and independent of the common and unique features. This leads to a fixed variance ratio between oddball and conventional alternatives:

$$R_{MNL-O} = \frac{\pi^2/3\theta^2}{\pi^2/6\theta^2} = 2 \tag{7}$$

In summary, despite its ability to specifically handle the oddball alternative, the MNL-O model still inherits some limitations from the MNL model. First, the identical and fixed perception variance among conventional alternatives is inadequate for modeling the heterogeneity issue among alternatives with distinct scales of utility. Second, the fixed variance ratio may not fully capture the heterogeneous contributions of common and unique features to the choice of an oddball alternative. Third, the issues discussed above result in a choice probability function based on absolute utility differences, which might generate unrealistic travel choice probabilities when applied to transportation networks with distinct trips lengths or service levels (Kitthamkesorn and Chen, 2013).

3. Weibit-based model for assessing travel choice with an oddball alternative

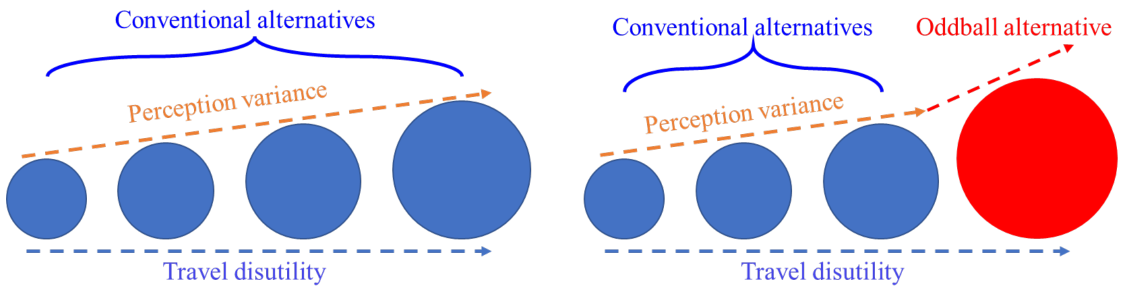
This study proposes a closed-form weibit-based model to explicitly address the inherent heterogeneity issues in the MNL-O model. In contrast to the additive utility assumed in the logit-based models, the weibit-based model adopts a multiplicative form of disutility function (Fosgerau and Bierlaire, 2009). Taking the basic multinomial weibit (MNW) model as an example, the random perceived disutility can be expressed by the multiplication of the deterministic system disutility and the random error:

$$V_k = v_k \cdot \varepsilon_k, \forall k \in A, \tag{8}$$

where v_k and ε_k denote the system disutility and random error term, respectively. The MNW model can be derived by assuming IID Weibull random errors with the same parameter β, i.e., $\varepsilon_k \sim W(\lambda, \alpha_k, \beta)$. λ, α_k, and β are location, scale, and shape parameters of the Weibull distribution. For simplicity, the location parameter is set to zero hereafter (Kitthamkesorn and Chen 2013, 2017). The MNW choice probability can then be expressed as (Castillo et al., 2008)

$$P_{MNW}\left(k\left|A\right.\right)=\frac{\left(v_k\cdot\alpha_k\right)^{-\beta}}{\sum_{l\in A}\left(v_l\cdot\alpha_l\right)^{-\beta}},\forall k\in A. \tag{9}$$

Thus, the properties of the Weibull distribution enable the MNW model to inherently address the heterogeneity issue by allowing disutility-dependent perception variances (Castillo et al., 2008). As shown in Figure 3(a), the perception variance of each alternative is proportional to its mean disutility, which leads to a relative difference-based choice probability function (Eq. (9)). However, the homogeneous proportionality between variance and mean disutility is unsuitable for distinguishing the oddball alternative from conventional alternatives. To address this issue, a multinomial weibit model with an oddball alternative (MNW-O) is developed in Section 3.1 to account for the oddball alternative's unique features and distinct proportionality between variance and mean disutility (as shown in Figure 3(b)).



(a) MNW model (b) MNW-O model

Figure 3. Choice set considered in different weibit-based models.

3.1. Model formulation

Consistent with the MNW model, the proposed MNW-O model is also based on a multiplicative specification of disutility perception. The conventional alternatives share the same disutility function for the MNW model, as given in Eq. (8), while the disutility function of the oddball alternative additionally includes the perceived disutility of its unique features, which is expressed as

$$V_r=\overline{V}_r\cdot\tilde{V}_r=\left(\overline{v}_r\cdot\varepsilon_r\right)\cdot\left(\tilde{v}_r\cdot\xi_r\right), \tag{10}$$

where $\overline{v}_r$ and ε_r denote the system disutility and random error, respectively, that are associated with the common attributes of the oddball alternative, and $\tilde{v}_r$ and ξ_r denote the system disutility and random error, respectively, that are associated with its unique attributes. In contrast to the additive oddball utility function in the MNL-O model (Eq. (3)), the perceived disutility associated with unique features is multiplied with the perceived disutility of common features. Following the principle of disutility minimization, the MNW-O choice probability of a conventional or oddball alternative is equivalent to the probability that the chosen alternative has lower disutility than all other alternatives, which can be expressed as

$$\begin{aligned} P_{MNW-O}\left(k\middle|A\right) &= P\left(v_k \cdot \varepsilon_k \le v_l \cdot \varepsilon_l, \forall l \ne k, r \in A; v_k \cdot \varepsilon_k \le v_r \cdot \varepsilon_r \cdot \xi_r\right) \\ &= P\left(\varepsilon_l \ge \frac{v_k \cdot \varepsilon_k}{v_l}, \forall l \ne k, r \in A; \varepsilon_r \ge \frac{v_k \cdot \varepsilon_k}{v_r \cdot \xi_r}\right) \end{aligned} , \tag{11}$$

$$\begin{aligned} P_{MNW-O}\left(r\middle|A\right) &= P\left(v_r \cdot \varepsilon_r \cdot \xi_r \le v_k \cdot \varepsilon_k, \forall k \ne r \in A\right) \\ &= P\left(\varepsilon_k \ge \frac{v_r \cdot \varepsilon_r \cdot \xi_r}{v_k}, \forall k \ne r \in A\right) \end{aligned} , \tag{12}$$

where $v_r = \overline{v}_r \cdot \tilde{v}_r$ denotes the total system disutility of the oddball alternative *r*. The random errors, including ε_k, ε_r, and ξ_r, are assumed to be IID Weibull variables with the same shape parameter, β : $\varepsilon_k \sim W\left(0, \alpha_k, \beta\right)$, $\varepsilon_r \sim W\left(0, \alpha_{r1}, \beta\right)$, and $\xi_r \sim W\left(0, \alpha_{r2}, \beta\right)$. Thus, by defining $\phi_r^W = \frac{\left(v_r \cdot \alpha_{r1} \cdot \alpha_{r2}\right)^{-\beta}}{\sum_{l \ne r \in A} \left(v_l \cdot \alpha_l\right)^{-\beta}}$, the following two propositions are reached.

Proposition 1. The choice probability of conventional alternative *k* from the MNW-O model is:

$$\begin{aligned} P_{MNW-O}\left(k\middle|A\right) &= \frac{\left(v_k \cdot \alpha_k\right)^{-\beta}}{\sum_{l \ne r \in A} \left(v_l \cdot \alpha_l\right)^{-\beta}} \cdot \left[1 - \phi_r^W \cdot e^{\phi_r^W} \cdot E_1\left(\phi_r^W\right)\right] \\ &= P_{MNW}\left(k\middle|A - r\right) \cdot \left[1 - \phi_r^W \cdot e^{\phi_r^W} \cdot E_1\left(\phi_r^W\right)\right], \forall k \ne r \in A \end{aligned} . \tag{13}$$

Proof. See Appendix A for the detailed proof.

Proposition 2. The choice probability of oddball alternative *r* from the MNW-O model is

$$P_{MNW-O}\left(r\middle|A\right) = \phi_r^W \cdot e^{\phi_r^W} \cdot E_1\left(\phi_r^W\right). \tag{14}$$

Proof. See Appendix B for the detailed proof.

Thus, Eqs. (13) and (14) demonstrate that the MNW-O choice probabilities have a similar form to probabilities of the MNL-O model given in Eqs. (4) and (5), which enables the proposed MNW-O model to address the oddball effect. The difference lies in the expression of term ϕ_r. The ϕ_r^L in the MNL-O model is dependent on an exponential-based utility function, which implies an absolute difference–based choice probability expression. By comparison, the MNW-O model adopts a power function–

based ϕ_r^W, implying that the MNW-O choice probability is dependent on the relative differences of travel disutility. This difference suggests that the MNW-O model inherits its ability to consider the heterogeneity issue from the MNW model (Kitthamkesorn and Chen, 2013). This property is discussed in detail in Section 3.2.

3.2. Model properties

This section discusses the properties of the proposed MNW-O model. First, the properties inherited from the MNL-O model are discussed, i.e., the logical consistency conditions and asymptotic values, in Section 3.2.1. Then, the perception variance of the conventional and oddball alternatives in the proposed MNW-O model are provided in Section 3.2.2. Finally, the theoretical advantages the proposed model gains from the properties of weibit-based models are discussed in Sections 3.2.3 to 3.2.5, including its more flexible perception variances, its ability to consider the heterogeneity issue, and its disutility-based model elasticities.

3.2.1. Logical consistency conditions and asymptotic values

This section shows the proposed MNW-O model has similar logical consistency and asymptotic properties to the MNL-O model. The exponential integral incorporated in the choice probability expression is known to have the following limits:

$$\lim_{x\to+\infty} E_1(x) = \frac{e^{-x}}{x}, \tag{15}$$

$$\lim_{x\to 0} E_1(x) = -\gamma - \ln x, \tag{16}$$

where γ is Euler's constant. Clearly, the MNW-O model can satisfy the logical consistency conditions for discrete choice models, i.e., for all alternative *k*, $0 \le P_{MNW-O}(k|A) \le 1$ and $\sum_{k\in A} P_{MNW-O}(k|A) = 1$. Furthermore, the MNW-O choice probabilities of the oddball alternative are derived at asymptotic values in two important cases.

Case 1. Disutility of oddball alternative approaches zero: $v_r \to 0^+$

In this case, the oddball alternative is extremely superior with negligible travel disutility. Because $\lim_{v_r\to 0^+} \phi_r^w \to +\infty$, the choice probability of the oddball alternative is

$$\lim_{v_r\to 0^+} P_{MNW-O}(r|A) = \lim_{\phi_r^w\to+\infty} \phi_r^w \cdot e^{\phi_r^w} \cdot \frac{e^{-\phi_r^w}}{\phi_r^w} = 1, \tag{17}$$

and the choice probabilities of the conventional alternatives are

$$\lim_{v_r \to 0^+} P(k|A) = \lim_{\phi_r^w \to +\infty} \frac{(v_k \cdot \alpha_k)^{-\beta}}{\sum_{l \neq r \in A} (v_l \cdot \alpha_l)^{-\beta}} \cdot \left(1 - \phi_r^w \cdot e^{\phi_r^w} \cdot \frac{e^{-\phi_r^w}}{\phi_r^w}\right) = 0 . \tag{18}$$

In this case, the choice probability of the oddball alternative approaches one, which is consistent with the conventional discrete choice models when there is an alternative dominating the choice set with extremely high utility.

Case 2. Disutility of oddball alternative approaches infinity: $v_r \to +\infty$

When the oddball alternative is extremely inferior with high disutility, i.e., $v_r \to +\infty$, then $\lim_{v_r +\infty} \phi_r^w \to 0^+$ and the choice probability for the oddball alternative becomes

$$\lim_{v_r \to +\infty} P_{MNW-O}(r|A) = \lim_{\phi_r^w \to 0^+} \phi_r^w \cdot e^{\phi_r^w} \cdot \left(-\gamma - \ln \phi_r^w\right) = 0 , \tag{19}$$

and the choice probabilities of the conventional alternatives are

$$\begin{aligned} \lim_{v_r \to +\infty} P_{MNW-O}(k|A) &= \lim_{\phi_r^w \to 0^+} \frac{(v_k \cdot \alpha_k)^{-\beta}}{\sum_{l \neq r \in A} (v_l \cdot \alpha_l)^{-\beta}} \cdot \left[1 - \phi_r^w \cdot e^{\phi_r^w} \cdot \left(-\gamma - \ln \phi_r^w\right)\right] \\ &= \frac{(v_k \cdot \alpha_k)^{-\beta}}{\sum_{l \neq r \in A} (v_l \cdot \alpha_l)^{-\beta}} \end{aligned} \tag{20}$$

In this case, because the IIA property is held within the set of conventional alternatives, the MNW-O model degenerates to the MNW model when the oddball alternative has extremely high disutility and negligible choice probability.

3.2.2. Perception variances

This section provides the perception variance of conventional and oddball alternatives in the proposed MNW-O model. The conventional alternatives in the MNW-O model share the same form of variance as in the MNW model. With the location parameter $\lambda = 0$, the variance of conventional alternative k can be expressed as

$$D(V_k) = E^2(V_k) \cdot \left[\kappa(\beta) - 1\right], \tag{21}$$

where $E(V_k)$ denotes the mean disutility and $\kappa(\beta) = \frac{\Gamma(1+2/\beta)}{\Gamma^2(1+1/\beta)}$.

Remark. The variance of the conventional alternative is proportional to the mean disutility. The proportionality is $\kappa(\beta) - 1$, which is dependent on the shape parameter β and remains constant for all the conventional alternatives.

Proposition 3. The perception variance of the oddball alternative in the MNW-O model is proportional to the mean disutility of both its common and unique features. The proportionality is dependent on the shape parameter β and differs from the proportionality of the conventional alternatives.

Proof. The perception variance of the oddball alternative can be expressed as

$$\begin{aligned} D_{MNW-O}\left(V_r\right) &= D\left(\bar{V}_r \cdot \tilde{V}_r\right) \\ &= E\left\{\left[\left(\bar{V}_r \cdot \tilde{V}_r\right) - E\left(\bar{V}_r \cdot \tilde{V}_r\right)\right]^2\right\} \end{aligned}. \tag{22}$$

With the independence assumption, Eq. (22) becomes

$$\begin{aligned} D_{MNW-O}\left(V_r\right) &= E\left(\bar{V}_r^2 \cdot \tilde{V}_r^2\right) - E^2\left(\bar{V}_r \cdot \tilde{V}_r\right) \\ &= E\left(\bar{V}_r^2\right) \cdot E\left(\tilde{V}_r^2\right) - E^2\left(\bar{V}_r\right) \cdot E^2\left(\tilde{V}_r\right) \\ &= \left[D\left(\bar{V}_r\right) + E^2\left(\bar{V}_r\right)\right] \cdot \left[D\left(\tilde{V}_r\right) + E^2\left(\tilde{V}_r\right)\right] - E^2\left(\bar{V}_r\right) \cdot E^2\left(\tilde{V}_r\right) \end{aligned}. \tag{23}$$

Substituting Eq. (21) into Eq. (23),

$$\begin{aligned} D_{MNW-O}\left(V_r\right) &= \left[E^2\left(\bar{V}_r\right) \cdot \kappa(\beta)\right] \cdot \left[E^2\left(\tilde{V}_r\right) \cdot \kappa(\beta)\right] - E^2\left(\bar{V}_r\right) \cdot E^2\left(\tilde{V}_r\right) \\ &= E^2\left(\bar{V}_r\right) \cdot E^2\left(\tilde{V}_r\right) \cdot \left[\kappa^2(\beta) - 1\right] \\ &= E^2\left(V_r\right) \cdot \left[\kappa^2(\beta) - 1\right] \end{aligned}. \tag{24}$$

Thus, the variance of the oddball alternative is a function of both the shape parameter and the mean disutility, and the proportionalities of the oddball and conventional alternatives are different, i.e., $\kappa^2(\beta) - 1 \neq \kappa(\beta) - 1$.

Proposition 4. The variance ratio between the oddball and conventional alternatives in the MNW-O model is flexible. The variance ratio is no lower than two when the oddball and conventional alternatives have the same mean disutility.

Proof. The variance ratio between the oddball and conventional alternatives in the MNW-O model is

$$R_{MNW-O} = \frac{E^2\left(V_r\right) \cdot \left[\kappa^2(\beta) - 1\right]}{E^2\left(V_k\right) \cdot \left[\kappa(\beta) - 1\right]}. \tag{25}$$

$R_{MNW\text{-}O}$ is therefore flexible, which is dependent on both the mean disutility of oddball and conventional alternatives and the shape parameter, β.

Then, the variance ratio is derived with the same mean disutility. By definition, β is positive, so $\Gamma(1+2/\beta) > 0$ and $\Gamma(1+1/\beta) > 0$. Therefore,

$$\frac{d\kappa(\beta)}{d\beta} = \frac{2}{\beta^2} \cdot \frac{\Gamma(1+2/\beta)}{\Gamma^2(1+1/\beta)} \cdot \left[\psi(1+1/\beta) - \psi(1+2/\beta)\right] < 0, \tag{26}$$

where $\psi(x)=\dfrac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. Thus, when $\beta>0$, $\kappa(\beta)$ is a decreasing function with respect to β. The asymptotic value of $\kappa(\beta)$ is

$$\lim_{\beta\to+\infty}\kappa(\beta)=\lim_{\beta\to+\infty}\frac{\Gamma(1+2/\beta)}{\Gamma^2(1+1/\beta)}=1. \tag{27}$$

Therefore, $\kappa(\beta)\geq 1$ within the domain of β. The variance ratio between the oddball and conventional alternatives with the same mean disutility is

$$R_{MNW-O}=\frac{\kappa^2(\beta)-1}{\kappa(\beta)-1}=\kappa(\beta)+1, \tag{28}$$

which is no lower than two, completing the proof.

The higher variance ratio between the oddball and conventional alternatives indicates that the oddball alternative is considered to have a higher uncertainty than conventional alternatives. Proposition 4 reveals that the proposed MNW-O model provides a disutility-dependent variance ratio, which highlights the heterogeneous perceptions of the common and unique features and is more realistic than the fixed variance ratio provided by the MNL-O model. When the oddball and conventional alternatives have the same disutility, the variance ratio in the MNW-O model decreases with the increase of β. This property is also intuitive, as a larger value of β can be interpreted as better knowledge of the transportation network, which reduces uncertainty.

3.2.3. Demonstration of the heterogeneous perception variances

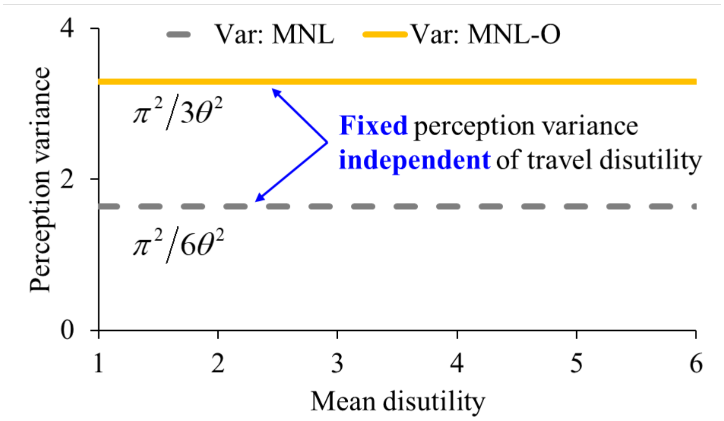
This section demonstrates the disutility-dependent perception variances and non-identical variance ratio provided in Propositions 3 and 4. The effects these properties have on handling the heterogeneity issue in choice modeling is then illustrated for networks with different scales of travel disutility.

3.2.3.1.Demonstration of perception variances with respect to alternative disutility

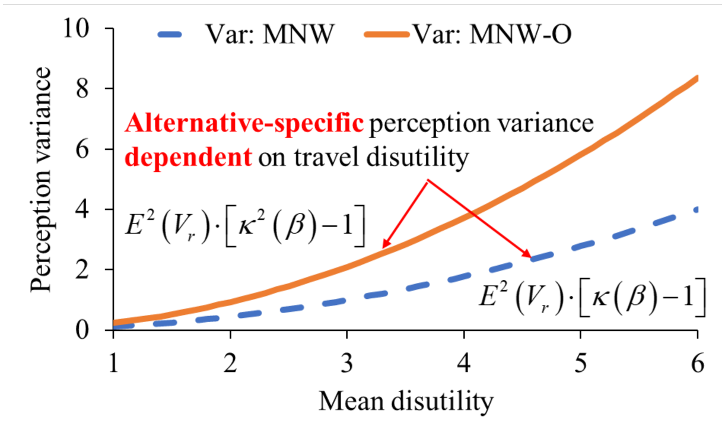
Figure 4 shows an example of the perception variance of an oddball alternative with respect to travel disutility in the MNL, MNL-O, MNW and MNW-O models. The following properties can be observed from Figure 4:

1. Given the scale parameter θ, constant and utility-independent perception variances are used in the logit-based models, which indicates its inability to reflect the heterogeneous perceptions of scales of utility for different alternatives or attributes.
2. By contrast, the weibit-based models have disutility-dependent perception variances. Specifically, in the MNW-O model, the oddball alternative has a different variance to the conventional alternatives, implying that the MNW-O model can

consider the heterogeneities with respect to these alternatives in different ways.



(a) Logit-based models with $\theta = 1$



(b) Weibit-based models with $\beta = 3.7$

Figure 4. Perception variance of an oddball alternative with respect to its mean disutility in different models.

3.2.3.2.Demonstration of perception variances with respect to the shape parameter

Figure 5 further explores the perception variance with respect to the shape parameter in weibit-based models and the variance ratio between oddball and conventional alternatives in the MNW-O model. Two conclusions can be drawn from Figure 5:

1. With the same disutility, the perception variance of an oddball alternative is always larger in the MNW-O model that in the MNW model. This indicates the MNW-O model considers the oddball alternative to have higher uncertainty than conventional alternatives, which is consistent with the feature of the MNL-O model.
2. An increasing shape parameter, β, causes the perception variances of the oddball alternatives in both models and the variance ratio in the MNW-O model to all decrease. When $\beta \to +\infty$, the variance ratio between the oddball and conventional alternatives approaches two, which is the variance ratio in the MNL-O model. The variances of both the MNW and MNW-O models approach zero with $\beta \to +\infty$, implying that when travelers have perfect knowledge of all travel alternatives then they deterministically choose the alternative with the lowest disutility.

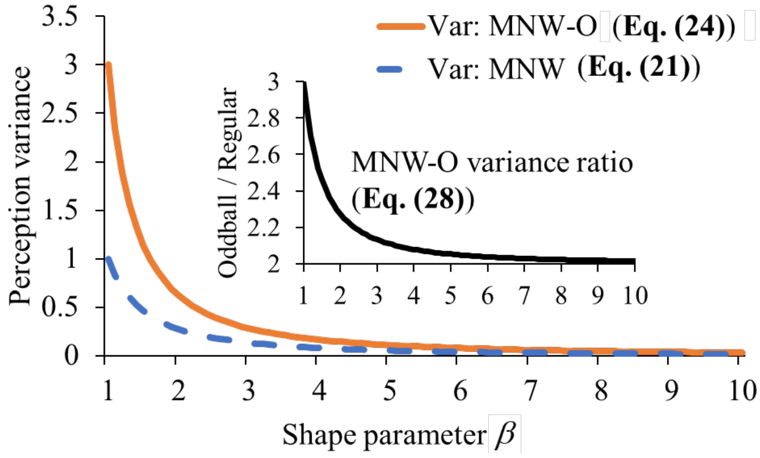


Figure 5. Perception variance of the oddball alternative with respect to the shape parameter in different models.

3.2.3.3. Effect of considering the heterogeneity issue

Based on the non-identical perception variances and flexible variance ratio given in Propositions 3 and 4, the MNW-O model inherits the ability to consider the heterogeneity issue from the MNW model and could outperform the MNL-O model in networks with distinct scales of travel disutility. Consider a trinomial choice problem with two conventional alternatives that have the same disutility v_k, and one oddball alternative with disutility v_r. Then, assume a constant disutility difference between the conventional and oddball alternatives: $v_r = v_k - 5$. Figure 6 shows the evolution of the oddball choice probabilities derived from different models with v_r varying from zero to 100, leading to the following observations:

1. The MNL-O model provides a constant choice probability for varying scales of disutility, which can be attributed to its absolute difference-based choice probability function. This result seems unrealistic in this case because the constant difference between the oddball and conventional alternatives should become increasingly negligible when the scale of disutility becomes larger.
2. The weibit models can indicate the effect of variation in the disutility scale via their variation in choice probability. This property makes the MNW-O model provide more realistic outcomes than the MNL-O model, which is consistent with the difference identified between MNW and MNL models in short and long networks (Kitthamkesorn and Chen, 2013).
3. Given the same system disutility, the MNW-O model tends to calculate a higher oddball choice probability than the MNW model, which is consistent with the difference between MNL-O and MNL models (Recker, 1995).

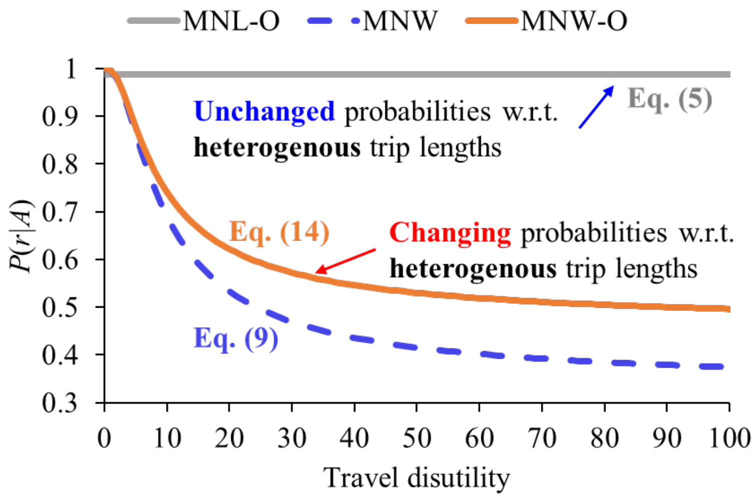


Figure 6. Effect of heterogeneity on the oddball choice probability.

3.2.4. Model elasticities

This section compares the elasticities of the proposed model with related models, including the MNL, MNL-O, and MNW models. The direct elasticity of $P(k|A)$ with respect to attribute i of alternative k can be expressed as

$$E_{\tau_k^i}^{P(k|A)} = \frac{\partial P(k|A)}{\partial \tau_k^i} \cdot \frac{\tau_k^i}{P(k|A)}, \forall \tau_k^i \in \boldsymbol{\tau}_{\mathbf{k}}, k \in A. \tag{29}$$

The cross elasticity of alternative k with respect to attribute i of alternative l is

$$E_{\tau_l^i}^{P(k|A)} \frac{\partial P(l|A)}{\partial \tau_l^i} \cdot \frac{\tau_l^i}{P(l|A)}, \forall \tau_k^i \in \boldsymbol{\tau}_{\mathbf{k}}, k, l \in A. \tag{30}$$

With $\frac{dE_1(x)}{dx} = -\frac{e^{-x}}{x}$, the direct and cross elasticities are derived for the MNL-O and MNW-O models that involve the exponential integral in their choice probability functions. Because the MNL-O and MNW-O models provide different probability expressions for the conventional and oddball alternatives, the direct and cross elasticities with respect to conventional and oddball alternatives are derived separately.

Table 1 compares the direct and cross elasticities of the MNL, MNL-O, MNW, and MNW-O models, highlighting the following properties:

1. Analogous to the MNW model, the elasticities of the MNW-O model are also dependent on the alternative disutility (Gu et al., 2022). This property makes an alternative with higher disutility less sensitive to an equivalent perturbation than alternatives with lower disutility, which indicates the ability of the MNW-O model to account for the heterogeneity issues, as discussed in Section 3.2.3. Moreover, the elasticities of both the MNW and MNW-O models share the opposite signs to their

logit-based counterparts. This is because the term v denotes utility in logit-based models, whereas it denotes disutility in weibit-based models.

2. Both the direct and cross elasticities of the MNW-O model involve terms related to the oddball alternative r. This indicates that the effects of an oddball alternative on the choice probabilities are specifically considered in the MNW-O model. By contrast, the MNL and MNW models do not make this distinction and only account for conventional alternatives.
3. In the MNL and MNW models, the cross elasticities for all alternatives k with respect to a change in alternative l remain constant, implying that the IIA property holds for all alternatives in the choice set. By comparison, the MNL-O and MNW-O models possess different cross elasticities when the oddball alternative r is involved, i.e., $E_{\tau_l^i}^{P(r|A)}$ differs from $E_{\tau_l^i}^{P(k|A)}$ and is dependent on a term related to oddball alternative r. This indicates that the IIA property is circumvented between the oddball and conventional alternatives.

Table 1. Comparison of direct and cross elasticities

Model	Direct elasticity	Cross elasticity
MNL	$\theta \cdot \omega^i \tau_k^i \cdot \left[1 - P_{MNL}\left(k \mid A\right)\right]$	$-\theta \cdot \omega^i \tau_l^i \cdot P_{MNL}\left(l \mid A\right)$
MNW	$-\left(v_k\right)^{-1} \cdot \beta \omega^i \tau_k^i \cdot \left[1 - P_{MNW}\left(k \mid A\right)\right]$	$\left(v_l\right)^{-1} \cdot \beta \cdot \omega^i \tau_l^i \cdot P_{MNW}\left(l \mid A\right)$
MNL-O	Direct elasticity of conventional alternative k, $E_{\tau_k^i}^{P(k\|A)}$: $\theta \cdot \omega^i \tau_k^i \cdot \left\{1 - P_{MNL}\left(k \mid A - r\right) \cdot \left[\left(1 + \phi_r^L\right) - \frac{P_{MNL-O}\left(r \mid A\right)}{1 - P_{MNL-O}\left(r \mid A\right)}\right]\right\}$	Cross elasticity of conventional alternative k with respect to attribute i of conventional alternative l, $E_{\tau_l^i}^{P(k\|A)}$: $-\theta \cdot \omega^i \tau_l^i \cdot P_{MNL}\left(l \mid A - r\right) \cdot \left[\left(1 + \phi_r^L\right) - \frac{P_{MNL-O}\left(r \mid A\right)}{1 - P_{MNL-O}\left(r \mid A\right)}\right]$
	Direct elasticity of oddball alternative r, $E_{\tau_r^i}^{P(r\|A)}$: $\theta \cdot \omega^i \tau_r^i \cdot \left[\left(1 + \phi_r^L\right) - \frac{\phi_r^L}{P_{MNL-O}\left(r \mid A\right)}\right]$	Cross elasticity of conventional alternative k with respect to attribute i of oddball alternative r, $E_{\tau_r^i}^{P(k\|A)}$: $\theta \cdot \omega^i \tau_r^i \cdot \left[\phi_r^L - \frac{P_{MNL-O}\left(r \mid A\right)}{1 - P_{MNL-O}\left(r \mid A\right)}\right]$
		Cross elasticity of oddball alternative r with respect to attribute i of conventional alternative l, $E_{\tau_l^i}^{P(r\|A)}$: $-\theta \cdot \omega^i \tau_l^i \cdot P_{MNL}\left(l \mid A - r\right) \cdot \left[\left(1 + \phi_r^L\right) - \frac{\phi_r^L}{P_{MNL-O}\left(r \mid A\right)}\right]$

MNW-O	Direct elasticity of conventional alternative k, $E_{\tau_k^i}^{P(k\|A)}$: $$-(v_k)^{-1}\cdot\beta\cdot\omega^i\tau_k^i\cdot\left\{1-P_{MNW}\left(k\|A-r\right)\cdot\left[\left(1+\phi_r^W\right)-\frac{P_{MNW-O}\left(r\|A\right)}{1-P_{MNW-O}\left(r\|A\right)}\right]\right\}$$ Direct elasticity of oddball alternative r, $E_{\tau_r^i}^{P(r\|A)}$: $$-(v_r)^{-1}\cdot\beta\cdot\omega^i\tau_r^i\cdot\left[\left(1+\phi_r^W\right)-\frac{\phi_r^W}{P_{MNW-O}\left(r\|A\right)}\right]$$	Cross elasticity of conventional alternative k with respect to attribute i of conventional alternative l, $E_{\tau_l^i}^{P(k\|A)}$: $$(v_l)^{-1}\cdot\beta\cdot\omega^i\tau_l^i\cdot P_{MNW}\left(l\|A-r\right)\cdot\left[\left(1+\phi_r^W\right)-\frac{P_{MNW-O}\left(r\|A\right)}{1-P_{MNW-O}\left(r\|A\right)}\right]$$ Cross elasticity of conventional alternative k with respect to attribute i of oddball alternative r, $E_{\tau_r^i}^{P(k\|A)}$: $$-(v_r)^{-1}\cdot\beta\cdot\omega^i\tau_r^i\cdot\left[\phi_r^W-\frac{P_{MNW-O}\left(r\|A\right)}{1-P_{MNW-O}\left(r\|A\right)}\right]$$ Cross elasticity of oddball alternative r with respect to attribute i of conventional alternative l, $E_{\tau_l^i}^{P(r\|A)}$: $$(v_l)^{-1}\cdot\beta\cdot\omega^i\tau_l^i\cdot P_{MNW}\left(l\|A-r\right)\cdot\left[\left(1+\phi_r^W\right)-\frac{\phi_r^W}{P_{MNW-O}\left(r\|A\right)}\right]$$

4. Empirical application

The proposed MNW-O model is applied to the Swissmetro data set (Bierlaire et al., 2001) that has been widely used in transportation studies (e.g., Fosgerau and Bierlaire, 2009; Li, 2011; Mabit, 2017; Sifringer et al., 2020; Han et al., 2022). The data set described mode choice scenarios among three alternative modes: train, car, and Swissmetro. We have selected the respondents who have access to all three mode alternatives in the choice set, where the Swissmetro can be considered as an oddball alternative with unique attributes (i.e., headway and seat availability). The number of observations is 3987. A brief description of the used data set can be found in Table 2.

Table 2. Description of the used data set

Alternatives	Attributes
Train	Travel cost, Travel time
Car	Travel cost, Travel time
Swissmetro (oddball)	Travel cost, Travel time, Headway, Seats availability

The alternative utility can be formulated as:

$$v_k = \bar{v}_r = \omega^c \cdot travel\ cost + \omega^t \cdot travel\ time\,, \tag{31}$$

$$\tilde{v}_r = \omega^h \cdot headway + \omega^s \cdot seats\ availability\,, \tag{32}$$

where ω are the parameters to be estimated.

4.1. Estimation results

We compare the estimation results of the MNW-O model against MNL and other advanced choice models, including the nested logit (NL) model that relaxes the independently distributed assumption, the MNL-O model for capturing one oddball alternative in the choice set, the heteroscedastic extreme value model (HEV; Bhat, 1995) that is flexible to relax the identically distributed assumption for all alternatives, and the MNW model that inherently allows disutility-dependent perception variances for all individuals. All the models are estimated using Apollo (Hess and Palma, 2019)[1]. Following Fosgerau and Bierlaire (2009), the coefficient of travel cost is normalized to minus unity and the scale/shape parameters of the logit/weibit-based models are estimated. Two socio-demographic attributes, namely gender (male or female) and age

[1] To estimate the HEV and MNW-O models, we extend the input code in Apollo to represent their choice probabilities by integrating external functions in the R library, i.e., the Gaussian quadrature function for the HEV model (following the estimation procedure suggested by Bhat, 1995) and the exponential integral function for the MNW-O model.

(over 54 or not), were considered in the model estimation. The alternative specific constants were not significantly estimated, therefore excluded. The estimation results are reported in Table 3.

Table 3. Estimation results

	MNL	NL	HEV	MNL-O	MNW	MNW-O
Attributes	Estimates (t-value)	Estimates (t-value)	Estimates (t-value)	Estimates (t-value)	Estimates (t-value)	Estimates (t-value)
Travel time	-1.582 (-16.91)	-1.026 (-13.29)	-1.681 (-10.88)	-1.298 (-14.02)	-1.655 (-14.00)	-1.793 (-9.59)
Headway	-0.028 (-8.02)	-0.019 (-4.93)	-0.035 (-5.95)	-0.054 (-10.56)	-0.024 (-7.32)	-0.036 (-15.73)
Seats availability	0.392 (2.24)	0.186 (-2.12)	0.350 (3.20)	0.599 (2.53)	0.329 (2.72)	0.755 (18.75)
Male	0.147 (1.62)	0.123 (2.93)	0.334 (3.09)	0.460 (4.94)	0.430 (7.86)	0.581 (10.85)
Age_Old	-0.197 (-3.09)	-0.069 (-1.70)	-0.104 (-1.61)	-0.441 (-5.03)	-0.229 (-4.09)	-0.302 (-3.60)
Scale	0.114 (18.57)	0.083 (14.86)	-	0.126 (16.67)	-	-
Scale_Car	-	-	0.097 (11.16)	-	-	-
Scale_Train	-	-	0.089 (12.70)	-	-	-
Scale_SM	-	-	0.052 (12.99)	-	-	-
Logsum	-	0.852 (13.47)	-	-	-	-
Shape	-	-	-	-	3.628 (33.10)	2.766 (28.71)
Model fit						
Final LL	-3625.77	-3620.26	-3589.91	-3609.07	-3562.96	-3552.31
Adj. rho-squared	0.171	0.172	0.179	0.175	0.185	0.188
BIC	7301.28	7298.56	7246.15	7267.88	7175.66	7154.36

Among the four logit-based models (MNL, NL, HEV, and MNL-O), the HEV model shows a better model fit than others. This result makes sense considering that the HEV model considers more generalized heteroscedasticity in the error variance between alternatives. The NL model allows covariances among similar alternatives and performs better than the basic MNL model. However, it still assumes an identical variance for all alternatives and is inadequate to account for the oddball effect. Although the MNL-O model also considers heteroscedasticity, it focuses on the non-identical variance of the oddball alternative and is less flexible than the HEV model. On the other hand, the weibit-based models (MNW and MNW-O) show better model fits than the logit-based models. This may be because the weibit-based models adopt the multiplicative error structure, which allows disutility-dependent perception variances (See Figure 4) and can better reflect the way travelers perceive travel disutility (Chakroborty et al., 2021; Nirmale and Pinjari, 2023). Finally, the proposed MNW-O model shows a better model fit than all other competing models. It suggests that the MNW-O model can successfully capture both the oddball effect and the heterogeneous perceptions of different alternatives and service features in the Swissmetro data set.

In addition, we applied statistical tests proposed by Vuong (1989) and Clarke (2003) to compare the models. Both tests are common in that their null hypothesis indicates the competing models are equally close to the actual model. On the other hand, they are distinguished in that the Vuong test assumes the asymptotic normality of the log-likelihood ratio between the two competing models, while the Clarke test is distribution-free considering the sum of the signs of the log-likelihood difference for each observation.

The statistic of the Voung test is

$$z_{Voung} = \frac{\sum_i \left[L_{mi}\left(\boldsymbol{\omega}_m\right) - L_{m'i}\left(\boldsymbol{\omega}_{m'}\right) \right]}{\sqrt{\frac{1}{n}\sum_i \left[L_{mi}\left(\boldsymbol{\omega}_m\right) - L_{m'i}\left(\boldsymbol{\omega}_{m'}\right) \right]^2 - \left\{ \frac{1}{n}\sum_i \left[L_{mi}\left(\boldsymbol{\omega}_m\right) - L_{m'i}\left(\boldsymbol{\omega}_{m'}\right) \right] \right\}^2}}, \tag{33}$$

where $L_{mi}\left(\boldsymbol{\omega}_m\right)$ is a loglikelihood value by a model *m* for an observation *i*.

On the other side, the Clarke test is

$$z_{Clarke} = \frac{2 \cdot \sum_i sgn\left[L_{mi}\left(\boldsymbol{\omega}_m\right) - L_{m'i}\left(\boldsymbol{\omega}_{m'}\right) \right] - n}{\sqrt{n}}, \tag{34}$$

where *n* is the number of observations and *sgn* denotes the sign function. Common to both tests, a large negative value of *z* means that model m' statistically outperforms model *m*.

Table 4 shows the values of test statistics z_{Voung} and z_{Clarke} for the comparison between the MNW-O model and other competing models. Consistent with the model fit results, both tests clearly confirm that the proposed MNW-O model is superior to all other models. These results demonstrate that the proposed MNW-O model has a clear advantage to simultaneously capture both the oddball effect and the disutility-dependent perception variance, while the competing models can either consider only one of the two issues (HEV, MNL-O, and MNW models) or none of them (NL and MNL models).

Table 4. Results of Voung and Clarke tests

Model 1	Model 2	Tests	
		Voung	Clarke
MNL	MNW-O	-85.229	-42.673
NL	MNW-O	-52.105	-30.988
HEV	MNW-O	-27.308	-20.330
MNL-O	MNW-O	-38.296	-36.934
MNW	MNW-O	-14.165	-12.283

4.2. Validation results

To validate the estimation results, a cross-validation test was conducted. First, we randomly created five subsets of data which includes about 20% of all observations. Of the five subsets, four were used as a training set to estimate models. Then the parameter estimates were applied to the remaining single subset (as a test set). This process is repeated 5 times to guarantee all subsets were used as the testing set.

Table 5. Results of cross-validation test

(a) Training sets

	Avg. Adj. rho-squared	Avg. BIC
MNL	0.169	5847.90
NL	0.171	5843.28
HEV	0.175	5800.24
MNL-O	0.174	5819.47
MNW	0.182	5745.64
MNW-O	0.186	5725.08

(b) Testing sets

	Avg. Adj. rho-squared	Avg. BIC
MNL	0.134	1465.15
NL	0.139	1463.47
HEV	0.148	1454.27
MNL-O	0.148	1455.19
MNW	0.157	1433.64
MNW-O	0.163	1425.77

Table 5 shows the results of the cross-validation test. The estimation results in training sets are consistent with the general interpretation of the results from the full data set. The MNW-O model is superior to all other models in terms of the model fit measures in all cases. The results in testing sets based on prediction performance also reveal the consistent outperformance of the MNW-O model against the competing models. In summary, we can conclude that the MNW-O model outperforms the competing models in the Swissmetro data set with respect to both model fit and prediction performance. It is therefore important to consider the heterogenous perceptions of not only the conventional alternatives but also the unique features of oddball alternative to better understand the choice behavior with an oddball alternative.

Table 6. Elasticities of MNW-O model[2]

Class 1	Alternatives	Elasticities		
		Train	Car	Swissmetro
Travel cost	Train	*-0.775*	0.239	0.147
	Car	0.094	*-0.581*	0.051
	Swissmetro	0.074	0.074	*-0.624*
Travel time	Train	*-0.174*	0.022	0.045
	Car	0.049	*-0.391*	0.033
	Swissmetro	0.055	0.055	*-0.410*
Headway	Swissmetro	0.204	0.204	*-1.308*
Seat availability	Swissmetro	0.181	0.181	*-1.026*

Table 6 reveals the results of elasticities in the MNW-O model based on Table 1. The direct elasticities are italicized in the table to distinguish from the cross elasticities. As discussed in Section 3.2.4., the IIA property is not valid between conventional and oddball alternatives. Therefore, the cross elasticities with respect to change in an attribute of oddball alternative are equal between conventional alternatives, while the cross elasticities with respect to change in an attribute of a conventional alternative is not the same between conventional and oddball alternatives. The results indicate that the choice probability to choose the oddball alternative (Swissmetro) is more sensitive

[2] Direct elasticities are italicized.

to the changes in the unique attributes (headway and seats availability) than in the common attributes (travel cost and travel time).

5. Conclusions

This study proposed a weibit-based model to assess choice behavior when there is an oddball alternative with heterogeneous features in the choice set. This proposed approach retains the closed-form choice probability expression, which ensures the computational efficiency of the probability evaluation and model estimation while facilitating model interpretation. Inspired by Recker's (1995) MNL-O model, the proposed MNW-O model retains the ability to specifically handle the oddball alternative as well as the asymptotic values and logical consistency conditions. However, the proposed MNW-O model also provides improved performance by enabling disutility-dependent perception variances for both the conventional and oddball alternatives and allowing a flexible variance ratio between them. These advantageous properties lead to relative difference-based choice probabilities, which allow the proposed model to further consider various heterogeneity issues that are ignored in the MNL-O model but are important for applications in modern transportation networks. The applicability of the proposed model is empirically demonstrated in a mode choice case study based on the Swissmetro data set (Bierlaire et al., 2001). The statistical results revealed that considering the advantageous properties of the MNW-O model is important for the understanding and prediction of choice behavior when an oddball alternative is included in choice set.

The proposed model has further numerous potential applications. In the context of mode choice, the MNW-O model is suitable for modeling the decision-making scenario where an emerging transportation mode is introduced to a multi-modal transportation system. The emerging mode is likely to be associated with new attributes that are not familiar to travelers, which brings additional uncertainty and should be treated as an oddball alternative (Song, 2019). For instance, connected and autonomous vehicles (CAVs) are planned to be introduced to transportation networks in the future. CAVs are expected to provide unprecedented service features, such as avoidance of potential crashes due to the smooth driving and effective utilization of in-vehicle travel time due to the fully autonomous driving technologies. Such unique attributes may provide incremental travel utility but also a higher perceived uncertainty. Thus, it would be suitable to model the CAVs as an oddball alternative and predict the market penetration via the proposed MNW-O model. Alternatively, the rapid development of urban agglomeration can make travel to locations in neighboring cities more frequent, providing travelers with extra opportunities via inter-city trips, which may also form an important oddball alternative in the destination choice problem (Huang et al., 2020). Additionally, the effect of heterogeneity has been extensively recognized as an important issue in destination choice behavior (e.g., Schmid et al., 2019). The proposed model is thus applicable to assessing travelers' decision-making among destination locations within an urban agglomeration that have distinct travel distances, different service levels, and heterogeneous opportunities.

The proposed model opens up the following potential research directions for future studies: (1) further handling the independence assumption and the IIA property remaining in the subset of conventional alternatives; (2) considering the oddball alternative with decremental utility perception of their unique features, e.g., the range anxiety of electric vehicle drivers, whereas the proposed model mainly reflects incremental utility perception with higher choice probability arising from the unique features of the oddball alternative; and (3) analogous to the development of mixed logit models (McFadden and Train, 2000), generalizing the proposed model by mixing the Weibull distributed error term with additional error terms with various distributional assumptions, e.g., the Fréchet distribution, the Log-logistic distribution, and the family of Log-normal distributions (Varela et al., 2018; Xie et al., 2020; Nirmale and Pinjari, 2023). On this basis, the outcome models may reproduce more general oddball effects based on the more flexible total error distributions while benefiting from the multiplicative error structure, which is consistent with the psychophysical laws of how individuals perceive attribute levels with varying magnitudes (Chakroborty et al., 2021).

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Appendix A. Derivation of choice probability of regular alternative

The MNW-O choice probability of regular alternative k expressed in Eq. (11) can be derived as follows based on the independence assumption:

$$P\left(k|A\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{f}_r(\xi_r)\cdot f_k(\varepsilon_k)\cdot\prod_{l\neq k,r\in A}P\left(\varepsilon_l\geq\frac{v_k\cdot\varepsilon_k}{v_l}\right)\cdot P\left(\varepsilon_r\geq\frac{v_k\cdot\varepsilon_k}{v_r\cdot\xi_r}\right)d\varepsilon_k d\xi_r$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{f}_r(\xi_r)\cdot f_k(\varepsilon_k)\cdot\prod_{l\neq k,r\in A}\left[1-F_l\left(\frac{v_k\cdot\varepsilon_k}{v_l}\right)\right]\cdot\left[1-\overline{F}_r\left(\frac{v_k\cdot\varepsilon_k}{v_r\cdot\xi_r}\right)\right]d\varepsilon_k d\xi_r \quad . \text{(A1)}$$

where f_k and $\tilde{f}_r$, are the PDFs of ε_k and ξ_r, F_l and $\overline{F}_r$ are the CDFs of ε_l and ε_r. With an additional random error term in the oddball alternative, the MNW-O choice probability is derived based on a double integral as in Eq. (A1), considering both the random errors with respect to common features and that with respect to unique features.

Without loss of generality, all random error terms are assumed to identically follow the Weibull distribution with location parameter $\eta_k=0$ and shape parameter β, $P\left(k|A\right)$ can be derived as

$$P\left(k|A\right)=\int_{0}^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}d\xi_r\int_{0}^{+\infty}\frac{\beta}{\alpha_k}\left(\frac{\varepsilon_k}{\alpha_k}\right)^{\beta-1}e^{-\left(\frac{\varepsilon_k}{\alpha_{r2}}\right)^{\beta}}\cdot e^{-\sum_{l\neq k,r\in A}\left(\frac{v_k\cdot\varepsilon_k}{v_l\cdot\alpha_l}\right)^{\beta}}\cdot e^{-\left(\frac{v_k\cdot\varepsilon_k}{v_r\cdot\xi_r\cdot\alpha_{r1}}\right)^{\beta}}d\varepsilon_k$$
. (A2)

The double integral in Eq. (A2) can be evaluated sequentially based on the integration by substitution. Let $u=e^{-\left(\frac{\varepsilon_k}{\alpha_{r2}}\right)^{\beta}}$, $du=\frac{\beta}{\alpha_k}\left(\frac{\varepsilon_k}{\alpha_k}\right)^{\beta-1}e^{-\left(\frac{\varepsilon_k}{\alpha_{r2}}\right)^{\beta}}d\varepsilon_k$, we have

$$P\left(k|A\right)=\int_{0}^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}d\xi_r\int_{0}^{1}u^{\sum_{l\neq k,r\in A}\left(\frac{v_k\cdot\varepsilon_k}{v_l\cdot\alpha_l}\right)^{\beta}}\cdot u^{\left(\frac{v_k\cdot\alpha_k}{v_r\cdot\xi_r\cdot\alpha_{r1}}\right)^{\beta}}du. \quad \text{(A3)}$$

Let $w=u^{1+\sum_{l\neq k,r\in A}\left(\frac{v_k\cdot\varepsilon_k}{v_l\cdot\alpha_l}\right)^{\beta}}=u^{\sum_{l\neq r\in A}\left(\frac{v_k\cdot\varepsilon_k}{v_l\cdot\alpha_l}\right)^{\beta}}$, $dw=\frac{1}{\sum_{l\neq r\in A}\left(\frac{v_k\cdot\varepsilon_k}{v_l\cdot\alpha_l}\right)^{\beta}}u^{\sum_{l\neq r\in A}\left(\frac{v_k\cdot\varepsilon_k}{v_l\cdot\alpha_l}\right)^{\beta}}du$, then

$$P\left(k\middle|A\right)=\int_0^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}d\xi_r\int_0^1\frac{1}{\sum_{l\neq r\in A}\left(\frac{v_k\cdot\varepsilon_k}{v_l\cdot\alpha_l}\right)^{\beta}}\cdot w^{\frac{\left[(v_k\cdot\alpha_k)/(v_r\cdot\xi_r\cdot\alpha_{r1})\right]^{\beta}}{\sum_{l\neq r\in A}\left(\frac{v_k\cdot\varepsilon_k}{v_l\cdot\alpha_l}\right)^{\beta}}}dw. \quad \text{(A4)}$$

With $\int_0^1 w^{\frac{\left[(v_k\cdot\alpha_k)/(v_r\cdot\xi_r\cdot\alpha_{r1})\right]^{\beta}}{\sum_{l\neq r\in A}\left(\frac{v_k\cdot\varepsilon_k}{v_l\cdot\alpha_l}\right)^{\beta}}}dw=\frac{1}{\frac{\left[(v_k\cdot\alpha_k)/(v_r\cdot\xi_r\cdot\alpha_{r1})\right]^{\beta}}{\sum_{l\neq r\in A}\left(\frac{v_k\cdot\varepsilon_k}{v_l\cdot\alpha_l}\right)^{\beta}}+1}$, Eq. (A4) can be written as

$$\begin{aligned}P\left(k\middle|A\right)&=\frac{\left(v_k\cdot\alpha_k\right)^{-\beta}}{\sum_{l\neq r\in A}\left(v_l\cdot\alpha_l\right)^{-\beta}}\cdot\int_0^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}\cdot\frac{1}{\frac{\left[(v_k\cdot\alpha_k)/(v_r\cdot\xi_r\cdot\alpha_{r1})\right]^{\beta}}{\sum_{l\neq r\in A}\left[(v_k\cdot\alpha_k)/(v_l\cdot\alpha_l)\right]^{\beta}}+1}d\xi_r\\&=\frac{\left(v_k\cdot\alpha_k\right)^{-\beta}}{\sum_{l\neq r\in A}\left(v_l\cdot\alpha_l\right)^{-\beta}}\cdot\int_0^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}\cdot\frac{\sum_{l\neq r\in A}\left(v_l\cdot\alpha_l\right)^{-\beta}}{\left(v_r\cdot\xi_r\cdot\alpha_{r1}\right)^{-\beta}+\sum_{l\neq r\in A}\left(v_l\cdot\alpha_l\right)^{-\beta}}d\xi_r\end{aligned}.$$

(A5)

Let $\phi_r^W=\frac{\left(v_r\cdot\alpha_{r1}\cdot\alpha_{r2}\right)^{-\beta}}{\sum_{l\neq r\in A}\left(v_l\cdot\alpha_l\right)^{-\beta}}$, with $\int_0^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}d\xi_r=1$, Eq. (A5) can be written as

$$\begin{aligned}P\left(k\middle|A\right)&=\frac{\left(v_k\cdot\alpha_k\right)^{-\beta}}{\sum_{l\neq r\in A}\left(v_l\cdot\alpha_l\right)^{-\beta}}\cdot\int_0^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}\cdot\left[1-\frac{\left(v_r\cdot\xi_r\cdot\alpha_{r1}\right)^{-\beta}}{\left(v_r\cdot\xi_r\cdot\alpha_{r1}\right)^{-\beta}+\sum_{l\neq r\in A}\left(v_l\cdot\alpha_l\right)^{-\beta}}\right]d\xi_r\\&=\frac{\left(v_k\cdot\alpha_k\right)^{-\beta}}{\sum_{l\neq r\in A}\left(v_l\cdot\alpha_l\right)^{-\beta}}\cdot\left[1-\int_0^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}\cdot\frac{\phi_r^W}{\phi_r^W+\left(\xi_r/\alpha_{r2}\right)^{\beta}}d\xi_r\right]\end{aligned}$$

. (A6)

Let $x=\phi_r^W+\left(\xi_r/\alpha_{r2}\right)^{\beta}$, $dx=\phi_r^W+\left(\xi_r/\alpha_{r2}\right)^{\beta}=\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}d\xi_r$, then

$$P\left(k\middle|A\right)=\frac{\left(v_k\cdot\alpha_k\right)^{-\beta}}{\sum_{l\neq r\in A}\left(v_l\cdot\alpha_l\right)^{-\beta}}\cdot\left[1-\int_{\phi_r^W}^{+\infty}e^{-\left(x-\phi_r^W\right)}\cdot\frac{\phi_r^W}{x}dx\right]. \quad \text{(A7)}$$

With $E_1(x)=\int_x^{+\infty}\frac{e^{-x}}{x}dx$, Eq. (A7) gives the MNW-O choice probability of the regular alternative in Eq. (13). This completes the proof.

Appendix B. Derivation of choice probability of oddball alternative

Based on the independence assumption, $P(r|A)$ expressed in Eq. (12) can be expressed as

$$\begin{aligned}P(r|A)&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f_r(\varepsilon_r)\cdot\tilde{f}_r(\xi_r)\cdot\prod_{k\neq r\in A}P\left(\frac{v_r\cdot\varepsilon_r\cdot\xi_r}{v_k}\right)d\varepsilon_r d\xi_r\\&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f_r(\varepsilon_r)\cdot\tilde{f}_r(\xi_r)\cdot\prod_{k\neq r\in A}\left[1-F_k\left(\frac{v_r\cdot\varepsilon_r\cdot\xi_r}{v_k}\right)\right]d\varepsilon_r d\xi_r\end{aligned}. \quad \text{(B1)}$$

Based on the identically Weibull distributed assumption, $P(r|A)$ can be derived as

$$P(r|A)=\int_0^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}d\xi_r\int_0^{+\infty}\frac{\beta}{\alpha_{r1}}\left(\frac{\varepsilon_r}{\alpha_{r1}}\right)^{\beta-1}e^{-\left(\frac{\varepsilon_r}{\alpha_{r1}}\right)^{\beta}}\cdot e^{-\sum_{k\neq r\in A}\left(\frac{v_r\cdot\varepsilon_r\cdot\xi_r}{v_k\cdot\alpha_k}\right)^{\beta}}d\varepsilon_r\,. \quad \text{(B2)}$$

Let $u=e^{-\varepsilon_r/\alpha_{r1}}$, Eq. (B2) can be written as

$$\begin{aligned}P(r|A)&=\int_0^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}d\xi_r\int_0^1 u^{\sum_{k\neq r\in A}\left(\frac{v_r\cdot\xi_r\cdot\alpha_{r1}}{v_k\cdot\alpha_k}\right)^{\beta}}du\\&=\int_0^{+\infty}\frac{\beta}{\alpha_{r2}}\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta-1}e^{-\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}}\frac{1}{1+\sum_{k\neq r\in A}\left(\frac{v_r\cdot\xi_r\cdot\alpha_{r1}}{v_k\cdot\alpha_k}\right)^{\beta}}d\xi_r\end{aligned}. \quad \text{(B3)}$$

Let $w=1+\sum_{k\neq r\in A}\left(\frac{v_r\cdot\xi_r\cdot\alpha_{r1}}{v_k\cdot\alpha_k}\right)^{\beta}=1+\frac{\sum_{k\neq r\in A}(v_k\cdot\alpha_k)^{-\beta}}{(v_r\cdot\alpha_{r1}\cdot\alpha_{r2})^{-\beta}}\cdot\left(\frac{\xi_r}{\alpha_{r2}}\right)^{\beta}$, Eq. (B3) can be expressed as

$$P(r|A)=\frac{(v_r\cdot\alpha_{r1}\cdot\alpha_{r2})^{-\beta}}{\sum_{k\neq r\in A}(v_k\cdot\alpha_k)^{-\beta}}\cdot\int_1^{+\infty}\frac{e^{-(w-1)\cdot\frac{(v_r\cdot\alpha_{r1}\cdot\alpha_{r2})^{-\beta}}{\sum_{k\neq r\in A}(v_k\cdot\alpha_k)^{-\beta}}}}{w}dw\,. \quad \text{(B4)}$$

With $\phi_r^W=\frac{(v_r\cdot\alpha_{r1}\cdot\alpha_{r2})^{-\beta}}{\sum_{l\neq r\in A}(v_l\cdot\alpha_l)^{-\beta}}$, $E_1(\mu)=\int_1^{+\infty}\frac{e^{-\mu\cdot x}}{x}dx$ (Gradshteyn and Ryzhik, 2007), Eq.

(B4) gives the MNW-O choice probability of the oddball alternative in Eq. (14). This completes the proof.