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# Maximum Capture Problem Based on Paired Combinatorial Weibit Model to Determine Park-and-Ride Facility Locations

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## Abstract

Park-and-ride (P&R) facilities are key components in encouraging people to use the transit system by allowing them to leave their private vehicles at certain locations. The well-known multinomial logit (MNL) model is often used to develop a random utility maximization–based mathematical programming formulation to determine P&R facility locations. According to the independently and identically distributed (IID) assumption, the MNL model cannot account for the route similarity and user heterogeneity. This study provides a new mixed integer linear programming (MILP) formulation by incorporating a newly developed paired combinatorial weibit (PCW) model to relax the IID assumption for determining the optimal P&R facility location. Specifically, the incorporation of a copula derived from a generalized extreme value (GEV) model addresses the issue of route overlap within the context of the PCW model. In addition, using the Weibull distribution permits the consideration of heterogeneous perception variance. Its two-level tree structure for evaluating the marginal and conditional probabilities allows a linearization scheme to obtain a set of linear constraints. Numerical examples reveal the influence of the IID assumption relaxation on the results. The two probabilities from the tree structure and the binary location variables are combined to present a corresponding PCW model under open/close P&R facility solution. According to the degree of route overlapping and route-specific perception variance, the fare structure, particularly the distance-based scheme, has an impact on the number of P&R users and location at optimum.

Keywords: Mixed integer linear programming; park and ride; paired combinatorial weibit; generalized extreme value; Weibull distribution

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# 1. Introduction

## 1.1 Overview

Public transit systems are crucial for sustainable development. By reducing carbon emission levels, transit systems can minimize the contribution of the transportation sector to climate change (Dalkmann et al., 2017). Within these systems, park-and-ride (P&R) facilities are a key component to promote the modal shift to public transit. In particular, the location of P&R facilities is a significant factor in encouraging commuters to use public transportation while leaving their private vehicles at such facilities (Noel, 1988; Cavadas and Antunes, 2019).

Several researchers have implemented mathematical programming (MP) to consider the optimal P&R facility location (Wang et al., 2004; 2014; Holguin-Veras et al., 2012; Farhan and Murray, 2008). Recently, the theoretical aspects and practical applications of the maximum capture problem has received attention (Freire et al., 2016). This problem extends the ordinary location problem to maximize the market share based on the customers or users' decision. In a pioneering study, ReVelle (1986) assumed an all-or-nothing assignment. In this framework, the users are deterministically assigned to the closest facility. To address the deterministic assumption, the random utility maximization (RUM) model is used. The users are assumed to choose the facility that maximizes their utility. Benati (1999) presented a nonlinear mixed integer programming framework based on the well-known multinomial logit (MNL) model. Moreover, several researchers linearized the MNL choice probability to create a mixed integer linear programming (MILP) in a multimodal hypernetwork. Benati and Hansen (2002) used a variable substitution. Haase (2009) used a constant substitution pattern, consistent with the adoption of the independence of irrelevant alternatives (IIA) property to linearize the MNL choice probability proposed by Aros-Vera et al. (2013) to determine the P&R facility location. The comparative analysis performed by Haase and Muller (2014) highlighted the advantages of the constant substitution pattern proposed by Haase (2009) and the use of the IIA property of the MNL model specified by Aros-Vera et al. (2013). However, the MNL model has two major drawbacks from the independently and identically distributed (IID) assumption. The independently distributed assumption prohibits the MNL model from accounting for overlapping between routes in the transportation network, whereas the identically distributed assumption prevents the MNL model from considering trips of different lengths (Sheffi, 1985).

Various studies were provided to relax the IID assumption in considering the P&R facility location in the literature. Since the perception heterogeneity in the P&R facility location is important (Pang and Khani, 2018), Kitthamkesorn et al. (2021) adopted the multinomial weibit

(MNW) choice model (Castillo et al., 2008) to relax the independently distributed assumption. Instead of the Gumbel distribution, the MNW model adopts the Weibull distribution to consider the perception variance as a function of the route travel cost (Kitthamkesorn and Chen, 2013; 2014). The corresponding IIA property was adopted to present a MILP. On the other hand, to relax the independently distributed assumption, several generalized extreme value (GEV)-based models (McFadden, 1978) and advanced choice models with the Gumbel random error can be used. For example, Dam et al. (2021) used the nested logit (NL) and cross NL (CNL) models. Haase et al. (2016) adopted a mixed logit (ML) model with a substitution pattern technique. Lui et al. (2018) incorporated the CNL model (Vovsha, 1997; Bekhor and Prashker, 1999; Kitthamkesorn et al., 2016) in a bi-level programming. The CNL two-level tree structure was used to construct a nonlinear MP formulation in a combined travel demand model. Note that the realization of NP-hard GEV-based MP necessitates the use of approximate algorithms for the computation (e.g., Feldman, 2017; Zhang et al., 2020). The ML-based programming requires simulation or approximation techniques. Since bi-level programming is frequently non-convex, tractability is compromise. None of the study relaxes both the independently distributed assumption and identically distributed assumption in considering the P&R facility location. Note further that embedding a choice model into an optimization problem is quite typical in many disciplines, e.g., network equilibrium models in transportation (Prashker and Bekhor, 2004), choice-based revenue management models in operations research and management science (Strauss et al., 2018), and facility location models in location science (Dam et al., 2023).

In this paper, we aim to provide a maximum capture problem to relax the IID assumption of the MNL model in determining P&R facility locations. Specifically, we adopt the paired combinatorial weibit (PCW) model due to its flexible two-level error structure for modeling the route choice problem. This flexibility allows route to belong to more than one nest, and the choice probability is calculated according to the two-level tree structure using the marginal and conditional probabilities. In addition, we make use of the two-level tree structure to linearize the nonlinear probabilities. The IIA of the PCW marginal and conditional probabilities and special ordered sets of type 2 (SOS2) are combined to provide a MILP.

### 1.2 Contributions

The objective of this study is to develop a random utility maximization–based MILP to simultaneously relax the independently distributed assumption and the identically distributed assumption in considering the park-and-ride (P&R) facility location. Three main contributions of this study are summarized as follows:

- A paired combinatorial weibit (PCW) model is developed from a joint Weibull distribution constructed by a copula. This closed-form choice model considers the route overlapping through the copula parameter and the route-specific perception variance via the Weibull distribution.
- A linearization of the PCW probability is provided. The marginal and conditional probabilities from the two-level tree structure of PCW and the binary location variables are combined to create a set of linear constraints.
- Numerical examples are also provided to illustrate several features of the PCW-based maximum capture problem. The results obtained using CPLEX indicate that the proposed MILP can consider both route overlapping and route-specific perception variance. The impact level of route overlapping and route-specific perception variance on the optimal solution depends on the fare structure scheme.

The remainder of this paper is organized as follows. Section 2 presents the background of the MNL-based and MNW-based maximum capture problem to examine the P&R facility location. Section 3 develops the PCW model. The proposed MILP is presented in Section 4. Section 5 illustrates several numerical examples, and Section 6 presents the concluding remarks.

## 2. Maximum capture problem for P&R facility location

This section clarifies the background of the two existing maximum capture problems to determine the P&R facility location, i.e., MNL-based and MNW-based models. First, we present the notation list and then describe the two existing problems.

### 2.1 Notation list

Two travel choice alternatives are considered, namely 1) private vehicles (auto) and 2) public transport via a P&R facility (transit) in a hypernetwork (Haase, 2009; Aros-Vera et al., 2013; Haase and Müller, 2014). The sets, variables, and parameters used to develop the proposed model are presented as follows.

| | | |
|---|---|---|
| Set | $IJ$ | Set of origin–destination (O–D) pairs |
| | $R_{ij}$ | Set of routes between O–D pair $ij \in IJ$ |
| | $N$ | Set of potential park-and-ride (P&R) facility locations |
| | $R_{ij}^T$ | Set of public transport routes via P&R facility between O–D pair $ij \in IJ$ ($R_{ij}^T \subset R_{ij}$) |

| | | |
|---|---|---|
| | $R_{ij}^{A}$ | Set of auto routes between O–D pair $ij{\in}IJ$ ($R_{ij}^{A}{\subset}R_{ij}$) |
| Variable | $P_{k}^{ij}$ | Probability of choosing route $k{\in}R_{ij}^{T}$ passing through P&R facility between O–D pair $ij{\in}IJ$ |
| | $P_{a}^{ij}$ | Probability of choosing route $a{\in}R_{ij}^{A}$ between O–D pair $ij{\in}IJ$ |
| | $x_n$ | Binary variable for P&R facility at location $n{\in}N$ |
| Parameter | $\theta$ | Multinomial logit (MNL) model dispersion parameter |
| | $\beta^{ij}$ | Multinomial weibit (MNW) model shape parameter between O–D pair $ij{\in}IJ$ |
| | $\zeta^{ij}$ | MNW model location parameter between O–D pair $ij{\in}IJ$ |
| | $\sigma_{kr}^{ij}$ | Parameter to consider the overlapping between route $k,r\in R_{ij}$, $ij{\in}IJ$ |
| | $\kappa_{ij}$ | Parameter to consider the degree of overlapping between O–D pair $ij{\in}IJ$ |
| | $g_{k}^{ij}$ | Travel cost for route $k{\in}R_{ij}$ between O–D pair $ij{\in}IJ$ |
| | $p$ | Number of P&R facilities |

### 2.2 MNL-based maximum capture problem

This subsection presents the background of the MNL-based maximum capture problem to determine the P&R facility location. First, we describe the MNL model, followed which we describe the nonlinear version provided by Haase (2009) and the MILP version proposed by Aros-Vera et al. (2013).

The MNL model is derived from the Gumbel distribution. Under the independence assumption and $\theta_{ijk} = \theta$, the MNL choice probability can be expressed as

$$P_{k}^{ij} = \frac{exp\left(-\theta \mathrm{g}_{\mathrm{k}}^{\mathrm{ij}}\right)}{\sum_{l\in R_{ij}} exp\left(-\theta \mathrm{g}_{l}^{\mathrm{ij}}\right)}. \tag{1}$$

Consider the following maximum capture problem:

$$\max \sum_{ij\in IJ} \sum_{k(n)\in R_{ij}^{T}} q_{ij} P_{k(n)}^{ij} \tag{2}$$

s.t.

$$\sum_{n\in N} x_n = p, \tag{3}$$

$$x_n \in \{0,1\}, \ \forall n \in N \tag{4}$$

$$P_{k(n)}^{ij} = \frac{x_n exp\left(-\theta \mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}\right)}{\sum_{m\in N}\sum_{r(m)\in R_{ij}^{T}} x_m \, exp\left(-\theta \mathrm{g}_{r(m)}^{\mathrm{ij}}\right) + \sum_{b\in R_{ij}^{A}} exp\left(-\theta \mathrm{g}_{b}^{\mathrm{ij}}\right)}, \quad (5)$$

where $q_{ij}$ is the travel demand for each O–D pair *ij*. The objective function in Eq. (2) is to maximize the market share through the P&R facility location or number of P&R users. Equation (3) constrains the number of P&R facilities. Eq. (4) specifies the binary variable. Equation (5) represents the probability of choosing route $k(n) \in R_{ij}^{T}$ passing through P&R facility $n \in N$ between O–D pair $ij \in IJ$, which depends on the state (open ($x_n = 1$) or close ($x_n = 0$)) of the P&R facility location.

The MNL model exhibits the IIA property, i.e.,

$$\frac{P_k^{ij}}{P_r^{ij}} = \frac{exp\left(-\theta \mathrm{g}_{\mathrm{k}}^{\mathrm{ij}}\right)}{\sum_{l\in R_{ij}} exp\left(-\theta \mathrm{g}_{l}^{\mathrm{ij}}\right)} \Big/ \frac{exp\left(-\theta \mathrm{g}_{\mathrm{r}}^{\mathrm{ij}}\right)}{\sum_{l\in R_{ij}} exp\left(-\theta \mathrm{g}_{l}^{\mathrm{ij}}\right)} = \frac{exp\left(-\theta \mathrm{g}_{\mathrm{k}}^{\mathrm{ij}}\right)}{exp\left(-\theta \mathrm{g}_{\mathrm{r}}^{\mathrm{ij}}\right)}. \quad (6)$$

Aros-Vera et al. (2013) adopted the IIA property of the MNL model to provide a MILP. Equation (5) can be replaced by a system of constraints:

$$P_l^{ij} \in [0,1], \ \forall l \in R_{ij}, ij \in IJ, \quad (7)$$

$$P_{k(n)}^{ij} \le x_n, \ \forall k(n) \in R_{ij}^{T}, n \in N, ij \in IJ, \quad (8)$$

$$\sum_{k(n)\in R_{ij}^{T}} P_{k(n)}^{ij} + \sum_{a\in R_{ij}^{A}} P_a^{ij} = 1, \forall ij \in IJ, \quad (9)$$

$$P_a^{ij} = \frac{exp\left(-\theta \mathrm{g}_{a}^{\mathrm{ij}}\right)}{exp\left(-\theta \mathrm{g}_{\mathrm{b}}^{\mathrm{ij}}\right)} P_b^{ij}, \ \forall a \ne b \in R_{ij}^{A}, ij \in IJ, \quad (10)$$

$$P_{k(n)}^{ij} \le \frac{exp\left(-\theta \mathrm{g}_{k(n)}^{\mathrm{ij}}\right)}{exp\left(-\theta \mathrm{g}_{\mathrm{b}}^{\mathrm{ij}}\right)} P_b^{ij}, \ \forall k(n) \in R_{ij}^{T}, n \in N, b \in R_{ij}^{A}, ij \in IJ, \quad (11)$$

$$P_a^{ij} \le \frac{exp\left(-\theta \mathrm{g}_{a}^{\mathrm{ij}}\right)}{exp\left(-\theta \mathrm{g}_{r(m)}^{\mathrm{ij}}\right)} P_{r(m)}^{ij} + (1 - x_m), \quad (12)$$

$$\forall k(n) \ne r(m) \in R_{ij}^{T}, n, m \in N, a \in R_{ij}^{A}, ij \in IJ,$$

$$P_{k(n)}^{ij} \le \frac{exp\left(-\theta \mathrm{g}_{k(n)}^{\mathrm{ij}}\right)}{exp\left(-\theta \mathrm{g}_{r(m)}^{\mathrm{ij}}\right)} P_{r(m)}^{ij} + (1 - x_m), \ \forall k(n) \ne r(m) \in R_{ij}^{T}, n, m \in N, ij \in IJ. \quad (13)$$

Equation (7) indicates that the choice probability lies between zero and one. The choice probability on route *k* through facility *n* between an O–D pair *ij* is greater than zero only when the facility is open, as shown in Eq. (8). Equation (9) presents the conservation constraint. Equations (10)–(13) indicate the exploitation of the IIA property of the MNL model for every possible route pair, specifically, the auto routes–auto routes, public transit routes–auto routes, auto routes–public transit routes, and public transit routes–public transit routes, respectively. The auto routes–auto routes relation in Eq. (10) has an equal sign (=) because $P_a^{ij}$ is always greater than zero. In contrast, $P_{k(n)}^{ij}$ is depended on $x_n$ in Eq. (8). Therefore, Eqs. (11)–(13) involve the less than or equal to sign ($\leq$). The location variable $x_n$ is adopted for a logical constraint, and the term $(1 - x_{\mathrm{m}})$ in Eqs. (12) and (13) is used to ensure logical reason. If $x_m = 1$, $P_{r(m)}^{ij} \geq 0$. If $x_m = 0$, $P_{r(m)}^{ij} = 0$, $P_a^{ij} \leq 1$, and $P_{k(n)}^{ij} \leq 1$, consistent with Eq. (7). Considering Eqs. (7)–(13), linear equations to determine $P_{k(n)}^{ij}$ based on $x_n$ can be obtained. Interested readers can refer to the work of Aros-Vera et al. (2013) and Haase and Müller (2014) for additional details.

Notably, the MNL model involves two drawbacks. According to the independently distributed assumption, the MNL model cannot account for the route correlation or route overlapping. Furthermore, the setting $\theta_{ijk} = \theta$ results in the identically distributed assumption, in which each route has the same and fixed perception variance of $\pi^2/6\theta^2$.

### 2.3 MNW-based maximum capture problem

To relax the identically distributed assumption, Kitthamkesorn et al. (2021) considered the MNW model (Castillo et al., 2008). This model is derived from the Weibull distribution, as presented in Table 1. This Weibull distribution has three parameters, namely the location parameter $\zeta_{ijk}$, scale parameter $\theta_{ijk}$, and shape parameter $\beta_{ijk}$.

Table 1: Weibull distribution

| Distribution | Weibull |
|---|---|
| Cumulative Distribution Function (CDF) | $1 - exp\left[-\left(\dfrac{t - \zeta_{ijk}}{\theta_{ijk}}\right)^{\beta_{ijk}}\right]$ |
| Mean ($g_k^{ij}$) | $\zeta_{ijk} + \theta_{ijk}\Gamma\left(1 + \dfrac{1}{\beta_{ijk}}\right)$ |
| Variance ($\upsilon_k^{ij}$) | $\theta_{ijk}^2\left[\Gamma\left(1 + \frac{2}{\beta_{ijk}}\right) - \Gamma^2\left(1 + \frac{1}{\beta_{ijk}}\right)\right]$ |

Under the independence assumption, $\zeta_{ijk} = \zeta_{ij}$ and $\beta_{ijk} = \beta_{ij}$, and the MNW model can be expressed as

$$P_k^{ij} = \frac{\left(\mathrm{g}_\mathrm{k}^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}{\sum_{l\in R_{ij}}\left(\mathrm{g}_\mathrm{l}^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}. \quad (14)$$

The probability of choosing the transit system via P&R facility $n$ can be expressed as

$$P_{k(n)}^{ij} = \frac{x_n\left(\mathrm{g}_{\mathrm{k}(n)}^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}{\sum_{m\in N}\sum_{r(m)\in R_{ij}^T} x_m\, exp\left(\mathrm{g}_{r(m)}^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}} + \sum_{b\in R_{ij}^A}\left(\mathrm{g}_b^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}, \quad (15)$$

Similar to the MNL model, the MNW model also exhibits the IIA property:

$$\frac{P_k^{ij}}{P_r^{ij}} = \frac{\left(\mathrm{g}_\mathrm{k}^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}{\sum_{l\in R_{ij}}\left(\mathrm{g}_\mathrm{l}^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}} \Bigg/ \frac{\left(\mathrm{g}_\mathrm{r}^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}{\sum_{l\in R_{ij}}\left(\mathrm{g}_\mathrm{l}^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}} = \frac{\left(\mathrm{g}_\mathrm{k}^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}{\left(\mathrm{g}_\mathrm{r}^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}. \quad (16)$$

To formulate the MNW-based maximum capture problem to determine the P&R facility location, Kitthamkesorn et al. (2021) replaced Eqs. (10)–(13) with a similar principle. The probability ratio $exp\left(-\theta \mathrm{g}_k^\mathrm{ij}\right)/exp\left(-\theta \mathrm{g}_r^\mathrm{ij}\right)$ in the MNL-based model is replaced by $\left(\mathrm{g}_k^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}} \Big/ \left(\mathrm{g}_r^\mathrm{ij} - \zeta_{ij}\right)^{-\beta_{ij}}$ to obtain the MNW choice probability at optimal.

Note that the MNW model includes the route-specific perception variance. According to the Weibull variance, which consists of the scale and shape parameters, setting $\zeta_{ijk} = \zeta_{ij}$ and $\beta_{ijk} = \beta_{ij}$ does not imply the identical distributed assumption. Thus, the MNW model includes the perception variance as a function of the route travel cost (Kitthamkesorn and Chen, 2013; 2014)

$$\upsilon_\mathrm{k}^\mathrm{ij} = \frac{\left(\mathrm{g}_\mathrm{k}^\mathrm{ij} - \zeta_{ij}\right)^2}{\Gamma\left(1 + \frac{1}{\beta_{ij}}\right)}\left[\Gamma\left(1 + \frac{2}{\beta_{ij}}\right) - \Gamma^2\left(1 + \frac{1}{\beta_{ij}}\right)\right]. \quad (17)$$

However, the MNW model still involves the independently distributed assumption and cannot account for route overlapping. Please refer to Gu et al. (2022a, b) for a detailed review of the MNW model and application to accessibility-based vulnerability analysis of multimodal transportation networks.

## 3. PCW model

As described in this section, a new choice model to account for both route overlapping and heterogeneous perception variance is developed. The paired combinatorial logit (PCL) model (Chu, 1989; Koppelman and Wen, 2000) is integrated with the Weibull distribution to create the PCW model. First, we review the PCL model and then describe the development of the

PCW model.

## 3.1 PCL model

The PCL model was developed to relax the independence assumption embedded in the MNL model. The route correlation from the transportation network topology or the route overlap is accounted for from the joint Gumbel distribution. The PCL choice probability can be derived from the GEV model (McFadden, 1978) through the following generation function:

$$G\left(y_{ij1},\ldots,y_{ij|R_{ij}|}\right)=\sum_{l=1}^{|R_{ij}|-1}\sum_{m=l+1}^{|R_{ij}|}\left(1-\sigma_{lm}^{ij}\right)\left[y_{ijl}^{\frac{1}{1-\sigma_{lm}^{ij}}}+y_{ijm}^{e\frac{1}{1-\sigma_{lm}^{ij}}}\right]^{1-\sigma_{lm}^{ij}}, \tag{18}$$

where $\sigma_{kr}^{ij}\in[0,1)$ is the similarity parameter to consider route overlapping, and $|R_{ij}|$ is the number of routes between O–D pair *ij*. With $y_{ijk}=exp\left(-\theta t_k^{ij}\right)$, a joint survival Gumbel distribution can be expressed as (Koppelman and Wen, 2000; Marzano and Papola, 2008)

$$\bar{H}_{ij}=exp\left[-\sum_{l=1}^{|R_{ij}|-1}\sum_{m=l+1}^{|R_{ij}|}\left(1-\sigma_{lm}^{ij}\right)\left[e^{\frac{\theta\{t_l^{ij}-\varsigma_{ijl}\}}{1-\sigma_{lm}^{ij}}}+e^{\frac{\theta\{t_m^{ij}-\varsigma_{ijm}\}}{1-\sigma_{lm}^{ij}}}\right]^{1-\sigma_{lm}^{ij}}\right], \tag{19}$$

Subsequently, the PCL probability can be expressed as

$$P_k^{ij}=\frac{\sum_{r\neq k}exp\left(\frac{-\theta g_k^{ij}}{1-\sigma_{kr}^{ij}}\right)\left(1-\sigma_{kr}^{ij}\right)\left[exp\left(\frac{-\theta g_k^{ij}}{1-\sigma_{kr}^{ij}}\right)+exp\left(\frac{-\theta g_r^{ij}}{1-\sigma_{kr}^{ij}}\right)\right]^{-\sigma_{kr}^{ij}}}{\sum_{l=1}^{|R_{ij}|-1}\sum_{m=l+1}^{|R_{ij}|}\left(1-\sigma_{lm}^{ij}\right)\left[exp\left(\frac{-\theta g_l^{ij}}{1-\sigma_{lm}^{ij}}\right)+exp\left(\frac{-\theta g_m^{ij}}{1-\sigma_{lm}^{ij}}\right)\right]^{1-\sigma_{lm}^{ij}}}. \tag{20}$$

The correlation between a route pair is considered through an overlapping section, resulting in a certain similarity of the unobserved characteristics between the route pair. The correlation and covariance can be determined from the bivariate distribution:

$$\bar{F}\left(t_l^{ij},t_m^{ij}\right)=exp\left[-\left(1-\sigma_{lm}^{ij}\right)\left[e^{\frac{\theta\{t_l^{ij}-\varsigma_{ijl}\}}{1-\sigma_{lm}^{ij}}}+e^{\frac{\theta\{t_m^{ij}-\varsigma_{ijm}\}}{1-\sigma_{lm}^{ij}}}\right]^{1-\sigma_{lm}^{ij}}\right]. \tag{21}$$

Note that the PCL correlation and covariance cannot be written in a closed form (Koppelman and Wen, 2000). Therefore, a numerical method or an estimation technique must be used (Marzano and Papola, 2008).

In practical scenarios, the parameter $\sigma_{kr}^{ij}$ can be related with the network topology as (Bekhor and Prashker, 1999)

$$\sigma_{kr}^{ij} = \left(\frac{l_{ijkr}}{\sqrt{L_{ijk}L_{ijr}}}\right)^{\kappa_{ij}}, \tag{22}$$

where $L_{ijk}$ is the length of route $k$ between O–D pair $ij$, $l_{ijrk}$ is the overlapping section between routes $k$ and $r$, and $\kappa_{ij} > 0$ is a parameter.

## 3.2 PCW model

To relax the identical assumption embedded in the PCL model, we consider the Weibull distribution. The PCL GEV generating function is used. Using the inversion method, a corresponding copula can be presented (Bhat, 2009; Fosgerau et al., 2013). The copula derived from the joint Gumbel distribution of the PCL model, defined in Eq. (19), can be expressed as

$$C_{ij} = exp\left[-\sum_{l=1}^{|R_{ij}|-1}\sum_{m=l+1}^{|R_{ij}|}\left(1-\sigma_{lm}^{ij}\right)\left[\left(-ln\, u_{ijl}\right)^{\frac{1}{1-\sigma_{lm}^{ij}}} + \left(-ln\, u_{ijm}\right)^{\frac{1}{1-\sigma_{lm}^{ij}}}\right]^{1-\sigma_{lm}^{ij}}\right], \tag{23}$$

where $u_{ijk}$, $\forall k \in R_{ij}, ij \in IJ$ is the marginal CDF of the perceived travel cost. Subsequently, the joint (survival) Weibull distribution can be expressed as

$$\bar{H}_{ij} = exp\left[-\sum_{l=1}^{|R_{ij}|-1}\sum_{m=l+1}^{|R_{ij}|}\left(1-\sigma_{lm}^{ij}\right)\left[\left(\frac{t_l^{ij}-\zeta_{ijl}}{\theta_{ijl}}\right)^{\frac{\beta_{ijl}}{1-\sigma_{lm}^{ij}}} + \left(\frac{t_m^{ij}-\zeta_{ijm}}{\theta_{ijm}}\right)^{\frac{\beta_{ijm}}{1-\sigma_{lm}^{ij}}}\right]^{1-\sigma_{lm}^{ij}}\right]. \tag{24}$$

The choice probability can be derived through integration (Ben-Akiva and Lerman, 1985)

$$P_k^{ij} = \int_{-\infty}^{\infty} H_{ijk}dx, \tag{25}$$

where $H_{ijk} = \partial H_{ij}/\partial t_{ijk}$. In this case,

$$\begin{aligned} P_k^{ij} = \int_{\zeta_{ijk}}^{\infty} -\sum_{k=1}^{|R_{ij}|-1}\sum_{r=k+1}^{|R_{ij}|}\frac{\beta_{ijk}}{\theta_{ijk}}\left(1-\sigma_{kr}^{ij}\right)\left(\frac{t_k^{ij}-\zeta_{ijk}}{\theta_{ijk}}\right)^{\frac{\beta_{ijk}}{1-\sigma_{kr}^{ij}}-1} \times \\ \left[\left(\frac{t_k^{ij}-\zeta_{ijk}}{\theta_{ijk}}\right)^{\frac{\beta_{ijk}}{1-\sigma_{kr}^{ij}}} + \left(\frac{t_r^{ij}-\zeta_{ijr}}{\theta_{ijr}}\right)^{\frac{\beta_{ijr}}{1-\sigma_{kr}^{ij}}}\right]^{-\sigma_{kr}^{ij}} \bar{H}_{ij}dt_{ijk}\,. \end{aligned} \tag{26}$$

Setting $\beta_{ijk} = \beta_{ij}$ and $\zeta_{ijk} = \zeta_{ij}$ yields

$$P_k^{ij} = \frac{\sum_{r\neq k}\left(\frac{1}{\theta_{ijk}}\right)^{\frac{\beta_{ij}}{1-\sigma_{kr}^{ij}}}\left(1-\sigma_{kr}^{ij}\right)\left[\left(\frac{1}{\theta_{ijk}}\right)^{\frac{\beta_{ij}}{1-\sigma_{kr}^{ij}}}+\left(\frac{1}{\theta_{ijr}}\right)^{\frac{\beta_{ij}}{1-\sigma_{kr}^{ij}}}\right]^{-\sigma_{kr}^{ij}}}{\sum_{l=1}^{|R_{ij}|-1}\sum_{m=l+1}^{|R_{ij}|}\left(1-\sigma_{lm}^{ij}\right)\left[\left(\frac{1}{\theta_{ijl}}\right)^{\frac{\beta_{ij}}{1-\sigma_{lm}^{ij}}}+\left(\frac{1}{\theta_{ijm}}\right)^{\frac{\beta_{ij}}{1-\sigma_{lm}^{ij}}}\right]^{1-\sigma_{lm}^{ij}}}. \tag{27}$$

By relating $\theta_{ijk}$ to the Weibull mean, the PCW probability expression can be expressed as

$$P_k^{ij} = \frac{\sum_{r\neq k}\left(\mathrm{g}_\mathrm{k}^\mathrm{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}}\left(1-\sigma_{kr}^{ij}\right)\left[\left(\mathrm{g}_\mathrm{k}^\mathrm{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}}+\left(\mathrm{g}_\mathrm{r}^\mathrm{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}}\right]^{-\sigma_{kr}^{ij}}}{\sum_{l=1}^{|R_{ij}|-1}\sum_{m=l+1}^{|R_{ij}|}\left(1-\sigma_{lm}^{ij}\right)\left[\left(\mathrm{g}_\mathrm{l}^\mathrm{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{lm}^{ij}}}+\left(\mathrm{g}_m^\mathrm{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{lm}^{ij}}}\right]^{1-\sigma_{lm}^{ij}}}. \tag{28}$$

The correlation and covariance can be determined from the bivariate distribution:

$$\bar{F}\left(t_l^{ij}, t_m^{ij}\right) = \exp\left[-\left(1-\sigma_{lm}^{ij}\right)\left[\left(\frac{t_l^{ij}-\zeta_{ij}}{\theta_{ijl}}\right)^{\frac{\beta_{ij}}{1-\sigma_{lm}^{ij}}}+\left(\frac{t_m^{ij}-\zeta_{ij}}{\theta_{ijm}}\right)^{\frac{\beta_{ij}}{1-\sigma_{lm}^{ij}}}\right]^{1-\sigma_{lm}^{ij}}\right]. \tag{29}$$

Similar to the PCL case, the correlation and covariance of the PCW model cannot be expressed in a closed form and numerical methods must be used. According to Eq. (29), the marginal distribution of the PCW model can be expressed as

$$\bar{F}\left(t_l^{ij}\right) = exp\left[-\left(\frac{t_l^{ij}-\zeta_{ij}}{\theta_{ijl}}\right)^{\beta_{ij}}\right], \tag{30}$$

which is consistent with the Weibull distribution presented in Table 1. In other words, the corresponding route-specific perception variance is identical to that of the MNW model, as defined in Eq. (17).

Furthermore, the PCW probability has a two-level tree structure:

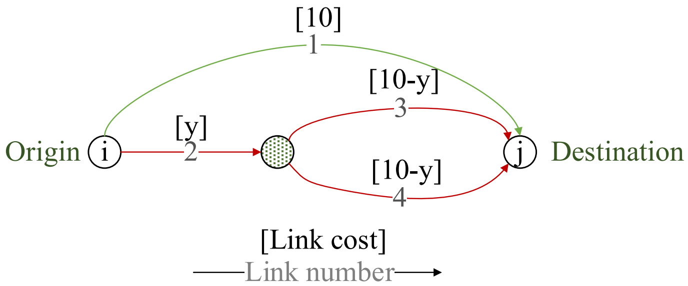
$$P_k^{ij} = \sum_{r\neq k} P_{kr}^{ij} P_{k|kr}^{ij}, \tag{31}$$

where the marginal and conditional probabilities can be respectively defined as follows:

$$P_{kr}^{ij} = \frac{\left(1-\sigma_{kr}^{ij}\right)\left[\left(\mathrm{g}_\mathrm{k}^\mathrm{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}}+\left(\mathrm{g}_\mathrm{r}^\mathrm{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}}\right]^{1-\sigma_{kr}^{ij}}}{\sum_{l=1}^{|R_{ij}|-1}\sum_{m=l+1}^{|R_{ij}|}\left(1-\sigma_{lm}^{ij}\right)\left[\left(\mathrm{g}_\mathrm{l}^\mathrm{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{lm}^{ij}}}+\left(\mathrm{g}_m^\mathrm{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{lm}^{ij}}}\right]^{1-\sigma_{lm}^{ij}}}, \tag{32}$$

$$P_{k|kr}^{ij} = \frac{\left(\mathrm{g}_{\mathrm{k}}^{\mathrm{ij}} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}}}{\left(\mathrm{g}_{\mathrm{k}}^{\mathrm{ij}} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}} + \left(\mathrm{g}_{\mathrm{r}}^{\mathrm{ij}} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}}} \quad . \tag{33}$$

Figure 1 depicts a graphical illustration of the two-level tree structure of PCW using the loop-hole network. The network has three routes. The upper route is truly independent, while the two lower routes are overlapped on link 2 with the travel cost of *y*. The marginal probability indicates the probability of selecting a route pair in the first level, whereas the conditional probability indicates the probability of selecting a certain route in a route pair in the second level. Notably, $P_{kr}^{ij} = P_{rk}^{ij}$ and $P_{k|kr}^{ij} = P_{k|rk}^{ij}$. We will use this two-level tree structure to obtain a MILP described in Section 4. It should be noted that this capability is not available for all GEV-based route choice models. Other GEV-based models' probability linearization can be difficult. Also, the path-size factor to account for route overlaps necessitates a complicated approximation approach (Liu et al., 2019).




| Route | Link |
|---|---|
| 1 | 1 |
| 2 | 2,3 |
| 3 | 2,4 |

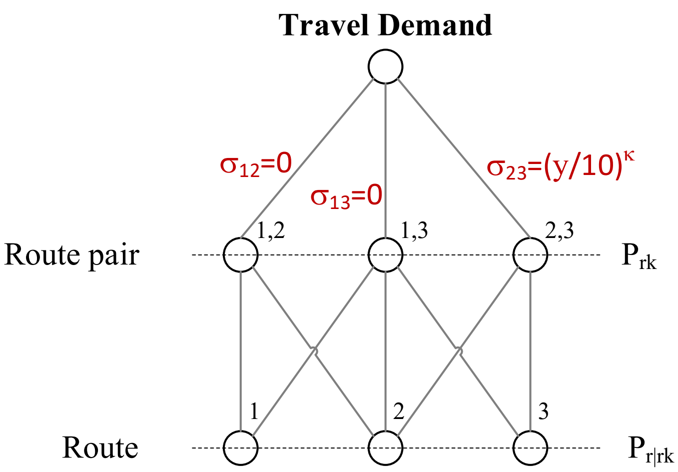



Figure 1: Modified loop-hole network and corresponding two-level tree structure (index *ij* is omitted for simplicity)

## 3.3 Comparison between the PCW model and some existing models

A comparison would be made for each assumption, including 1) identically distributed assumption, 2) independently distributed assumption, and 3) independently and identically distributed (IID) assumption. The comparisons are made between the PCW model and some existing models, including the path-size logit (PSL) model (Ben-Akiva and Bierlaire, 1999), adaptive path-size logit (APSL) model (Duncan et al., 2020), PCL model, cross nested logit (CNL) model (Bekhor and Prashker, 1999), and path-size weibit (PSW) model (Kitthamkesorn and Chen, 2013). For comparison purpose, the $\theta = 0.1$, $\beta_{ij} = 3.7$, $\zeta_{ij} = 0$, and $\kappa_{ij} = 1$. The APSL probability is computed by a fixed-point problem.

### 3.3.1 Identically distributed assumption

A two-route network in Figure 2 is adopted to illustrate the capability of the PCW model in relaxing the identically distributed assumption. In this network, the upper route is longer than the lower route by 5 units. At $y$ = 5, the upper route is two times longer than the lower route. Meanwhile, at $y$ = 125, the upper route is only 4 percent longer than the lower route.

According to the identical perception variance of $\pi^2/6\theta^2$, the logit models produce the same choice probability for all variable cost $y$s, regardless of the trip length. In contrast, the MNW, PSW, and PCW models have the perception variance as a function of the route travel cost (see Eq. (17)). These three weibit models can account for the trip with different lengths. The length between the upper and lower routes becomes more similar as $y$ increases, and the probability of choosing each route becomes more comparable.

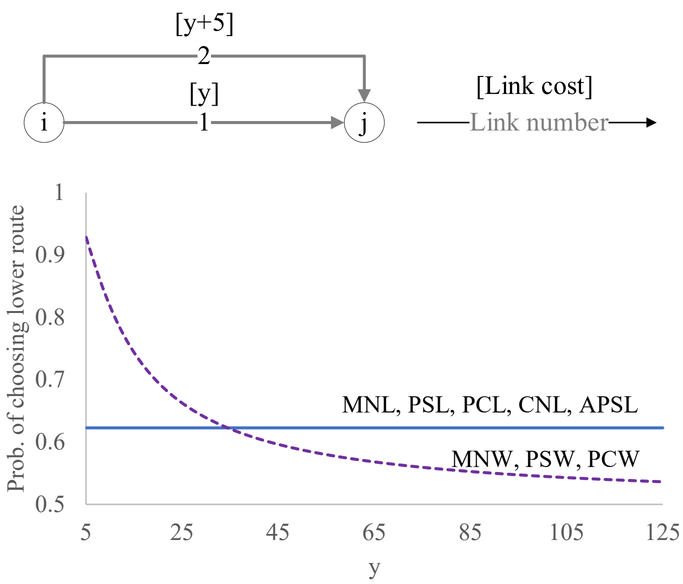



Figure 2: Two-route network and probability of choosing the lower route

It is acknowledged that the implementation of cost transformation techniques can potentially mitigate certain limitations associated with the logit model. Utilizing the logarithmic transformation of the cost of the route is a viable approach. The probability would exhibit similar results compared to those of the weibit models. The rationale behind this assertion is rooted in the log transformation's ability to establish a connection between the logit and weibit models, as shown by previous studies (Castillo et al., 2008; Li, 2011; Kitthamkesorn and Chen, 2013; 2014). It is noteworthy that MNL scaling can only be done at the OD level to modify the choice probability, it is not applicable at the route level (Chen et al., 2012). In other words, each route still has the homogeneous perception variance of $\pi^2/6\theta^2$.

According to Li (2011), it is hard to believe that a single statistical distribution can accommodate various transportation applications. Li (2011) proposed a semi-parametric approach to generate closed-form choice models based on distributions with the min-stable property. Mattsson et al. (2014) proposed a general framework to generate probabilistic models and specifically obtained the choice models based on the extreme value distributions. These route choice models are related to the logit model through some transformations. Nonetheless, the route-specific perception variance can be presented explicitly. The perception variance can be a function of the route travel cost and distribution parameter, which could be identified based on a particular distribution.

### 3.3.2 Independently distributed assumption

The loop-hole network in Figure 1 is used to demonstrate the capability of the PCW model in relaxing the independently distributed assumption. All three routes in the loop-hole network have the same cost of 10 units. The upper route is truly independent. Meanwhile the two lower routes are correlated with the overlapping section of $y$. According to the independently distributed assumption, the MNL and MNW models give the same result for all $y$s, regardless of the overlapping section. Note that other models (e.g., PSL, APLS, PSW, and CNL) can consider the route overlapping. At $y = 0$, all three routes are independent, and these models generate the same probability. As $y$ increases, the overlapping section makes the two lower routes to become more similar. These models present a larger probability of choosing the upper route. The PSL, APSL, PSW, and CNL models present a convex curve while the PCL and PCW models show a concave curve. Only two routes are possible for $y = 10$, and all these models except for the CNL model produce equal probability of choosing each route.

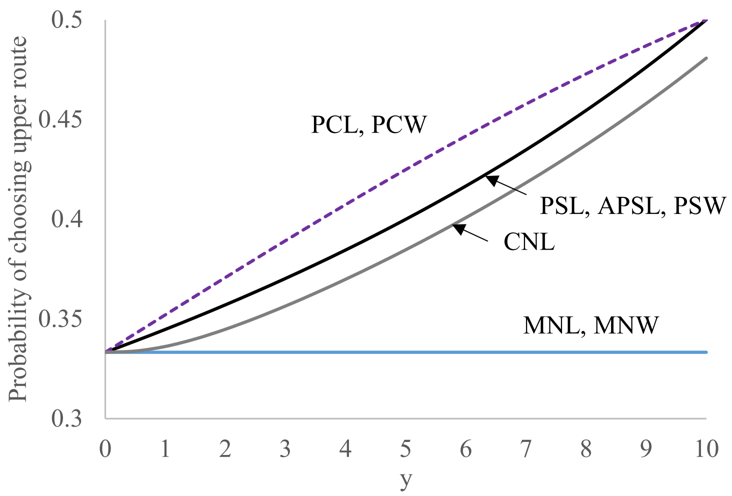
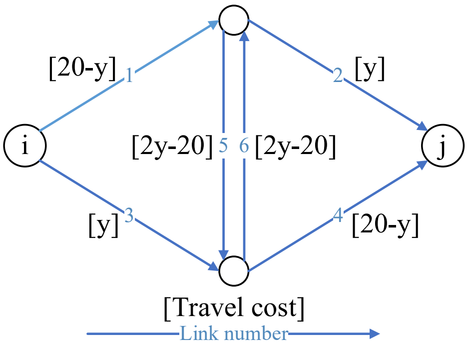



Figure 3: Loop-hole network and probability of choosing the upper route

#### 3.3.3 Independently and identically distributed (IID) assumption

A switching route network in Figure 4 has four routes. Routes 1 to 3 have the same travel cost of 10 units, whereas the travel cost on route 4 can be longer, dependent on $y$. The multinomial probit (MNP) model is used as a benchmark. The route coefficient of variation (CV) is assumed to be 0.3. The MNP model uses a joint Normal distribution to address the IID assumption. Its probability is produced by the Monte Carlo simulation (Sheffi, 1985) with 1,000 draws.

At $y$ = 10, all routes have the same cost, and all models except for the APSL model give the same result. As $y$ increases, the PSL and PSW models provide a higher probability of choosing route 2. This is due to the drawback of using a path-size factor in the switching network as identified by Prashker and Bekhor (2004). The PCL, CNL, and PCW models generate similar probability of choosing route 3 to the MNP model. Comparing between the PCL and PCW models, the PCW model produces a closer result to the MNP model according to the PCW parameter setting $\beta_{ij}$ = 3.7 and $\zeta_{ij}$ = 0, which corresponds to the assumed CV of 0.3 for each route (Kitthamkesorn and Chen, 2013; 2014).




| Route | Link | Route travel cost |
|---|---|---|
| 1 | 1, 2 | 20 |
| 2 | 3, 4 | 20 |
| 3 | 1, 5, 4 | 20 |
| 4 | 3, 6, 2 | 4y-20 |

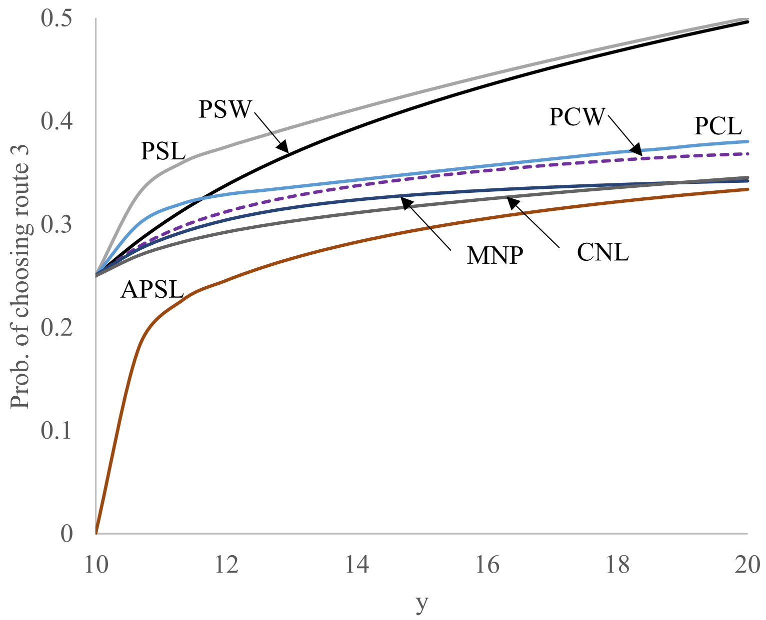



Figure 4: Switching route network and probability of choosing route 3

Note that, while the PCW model can relax the assumption of independent and identically distributed (IID), the MNP model offers a higher flexibility in achieving this relaxation. The PCW model accounts for the route overlapping and heterogeneous perception variance as a function of the route travel cost. The choice probability for this closed-form model is influenced by the route overlapping and route-specific perception variance to a greater extent when the route travel cost is larger. The MNP model can incorporate the consideration of both route overlapping and heterogeneous perception variance from any function, rather than being solely dependent on route travel cost. The absence of a closed-form solution renders the computation of the MNP model both costly and challenging, particularly when attempting to integrate it into a location problem.

## 4. PCW-based maximum capture problem for P&R facility location

A maximum capture problem is formulated to examine the P&R facility location considering the PCW choice behavior. Because the PCW model does not exhibit the IIA property, its two-level tree structure is used to linearize the nonlinear probability.

### 4.1 Nonlinear PCW choice probability for the P&R facility location

The probability of travelers traveling through a P&R facility potential site under the PCW choice behavior can be defined as follows:

$$
P^{ij}_{k(n)} = \frac{
\begin{array}{c}
\underbrace{\sum_a x_n\left(g^{ij}_{k(n)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)a}}}\left(1-\sigma^{ij}_{k(n)a}\right) \times\left[x_n\left(g^{ij}_{k(n)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)a}}}+\left(g^{ij}_{a}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)a}}}\right]^{-\sigma^{ij}_{k(n)a}}}_{\text{Transit–Auto}} \\
\underbrace{+\sum_{r(m)\neq k(n)} x_n\left(g^{ij}_{k(n)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ijk(n)r(m)}}}\left(1-\sigma^{ij}_{k(n)r(m)}\right) \times\left[x_n\left(g^{ij}_{k(n)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)r(m)}}}+x_m\left(g^{ij}_{r(m)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)r(m)}}}\right]^{-\sigma^{ij}_{k(n)r(m)}}}_{\text{Transit–Transit}}
\end{array}
}{
\begin{array}{c}
\underbrace{\sum_{a=1}^{\left|R^A_{ij}\right|-1}\sum_{b=a+1}^{\left|R^A_{ij}\right|}\left(1-\sigma^{ij}_{ab}\right)\left[\left(g^{ij}_{a}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{ab}}}+\left(g^{ij}_{b}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{ab}}}\right]^{1-\sigma^{ij}_{ab}}}_{\text{Auto–Auto}} \\
\underbrace{+\sum_{k(n)=1(1)}^{\left|R^T_{ij}\right|}\sum_{a=1}^{\left|R^A_{ij}\right|}\left(1-\sigma^{ij}_{k(n)a}\right) \times\left[x_n\left(g^{ij}_{k(n)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)a}}}+\left(g^{ij}_{a}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)a}}}\right]^{-\sigma^{ij}_{k(n)a}}}_{\text{Transit–Auto}} \\
\underbrace{+\sum_{k(n)=1(1)}^{\left|R^T_{ij}\right|-1}\sum_{r(m)=k(n)+1}^{\left|R^T_{ij}\right|}\left(1-\sigma^{ij}_{k(n)r(m)}\right) \times\left[x_n\left(g^{ij}_{k(n)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)r(m)}}}+x_m\left(g^{ij}_{r(m)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)r(m)}}}\right]^{-\sigma^{ij}_{k(n)r(m)}}}_{\text{Transit–Transit}}
\end{array}
} \tag{34}
$$

As the PCW model considers each route pair in a hypernetwork, Eq. (34) must account for every possible route pair combination in a choice set, including the pairs of auto route and auto route (Auto–Auto), the auto route and transit route (Auto-Transit or Transit–Auto), and transit route and transit route (Transit–Transit). For probability $P^{ij}_{k(n)}$, the numerator includes the Transit–Auto and Transit–Transit route pairs associated with route $k(n)$ between O–D pair $ij$. The denominator includes all possible route pairs between O–D pair $ij$. The binary variable is added to each term related to the transit route, i.e., $x_n\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ka}^{ij}}}$. In this manner, the relation between the binary variable and choice probability is maintained, as in the MNL case in Eq. (5). Notably, this probability is a nonlinear function; it includes several binary variables for each transit route, rendering it more complex than the MNL models. Furthermore, the PCW model does not exhibit the IIA property, and the same linearization approach cannot be applied.

### 4.2 Two-level tree structure to linearize the PCW choice probability

To address the abovementioned problem, we use the PCW model two-level tree structure to linearize its probability. According to Eqs. (32) and (33) and the parameter setting in Eq. (22), both marginal and conditional probabilities exhibit the independence of irrelevant alternatives, wherein any change in the other routes does not influence the route pair probability ratio under consideration:

$$\frac{P_{kr}^{ij}}{P_{ls}^{ij}}=\frac{\left(1-\sigma_{kr}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}}+\left(\mathrm{g}_{\mathrm{r}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}}\right]^{1-\sigma_{kr}^{ij}}}{\left(1-\sigma_{ls}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{l}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}}+\left(\mathrm{g}_{s}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}}\right]^{1-\sigma_{ls}^{ij}}}\ ,\forall k,r,l,s\in R_{ij},ij\in IJ, \tag{35}$$

$$\frac{P_{k|kr}^{ij}}{P_{r|kr}^{ij}}=\frac{\left(\mathrm{g}_{\mathrm{k}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ijkr}}}}{\left(\mathrm{g}_{\mathrm{r}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{kr}^{ij}}}}\ ,\forall k,r\in R_{ij},ij\in IJ. \tag{36}$$

Similar to the MNL-based and MNW-based models, all possible pairings must be considered as a constraint. The binary variable $x_n$ is used to ensure the logic of each choice probability ratio. The following constraints are used to replace Eqs. (7)–(13) in the MNL-based model to clarify the PCW travel choice behavior.

#### 4.2.1 Marginal probability

First, we consider the case of the marginal probability. The conservation constraint can be expressed as

$$\sum_{l=1}^{|R_{ij}|-1}\sum_{s=l+1}^{|R_{ij}|}P_{ls}^{ij}=1\ ,\forall l,s\in R_{ij},ij\in IJ, \tag{37}$$

in which all marginal probabilities lie between 0 and 1:

$$0\le P_{ls}^{ij}\le 1\ ,\forall l,s\in R_{ij},ij\in IJ. \tag{38}$$

All the marginal probabilities are greater than zero, except that associated with a transit route pair. The marginal probability for a transit route pair can be zero when all corresponding P&R facility potential sites are closed. In this case, the logic and relationship through the binary variable are introduced:

$$P_{k(n)r(m)}^{ij} \leq x_n + x_m, \forall k(n) \neq r(m) \in R_{ij}^{T}, n, m \in N, ij \in IJ. \tag{39}$$

Equation (39) ensures that $P_{k(n)r(m)}^{ij}$ is zero when the corresponding P&R sites $n$ and $m$ are closed. If one (or both) corresponding P&R site(s) is (are) opened, Equation (38) governs the probability value. All possible pairings are shown in Figure 5, and all corresponding equations are presented in **Appendix A**. These constraints include the following pairings: 1) Auto–Auto Auto–Auto 2) Auto–Auto Transit–Auto 3) Auto–Auto Transit–Transit 4) Transit–Auto Auto–Auto 5) Transit–Auto Transit–Auto 6) Transit–Auto: Transit–Transit 7) Transit–Transit Auto–Auto 8) Transit–Transit Transit–Auto and 9) Transit–Transit Transit–Transit

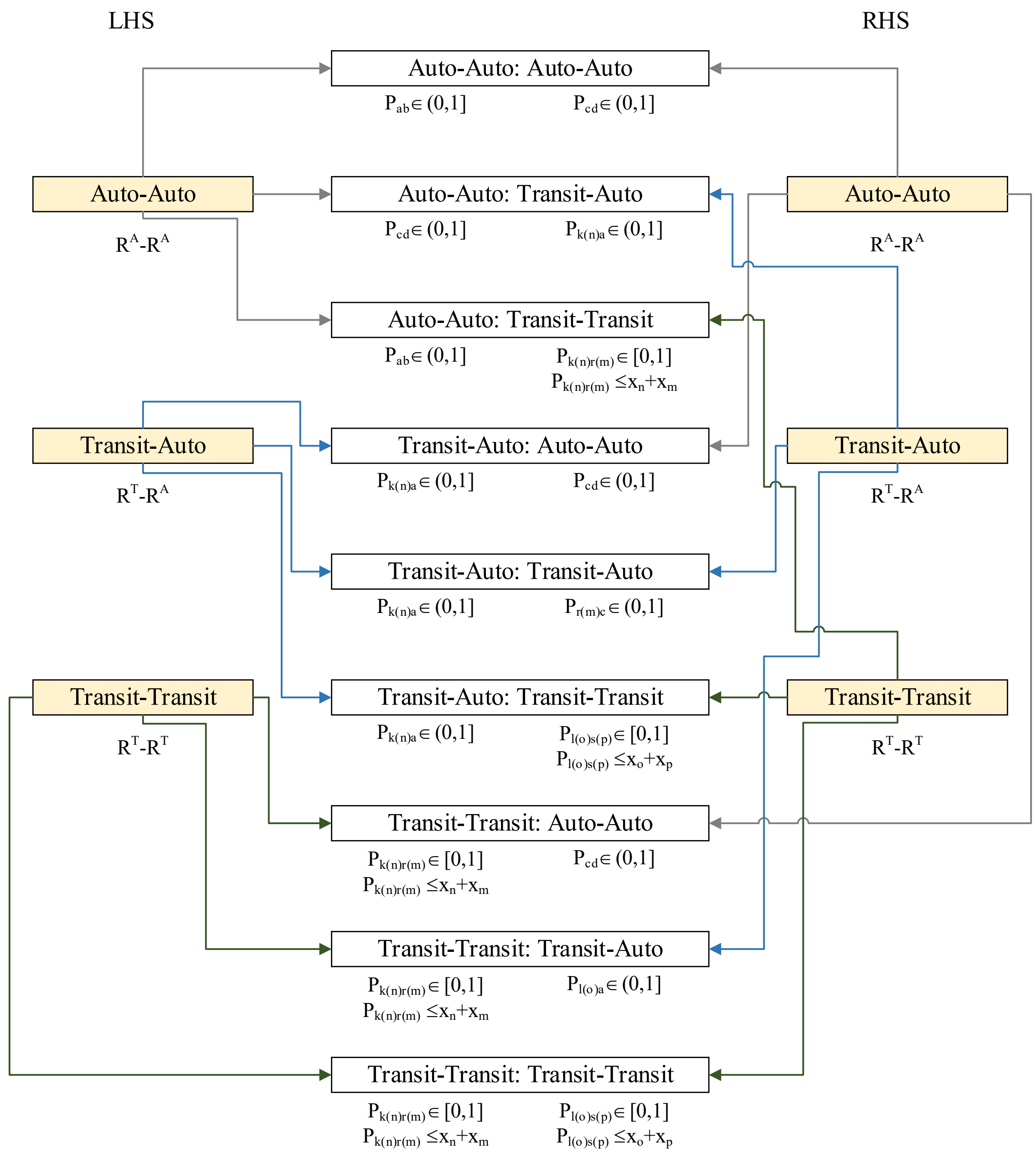



Figure 5: Logic and relationship for all possible pairings at the marginal probability level (index *ij* is omitted for simplicity)

All possible pairings according to the two-level tree structure for the marginal probability are summarized in Table 2. The main difference between each constraint pertains to the probability ratio on the right-hand side (RHS). The numerator and denominator are considered to cover all possible circumstances for each pairing. The terms "$+x_n$" and "$+(1-x_n)$" are used to ensure the logic of each choice probability ratio with every possible circumstance that includes the transit route. This configuration helps ensure that the marginal probability on the left-hand side is in the range of zero and one.

Table 2: Constraints based on the PCW marginal probability

| Pairing | Numerator on the RHS | | | | Denominator on the RHS | | | | Logical term | Eq. |
|---|---|---|---|---|---|---|---|---|---|---|
| | Auto | Auto | Transit [$n$] | Transit [$m$] | Auto | Auto | Transit [$o$] | Transit [$p$] | | |
| Auto–Auto: Auto–Auto | • | • | | | • | • | | | - | (48) |
| Auto–Auto: Transit–Auto | • | • | | | • | | • | | $(1-x_o)$ | (49) |
| | • | • | | | • | | | | $x_o$ | (50) |
| Auto–Auto: Transit–Transit | • | • | | | | | • | • | $(1-x_o)+(1-x_p)$ | (51) |
| | • | • | | | | | • | | $(1-x_o)+x_p$ | (52) |
| Transit–Auto: Auto–Auto | • | | • | | • | • | | | $(1-x_n)$ | (53) |
| | • | | | | • | • | | | $x_n$ | (54) |
| Transit–Auto: Transit–Auto | • | | • | | • | | • | | $(1-x_n)+(1-x_o)$ | (55) |
| | • | | • | | • | | | | $(1-x_n)+x_o$ | (56) |
| | • | | | | • | | • | | $x_n+(1-x_o)$ | (57) |
| | • | | | | • | | | | $x_n+x_o$ | (58) |

| Pairing | Numerator on the RHS | | | | Denominator on the RHS | | | | Logical term | Eq. |
|---|---|---|---|---|---|---|---|---|---|---|
| | Auto | Auto | Transit $[n]$ | Transit $[m]$ | Auto | Auto | Transit $[o]$ | Transit $[p]$ | | |
| Transit–Auto: Transit–Transit | • | | • | | | | • | • | $(1-x_n)+(1-x_o)+(1-x_p)$ | (59) |
| | • | | • | | | | • | | $(1-x_n)+(1-x_o)+x_p$ | (60) |
| | • | | | | | | • | • | $x_n+(1-x_o)+(1-x_p)$ | (61) |
| | • | | | | | | • | | $x_n+(1-x_o)+x_p$ | (62) |
| Transit–Transit: Auto–Auto | | | • | • | • | • | | | $(1-x_n)+(1-x_m)$ | (63) |
| | | | • | | • | • | | | $(1-x_n)+x_m$ | (64) |
| Transit–Transit: Transit–Auto | | | • | • | • | | • | | $(1-x_n)+(1-x_m)+(1-x_o)$ | (65) |
| | | | • | • | • | | | | $(1-x_n)+(1-x_m)+x_o$ | (66) |
| | | | • | | | | | | $(1-x_n)+x_m+x_o$ | (67) |
| Transit–Transit: Transit–Transit | | | • | • | | | • | • | $(1-x_n)+(1-x_m)+(1-x_o)$ $+(1-x_p)$ | (68) |
| | | | • | • | | | • | | $(1-x_n)+x_m+(1-x_o)+x_p$ | (69) |
| | | | • | | | | • | • | $(1-x_n)+x_m+(1-x_o)$ $+(1-x_p)$ | (70) |
| | | | • | | | | • | | $(1-x_n)+x_m+(1-x_o)+x_p$ | (71) |

4.2.2 Conditional probability

Next, the pairing between the conditional probabilities is considered. First, the probability is bounded between zero and one:

$$0 \leq P^{ij}_{l|ls} \leq 1, \forall l \neq s \in R_{ij}, ij \in IJ. \tag{40}$$

The relationship between the conditional probability and $x_n$ can be expressed as

$$P^{ij}_{k(n)|k(n)l} \leq x_n, \forall k(n) \in R^T_{ij}, n \in N, l \in R_{ij}, ij \in IJ, \tag{41}$$

Generally, the summation of each pair of the conditional probability should equal one. The summation between $P^{ij}_{k(n)|k(n)r(m)}$ and $P^{ij}_{r(m)|k(n)r(m)}$ depends on the binary variables. If only one site is opened, the summation must equal one. If both sites are closed, the summation must be zero. In this scenario, the logic and relationship can be presented as

$$P^{ij}_{k(n)|k(n)r(m)} + P^{ij}_{r(m)|k(n)r(m)} \leq x_n + x_m, \forall k(n) \neq r(m) \in R^T_{ij}, n, m \in N, ij \in IJ, \tag{42}$$

$$P^{ij}_{k(n)|k(n)r(m)} + P^{ij}_{r(m)|k(n)r(m)} \geq x_n, \forall k(n) \neq r(m) \in R^T_{ij}, n, m \in N, ij \in IJ, \tag{43}$$

$$P^{ij}_{k(n)|k(n)r(m)} + P^{ij}_{r(m)|k(n)r(m)} \geq x_m, \forall k(n) \neq r(m) \in R^T_{ij}, n, m \in N, ij \in IJ. \tag{44}$$

All possible pairings of the PCW conditional probability are shown in Figure 6, and all corresponding equations are presented in **Appendix B**. The constraints include those of 1) Auto: Auto, 2) Auto: Transit, 3) Transit: Auto, and 4) Transit: Transit. The term "$(1 - x_n)$" is added to the constraint with the transit route on the RHS. As in the MNL-based and MNW-based models, this term helps ensure that the conditional probability is in the range of zero and one, even if the site on the RHS is closed.

According to the marginal and conditional probabilities, the PCW probability can be computed using Eq. (31). To linearize this equation, we reformulate the problem as follows. Let $y_{ijkr}$ and $z_{ijkr}$ be a continuous variable, where

$$y_{ijkr} = \frac{1}{2}\left(P^{ij}_{kr} + P^{ij}_{k|kr}\right), \forall k \neq r \in R_{ij}, ij \in IJ, \tag{45}$$

$$z_{ijkr} = \frac{1}{2}(P^{ij}_{kr} - P^{ij}_{k|kr}), \quad \forall k \neq r \in R_{ij}, ij \in IJ, \tag{46}$$

In this case,

$$P^{ij}_{kr} P^{ij}_{k|kr} = \left(y_{ijkr}\right)^2 - \left(z_{ijkr}\right)^2, \quad \forall k \neq r \in R_{ij}, ij \in IJ, \tag{47}$$

Both $\left(y_{ijkr}\right)^2$ and $\left(z_{ijkr}\right)^2$ can be determined through SOS2. The use of SOS2 in transportation analysis has been reported by Luathep et al. (2011).

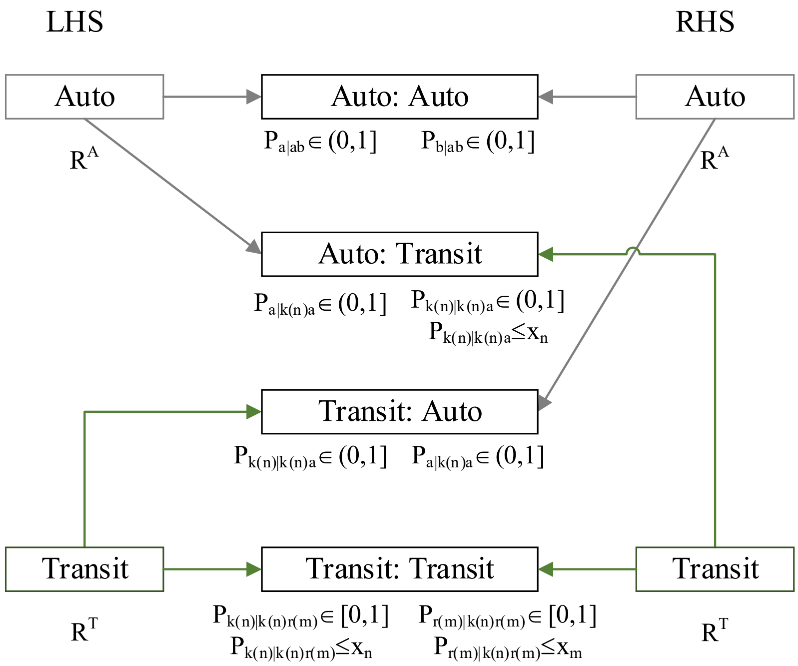



Figure 6: Logic and relationship for all possible pairings at the conditional probability level (index *ij* is omitted for simplicity)

**Proposition 1.** The MILP formulation in Eqs. (2)–(4) and Eqs. (37)–(67) generates the maximum number of P&R users under the PCW travel choice behavior.

**Proof.** Assume that there exist at least two routes connecting each O–D pair. For simplicity, three cases are considered: (a) all P&R potential sites between an O–D pair *ij* are closed, (b) only one P&R potential site between an O–D pair *ij* is open, and (c) all P&R potential sites between an O–D pair *ij* are open. Detailed information can be found in **Appendix C**.

(a) All P&R potential sites between an O–D pair *ij* are closed. In this case, the binary variables $x_n = 0$ and $x_m = 0$. Only one equation from each pairing category is active. The other cases are governed by Eqs. (38) and (40). The equality for the marginal probability can be expressed as follows:

$$P_{ab}^{ij} = \frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}} P_{cd}^{ij},\ \forall a \neq b, c \neq d \in R_{ij}^{A}, ij \in IJ.$$

$$P_{k(n)r(m)}^{ij} = 0\ , \forall k(n) \neq r(m) \in R_{ij}^{T}, n, m \in N, ij \in IJ.$$

$$P_{cd}^{ij} = \frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{k(n)a}^{ij},$$

$$\forall k(n) \in R_{ij}^{T}, n \in N, a, c \neq d \in R_{ij}^{A}, ij \in IJ, \text{ and}$$

$$P_{k(n)a}^{ij} = \frac{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(g_c^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)c}^{ij}, \forall k(n), l(o) \in R_{ij}^{T}, n, o \in N, a, c \in R_{ij}^{A}, ij \in IJ.$$

According to this expression,

$$P_{ab}^{ij} \frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}} + ..$$

$$+ P_{ab}^{ij} \frac{\left(g_e^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}} + .. = 1 .$$

The marginal probability for the auto–auto pair can be expressed as

$$P_{ab}^{ij} = \frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\sum_{l=1}^{|R_{ij}|-1}\sum_{s=l+1}^{|R_{ij}|}\left(1-\sigma_{ls}^{ij}\right)\left[\left(g_l^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}}+\left(g_s^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}}\right]^{1-\sigma_{ls}^{ij}}}, \forall a \neq b \in R_{ij}^{A}, ij \in IJ.$$

Similarly, the marginal probability for transit–auto can be expressed as

$$P_{k(n)a}^{ij} = \frac{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\sum_{l=1}^{|R_{ij}|-1}\sum_{s=l+1}^{|R_{ij}|}\left(1-\sigma_{ls}^{ij}\right)\left[\left(g_l^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}}+\left(g_s^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}}\right]^{1-\sigma_{ls}^{ij}}},$$

$$\forall k(n) \in R_{ij}^{T}, n \in N, a \in R_{ij}^{A}, ij \in IJ.$$

The expressions for the other pairs can be derived in a similar manner, consistent with Eq. (32).

The conditional probability can be expressed as

$$P_{a|ab}^{ij} = \frac{\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}{\left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}} P_{b|ab}^{ij} \quad, \forall a, b \in R_{ij}^{A}, ij \in IJ,$$

$P_{a|k(n)a}^{ij} = 1,\ \forall k(n) \in R_{ij}^{T}, n \in N, a \in R_{ij}^{A}, ij \in IJ,$

$P_{k(n)|k(n)a}^{ij} = 0, \forall k(n) \in R_{ij}^{T}, n \in N, a \in R_{ij}^{A}, ij \in IJ,$ and

$P_{k(n)|k(n)r(m)}^{ij} = 0, \forall k(n) \neq r(m) \in R_{ij}^{T(n)}, n, m \in N, ij \in IJ.$

In this case,

$$P_{a|ab}^{ij} + P_{a|ab}^{ij} \frac{\left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}{\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}} = 1$$

$$P_{a|ab}^{ij} = \frac{\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}{\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}} + \left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}, \forall a, b \in R_{ij}^{A}, ij \in IJ.$$

According to Eq. (31), the abovementioned equality constraints of the marginal and conditional probabilities yield the PCW model defined in Eq. (34).

(b) Only one P&R potential site between an O–D pair *ij* is open, i.e., $x_n = 1$. The following equations can be derived, i.e.,

*Marginal probability*

$$P_{ab}^{ij} = \frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}} + \left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}} + \left(g_d^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}} P_{cd}^{ij},\ \forall a \neq b, c \neq d \in R_{ij}^{A}, ij \in IJ.$$

$$P_{cd}^{ij} = \frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}} + \left(g_d^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(g_{k(n)}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}} + \left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}} P_{k(n)a}^{ij},$$

$$\forall k(n) \in R_{ij}^{T}, n \in N, a, c \neq d \in R_{ij}^{A}, ij \in IJ,$$

$$P_{k(n)r(m)}^{ij} = \frac{\left(g_{k(n)}^{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}} + \left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}} P_{ab}^{ij},$$

$$\forall k(n) \neq r(m) \in R_{ij}^T, n, m \in N, a, b \in R_{ij}^A, ij \in IJ,$$

$$P_{k(n)a}^{ij} = \frac{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(g_{k(n)}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}} + \left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}}{\left(g_c^{ij} - \zeta_{ij}\right)^{-\beta_{ij}}} P_{cl(o)}^{ij},$$

$$\forall k(n), l(o) \in R_{ij}^T, n, o \in N, a, c \in R_{ij}^A, ij \in IJ,$$

$$P_{k(n)r(m)}^{ij} = \frac{\left(g_{k(n)}^{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}{\left(g_a^{ij} - \zeta_{ij}\right)^{-\beta_{ij}}} P_{al(o)}^{ij}, \forall k(n) \neq r(m), l(o) \in R_{ij}^T, n, m, o \in N, a \in R_{ij}^A, ij \in IJ.$$

*Conditional probability*

$$P_{a|ab}^{ij} = \frac{\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}{\left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}} P_{b|ab}^{ij}, \forall a, b \in R_{ij}^A, ij \in IJ,$$

$$P_{a|k(n)a}^{ij} = \frac{\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}}{\left(g_{k(n)}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}} P_{k(n)|k(n)a}^{ij}, \forall k(n) \in R_{ij}^T, n \in N, a \in R_{ij}^A, ij \in IJ,$$

$$P_{k(n)|k(n)r(m)}^{ij} = 1, \forall k(n), r(m) \in R_{ij}^T, n \neq m \in N, ij \in IJ.$$

Following a similar procedure as in the previous scenario, we can conclude that the PCW choice behavior pertains to Eq. (34).

(c) All P&R potential sites between an O–D pair *ij* are open. The following equalities can be derived:

$$P_{ab}^{ij} = \frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}} + \left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}} + \left(g_d^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}} P_{cd}^{ij}, \quad \forall a \neq b, c \neq d \in R_{ij}^A, ij \in IJ.$$

$$P_{cd}^{ij} = \frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(\mathrm{g}_{c}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(\mathrm{g}_{\mathrm{d}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}+\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}} P_{k(n)a}^{ij},$$

$$\forall k(n) \in R_{ij}^{T}, n \in N, a, c \neq d \in R_{ij}^{A}, ij \in IJ,$$

$$P_{ab}^{ij} = \frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(\mathrm{g}_{a}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(\mathrm{g}_{\mathrm{b}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(1-\sigma_{k(n)r(m)}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}+\left(\mathrm{g}_{\mathrm{r(m)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}} P_{k(n)r(m)}^{ij},$$

$$\forall k(n) \neq r(m) \in R_{ij}^{T}, n, m \in N, a \neq b \in R_{ij}^{A}, ij \in IJ,$$

$$P_{k(n)a}^{ij} = \frac{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}+\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}}{\left(1-\sigma_{l(o)c}^{ij}\right)\left[\left(\mathrm{g}_{c}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)c}^{ij}}}+\left(\mathrm{g}_{\mathrm{l(o)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)c}^{ij}}}\right]^{1-\sigma_{l(o)c}^{ij}}} P_{l(o)c}^{ij},$$

$$\forall k(n), l(o) \in R_{ij}^{T}, n, o \in N, a, c \in R_{ij}^{A}, ij \in IJ,$$

$$P_{l(o)a}^{ij} = \frac{\left(1-\sigma_{l(o)a}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)a}^{ij}}}+\left(\mathrm{g}_{\mathrm{l(o)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)a}^{ij}}}\right]^{1-\sigma_{l(o)a}^{ij}}}{\left(1-\sigma_{k(n)r(m)}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}+\left(\mathrm{g}_{\mathrm{r(m)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}} P_{k(n)r(m)}^{ij},$$

$$\forall k(n) \neq r(m), l(o) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ,$$

$$P_{k(n)r(m)}^{ij} = \frac{\left(1-\sigma_{k(n)r(m)}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}+\left(\mathrm{g}_{\mathrm{r(m)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}}{\left(1-\sigma_{l(o)s(p)}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{l(o)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)s(p)}^{ij}}}+\left(\mathrm{g}_{\mathrm{s(p)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)s(p)}^{ij}}}\right]^{1-\sigma_{l(o)s(p)}^{ij}}} P_{l(o)s(p)}^{ij},$$

$$\forall k(n) \neq r(m), l(o) \neq s(p) \in R_{ij}^{T}, n, m, o, p \in N, ij \in IJ,$$

$$P^{ij}_{a|ab} = \frac{\left(g^{ij}_a - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{ab}}}}{\left(g^{ij}_b - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{ab}}}} P^{ij}_{b|ab}, \forall a, b \in R^{A}_{ij}, ij \in IJ,$$

$$P^{ij}_{a|k(n)a} = \frac{\left(g^{ij}_a - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)a}}}}{\left(g^{ij}_{k(n)} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)a}}}} P^{ij}_{k(n)|k(n)a}, \forall k(n) \in R^{T}_{ij}, n \in N, a \in R^{A}_{ij}, ij \in IJ,$$

$$P^{ij}_{k(n)|k(n)r(m)} = \frac{\left(g^{ij}_{k(n)} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)r(m)}}}}{\left(g^{ij}_{r(m)} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)r(m)}}}} P^{ij}_{r(m)|k(n)r(m)},$$

$$\forall k(n), r(m) \in R^{T(n)}_{ij}, n \neq m \in N, ij \in IJ,$$

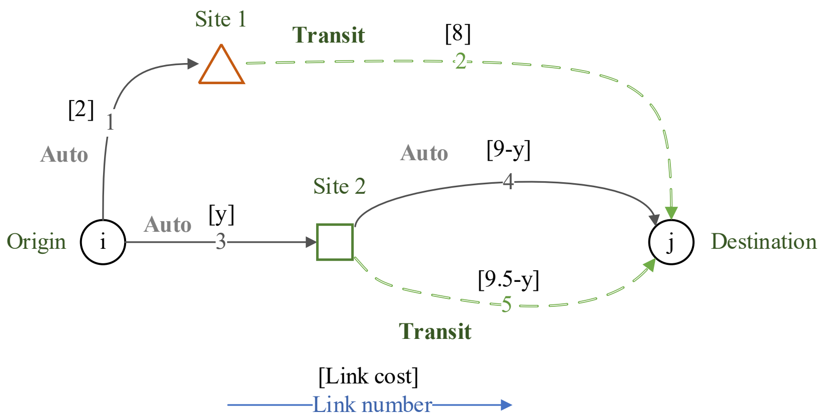
Following a similar procedure as in the previous cases, it can be concluded that the PCW choice behavior pertains to Eq. (34). Thus, the MILP defined in Eqs. (2)–(4) and Eqs. (37)–(67) generates the maximum number of P&R users under the PCW travel choice behavior. This completes the proof. □

## 5. Numerical examples

Two numerical examples are presented to illustrate several features of the proposed PCW-based p-Hub location model for determining the P&R facility location. The results are compared with those obtained using two existing maximum capture problem for determining P&R facility locations, namely the MNL-based model (Aros-Vera et al., 2013) and MNW-based model (Kitthamkesorn et al., 2021). For comparison, the parameters are set as $\theta = 0.1$, $\beta_{ij} = 3.7$, $\zeta_{ij} = 0$, and $\kappa_{ij} = 1$ , unless specified otherwise. IBM-ILOG CPLEX 12.10.0 is used to solve the problem.

### 5.1 Modified loop-hole network

A modified loop-hole network, as shown in Figure 7, is used to compare the solution with those obtained using existing models. This network has one O–D pair, two P&R facility potential sites, and three routes. The O–D demand is 1,000. The P&R facility potential sites are on routes 1 and 2. Route 1 is a truly independent route with a travel cost of 10. Routes 2 and 3 are shorter (i.e., travel cost of 9.5 and 9, respectively) than route 1 with an overlapping section $y$. Route 3 is an auto route.




| Route | Link |
|---|---|
| 1 | 1,2 |
| 2 | 3,5 |
| 3 | 3,4 |

Figure 7: Modified loop-hole network

Figure 8 depicts the result of each model at $p$ = 1. Both MNL-based and MNW-based models select site 2 as the location of the P&R facility, without considering the overlapping section $y$. This is because the independently distributed assumption is embedded in the MNL and MNW models. The result difference between the MNL-based and MNW-based models is caused by the identically distributed assumption included in the MNL model. On the other hand, the PCW model can relax both independently and identically distributed assumption as presented in Section 3.3. The PCW-based model produces results according to the overlapping section $y$ and its overall trip length. At $y = 0$ (no overlap), the PCW-based model selects site 2 as the location of the P&R facility, with the same number of P&R users as in the MNW-based model. As $y$ increases, the two lower routes are increasingly similar, and the number of P&R users at site 2 decreases accordingly. As a result, the P&R facility is located at site 1 when $y >$ 5.89 (i.e., route 2 is no longer attractive owing to the overlapping section).

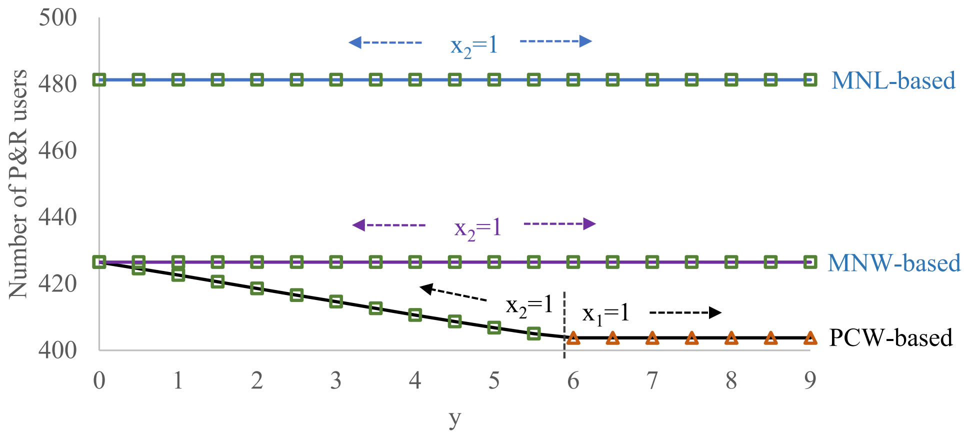



Figure 8: Location of the P&R facility for the modified loop-hole network

The number of P&R users in site 2 decreases as $y$ increases. This phenomenon occurs because $P_{23}^{ij}$ and $P_{2|23}^{ij}$ are smaller for a higher $\sigma_{23}^{ij}$, as presented in Figure 9. At $y = 5.89$, the probability of selecting route 2 is less than that of selecting route 1, as an independent route. Thus, the PCW-based model selects site 1 as the location of the P&R facility when $y \geq 5.89$. The PCW marginal and conditional probabilities remain the same for $y \geq 5.89$ as $x_2 = 0$ and routes 1 and 3 are independent. When $\rho_{kr}^{ij} = 1$, the PCW model collapses to the MNW model.

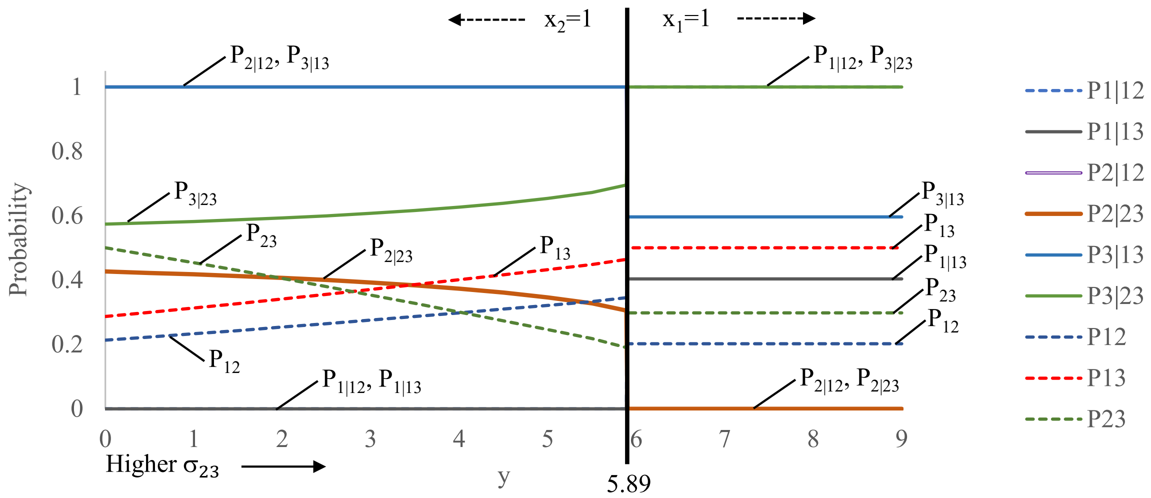



Figure 9: PCW marginal and conditional probabilities for the modified loop-hole network (index $ij$ is omitted for simplicity)

Subsequently, we investigate the influence of the parameter $\beta_{ij}$ on the solution. The interchanging point ($I_p$) from site 2 to site 1 is shown in Figure 10. For a smaller $\beta_{ij}$, a higher perception variance is obtained for each route (according to Eq. (17)). Travelers are assumed to be less sensitive to the travel cost difference. As $\beta_{ij}$ increases, $I_p$ increases. A larger $\beta_{ij}$ generates a smaller route-specific perception variance. Travelers are more sensitive to the route travel cost difference (Kitthamkesorn and Chen, 2013; 2014).

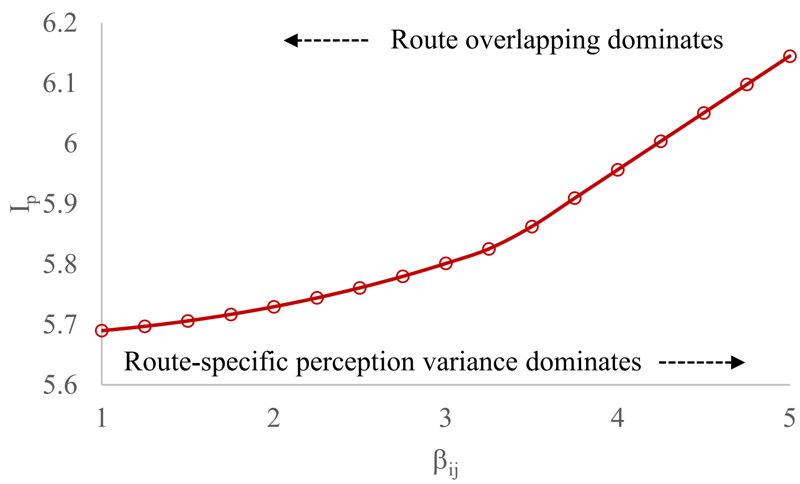


Figure 10: Interchanging point for the P&R site selection

## 5.2 Sioux Falls network

The Sioux Falls network shown in Figure 11 is considered to investigate the influence of the transportation policy. This network contains 24 nodes (all of which are origins and destinations) and 76 links. We assume that there are two light rail transit (LRT) lines: the North–South line and East–West line. Each node on the LRT system is a potential P&R facility site. The impact of fare structure as a significant component of the transit level of service (De Ona and De Ona, 2015) is investigated. Two transit fare structures are considered. The flat-rate scheme charges each transit user 10 units for one trip. The distance-based scheme has an entrance fee of 7.50 units and a distance fee of 0.20 unit for each unit of length traveled.

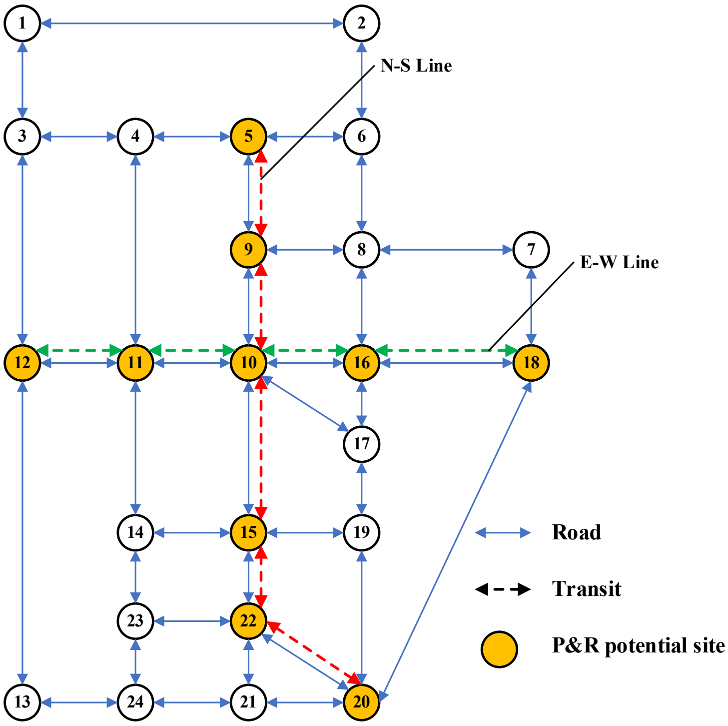
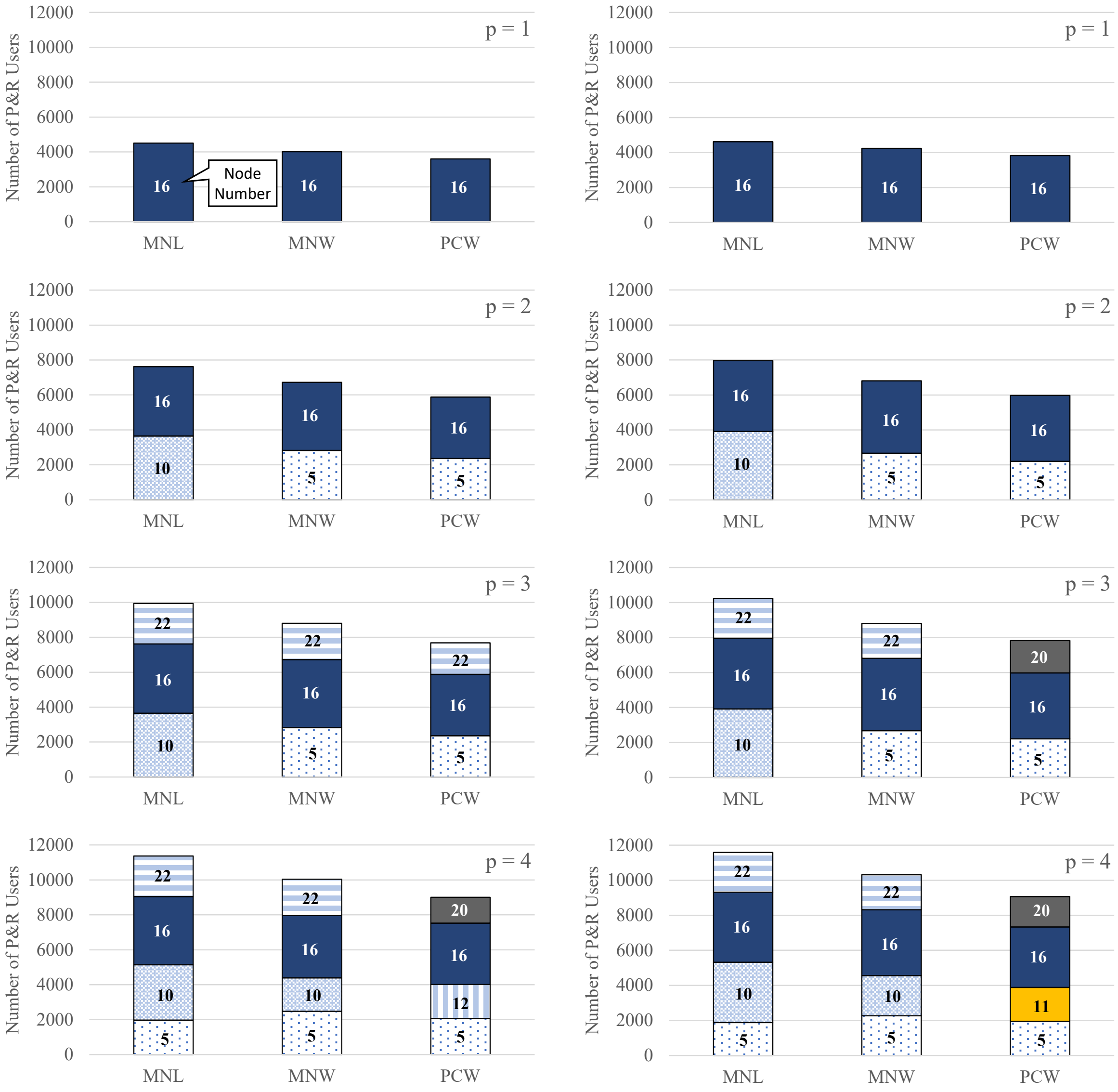



Figure 11: Sioux Falls network

The P&R facility location results for the two fare structures are presented in Figure 12. The number of P&R users increases with $p$. Each model yields different optimal P&R facility locations. The PCW-based model appears to be highly sensitive to the transit fare scheme. This phenomenon occurs because the PCW model considers both route overlapping and route-specific perception variance. In contrast, the MNL model cannot consider any aspect, and the MNW model cannot consider route overlapping.




a) Flat-rate scheme

b) Distance-based scheme

Figure 12: P&R facility locations under two transit fare schemes for the Sioux Falls network

Furthermore, the impact of the PCW-based model parameter $\kappa_{ij}$ on the P&R facility location ($p$ = 4) is investigated as shown in Figure 13. As $\kappa_{ij}$ increases, the number of P&R users decreases. This phenomenon occurs because most auto-only routes are shorter than the transit routes. Although the flat-rate scheme yields almost the same P&R facility location combinations, the distance-based scheme exhibits different P&R facility location combinations for larger $\kappa_{ij}$ values. It is a fact that the flat-rate scheme incentivizes users to engage in longer journeys, while the distance-based scheme discourages the extent of travel on the public transit. The distance-based scheme would impact the PCW choice probability more when changing $\kappa_{ij}$. In terms of the route overlapping, we can consider Eq. (22), i.e., $\sigma_{kr}^{ij} = \left(\frac{l_{ijkr}}{\sqrt{L_{ijk}L_{ijr}}}\right)^{\kappa_{ij}}$. While the flat-rate scheme considered the transit network, the transit route overlapping section in the distance-based scheme will be changed significantly according to the P&R location selection. When $\kappa_{ij}$ increases, this impact increases and the change is more obvious. In terms of the heterogeneous perception variance, the transit travel cost under the distance-based scheme will significantly change according to the P&R location selection as well.

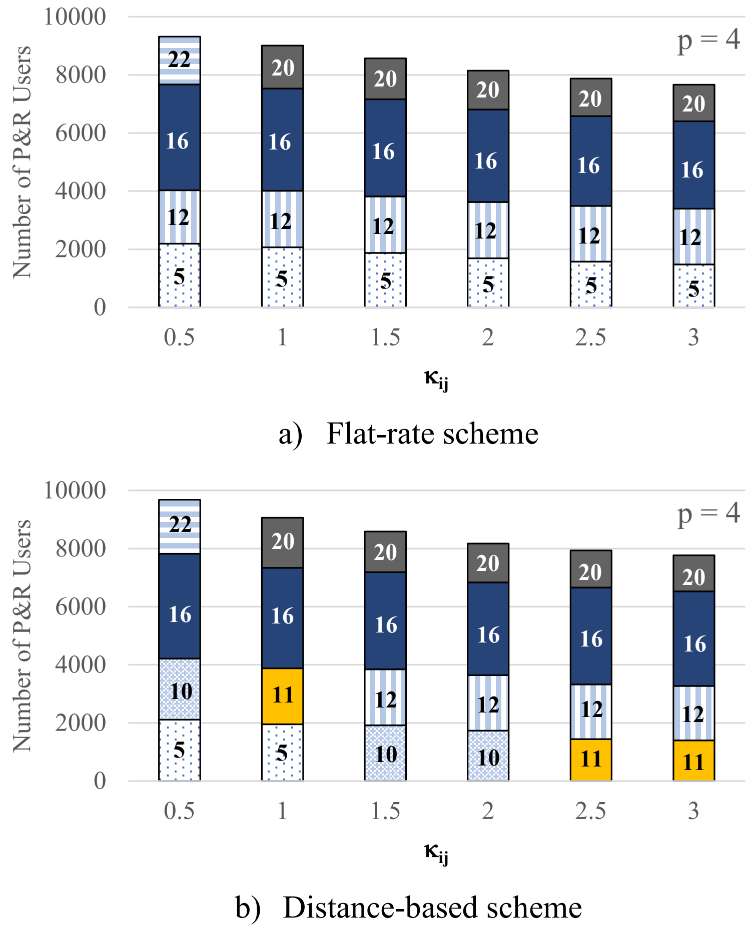
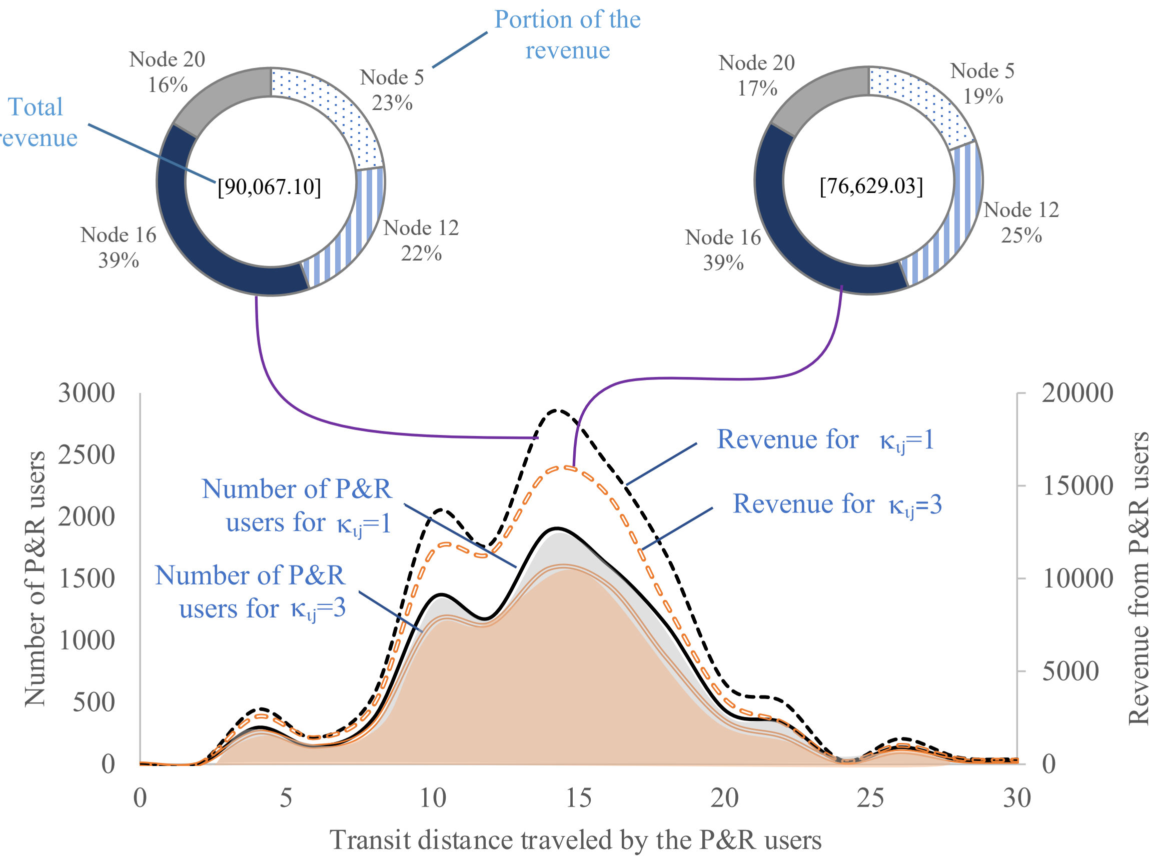
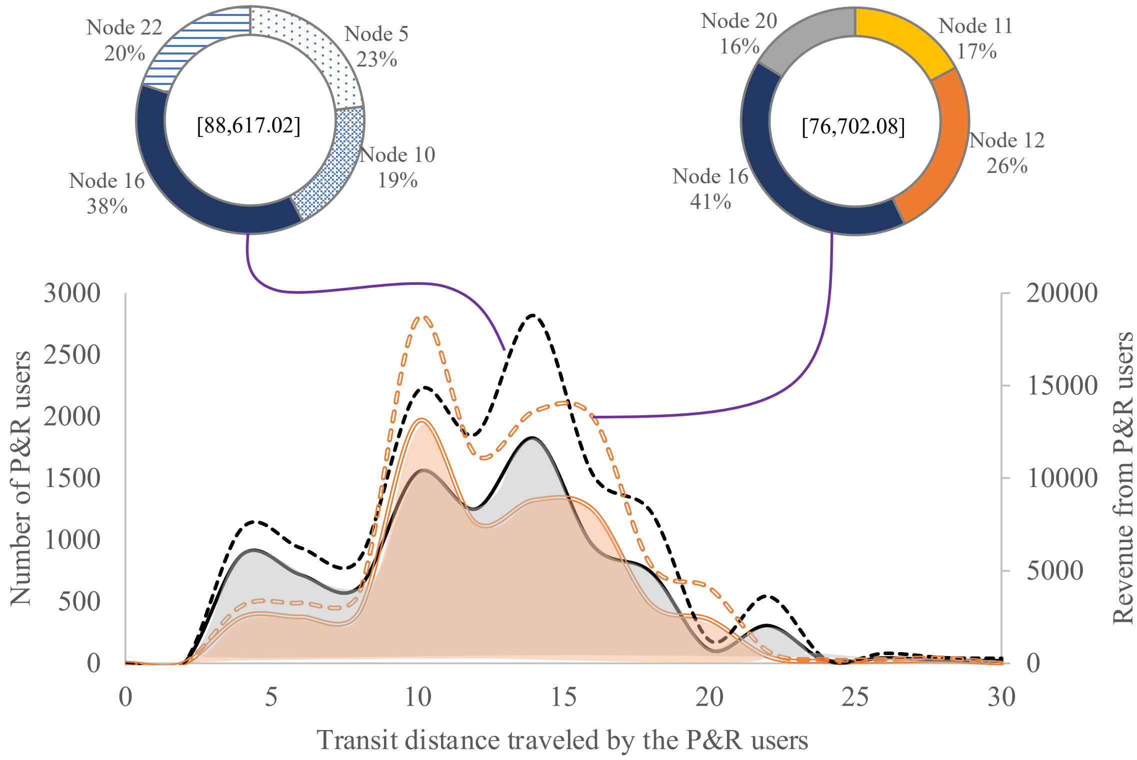



Figure 13: Impact of $\kappa_{ij}$ on the P&R facility locations for $p$=4 under two transit fare schemes

The two charging schemes correspond to different distributions of the number of P&R users with respect to the transit distance traveled as shown in Figure 14. The flat-rate scheme encourages users to travel farther, whereas the distance-based scheme discourages the travel distance in transit. In terms of the revenue distribution, the flat-rate scheme produces a similar pattern for $\kappa_{ij}$ = 1 and $\kappa_{ij}$ = 3. Although the total revenue is higher in the case of $\kappa_{ij}$ = 1, the distance-based scheme yields considerably different revenue patterns. As $\kappa_{ij}$ increases, the P&R users seem to travel less on the transit system. In addition, the revenue portion from the P&R facility location shifts toward the terminal station for which the overlapped route is smaller. Note that the variation in the number of P&R users appears to be comparable between the flat-rate and distance-based pricing models. This is due to the network topology of Sioux Falls. This transportation network exhibits marginal difference in travel demand between the central region (proximate to node 10) and the peripheral area (encompassing nodes 1, 2, 3, 6, 13, 20, 21, and 24). The implementation of a flat-rate pricing scheme yielded a P&R facility location solution in the outer area, while the utilization of a distance-based pricing scheme resulted in a location solution in the center. This aligns with the engagement of each pricing scheme.




a) Flat-rate scheme




b) Distance-based scheme

Figure 14: Transit distance traveled by the P&R users and the corresponding revenue for $p$=4 under two transit fare schemes

## 6. Conclusions and suggestions

This study developed a mixed integer linear programming (MILP) to relax the independently and identically distributed (IID) assumption for determining the park-and-ride (P&R) facility location in a hypernetwork. The paired combinatorial weibit (PCW) model was provided by integrating a Generalized Extreme Value model and the Weibull random error. The route overlapping can be accounted for through the GEV model, and the heterogeneous perception variance can be considered using the Weibull distribution. The PCW model's two-level tree structure was adopted to linearize its probability. The ratio of both PCW marginal and conditional probabilities and the binary location variable are combined to present a linear constraint set in a maximum capture problem. Numerical examples revealed a significant impact of the route overlapping and route-specific perception variance on the optimal P&R facility location. The examples also showed that the public transit fare structure influences the P&R facility location solution. The variation in the distance-based fare scheme was considerably influenced by the degree of route overlapping and heterogeneous perception variance.

The proposed PCW-based MILP involves certain limitations. As the PCW model considers route overlapping through a route-pair combination, the number of constraints associated with the marginal and conditional probabilities increases exponentially with the number of P&R facility potential sites. Future works can focus on extending the model to consider the budget and capacity constraints of the P&R facility, developing conic reformulation (e.g., Altekin et al., 2021) to consider congestion, and investigating the PCW model as a random utility model (e.g., Koppelman and Wen, 2000) with empirical results to support model predictions and policy decisions.

## Acknowledgements

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**Appendix A**

All possible pairings for the PCW marginal probability.

Auto-Auto: Auto-Auto

Since the auto route is always opened, the equality constraint can be adopted, i.e.,

$$P_{ab}^{ij} = \frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}} P_{cd}^{ij}, \quad \forall a \neq b, c \neq d \in R_{ij}^{A}, ij \in IJ. \tag{48}$$

Auto-Auto: Transit-Auto

Both marginal probabilities $P_{cd}^{ij}$ and $P_{k(n)a}^{ij}$ are always greater than 0. Note that $P_{k(n)a}^{ij}$ is conditioned on the P&R facility site *n* whether it is selected to open or not. Further, $\left(g_l^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}} > 0$ for all routes. Then, we have

- Site *n* is opened

$$P_{cd}^{ij} \leq \frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(1-\sigma_{k(o)a}^{ij}\right)\left[\left(g_{k(o)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(o)a}^{ij}}}+\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(o)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}} P_{k(o)a}^{ij} + (1-x_o), \tag{49}$$

$$\forall k(o) \in R_{ij}^{T}, o \in N, a, c \neq d \in R_{ij}^{A}, ij \in IJ,$$

- Site n is not opened

$$P_{cd}^{ij} \leq \frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{k(o)a}^{ij} + x_o, \tag{50}$$

$$\forall k(o) \in R_{ij}^{T}, o \in N, a, c \neq d \in R_{ij}^{A}, ij \in IJ,$$

Note that only one of the above two equations will be active. If $x_o = 1$, Eq. (49) is active and $P_{cd}^{ij}$ in Eq. (50) will be governed by Eq. (38). If $x_n = 0$, Eq. (50) is adopted.

Auto-Auto: Transit-Transit

Unlike the above marginal probabilities, the marginal probability for a transit route pair

$P_{k(n)r(m)}^{ij}$ can be zero. Two cases can be considered, i.e.,

- Both sites are opened

$$P_{ab}^{ij} \le \frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(1-\sigma_{k(o)r(p)}^{ij}\right)\left[\left(g_{k(o)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(o)r(p)}^{ij}}}+\left(g_{r(p)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(o)r(p)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}} P_{k(o)r(p)}^{ij} \tag{51}$$

$$+(1-x_o)+\left(1-x_p\right),$$

$$\forall k(o) \neq r(p) \in R_{ij}^{T}, o, p \in N, a \neq b \in R_{ij}^{A}, ij \in IJ,$$

- One site is not opened

$$P_{ab}^{ij} \le \frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(g_{k(o)}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{k(o)r(p)}^{ij} + (1-x_o) + x_p, \tag{52}$$

$$\forall k(o) \neq r(p) \in R_{ij}^{T}, o, p \in N, a \neq b \in R_{ij}^{A}, ij \in IJ.$$

Eq. (51) and Eq. (52) guarantee that $P_{ab}^{ij} \in [0,1]$ even if $P_{k(o)r(p)}^{ij} = 0$.

<u>Transit-Auto: Auto-Auto</u>

Both $P_{k(n)a}^{ij}$ and $P_{cd}^{ij}$ are always greater than zero. Two cases are considered, i.e.,

- Site *n* is opened

$$P_{k(n)a}^{ij} \le \frac{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}+\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}}{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}} P_{cd}^{ij} + (1-x_n), \tag{53}$$

$$\forall k(n) \in R_{ij}^{T}, n \in N, a, c \neq d \in R_{ij}^{A}, ij \in IJ,$$

- Site *n* is not opened

$$P_{k(n)a}^{ij} \leq \frac{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}} P_{cd}^{ij} + x_n, \tag{54}$$

$$\forall k(n) \in R_{ij}^T, n \in N, a, c \neq d \in R_{ij}^A, ij \in IJ,$$

Transit-Auto: Transit-Auto

Again, both $P_{k(n)a}^{ij}$ and $P_{l(o)c}^{ij}$ are always greater than zero. The binary variables are used to identify whether site *n* and site *o* are open. Four cases are considered, i.e.,

- The site on RHS is opened, and the site on LHS is opened

$$P_{k(n)a}^{ij} \leq \frac{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}+\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}}{\left(1-\sigma_{l(o)c}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)c}^{ij}}}+\left(g_{l(o)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)c}^{ij}}}\right]^{1-\sigma_{l(o)c}^{ij}}} P_{l(o)c}^{ij} \tag{55}$$

$$+(1-x_n)+(1-x_o),$$

$$\forall k(n), l(o) \in R_{ij}^T, n, o \in N, a, c \in R_{ij}^A, ij \in IJ,$$

- The site on RHS is opened, and the site on LHS is not opened

$$P_{k(n)a}^{ij} \leq \frac{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}+\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}}{\left(g_c^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)c}^{ij} \tag{56}$$

$$+(1-x_n)+x_o,$$

$$\forall k(n), l(o) \in R_{ij}^T, n, o \in N, a, c \in R_{ij}^A, ij \in IJ,$$

- The site on RHS is not open, and the site on LHS is opened

$$P_{k(n)a}^{ij} \leq \frac{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{l(o)c}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)c}^{ij}}}+\left(g_{l(o)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)c}^{ij}}}\right]^{1-\sigma_{l(o)c}^{ij}}} P_{l(o)c}^{ij} + x_n + (1-x_o), \tag{57}$$

$$\forall k(n), l(o) \in R_{ij}^T, n, o \in N, a, c \in R_{ij}^A, ij \in IJ,$$

- The site on RHS is not opened, and the site on LHS is not opened

$$P_{k(n)a}^{ij} \leq \frac{\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(\mathrm{g}_{\mathrm{c}}^{\mathrm{ij}}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)c}^{ij} + x_n + x_o, \forall k(n), l(o) \in R_{ij}^{T}, n, o \in N, a, c \in R_{ij}^{A}, ij \in IJ, \quad (58)$$

<u>Transit-Auto: Transit-Transit</u>

There are four cases as follows.

- The site on LHS is opened, and both site on RHS is opened

$$P_{l(n)a}^{ij} \leq \frac{\left(1-\sigma_{l(n)a}^{ij}\right)\left[\left(g_{a}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(n)a}^{ij}}}+\left(g_{l(n)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(n)a}^{ij}}}\right]^{1-\sigma_{l(n)a}^{ij}}}{\left(1-\sigma_{k(o)r(p)}^{ij}\right)\left[\left(g_{k(o)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(o)r(p)}^{ij}}}+\left(g_{r(p)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(o)r(p)}^{ij}}}\right]^{1-\sigma_{k(o)r(p)}^{ij}}} P_{k(o)r(p)}^{ij} + \quad (59)$$

$$(1-x_n) + (1-x_o) + \left(1-x_p\right),$$

$$\forall k(o) \neq r(p), l(n) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ,$$

- The site on LHS is opened, and a site on RHS is not opened

$$P_{l(n)a}^{ij} \leq \frac{\left(1-\sigma_{l(n)a}^{ij}\right)\left[\left(g_{a}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(n)a}^{ij}}}+\left(g_{l(n)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(n)a}^{ij}}}\right]^{1-\sigma_{l(n)a}^{ij}}}{\left(g_{k(o)}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{k(o)r(p)}^{ij} \quad (60)$$

$$+(1-x_n) + x_o + \left(1-x_p\right),$$

$$\forall k(o) \neq r(p), l(n) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ,$$

- The site on LHS is not opened, and both site on RHS is opened

$$P_{l(n)a}^{ij} \leq \frac{\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{k(o)r(p)}^{ij}\right)\left[\left(g_{k(o)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(o)r(p)}^{ij}}}+\left(g_{r(p)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(o)r(p)}^{ij}}}\right]^{1-\sigma_{k(o)r(p)}^{ij}}} P_{k(o)r(p)}^{ij} \quad (61)$$

$$+x_n + (1-\mathrm{x_o}) + \left(1-\mathrm{x_p}\right),$$

$$\forall k(o) \neq r(p), l(n) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ,$$

- The site on LHS is not opened, and a site on RHS is not opened

$$P_{l(n)a}^{ij} \leq \frac{\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(g_{k(o)}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{k(o)r(p)}^{ij} + x_n + (1-x_o) + x_p, \quad (62)$$

$$\forall k(o) \neq r(p), l(n) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ,$$

Transit-Transit: Auto-Auto

$P_{ab}^{ij}$ is always greater than zero. If site *n* and site *m* are closed, $P_{k(n)r(m)}^{ij}$ on the LHS is equal to zero from Eq. (39). If not, there are two cases, i.e.,

- Both sites are opened

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(1-\sigma_{k(n)r(m)}^{ij}\right)\left[\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}+\left(g_{r(m)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}}{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_{a}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_{b}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}} P_{ab}^{ij} + (1-x_n) + (1-x_m), \tag{63}$$

$$\forall k(n) \neq r(m) \in R_{ij}^{T}, n, m \in N, a \neq b \in R_{ij}^{A}, ij \in IJ,$$

- One site is not opened

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_{a}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_{b}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}} P_{ab}^{ij} + (1-x_n) + x_m, \tag{64}$$

$$\forall k(n) \neq r(m) \in R_{ij}^{T}, n, m \in N, a, b \in R_{ij}^{A}, ij \in IJ,$$

Transit-Transit: Transit-Auto

Similarly, $P_{l(o)a}^{ij}$ is always greater than zero. However, the denominator on the RHS depend on site *o*.

- Both sites on LHS are opened, and the site on RHS is opened

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(1-\sigma_{k(n)r(m)}^{ij}\right)\left[\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}+\left(g_{r(m)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}}{\left(1-\sigma_{l(o)a}^{ij}\right)\left[\left(g_{a}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)a}^{ij}}}+\left(g_{l(o)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)a}^{ij}}}\right]^{1-\sigma_{l(o)a}^{ij}}} P_{l(o)a}^{ij} + (1-x_n) + (1-x_m) + (1-x_o), \tag{65}$$

$$\forall k(n) \neq r(m), l(o) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ,$$

- Both sites on LHS are opened, and the site on RHS is closed

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(1-\sigma_{k(n)r(m)}^{ij}\right)\left[\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}+\left(g_{r(m)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}}{\left(g_{a}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)a}^{ij} + (1-x_n)+(1-x_m)+x_o, \tag{66}$$

$$\forall k(n), r(m), l(o) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ,$$

- One site on LHS is not opened, and the site on RHS is not opened

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(g_{a}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)a}^{ij} + (1-x_n)+x_m+x_o, \tag{67}$$

$$\forall k(n) \neq r(m), l(o) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ,$$

<u>Transit-Transit: Transit-Transit</u>

Both LHS and RHS depend on the site selection. Three cases are considered, i.e.,

- Both sites on RHS are opened, and both sites on LHS are opened

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(1-\sigma_{k(n)r(m)}^{ij}\right)\left[\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}+\left(g_{r(m)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}}{\left(1-\sigma_{l(o)s(p)}^{ij}\right)\left[\left(g_{l(o)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)s(p)}^{ij}}}+\left(g_{s(p)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)s(p)}^{ij}}}\right]^{1-\sigma_{l(o)s(p)}^{ij}}} P_{l(o)s(p)}^{ij} + (1-x_n)+(1-x_m)+(1-x_o)+(1-x_p), \tag{68}$$

$$\forall k(n) \neq r(m), l(o) \neq s(p) \in R_{ij}^{T}, n, m, o, p \in N, ij \in IJ,$$

- One site on RHS is opened, and one site on LHS is not opened

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(g_{l(o)}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)s(p)}^{ij} + (1-x_n)+x_m+(1-x_o)+x_p, \tag{69}$$

$$\forall k(n) \neq r(m), l(o) \neq s(p) \in R_{ij}^{T}, n, m, o, p \in N, ij \in IJ,$$

- One site on RHS is not opened, and both sites on LHS are opened

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{l(o)s(p)}^{ij}\right)\left[\left(g_{l(o)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)s(p)}^{ij}}}+\left(g_{s(p)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)s(p)}^{ij}}}\right]^{1-\sigma_{l(o)s(p)}^{ij}}} P_{l(o)s(p)}^{ij} \tag{70}$$

$$+(1-x_n)+x_m+(1-x_o)+(1-x_p),$$

$$\forall k(n) \neq r(m), l(o) \neq s(p) \in R_{ij}^{T}, n, m, o, p \in N, ij \in IJ,$$

- One site on RHS is not opened, and one site on LHS is not opened

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(g_{l(o)}^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)s(p)}^{ij} \tag{71}$$

$$+(1-x_n)+x_m+(1-x_o)+x_p,$$

$$\forall k(n) \neq r(m), l(o) \neq s(p) \in R_{ij}^{T}, n, m, o, p \in N, ij \in IJ,$$

**Appendix B**

All possible pairings for the PCW conditional probability.

<u>Auto-Auto</u>

$P_{a|ab}^{ij}$ and $P_{b|ab}^{ij}$ are always greater than zero. An equality constraint and a conservation constraint can respectively be expressed, i.e.,

$$P_{a|ab}^{ij} = \frac{\left(g_{a}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}{\left(g_{b}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}} P_{b|ab}^{ij}, \forall a, b \in R_{ij}^{A}, ij \in IJ, \tag{72}$$

$$P_{a|ab}^{ij} + P_{b|ab}^{ij} = 1, \forall a \neq b \in R_{ij}^{A}, ij \in IJ. \tag{73}$$

<u>Auto-Transit</u>

Since $P_{a|k(n)a}^{ij}$ and $P_{k(n)|k(n)a}^{ij}$ depend on $x_n$, the term $(1-x_n)$ is added in the RHS, i.e.,

$$P_{a|k(n)a}^{ij} \leq \frac{\left(g_{a}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}}{\left(g_{k(n)}^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}} P_{k(n)|k(n)a}^{ij} + (1-x_n), \tag{74}$$

$$\forall k(n) \in R_{ij}^{T}, n \in N, a \in R_{ij}^{A}, ij \in IJ,$$

$$P_{k(n)|k(n)a}^{ij} + P_{a|k(n)a}^{ij} = 1, \forall k(n) \in R_{ij}^{T}, n \in N, a \in R_{ij}^{A}, ij \in IJ, \tag{75}$$

Note that if $x_n = 0$, $P_{k(n)|k(n)a}^{ij} = 0$ and $P_{a|k(n)a}^{ij} = 1$.

Transit-Auto

$$P^{ij}_{k(n)|k(n)a} \le \frac{\left(g^{ij}_{k(n)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)a}}}}{\left(g^{ij}_{a}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)a}}}} P^{ij}_{a|k(n)a}, \forall k(n) \in R^{T}_{ij}, n \in N, a \in R^{A}_{ij}, ij \in IJ, \tag{76}$$

The conservation constraint is the same as Eq. (75).

Transit-Transit

$$P^{ij}_{k(n)|k(n)r(m)} \le \frac{\left(g^{ij}_{k(n)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)r(m)}}}}{\left(g^{ij}_{r(m)}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{k(n)r(m)}}}} P^{ij}_{r(m)|k(n)r(m)} + (1-x_m)$$
$$, \forall k(n), r(m) \in R^{T(n)}_{ij}, n \neq m \in N, ij \in IJ, \tag{77}$$

**Appendix C**

**Proof of Proposition 1.** Assuming that there are at least two routes connecting each O-D pair. For simplicity, three cases are considered, including (a) all P&R potential sites between an O-D pair *ij* are not opened, (b) only one P&R potential site between an O-D pair *ij* is opened, and (c) all P&R potential sites between an O-D pair *ij* are opened.

(a) All P&R potential sites between an O-D pair *ij* are not opened. The binary variable $x_n = 0$ and $x_m = 0$. Only one equation from each pairing category would be active. The others would be governed by Eq. (38) and Eq. (40).

Auto-Auto: Auto-Auto

$$P^{ij}_{ab} = \frac{\left(1-\sigma^{ij}_{ab}\right)\left[\left(g^{ij}_{a}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{ab}}}+\left(g^{ij}_{b}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{ab}}}\right]^{1-\sigma^{ij}_{ab}}}{\left(1-\sigma^{ij}_{cd}\right)\left[\left(g^{ij}_{c}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{cd}}}+\left(g^{ij}_{d}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma^{ij}_{cd}}}\right]^{1-\sigma^{ij}_{cd}}} P^{ij}_{cd}, \ \forall a \neq b, c \neq d \in R^{A}_{ij}, ij \in IJ.$$

Auto-Auto: Transit-Auto

$$P_{cd}^{ij} \le \frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{k(n)a}^{ij},$$

$$\forall k(n) \in R_{ij}^T, n \in N, a, c \ne d \in R_{ij}^A, ij \in IJ,$$

Auto-Auto: Transit-Transit

$$P_{ab}^{ij} \in (0,1], \forall a \ne b \in R_{ij}^A, ij \in IJ.$$

Transit-Auto: Auto-Auto

$$P_{k(n)a}^{ij} \le \frac{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}} P_{cd}^{ij},$$

$$\forall k(n) \in R_{ij}^T, n \in N, a, c \ne d \in R_{ij}^A, ij \in IJ,$$

Transit-Auto: Transit-Auto

$$P_{k(n)a}^{ij} \le \frac{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(g_c^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)c}^{ij}, \forall k(n), l(o) \in R_{ij}^T, n, o \in N, a, c \in R_{ij}^A, ij \in IJ,$$

Auto-Transit: Transit-Transit

$$P_{k(n)a}^{ij} \in (0,1], \forall k(n) \in R_{ij}^T, n \in N, a \in R_{ij}^A, ij \in IJ,$$

Transit-Transit: Auto-Auto, Transit-Transit: Transit-Auto, and Transit-Transit: Transit-Transit

$$P_{k(n)r(m)}^{ij} = 0, \forall k(n) \ne r(m) \in R_{ij}^T, n, m \in N, ij \in IJ.$$

From the above equation system, we have equality constraints, i.e.,

$$P_{cd}^{ij} = \frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{k(n)a}^{ij},$$

$$\forall k(n) \in R_{ij}^T, n \in N, a, c \ne d \in R_{ij}^A, ij \in IJ, \text{ and}$$

$$P_{k(n)a}^{ij} = \frac{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(g_c^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)c}^{ij}, \forall k(n), l(o) \in R_{ij}^T, n, o \in N, a, c \in R_{ij}^A, ij \in IJ.$$

Then, we have

$$P_{ab}^{ij}\frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(g_d^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}+..$$

$$+P_{ab}^{ij}\frac{\left(g_e^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}+..=1\ .$$

The marginal probability for auto-auto pair can be expressed as

$$P_{ab}^{ij}=\frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\sum_{l=1}^{\left|R_{ij}\right|-1}\sum_{s=l+1}^{\left|R_{ij}\right|}\left(1-\sigma_{ls}^{ij}\right)\left[\left(g_l^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}}+\left(g_s^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}}\right]^{1-\sigma_{ls}^{ij}}}\ ,\forall a\neq b\in R_{ij}^{A}, ij\in IJ.$$

Similarly, the marginal probability for auto-transit can be expressed as

$$P_{k(n)a}^{ij}=\frac{\left(g_a^{ij}-\zeta_{ij}\right)^{-\beta_{ij}}}{\sum_{l=1}^{\left|R_{ij}\right|-1}\sum_{s=l+1}^{\left|R_{ij}\right|}\left(1-\sigma_{ls}^{ij}\right)\left[\left(g_l^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}}+\left(g_s^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ls}^{ij}}}\right]^{1-\sigma_{ls}^{ij}}}\ ,\forall a\in R_{ij}^{A}, ij\in IJ.$$

The other pairs can be written similarly, which is consistent with Eq. (32). For the conditional probability, we have

Auto-Auto

$$P_{a|ab}^{ij}=\frac{\left(g_a^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}{\left(g_b^{ij}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}P_{b|ab}^{ij}\ ,\forall a,b\in R_{ij}^{A}, ij\in IJ,$$

Auto-Transit

$$P_{a|k(n)a}^{ij}=1,\ \forall k(n)\in R_{ij}^{T}, n\in N, a\in R_{ij}^{A}, ij\in IJ,$$

Transit-Auto

$$P_{k(n)|k(n)a}^{ij}=0,\forall k(n)\in R_{ij}^{T}, n\in N, a\in R_{ij}^{A}, ij\in IJ,$$

Transit-Transit

$$P_{k(n)|k(n)r(m)}^{ij}=0,\forall k(n)\neq r(m)\in R_{ij}^{T(n)}, n,m\in N, ij\in IJ.$$

Then, we have

$$P_{a|ab}^{ij} + P_{a|ab}^{ij} \frac{\left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}{\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}} = 1$$

$$P_{a|ab}^{ij} = \frac{\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}{\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}} + \left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}, \forall a, b \in R_{ij}^A, ij \in IJ.$$

From Eq. (31), the above equality constraints of the marginal and conditional probabilities give the PCW model in Eq. (34) for this case.

(b) Only one P&R potential site between an O-D pair *ij* is opened, i.e., $x_n = 1$. The following equation system are active.

Auto-Auto: Auto-Auto

$$P_{ab}^{ij} = \frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}} + \left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}} + \left(g_d^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}} P_{cd}^{ij},\ \forall a \neq b, c \neq d \in R_{ij}^A, ij \in IJ.$$

Auto-Auto: Transit-Auto

$$P_{cd}^{ij} \leq \frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(g_c^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}} + \left(g_d^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(1-\sigma_{ijk(n)a}\right)\left[\left(g_{ijk(n)} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ijk(n)a}}} + \left(g_{ija} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ijab}}}\right]^{1-\sigma_{ijk(n)a}}} P_{k(n)a}^{ij},$$

$$\forall k(n) \in R_{ij}^T, n \in N, a, c \neq d \in R_{ij}^A, ij \in IJ,$$

Auto-Auto: Transit-Transit

$$P_{ab}^{ij} \leq \frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_a^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}} + \left(g_b^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(g_{k(n)}^{ij} - \zeta_{ij}\right)^{-\beta_{ij}}} P_{k(n)r(m)}^{ij},$$

$$\forall k(n) \neq r(m) \in R_{ij}^T, n, m \in N, a \neq b \in R_{ij}^A, ij \in IJ.$$

Transit-Auto: Auto-Auto

$$P_{k(n)a}^{ij} \leq \frac{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}+\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}}{\left(1-\sigma_{cd}^{ij}\right)\left[\left(\mathrm{g}_{c}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(\mathrm{g}_{\mathrm{d}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}} P_{cd}^{ij},$$

$$\forall k(n) \in R_{ij}^{T}, n \in N, a, c \neq d \in R_{ij}^{A}, ij \in IJ,$$

<u>Transit-Auto: Transit-Auto</u>

$$P_{k(n)a}^{ij} \leq \frac{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}+\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}}{\left(\mathrm{g}_{\mathrm{c}}^{\mathrm{ij}}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)c}^{ij},$$

$$\forall k(n), l(o) \in R_{ij}^{T}, n, o \in N, a, c \in R_{ij}^{A}, ij \in IJ.$$

<u>Auto-Transit: Transit-Transit</u>

$$P_{l(o)a}^{ij} \leq \frac{\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{k(n)r(m)}^{ij}, \forall k(n) \neq r(m), l(o) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ,$$

<u>Transit-Transit: Auto-Auto</u>

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{ab}^{ij}\right)\left[\left(\mathrm{g}_{a}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(\mathrm{g}_{\mathrm{b}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}} P_{ab}^{ij},$$

$$\forall k(n) \neq r(m) \in R_{ij}^{T}, n, m \in N, a, b \in R_{ij}^{A}, ij \in IJ,$$

<u>Transit-Transit: Transit-Auto</u>

$$P_{k(n)r(m)}^{ij} \leq \frac{\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{-\beta_{ij}}}{\left(\mathrm{g}_{a}^{\mathrm{ij}}-\zeta_{ij}\right)^{-\beta_{ij}}} P_{l(o)a}^{ij}, \forall k(n) \neq r(m), l(o) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ.$$

<u>Transit-Transit: Transit-Transit</u>

$$P_{k(n)r(m)}^{ij} \in [0,1] \ \forall k(n) \neq r(m) \in R_{ij}^{T}, n, m \in N, ij \in IJ.$$

Then, we have equality constraints as follows.

$$P_{cd}^{ij} = \frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(\mathrm{g}_{c}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(\mathrm{g}_{\mathrm{d}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}+\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}} P_{k(n)a}^{ij},$$

$$\forall k(n) \in R_{ij}^{T}, n \in N, a, c \neq d \in R_{ij}^{A}, ij \in IJ,$$

$$P_{k(n)r(m)}^{ij} = \frac{\left(g_{k(n)}^{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}{\left(1-\sigma_{ab}^{ij}\right)\left[\left(g_{a}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}} + \left(g_{b}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}} P_{ab}^{ij},$$

$$\forall k(n) \neq r(m) \in R_{ij}^{T}, n, m \in N, a, b \in R_{ij}^{A}, ij \in IJ,$$

$$P_{k(n)a}^{ij} = \frac{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(g_{k(n)}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}} + \left(g_{a}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}}{\left(g_{c}^{ij} - \zeta_{ij}\right)^{-\beta_{ij}}} P_{cl(o)}^{ij},$$

$$\forall k(n), l(o) \in R_{ij}^{T}, n, o \in N, a, c \in R_{ij}^{A}, ij \in IJ,$$

$$P_{k(n)r(m)}^{ij} = \frac{\left(g_{k(n)}^{ij} - \zeta_{ij}\right)^{-\beta_{ij}}}{\left(g_{a}^{ij} - \zeta_{ij}\right)^{-\beta_{ij}}} P_{al(o)}^{ij}, \forall k(n) \neq r(m), l(o) \in R_{ij}^{T}, n, m, o \in N, a \in R_{ij}^{A}, ij \in IJ.$$

For the conditional probability, we have

<u>Auto-Auto</u>

$$P_{a|ab}^{ij} = \frac{\left(g_{a}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}{\left(g_{b}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}} P_{b|ab}^{ij}, \forall a, b \in R_{ij}^{A}, ij \in IJ,$$

<u>Auto-Transit and Transit-Auto</u>

$$P_{a|k(n)a}^{ij} = \frac{\left(g_{a}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}}{\left(g_{k(n)}^{ij} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}} P_{k(n)|k(n)a}^{ij}, \forall k(n) \in R_{ij}^{T}, n \in N, a \in R_{ij}^{A}, ij \in IJ,$$

<u>Transit-Transit</u>

$$P_{k(n)|k(n)r(m)}^{ij} = 1, \forall k(n), r(m) \in R_{ij}^{T}, n \neq m \in N, ij \in IJ.$$

Following the similar procedure as the previous scenario, we have the PCW choice behavior in Eq. (34) for this case.

(c) All P&R potential sites between an O-D pair *ij* are opened. We have the following equations, i.e.,

$$P_{ab}^{ij}=\frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(\mathrm{g}_{a}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(\mathrm{g}_{\mathrm{b}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(1-\sigma_{cd}^{ij}\right)\left[\left(\mathrm{g}_{c}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(\mathrm{g}_{\mathrm{d}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}P_{cd}^{ij},\quad \forall a\neq b,c\neq d\in R_{ij}^{A},ij\in IJ.$$

$$P_{cd}^{ij}=\frac{\left(1-\sigma_{cd}^{ij}\right)\left[\left(\mathrm{g}_{c}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}+\left(\mathrm{g}_{\mathrm{d}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{cd}^{ij}}}\right]^{1-\sigma_{cd}^{ij}}}{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}+\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}}P_{k(n)a}^{ij},$$

$$\forall k(n)\in R_{ij}^{T},n\in N,a,c\neq d\in R_{ij}^{A},ij\in IJ,$$

$$P_{ab}^{ij}=\frac{\left(1-\sigma_{ab}^{ij}\right)\left[\left(\mathrm{g}_{a}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}+\left(\mathrm{g}_{\mathrm{b}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}\right]^{1-\sigma_{ab}^{ij}}}{\left(1-\sigma_{k(n)r(m)}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}+\left(\mathrm{g}_{\mathrm{r(m)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}}P_{k(n)r(m)}^{ij},$$

$$\forall k(n)\neq r(m)\in R_{ij}^{T},n,m\in N,a\neq b\in R_{ij}^{A},ij\in IJ,$$

$$P_{k(n)a}^{ij}=\frac{\left(1-\sigma_{k(n)a}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}+\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}\right]^{1-\sigma_{k(n)a}^{ij}}}{\left(1-\sigma_{l(o)c}^{ij}\right)\left[\left(\mathrm{g}_{c}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)c}^{ij}}}+\left(\mathrm{g}_{\mathrm{l(o)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)c}^{ij}}}\right]^{1-\sigma_{l(o)c}^{ij}}}P_{l(o)c}^{ij},$$

$$\forall k(n),l(o)\in R_{ij}^{T},n,o\in N,a,c\in R_{ij}^{A},ij\in IJ,$$

$$P_{l(o)a}^{ij}=\frac{\left(1-\sigma_{l(o)a}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{a}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)a}^{ij}}}+\left(\mathrm{g}_{\mathrm{l(o)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)a}^{ij}}}\right]^{1-\sigma_{l(o)a}^{ij}}}{\left(1-\sigma_{k(n)r(m)}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}+\left(\mathrm{g}_{\mathrm{r(m)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}}P_{k(n)r(m)}^{ij},$$

$$\forall k(n)\neq r(m),l(o)\in R_{ij}^{T},n,m,o\in N,a\in R_{ij}^{A},ij\in IJ,$$

$$P_{k(n)r(m)}^{ij} = \frac{\left(1-\sigma_{k(n)r(m)}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{k(n)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}+\left(\mathrm{g}_{\mathrm{r(m)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}\right]^{1-\sigma_{k(n)r(m)}^{ij}}}{\left(1-\sigma_{l(o)s(p)}^{ij}\right)\left[\left(\mathrm{g}_{\mathrm{l(o)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)s(p)}^{ij}}}+\left(\mathrm{g}_{\mathrm{s(p)}}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{l(o)s(p)}^{ij}}}\right]^{1-\sigma_{l(o)s(p)}^{ij}}} P_{l(o)s(p)}^{ij},$$

$$\forall k(n) \neq r(m), l(o) \neq s(p) \in R_{ij}^{T}, n, m, o, p \in N, ij \in IJ,$$

$$P_{a|ab}^{ij} = \frac{\left(\mathrm{g}_{a}^{\mathrm{ij}} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}}{\left(\mathrm{g}_{b}^{\mathrm{ij}} - \zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{ab}^{ij}}}} P_{b|ab}^{ij}, \forall a, b \in R_{ij}^{A}, ij \in IJ,$$

$$P_{a|k(n)a}^{ij} = \frac{\left(\mathrm{g}_{a}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}}{\left(\mathrm{g}_{k(n)}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)a}^{ij}}}} P_{k(n)|k(n)a}^{ij}, \forall k(n) \in R_{ij}^{T}, n \in N, a \in R_{ij}^{A}, ij \in IJ,$$

$$P_{k(n)|k(n)r(m)}^{ij} = \frac{\left(\mathrm{g}_{k(n)}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}}{\left(\mathrm{g}_{r(m)}^{\mathrm{ij}}-\zeta_{ij}\right)^{\frac{-\beta_{ij}}{1-\sigma_{k(n)r(m)}^{ij}}}} P_{r(m)|k(n)r(m)}^{ij},$$

$$\forall k(n), r(m) \in R_{ij}^{T(n)}, n \neq m \in N, ij \in IJ,$$

Following the similar procedure, we have the PCW choice behavior in Eq. (34) for this case. Thus, the MILP in Eq. (2) – Eq. (4) and Eq. (37) – Eq. (67) generates the maximum number of P&R users under the PCW travel choice behavior. This completes the proof. □

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